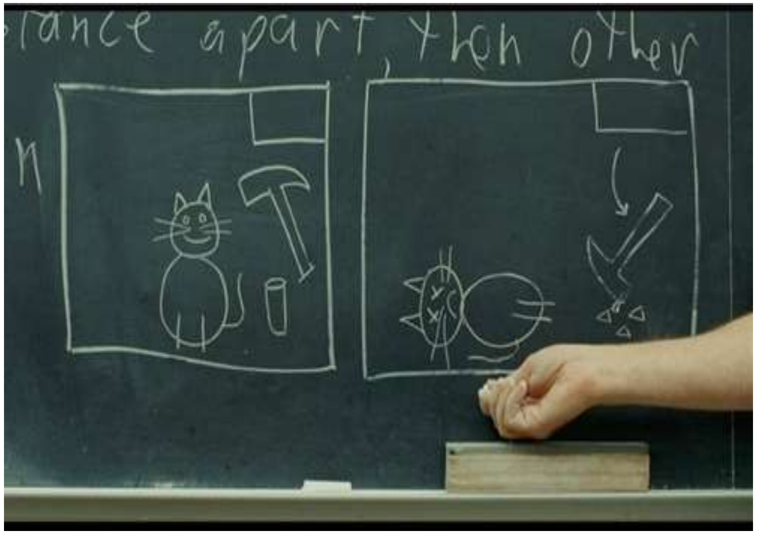
Quantum Geometry:

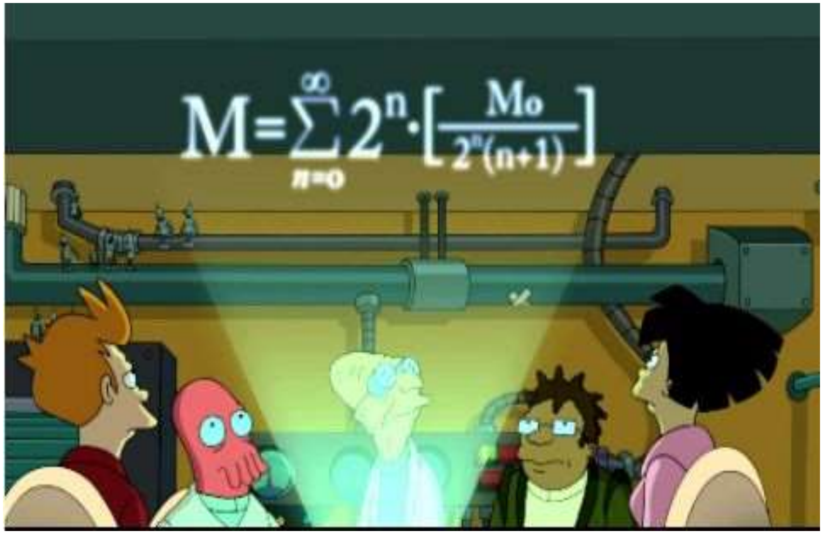
A reunion of math and physics

Physics and Math are quite different:

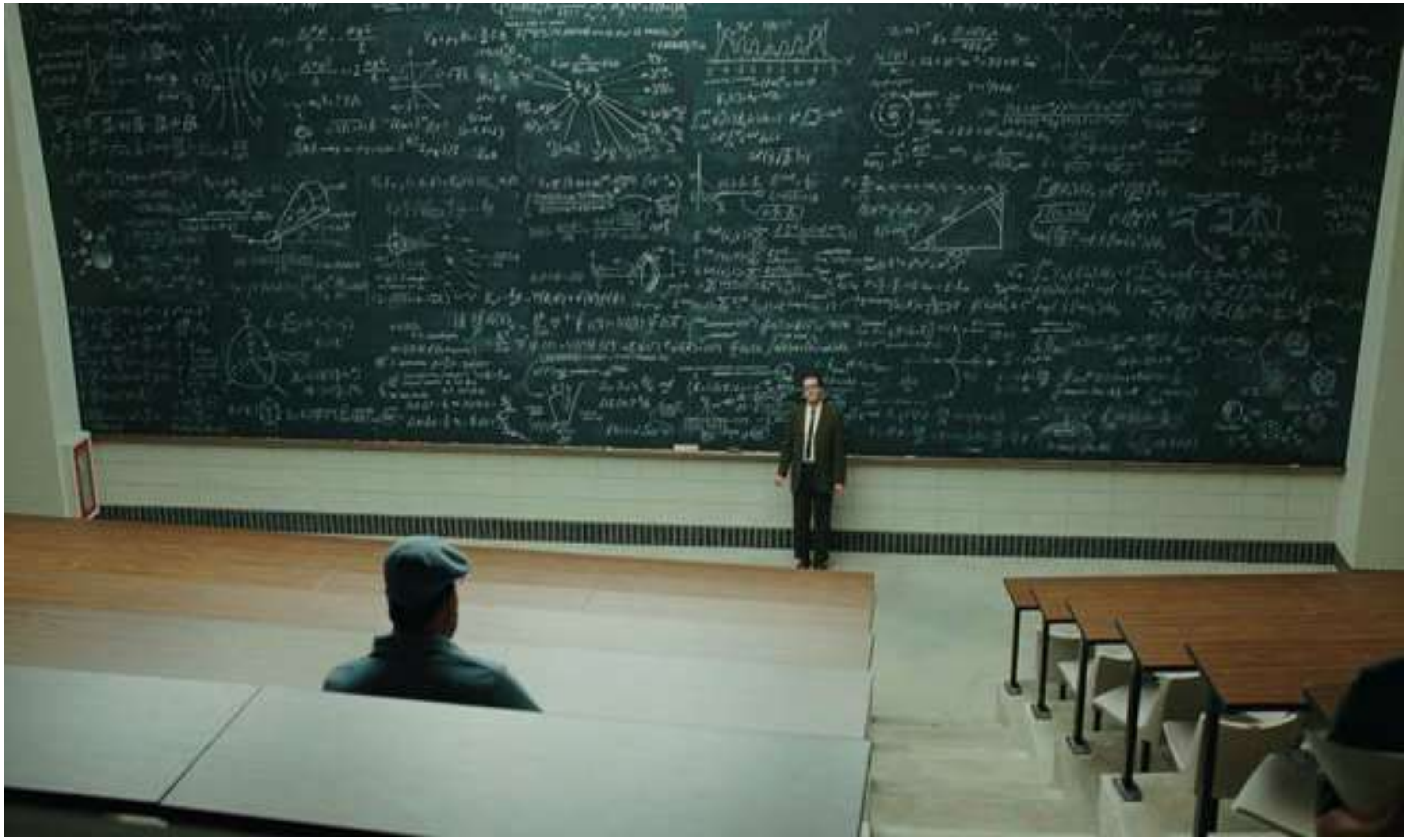
Physics



Math



Although to an uninitiated eye
they may appear indistinguishable



- Math: deals with abstract ideas which exist independently of us, our practice, or our world (Plato)

- Physics: the study of the most fundamental properties of the real world, especially motion and change (Aristotle)

- Mathematicians prove theorems and value rigorous proofs.

  E.g. Jordan curve theorem:

  “Every closed non-self-intersecting curve on a plane has an inside and an outside.”

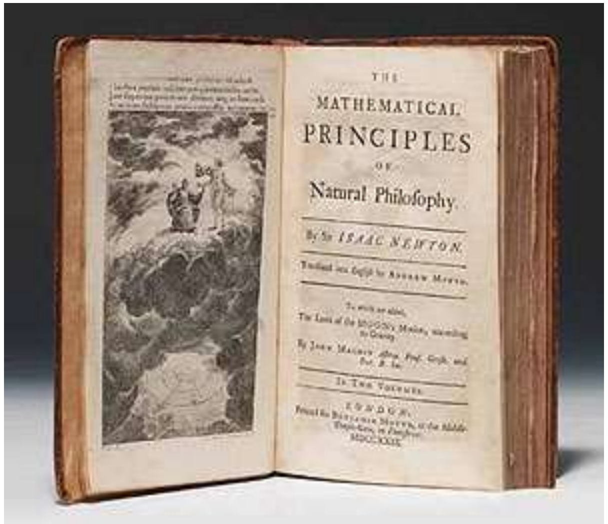
  Seems evident but is not easy to prove.

- Physicists are more relaxed about rigor.

YOU WANT PROOF?
I'LL GIVE YOU PROOF!

Since the times of Isaac Newton, physics is impossible without math:

Laws of Nature are most usefully expressed in mathematical form.



$$E = mc^2$$

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$H(t)|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle$$

$$\nabla \cdot \mathbf{B} = 0$$

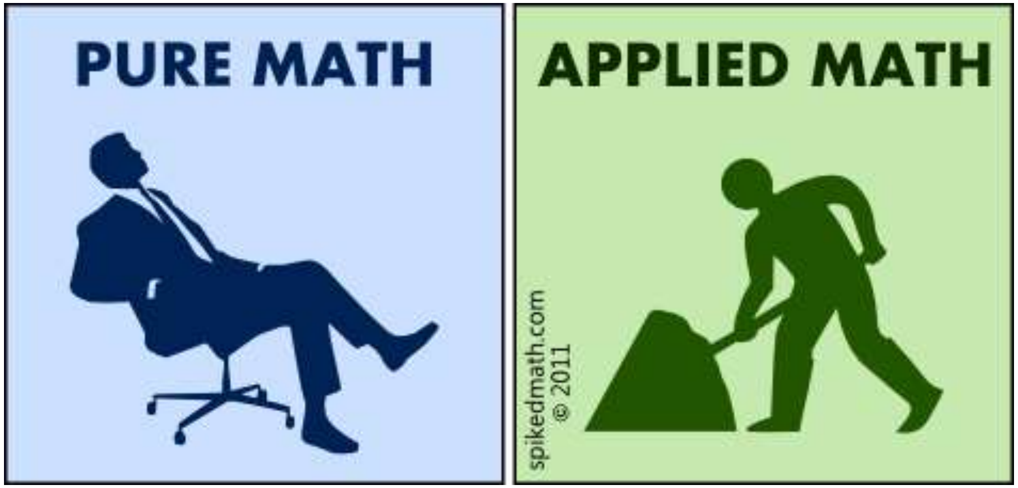
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 4\pi\, \rho(\mathbf{r},\ \mathbf{t})$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}(\mathbf{r},\ t) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$i\hbar\gamma^\mu \partial_\mu = mc\psi$$

In a sense, physics is applied math.



Most of the time, physicists are “consumers” of math:
They do not invent new mathematical concepts.

And mathematicians usually do not need physics.

$$M \rightarrow \Phi$$

But once in a while physicists have
to invent new math concepts to describe what
they see around them.

Isaac Newton had to invent calculus to be able to formulate laws of motion.

F=ma

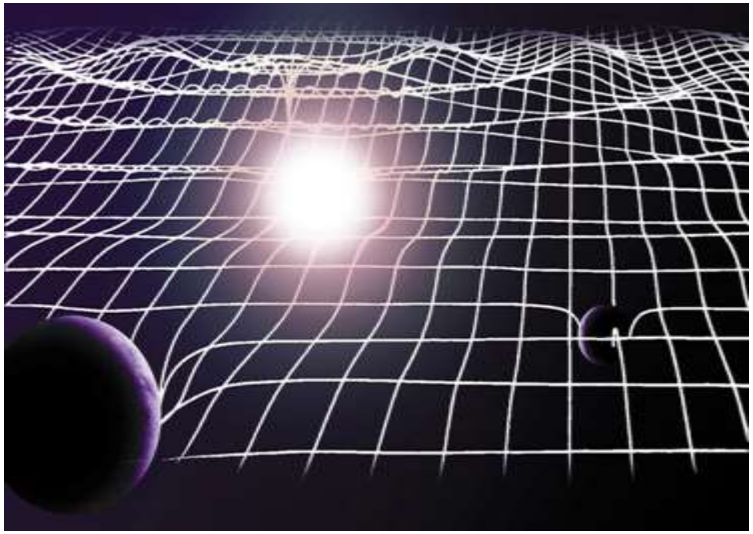
Here **a** is time derivative of velocity.

The invention of calculus was a revolution in mathematics.

Relativity theory of Einstein did not
lead to a mathematical revolution.

It used the tools which were already available:

The geometry of curved space created by Riemann.



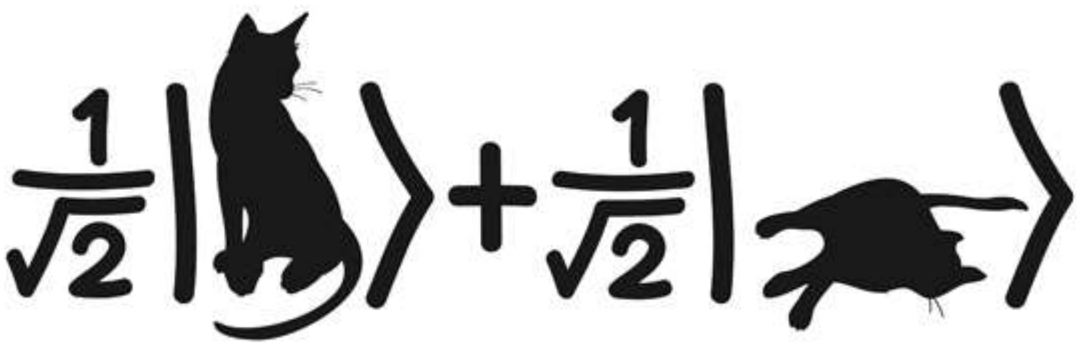
But quantum mechanics does require radically new mathematical tools.

Some of these have been invented by mathematicians inspired by physical problems.

Some were intuited by physicists.

Some remain to be discovered.

What sort of math does one need for Quantum Physics?



Classical Mechanics

- Observables (things we can measure) are real numbers
- Determinism
- Positions and velocities are all we need to know

Quantum Mechanics

- Observables are not numbers: they do not have particular values until we measure them.
- Outcomes are inherently uncertain, physical theory can only predict probabilities of various outcomes.
- Cannot measure positions and velocities at the same time (Heisenberg's uncertainty principle).

Heisenberg's Uncertainty Principle

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Δx is the uncertainty of position

Δp is the uncertainty of momentum (p=mv)

$\hbar=6.626 \cdot 10^{-34}$ kg·m^2/sec is Planck's constant

The better you know the position of a particle, the less you know about its momentum. And vice versa:

How can we describe this strange property mathematically?

The answer is surprising:

Quantum position and quantum momentum are entities which violate a basic rule of elementary math: commutativity of multiplication

$$X\,P \neq P\,X$$

Recall that ordinary multiplication of numbers is commutative:

$$a\ b = b\ a$$

and associative:

$$a\ (b\ c) = (a\ b)\ c$$

One can often define multiplication of other entities. It is usually associative, but in many cases fails to be commutative.

Which other entities can be multiplied?

Example 1: functions on a set **X**.

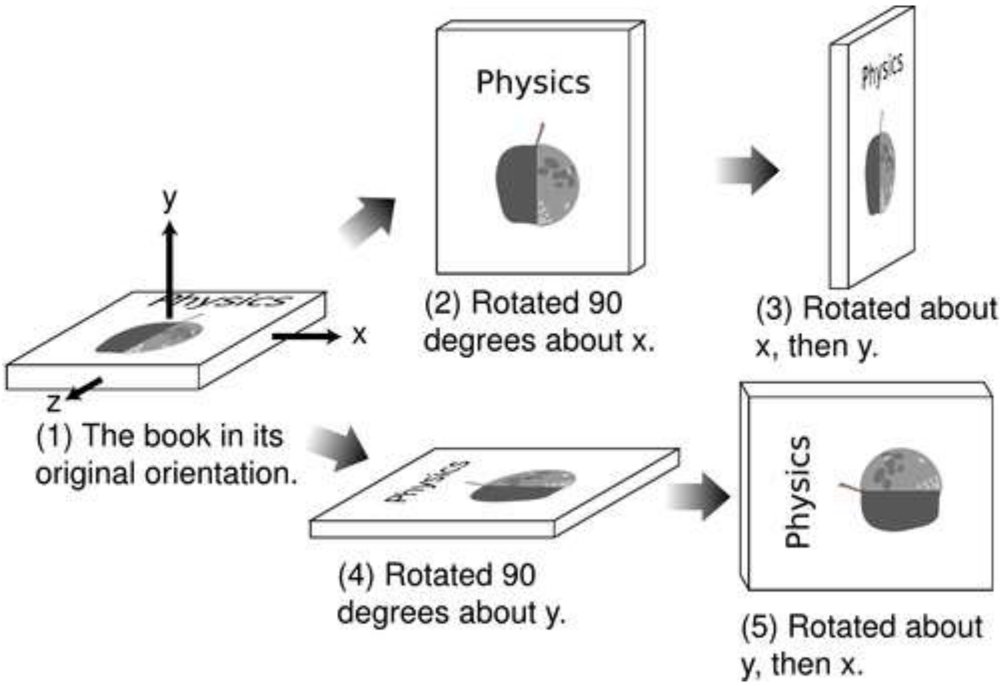
A function **f** attaches a number **f(x)** to every element **x** of the set **X.**

The product of functions **f** and **g** is a function which attaches the number **f(x)· g(x)** to **x**.

This multiplication is commutative and associative.

Example 2: rotations in space.

Multiplying two rotations is the same as doing them in turn. One can show that the result is again a rotation. This operation is associative but not commutative.



Another difference between the two examples is that functions on a set **X** can be both added and multiplied, but rotations can be only multiplied.

When some entities can be both added and multiplied, and all the usual rules hold, mathematicians say these entities form a commutative algebra.

Functions on a set **X** form a commutative algebra.

When all rules hold, except commutativity, mathematicians say the entities form a non-commutative algebra.

Quantum observables form a non-commutative algebra!

This is a mathematical reflection of the Heisenberg Uncertainty Principle.

But there are many more non-commutative algebras than commutative ones.

Just like there are more not-bananas than bananas.

Bananas

Not bananas

There are many special cases, where we know the answer. Say, for a particle moving on a line, we have position X and momentum P.

Their algebra is determined by the following "commutation relation"

$$XP - PX = i\hbar$$

where i is the imaginary unit, $i^2 = -1$.

But how do we find suitable multiplication rules in other situations?

To find the right algebra, we can try to use the Correspondence Principle of Niels Bohr:

Quantum physics should become approximately classical as $\hbar$ becomes very small.

Slight difficulty: $\hbar$ has a particular value, how can one make it smaller or larger?

But this is easy: imagine you are a god and can choose the value of $\hbar$ when creating the Universe.

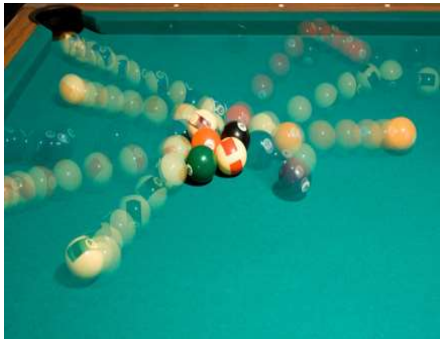
A Universe with a larger $\hbar$ will be more quantum.

A Universe with a smaller $\hbar$ will be more classical.

Tuning $\hbar$ to zero will make the Universe completely classical.

Conversely, we can try to start with a classical system and turn it into a quantum one, by "cranking up" $\hbar$.

Classical



correspondence

←

→

quantization

Quantum

This is called quantization.

Let's recap.

To describe a quantum system mathematically, we need to find the right non-commutative algebra.

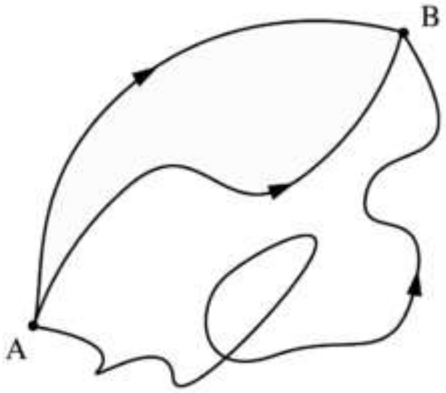
We can start with the mathematical description of a classical system and try to “quantize” it by cranking up $\hbar$. This is called quantization.

But is there enough information in classical physics to figure out how to quantize it?

R. Feynman argued that the answer is “yes”.

His argument relied on something called the “path-integral”.

Roughly:
Quantum answer is obtained by summing contributions from all possible classical trajectories.



Each trajectory contributes $e^{iS/\hbar}$

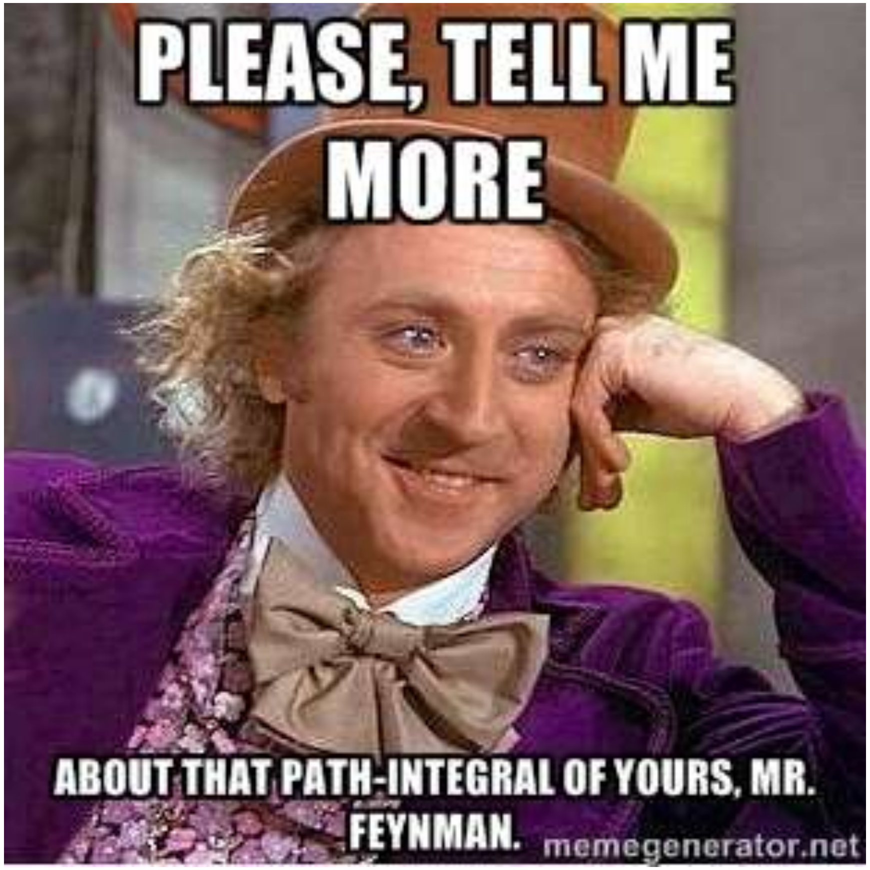
Some call it sum over histories.

This argument made most physicists happy.

In fact, physicists use Feynman's path-integral all the time.

But there is a problem: it makes no mathematical sense.

Until recently, most mathematicians regarded path-integral with skepticism.



This did not bother physicists, because their mathematically suspect theories produced predictions which agreed with experiment, sometimes with an unprecedented accuracy.

For example, the gyromagnetic ratio for the electron:

g_{exp}=2.00231930436... experiment

g_{theor}=2.00231930435... theory

On the other hand, in the 1950s and 1960s, there was a revolution in mathematics associated with the names of Grothendieck, Serre, Hirzebruch, Atiyah, and others.

Alexander Grothendieck (1928-2014)

Physicists paid no attention to it whatsoever.

“In the thirties, under the demoralizing influence of quantum-theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets.”

(R. Jost, a noted mathematical physicist.)

“Dear John, I am not interested in what today's mathematicians find interesting.”

(R. Feynman, in response to an invitation from J. A. Wheeler to attend a math-physics conference in 1966.)

"I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce."

Freeman Dyson, in a 1972 lecture.

But soon afterwards, things began to change.

1978: mathematicians Atiyah, Drinfeld, Hitchin and Manin used sophisticated algebraic geometry to solve instanton equations, which are important in physics.

1984: physicists Belavin, Polyakov, and Zamolodchikov used representation theory of Lie algebras to learn about phase transitions in 2d systems.

Physicists started to pay attention.

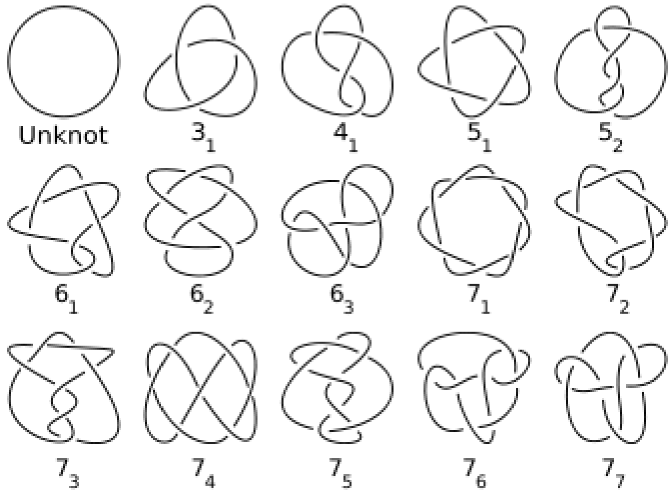
The advent of supersymmetry (1972) and modern string theory (1984) further contributed to the flow of ideas from math to physics.

1986: Calabi-Yau manifolds are important for physics, to study them one needs tools from modern differential geometry and algebraic geometry.

E. Witten, A. Strominger, G. Horowitz, P. Candelas,
J. Polchinski, J. Harvey, C. Vafa, P. Ginsparg, and others.

The turning point came about 1989.

E. Witten uses quantum theory to define invariants of knots.



Mirror symmetry for Calabi-Yau manifolds is discovered (various authors).

Mathematicians started to pay attention.

All these results were deduced by thinking about path-integrals.

So perhaps one can make sense of the path-integral at least in some situations?

This is when Maxim Kontsevich burst onto the scene.

Maxim took the path-integral seriously and showed how to use it to derive new mathematical results.

I will focus on one striking example: the solution of the quantization problem for Poisson manifolds.

The next portion of the talk will be more technical...

The quantization problem

- Start with a classical system described by a commutative algebra A
- Use the information contained in the classical system to turn it into a non-commutative algebra B
- Correspondence principle: B depends on a parameter $\hbar$ so that for $\hbar$=0 it becomes A

Here is a motivating example: start with an algebra of functions of two variables X and P.

The functions must be nice: polynomials, or functions which have derivatives of all orders.

Then postulate a multiplication rule such that $XP-PX=i\hbar$.

If we denote $AB-BA=[A,B]$, then $[X,P]=i\hbar$.

[A,B] is called the commutator of A and B.

If it vanishes for all A and B, the multiplication rule is commutative.

Otherwise, it is non-commutative.

In classical theory, [A,B]=0 for all observables A and B.

In quantum theory it is not true. So how can we figure out which observables cease to commute after quantization?

Additional information used: ``Poisson bracket''.

The set of functions of classical observables X and P has a ``bracket operation”:

To a pair of functions f(X,P), g(X,P) one associates a new function

$$\{f, g\} = \frac{\partial f}{\partial X}\frac{\partial g}{\partial P} - \frac{\partial g}{\partial X}\frac{\partial f}{\partial P}$$

In particular, we can take f(X,P) and g(X,P) to be simply X and P. Then

$$\{X, X\} = 0, \ \{P, P\} = 0, \ \{X, P\} = 1.$$

Now let us apply the substitution rule:

$$\{\ ,\ \} \to \frac{1}{i\hbar}[\ ,\ ]$$

Get [X.X]=0, [P,P]=0, [X,P]=i$\hbar$,

which is exactly right.

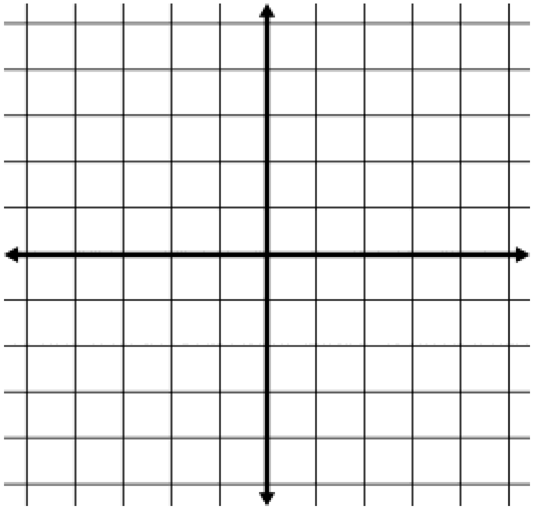
Poisson bracket thus seems to provide the information needed to ``deform'' the algebra of classical observables (functions of X and P) into a non-commutative algebra of quantum X and P.

But does it, really?

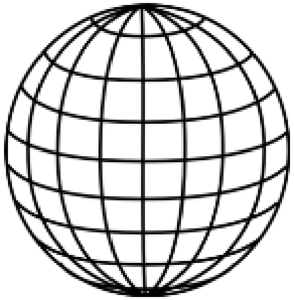
Problem is, X and P are not available, in general.

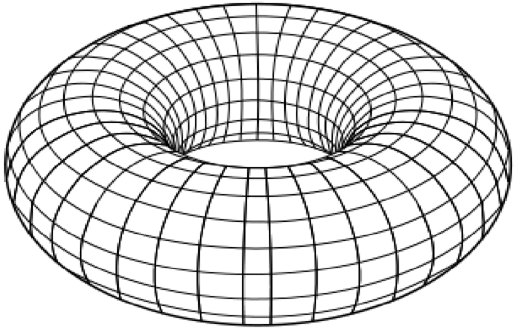
Instead, one has a space whose points are possible states of a classical system (so called Phase Space).

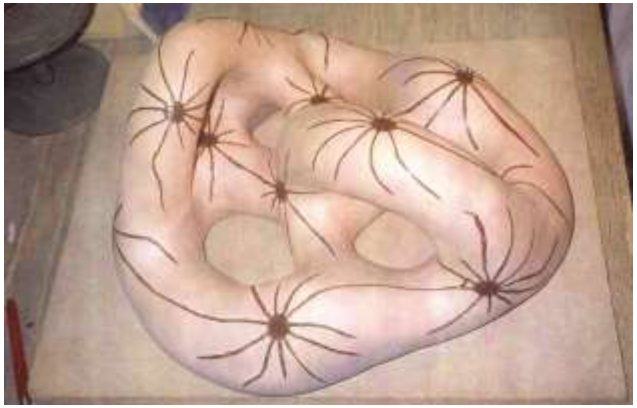
Phase space can be flat:



But it can also be curved:







What are X and P here?

In general, X and P are just some local coordinates on our phase space M. There are lots of possible choices for them locally, but no good choice globally.

Instead of a simple formula for a Poisson bracket, we have some generic bracket operation taking two function f and g as arguments and spitting out a third function.

$$f, g \mapsto \{f, g\}$$

This bracket operation is called the Poisson bracket.
It is needed to write down equations of motion in classical mechanics:

$$\frac{df(X,P)}{dt} = \{H(X,P), f(X,P)\}$$

Here H(X,P) is the Hamiltonian of the system (i.e. the energy function).

The Poisson bracket must have a number of properties ensuring that equations make both mathematical and physical sense.

Properties of the Poisson bracket

- $\{f,g\}$ is linear in both f and g.
- $\{f,g\}=-\{g,f\}$
- $\{f\cdot g,h\}=f\cdot\{g,h\}+g\cdot\{f,h\}$ (Leibniz rule)
- $\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,f\},h\}=0$ (Jacobi identity)

So, can one take an arbitrary phase space, with an arbitrary Poisson bracket, and quantize it?

That is, can one find a non-commutative but associative multiplication rule such that

$$f \star g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \text{terms of order } \hbar^2 \text{ and higher}$$

This is the basic problem of Deformation Quantization.

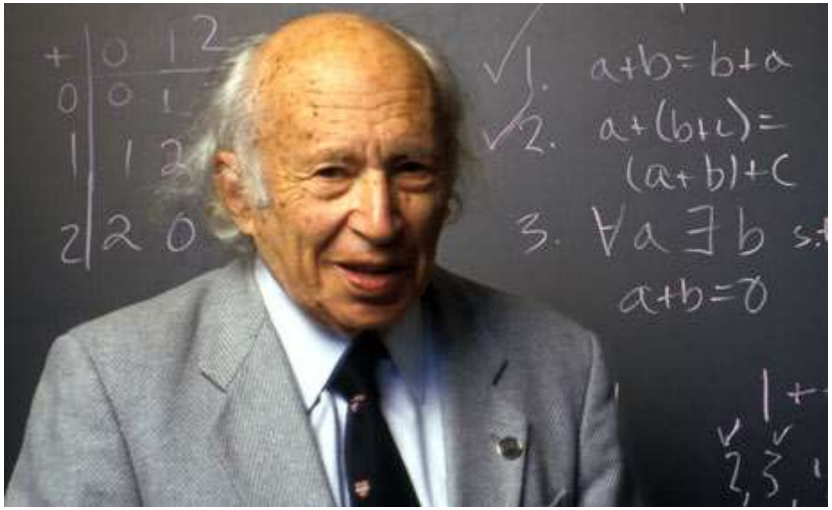
A space X with a non-commutative rule for multiplying functions on X is an example of a quantum space (or non-commutative space).

Quantum geometry is the study of such “spaces”.

The goal of Deformation Quantization is to turn a Poisson space (a space with a Poisson bracket) into a non-commutative space.

The idea to replace a commutative algebra of functions with a non-commutative one and treat it as the algebra of functions on a non-commutative space has been very fruitful.

The motivation comes from the work of I. M. Gelfand and M. A. Naimark in functional analysis (1940s) and A. Grothendieck in algebraic geometry (1950s).



Israel Gelfand (1913-2009) was a famous Soviet mathematician and Maxim's mentor.

Somewhat atypically for pure mathematicians of his era, Gelfand maintained a life-long interest in physics.

(In fact, this was less atypical in the Soviet Union: other names which could be mentioned are V. I. Arnold, S. P. Novikov, and Yu. I. Manin.)

Some Poisson spaces look locally like a flat phase space with its Poisson bracket. That is, around every point there are local coordinates X^i and P_i such that

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial X^i}\frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial X^i}\frac{\partial f}{\partial P_i} \right)$$

For such spaces (called symplectic) existence of deformation quantization was proved by De Wilde and Lecomte (1983) and Fedosov (1994).

For symplectic spaces the existence of quantization is very plausible on physical grounds.

But the general case seems much more difficult, because Poisson bracket may ``degenerate'' at special loci. It is not even clear why quantization should exist.

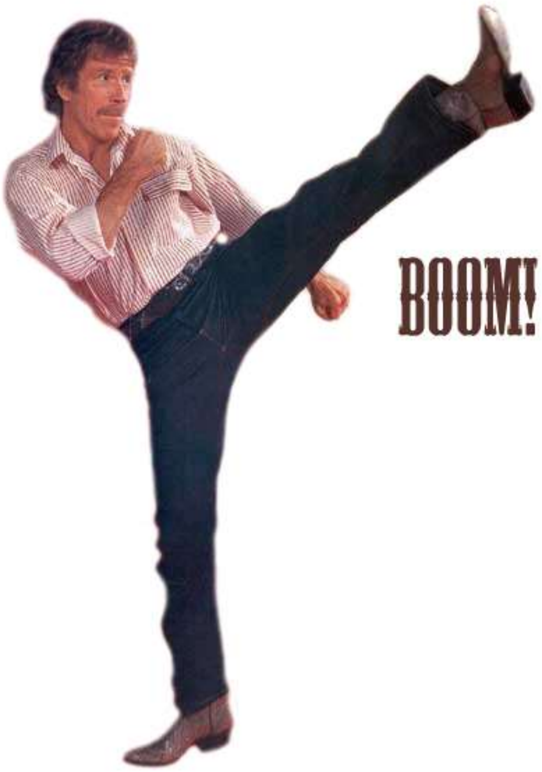
That is why it was a big surprise when Maxim proved in 1997 that every Poisson manifold can be quantized:

M. Kontsevich, *Deformation Quantization of Poisson manifolds,* Lett. Math. Phys. **66** (2003) 157-216.

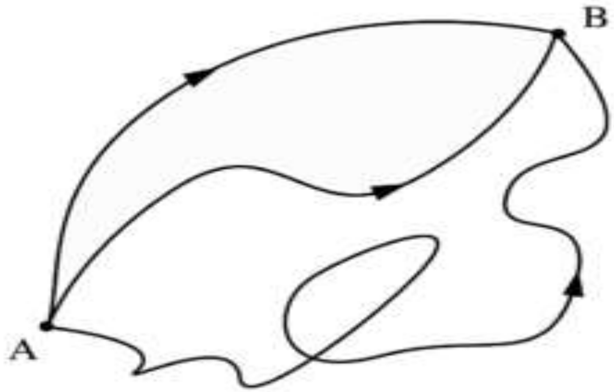
Maxim deduced this from his Formality Theorem, which I do not have time to explain.

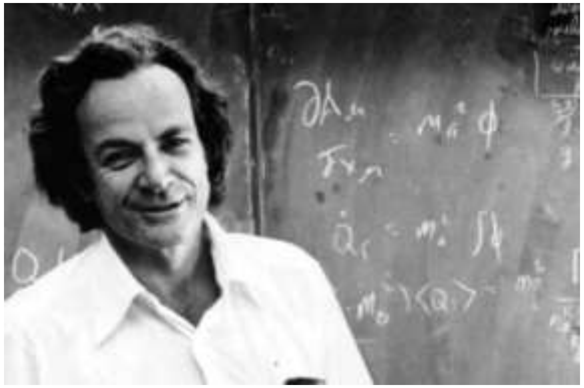
How did Maxim do it???

His signature move: Feynman diagrams.



Feynman told us that to do quantum mechanics
one has to compute the path-integral (sum over histories)





But Feynman not only invented the path-integral.

He also proposed a method to compute it.

The idea is to disregard interactions of particles, at least in the beginning.

Then the path-integral is “easy” to compute.

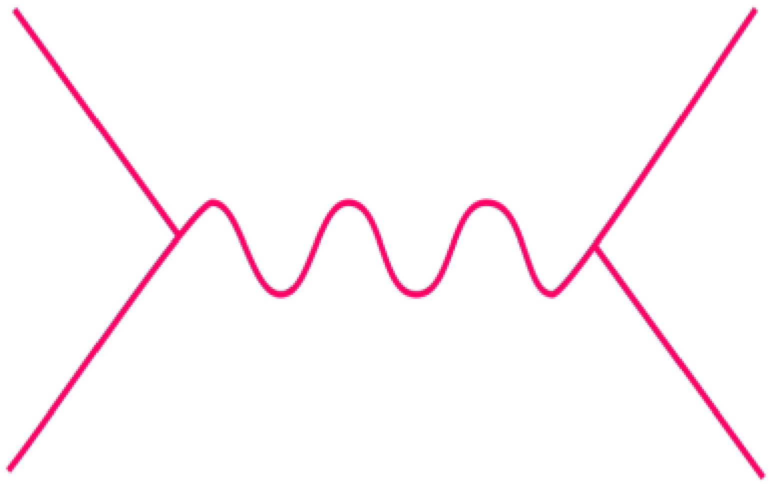
But the result is not very accurate, because we completely neglected all interactions.

Next we assume that particles have interacted at most once. The calculation is a bit more difficult, we get a more accurate result.

Next we assume that particles have interacted at most twice. This is an even harder calculation.

And so on. This is called perturbation theory.

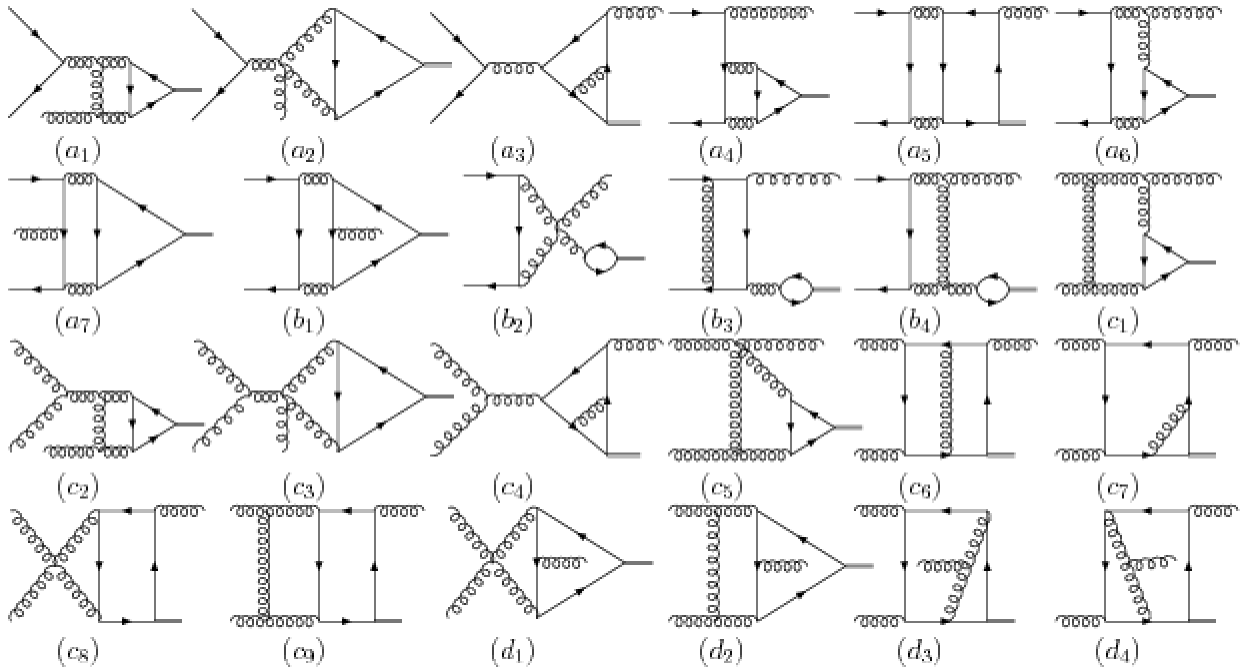
Feynman's genius was to realize that each possible way for particle to interact can be represented by a picture.



After one draws all possible pictures, one computes a mathematical expression for each picture, following specific rules (Feynman rules).

Particle physics is, to a large extent, the art of computing Feynman diagrams.

Sometimes, physicists need to evaluate hundreds or thousands Feynman diagrams.



Some features of quantum systems are not captured by Feynman diagrams.

They go beyond perturbation theory and therefore are called non-perturbative features.

But the best understood part of quantum theory is still perturbation theory, and all physicists (but hardly any mathematicians) learn it.

Back to Deformation Quantization!

Maxim had the idea that the non-commutative multiplication rule can be obtained from Feynman diagrams.

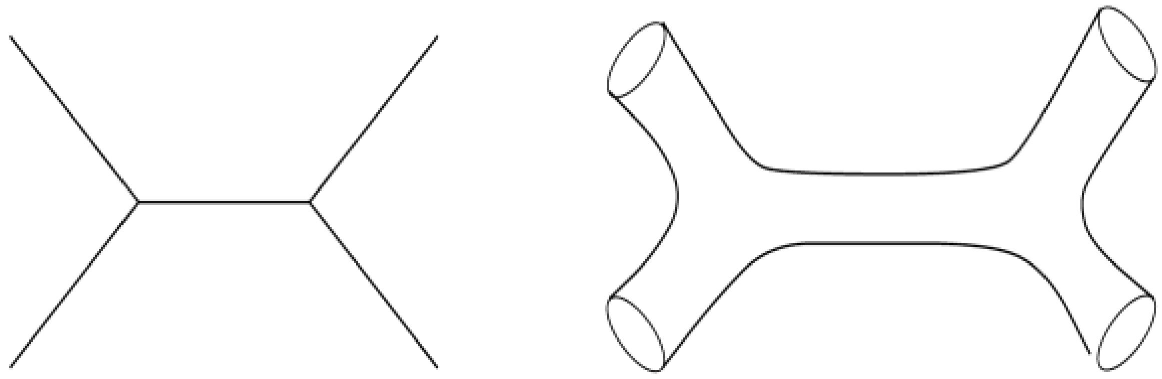
The magic of the path-integral ensures that the rule is associative, but not commutative.

A further twist: the path-integral that needs to be turned into diagrams describes not particles, but strings!

String theory has a reputation of being very complicated and, somehow, new agey.

In fact, string theory has nothing to do with yoga, auras and alternative medicine.

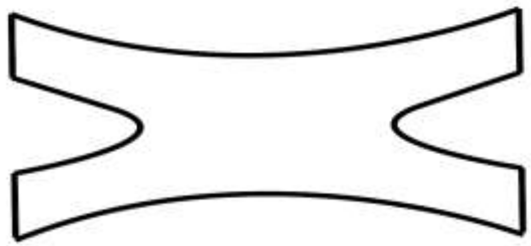
And the basic idea of string theory is simple:
take Feynman diagrams and thicken every particle trajectory into a trajectory swept out by a string.



The picture on the right describes the history of two loops of string merging into a single loop and them parting their ways again.

There are stringy Feynman rules which translate the picture on the right into a mathematical formula for the probability of this process.

One can also have bits of string instead of loops. In this picture two bits merge into one and then break apart again:



Bits of string are called open strings, loops are called closed strings.

For Deformation Quantization, one needs to use open strings.

In the “usual” string theory, strings move in physical space, perhaps with some extra hidden dimensions added.

Maxim's idea was to consider strings moving in the phase space of the classical system to be quantized.

In his paper Maxim did not explain this, but just wrote down the stringy Feynman rules.

The fact that these rules give rise to associative product looked like magic.

Later A. Cattaneo and G. Felder showed how to derive these Feynman rules from a path-integral for open strings.

String theory   Feynman rules

↘ ↙

Deformation Quantization

Varieties of Quantum Geometry

- Non-commutative geometry:
  - Quantization of phase space ⋎
  - Hidden dimensions may be non-commutative (A. Connes)
- Stringy geometry
  - Mirror symmetry ⋎
  - Hidden dimensions in non-perturbative string theory (M-theory, strings in low dimensions) ⋎

Quantized space-time

The idea that physical space-time should be quantized and perhaps non-commutative is attractive.

Motivation: in quantum gravity, one cannot measure distances shorter than some minimal length.

Reason: achieving a very accurate length measurement requires a lot of energy, which may curve the space-time and distort the result.

What is the structure of space-time at very short length and time scales?

Is it non-commutative? Is it stringy?

It is a safe bet that answering these physical questions will require entirely new math.

The End