Quantization as a Kan extension

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Abstract

'All concepts are Kan extensions'–Saunders Mac Lane. In this note we work out a suggestion by Urs Shreiber and prove that, in a suitably discretized setup, quantization can be understood as Kan extension of the classical action.

Introduction

In 2007 Urs Schreiber mused on the blog 'The n-Category Café' that:

> Quantum mechanics, which has historically been considered as a structure internal to 0**Cat**, has a more natural formulation as a structure internal to 1**Cat**. This involves refining functions to "bundles of numbers" (namely fibered categories, or, dually, refining (action) functors to pseudofunctors) and it involves refining linear maps by spans of groupoids.
>
> In this natural formulation, quantization will no longer be a mystery – but a pushforward: the quantum propagator ($t \mapsto U(t)$) is the pushforward to a point of the classical action (pseudo)-functor.[1]

Schreiber has since collected many toy examples supporting this thesis. In this note we add the following substantial fact: It is indeed true that the quantum propagator arises as a pushforward (pushforward being Kan extension), at least in a suitably discretized setup. Or rather, we prove a very general mathematical theorem ('Theorem 1') about Kan extension of a **Vect**-valued functor defined on a groupoid. Inserting suitable functors and categories into this theorem, we get formulas that suggest we are looking at the abstract nonsense behind path integral quantization.

To explain the physical content as well as the philosophy behind our result, consider the following example. Let (M, ds^2) be a lorentzian manifold and let $\mathcal{P}_1(M)$ be the category which has as objects points $x \in M$ and as morphisms $x \to x'$ thin homotopy classes of paths from x to x' in M.[2] The classical dynamics can be encoded in a function S, called an 'action', on morphisms in $\mathcal{P}_1(M)$. Usually $S(\gamma)$ has the form of an integral over the path γ, in which case S will behave additively with respect to composition; $S(\gamma \circ \gamma') = S(\gamma) + S(\gamma')$. Because of this additive property it will induce a functor

$$e^{iS} : \mathcal{P}_1(M) \longrightarrow \mathbf{Vect},$$

which maps every point x to $\mathbf{C}$ and a path $\gamma : x \to x'$ to the linear map which is multiplication by the unit modulus complex number $\exp iS(\gamma)$. The solutions to the classical equations of motion are the paths γ which minimize the action, i.e. the paths such that $\delta[\exp iS(\gamma)] = 0$. In quantum mechanics the configurations of a particle are no longer simply positions (points) in M, but are instead wave functions $\psi : M \to \mathbf{C}$, i.e. points in the function space $\mathbf{C}^M$. The classical equations of motion are replaced by linear maps $U(T) : \mathbf{C}^M \to \mathbf{C}^M$, giving the proper time T-evolutions of configurations according to the functional integral formula

$$(U(T)\psi)(y) = \int_M dx \int D\gamma \, e^{iS(\gamma)} \psi(x).$$

The functional integral is over all paths γ from x to y with proper time $\tau(\gamma) := \int_\gamma ds = T$. Replacing $\mathbf{C}^M$ by a smaller vector space $\mathcal{H}$, the maps $U(T)$ ($T \in \mathbf{R}$) assemble to define a functor

$$Z : \mathcal{P}_1(\mathbf{R}) \longrightarrow \mathbf{Vect},$$

[1] Schreiber 2007.

[2] The reason one wants to consider paths up to thin homotopy is that the composition of two smooth paths need not be smooth, but it is always thinly homotopic to a smooth path.

which sends every 'time instant' $s \in \mathbf{R}$ to the vector space $\mathcal{H}$ of configurations and sends a path $s \to s+T$ to the linear map $U(T)$. The passage from $\exp iS$ to Z is known as (path integral) quantization. Looking at the formulas, one sees that Z is defined on a much smaller category than the classical action $\exp iS$ – a lot of information is forgotten in the quantization. On the other hand, the configuration space $\mathcal{H} \subset \mathbf{C}^M$ looks almost like the sum $\bigoplus_{x \in M} \exp iS(x) = \bigoplus_x \mathbf{C}$ and the propagators $U(T)$ have the form of 'sums' $\sum \exp iS(\gamma)$. This summing over all possible choices behaves mathematically as if it Z is trying to compensate for the loss of information in the passage from $\mathcal{P}_1(M)$ to $\mathcal{P}_1(\mathbf{R})$. Category-theoretically, Z looks like a colimit, or Kan extension.

Let $\tilde{X}$ be the groupoid which has as objects pairs (x, s), where $x \in M$ and $s \in \mathbf{R}$, and as morphisms $(x, s) \to (x', s')$ paths $\gamma : x \to x' \in \mathcal{P}_1(M)$ of proper time $\tau(\gamma) = s' - s$. We think of an object (x, s) as a particle at position x at proper time s, and of a morphism $(x, s) \to (x', s')$ as a hypothetical evolution of the particle. The proper time τ extends to a functor from $\tilde{X}$ to $\mathcal{P}_1(\mathbf{R})$, mapping a diagram $(x, s) \to (x', s+T)$ to $T : s \to s+T$. Reinterpreting the classical action $\exp iS : \mathcal{P}_1(M) \to \mathbf{Vect}$ as a functor on $\tilde{X}$ in the obvious way, we get a diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{e^{iS}} & \mathbf{Vect} \\ \tau \big\downarrow & & \\ \mathcal{P}_1(\mathbf{R}). & & \end{array}$$

The quantum theory is supposed to be a functor $Z : \mathcal{P}_1(\mathbf{R}) \to \mathbf{Vect}$. One could of course proceed to directly write down a functor Z, using physical arguments to define its form, but this would be very ad hoc mathematically. We show that there is another way, a way which maybe is physically ad hoc but is mathematically canonical.

Given a diagram of functors

$$\begin{array}{ccc} C & \xrightarrow{F} & C'' \\ G \big\downarrow & & \\ C', & & \end{array}$$

there are really only two distinguished ways to produce a functor $H : C' \longrightarrow C''$, namely the left Kan extension

$$\mathrm{Lan}_G F$$

of F by G or the right Kan extension

$$\mathrm{Ran}_G F$$

of F by G. The left Kan extension is, if it exists, defined by the property that there exists a natural transformation

$$\lambda : F \to \mathrm{Lan}_G F \circ G,$$

universal in the sense that for any other functor $H : C' \longrightarrow C''$ and natural transformation $h : F \to H \circ G$, there exists a unique transformation $\tilde{h} : \mathrm{Lan}_G F \to H$ such that $h = \tilde{h}_G \circ \lambda$. The right Kan extension is dually characterized by the existence of a universal natural transformation

$$\rho : \mathrm{Ran}_G F \circ G \to F.$$

Both versions of Kan extension can be thought of as providing 'best approximations' to the problem of finding a functor H such that $H \circ G = F$, i.e. to the problem of extending the domain of F from C to C' by G, or to 'pushing forward' F along G to a functor defined on C'. One may think of the left extension as an 'approximation from below' and of the right extension as 'approximation from above'. (The precise meaning of from 'above' and from 'below' are contained in the universal natural transformations.)

With $\tilde{X}$, τ and $\exp iS$ as above, we prove the following: Let X be a finite subgroupoid of $\tilde{X}$. Then

> *Both Kan extensions* $\mathrm{Lan}_\tau \exp iS$ *and* $\mathrm{Ran}_\tau \exp iS$ *of the classical action* $\exp iS$ *(restricted to X) along the proper time projection* $\tau : X \to \mathcal{P}_1(\mathbf{R})$ *are equal to a functor*
>
> $$Z : \mathcal{P}_1(\mathbf{R}) \longrightarrow \mathbf{Vect}$$

which maps an 'instant' $s \in \mathbf{R}$ to a certain subspace $Z(s)$ of $\mathbf{C}^{\mathrm{Ob}X}$ and an 'evolution' $T : s \to s + T$ to the linear map which maps $(\psi(x))_{x\in X}$ to the vector with components

$$(Z(T)\psi)(y) = \sum_{x\in X} \sum_{\substack{\gamma:x\to y \\ \tau(\gamma)=T}} w_T e^{iS(\gamma)} \psi(x).$$

The subspace $Z(s)$ of $\mathbf{C}^{\mathrm{Ob}X}$ is the space of functions ψ invariant under $T = 0$ propagation, i.e. the space of ψ satisfying $Z(0)\psi = \psi$. The factor w_T is a weight factor; essentially a normalization constant.

Hence, at least in a discretized setup, the quantum theory (as encoded by Z) is in a precise mathematical sense the best approximation that there is to the problem of formulating the classical dynamics as a theory solely formulated on the worldline of the particle. The mathematical result ('Theorem 1') underpinning the above result is general enough to produce similar 'quantization formulas' for discretized field theories as well (see the section 'Examples'), yieldig similar interpretations of what quantization is supposed to be mathematically. We believe that our interpretation of the moral of quantization, suggested by the Kan extension formula, is quite general. Usually, one has a classical 'action' of some kind defined for manifolds *with some extra structure*, e.g. a riemannian metric, a symplectic form, a principal bundle, or etc. Quantization is what happens when one tries to assign that same action to a manifold that *does not have that structure*! Hence one has to mathematically compensate for this by *summing over all possible structures* of the specified type.

The contents of the paper are as follows. The first section, 'Section 1', is devoted to a proof of our main mathematical result, which is a general result concerning Kan extension of a **Vect**-valued functor defined on a groupoid. 'Section 2' contains a number of examples elucidating the mathematical constructions and its relation to quantization. In 'Section 3' we again return to purely mathematical considerations, discussing an alternative way to arrive at the Kan extension formulas of 'Section 1'. At the end, in the section 'Conclusions', we try to put our results in some perspective and sketch some venues of possible further developement.

1 The main theorem

This section is devoted to a proof of our main mathematical result.

Consider a diagram of functors

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f} & \mathbf{Vect} \\ {\scriptstyle g}\big\downarrow & & \\ \mathscr{B}, & & \end{array}$$

where $\mathscr{A}$ is a groupoid satisfying the following two conditions:

(i) The set theoretic fiber $g^{-1}(b)$ is a finite set for every $b \in \mathscr{B}$.

(ii) For every object $a \in \mathscr{A}$ and morphism $\beta \in \mathrm{Ar}\mathscr{B}$ with $\mathrm{cod}(\beta) = g(a)$, the number $w_\beta(a)$ defined by the relation

$$w_\beta(a)^{-1} := \sum_{a'\in\mathscr{A}} |\{\alpha : a' \to a \mid g(\alpha) = \beta\}|$$

is a finite number. (Here, and elsewhere, $|\cdot|$ denotes the cardinality of a set.)

Under the above assumptions, define for each object b in $\mathscr{B}$ a vector space

$$\tilde{Z}(b) := \bigoplus_{a\in g^{-1}(b)} f(a),$$

and for each morphism $\beta : b \to b'$ a linear map $\tilde{Z}(\beta) : \tilde{Z}(b) \to \tilde{Z}(b')$, defined by the equation

$$\pi_{f(a')} \circ \tilde{Z}(\beta) = \sum_{a\in g^{-1}(b)} \sum_{\substack{\alpha:a\to a' \\ g(\alpha)=\beta}} w_\beta(a') f(\alpha),$$

with $\pi_{f(a')}$ the projection of $\tilde{Z}(b')$ onto the $f(a')$-component. For each $b \in \mathscr{B}$, put $\wp_b := \tilde{Z}(1_b)$.

It is easily verified that $\wp_b : \tilde{Z}(b) \to \tilde{Z}(b)$ is idempotent (for all b), and that for all morphisms $\beta : b \to b'$, $\tilde{Z}(\beta) \circ \wp_b = \tilde{Z}(\beta)$ and $\wp_{b'} \circ \tilde{Z}(\beta) = \tilde{Z}(\beta)$. It follows that $\tilde{Z}$ induces a functor

$$Z : \mathscr{B} \longrightarrow \mathbf{Vect},$$

by $Z(b) := \mathrm{Im}(\wp_b) \subset \tilde{Z}(b)$ and $Z(\beta) := \tilde{Z}(\beta)|_{Z(b)}$.

With notation and assumptions as above, we prove the following theorem.

Theorem 1. The two Kan extensions

$$\mathrm{Lan}_g f \qquad \text{and} \qquad \mathrm{Ran}_g f$$

of f along g coincide as functors – both are equal to the functor Z. The components

$$\lambda(a) : f(a) \longrightarrow Z \circ g(a),$$

of the universal natural transformation $\lambda : f \to Z \circ g$ (witnessing Z as the left Kan extension) are given as $\lambda(a) = \wp_{g(a)} \circ \iota_{f(a)}$. Dually, the components of the universal natural transformation $\rho : Z \circ g \to f$ are $\rho(a) = \pi_{f(a)} \circ \wp_{g(a)}$.

1.1 The left Kan extension

In this section we prove that the left Kan extension of f along g is as described in 'Theorem 1'.

The left Kan extension $\mathrm{Lan}_g f$ is given pointwise by the formula

$$\mathrm{Lan}_g f(b) = \mathrm{colim}\big((g \downarrow b) \xrightarrow{k} \mathscr{A} \xrightarrow{f} \mathbf{Vect}\big),$$

where $(g \downarrow b)$ is the comma category of arrows from g to b and k is the canonical projection.

For each object $(a, \beta : g(a) \to b)$ in $(g \downarrow b)$, define

$$\tilde{\lambda}_{(a,\beta)} : k^* f(a,\beta) = f(a) \longrightarrow \tilde{Z}(b),$$
$$\tilde{\lambda}_{(a,\beta)} = \sum_{a' \in g^{-1}(b)} \sum_{\substack{\alpha : a \to a' \\ g(\alpha) = \beta}} w_\beta f(\alpha).$$

We show how the colimit colim $k^* f$ can be deduced from the family of maps $(\tilde{\lambda}_{(a,\beta)})$ in a series of lemmas.

Lemma 1. The maps $(\tilde{\lambda}_{(a,\beta)})$ are the components of a cocone from $k^* f$.

Proof. Let $\alpha : (a,\beta) \to (a',\beta')$ be a morphism in $(g \downarrow b)$. By linearity and functoriality,

$$\tilde{\lambda}_{(a',\beta')} \circ f(a) = \sum_{a''} \sum_{\substack{\alpha' : a' \to a'' \\ g(\alpha') = \beta'}} w_{\beta'} f(\alpha' \circ \alpha).$$

Since $\mathscr{A}$ is a groupoid, there is for every $\alpha'' : a \to a''$ a unique α' such that $\alpha' \circ \alpha = \alpha''$, namely $\alpha' = \alpha'' \circ \alpha^{-1}$. By functoriality of g, if $g(\alpha'') = \beta$, then for $\alpha' = \alpha'' \circ \alpha^{-1}$ we have $g(\alpha') = g(\alpha)^{-1} \circ \beta = \beta'$. Hence the above equals

$$\sum_{a''} \sum_{\substack{\alpha'' : a \to a'' \\ g(\alpha) = \beta}} w_\beta f(\alpha''),$$

which is $\tilde{\lambda}_{(a,\beta)}$. □

Lemma 2. Every other cocone factors through $(\tilde{\lambda}_{(a,\beta)})$.

Proof. Let $\mu_{(a,\beta)} : f(a) \to M$ be the components of another cocone. Note that this implies that

$$\mu_{(a,\beta)} = \mu_{(a',1_b)} \circ f(\alpha)$$

for all $\alpha : a \to a'$ with $a' \in g^{-1}(b)$ and $g(\alpha) = \beta$. Using this, define a map $\mu : \tilde{Z}(b) \to M$ by $\mu|_{f(a')} := \mu_{(a',1_b)}$. Using the definition of w_β it is straightforward to verify that $\mu \circ \tilde{\lambda}_{(a,\beta)} = \mu_{(a,\beta)}$. □

Lemma 3. The cocone $(\tilde{\lambda}_{(a,\beta)})$ factors through the inclusion $Z(b) \subset \tilde{Z}(b)$. Moreover, $Z(b)$ is the minimal subspace with this property.

Proof. Since $\wp_b$ is idempotent, each component $\tilde{\lambda}_{(a,\beta)}$ factors through $Z(b) \subset \tilde{Z}(b)$ iff $\wp_b \circ \tilde{\lambda}_{(a,\beta)} = \tilde{\lambda}_{(a,\beta)}$. This relation is easily verified, again using the definition of w_β. To show that $Z(b)$ is the minimal subspace, note that

$$\pi_{f(a)} \circ \wp_b = \sum_{a' \in g^{-1}} \pi_{f(a)} \circ \tilde{\lambda}_{(a',1_b)}.$$

□

It follows that $Z(b) = \mathrm{Lan}_g f(b) = \mathrm{colim}_{(g \downarrow b)} k^* f$ and that the colimiting cone is given by the maps $\lambda_{(a,\beta)}$ trivially induced by the maps $\tilde{\lambda}_{(a,\beta)}$. To deduce the action of $\mathrm{Lan}_g f$ on morphisms, one may argue as follows. Let $\beta : b \to b'$ be a morphism in $\mathscr{B}$. Given (a', β') in $(g \downarrow b)$, $(a', \beta \circ \beta')$ will be an object in $(g \downarrow b')$. By the universality of the colimiting cones, there is a unique morphism $\mathrm{Lan}_g f(\beta)$ which makes the following diagram commute (for all choices of (a', β')):

$$\begin{array}{ccc} f(a') & = & f(a') \\ \lambda_{(a',\beta')} \downarrow & & \downarrow \lambda_{(a',\beta\circ\beta')} \\ Z(b) & \xrightarrow{\mathrm{Lan}_g f(\beta)} & Z(b'). \end{array}$$

It is easily verified that $\tilde{Z}(\beta)$ satisfies the analogous diagram, with $Z(b)$ replaced by $\tilde{Z}(b)$, $\lambda_{(a',\beta')}$ replaced by $\tilde{\lambda}_{(a',\beta')}$, and etc. Moreover, $\tilde{Z}(\beta) \circ \wp_b = \tilde{Z}(\beta)$ and $\wp_{b'} \circ \tilde{Z}(\beta) = \tilde{Z}(\beta)$, so $\mathrm{Lan}_g f(\beta)$ must be the map $\tilde{Z}(\beta)$ restricted to $Z(b)$.

1.2 The right Kan extension

The deduction of the formula for the right Kan extension $\mathrm{Ran}_g f$ is very similar to the calculation of the left extension, so we shall allow ourselves to be brief.

We start from the pointwise formula

$$\mathrm{Ran}_g f(b) = \lim\big((b \downarrow g) \xrightarrow{j} \mathscr{A} \xrightarrow{f} \mathbf{Vect}\big),$$

where $(b \downarrow g)$ is the comma category of arrows from b to g and j is the canonical projection.

For each object $(a, \beta : b \to g(a))$ in $(b \downarrow g)$, define

$$\tilde{\rho}_{(a,\beta)} : \tilde{Z}(b) \longrightarrow j^* f(a, \beta) = f(a),$$

$$\tilde{\rho}_{(a,\beta)} := \sum_{a' \in g^{-1}(b)} \sum_{\substack{\alpha : a' \to a \\ g(\alpha) = \beta}} w_\beta f(\alpha).$$

Repeating the analogous argument for the left extension, we deduce that $(\tilde{\rho}_{(a,\beta)})$ is a cone to $j^* f$, i.e. that $f(\alpha) \circ \tilde{\rho}_{(a,\beta)} = \tilde{\rho}_{(a',\beta')}$ for all $\alpha : a \to a'$ such that $\beta' = g(\alpha) \circ \beta$.

If $\mu_{(a,\beta)} : M \to j^* f(a, \beta)$ are the components of another cone, then we may define a linear map $\mu : M \to \tilde{Z}(b)$ by $\mu(m) = (\mu_{(a,1_b)}(m))_{a \in g^{-1}(b)}$ $(m \in M)$. This map is such that

$$\tilde{\rho}_{(a,\beta)} \circ \mu = \sum_{a'} \sum_{\substack{\alpha : a' \to a \\ g(\alpha) = \beta}} w_\beta \underbrace{f(\alpha) \circ \mu_{(a',1_b)}}_{= \mu_{(a,\beta)}} = \mu_{(a,\beta)}.$$

Hence every cone factors through $(\tilde{\rho}_{(a,\beta)})$.

One easily verifies that $\tilde{\rho}_{(a,\beta)} \circ \wp_b = \tilde{\rho}_{(a,\beta)}$, so restricting each component $\tilde{\rho}_{(a,\beta)}$ to $Z(b) \subset \tilde{Z}(b)$ we get a new cone $(\rho_{(a,\beta)} : Z(b) \to j^* f(a, \beta))$, still with the property that every other cone factors through it. The vector space $Z(b)$ is minimal with this property since $\pi_{f(a)} \circ \wp_b = \tilde{\rho}_{(a,1_b)}$.

Arguing as in the case of the left extension, one deduces that $\mathrm{Ran}_g f(\beta) = Z(\beta) := \tilde{Z}(\beta)|_{Z(b)}$ for a morphism β in $\mathscr{B}$.

2 Examples

Example 1. Our first example is the setup described in the 'Introduction'.

Let X be a finite subgroupoid of the path groupoid $\mathcal{P}_1(M)$ of a lorentzian manifold (M, ds^2). Let $\mathscr{B} = \mathcal{P}_1(\mathbf{R}) = \mathbf{R}//\mathbf{R}$, and let $\mathscr{A}$ be the groupoid which has as objects pairs $(x, s) \in \mathrm{Ob}(X) \times \mathbf{R}$ and as morphisms $(x, s) \to (x', s')$ curves $\gamma : x \to x'$ in X of proper time $\tau(\gamma) := \int_\gamma ds = s' - s$. Let $f = \exp iS : \mathscr{A} \to \mathbf{Vect}$ be an 'exponentiated action', mapping every object to a copy of $\mathbf{C}$ and a curve (morphism) γ to $\exp iS(\gamma)$, where S is additive with respect to composition of paths. Let $g = \tau : \mathscr{A} \to \mathscr{B}$ be the functor which maps a diagram $(x, s) \to (x', s')$ to the diagram $(s' - s) : s \to s'$. We get that

$$\tilde{Z}(s) = \mathbf{C}^{\mathrm{Ob}X}$$

and that $Z(T) : Z(s) \to Z(s + T)$ is the linear map that takes $(\psi(x))_{x \in X}$ to the vector with components

$$(Z(T)(\psi))(y) = \sum_{x \in X} \sum_{\substack{\gamma : x \to y \\ \tau(\gamma) = T}} w_T e^{iS(\gamma)} \psi(x).$$

We can extract from this the kernel $K(x, y; T)$, such that $(Z(T)\psi)(y) = \sum_x K(x, y, T)\psi(x)$. With the obvious discrete measures we can write it suggestively as an integral:

$$K(x, y; T) = \int_{\gamma(0)=x}^{\gamma(T)=y} D\gamma \, e^{iS(\gamma)}.$$

In the continuum limit, if such a limit can be shown to exist, this may be expected to reproduce relativistic quantum mechanics, perhaps modulo some scaling factor.

The projector $\wp$ projects out the states $\psi \in \mathbf{C}^{\mathrm{Ob}X}$ that are invariant under $T = 0$ propagators, i.e. $Z(s) = \mathrm{Im}(\wp_s)$ consists only of what is sometimes called 'physical states'.

Example 2. This example is an attempt to formulate something with the flavour of Chern-Simons theory.

Let Σ be a finite version of the path groupoid $\mathcal{P}_1(S)$ of a manifold S, and let X be the finite groupoid. One may regard an n-dimensional quantum field theory with worldvolume $\Sigma \times \mathcal{P}_1(\mathbf{R})$ and target X as a $0+1$-dimensional field theory on $\mathcal{P}_1(\mathbf{R})$ with target X^Σ. Motivated by this we let $\mathscr{A} = X^\Sigma \times \mathcal{P}_1(\mathbf{R})$ and $g : \mathscr{A} \to \mathscr{B} = \mathcal{P}_1(\mathbf{R})$ be the cartesian projection. Let $f = \chi : \mathscr{A} \to \mathbf{Vect}$ be any functor. We then get

$$\tilde{Z}(s) = \bigoplus_{\Phi : \Sigma \to X} \chi(\Phi, s),$$

and for $T : s \to s + T$

$$\pi_{(\Phi', s+T)} \circ Z(T) = \sum_{\Phi} \sum_{\eta : \Phi \to \Phi'} \mu_{X^\Sigma} \chi(\eta, T).$$

Here μ_{X^Σ} is the Leinster measure (aka 'the groupoid measure') on the groupoid X^Σ. Note that a natural transformation $\eta : \Phi \to \Phi'$ can be regarded as a functor $\Sigma \times I \to X$, for I the 'interval category' $\{a \to b\}$.

For X equal to the one-object groupoid BG of a finite group G, we get a Dijkgraaf-Witten-type theory, which is a finite toy model of Chern-Simons theory.[3] It is interesting that the Dijkgraaf-Witten measure μ_{BG^Σ} appears here by pure abstract nonsense, as opposed to being inserted by hand (as it was in the original construction by Dijkgraaf and Witten). Specifying further to $\Sigma = \{\mathrm{pt}\}$ (so that we are effectively looking at a 1-dimensional QFT with gauge group G), and letting χ be the composite of a character $BG \to BU(1)$ on G and the representation $BU(1) \to \mathbf{Vect}$ of $U(1)$ in $\mathbf{C}$, we get $\tilde{Z}(s) = \mathbf{C}$ and

$$Z(T) = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

This toy example has been described elsewhere, see e.g. p. 4 of Freed et al. 2009.

[3]See, e.g. Freed and Quinn 1992 or the detailed discussion beginning on p.67 of Bartlett 2005.

Example 3. Let X be an oriented four-manifold and let $\mathcal{P}_4(X)$ be the groupoid which has as objects smooth three-dimensional submanifolds $\Sigma \subset X$ and as morphisms $\Sigma \to \Sigma'$ submanifolds M of X with boundary $(-\Sigma) \sqcup \Sigma'$ (or such submanifolds M defined up to smooth homotopy, suitably defined). We think of the objects as 'simultaneity slices' of a spacetime, and of the morphisms as (sub-)spacetimes between such slices. Let $\mathscr{B}$ be a groupoid obtained by first choosing a finite subcategory of $\mathcal{P}_4(X)$ and then discretizing all its objects Σ and morphisms M to finite groupoids. Let BG be the one-object groupoid of a finite group G and let $\mathscr{A}$ be the groupoid which has as objects pairs (Σ, A) with $\Sigma \in \mathscr{B}$ and $A : \Sigma \to BG$, and as morphisms $(\Sigma_{in}, A_{in}) \to (\Sigma_{out}, A_{out})$ pairs (M, A) where $M : \Sigma_{in} \to \Sigma_{out}$ is a morphism in $\mathscr{B}$ and $A \in BG^M$ satisfies $A|\Sigma_{in} = A_{in}$, $A|\Sigma_{out} = A_{out}$. Morally, $\mathscr{A}$ is the category whose objects are simultaneity slices with a specified G-connection and whose morphisms are spacetime bordisms with compatible connections. Physically, the G-connections are gauge-fields.

Let $f = I : \mathscr{A} \to \mathbf{Vect}$ be any functor and led $g : \mathscr{A} \to \mathscr{B}$ be the forgetful functor that forgets the gauge field. We get

$$\tilde{Z}(\Sigma) = \bigoplus_{A \in BG^{\Sigma}} I(\Sigma, A),$$

and, for $M : \Sigma_{in} \to \Sigma_{out}$,

$$\pi_{A_{out}} \circ Z(M) = \sum_{A_{in}} \sum_{\substack{A \in BG^M \\ A|\Sigma_{in} = A_{in} \\ A|\Sigma_{out} = A_{out}}} w_M I(M, A).$$

If I has the form of an exponentiated action (say defined by integration over M of a lagrangian density), then, as in Example 1, we can introduce a discrete measure and write

$$K(A_{in}, A_{out}; M) = \int_{A_{in}}^{A_{out}} DA\, e^{iS_M(A)},$$

which suggests a quantized gauge theory.

3 Kan extension as limits over weak sections

Consider the Kan extension diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f} & \mathbf{Vect} \\ {\scriptstyle g}\big\downarrow & \nearrow {\scriptstyle L_g f} & \\ \mathscr{B}. & & \end{array}$$

Here $L_g f$ is the Left Kan extension $\mathrm{Lan}_g f$. Let $\lambda : f \to L_g f \circ g$ be the natural transformation encoding the universality of $L_g f$. Recall the following terminology:

Definition 1. The **category of** (weak) **left sections** of g is the category $\Gamma_l(g)$ which has

- as objects pairs (ϕ, η_ϕ) where ϕ is a functor from $\mathscr{B}$ to $\mathscr{A}$ and η_ϕ is a natural transformation $\phi^* g \to 1_{\mathscr{B}}$,
- and as morphisms $(\phi, \eta_\phi) \to (\phi', \eta_{\phi'})$ natural transformations $\nu : \phi \to \phi'$ with the property that $\eta_{\phi'} \circ g\nu = \eta_\phi$.

There is an obvious forgetful functor $U : \Gamma_l(g) \to [\mathscr{B}, \mathscr{A}]$, where $[\mathscr{B}, \mathscr{A}]$ is the category of functors from $\mathscr{B}$ to $\mathscr{A}$. Using this functor we can transgress the functor f on $\mathscr{A}$ to the category of sections $\Gamma_l(g)$, forming the functor

$$tg_l f := [\mathscr{B}, f] \circ U : \Gamma_l(g) \longrightarrow [\mathscr{B}, \mathbf{Vect}].$$

We claim that the Kan extension $L_g f$ can be computed as a colimit of this transgressed functor.

As a prelimiary step, note that the natural transformation $\lambda : f \to L_g f \circ g$ induces a family of natural transformations

$$\lambda_{(\phi, \eta_\phi)} : tg_l f(\phi, \eta_\phi) = \phi^* f \longrightarrow L_g f \qquad ((\phi, \eta_\phi) \in \Gamma_l(g))$$

by

$$\lambda_{(\phi,\eta_\phi)}(b) := L_g f(\eta_\phi(b)) \circ \lambda(\phi(b)).$$

(Drawing the relevant 2-cell diagrams makes this construction obvious.)

Remark 1. The transformations $\lambda_{(\phi,\eta_\phi)}$ form the components of a cocone under $tg_l f$.

Proof. Let $\nu : (\phi, \eta_\phi) \to (\phi', \eta_{\phi'})$ be a morphism in $\Gamma_l(g)$ and pick $b \in \mathscr{B}$. We have

$$\begin{aligned}
\big(\lambda_{(\phi',\eta_{\phi'})} \circ tg_l f(\nu)\big)(b) &= L_g f(\eta_{\phi'}(b)) \circ \lambda(\phi'(b)) \circ f(\nu(b)) \\
&= L_g f\big(\eta_{\phi'}(b) \circ g(\nu(b))\big) \circ \lambda(\phi(b)) \\
&= L_g f(\eta_\phi(b)) \circ \lambda(\phi(b)) \\
&= \lambda_{(\phi,\eta_\phi)}(b).
\end{aligned}$$

This proves the claim. □

Note that, in the notation of the preceeding sections of the paper,

$$\lambda_{(\phi,\eta_\phi)}(b) = \lambda_{(\phi(b),\eta_\phi(b))} = \sum_a \sum_{\substack{\alpha:\phi(b)\to a \\ g(\alpha)=\eta_\phi(b)}} w_{\eta_\phi(b)} f(\alpha).$$

This means that for each fixed b in $\mathscr{B}$, the cocone

$$\begin{array}{ccc}
f(\phi(b)) & \xrightarrow{f(\nu(b))} & f(\phi'(b)) \\
{\scriptstyle \lambda_{(\phi,\eta_\phi)}(b)} \searrow & & \swarrow {\scriptstyle \lambda_{(\phi',\eta_{\phi'})}(b)} \\
& L_g f(b) &
\end{array} \qquad ((\phi,\eta_\phi) \in \Gamma_l(g))$$

is universal.

Lemma 4. The cocone $(\lambda_{(\phi,\eta_\phi)})$ is the colimiting cone of $tg_l f$.

Proof. By the remarks above, we have a pointwise equality $L_g f(b) = tg_l f(b)$. We must only show equality on morphisms.

Let $\beta : b \to b'$ be a morphism in $\mathscr{B}$. We get a cocone

$$\begin{array}{ccc}
f(\phi(b)) & \xrightarrow{f(\nu(b))} & f(\phi'(b)) \\
{\scriptstyle \lambda_{(\phi,\eta_\phi)}(b') \circ f(\phi(\beta))} \searrow & & \swarrow {\scriptstyle \lambda_{(\phi',\eta_{\phi'})}(b') \circ f(\phi'(\beta))} \\
& L_g f(b') &
\end{array}$$

under $\{tg_l f(\phi,\eta_\phi)(b)\}$. By universality of the cocone $\{\lambda_{(\phi,\eta_\phi)}(b) : tg_l f(\phi,\eta_\phi)(b) \to L_g f(b)\}$, there exists a unique morphism $tg_l f(\beta) : L_g f(b) \to L_g f(b')$ such that

$$\lambda_{(\phi,\eta_\phi)}(b') \circ f(\phi(\beta)) = tg_l f(\beta) \circ \lambda_{(\phi,\eta_\phi)}(b).$$

Our claim is that $tg_l f(\beta) = L_g f(\beta) = Z(\beta)$. Inserting $tg_l f = Z(\beta)$ in above equation, the right hand side equals

$$\sum_a \Bigg(\sum_{a'} \sum_{\substack{\alpha':\phi(b)\to a' \\ g(\alpha')=\eta_\phi(b)}} \sum_{\substack{\alpha:a'\to a \\ g(\alpha)=\beta}} w_{\eta_\phi(b)} w_\beta f(\alpha \circ \alpha') \Bigg),$$

while the left hand side equals

$$\sum_a \Bigg(\sum_{\substack{\alpha'':\phi(b')\to a \\ g(\alpha'')=\eta_\phi(b')}} w_{\eta_\phi(b')} f(\alpha'' \circ \phi(\beta)) \Bigg).$$

A simple counting argument, repeating the argument in 'Lemma 1', 'Section 1.1', and using the definition of the weights w shows that these two expressions are equal. □

Corollary 1. The Kan extension $\mathrm{Lan}_g f$ is equal to the colimit $\mathrm{colim}_{\Gamma_l(g)} tg_l f$ of the transgression of f over the category of weak left sections.

There is a similar relation for the right Kan extension. Let $R_g f := \mathrm{Ran}_g f$ be the right Kan extension of f along g, and let $\rho : R_g f \circ g \to f$ be the associated universal transformation.

Definition 2. The **category of** (weak) **right sections** of g is the category $\Gamma_r(g)$ which has

- as objects pairs (ψ, ξ_ψ) consisting of a functor $\psi : \mathscr{B} \to \mathscr{A}$ and a transformation $\xi_\psi : 1_{\mathscr{B}} \to \xi_\psi \circ g$
- and as morphisms $(\psi, \xi_\psi) \to (\psi', \xi_{\psi'})$ natural transformations $\nu : \psi \to \psi'$ such that $g\nu \circ \xi_\psi = \xi_{\psi'}$.

The transformation ρ defines a family of arrows

$$\rho_{(\psi,\xi_\psi)} : R_g f \longrightarrow \psi^* f \qquad ((\psi, \xi_\psi) \in \Gamma_r(g))$$

by

$$\rho_{(\psi,\xi_\psi)}(b) := \rho(\psi(b)) \circ R_g f(\xi_\psi(b)).$$

These constitute the components of a cone to

$$tg_r f := [\mathscr{B}, f] \circ U : \Gamma_r(g) \longrightarrow [\mathscr{B}, \mathbf{Vect}],$$

for $U : (\psi, \xi_\psi) \mapsto \psi$ the canonical forgetful functor. This cone can be shown to be universal, noting that $\rho_{(\psi,\xi_\psi)}(b) = \rho_{(\phi(b),\xi_\psi(b))}$ and repeating the arguments for the colimit over the category of left sections. Hence

$$R_g f = \lim_{\Gamma_r(g)} tg_r f.$$

Let us summarize the content of the above discussion.

Given $b \in \mathscr{B}$, let $(\cdot)|_b : \Gamma_l(g) \to (g \downarrow b)$ be the functor $(\phi, \eta_\phi) \mapsto (\phi(b), \eta_\phi(b))$. There for ay category $\mathscr{C}$, there is a natural functor $\mathscr{C}^{\mathscr{B}} \to \mathscr{C}^{\{b\}}$ which we shall also denote as $(\cdot)|_b$. There is a commuting diagram

$$\begin{array}{ccccc}
(g \downarrow b) & \xleftarrow{(\cdot)|_b} & \Gamma_l(g) & \xrightarrow{(\cdot)|_{b'}} & (g \downarrow b') \\
\downarrow k & & \downarrow U & & \downarrow k' \\
\mathscr{A}^{\{b\}} & \xleftarrow{(\cdot)|_b} & \mathscr{A}^{\mathscr{B}} & \xrightarrow{(\cdot)|_{b'}} & \mathscr{A}^{\{b'\}} \\
\downarrow f & & \downarrow f^{\mathscr{B}} & & \downarrow f \\
\mathbf{Vect}^{\{b\}} & \xleftarrow{(\cdot)|_b} & \mathbf{Vect}^{\mathscr{B}} & \xrightarrow{(\cdot)|_b} & \mathbf{Vect}^{\{b'\}}.
\end{array}$$

Taking the colimit of the vertical maps calculates the left Kan extension: the colimit over the left- and rightmost columns calculates the spaces $Z(b)$ and $Z(b')$, while the colimit of the centermost column gives the morphisms $Z(\beta)$. There is a corresponding diagram for the right Kan extension.

We feel that above diagram may provide an important clue as to our Kan extension formulas should generalize to higher categories. Namely, consider a scenario where $\mathscr{B} = \Sigma$ is a groupoid. Then the notions of (weak) left and right sections coincide, since any natural transformation $\phi^* g \to 1$ can be inverted. In other words, we can define a category $\Gamma(g) = \Gamma_l(g) = \Gamma_r(g)$ of weak sections of g. Given a cospan of groupoids (to be thought of as a bordism)

$$\Sigma_{in} \to \Sigma \leftarrow \Sigma_{out},$$

we get a diagram

$$\begin{array}{ccccc}
\Gamma(\Sigma_{in}, g) & \longleftarrow & \Gamma(\Sigma, g) & \longrightarrow & \Gamma(\Sigma_{out}, g) \\
tg\, f|_{\Sigma_{in}} \downarrow & & tg\, f \downarrow & & \downarrow tg\, f|_{\Sigma_{out}} \\
[\Sigma_{in}, \mathbf{Vect}] & & [\Sigma, \mathbf{Vect}] & & [\Sigma_{out}, \mathbf{Vect}].
\end{array}$$

A higher-categorical version of our Kan extension formula would, maybe, involve taking colimits (or limits) of the vertical maps in such diagrams. If Σ_{in} and Σ_{out} are both copies of the final category $\mathbf{1}$, then this prescription calculates the Kan extension (both left and right) of f along g, as we have shown. When Σ_{in} and Σ_{out} are not of that simple form, then we ought to get something like a higher categorical version of Kan extension. A very similar proposal for a categorical definition of path integrals in higher categorical (extended) QFTs appeared recently in Freed et al. 2009 (see p.12).

Conclusions

There are two immediate directions in which the result we have proven begs to be generalized. Firstly, the result should extend to higher categories (the categorified versions of n-dimensional quantum field theories). Secondly, it ought to be possible to find at least traces of the universality described by the Kan extension formula also in the continuum (i.e. non-discretized) theory of particle quantum mechanics.

One of the main difficulties with the first extension of our result, that to higher categories, seems to be to know what the right definitions to start from are. For example, it seems both desirable and unavoidable to have the freedom to allow the background field ('action') f to be a pseudofunctor. (Dijkgraaf-Witten theory for example uses a group cocycle as a backgroud field.) The difficulties with the second route of generalization also involve knowing what definitions to start from, but it also involves making sense of the full-fledged path integral, which is a very thorny business. Maybe the universal categorical properties of Kan extension in conjunction with a generalization of the Leinster measure can help us to finally nail the *right* definition of the (non-discrete) path integral?

A related but purely mathematical problem is the following: Given a manifold M, a vector bundle $E \to M$ with connection on M induces a **Vect**-valued functor on $\mathcal{P}_1(M)$, assigning fibers of E to points and parallel transport maps to paths. Given a smooth map $\phi : M \to N$, can one Kan extend the 'transport functor' on $\mathcal{P}_1(M)$ along $\phi : \mathcal{P}_1(M) \to \mathcal{P}_1(N)$ to obtain a transport functor on $\mathcal{P}_1(N)$ equivalent to a bundle with connection on N? If so, then this would be a new and possibly interesting mathematical operation on vector bundles.

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