Homotopy Type of Graph Configuration Spaces

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(joint work with M. Bouzouita)

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Graph Configuration Spaces

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an abstract graph with

- Vertices $V(\Gamma) = \{v_1, \ldots, v_n\}$ (finite set)
- Edges $E(G)$. An element of $E(G)$ is of the form $\{v_i, v_j\}, i \neq j$.

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- Different labeling of vertices produce homeomorphic spaces.
- WLOG we can assume our graphs to be connected, since

$$\text{Conf}_{\Gamma_1 \sqcup \Gamma_2}(X) \cong \text{Conf}_{\Gamma_1}(X) \times \text{Conf}_{\Gamma_2}(X)$$

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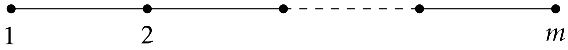

$$\mathrm{Conf}_{L_m}(X) = \{(x_1, x_2, \ldots, x_m) \in X^m \mid x_i \neq x_{i+1}\ , 1 \leqslant i < m\}$$

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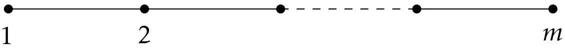

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We have a homeomorphism

$$\mathrm{Conf}_{L_m}(\mathbb{R}^N) \xrightarrow{\cong} \mathbb{R}^N \times (\mathbb{R}^N - \{0\}) \times \cdots \times (\mathbb{R}^N - \{0\})$$
$$(x_1, \ldots, x_m) \longmapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_m - x_{m-1})$$

so

$$\boxed{\mathrm{Conf}_{L_m}(\mathbb{R}^N) \simeq \prod^{m-1} S^{N-1}}$$

If $\mathcal{G}$ is the category whose objects are undirected simple graphs, which are vertex labeled and whose morphisms are the graph homomomorphisms.

Fix $X = \mathbb{R}^N$ (or any other good enough space).

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The study of $\text{Conf}_\Gamma(X)$ was motivated by the study of INVARIANTS OF GRAPHS in data analysis.

The Chromatic Polynomial

Let Γ be a graph, and $\lambda \in \mathbb{N}$. A mapping $f : V(G) \to \{1, 2, \ldots, \lambda\}$ is called a λ-colouring of Γ if $f(i) \neq f(j)$ whenever $\{i, j\} \in E(\Gamma)$.

The number of distinct λ-colourings of Γ is denoted by $\pi_\Gamma(\lambda)$, and this is a polynomial in λ (the chromatic polynomial).

Examples:

- If $\Gamma = L_n$ is the line graph with n vertices, then $\pi_{L_n}(\lambda) = \lambda^n(\lambda - 1)^{n-1}$.
- If $\Gamma = K_n$ is the complete graph on n-vertices, then $\pi_{K_n}(\lambda)$ is the falling factorial $\pi_{K_n}(\lambda) = \lambda(\lambda - 1)\cdots(\lambda - n + 1)$

Let $\chi_c(X)$ denote the Euler characteristic of a space X (with compact supports).

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Theorem: (Eastwood-Huggett, Kallel-Taamallah)

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Corollary:

$$\chi(\mathrm{Conf}_\Gamma(\mathbb{R}^n)) = (-1)^{n|V|}\pi_\Gamma((-1)^n)$$

This key result is the starting point of this work.

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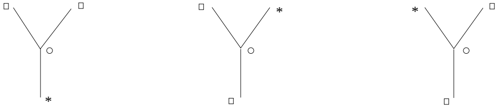

Different coloring configurations of the Y-graph.

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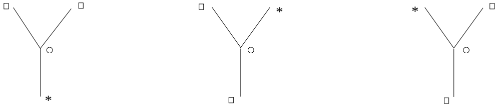

Different coloring configurations of the Y-graph.

Conf(X,T) consists of all tuples (x_1,x_2,x_3,x_4) with $x_i \neq x_1$

This space stratify as follows

$$\{(y,x,x,x)\}\ ,\ \{(y,x,x,z)\}\ ,\{(y,x,z,x)\}\ ,\{(y,x,z,z)\}\ ,\ \{(y,x,z,t)\}$$

where different letters mean *distinct* entries in X.

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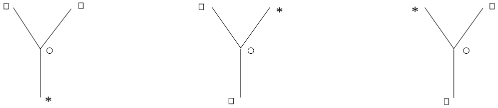

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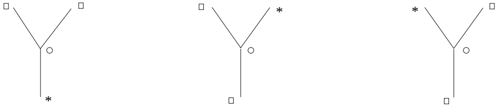

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We have the stratification

$$\mathrm{Conf}(X, \Gamma) \doteqdot \mathrm{Conf}(X, 2) \sqcup 3\mathrm{Conf}(X, 3) \sqcup \mathrm{Conf}(X, 4)$$

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$$\pi_\Gamma(\lambda) = \sum_{i=1}^{n} (-1)^{n-i} a_i(\Gamma) \lambda^i \tag{1}$$

where the coefficient $a_i(\Gamma)$ for $0 < i < n$ counts the number of spanning subgraphs of Γ that have exactly $n - i$ edges and that contain no broken circuits.

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The Linear Term: The number $a_1(\Gamma)$ has several interpretations. It is the number of "spanning trees with no broken circuits" of Γ.

Results:

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- An orientation of a graph $\Gamma = (V, E)$ is an assignment of a direction (i.e. arrow) to each edge $\{i, j\}$, denoted by $i \to j$ or $j \to i$, as the case may be.
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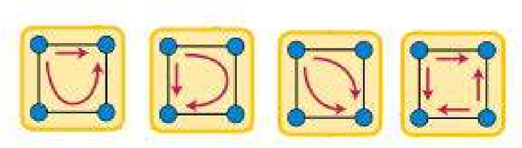
Theorem (Stanley):
Let $|A(\Gamma, v_0)|$ be the number of all acyclic orientations with a unique sink (or source) v_0. Then

$$|A(\Gamma, v_0)| = a_1(\Gamma)$$

and this number is independent of the choice of v_0.

Example: Let C_4 be the square graph. The acyclic orientations of C_4 with a single source are displayed below



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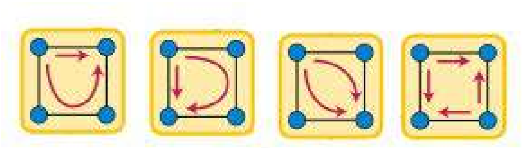

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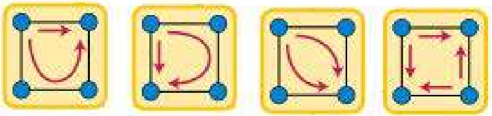

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The chromatic polynomial of C_4 is

$$\chi_{C_4}(\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

so $\boxed{a_1 = 3}$.

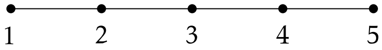
Bond Partitions

Given a graph Γ with vertex set $V(\Gamma)$, a "connected partition" or "a bond partition" of Γ is any unordered set partition of $V(\Gamma)$, written $B_1|B_2|\cdots|B_k$, where the B_i's are the blocks which are assumed to be the vertices of a connected subgraph Γ_i of Γ.

The orderof the B_i's appearing in this notation is immaterial.

The integer k, $1 \leqslant k \leqslant |V(\Gamma)|$, is the length of the partition.

Example: Consider the line graph L_5 on 5 vertices labeled $1, 2, \ldots, 5$.



The bond partitions of length 3 of L_5 are listed lexicographically as follows:

$$1|2|345\ ,\ 1|5|234\ ,\ 4|5|123\ ,\ 1|23|45\ ,\ 3|12|45\ ,\ 5|12|34$$

Denote the set of all connected partitions of Γ having length k by $\mathcal{B}_k(\Gamma)$.

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Finally, write the set of all subgraph partitions of Γ as

$$\boxed{\mathcal{B}(\Gamma) = \bigcup_{1\leqslant k\leqslant n} \mathcal{B}_k(\Gamma)}$$

Main Result

Theorem (S.K and M. Bouzouita):
Let Γ be a finite simple graph, $N \geqslant 2$ and $m = |V(\Gamma)|$. Then stably (after one suspension)

$$\mathit{Conf}_\Gamma(\mathbb{R}^N)_+ \simeq_s \bigvee_{B \in \mathcal{B}(\Gamma)} \left(S^{(m-|B|)(N-1)}\right)^{\bigvee a_1(B)}$$

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Consequences:

- The homology of the configuration space is **torsion free** and it is concentrated in degrees that are a multiple of $N-1$.

- The first non-zero betti number is $b_{N-1} = |E(\Gamma)|$ (number edges).

Corollary: *The Poincaré polynomial of* $Conf_\Gamma(\mathbb{R}^N)$ *is*

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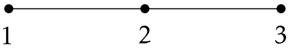
The Poincaré series is not computed and it is not readily obtainable.

Simple Example: Consider the line graph with THREE nodes labeled $1, 2, 3$. The graph is in $\mathbb{R}^3$ ($N = 3$)



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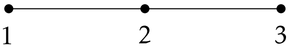

Bond	**length**	**Whitney number**	**Sphere summand**
B	$\lvert B\rvert$	$a_1(B)$	$S^{(3-\lvert B\rvert)(N-1)}$
1|2|3	3	1	S^0
12|3	2	1	S^2
1|23	2	1	S^2
123	1	1	S^4

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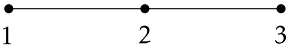

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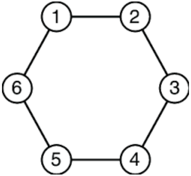
Consequently $\boxed{\mathrm{Conf}_{L_3}(\mathbb{R}^3) \simeq_s S^2 \vee S^2 \vee S^4}$

This is consistent with the earlier result $\mathrm{Conf}_{L_3}(\mathbb{R}^3) \cong S^2 \times S^2$.

Second Example: The Cyclic Graphs

Let $\Gamma = C_m$ be the cyclic graph on m-vertices.



The associated configuration space is called the "cyclic configuration space"

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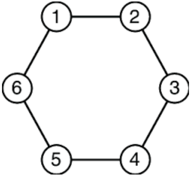

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Literature: This space has been studied by M. Farber and S. Tabachnikov in connection with the problem of finding upper bounds to the number of periodic trajectories of high dimensional billiard problems.

The following corollary was originally obtained using sophisticated methods (Leray Spectral Sequence).

Theorem (Farber-Tabashnikov):

$$P_{C_m}(\mathbb{R}^N) = (t^{N-1} + 1)^m - t^{(m-1)(N-1)} - t^{m(N-1)}$$

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Let $\mathcal{B}_k(C_m)$ be the set of all bond partitions of the cyclic graph C_m, $k \geqslant 2$. Then $|\mathcal{B}_k(C_m)| = \binom{m}{k}$.

Third Example: The complete graph

Let $\Gamma = K_m$ be the complete graph on m-vertices. Then one recovers the following well-known result.

Theorem (Stable Arnold-Cohen):

$$\mathit{Conf}_{K_m}(\mathbb{R}^N)_+ \simeq_s \bigvee_{k=0}^{m-1} \left(S^{k(N-1)}\right)^{\left[{m \atop m-k}\right]}$$

where $\left[{m \atop m-k}\right]$ *is the unsigned Stirling numbers of the first kind corresponding to the number of permutations of* m *elements with* k *disjoint cycles.*

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How do we get this splitting from our main result?

• When $\Gamma = K_m$ is a complete graph, any collection of vertices makes up a complete subgraph of Γ.

• For a complete graph K_r, $a_1(K_r) = (r-1)!$

Remark: The stable Arnold-Cohen splitting is classically deduced from the fact that the homology of $\mathrm{Conf}_m(\mathbb{R}^N)$ is generated by classes of products of spheres (Cohen-Taylor, Fadell-Husseini, Salvatore), so that

$$\mathrm{Conf}_m(\mathbb{R}^N)_+ \simeq_s \prod_{k=1}^{m-1} \left(S^{N-1}\right)^{\vee k}$$

Poset topology

This is an extremely powerful area of (combinatorial) topology.

First initiated by H. Whitney and G. Rota in the 50s, it was later vastly developed by R.P. Stanley in the 70s, and a bit later by A. Bjorner, M. Wachs and many others.

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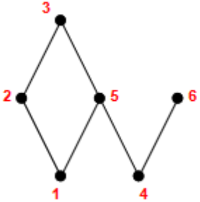
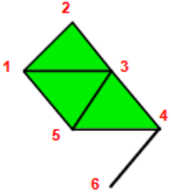
The **order complex** of P is the simplicial complex whose simplices are the chains of P

$$\Delta(P) = \{\{i_1, \dots, i_k\}, i_1 < i_2 < \cdots < i_k \text{ in } P\}$$



P



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Bond Poset

The connection with graph arrangements is through the bond lattice (A lattice is a poset where every two points have a meet and a join).

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The connection with graph arrangements is through the bond lattice (A lattice is a poset where every two points have a meet and a join).

Let Γ be a graph. Construct the poset Π_Γ:

- Every bond partition $B := B_1|B_2|\cdots|B_k$ is an element of $\Pi(\Gamma)$.
- $B \leqslant B'$ if B' is a coarsening of B.

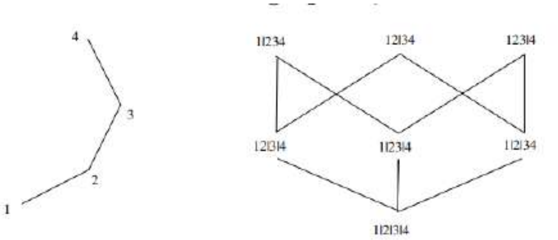


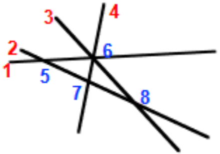
Figure: The Bond lattice Π_{L_4} of the line graph L_4.

The Graphic Arrangements

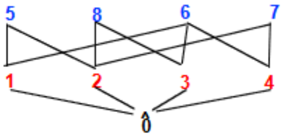
An arrangement of affine (or linear subspaces) A_i in $\mathbb{R}^N$ is any finite collection of such: $A = \{A_i\}_{i \in I}$.

An arrangement A gives rise to a poset of intersections (also called the “intersection semi-lattice”) $\mathcal{L}(A)$.

The elements of $\mathcal{L}(A)$ are the A_i’s and their intersections. The order is given by REVERSE inclusion so that $x \leqslant y$ if $y \subset x$.



A



$L(A)$

Let Γ be a simple graph, and $|V(\Gamma)| = m$.

Then obviously $\mathrm{Conf}_\Gamma(\mathbb{R}^n)$ is the complement of a subspace arrangement

$$(\mathbb{R}^n)^m - \bigcup A_{ij}$$

where

$$A_{ij} = \{(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m \mid x_i \neq x_i, \text{ if } \{i, j\} \in E(\Gamma)\}$$

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Observation: *The bond lattice of Γ is isomorphic to the intersection lattice of $\mathcal{L}(A)$.*

We can now use the theory subspace arrangements to compute the homology of the graph configuration spaces.

This uses a formula by Goresky and MacPherson

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Theorem: Goresky-MacPherson Formula

$$\tilde{H}^i(\text{Conf}(\mathbb{R}^N,\Gamma);\mathbb{Z}) \cong \bigoplus_{x\in\mathcal{L}(A)\setminus\{\hat{0}\}} \tilde{H}_{mN-i-\dim B(x)-2}(\Delta(\hat{0},x);\mathbb{Z})$$

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Remark: The streamlined way to think about this formula is as a direct application of Alexander duality.

Associated to the subspace arrangement $A = \{A_i\}$ in $\mathbb{R}^N$ is the *singularity link*

$$\mathcal{V}_A^0 := S^{N-1} \cap \bigcup_i A_i$$

and Alexander duality gives that

$$H^i(\mathcal{M}_A; \mathbb{F}) \cong H_{n-2-i}(\mathcal{V}_A^0; \mathbb{F}) \quad (\mathbb{F} \text{ is any field})$$

Literature: The (stable) homotopy type of $\mathcal{V}_A^0$ was computed by Ziegler and Zivaljevic (and Kozlov). The answers are phrased in terms of the lower intervals in the intersection lattice L_A of the subspace arrangement A. For general subspace arrangements, these lower intervals in L_A can have arbitrary homotopy type.

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To prove our Theorem, we first get the homology, and to that end, we need understand the homology of the intervals!

Strategy:

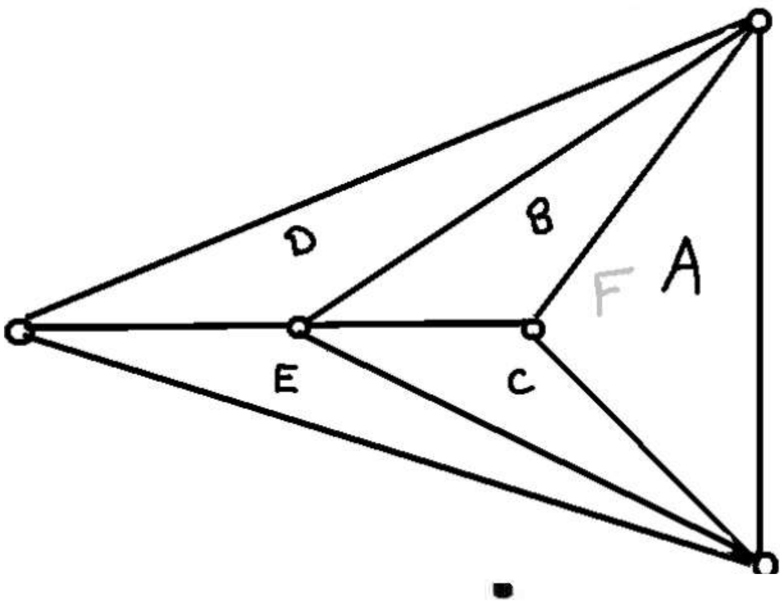
- *Show that each interval is the homotopy type of a wedge of spheres (This uses shellability).*
- *Knowing the dimension of those spheres, we can deduce their number from the Euler characteristic (so link to chromatic).*

Shellability

A facet is a maximal face of a simplicial complex.

A simplex is pure if all facets have the same dimension.

A shelling is a linear order on the facets with a special condition: Pick a first facet. Then each new facet added to the list must meet the old complex at a nonempty union of MAXIMAL proper faces.



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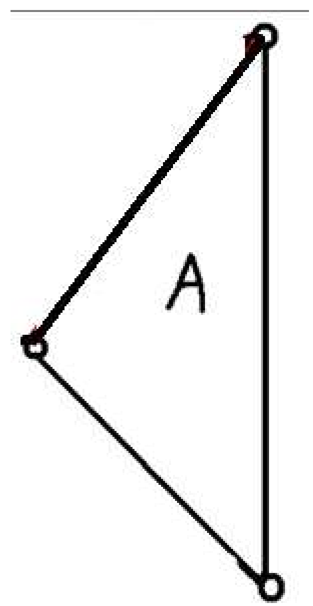

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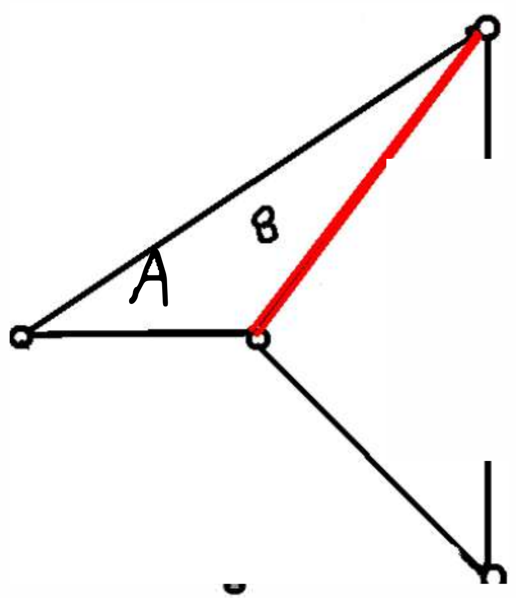

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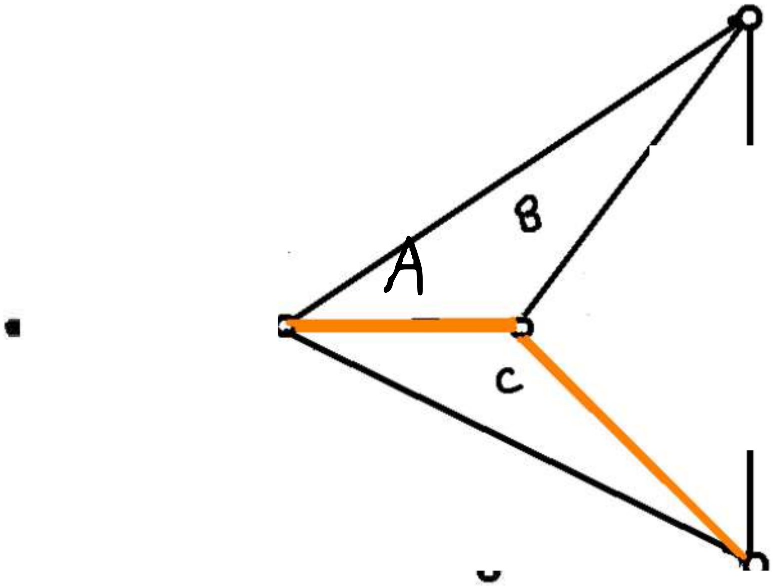

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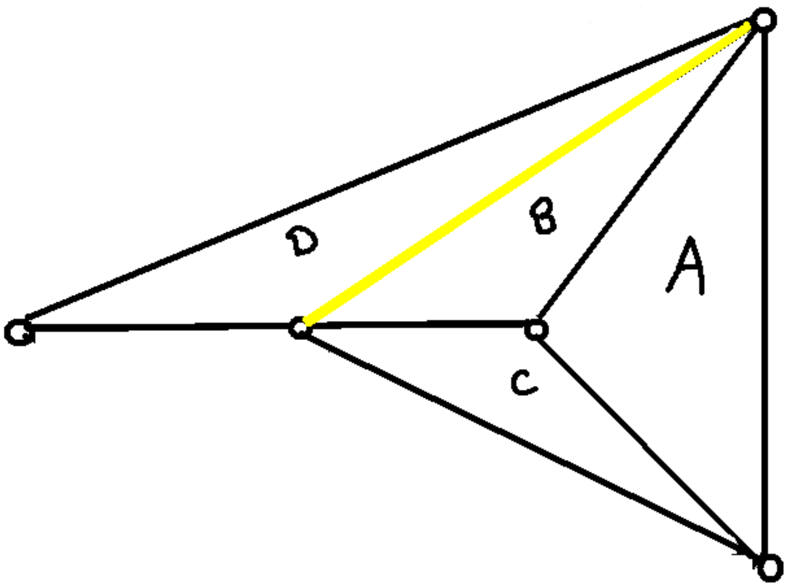

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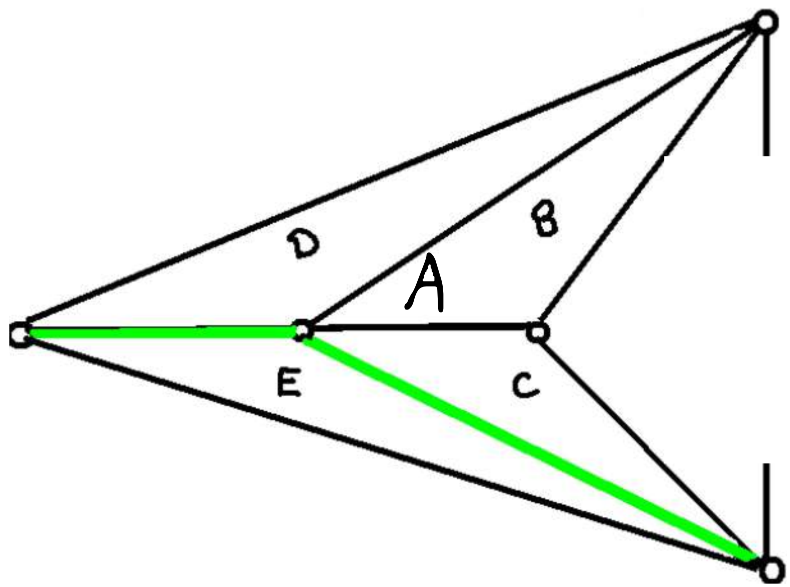

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More precisely, let $\mu(P) = \mu(\hat{0}, \hat{1})$ be the Mobius function of the poset.

Theorem: *Let $P = \mathring{P} \cup \{\hat{0}, \hat{1}\}$ be a bounded and ranked poset ($\mathring{P}$ is its proper part), and suppose that $\Delta(\mathring{P})$ is shellable. Then $\Delta(\mathring{P})$ is a wedge of $(-1)^d \mu(P)$ spheres of dimension $d = rk(P) - 2$.*

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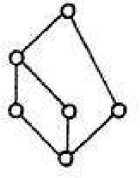
The rank of a ranked poset is the length of a maximal chain (this is well defined if P is ranked).

Example: Below is an example of a poset that is not ranked



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As a result we get the following main consequence.

Proposition: *Let Γ be a simple graph with n vertices, and let $\mathring{\Pi}_\Gamma$ be the proper part of the bond lattice. Then* $|\mathring{\Pi}_\Gamma| \simeq \bigvee^{\pm\mu(\Pi_\Gamma)} S^{n-3}$.

SO TO RECAPITULATE !

- The bond lattice is nice (ranked, shellable).
- The intervals of the bond lattice are also ranked and shellable.
- The interval $(\hat{0}, x)$, $x = B_1|\ldots|B_k$, is the POSET PRODUCT of the bond lattices of the Γ_i's (where Γ_i is the connected graph corresponding to B_i).
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BUT, where does the $a_1(\Gamma)$ (linear term of the chromatic polynomial) come from?

Mobius Inversion Let P be a finite poset and $f, g : P \to \mathbb{R}$ (or $\mathbb{Z}$). Suppose that for all $x \in P$ we have

$$f(x) = \sum_{y \geqslant x} g(y) \implies g(x) = \sum_{y \geqslant x} \mu(x, y) f(y).$$

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In our case, let P be the poset of intersections $\{A_\alpha\}$ of an arrangement $\mathcal{A}$ in X, with $\alpha < \beta$ if $A_\alpha \supset A_\beta$. Set

$$g(A_\alpha) = \chi_c \left(A_\alpha - \bigcup_{\alpha < \beta} A_\beta \right)$$

$$f(A_\alpha) = \chi_c(A_\alpha)$$

Since

$$f(A_\alpha) = \chi_c(A_\alpha) = \sum_{\beta \geqslant \alpha} g(A_\beta)$$

It follows by Mobius inversion that

Proposition:

$$\chi_c\left(X \backslash \bigcup A_\alpha\right) = \sum_{\hat{0} \leqslant \alpha \leqslant \hat{1}} \mu(\hat{0}, \alpha)\chi_c(A_\alpha)$$

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Apply this to the graph configuration space and its bond poset, we get

Corollary (Rota): *The characteristic polynomial of* Π_Γ *coincides with the chromatic polynomial*

$$\chi_\Gamma(\lambda) = \sum_{x\in\Pi_\Gamma} \mu(\hat{0},x)\lambda^{\rho(x)} \quad (2)$$

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By comparing Whitney's and Rota's formulas, we see that

$$\boxed{\mu(\Pi_\Gamma) = (-1)^{n-1} a_1(\Gamma)} \tag{3}$$

Application (Configuration Spaces with Obstacles)

For $\zeta = (p_1, \ldots, p_n) \in \text{Conf}_n(X)$.

Consider the following configuration space of points

$$\text{Conf}(X, \zeta) := \{(x_1, \ldots, x_n) \mid x_i \neq x_j, i \neq j \text{ and } x_i \neq p_i, \forall i\}$$

These are configuration of pairwise distinct points $(x_1, \ldots, x_n)$ in X such that x_i avoid p_i, for all i.

Theorem: (K - Bouzouita)

$$P_t(\mathit{Conf}(\mathbb{R}^N,\zeta)) = \frac{P_t(\mathit{Conf}(\mathbb{R}^N, K_n \square K_2))}{P_t(\mathit{Conf}(\mathbb{R}^N, K_n))}$$

In particular, the first non-trivial positive betti number is $b_{N-1} = \binom{n}{2} + n$.

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Proposition:
Suppose H *is relatively complete in* G*. Then the projection map* $\mathit{Conf}_G(\mathbb{R}^N) \longrightarrow \mathit{Conf}_H(\mathbb{R}^N)$ *is a bundle projection*.

Thank you