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Operations on continuous bundles of C*-algebras

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1 Introduction

Recently several authors [2, 3, 8, 14] have considered bundles of C*-algebras obtained by applying to certain continuous fields of C*-algebras product operations such as taking a tensor product with a fixed C*-algebra or a crossed product by a group acting fibrewise. We consider the fundamental question, touched on in these articles, of the continuity of the resulting product bundles.

To describe the problem we will use the following idea of a continuous bundle of C*-algebras over a locally compact Hausdorff space.

Definition 1.1 A **bundle** of C*-algebras, or **C*-bundle** over a locally compact Hausdorff space X is a triple $\mathscr{A} = (X, \pi_x : A \to A_x, A)$, where A is a C*-algebra, the **bundle C*-algebra**, and for each $x \in X$, A_x is a C*-algebra and π_x a $*$-epimorphism of A onto A_x such that

(i) $\{\pi_x : x \in X\}$ is faithful, i.e. $\|a\| = \sup_{x\in X} \|a_x\|$, where $a_x = \pi_x(a)$ for each x;

(ii) for $f \in C_0(X)$ and $a \in A$, there is an element $fa \in A$ such that $(fa)_x = f(x)a_x$ for $x \in X$. When it is clear what the maps π_x are, $\mathscr{A}$ will be written simply (X, A_x, A). A **continuous bundle** of C*-algebras is a C*-bundle $\mathscr{A} = (X, A_x, A)$ which also satisfies

(iii) for $a \in A$, the function $N(a) : x \to \|a_x\|$ is in $C_0(X)$.

This definition is equivalent to the classical definition of a continuous field of C*-algebras [5, 10.3.1] when X is compact, but has the conceptual and notational advantage that the algebra of sections is a C*-algebra even when X is not compact. We can identify A with the *-algebra of elements γ in the cartesian product $\prod_{x\in X} A_x$ for which there is an $a \in A$ with $\gamma_x = \pi_x(a)$ for $x \in X$. If Γ is the *-algebra of elements of $\prod_{x\in X} A_x$ which coincide on compact subsets of X with elements of A, the triple (X, A_x, Γ) is a continuous field of C*-algebras in the sense of [5], and $C_0(\Gamma) = A$. Conversely, if (X, A_x, Γ)

is a continuous field of C*-algebras on X and A is the *-algebra of $\gamma \in \Gamma$ such that the function $x \rightarrow \|\gamma_x\|$ is in $C_0(X)$, then A is a C*-algebra and $(X, \pi_x : A \rightarrow A_x, A)$, where the π_x are the obvious coordinate morphisms, is a continuous bundle in the sense of (1.1), with $A = C_0(\Gamma)$. Thus the concepts of continuous bundle and continuous field of C*-algebras are essentially equivalent for locally compact base spaces. The term *bundle* seems to us more appropriate than *field* in view of the increasing interplay between C*-algebra theory and topology.

If $\mathscr{A} = (X, A_x, A)$ is a continuous bundle of C*-algebras, and B is another C*-algebra, there are natural bundles $\mathscr{A} \otimes B$ and $\mathscr{A} \otimes_{max} B$ over X with bundle algebras $A \otimes_{min} B$ and $A \otimes_{max} B$, respectively, defined as follows.

1. Since $\{\pi_x : x \in X\}$ is a faithful family of morphisms on A, the family $\{\pi_x \otimes \mathrm{id}: A \otimes B \rightarrow A_x \otimes B; x \in X\}$ is faithful on $A \otimes B$ (where, from now on, $\otimes$ denotes $\otimes_{min}$). Thus $A \otimes B$ can be identified with a *-algebra of bounded elements of $\prod_{x \in X}(A_x \otimes B)$. If fc is defined for $f \in C_0(X)$ and $c \in A \otimes B$ so that when $c = a \otimes b$ with $a \in A$ and $b \in B$, $f(a \otimes b) = (fa) \otimes b$, then the triple $\mathscr{A} \otimes B = (X, A_x \otimes B, A \otimes B)$ satisfies (i) and (iii) of Definition 1.1, but it is not clear that (ii) always holds. It is, in fact, always true that for $c \in A \otimes B$, $x \rightarrow \|(\pi_x \otimes \mathrm{id})(c)\|$ is lower semicontinuous (see sect. 2), so that (ii) holds exactly when this function is also upper semicontinuous for all c.

2. Since A is an ideal of its multiplier algebra $M(A)$, and B is an ideal of its unitization $\tilde{B}$, $A \otimes_{max} B$ is an ideal of $M(A) \otimes_{max} \tilde{B}$, and any irreducible representation π of $A \otimes_{max} B$ extends to an irreducible representation $\tilde{\pi}$ of $M(A) \otimes_{max} \tilde{B}$. The algebra $C_0(X)$ is identified with a C*-subalgebra of the centre $Z(M(A))$ of $M(A)$, and so $C_0(X) \otimes 1 \subseteq Z(M(A) \otimes_{max} \tilde{B})$. Thus $\tilde{\pi}|_{C_0(X) \otimes 1}$ is a character of $C_0(X) \otimes 1$, corresponding to evaluation at some point $x \in X$ which depends only on π. Let $J_x = \{a \in A : a_x = 0\}$. Then $J_x = \overline{C_{0,x}(X)A}$, where $C_{0,x}(X) = \{f \in C_0(X) : f(x) = 0\}$, and $\tilde{\pi}(J_x \otimes 1) = 0$. Since $(A \otimes_{max} B)/(J_x \otimes_{max} B) \cong A_x \otimes_{max} B$, it follows that π factors through $A_x \otimes_{max} B$ via the quotient map $\pi_x \otimes_{max} \mathrm{id} : A \otimes_{max} B \rightarrow A_x \otimes_{max} B$. If $c \in A \otimes_{max} B$, then $\|c\| = \sup_\pi \|\pi(c)\|$, where the supremum is over all irreducible representations of $A \otimes_{max} B$. Thus $\|c\| = \sup_{x \in X} \|(\pi_x \otimes_{max} \mathrm{id})(c)\|$, and $A \otimes_{max} B$ can be identified with a *-subalgebra of $\prod_{x \in X}(A_x \otimes_{max} B)$. If the module action of $C_0(X)$ on $A \otimes_{max} B$ is defined so that $f(a \otimes b) = (fa) \otimes b$ for $f \in C_0(X), a \in A$ and $b \in B$, and is extended to arbitrary $c \in A \otimes_{max} B$ by linearity and continuity, conditions (i) and (iii) of Definition (1.1) again hold, and it is always true that for $c \in A \otimes_{max} B$, the function $x \rightarrow \|(\pi_x \otimes_{max} \mathrm{id})(c)\|$ is upper semicontinuous. Thus (ii) will hold if and only if this function is lower semicontinuous. The bundle $(X, \pi_x \otimes_{max} \mathrm{id} : A \otimes_{max} B \rightarrow A_x \otimes_{max} B, A \otimes_{max} B)$ will be denoted by $\mathscr{A} \otimes_{max} B$.

Before we embarked on this investigation it was widely believed that a product bundle of form $\mathscr{A} \otimes B$ would always satisfy (ii) for continuous $\mathscr{A}$. Surprisingly, we have found that there are instances where upper semicontinuity can fail. Similarly lower semicontinuity can fail for bundles of form $\mathscr{A} \otimes_{max} B$.

For a continuous bundle $\mathscr{A} = (X, \pi_x : A \to A_x, A)$ with $J_x = \ker \pi_x$ for $x \in X$, the continuity of $\mathscr{A} \otimes B$ is closely linked to exactness for the sequences

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \qquad (*)$$

(Rieffel [14] had already observed this). In general exactness of a short exact sequence of C*-algebras need not be preserved on taking a minimal C*-tensor product with another C*-algebra [16], [18], and this fact turns out to be the key to producing examples of continuous C*-bundles $\mathscr{A}$ and C*-algebras B such that $\mathscr{A} \otimes B$ does not satisfy (ii). We actually go further, and give a new characterisation of exact C*-algebras (Theorem 4.5), which may be summarised as

Theorem A *Let B be a C*-algebra. Then B is exact if and only if for any continuous bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_t, A)$ of C*-algebras on the one-point compactification $\hat{\mathbb{N}}$ of $\mathbb{N}$, with A separable, $\mathscr{A} \otimes B$ is continuous.*

A sufficient condition for $(*)$ to be exact for arbitrary B is that A be an exact C*-algebra. This implies [10] that each A_x is exact, and the question arises whether, conversely, exactness of each A_x implies that of A. Again, this is not always true, and we have the following criterion (see Theorem 4.6) for exactness of A:

Theorem B *Let $\mathscr{A} = (X, A_x, A)$ be a continuous bundle of C*-algebras with each A_x exact. Then the following conditions are equivalent:*
(i) *A is exact;*
(ii) *for any C*-algebra B, the bundle $\mathscr{A} \otimes B$ is continuous;*
(iii) *each of the quotient maps $\pi_x : A \to A_x$ is locally liftable (in the sense of [11]).*

Using results from [11], we construct examples of bundles (X, A_x, A), with each A_x exact, for which one of the π_x's has no local lifting. It then follows by Theorem B that A is not exact.

Turning to an analogous question for the maximal C*-tensor product, we obtain

Theorem C *Let B be a C*-algebra. Then B is nuclear if and only if for any continuous bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_t, A)$ of C*-algebras, with A separable, the bundle $\mathscr{A} \otimes_{max} B$ is continuous.*

The plan of the paper is as follows. Section 2 is devoted to preliminaries on bundles of C*-algebras, in particular, criteria for upper and lower semicontinuity. In Sect. 3 we consider the continuity question for product bundles of form $\mathscr{A} \otimes_{max} B$, where $\mathscr{A}$ is a continuous C*-bundle. As well as proving Theorem C, we give an explicit example of a product bundle of this type which is not continuous. In Sect. 4 we examine analogous questions for product bundles of form $\mathscr{A} \otimes B$. We give several examples of discontinuous bundles of this type and establish the characterisations summarised above as Theorems A and B.

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2 Continuity criteria for bundles of C*-algebras

2.1 Upper and lower semicontinuity for a bundle of C-algebras*

Let (X, A_x, A) be a continuous bundle of C*-algebras on a locally compact Hausdorff space X. Condition (iii) of Definition (1.1) says that A is a $C_0(X)$-module. The following nondegeneracy property holds.

Lemma 2.1 *A is the closure of $C_0(X)A$.*

Proof. (i) $C_0(X)A \subseteq A$.

(ii) Let $0 \neq a \in A, \varepsilon > 0$, and let $Y = \{x \in X : \|a_x\| \geqq \varepsilon\}$ and $Y' = \{x \in X : \|a_x\| > \varepsilon/2\}$. Then $Y \subseteq Y'$, and Y and $\overline{Y'}$ are compact. Let $f : X \to [0,1]$ be a continuous function such that $f|_Y = 1, f|_{X-Y'} = 0$. Then for $x \notin Y$,

$$\|(fa - a)_x\| \leqq (f(x) + 1)\|a_x\| < 2\varepsilon .$$

Thus $\|fa - a\| < 2\varepsilon$, and, since $fa \in C_0(X)A$ and ε is arbitrary, the result follows. □

If A is a C*-algebra which is a $C_0(X)$-module, for a locally compact Hausdorff space X, then $C_0(X) \subseteq Z(M(A))$. More generally, let A be a C*-algebra, suppose $C_0(X) \subseteq Z(M(A))$ for such an X, with $\overline{C_0(X)A} = A$, and let $\pi_x : A \to A_x$ be a faithful family of $*$-epimorphisms such that for each $x \in X$,

$$\pi_x(fa) = f(x)\pi_x(a) \quad (a \in A) .$$

Then $(X, \pi_x : A \to A_x, A)$ satisfies conditions (i) and (iii) of Definition 1.1, but in general the function $x \to \|\pi_x(a)\|$ will not be continuous for $a \in A$. The next lemma gives a criterion for the lower semicontinuity of this function.

Lemma 2.2 *The function $x \to \|\pi_x(a)\|$ is lower semicontinuous for $a \in A$ if and only if, for any closed subset $X' \subseteq X$ and dense subset D of X', the morphisms $\pi_D = \bigoplus_{x \in D} \pi_x$ and $\pi_{X'} = \bigoplus_{x \in X'} \pi_x$ have the same kernel.*

Proof. $\Rightarrow$: Let X' be a closed subset of X, let D be dense in X', and let $a \in \ker \pi_D$. For $x \in X'$ and $\varepsilon > 0$, there is a neighbourhood U of x in X such that $\|\pi_y(a)\| \geqq \|\pi_x(a)\| - \varepsilon$ for $y \in U$. Now $D \cap U \neq \phi$, and for $y \in D \cap U$

$$\|\pi_x(a)\| \leqq \|\pi_y(a)\| + \varepsilon = \varepsilon .$$

Since ε is arbitrary, $\|\pi_x(a)\| = 0$ and $a \in \ker \pi_{X'}$.

$\Leftarrow$: Suppose that $x \to \|\pi_x(a)\|$ is not lower semicontinuous for some a at some $x \in X$. Then for some $\varepsilon > 0$, there is a net $\{x_\lambda\}$ in X such that

$x = \lim_\lambda x_\lambda$ and $\|\pi_{x_\lambda}(a)\| < \|\pi_x(a)\| - \varepsilon$ for all λ. Replacing a with $|a|$ and using functional calculus, we can assume that $\pi_{x_\lambda}(a) = 0$ for all λ and $\pi_x(a) \neq 0$. Letting $D = \{x_\lambda : \lambda \in \Lambda\}$ and $X' = \bar{D}$, $a \in \ker \pi_D$, but $a \notin \ker \pi_{X'}$, a contradiction. □

Let us now take A and X as in the previous lemma, and let J_x denote the ideal $\overline{C_{0,x}A}$ of A; clearly $J_x \subseteq \ker \pi_x$ for $x \in X$. A useful (and well-known) criterion for the upper semicontinuity of the functions $x \to \|\pi_x(a)\|$ is

Lemma 2.3 *The function $x \to \|\pi_x(a)\|$ is upper semicontinuous on X for all $a \in A$ if and only if $J_x = \ker \pi_x$ for $x \in X$.*

Proof. $\Rightarrow$: Suppose $a \in \ker \pi_x$. Given $\varepsilon > 0$, there is a neighbourhood U of x such that $\|\pi_y(a)\| \leqq \varepsilon$ for $y \in U$. Let $f : X \to [0,1]$ be a continuous function such that $f(x) = 1$ and $f(y) = 0$ $(y \notin U)$. Then

$$\begin{aligned}\|a - (1-f)a\| &= \sup_{y\in X} \|\pi_y(a) - (1 - f(y))\pi_y(a)\| \\ &= \sup_{y\in X} |f(y)|\|\pi_y(a)\| \\ &\leqq \varepsilon\,.\end{aligned}$$

Since ε is arbitrary, $a \in J_x$. Since $J_x \subseteq \ker \pi_x, J_x = \ker \pi_x$.

$\Leftarrow$: Let $a \in A$, $x \in X$ and $\varepsilon > 0$. If $b \in A$ and $f \in C_0(X)$ are chosen so that $f(x) = 0$ and $\|a - fb\| \leqq \|\pi_x(a)\| + \varepsilon/2$, there is a neighbourhood U of x such that $|f(y)|\|b\| \leqq \varepsilon/2$ for $y \in U$. For such y,

$$\begin{aligned}\|\pi_y(a)\| &\leqq \|\pi_y(a) - f(y)\pi_y(b)\| + \|f(y)\pi_y(b)\| \\ &\leqq \|a - fb\| + \varepsilon/2 \\ &\leqq \|\pi_x(a)\| + \varepsilon\,.\end{aligned}$$

Since ε is arbitrary, the map $x \to \|\pi_x(a)\|$ is upper semicontinuous at x. □

2.2 Continuity of tensor and crossed product bundles

Throughout this section X will be a locally compact Hausdorff space and $\mathscr{A} = (X, \pi_x : A \to A_x, A)$ a continuous bundle of C*-algebras on X. If B is another C*-algebra, the product bundles $\mathscr{A} \otimes B$ and $\mathscr{A} \otimes_{max} B$ were defined in the introduction.

Now let G be a locally compact group, and let α_x be a continuous action of G on A_x (by *-automorphisms) for $x \in X$. If for each $a \in A$ and $g \in G$ there is an element $\alpha^g(a) \in A$ such that $\pi_x(\alpha^g(a)) = \alpha_x^g(a_x)$ for $x \in X$, the action α of G on A is continuous, and $\{\alpha_x\}_{x\in X}$ is said to be a *continuous field of actions* of G. In this case we can define natural bundle structures on X associated with the full and reduced crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$ analogous to the tensor product bundles. For $x \in X$ the morphism $\pi_x : A \to A_x$ is a G-map (relative to the actions α and α_x), so that π_x extends naturally to

$*$-epimorphisms $\pi_x^G : A \rtimes_\alpha G \to A_x \rtimes_{\alpha_x} G$ and $\pi_x^{G,r} : A \rtimes_{\alpha,r} G \to A_x \rtimes_{\alpha_x,r} G$. We thus obtain bundles $\mathscr{A} \rtimes_\alpha G = (X, A_x \rtimes_{\alpha_x} G, A \rtimes_\alpha G)$ and $\mathscr{A} \rtimes_{\alpha,r} G = (X, A_x \rtimes_{\alpha_x,r} G, A \rtimes_{\alpha,r} G)$ which satisfy (i) and (iii) of Definition (1.1). When all the actions α_x are trivial, these bundles reduce to the tensor product bundles $\mathscr{A} \otimes_{max} C^*(G)$ and $\mathscr{A} \otimes C_r^*(G)$, respectively.

As is well known, the bundles $\mathscr{A} \otimes_{max} B$ and $\mathscr{A} \otimes_\alpha G$ are upper semicontinuous (cf [14]). Short proofs of this fact follow easily from Lemma 2.3, and indeed for the maximal tensor product bundle, the result is a special case of the next lemma. The crossed product result is proved analogously. Let v be a C*-norm on the algebraic tensor product $A \odot B$, and let $A \otimes_v B$ denote the completion. For $x \in X$, the closure $J_x \otimes_v B$ of $J_x \odot B$ in $A \otimes_v B$ is an ideal. The quotient of $A \otimes_v B$ by this ideal is naturally isomorphic to the completion of $A_x \odot B$ with respect to some C*-norm v_x. Let $\pi_x \otimes_{v_x} \mathrm{id}$ be the corresponding quotient morphism.

Lemma 2.4 *If $c \in A \otimes_v B$, then the function $x \to \|(\pi_x \otimes_{v_x} id)(c)\|$ is upper semicontinuous.*

Proof. Since $\ker(\pi_x \otimes_{v_x} \mathrm{id}) = J_x \otimes_v B = \overline{C_{0,x}(X)A} \otimes_v B = \overline{C_{0,x}(X)(A \otimes_v B)}$, the result is immediate from Lemma 2.3. □

Turning to the bundles $\mathscr{A} \otimes B$ and $\mathscr{A} \rtimes_{\alpha,r} G$, these are known to be lower semicontinuous [14]; we include a short proof for the tensor product case which uses Lemma 2.2.

Lemma 2.5 *If $c \in A \otimes B$, then the function $x \to \|(\pi_x \otimes id)(c)\|$ is lower semicontinuous.*

Proof. Letting $D \subseteq X$, $X' = \bar{D}$ and $\bar{A} = \pi_{X'}(A)$, for $x \in X', \pi_x$ factors through $\bar{A}$, that is, $\pi_x = \bar{\pi}_x \circ \pi_{X'}$, where $\bar{\pi}_x$ is a *-epimorphism from $\bar{A}$ to A_x. Let $\bar{a} \in \bar{A}$ with $\bar{\pi}_x(\bar{a}) = 0$ for $x \in D$. Then $\bar{a} = \pi_{X'}(a)$ for some $a \in A$, and $\|\pi_x(a)\| = 0$ for $x \in D$, and hence for $x \in X'$, since $x \to \|\pi_x(a)\|$ is continuous. Thus $\bar{a} = 0$, and $\{\bar{\pi}_x : x \in D\}$ is a faithful family of morphisms on $\bar{A}$, so that $\{\bar{\pi}_x \otimes \mathrm{id} : x \in D\}$ is a faithful family of morphisms on $\bar{A} \otimes B$. Suppose $c \in \ker(\pi_x \otimes \mathrm{id})$ for $x \in D$. Then $(\bar{\pi}_x \otimes \mathrm{id})((\pi_{X'} \otimes \mathrm{id})(c)) = 0$ for $x \in D$, so that $(\pi_{X'} \otimes \mathrm{id})(c) = 0$, and $(\pi_x \otimes \mathrm{id})(c) = 0$ for $x \in X'$. The result now follows from Lemma 2.2. □

Remarks 2.6 1. When B is a nuclear C*-algebra, the norms $\|\cdot\|_{max}$ and $\|\cdot\|_{min}$ coincide on $A \odot B$ and $A_x \odot B$ $(x \in X)$, so that the bundles $\mathscr{A} \otimes_{max} B$ and $\mathscr{A} \otimes B$ coincide. For $c \in A \otimes B$, the function $x \to \|c_x\|$ is upper semicontinuous by Lemma 2.4 and lower semicontinuous by Lemma 2.5, hence continuous.

2. When G is an amenable group, $B \rtimes_\alpha G = B \rtimes_{\alpha,r} G$ for any C*-algebra B [13], so that the bundles arising from the full and reduced crossed products of A by G coincide. Since one is upper, and the other lower, semicontinuous, the continuity of $\mathscr{A} \rtimes_\alpha G$ follows.

If we apply the discussion preceding Lemma 2.4 to the norm $\|\cdot\|_{min}$ on $A \odot B$, each quotient $(A \otimes B)/(J_x \otimes B)$ is naturally isomorphic to some completion

$A_x \otimes_{\nu_x} B$ of $A_x \odot B$. The norms $\|\cdot\|_{\nu_x}$ and $\|\cdot\|_{min}$ coincide exactly when the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \qquad (*)$$

is exact. This leads to the following useful criterion for $\mathscr{A} \otimes B$ to be continuous.

Proposition 2.7 *The bundle $\mathscr{A} \otimes B$ is continuous if and only if* $(*)$ *is exact for $x \in X$.*

Proof. $\Leftarrow$: Since exactness of $(*)$ implies $\|\cdot\|_{\nu_x} = \|\cdot\|_{min}$ for $x \in X$, the bundles $(X, A_x \otimes_{\nu_x} B, A \otimes B)$ and $(X, A_x \otimes B, A \otimes B)$ coincide. Since these bundles are upper and lower semicontinuous, respectively, their continuity follows.

$\Rightarrow$: Let $c \in A \otimes B$, $x \in X$ and let U be a neighbourhood of x. If $f : X \to [0,1]$ is a continuous function such that $f(x) = 1$ and $f|_{X \setminus U} \equiv 0$,

$$\begin{aligned} \|fc\| &= \sup_{y \in X} \|(\pi_y \otimes \mathrm{id})(fc)\| \\ &= \sup_{y \in X} f(y) \|(\pi_y \otimes \mathrm{id})(c)\| \\ &\leqq \sup_{y \in U} \|(\pi_y \otimes \mathrm{id})(c)\| \end{aligned}$$

and

$$\begin{aligned} \|(\pi_x \otimes_{\nu_x} \mathrm{id})(c)\| &= \|(\pi_x \otimes_{\nu_x} \mathrm{id})(fc)\| \\ &\leqq \|fc\| . \end{aligned}$$

Thus

$$\|(\pi_x \otimes_{\nu_x} \mathrm{id})(c)\| \leqq \sup_{y \in U} \|(\pi_y \otimes \mathrm{id})(c)\|$$

and, since U is an arbitrary neighbourhood of x, the continuity assumption implies $\|(\pi_x \otimes_{\nu_x} \mathrm{id})(c)\| \leqq \|(\pi_x \otimes \mathrm{id})(c)\|$, whence $\|\cdot\|_{\nu_x} = \|\cdot\|_{min}$. □

Corollary 2.8 *Let $\mathscr{A} = (X, A_x, A)$ be a continuous bundle of C*-algebras. For a C*-algebra B, the bundle $\mathscr{A} \otimes B$ is continuous when either A or B is exact.*

Proof. When B is exact, it is immediate that $(*)$ is exact for all x, so that the result follows. If A is exact, it has property C of Archbold and Batty by results of [1] and [11], and this implies that $(*)$ is exact for arbitrary B by [1, Theorem 3.4]. □

Remarks 2.9 1. We shall give a converse to this in Sect. 4.
2. If A is exact, so is A_x for $x \in X$, by [10].

3 The operation $\cdot \otimes_{max} B$

Recall that if A, B and C are C*-algebras such that $A \subseteq B$, the inclusion map $A \odot C \to B \odot C$ extends to a $*$-homomorphism $\iota : A \otimes_{max} C \to B \otimes_{max} C$ which is, in general, not injective. In what follows we write $A \otimes_{max} C \subseteq B \otimes_{max} C$ to indicate that ι is injective.

If C is a nuclear C*-algebra, $\|\cdot\|_{max} = \|\cdot\|_{min}$ on $A \odot C$ and $B \odot C$, so that ι is injective, since, by a basic property of the minimal norm, $A \otimes C \subseteq B \otimes C$. It is known, conversely, that if, for a given C, ι is injective for any B and $A \subseteq B$, then C is nuclear ([7], see also the proof of Theorem 3.1 below). It follows, that if C is any nonnuclear C*-algebra, there are B and $A \subseteq B$ such that the corresponding map $\iota : A \otimes_{max} C \to B \otimes_{max} C$ is not a monomorphism. We now construct, given such A, B and C, a continuous bundle $\mathscr{D} = ([0,1], \pi_x : D \to D_x, D)$ such that the bundle $\mathscr{D} \otimes_{max} C$ is not continuous.

Fix $\alpha \in [0,1]$ and let $D_\alpha = \{f \in C([0,1],B) : f(\alpha) \in A\}$, $D_x = B$ for $x \in [0,1] \backslash \{\alpha\}$; $D_\alpha = A$ and let $\pi_x : D \to D_x$ be the map $f \to f(x)$ $(x \in [0,1])$. Then $\mathscr{D}_\alpha = ([0,1], D_x, D_\alpha)$ is a continuous bundle of C*-algebras (it is, in fact, a subbundle of the trivial bundle $([0,1], B, C([0,1],B))$. By our assumption on A, B and C, there is an h in $A \odot C$ such that $\|h\|_{A \otimes_{max} C} > \|h\|_{B \otimes_{max} C}$. Let $h = \sum_1^m a_i \otimes c_i$ and let $\bar{a}_i$ be the function on $[0,1]$ taking the constant value a_i $(i = 1,2,\ldots,m)$. Then each $\bar{a}_i \in D$, $\bar{h} = \sum_1^m \bar{a}_i \otimes c_i \in D \odot C \subseteq D \otimes_{max} C$ and

$$\|(\pi_x \otimes_{max} \mathrm{id})(\bar{h})\| = \begin{cases} \|h\|_{B \otimes_{max} C} & (x \neq \alpha) \\ \|h\|_{A \otimes_{max} C} & (x = \alpha)\,. \end{cases}$$

Thus $\|\bar{h}_\alpha\| > \|\bar{h}_x\|$ for $x \neq \alpha$, and $\|\bar{h}_x\|$ is constant on the set $[0,1] \setminus \{\alpha\}$, so that the function $x \to \|\bar{h}_x\|$ is not lower semicontinuous at $x = \alpha$ (it is clearly upper semicontinuous on $[0,1]$). It follows that the product bundle $\mathscr{D}_\alpha \otimes_{max} C$ is not continuous.

Examples 3.1 1. To exhibit concrete examples of such bundles $\mathscr{D}_\alpha$, we take $A = C = C_r^*(\mathbb{F}_2)$, the regular C*-algebra of the free group $\mathbb{F}_2$ on two generators, and B the Cuntz algebra $\mathcal{O}_2$. By a well-known result of Choi [4], $\mathcal{O}_2$ has a unital C*-subalgebra $*$-isomorphic to $C_r^*(PSL_2(\mathbb{Z})) = C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$. As many people have observed, since $\mathbb{Z}_2 * \mathbb{Z}_3$ has a subgroup isomorphic to $\mathbb{F}_2$,[1] $C_r^*(\mathbb{F}_2)$ is $*$-isomorphic to a C*-subalgebra of $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ and hence of $\mathcal{O}_2$. Accordingly we identify A with a closed $*$-subalgebra of B. Since $\mathcal{O}_2$ is nuclear, but $\|\cdot\|_{max} \neq \|\cdot\|_{min}$ on $C_r^*(\mathbb{F}_2) \odot C_r^*(\mathbb{F}_2)$ [15], the corresponding map $\iota : A \otimes_{max} C \to B \otimes_{max} C = B \otimes C$ is not injective. In fact an explicit example of an $h \in A \odot C$ such that $\|h\|_{max} \neq \|h\|_{min}$ is given in [17]. In the corresponding product bundle $\mathscr{D}_\alpha \otimes_{max} C$, the function $x \to \|\bar{h}_x\|$ is not lower semicontinuous at $x = \alpha$.

2. Example 1 can be modified so that the resulting bundle on $[0,1]$ has the additional property that the fibre algebras are constant. Let $E_{2i} = C_r^*(\mathbb{F}_2)$, $E_{2i-1} = \mathcal{O}_2$ $(i = 1,2,\ldots)$ and let E be the infinite spatial tensor product $\bigotimes_{i=1}^\infty E_i$. Letting $A_1 = E \otimes A$ and $B_1 = E \otimes B$, A_1 is a C*-subalgebra of B_1 and $E \cong E \otimes A_1 \cong E \otimes B_1$. Let $h \in A \odot C$ be such that

[1] If a and b are the generators of $\mathbb{Z}_2 * \mathbb{Z}_3$, with $a^2 = b^3 = 1$, the elements bab and $ababa$ generate such a subgroup

$\|h\|_{A\otimes_{max}C} > \|h\|_{B\otimes_{max}C}$ and let $k = 1_E \otimes h \in A_1$. Then $\|k\|_{A_1\otimes_{max}C} = \|h\|_{A\otimes_{max}C}$, since $A \otimes_{max} C \cong (1_E \otimes A) \otimes_{max} C \subseteq (E \otimes A) \otimes_{max} C = A_1 \otimes_{max} C$ canonically. Similarly, $\|k\|_{B_1\otimes_{max}C} = \|h\|_{B\otimes_{max}C}$, so that $\|k\|_{A_1\otimes_{max}C} > \|k\|_{B_1\otimes_{max}C}$. The above construction now yields a bundle $\mathscr{B}_\alpha = ([0,1], D_x, D_\alpha)$ with $D_x \cong E$ for $0 \leqq x \leqq 1$. The element $\bar{k} \in D_\alpha \otimes_{max} C$ constructed from k as above has the property that the function $x \to \|\bar{k}_x\|$ is not lower semicontinuous at $x = \alpha$.

3. By taking the restricted direct product of the bundles $\mathscr{B}_\alpha$ for all α in $[0,1]$, we obtain a continuous bundle $([0,1], D_x, D)$ with constant fibre such that $D \otimes_{max} C$ contains, for each $\alpha \in [0,1]$, an h such that $x \to \|h_x\|$ is not lower semicontinuous at $x = \alpha$. If instead of taking the direct sum we take the infinite spatial tensor product of the $\mathscr{B}_\alpha$, the resulting bundle has the same properties, and in addition each fibre is simple (since $\mathcal{O}_2, C_r^*(\mathbb{F}_2)$ and hence E are simple).

The method used in constructing these examples also yields the following characterisation of nuclearity.

Theorem 3.2 *For a C*-algebra C the following conditions are equivalent*:
(i) *C is nuclear*;
(ii) *if $A \subseteq B$ are C*-algebras, then $A \otimes_{max} C \subseteq B \otimes_{max} C$*;
(iii) *if $A \subseteq B$ are separable C*-algebras, then $A \otimes_{max} C \subseteq B \otimes_{max} C$*;
(iv) *for every continuous C*-bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_x, A)$, with A separable, the bundle $\mathscr{A} \otimes_{max} C$ is continuous*;
(v) *for every continuous C*-bundle $\mathscr{A} = (X, A_x, A)$ over a locally compact space X, the bundle $\mathscr{A} \otimes_{max} C$ is continuous.*

Proof. (i) $\Rightarrow$ (v): This follows immediately from Remark 2.6(1).

(v) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (iii): Let $A \subseteq B$ be separable C*-algebras, and let D be the algebra of sequences $\{(a_i) : a_i \in B\ (i = 1,2,\ldots), \lim_{i\to\infty} a_i$ exists and is in $A\}$. Then D is a C*-algebra, and putting $D_i = B\ (i \in \mathbb{N})$, $D_\infty = A$, and taking $\pi_i : D \to D_i$ and $\pi_\infty : D \to D_\infty$ to be the morphisms $(a_j) \to a_i$ and $(a_j) \to \lim_{i\to\infty} a_i$, respectively, $\mathscr{D} = (\hat{\mathbb{N}}, \pi_n : D \to D_n, D)$ is a continuous C*-bundle over $\hat{\mathbb{N}}$ with separable bundle algebra D.

By hypothesis the product bundle $\mathscr{D} \otimes_{max} C$ is continuous. Let $d \in A \odot C$, so that $d = \sum_{j=1}^n a^j \otimes c^j$ for suitable $a^j \in A$ and $c^j \in C$, let $\bar{a}^j$ be the element of D with constant entry a^j, for $1 \leqq j \leqq m$, and let $\bar{d}$ be the element of the product bundle with constant entry d. Then

$$\begin{aligned}\|(\pi_n \otimes \mathrm{id})(\bar{d})\| &= \|(\pi_n \otimes \mathrm{id})(\textstyle\sum \bar{a}^j \otimes c^j)\| \\ &= \begin{cases} \|\sum a^j \otimes c^j\|_{B\otimes_{max}C} & (n = 1,2,\ldots) \\ \|\sum a^j \otimes c^j\|_{A\otimes_{max}C} & (n = \infty)\,. \end{cases}\end{aligned}$$

By the continuity of the function $n \to \|(\pi_n \otimes \mathrm{id})(\bar{d})\|$ at ∞, it follows that

$$\|d\|_{A\otimes_{max}C} = \|d\|_{B\otimes_{max}C}$$

as required.

(iii) $\Rightarrow$ (ii): The separable C*-subalgebras of B form a net $\{B_\lambda\}_{\lambda \in \Lambda}$ partially ordered by inclusion. If $B_\lambda \subseteq B_\mu$, then $B_\lambda \otimes_{max} C \subseteq B_\mu \otimes_{max} C$ by hypothesis. We can thus form the inductive limit C*-algebra $\lim_{\rightarrow}(B_\lambda \otimes_{max} C)$, which is the completion $B \otimes_\nu C$ of $B \odot C$ relative to some C*-norm ν. For $x \in B \odot C$ there is a B_λ such that $x \in B_\lambda \odot C$. If $A \subseteq B$ and $x \in A \odot C$, we can choose B_λ to be contained in A. Thus

$$\|x\|_{A \otimes_{max} C} \leqq \|x\|_{B_\lambda \otimes_{max} C} = \|x\|_{B \otimes_\nu C} \leqq \|x\|_{B \otimes_{max} C},$$

since $B_\lambda \odot C \subseteq A \odot C$. Since $A \odot C \subseteq B \odot C$, $\|x\|_{B \otimes_{max} C} \leqq \|x\|_{A \otimes_{max} C}$. Thus x has the same norm considered either as an element of $A \otimes_{max} C$ or $B \otimes_{max} C$. It follows that $A \otimes_{max} C \subseteq B \otimes_{max} C$.

(ii) $\Rightarrow$ (i): This follows from [7, Theorem 6.2], but for completeness we outline a short proof. If π is a faithful nondegenerate representation of C on a Hilbert space H, let $A = \pi(C)'$ and $B = B(H)$. The map $a \otimes c \to a\pi(c)$ extends by linearity and continuity to a representation $\bar{\pi}$ of $A \otimes_{max} C$ on H. Since by hypothesis $A \otimes_{max} C \subseteq B \otimes_{max} C$, there is a representation σ of $B \otimes_{max} C$ on a Hilbert space $K \supseteq H$ such that for $x \in A \otimes_{max} C$, $E_H \sigma(x) E_H|_H = \bar{\pi}(x)$, where E_H is the orthogonal projection of K onto H. The map ρ of $B(H)$ into itself given by $\rho(x) = E_H \sigma(x \otimes 1) E_H|_H$ is then a norm one projection of $B(H)$ onto A. Thus A, and hence $A' = \pi(C)''$ are injective. In particular, C^{**} is injective, and so C is nuclear. □

Remarks 3.3 A further condition (iv)′, obtained by replacing $\hat{\mathbb{N}}$ by $[0,1]$ in condition (iv) of Theorem 3.2, is easily seen to be equivalent to conditions (i) – (iv) of the theorem using the construction preceding Examples 3.1.

4 The operation $\cdot \otimes_{min} B$

4.1 Constructing separable continuous bundles of C-algebras*

Let $B_1, B_2, \ldots$ be a sequence of separable unital C*-algebras and A a separable unital C*-subalgebra of $\bigoplus_{i=1}^{\infty} B_i = \{(b_i) : b_i \in B_i \ (i = 1,2,\ldots), \sup \|b_i\| < \infty\}$ containing the ideal $I_0 = \{(a_i) \in \bigoplus B_i : \lim_{i \to \infty} \|a_i\| = 0\}$ of zero sequences. Let π_i be the coordinate projection of $\bigoplus B_i$ onto B_i $(1 = 1,2,\ldots)$.

Lemma 4.1 *There is a sequence $k_1 < k_2 < \ldots$ of natural numbers such that if $\sigma_i = \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l$, then $\lim_{i \to \infty} \|\sigma_i(a)\|$ exists for $a \in A$.*

Proof. Let $A_\infty = A/I_0$, with quotient map $\pi : A \to A_\infty$, and let $a^1, a^2, \ldots$ be a sequence dense in A. We define the sequence $k_1 < k_2 < \ldots$ of natural numbers so that for $i = 1,2,\ldots,$

$$\left| \left\| \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l(a^j) \right\| - \|\pi(a^j)\| \right| \leqq (1/i)\|a^j\|$$

for $1 \leqq j \leqq i$. To do this we actually define inductively two sequences k_i and k_i', with $1 = k_0 = k_0' < k_1 < k_1' < k_2 < k_2' < \ldots$, such that for $i \geqq 1$ and $1 \leqq j \leqq i$

$$\left\| \bigoplus_{l=k_i}^{\infty} \pi_l(a^j) \right\| - \|\pi(a^j)\| \leqq (1/i)\|a^j\|$$

and

$$\left\| \bigoplus_{l=k_i}^{\infty} \pi_l(a^j) \right\| - \left\| \bigoplus_{l=k_i}^{r} \pi_l(a^j) \right\| \leqq (1/i)\|a^j\|$$

for $r \geqq k_i'$. If k_i and k_i' are defined, since $\|\pi(a)\| = \limsup \|\pi_i(a)\|$ for $a \in \bigoplus B_i$, choose $k_{i+1} > k_i'$ such that

$$\sup_{l \geqq k_{i+1}} \|\pi_l(a^j)\| \leqq \left(1 + \frac{1}{(i+1)}\right) \|\pi(a^j)\|$$

for $1 \leqq j \leqq i+1$, and choose $k_{i+1}' > k_{i+1}$ so that if $r \geqq k_{i+1}'$,

$$\left\| \bigoplus_{l=k_{i+1}}^{\infty} \pi_l(a^j) \right\| - \left\| \bigoplus_{l=k_{i+1}}^{r} \pi_l(a^j) \right\| \leqq \frac{1}{i+1}\|a^j\|$$

for $1 \leqq j \leqq i$.

For $i = 1, 2, \ldots$ let $\sigma_i = \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l$. From the construction of the sequence $\{k_i\}$ it follows that for $i = 1, 2, \ldots$, $\lim_{j\to\infty} \|\sigma_j(a^i)\|$ exists and equals $\|\pi(a^i)\|$. Thus $\lim_{j\to\infty} \|\sigma_j(a)\|$ exists for $a \in A$. □

Let $A_i = \sigma_i(A) = \bigoplus_{l=k_i}^{k_{i+1}-1} B_l$. We can regard A as a C*-subalgebra of $\bigoplus_{i=1}^{\infty} A_i$. Since A contains the ideal I_0 of zero-sequences in $\bigoplus B_i$, I_0 coincides with the ideal of sequences in $\bigoplus_{i=1}^{\infty} A_i$ tending to zero at infinity. Since A is unital, it follows that $C(\hat{\mathbb{N}}) \subseteq A$, where $C(\hat{\mathbb{N}})$ is identified with the algebra of sequences in $\bigoplus A_i$ whose entries are scalar multiples of the identity. Letting $A_\infty = A/I_0$ and $\sigma_\infty = \pi$, we have

Corollary 4.2 *The algebra A is a $C(\hat{\mathbb{N}})$-module, and $(\hat{\mathbb{N}}, \sigma_n : A \to A_n, A)$ is a continuous bundle of C*-algebras.*

Let M_i be the C*-algebra of $i \times i$ complex matrices, let $M = \bigoplus_{i=1}^{\infty} M_i$ and let I_0 be the ideal of zero-sequences in M. It was shown in [14] that a C*-algebra C is exact if and only if the sequence

$$0 \to I_0 \otimes C \to M \otimes C \to (M/I_0) \otimes C \to 0 \qquad (*)$$

is exact.

Proposition 4.3 *Let C be an inexact C*-algebra. Then there is a continuous C*-bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_n, A)$ with A separable such that $\mathscr{A} \otimes C$ is not continuous.*

Proof. If C is not exact, $(*)$ is not exact, which implies that there is an $x \in \ker(\pi \otimes \mathrm{id})$ such that $x \notin I_0 \otimes C$, where $\pi : M \to M/I_0$ is the quotient map. Let A be a separable unital C*-subalgebra of M containing I_0 and such that $x \in A \otimes C$. Applying Lemma 4.1 and Corollary 4.2 with $B_i = M_i$ $(i = 1, 2, \ldots)$, there is a continuous bundle $(\hat{\mathbb{N}}, \sigma_n : A \to A_n, A)$ such that $I_0 = \ker \sigma_\infty$. Since

$$0 < \mathrm{dist}(x, I_0 \otimes C) = \limsup_n \|(\sigma_n \otimes \mathrm{id})(x)\| ,$$

and $(\sigma_\infty \otimes \mathrm{id})(x) = 0$, the function $n \to \|(\sigma_n \otimes \mathrm{id})(x)\|$ is not upper semicontinuous at ∞. □

Examples 4.4 1. Let G be a countable residually finite group with property T, for example $SL_3(\mathbb{Z})$. Then there is a sequence $\pi_1, \pi_2, \ldots$ of mutually inequivalent irreducible unitary representations on finite dimensional Hilbert spaces $H_1, H_2, \ldots$ such that any finite dimensional irreducible unitary representation of G is equivalent to one of the π_i. If $C = \bigoplus_{i=1}^{\infty} B(H_i)$, then the C*-subalgebra $A = C^*((\bigoplus \pi_k)(G))$ of C contains the ideal J_0 of zero sequences in C, and the sequence

$$0 \to J_0 \otimes A \to A \otimes A \to (A/J_0) \otimes A \to 0$$

is not exact (see [18] for details). Taking $B_i = B(H_i)$ $(i = 1, 2, \ldots)$, the above construction yields a continuous bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_n, A)$ with each A_i a direct sum of finitely many of the B_j, for $i < \infty$, and $A_\infty = A/J_0$, such that $\mathscr{A} \otimes A$ is not continuous. It is unknown whether, for a specific choice of G such as $SL_3(\mathbb{Z})$, the representations $\pi_1, \pi_2, \ldots$ can be chosen in such a way that for all $a \in A$, $\lim_{i \to \infty} \|\pi_i(a)\|$ exists. If this is the case, the sequences $\{\pi_i\}$ and $\{\sigma_i\}$ coincide. In this example all the A_i for $i < \infty$ are finite dimensional, hence exact. It is not at present known whether A/J_0 is exact for any choice of G. It is known that $C_r^*(SL_n(\mathbb{Z}))$ is a subalgebra of a nuclear C*-algebra, for $n \geqq 2$ (cf [11]), and it can be shown that for any G of the type under consideration, the canonical morphism $C^*(G) \to C_r^*(G)$ has a factorisation

$$C^*(G) \to A \to A/J_0 \to C_r^*(G) .$$

2. To obtain an example of a continuous bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_n, A)$ with all the A_n exact but $\mathscr{A} \otimes B$ not continuous for some B, we use results from [11]. Recall [11] that for any C*-algebra A, $C(A)$ =cone(A) is the unitization of the C*-algebra $C_0((0,1]) \otimes A$. Thus $C(A) \subseteq C([0,1]) \otimes \tilde{A}$. If A is exact, then so are $\tilde{A}$, $C([0,1]) \otimes \tilde{A}$ and hence $C(A)$. When G is a countable group, the regular C*-algebra $C_r^*(G)$ is known to be exact in many cases, in particular when G is amenable, G is a free group $\mathbb{F}_n$ $(n = 2, 3, \ldots, \infty)$ or $G \cong SL_n(\mathbb{Z})$ $(n = 2, 3, \ldots)$ [11]. (It is conjectured, but, so far as we know, unproved, that $C_r^*(G)$ is exact for all countable G).

Let G be a countable, nonamenable residually finite group. By [11, Sect. 7] (see also [16]) there is an ideal J of the full group C*-algebra $D = C^*(\mathbb{F}_\infty)$ such that $D/J \cong C_r^*(G)$ $(= B$, say), and such that the sequence

$$0 \to J \otimes D \to D \otimes D \to B \otimes D \to 0$$

is not exact. By [11, Proposition 5.1 and Lemma 6.1], there are a sequence $n_1, n_2, \ldots$ in $\mathbb{N}$ and a C*-subalgebra $A \subseteq M = \bigoplus_{i=1}^{\infty} M_{n_i}$ such that, if I_0 is the ideal of zero sequences in M, $I_0 \subseteq A$, $A/I_0 \cong C(B)$ and the sequence

$$0 \to I_0 \otimes D \to A \otimes D \to C(B) \otimes D \to 0$$

is not exact. Applying the above construction, we obtain a continuous C*-bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_n, A)$ on $\hat{\mathbb{N}}$, with the A_n finite dimensional for $n < \infty$ and $A_\infty \cong C(B)$ such that $\mathscr{A} \otimes D$ is not continuous. Choosing G so that $C_r^*(G)$ is exact, for example taking $G = SL_2(\mathbb{Z})$, all the fibre algebras A_n ($n = 1, 2, \ldots, \infty$) are exact. Since the product bundle is not exact, it follows by Corollary 2.8, that A is not exact. Thus for a continuous C*-bundle (X, A_x, A), exactness of A_x for all x does not imply that A is exact, in general. We shall give necessary and sufficient conditions under which exactness of A_x for all x does imply exactness of A in Theorem 4.6.

3. Example 2 can be modified to give a continuous C*-bundle $\bar{\mathscr{A}} = (\hat{\mathbb{N}}, \bar{A}_n, \bar{A})$ with constant exact fibre algebra such that $\bar{\mathscr{A}} \otimes D$ is not continuous. Let D_i ($i = 1, 2, \ldots$) be a sequence of C*-algebras consisting of $A_1, A_2, \ldots, A_\infty$, each occurring an infinite number of times. Since spatial tensor products and inductive limits of exact C*-algebras are exact, it follows that the algebras $C = \bigotimes_{i=1}^{\infty} D_i$, $\bar{A}_n = C \otimes A_n$ ($n \in \hat{\mathbb{N}}$) and $\bar{A} = C \otimes A$ are exact. Then $\bar{A}_n \cong C$ ($n = 1, 2, \ldots, \infty$) and $\bar{\mathscr{A}} = C \otimes \mathscr{A} = (\hat{\mathbb{N}}, \mathrm{id}_C \otimes \pi_n : \bar{A} \to \bar{A}_n, \bar{A})$ is continuous, by Lemma 2.8, but $\bar{\mathscr{A}} \otimes D$ is not continuous, since it contains the discontinuous bundle of Example 2 as a subbundle.

4.2 *A characterisation of exactness*

The techniques of the previous section give the following general result.

Theorem 4.5 *For a C*-algebra C the following conditions are equivalent*:
(i) *C is exact*;
(ii) *The sequence*

$$0 \to I_0 \otimes C \to M \otimes C \to (M/I_0) \otimes C \to 0$$

is exact;
(iii) *for every continuous C*-bundle $\mathscr{A} = (X, A_x, A)$ with X locally compact, the bundle $\mathscr{A} \otimes C$ is continuous*;
(iv) *for every continuous C*-bundle $\mathscr{A} = (\hat{\mathbb{N}}, A_x, A)$ with A separable, the bundle $\mathscr{A} \otimes C$ is continuous.*

Proof. (ii) $\Leftrightarrow$ (i): this is just [9, Theorem 1.1].
(i) $\Rightarrow$ (iii): This follows by Corollary 2.8.
(iii) $\Rightarrow$ (iv): Trivial.
(iv) $\Rightarrow$ (i): This follows immediately from Proposition 4.3. □

4.3 A characterisation of continuous C^*-bundles with exact bundle C^*-algebras

Let $(X, \pi_x : A \to A_x, A)$ be a continuous bundle of C*-algebras. If A is exact, then by Remark 2.9(2), each A_x is exact, and, as remarked in the proof of Corollary 2.8, for any C*-algebra B and $x \in X$, the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \qquad (*)$$

is exact. Effros and Haagerup [6, Theorem 3.2] showed that $(*)$ is exact for arbitrary B precisely when the quotient map π_x is *locally liftable*, that is, for any finite dimensional operator subsystem $Z \subseteq A_x$, there is a completely positive isometry $\iota_{Z,x} : Z \to A$ such that $\pi_x \circ \iota_{Z,x} = \mathrm{id}_Z$.

If, conversely, each A_x is exact, we have already seen that A need not be exact, and the question that arises is under what circumstances we can conclude that A is exact. This is answered by the following theorem. Here the amalgamated minimal tensor product $A \otimes_{C_0(X)} B$ of the C*-algebras of continuous bundles $(X, \pi_x : A \to A_x, A)$ and $(X, \sigma_x : B \to B_x, B)$ over the same space X is the algebra $(\bigoplus_{x \in X}(\pi_x \otimes \sigma_x))(A \otimes B)$.

Theorem 4.6 *Let $\mathscr{A} = (X, A_x, A)$ be a continuous bundle of C^*-algebras on a locally compact Hausdorff space X, such that each A_x is exact. The following conditions are equivalent*:
(i) *the algebra A is exact*;
(ii) *the map $\pi_x : A \to A_x$ is locally liftable for $x \in X$*;
(iii) *for every C^*-algebra B, the bundle $\mathscr{A} \otimes B$ is continuous*;
(iv) *for every continuous C^*-bundle $\mathscr{B} = (X, \sigma_x : B \to B_x, B)$ on X, the bundle $\mathscr{A} \otimes_{C_0(X)} \mathscr{B} = (X, \pi_x \otimes \sigma_x : A \otimes_{C_0(X)} B \to A_x \otimes B_x, A \otimes_{C_0(X)} B)$ is continuous*;
(v) *for every continuous C^*-bundle $\mathscr{B} = (Y, \sigma_y : B \to B_y, B)$ on a locally compact space Y, the bundle $\mathscr{A} \otimes \mathscr{B} = (X \times Y, \pi_x \otimes \sigma_y : A \otimes B \to A_x \otimes B_y, A \otimes B)$ is continuous.*

To prove this theorem we need a number of preparatory results.

Lemma 4.7 *Let A and B be C^*-algebras. For $c \in A \odot B$, $\|c\|_{min}$ is the supremum of*

$$\left[\frac{(f \otimes g)(d^*c^*cd)}{(f \otimes g)(d^*d)}\right]^{\frac{1}{2}}$$

over all pure states $g \in P(A)$, $g \in P(B)$ and $d \in A \odot B$ such that $(f \otimes g)(d^*d) \neq 0$.

Proof. Let (π_f, H_f, ξ_f) be the triple consisting of an irreducible representation of A on a Hilbert space H_f with cyclic unit vector ξ_f corresponding to $f \in P(A)$ by the GNS construction. Since the families $\{\pi_f : f \in P(A)\}$ and $\{\pi_g : g \in P(B)\}$ of irreducible representations of A and B, respectively, are faithful, the family

$$\{\pi_f \otimes \pi_g : f \in P(A), g \in P(B)\}$$

is faithful on $A \otimes B$ (cf [15]). If $0 \neq c \in A \odot B$, and $\varepsilon > 0$, there are $f \in P(A)$ and $g \in P(B)$ such that $\|(\pi_f \otimes \pi_g)(c)\| \geqq \|c\|_{min} - \varepsilon/2$. There is a $d \in A \odot B$ such that $\xi = ((\pi_f \otimes \pi_g)(d))(\xi_f \otimes \xi_g)$ is a unit vector and

$$\|((\pi_f \otimes \pi_g)(c))\xi\| \geqq \|(\pi_f \otimes \pi_g)(c)\| - \varepsilon/2 \geqq \|c\|_{min} - \varepsilon .$$

Thus $(f \otimes g)(d^*d) = 1$ and

$$\|c\|^2_{min} \geqq (f \otimes g)(d^*c^*cd) \geqq (\|c\|_{min} - \varepsilon)^2 .$$

Since ε is arbitrary, the result follows. □

Lemma 4.8 *Let $(X, \pi_x : A \to A_x, A)$ be a continuous bundle of C^*-algebras on a locally compact space X, and let $D \subseteq X$. If $x_0 \in \bar{D}$, $f \in P(A_{x_0})$, $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, then there are $x \in D$ and $\bar{f} \in S(A_x)$ such that*

$$|\bar{f}(\pi_x(a_i)) - f(\pi_{x_0}(a_i))| \leqq \varepsilon \quad (1 \leqq i \leqq n) .$$

Proof. Letting $A_D = \pi_D(A)$ and defining $\bar{\pi}_x : A_D \to A_x$ by $\bar{\pi}_x \circ \pi_D = \pi_x$ for $x \in D$, $\lim_{x \to x_0} \|\pi_x(a)\| = \|\pi_{x_0}(a)\|$ for $a \in A$. Thus A_{x_0} is a quotient of A_D. Let $\pi_0 : A_D \to A_{x_0}$ be the quotient map. Taking $S \subseteq S(A_D)$ to be the set of states on A_D of form $\varphi \circ \bar{\pi}_x$ for $x \in D$, $\varphi \in S(A_x)$, then for $a \in A_D$, $a \geqq 0 \Leftrightarrow f(a) \geqq 0$ $(f \in S)$. If f is a pure state of A_{x_0}, then $f \circ \pi_0$ is a pure state of A_D and, by [5, 3.4.1], $f \circ \pi_0$ is in the weak closure of S. Thus there are an $x \in D$ and an $\bar{f} \in S(A_x)$ with the required properties. □

Proposition 4.9 *Let $(X, \pi_x : A \to A_x, A)$ and $(Y, \sigma_y : B \to B_y, B)$ be continuous bundles of C^*-algebras over the locally compact Hausdorff spaces X and Y. Then the function $(x, y) \to \|(\pi_x \otimes \sigma_y)(c)\|$ is lower semicontinuous on $X \times Y$ for each $c \in A \otimes B$.*

Proof. Suppose that $\|(\pi_x \otimes \sigma_y)(c)\|$ is not lower semicontinuous for some $c \in A \otimes B$ at $z_0 = (x_0, y_0)$. Then there are an $\varepsilon > 0$ and a net $\{z_\lambda\}_{\lambda \in \Lambda}$ converging to z_0 such that $\|c_{z_\lambda}\| \leqq \|c_{z_0}\| - 3\varepsilon$ for all λ. Let $d \in A \odot D$ with $\|c - d\| \leqq \varepsilon$. Then

$$\|d_{z_\lambda}\|_{min} \leqq \|d_{z_0}\|_{min} - \varepsilon \quad (\lambda \in \Lambda) . \qquad (*)$$

By Lemma 4.7 there are $h_0 \in A_{x_0} \odot B_{y_0}$, $f \in P(A_{x_0})$ and $g \in P(B_{y_0})$ such that $(f \otimes g)(h_0^* h_0) = 1$ and

$$[(f \otimes g)(h_0^* d_{z_0}^* d_{z_0} h_0)]^{\frac{1}{2}} \geqq \|d_{z_0}\| - \varepsilon/3 .$$

Choose $h \in A \odot B$ such that $h_{z_0} = h_0$. There are $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_n \in B$ such that

$$h^*d^*dh = \sum_{i=1}^{m} a_i \otimes b_i \quad \text{and} \quad h^*h = \sum_{i=m+1}^{n} a_i \otimes b_i .$$

Let $\varepsilon' > 0$, where a suitably small value of ε' will be chosen later in the proof. If $z_\lambda = (x_\lambda, y_\lambda)$ for $\lambda \in \Lambda$, then $\lim_\lambda x_\lambda = x_0$. If U is a neighbourhood of x_0 in X and D is the set $\{x_\lambda : x_\lambda \in U\}$, applying Lemma 4.8, there are an $x \in D$ and $\bar{f} \in S(A_x)$ such that

$$|\bar{f}(\pi_x(a_i)) - f(\pi_{x_0}(a_i))| \leqq \varepsilon' \quad (1 \leqq i \leqq n).$$

It follows that there is a refinement $\{z_\lambda\}_{\lambda\in\Lambda'}$ of the net $\{z_\lambda\}_{\lambda\in\Lambda}$, with $\Lambda' \subseteq \Lambda$, converging to z_0 and such that for each $\lambda \in \Lambda'$ there is an $f_\lambda \in S(A_{x_\lambda})$ satisfying

$$|f_\lambda(\pi_{x_\lambda}(a_i)) - f(\pi_{x_0}(a_i))| \leqq \varepsilon' \quad (1 \leqq i \leqq n).$$

Repeating this argument for subsets $D \subseteq Y$ of form $\{y_\lambda : y_\lambda \in V\}$, where V is a neighbourhood of y_0 in Y, we get a refinement $\{z_\lambda\}_{\lambda\in\Lambda''}$ of the net $\{z_\lambda\}_{\lambda\in\Lambda'}$, with $\Lambda'' \subseteq \Lambda'$, converging to z_0 and such that for each $\lambda \in \Lambda''$ there is a $g_\lambda \in S(B_{x_\lambda})$ satisfying

$$|g_\lambda(\sigma_{y_\lambda}(b_i)) - g(\sigma_{y_0}(b_i))| \leqq \varepsilon' \quad (1 \leqq i \leqq n).$$

It is now easy to see that, for a suitable choice of ε', determined by $\|a_1\|,\ldots,\|a_n\|$, $\|b_1\|,\ldots,\|b_n\|$, $(f\otimes g)(h_0^* d_0^* dh)$ and ε, for any $\lambda \in \Lambda''$

$$\left[\frac{(f_\lambda\otimes g_\lambda)(h_{z_\lambda}^* d_{z_\lambda}^* d_{z_\lambda} h_{z_\lambda})}{(f_\lambda\otimes g_\lambda)(h_{z_\lambda}^* h_{z_\lambda})}\right]^{\frac{1}{2}} \geqq \left[\frac{(f\otimes g)(h_0^* d_{z_0}^* d_{z_0} h_0)}{(f\otimes g)(h_0^* h_0)}\right]^{\frac{1}{2}} - \varepsilon/3 .$$

It follows that $\|d_{z_\lambda}\|_{min} \geqq \|d_{z_0}\|_{min} - 2\varepsilon/3$, which contradicts $(*)$. Thus the given function is lower semicontinuous. □

Lemma 4.10 *Let $\mathscr{A} = (X, \pi_x : A \to A_x, A)$ and $\mathscr{B} = (Y, \sigma_y : B \to B_y, B)$ be continuous bundles of C^*-algebras.*

1. *If $\mathscr{A}\otimes\mathscr{B} = (X\times Y, \pi_x\otimes\sigma_y : A\otimes B \to A_x\otimes B_y, A\otimes B)$ is a continuous bundle, so is $\mathscr{A}\otimes B$.*
2. *If A_x is exact for $x\in X$ and $\mathscr{A}\otimes B$ is a continuous bundle, then the bundle $\mathscr{A}\otimes\mathscr{B}$ is continuous.*

Proof. 1. Let $c \in A\otimes B$. The function $(x,y) \to \|(\pi_x\otimes\sigma_y)(c)\|$ is in $C_0(X\times Y)$ and for fixed x,

$$\sup_{y\in Y} \|(\pi_x\otimes\sigma_y)(c)\| = \|(\pi_x\otimes \mathrm{id})(c)\| .$$

If $\varepsilon > 0$, a simple compactness argument shows that there is a neighbourhood U of x such that

$$\|(\pi_{x'}\otimes\sigma_y)(c)\| \leqq \|(\pi_x\otimes\sigma_y)(c)\| + \varepsilon$$

for $x' \in U$ and $y \in Y$. Thus

$$\|(\pi_{x'}\otimes \mathrm{id})(c)\| \leqq \|(\pi_x\otimes \mathrm{id})(c)\| + \varepsilon$$

for $x' \in U$. This shows that the map $x \to \|(\pi_x\otimes \mathrm{id})(c)\|$ is upper semicontinuous. Since this map is also lower semicontinuous, by Lemma 2.5, it is continuous.

2. For fixed x exactness of A_x implies that $A_x \otimes \mathscr{B}$ is continuous, by Corollary 2.8. For $c \in A \otimes B$, the function $(x, y) \rightarrow \|(\pi_x \otimes \sigma_y)(c)\|$ is lower semicontinuous, by Proposition 4.9. So to prove (2), it is enough to show that this function is also upper semicontinuous for all c.

Suppose that for some c the function is not upper semicontinuous at (x_0, y_0). Then there are an $\varepsilon > 0$ and a net $\{(x_\lambda, y_\lambda)\}_{\lambda \in \Lambda}$ such that $(x_0, y_0) = \lim_\lambda (x_\lambda, u_\lambda)$ and

$$\|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \geqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon$$

for $\lambda \in \Lambda$. Since the bundle $A_{x_0} \otimes \mathscr{B}$ is continuous, there is a neighbourhood U of y_0 in Y such that

$$\|(\pi_{x_0} \otimes \sigma_y)(c)\| \leqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon/3$$

for $y \in U$. Let $f : Y \rightarrow [0, 1]$ be a continuous function such that $f|_{Y \setminus U} \equiv 0$, $f(y_0) = 1$. Replacing c by $(1 \otimes f)c$ and refining the net $\{(x_\lambda, y_\lambda)\}$, we can assume that

$$\sup_{y \in Y} \|(\pi_{x_0} \otimes \sigma_y)(c)\| \leqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon/3$$

and

$$\|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \geqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 3\varepsilon/4$$

for $\lambda \in \Lambda$. For some neighbourhood V of x_0 in X

$$\begin{aligned} \|(\pi_x \otimes \mathrm{id})(c)\| &\leqq \|(\pi_{x_0} \otimes \mathrm{id})(c)\| + \varepsilon/3 \\ &\leqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 2\varepsilon/3 \end{aligned}$$

for $x \in V$. Choose λ so that $(x_\lambda, y_\lambda) \in V \times U$. Then

$$\begin{aligned} \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 3\varepsilon/4 &\leqq \|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \\ &\leqq \|(\pi_{x_\lambda} \otimes \mathrm{id})(c)\| \\ &\leqq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 2\varepsilon/3 . \end{aligned}$$

This is a contradiction. So $(x, y) \rightarrow \|(\pi_x \otimes \sigma_y)(c)\|$ is upper semicontinuous for all c, and (2) follows. □

Proposition 4.11 *Let $(X, \pi_x : A \rightarrow A_x, A)$ be a continuous bundle of C*-algebras such that each A_x is exact and, for any C*-algebra B, the sequence*

$$0 \rightarrow J_x \otimes B \rightarrow A \otimes B \rightarrow A_x \otimes B \rightarrow 0 \qquad (*)$$

is exact. Then A is exact.

Proof. 1. We prove first that if $\pi : A \to B(H)$ is any factor representation of A and $J = \ker \pi$, then A/J is exact and the sequence

$$0 \to J \otimes B \to A \otimes B \to (A/J) \otimes B \to 0 \qquad (**)$$

is exact for any B. Extending π to a representation $\bar{\pi}$ of $M(A)$ on H, $\bar{\pi}$ restricted to $C_0(X)$ is a character of $C_0(X)$ since π is a factor representation of A. This implies that for some $x \in X$, $J_x \subseteq J$. Thus A/J is a quotient of A_x. Since the latter is exact, so is A/J.

Let c be in the kernel of the quotient morphism $J \otimes B \to (J/J_x) \otimes B$. Since $(J/J_x) \otimes B \subseteq (A/J_x) \otimes B = A_x \otimes B$, $c \in \ker(\pi_x \otimes \mathrm{id}) = J_x \otimes B$, since $(*)$ is exact. It follows that in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J_x \otimes B & \to & J \otimes B & \to & (J/J_x) \otimes B & \to & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \to & J_x \otimes B & \to & A \otimes B & \to & (A_x) \otimes B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & (A/J) \otimes B & \xrightarrow{id} & (A/J) \otimes B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the rows and and outer columns are exact. (The exactness of the right-hand column follows from [1, Theorem 3.4], since A_x, being exact, has property C [10]). It is now easy to see, by a standard diagram-chasing argument, [2] that the centre column, i.e. $(**)$, is exact.

2. Let

$$0 \to K \to B \to B/K \to 0$$

be a short exact sequence of C*-algebras. If π is an irreducible representation of $A \otimes B$ such that $\pi(A \otimes K) = \{0\}$, the restrictions π_1 and π_2 of π to A and B, respectively, are factor representations, and $\pi_2(K) = \{0\}$. Let $J = \ker \pi_1$. Then $J \otimes B \subseteq \ker \pi$ and, since the sequence $(**)$ is exact,

$$(A \otimes B)/(J \otimes B) \cong (A/J) \otimes B$$

canonically, and $\pi = \tilde{\pi} \circ (\varphi \otimes \mathrm{id}_B)$, where $\varphi : A \to A/J$ is the quotient morphism and $\tilde{\pi}$ is an irreducible representation of $(A/J) \otimes B$. By part (1) of the proof $A/J \cong \pi_1(A)$ is exact, and moreover $A \otimes K \subseteq \ker \pi$, which implies that $(A/J) \otimes K \subseteq \ker \tilde{\pi}$. Thus the sequence

$$0 \to (A/J) \otimes K \to (A/J) \otimes B \to (A/J) \otimes (B/K) \to 0$$

[2] Known variously as the 3×3- and 9-lemma

is exact, and so $\bar{\pi} = \bar{\bar{\pi}} \circ (\mathrm{id}_{A/J} \otimes \psi)$, where $\psi : B \to B/K$ is the quotient morphism and $\bar{\bar{\pi}}$ is an irreducible representation of $(A/J) \otimes (B/K)$.

Now $\varphi \otimes \psi = (\mathrm{id}_{A/J} \otimes \psi) \circ (\varphi \otimes \mathrm{id}_B) = (\varphi \otimes \mathrm{id}_{B/K}) \circ (\mathrm{id}_A \otimes \psi)$, and so $\pi = \bar{\bar{\pi}} \circ (\mathrm{id}_{A/J} \otimes \psi) \circ (\varphi \otimes \mathrm{id}_B) = \big(\bar{\bar{\pi}} \circ (\varphi \otimes \mathrm{id}_{B/K})\big) \circ (\mathrm{id}_A \otimes \psi)$. If $x \in (A \otimes B) \setminus (A \otimes K)$, we can choose π so that $\pi(x) \neq 0$ and $A \otimes K \subseteq \ker \pi$. Then $\big(\bar{\bar{\pi}} \circ (\varphi \otimes \mathrm{id}_{B/K})\big)((\mathrm{id}_A \otimes \psi)(x)) = \pi(x) \neq 0$, which implies that $(\mathrm{id}_A \otimes \psi)(x) \neq 0$, i.e $x \notin \ker(\mathrm{id}_A \otimes \psi)$. Since x is arbitrary, it follows that $\ker(\mathrm{id}_A \otimes \psi) = A \otimes K$, which is equivalent to the exactness of the sequence

$$0 \to A \otimes K \to A \otimes B \to A \otimes (B/K) \to 0.$$

Since B and $K \lhd B$ are arbitrary, A is exact. □

Proof of Theorem 4.6 (i) ⇒ (iii): By [10], if A is exact it has property C. This implies [1, Theorem 3.4] that the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \qquad (*)$$

is exact for any B and $x \in X$. By Proposition 2.7 the bundle $\mathscr{A} \otimes B$ is continuous.

(iii) ⇒ (v): This follows immediately from Lemma 4.10(2).

(v) ⇒ (iv): Since the bundle $\mathscr{A} \otimes_{C_0(X)} \mathscr{B}$ is obtained from the bundle $\mathscr{A} \otimes \mathscr{B}$ by restriction to the diagonal $\{(x,x) : x \in X\} \subseteq X \times X$, the result is immediate.

(iv) ⇒ (iii): Let $\mathscr{B}$ be the trivial C*-bundle on X with fibre B. Then the product bundle $\mathscr{A} \otimes_{C_0(X)} \mathscr{B}$, which is continuous by assumption, is just $\mathscr{A} \otimes B$.

(iii) ⇒ (i): By Proposition 2.7, the continuity of the bundle $\mathscr{A} \otimes B$ implies the exactness of the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \qquad (*)$$

for each $x \in X$. Since each A_x is exact, Proposition 4.11 implies that A is exact.

(iii) ⇔ (ii): As remarked earlier, the exactness of $(*)$ for all B is equivalent to the map π_x being locally liftable, by [6, Theorem 3.2]. □

Remark 4.12 Another route from (v) to (i) is the following. If $\mathscr{B} = (Y, \sigma_y : B \to B_y, B)$ is a continuous bundle, then the continuity of the product bundle $\mathscr{A} \otimes \mathscr{B}$ implies that of the bundle $A \otimes \mathscr{B}$, by Lemma 4.10(1). By Theorem 4.5 A is exact.

5 Concluding remarks

1. Recall that a continuous C*-bundle $\mathscr{B} = (X, B_x, B)$ is *trivial* when $B = C_0(X,A)$, $B_x = A$ $(x \in X)$ for some C*-algebra and $\pi_x(f) = f(x)$ for $f \in C_0(X)$. Thus $\mathscr{B} = \lhd_X \otimes A$, where $\lhd_X$ is the trivial bundle on X with fibre $\lhd$. For any C*-algebra C, $\mathscr{B} \otimes C \cong (A \otimes \lhd_X) \otimes C \cong \lhd_X \otimes (A \otimes C)$, so that $\mathscr{B} \otimes C$ is itself continuous. If $\mathscr{B}_1$ is a subbundle of $\mathscr{B}$, it follows that

$\mathscr{B}_1 \otimes C$ is also continuous. If $\mathscr{B}_1$ is a continuous bundle for which $\mathscr{B}_1 \otimes C$ is not continuous for some C, as is the case for the bundles of Examples 4.4, $\mathscr{B}_1$ cannot be subtrivial, i.e. a subbundle of a trivial bundle. In particular, if $\mathscr{B}_1 = (X, \pi_x : A \to A_x, A)$, there cannot exist a family σ_x of representations of the algebras A_x on a fixed Hilbert space H such that the function $x \to \sigma_x(\pi_x(a))$ is norm-continuous. This contrasts with a recent result of Blanchard [3] that for any continuous C*-bundle $\mathscr{B} = (X, A_x, A)$ with A separable, there is a family of representations $\sigma_x : A_x \to B(H)$ such that the function $x \to \sigma_x(\pi_x(a))$ is strongly continuous for $a \in A$. It would be interesting to know under what circumstances a continuous bundle will be subtrivial. There is some evidence to suggest that this will be the case if all the fibre algebras are nuclear.

2. We have not considered continuity conditions for crossed product bundles of form $\mathscr{A} \rtimes_\alpha G$ and $\mathscr{A} \rtimes_{\alpha,r} G$. As noted in Sect. 2, when the actions are trivial, these bundles reduce to the tensor product bundles $\mathscr{A} \otimes_{max} C^*(G)$ and $\mathscr{A} \otimes C^*_r(G)$, respectively. By Theorem 3.2, $\mathscr{A} \otimes_{max} C^*(G)$ is continuous for arbitrary continuous $\mathscr{A}$ if and only if $C^*(G)$ is nuclear, which is the case for discrete G if and only if G is amenable, by [12, Theorem 4.2]. By Remark 2.6 (2), amenability of G implies the continuity of any crossed product bundle $\mathscr{A} \rtimes_\alpha G = \mathscr{A} \rtimes_{\alpha,r} G$.

The position for reduced crossed products is less clear. By Theorem 4.5, $\mathscr{A} \otimes C^*_r(G)$ is continuous for arbitrary continuous $\mathscr{A}$ if and only if $C^*_r(G)$ is exact. As we mentioned earlier, it appears to be still unknown whether $C^*_r(G)$ is exact for all discrete G. If $\mathscr{A} = (X, A_x, A)$ is a continuous C*-bundle and $x \to \alpha_x$ is a continuous field of actions of a discrete group G on $\mathscr{A}$, then by reasoning similar to the proof of Proposition 2.7 it can be shown that the bundle $\mathscr{A} \rtimes_{\alpha,r} G$ is continuous if and only if each of the sequences

$$0 \to J_x \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G \to A_x \rtimes_{\alpha_x,r} G \to 0$$

is exact. Exactness questions of this type will be considered in another paper.

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