3 Around the Chern–Simons Functional

In this section we move from surfaces to threefolds, along with connections and bundles on them. By studying the topological Chern–Simons action functional on the space of connections on a three-dimensional manifold with boundary, we recover the definition of the symplectic structure on the moduli space of flat connections on a compact Riemann surface. Similarly, the holomorphic Chern–Simons functional on $\bar{\partial}$-connections over three-dimensional Fano manifolds is related to the holomorphic symplectic structure on the moduli spaces of stable bundles over $K3$ or abelian surfaces.

Furthermore, the corresponding path integrals for these Chern–Simons functionals in the abelian case can be used to define the Gauss linking number of oriented curves in three-dimensional space and its holomorphic analogue, the polar linking number of holomorphic curves.

3.1 A Reminder on the Lagrangian Formalism

A motion of a particle on a manifold can be described by the least action principle. Consider an action functional

$$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt$$

defined on the space $\mathcal{C}[t_0, t_1]$ of smooth maps $q : [t_0, t_1] \to M$ of the interval $[t_0, t_1]$ to the manifold M. Here L is a (time-dependent) Lagrangian function, $L : TM \times \mathbb{R} \to \mathbb{R}$, which we assume to depend only on t, q, and its first derivative $\dot{q} := dq/dt$.

For a path variation δq one can find the corresponding *variation of the action functional*, i.e., the linear-in-δq term of the difference $S[q + \delta q] - S[q]$:

$$\delta S[q] = \int_{t_0}^{t_1} E\, \delta q\, dt + p\, \delta q|_{t_0}^{t_1}\ ,$$

where

$$E := \frac{\partial L(q, \dot{q}, t)}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}}$$

and

$$p := \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}}\ .$$

(Here and below we assume the summation over the coordinates $q = (q^1, \ldots, q^d)$: $p\delta q := \sum_j p_j \delta q^j, \quad p_j := \partial L(q, \dot{q}, t)/\partial \dot{q}^j$, etc.)

Exercise 3.1 Prove the variation formula. (Hint: use integration by parts.)

This way the variation δS can be regarded as a 1-form on the infinite-dimensional space $\mathcal{C}[t_0, t_1]$ of "virtual trajectories" of the particle.

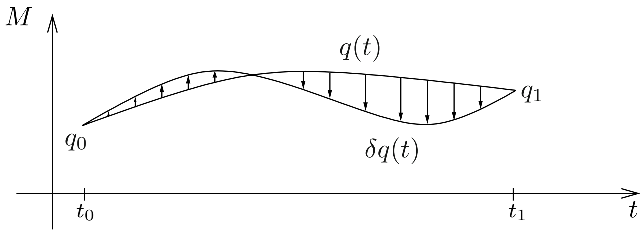

Fig. 3.1. A small variation of the path $q(t)$ with fixed endpoints.

Definition 3.2 The *least action principle* states that the actual trajectories of the particle are the critical points of this action functional: $\delta S[q] = 0$.

By confining ourselves to variations with fixed ends, $\delta q(t_0) = \delta q(t_1) = 0$, we come to a necessary condition on the extremals. Namely, actual particle trajectories satisfy the *Euler–Lagrange equation* $E = 0$, i.e.,

$$\frac{\partial L(q, \dot{q}, t)}{\partial q} - \frac{d}{dt}\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} = 0\,.$$

Denote by $\mathcal{E}[t_0, t_1]$ the space of all solutions to the Euler–Lagrange equation, i.e., the space of such trajectories.

Exercise 3.3 A free particle of mass m moving in the space $\mathbb{R}^d$ with a potential energy $V : \mathbb{R}^d \to \mathbb{R}$ has the Lagrangian $L(q, \dot{q}, t) = m|\dot{q}|^2/2 - V(q)$, the difference of its kinetic and potential energies. Prove that the Euler–Lagrange equation for this L gives the *Newton equation* of motion:

$$m\ddot{q} = -\mathrm{grad}\; V(q)\,.$$

Now we restrict the variation 1-form δS to the space of extremals $\mathcal{E}[t_0, t_1]$, which is singled out by the Euler–Lagrange equation. On this space of "trajectories with free ends" we obtain

$$\delta S = p\,\delta q|_{t_0}^{t_1} = \sigma_1 - \sigma_0\,, \tag{3.5}$$

where $\sigma_i := p\,\delta q|_{t_i}$, $i = 0, 1$ are the corresponding 1-forms on $\mathcal{C}[t_0, t_1]$. One can regard the above as a relation between these three 1-forms: σ_0, σ_1, and δS, which holds for their restrictions to the space of extremals $\mathcal{E}[t_0, t_1]$.

Now, by applying the exterior differential δ (on the infinite-dimensional manifold $\mathcal{C}[t_0, t_1]$) to both sides of the relation (3.5) above and using $\delta^2 = 0$, we obtain $\delta\sigma_0 = \delta\sigma_1$, which holds on $\mathcal{E}[t_0, t_1]$. This means that the space $\mathcal{E}[t_0, t_1]$ turns out to be naturally equipped with a closed 2-form ω defined by

$$\omega := \delta\sigma_0 = \delta\sigma_1\,.$$

Definition 3.4 A manifold N equipped with a closed 2-form ω (not necessarily nondegenerate) is called *presymplectic*.

Consider the distribution of null-spaces of this 2-form in N.

Exercise 3.5 (i) Assuming that this distribution has constant rank, prove that it is integrable, i.e., it is tangent to a foliation in N.

(ii) Assuming that this null-foliation is a fibration $\pi : N \to N'$, prove that the base of this fibration carries a natural symplectic structure, i.e., (N', ω') is a symplectic manifold such that $\pi^*\omega' = \omega$.

The above discussion shows that whenever the space of extremals $\mathcal{E}[t_0, t_1]$ is a manifold, it is in fact a *presymplectic* manifold. However, the 2-form ω is often degenerate. The *phase space* $\mathcal{P}$ of the particle can be described as the corresponding *symplectic* manifold. (Here we implicitly assume that various regularity conditions are satisfied to guarantee that both $\mathcal{E}[t_0, t_1]$ and the phase space are smooth manifolds.)

Exercise 3.6 Check that for the above example of a particle motion in $\mathbb{R}^d$ this definition of the phase space $\mathcal{P}$ coincides with $T^*\mathbb{R}^d$ equipped with the natural symplectic structure.

Remark 3.7 [393, 79, 341] The discussed Lagrangian formalism can be generalized to infinite-dimensional target manifolds M or to higher-dimensional domains instead of the interval $[t_0, t_1]$. These are the objects that a field theory deals with. Consider, for example, a local action functional

$$S[\varphi] = \int_N L(\varphi(x), \partial\varphi(x))\, d^n x$$

describing a field theory on an n-dimensional manifold N with boundary ∂N. Here $x = (x_1, \ldots, x_n)$ are local coordinates on N, φ is a map from N to a target manifold M or a section of some bundle on N, $\partial\varphi$ are the first derivatives of φ, while the Lagrangian L can depend on additional structures on N. As in the one-dimensional situation described above, one can pose a variational problem $\delta S[\varphi] = 0$, which leads to the Euler–Lagrange equations.

Suppose first that $N = I \times \Sigma$, where I is an interval and a manifold Σ has dimension $n-1$. One can consider $t \in I$ as the time variable and identify the field theory with an infinite-dimensional classical mechanics, where the space of maps $\varphi : \Sigma \to M$ plays the role of the target. In particular, one has a presymplectic manifold of extremals $\mathcal{E}_N$ and the symplectic phase space $\mathcal{P}$ associated to N (or, rather, to Σ).

Alternatively, one can associate the phase spaces $\mathcal{P}_0$ and $\mathcal{P}_1$ to the corresponding boundary components $\partial N = \Sigma_1 - \Sigma_0$ of N, and equip the total phase space $\mathcal{P}_0 \times \mathcal{P}_1$ with the product symplectic structure. There is a natural projection α_N of the space $\mathcal{E}_N$ of extremals into the product $\mathcal{P}_0 \times \mathcal{P}_1$, since it "tautologically" projects to each factor: one describes the extremals via different boundary components, taking the orientation of the latter into account. Then the relation $0 = \delta^2 S = \delta\sigma_1 - \delta\sigma_0$ that held on $\mathcal{E}_N$ now reads that the image $\alpha_N(\mathcal{E}_N)$ of $\mathcal{E}_N$ is an *isotropic submanifold* in the symplectic manifold $\mathcal{P}_0 \times \mathcal{P}_1$.

Definition 3.8 A submanifold of a symplectic manifold is *isotropic* if the restriction of the symplectic form to this submanifold is zero.

Exercise 3.9 Let $f : (N_1, \omega_1) \to (N_2, \omega_2)$ be a diffeomorphism between two symplectic manifolds. Prove that f is a symplectic map, i.e., $\omega_1 = f^*\omega_2$, if and only if the graph of f is an isotropic submanifold in the symplectic manifold $(N_1 \times N_2,\, \omega_1 \ominus \omega_2)$.

(An isotropic submanifold of maximal possible dimension, which is equal to half the dimension of the symplectic manifold, is called a *Lagrangian submanifold*; cf. Section I.4.5. This is the case for the graph of f.)

One can see that the image $\alpha_N(\mathcal{E}_N)$ is indeed isotropic in $\mathcal{P}_0 \times \mathcal{P}_1$, since the 2-form $\delta\sigma_1 - \delta\sigma_0$ is exactly the restriction of the product symplectic structure of $\mathcal{P}_0 \times \mathcal{P}_1$ (with different orientations of the boundary components) to this image.

The latter formulation of the presymplectic/isotropic properties of the space of extremals $\mathcal{E}_N$ extends naturally to the general case of a manifold N with boundary consisting of several components $\Sigma_1, \ldots, \Sigma_k$. Associate the phase space $\mathcal{P}_j$ to each component Σ_j, thinking of a neighborhood of Σ_j in N as a product $I \times \Sigma$. One has the relations $\delta S = \sigma_1 + \cdots + \sigma_k$ and $\delta\sigma_1 + \cdots + \delta\sigma_k = 0$ on the space of extremals $\mathcal{E}_N$, where σ_j stands for the contribution of the corresponding boundary component. The latter shows that the image $\alpha_N(\mathcal{E}_N)$ under the natural map $\alpha_N : \mathcal{E}_N \to \mathcal{P}_1 \times \cdots \times \mathcal{P}_k$ is *isotropic* with respect to the product symplectic structure on the phase space $\mathcal{P}_1 \times \cdots \times \mathcal{P}_k$. We refer to [341, 79] for more details.

Remark 3.10 The philosophy of holomorphic orientation (see Sections 2.2 and 2.3) can be applied to field-theoretic notions in the following way. Suppose we have an action functional

$$\mathcal{S}[\varphi] = \int_M L(\varphi, \partial\varphi)\, d^n x$$

on *smooth* fields φ (e.g., functions, connections, etc.) on a *real* (oriented) manifold M, and this functional is defined by an n-form $L\, d^n x$, which depends on the fields and their derivatives.

Then one can suggest the following complex analogue $\mathcal{S}_{\mathbb{C}}$ of the action functional $\mathcal{S}$ for a *complex* n-dimensional manifold X equipped with a "polar orientation," i.e., with a holomorphic or meromorphic n-form μ:

$$\mathcal{S}_{\mathbb{C}}[\varphi] := \int_X \mu \wedge L(\varphi, \bar\partial\varphi)\, d^n \bar{x}\,.$$

Here φ stands for *smooth* fields on a complex manifold X. Now the $(0,n)$-form $L\, d^n\bar{x}$ is integrated against the holomorphic orientation μ over X.

Furthermore, the interrelation between the extremals of the real functional $\mathcal{S}[\varphi]$ (on smooth fields) on the real manifold M and the boundary values of those fields on ∂M is replaced by the analogous interrelation for the complex functional $\mathcal{S}_{\mathbb{C}}[\varphi]$ (still on smooth fields) on a complex manifold X (equipped with an n-form μ) and on the polar divisor $Y := \operatorname{div}_\infty \mu \subset X$ (equipped with the residue $(n-1)$-form $\nu := \operatorname{res} \mu$).

The above discussion will allow us to see in the next two sections how the symplectic structures on the moduli of flat connections and holomorphic bundles on surfaces arise naturally from the Lagrangian formalism related to the topological and holomorphic Chern–Simons functionals.

3.2 The Topological Chern–Simons Action Functional

Let N be a real compact oriented three-dimensional manifold with boundary $\partial N = \Sigma$. As usual in the "real case," we take G to be a compact simply

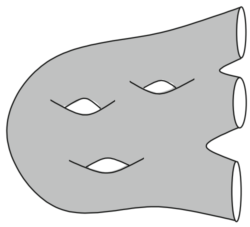

Fig. 3.2. Three-dimensional manifold N with boundary $\partial N = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

connected simple Lie group with the corresponding Lie algebra $\mathfrak{g}$. Denote the nondegenerate invariant (Killing) bilinear form on $\mathfrak{g}$ by $\operatorname{tr}(XY) := \langle X, Y\rangle$. Fix a trivial G-bundle E over N and let $\mathcal{A}$ denote the space of connections in the bundle E. Upon fixing a reference flat connection, we think of $\mathcal{A}$ as the space $\Omega^1(N, \mathfrak{g})$.

Definition 3.11 The topological *Chern–Simons action functional* is the following real-valued function on the space of connections $\mathcal{A}$:

$$\mathrm{CS}(A) := \int_N \operatorname{tr}(A \wedge dA) + \frac{2}{3}\int_N \operatorname{tr}(A \wedge A \wedge A)\,,$$

where a connection $A \in \mathcal{A}$ is understood as a $\mathfrak{g}$-valued 1-form on N.

Proposition 3.12 *The set of extremals, i.e., solutions of the Euler–Lagrange equation, for the Chern–Simons functional* CS *is the space of flat connections in the G-bundle E over the manifold N.*

PROOF. For a small variation δA of a connection $A \in \mathcal{A}$ the corresponding variation of the functional is

$$\begin{aligned}\delta\,\mathrm{CS} &= \int_N \mathrm{tr}(\delta A \wedge dA) + \int_N \mathrm{tr}(A \wedge d\delta A) + 2\int_N \mathrm{tr}(\delta A \wedge A \wedge A)\\ &= \int_N d\ \mathrm{tr}(A \wedge \delta A) + 2\int_N \mathrm{tr}\left(\delta A \wedge (dA + A \wedge A)\right)\\ &= \int_{\partial N} \mathrm{tr}(A \wedge \delta A) + 2\int_N \mathrm{tr}\left(\delta A \wedge (dA + A \wedge A)\right)\,,\end{aligned}$$

where at the last step we used the Stokes formula.

By imposing the boundary condition $\delta A|_{\partial N} = 0$ on variations δA, we obtain the Euler–Lagrange equation

$$dA + A \wedge A = 0\,,$$

i.e., the equation of vanishing curvature $F(A) = 0$ on N. Hence the space of solutions of this equation is exactly the space of flat connections on the real threefold N. □

The first term in the above calculation of $\delta\,\mathrm{CS}$ gives the boundary contribution, the 1-form $\sum \sigma_j$ on the extremals, where the summation is taken over the boundary components of ∂N. Take $N = I \times \Sigma$ to be a finite cylinder over a closed two-dimensional surface Σ. Then the presymplectic structure on the space of flat connections on N, i.e., on the extremals for our action functional, is $\omega = \delta\sigma$ for

$$\sigma := \int_\Sigma \mathrm{tr}(a \wedge \delta a)\,,$$

where $a := A|_\Sigma$ denotes the restriction of a flat connection A from the manifold N to either of its boundary components Σ. (Here we omit the index $j = 0, 1$ for σ_j, since $\omega = \delta\sigma_0 = \delta\sigma_1$.)

Exercise 3.13 Verify that the 2-form $\omega = \delta\sigma$ is degenerate on the space of flat connections on the surface Σ exactly along the gauge equivalence classes of the connections $\{a\}$. (Hint: the 2-form $\delta\sigma = \int_\Sigma \mathrm{tr}(\delta a \wedge \delta a)$ is the restriction of the canonical 2-form ω from the set of all connections to the subset of flat connections on Σ; cf. Definition 2.1.)

Thus the moduli space of flat connections $\mathcal{M}^\Sigma$ on the surface Σ appears as the natural symplectic (or phase) space for this presymplectic space of flat connections on Σ, and we obtain yet another definition of the symplectic structure on $\mathcal{M}^\Sigma$ from Section 2.1.

Corollary 3.14 *The moduli space $\mathcal{M}^\Sigma$ of flat connections on a surface Σ is naturally symplectic as the phase space for extremals of the Chern–Simons action functional for connections on the threefold $N = I \times \Sigma$.*

Remark 3.15 To see why this action functional is called *topological* we now check the invariance property of the Chern–Simons action with respect to gauge transformations of the connections. Let M be a compact three-dimensional manifold *without boundary* and suppose that A and $\widetilde{A}$ are connections in a G-bundle over M that are sent to each other by a gauge transformation g:

$$\widetilde{A} = gAg^{-1} - dgg^{-1} .$$

Then the Chern–Simons actions for them are related as follows:

$$\mathrm{CS}(\widetilde{A}) = \mathrm{CS}(A) + \frac{1}{3}\int_M \mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) .$$

Recall that the 3-form $\frac{1}{24\pi^2}\,\mathrm{tr}(g^{-1}dg)^{\wedge 3}$ is the pullback under the map $g : M \to G$ of an integral closed 3-form η on the compact simply connected simple Lie group G (see Proposition 2.16 in Appendix A.2; cf. Section II.1.3). Thus the integral of this form depends only on topological properties of the map g and can be expressed as

$$\frac{1}{24\pi^2}\int_M \mathrm{tr}(g^{-1}dg)^{\wedge 3} = \int_M g^*\eta \,,$$

which is an integer, since the 3-form η generates $H^3(G,\mathbb{Z})$. The latter implies that the exponential $\exp\left(\frac{i}{4\pi}\,\mathrm{CS}(A)\right)$ is gauge invariant:

$$\frac{i}{4\pi}\,\mathrm{CS}(\widetilde{A}) - \frac{i}{4\pi}\,\mathrm{CS}(A) = 2\pi i \cdot \frac{1}{24\pi^2}\int_M \mathrm{tr}(g^{-1}dg)^{\wedge 3} \in 2\pi i \cdot \mathbb{Z} .$$

Remark 3.16 An interesting integer-valued invariant for a homology 3-sphere M was introduced by Casson and is closely related to the gauge-theoretic constructions above [70]. Roughly speaking, the *Casson invariant* $\mathrm{Cas}(M)$ is defined as the algebraic number of the conjugacy classes of irreducible SU(2)-representations of the fundamental group $\pi_1(M)$. In other words, it counts the number of irreducible flat SU(2)-connections on M modulo conjugation. The homology restriction on the threefold M is related to the fact that if $H_1(M) \neq 0$, then the moduli space of flat connections on M might not be zero-dimensional, and in particular, it would not consist of a finite number of points. The reason for restricting to SU(2) is clarified in the following exercise.

Exercise 3.17 Show that the only reducible representation $\rho : \pi_1(M) \to \mathrm{SU}(2)$ is the trivial one. (Hint: Reducible representations of $\pi_1(M)$ in SU(2) are necessarily abelian and hence factor through the homology $H_1(M)$. This homology group is trivial for a homology 3-sphere.)

Now consider a Heegaard splitting of M into two handlebodies $M = M_1 \cup_\Sigma M_2$ glued together along their common boundary, an embedded surface $\Sigma \subset M$. Consider the moduli space $\mathcal{M}^\Sigma$ of flat connections in the trivial

SU(2)-bundle on the surface Σ. Define two submanifolds L_1 and L_2 of the symplectic manifold $\mathcal{M}^\Sigma$ as those (equivalence classes of) flat connections on the surface Σ that extend to M_1 and M_2 respectively. One can show that these submanifolds are Lagrangian. Their intersection points $L_1 \cap L_2$ correspond to flat connections extendable to the whole of M. Thus the Casson invariant is defined as the intersection number of these submanifolds,

$$\mathrm{Cas}(M) = \#(L_1, L_2)\,,$$

where we assume that the submanifolds intersect transversally, and exclude the intersection corresponding to the trivial representation; see details, for example, in [364].

3.3 The Holomorphic Chern–Simons Action Functional

A complex three-dimensional manifold X equipped with a nowhere vanishing meromorphic 3-form μ can be regarded as a complex analogue of a real oriented manifold with boundary, following the general philosophy that we adopted in Sections 2.2 and 2.3. Accordingly, one can complexify the Lagrangian formalism to this situation. Here we define a holomorphic analogue of the Chern–Simons action functional for (X, μ) and relate it to Mukai's holomorphic symplectic structures on moduli of holomorphic bundles over complex surfaces, following [195, 85].

Let $G_\mathbb{C}$ be a complex simple and simply connected Lie group and $E_\mathbb{C}$ a complex $G_\mathbb{C}$-bundle over the manifold X. As before, let us denote by $\mathcal{A}^X_\mathbb{C}$ the space of $(0,1)$-connections in the bundle $E_\mathbb{C}$.

Definition 3.18 The *holomorphic Chern–Simons action functional* $\mathrm{CS}_\mathbb{C} : \mathcal{A}^X_\mathbb{C} \to \mathbb{C}$ is defined via

$$\mathrm{CS}_\mathbb{C}(A) := \int_X \mu \wedge \left(\langle A \wedge \bar\partial A \rangle + \frac{2}{3} \langle A \wedge A \wedge A \rangle \right)$$

for any $(0,1)$-connection $A \in \mathcal{A}^X_\mathbb{C}$ thought of as a $\mathfrak{g}_\mathbb{C}$-valued $(0,1)$-form on X. As usual, we assume that the 3-form μ has only first-order poles, and hence the integral above is well defined.

Proposition 3.19 *The extremals of the holomorphic Chern–Simons functional are holomorphic structures in the complex bundle $E_\mathbb{C}$.*

Proof. Indeed, in the same way as in the real case and by using the Cauchy–Stokes formula we come to the Euler–Lagrange equation

$$\bar\partial A + A \wedge A = 0$$

in the holomorphic setting. Its solutions are $(0,1)$-connections A with vanishing $(0,2)$-curvature, $F^{0,2}(A)=0$, and each such connection defines the corresponding holomorphic structure in the complex bundle $E_{\mathbb{C}}$. □

Consider now the "boundary term" of the variation $\delta\,\mathrm{CS}_{\mathbb{C}}$, which now descends to the polar divisor of the meromorphic 3-form μ. Denote this polar divisor by $Y := \mathrm{div}_{\infty}\mu \subset X$. Note that the residue $\nu := \mathrm{res}_Y\,\mu$ is a nonvanishing 2-form on the divisor Y, since μ itself is nonvanishing (see Exercise 2.19). In particular, the canonical bundle of Y has to be trivial, so that Y is either a $K3$ surface or a complex torus.

To define the presymplectic structure in the real case we considered a cylinder $M = I \times \Sigma$ over a Riemann surface Σ. Here we look at the complex analogue of such a cylinder. Namely, let $X = \mathbb{CP}^1 \times Y$ be the product of $\mathbb{CP}^1$ and a $K3$ surface or abelian surface Y. Suppose that Y is endowed with a holomorphic (necessarily nonvanishing) 2-form ν, and consider the meromorphic 3-form $\mu = (dz/z)\wedge\nu$ on X, where dz/z is a 1-form on the complex line $\mathbb{CP}^1$. One can see that $\nu = \mathrm{res}_{z=0}\,\mu = -\,\mathrm{res}_{z=\infty}\,\mu$.

Now the variation of the holomorphic Chern–Simons functional satisfies the relation $\delta\,\mathrm{CS}_{\mathbb{C}} = \sigma_{0,\mathbb{C}} + \sigma_{\infty,\mathbb{C}}$ on the space of extremals, which are the integrable $(0,1)$-connections on X, i.e., the connections with vanishing $(0,2)$-curvature. Here $\sigma_{0,\mathbb{C}}$ and $\sigma_{\infty,\mathbb{C}}$ stand for the contributions of the corresponding components $z=0$ and $z=\infty$ of the polar divisor of μ.

This allows us to introduce the *holomorphic presymplectic* structure $\omega_{\mathbb{C}} = \delta\sigma_{\mathbb{C}}$ on the "boundary values" of the extremals, i.e., on the space of integrable connections on the surface Y. Explicitly, the holomorphic 1-form $\sigma_{\mathbb{C}}$ is

$$\sigma_{\mathbb{C}} := \int_Y \nu \wedge \mathrm{tr}(a \wedge \delta a)\,,$$

where $a := A|_{z=0}$ is the restriction of a $(0,1)$-connection A in $E_{\mathbb{C}}$ from the threefold X to the surface Y (understood as one component $\{z=0\}\times Y \subset X$ of the polar divisor of μ), δa is the corresponding variation of a, and $\nu = \mathrm{res}_{z=0}\,\mu$ is a holomorphic 2-form on Y.

One can show that, similarly to the real case, the presymplectic structure $\omega_{\mathbb{C}}$ is degenerate along the orbits of the action of the complex group of gauge transformations $G^Y_{\mathbb{C}}$ on integrable $(0,1)$-connections (i.e., holomorphic structures) in the bundle $E_{\mathbb{C}}$ over Y. After taking the quotient with respect to the group action, we obtain a nondegenerate holomorphic symplectic structure on the moduli space of (stable) holomorphic bundles on the $K3$ or abelian surface Y. (Here, as usual, we are concerned with the moduli space only locally around a smooth point.) Thus the holomorphic Lagrangian formalism gives an alternative approach to Mukai's result discussed before:

Theorem 3.20 ([283]) *There exists a holomorphic symplectic structure $\omega_{\mathbb{C}}$ on the moduli space $\mathcal{M}_Y$ of stable holomorphic $G_{\mathbb{C}}$-bundles over a $K3$ or abelian surface Y.*

Remark 3.21 It turns out that there exists a holomorphic analogue of the Casson invariant for a Calabi–Yau manifold X; see [85, 366]. Instead of a Heegaard splitting of a real manifold, one considers a degeneration of this CY manifold to an intersection of two Fano manifolds. The divisor of intersection is a $K3$ or abelian surface, and one counts in a special way the holomorphic bundles over Y extendable to both of these two Fano manifolds.

We also note that the holomorphic Chern–Simons action functional has more complicated transformation properties with respect to gauge transformations. After a "large" gauge transformation, the value of the functional differs by a multiple of the integrals $\int_X \mu \wedge g^*\eta$. The latter can be viewed as the integrals of the meromorphic 3-form μ over the three-cycles in X that are Poincaré dual to the 3-form $g^*\eta$ for various maps $g : X \to G_{\mathbb{C}}$. The values of these integrals can form a lattice or even a dense set in $\mathbb{C}$; hence considering the exponential similar to $\exp\left(\frac{i}{4\pi}CS(A)\right)$ does not allow one to extract a gauge-invariant quantity in the holomorphic setting.

3.4 A Reminder on Linking Numbers

Let M be a simply connected oriented manifold and let γ_1 and γ_2 be two nonintersecting oriented closed curves in M. Pick an oriented surface $D_1 \subset M$ (a Seifert surface for the curve γ_1) such that the curve γ_1 is the oriented boundary of the surface D_1 and such that D_1 and γ_2 intersect transversally.

Definition 3.22 The *linking number* $\mathrm{lk}(\gamma_1, \gamma_2)$ of the curves γ_1 and γ_2 is the intersection number of the surface D_1 and the curve γ_2, i.e., the number of intersections of the curve γ_2 with the surface D_1 counted with orientation (see Figure 3.3):

$$\mathrm{lk}(\gamma_1, \gamma_2) = \#(D_1, \gamma_2).$$

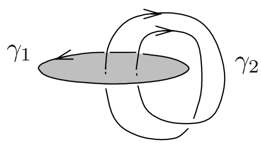

Fig. 3.3. Linking of two oriented curves.

The sign at each intersection point is obtained by forming there a frame from the orientation frames for D_1 and γ_2, and comparing it with the orientation of the ambient manifold M.

Proposition 3.23 *The linking number* $\mathrm{lk}(\gamma_1, \gamma_2)$ *is*

(i) independent of the choice of a Seifert surface D_1*,*

(ii) symmetric in γ_1 *and* γ_2*,*

(iii) invariant with respect to isotopy of the curves, provided they do not intersect each other,

(iv) well defined in any (not necessarily simply connected) oriented three-dimensional manifold M*, provided that both curves* γ_1 *and* γ_2 *are homologous to* 0 *in* M*.*

Note that if the manifold M is not simply connected and only one of the curves is homologous to 0 in M, but the other is not, the linking number might not be well defined. For instance, take $M = \mathbb{T}^3$ and two curves, one of which is homologous to 0, while the other is a generator in $H_1(\mathbb{T}^3, \mathbb{Z})$. Then by taking different Seifert surfaces for the first curve one obtains either 0 or 1 for their linking number; see Figure 3.4.

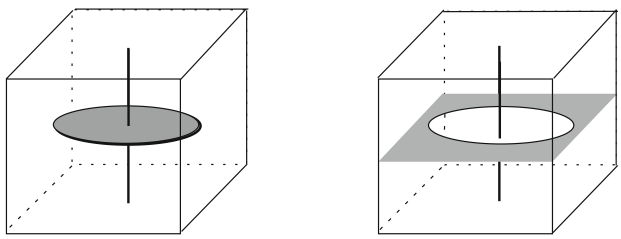

Fig. 3.4. Two Seifert surfaces for the horizontal circle in the cube-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, one "inside" and one "outside," give linking numbers ± 1 and 0, respectively, for the intersection with the "vertical" cycle.

Exercise 3.24 Prove the above proposition. Furthermore, show also that the linking number is actually invariant when the curve γ_1 changes to a curve (or a collection of curves) $\widetilde{\gamma}_1$ homologous to γ_1 in the complement $M \setminus \gamma_2$ (see Figure 3.5).

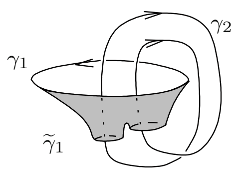

Fig. 3.5. The homologous curves γ_1 and $\widetilde{\gamma}_1$ have the same linking number with the curve γ_2.

Needless to say, the linking number easily generalizes to manifolds of any dimension n, provided that the linking submanifolds are homologous to zero and have "linking dimensions": the sum of their dimensions equals $n-1$.

Remark 3.25 There exists the Gauss integral formula for the linking number of two curves γ_1 and γ_2 in $\mathbb{R}^3$. Recall it here in a somewhat "symbolic form," which we need further.

Let $\mathit{\Delta} \subset M \times M$ denote the diagonal in $M \times M$ and let δ stand for its Poincaré dual current, a closed 3-form supported on the diagonal, $[\delta] \in H^3(M \times M, \mathbb{R})$. Then we can write

$$\mathrm{lk}(\gamma_1, \gamma_2) = \#(\mathit{\Delta}\,, D_1 \times \gamma_2) = \int_{x \in D_1} \int_{y \in \gamma_2} \delta(x, y)\,, \tag{3.6}$$

where $\#(\mathit{\Delta}\,, D_1 \times \gamma_2)$ is the intersection number of $\mathit{\Delta}$ and $D_1 \times \gamma_2 \subset M \times M$. One can split the 3-form δ into the homogeneous components

$$\delta = \delta^{3,0} + \delta^{2,1} + \delta^{1,2} + \delta^{0,3}\ ,$$

where $\delta^{i,j}$ denotes the component that is an i-form on the first factor of $M \times M$, and a j-form on the second factor. Note that in equation (3.6) we had to integrate only over the component $\delta^{2,1}$, since all the other integrals vanish. This component is an exact 2-form in x on D_1, which allows us to apply the Stokes formula:

$$\mathrm{lk}(\gamma_1, \gamma_2) = \int_{x \in D_1} \int_{y \in \gamma_2} \delta^{2,1}(x, y) = \int_{x \in \gamma_1} \int_{y \in \gamma_2} d_x^{-1} \delta^{2,1}(x, y)\ .$$

For $\mathbb{R}^3$ the $(1,1)$-form $d_x^{-1}\delta^{2,1}(x, y)$ on the torus $\gamma_1 \times \gamma_2$ assumes the standard *Gauss form*

$$\frac{1}{4\pi} \cdot \frac{(\overrightarrow{x-y}, \overrightarrow{dx}, \overrightarrow{dy})}{\|\overrightarrow{x-y}\|^3}\,,$$

where $(\cdot,\cdot,\cdot)$ is the mixed product of three vectors in $\mathbb{R}^3$.

Remark 3.26 In what follows we need a bit of calculus of such δ-type forms. Let δ_γ be the Dirac δ-type 2-form supported on a closed oriented curve γ in a simply connected threefold M. (Alternatively, the curve γ can be regarded as a de Rham current, a linear functional on 1-forms on M, whose value is the integral of the 1-form over γ.) The integral of this 2-form δ_γ over a two-dimensional surface counts the intersection number of this surface with the curve γ. Then by using the decomposition of the diagonal 3-form δ into the homogeneous components, we can express

$$\delta_\gamma(x) = \int_{y \in \gamma} \delta^{2,1}(x, y)\,,$$

where we denote the coordinates on the first and the second factors of $M \times M$ by x and y respectively. Choose a surface $D \subset M$ whose boundary is $\gamma = \partial D$. Similarly, we can define the δ-type 1-form supported on the surface D by

$$\delta_D(x) = \int_{y \in D} \delta^{1,2}(x, y) \,.$$

The relation $\partial D = \gamma$ is equivalent to the relation between the corresponding δ-forms: $d_x \delta_D(x) = \delta_\gamma(x)$, due to the Stokes theorem, or more explicitly,

$$\delta_\gamma(x) = \int_{y \in D} d_x(\delta^{1,2})(x, y) \ ,$$

where d_x denotes the exterior derivative applied to the x-coordinates only. Finally, if γ_1 and γ_2 are two nonintersecting curves, we have

$$\mathrm{lk}(\gamma_1, \gamma_2) = \int\limits_{x \in D_1} \delta_{\gamma_2}(x) = \int\limits_M \delta_{D_1}(x) \wedge \delta_{\gamma_2}(x) = \int\limits_M \delta_{D_1} \wedge d\delta_{D_2} \,, \tag{3.7}$$

where $\partial D_2 = \gamma_2$. The latter form suggests a common nature of the linking number and the $A \wedge dA$-part of the Chern–Simons functional, which we are going to study below.

3.5 The Abelian Chern–Simons Path Integral and Linking Numbers

We start with a reminder on finite-dimensional Gaussian integrals. Let (x, Qx) be a symmetric negative-definite form in the Euclidean $\mathbb{R}^n$. The classical Gauss integral

$$\int_{\mathbb{R}} \exp(-qx^2/2)dx = \sqrt{2\pi/q}$$

has the multidimensional analogue

$$\int_{\mathbb{R}^n} e^{\frac{1}{2}(x,Qx)}\, d^n x = \left(\frac{(2\pi)^n}{\det(-Q)}\right)^{\frac{1}{2}} .$$

Now fix a vector $J \in \mathbb{R}^n$ and consider the integral

$$Z_Q(J) := \int_{\mathbb{R}^n} e^{\frac{1}{2}(x,Qx)+(x,J)}\, d^n x = \int_{\mathbb{R}^n} e^{S_J(x)}\, d^n x$$

corresponding to the shift $S_J(x) := \frac{1}{2}(x, Qx) + (x, J)$ of the quadratic form by a linear term. (The initial integral is $Z_Q(0)$.) This integral can easily be solved by completing the square. Indeed, let x_0 be a solution of the equation $Qx_0 + J = 0$, i.e., $x_0 = -Q^{-1}J$. Then by introducing a shifted variable $\widetilde{x} = x - x_0$ and using the translation invariance of the measure $d^n x$, we obtain

$$\begin{aligned} Z_Q(J) &= \int_{\mathbb{R}^n} e^{S_J(\widetilde{x}+x_0)}\, d^n x \\ &= \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2}(\widetilde{x}+x_0, Q(\widetilde{x}+x_0)) + (\widetilde{x}+x_0, J)\right\} d^n x \\ &= \int_{\mathbb{R}^n} \exp\left\{\frac{1}{2}(\widetilde{x}, Q\widetilde{x}) + \frac{1}{2}(x_0, Qx_0) + (x_0, J)\right\} d^n \widetilde{x} \\ &= e^{S_J(x_0)} \int_{\mathbb{R}^n} e^{\frac{1}{2}(\widetilde{x}, Q\widetilde{x})}\, d^n\widetilde{x} = e^{\frac{1}{2}(x_0,J)}\, Z_Q(0)\,. \end{aligned}$$

Thus, we have

$$\frac{Z_Q(J)}{Z_Q(0)} = e^{S_J(x_0)} = e^{\frac{1}{2}(x_0,J)} = e^{-\frac{1}{2}(Q^{-1}J,J)}\,. \tag{3.8}$$

Remark 3.27 When the space $\mathbb{R}^n$ is replaced by some infinite-dimensional vector space, the integrals defining $Z_Q(0)$ and $Z_Q(J)$ usually do not make sense. However, one can "calculate" their ratio, which often turns out to be well defined. Note that the second of the equivalent expressions for the ratio $Z_Q(J)/Z_Q(0)$ in formula (3.8) has the form $\exp(\frac{1}{2}(x_0,J)) = \exp(-\frac{1}{2}(x_0,Qx_0))$, which allows us to avoid looking for the inverse Q^{-1} of the corresponding operator in the infinite-dimensional space.

Consider an application of this idea to the abelian Chern–Simons path integral. Let $\mathcal{A}$ be the space of connections in a U(1)-bundle over a real three-dimensional simply connected manifold M without boundary. We can think of such connections as real-valued 1-forms on M. Denote by $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ the Chern–Simons action functional on $\mathcal{A} = \Omega^1(M,\mathbb{R})$, which now becomes a quadratic form

$$\mathrm{CS}(A) = \int_M A \wedge dA\,,$$

since the group U(1) is abelian and the cubic term $A\wedge A\wedge A$ vanishes. Note that the kernel of this quadratic form is the space of exact 1-forms $d\Omega^0 \subset \Omega^1(M,\mathbb{R})$.

Fix some linear functional J on $\Omega^1(M,\mathbb{R})$, i.e., a de Rham current on this space, and define

$$S_J(A) := \frac{1}{2}\int_M A\wedge dA + \int_M A\wedge J\ .$$

for $A \in \Omega^1(M,\mathbb{R})$. We also impose the condition $dJ = 0$, so that the linear term $\int_M A\wedge J$ is well defined on the quotient $\Omega^1(M)/d\Omega^0(M)$. Now make the following "formal" definition.

Definition 3.28 The *abelian Chern–Simons path integral* is the expression

$$Z_{\mathrm{CS}}(J) := \int_{\Omega^1/d\Omega^0} e^{S_J(A)}\, DA\,,$$

where DA stands for a translation-invariant measure on the infinite-dimensional space $\Omega^1(M)/d\Omega^0(M)$.

Rather than trying to define the measure and the path integral precisely, we are going to see what the above formal manipulations with Gaussian integrals give us in this situation, where, in a sense, the operator Q is replaced by the outer derivative d. By formula (3.8) for the ratio $Z_Q(J)/Z_Q(0)$ we obtain

$$\frac{Z_{\mathrm{CS}}(J)}{Z_{\mathrm{CS}}(0)} = e^{S_J(A_0)} = e^{\frac{1}{2}\int_M A_0\wedge J}\ ,$$

where A_0 is a solution of the equation $dA_0 + J = 0$. (Recall that J is a closed current on a simply connected M, and hence it is exact, i.e., this equation formally has a solution.)

Now we would like to specify the functional J on 1-forms $A \in \Omega^1(M, \mathbb{R})$ to be the integral of the form over a collection of curves in the simply connected manifold M. Let γ_i, $i = 1, \dots, k$, be closed oriented nonintersecting curves in the manifold M. We set $J = \sum_i q_i\, \delta_{\gamma_i}$, where δ_{γ_i} is the δ-type 2-form on M supported on the curve γ_i, while q_i are real parameters. By applying the calculus of δ-forms (see Remark 3.26) we obtain that the ratio $Z_{\mathrm{CS}}(J)/Z_{\mathrm{CS}}(0)$ assumes the following explicit form:

$$\begin{aligned}
\frac{Z_{\mathrm{CS}}(J)}{Z_{\mathrm{CS}}(0)} &= \exp\left\{\frac{1}{2}\int_M A_0\wedge J\right\} = \exp\left\{\frac{1}{2}\int_M A_0 \wedge \sum_i q_i \int_{y\in\gamma_i} \delta^{2,1}(x,y)\right\}\\
&= \exp\left\{\frac{1}{2}\int_M A_0\wedge\sum_i q_i\int_{y\in D_i} d_x\delta^{1,2}(x,y)\right\}\\
&= \exp\left\{\frac{1}{2}\int_M -dA_0\wedge\sum_i q_i\int_{y\in D_i}\delta^{1,2}(x,y)\right\}\\
&= \exp\left\{\frac{1}{2}\int_M \left(\sum_j q_j\int_{z\in\gamma_j}\delta^{2,1}(x,z)\right)\wedge\left(\sum_i q_i\int_{y\in D_i}\delta^{1,2}(x,y)\right)\right\}\\
&= \exp\left\{\frac{1}{2}\sum_{i,j} q_i q_j\int_M \delta_{\gamma_j}(x)\wedge\delta_{D_i}(x)\right\} = \exp\left\{\frac{1}{2}\sum_{i,j} q_i q_j\, lk(\gamma_j,\gamma_i)\right\}
\end{aligned}$$

Here we have used the Stokes theorem, as well as the definition of A_0 as a solution of the equation $dA_0 + J = 0$.

Corollary 3.29 ([340, 318]) *For the functional J defined as the integral of 1-forms over a collection of curves in a threefold, the ratio $Z_{\mathrm{CS}}(J)/Z_{\mathrm{CS}}(0)$ counts the pairwise linking numbers of these curves.*

Note also that above, in the latter sum, we had to assume that $i \neq j$, so that the linking number was defined. The case of self-linking is much more subtle. It leads to divergences of the path integral and requires some additional specifications, such as framing, for its normalization; see [54]. The value $Z_{\mathrm{CS}}(0)$ in the case without any curve corresponds to the Ray–Singer torsion of the manifold M [340].

The topological Chern–Simons path integral has a holomorphic analogue.

Definition 3.30 (cf. [390]) For a three-dimensional Calabi–Yau manifold X with a holomorphic 3-form μ the *holomorphic abelian Chern–Simons path integral* is the expression

$$Z_{\mathbb{C}S}(J) := \int_{\Omega^{0,1}/\bar\partial\Omega^{0,0}} e^{S_{\mathbb{C}J}(A)}\, DA\,,$$

where

$$S_{\mathbb{C}J}(A) := \frac{1}{2}\int_X \mu \wedge A \wedge \bar\partial A + \langle \mathbb{C}J, A\rangle$$

is the quadratic form shifted by the linear functional $\mathbb{C}J$ on the space of $(0,1)$-connections $A \in \Omega^{0,1}(X,\mathbb{C})$.

Remark 3.31 For a complex curve $C \subset X$ equipped with a holomorphic 1-form α define the linear functional on $(0,1)$-connections A by assigning $\langle \mathbb{C}J_C, A\rangle := \int_C \alpha \wedge A$. Similarly to the topological case, if such a functional $\mathbb{C}J$ corresponds to a collection of complex curves, the holomorphic abelian Chern–Simons path integral can be described in terms of the *polar linking number*, a holomorphic analogue of the Gauss linking number, which we define in Section 4.3. The relation of this functional with the holomorphic analogue of linking was established in [134, 195, 366].

The abelian theory is a particular case of the general Chern–Simons path integral. In the topological case we consider a link $L = \cup_i \gamma_i$ in a compact real threefold M. Let $\mathcal{A}$ be the affine space of all connections in the (trivial) G-bundle over M for a compact simply connected simple Lie group G. We identify $\mathcal{A}$ with the space $\Omega^1(M,\mathfrak{g})$ of 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. Finally, let $G^M = C^\infty(M,G)$ be the group of gauge transformations in the bundle.

Definition 3.32 The *nonabelian Chern–Simons path integral* for a link $L \subset M$ is the following function of a parameter k:

$$Z_{\mathrm{CS}}(L;k) = \int\limits_{\mathcal{A}/G^M} \Big\{ \exp\Big\{ ik \int_M \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big) \Big\} \\ \times \prod_{\gamma_i \subset L} \mathrm{tr}\Big(P \exp \int_{\gamma_i} A \Big) \Big\}\, DA,$$

where $P\exp$ is the path-ordered exponential integral of a nonabelian connection A over γ_i, and DA is an appropriate measure on the moduli space of the connections $\mathcal{A}/G^M$.

Remark 3.33 Witten showed in [389] that for $M = S^3$ and $G = \mathrm{SU}(2)$ this path integral leads to the Jones polynomial for the link L. Other link or knot invariants can be obtained by changing the group. Note that they are always Vassiliev-type invariants of finite order [35, 36]. There are various ways to give $Z_{\mathrm{CS}}(L;k)$ and the corresponding link invariants rigorous definitions (see, e.g., the combinatorial [327] or probabilistic [5] approaches).

The extension of these results to a holomorphic version of the nonabelian Chern–Simons path integral is an intriguing open problem. The more complicated gauge transformation property of the holomorphic Chern–Simons action functional already makes the first step, writing out the corresponding path integral for an arbitrary collection of complex curves in a Calabi–Yau threefold, a serious problem; see some discussion in [391, 134, 366].

3.6 Bibliographical Notes

The Chern–Simons functional was introduced in [72]. For the relation of the abelian Chern–Simons functional to linking numbers we refer to [340, 318]. The appearance of the Jones polynomial and other knot invariants from the Chern–Simons functional was discovered by Witten [389]; see more details in [35, 210]. An excellent account of the relation between this functional to knot theory is contained in the book by Atiyah [27]. The relation between the Chern–Simons functional and the Vassiliev knot invariants is described in [36, 210].

The holomorphic Chern–Simons functional was introduced in [390] and studied in a number of papers [85, 134, 195, 196, 367]. For a higher-dimensional version of the Chern–Simons functional and its relation to linking numbers of several submanifolds see [124].

The classical Lagrangian formalism can be found, for example, in [18]. The formalism of the Lagrangian field theory was described in [393]; see also the presentations in [79, 341] for more details and examples. For preliminaries on linking numbers one can look at any book on differential topology, e.g., [162]. The question of when the space of extremals (more precisely, geodesics on a manifold) is a smooth manifold by itself is addressed, for example, in [38, 39].

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