Algebraic Topology

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Contents

0 Introduction

These notes are taken from a year-long course in algebraic topology taught by Dr. Thomas Mark at the University of Virginia in the spring and fall of 2016. The main topics covered are homotopy theory, homology and cohomology, including:

- Homotopy and the fundamental group
- Covering spaces and covering transformations
- Universal covering spaces and the 'Galois group' of a space
- Graphs and subgroups of free groups
- The fundamental group of surfaces
- Chain complexes
- Simplicial and singular homology
- Exact sequences and excision
- Cellular homology
- Cohomology theory
- Cup and cap products
- Poincaré duality.

The companion text for the course is Bredon's *Topology and Geometry*.

0.1 Differential Forms

As a motivation for the study of homology in algebraic topology, we begin by discussing differential forms with the goal of constructing de Rham cohomology. This material provides a great case study which one can return to in later sections to apply results in the general homology theory.

Definition. *If V is a finite dimensional real vector space, the space $\bigwedge^p V^*$ is called the* p**th exterior algebra** *of V. The elements of $\bigwedge^p V^*$ are called* **exterior p-forms**.

Example 0.1.1. For $p = 0$, $\bigwedge^0 V^* = \mathbb{R}$, the underlying field. For $p = 1$, $\bigwedge^1 V^* = V^*$, the vector space dual to V, which one may recall is defined as $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. For $p \geq 2$, $\bigwedge^p V^*$ may be viewed as the space of functions

$$\omega : \underbrace{V \times \cdots \times V}_{p} \longrightarrow \mathbb{R}$$

that satisfy the following properties:

(a) (Multilinearity) $\omega(\ldots, v + cw, \ldots) = \omega(\ldots, v, \ldots) + c\omega(\ldots, w, \ldots)$.

(b) (Alternating) $\omega(\ldots, w, v, \ldots) = -\omega(\ldots, v, w, \ldots)$.

The alternating property may be written in a more general form:

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) = (-1)^{|\sigma|}\omega(v_1, \ldots, v_p),$$

where σ is a permutation on p symbols and $|\sigma|$ denotes its sign (-1 if odd and $+1$ if even). The alternating property further implies that $\omega(v, v, \ldots) = 0$, i.e. repeated terms produce 0 when an exterior form is applied to a vector.

Example 0.1.2. If $\dim V = n$ and a basis of V is given, we can view an n-form $\omega \in \bigwedge^n V^*$ in terms of a determinant:

$$\omega(v_1, \ldots, v_n) = \det(v_1 \mid \cdots \mid v_n).$$

Definition. *If $\omega \in \bigwedge^p V^*$ and $\eta \in \bigwedge^q V^*$, their* **exterior product** *(or* **wedge product***) is an exterior form $\omega \wedge \eta \in \bigwedge^{p+q} V^*$ defined by*

$$(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_{\sigma} (-1)^{|\sigma|}\omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$

where the sum is over all (p, q)-shuffles σ, i.e. permutations satisfying $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p + 1) < \ldots < \sigma(p + q)$.

Remark. One can alternatively write the wedge product formula as

$$(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^{|\sigma|}\omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}).$$

The wedge product defines an associative multiplication on the vector space

$$\bigwedge{}^* V^* = \bigoplus_{p \geq 0} \bigwedge{}^p V^*,$$

called the **exterior algebra** of V.

Example 0.1.3. Take some 1-forms $\omega_1, \ldots, \omega_p$ on $\mathbb{R}^n$ and vectors $v_1, \ldots, v_p$. We claim that

$$(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, \ldots, v_p) = \det(\omega_i(v_j)).$$

Recall by the Leibniz rule for determinants that if $B = (b_{ij})$ is a $p \times p$ matrix then the formula for its determinant is

$$\det B = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) \cdot b_{\sigma(1)1} \cdots b_{\sigma(p)p}$$

where S_p is the symmetric group on p symbols. Accordingly, the right hand side of the equation in the problem reads

$$\det[\omega_i(v_j)] = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) \cdot \omega_{\sigma(1)}(v_1) \cdots \omega_{\sigma(n)}(v_n).$$

By definition of the wedge product,

$$(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, \ldots, v_p) = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) \cdot \omega_1(v_{\sigma(1)}) \cdots \omega_p(v_{\sigma(p)}).$$

Now this looks like we have the indices irreparably reversed, but in fact this formula is equal to $\det[\omega_j(v_i)]$, the determinant of the *transpose* of $[\omega_i(v_j)]$, which we know from linear algebra is equal to the determinant of $[\omega_i(v_j)]$ anyways. So the formula holds.

Example 0.1.4. If $\{e_1, \ldots, e_n\}$ is a basis of V, there are corresponding linear functionals $dx_1, \ldots, dx_n : V \to \mathbb{R}$ uniquely defined by $dx_i(e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. The dx_i are in fact the dual basis to the e_i, that is $dx_i \in V^* = \bigwedge^1 V^*$, meaning the dx_i are examples of exterior 1-forms. The wedge of two 1-forms looks like

$$(dx_i \wedge dx_j)(v, w) = dx_i(v)dx_j(w) - dx_i(w)dx_j(v) = v_i w_j - v_j w_i,$$

where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. For example, in $\mathbb{R}^3$ the standard basis gives 3 linear functionals dx, dy and dz, which act on 3-vectors in the following way:

$$dx \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \qquad dy \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \qquad dz \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_3.$$

Lemma 0.1.5. *If $\omega_1, \ldots \omega_p$ are 1-forms on V and $v_1, \ldots, v_p \in V$ then*

$$(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, \ldots, v_p) = \det(\omega_j(v_i))_{ij}.$$

Proof. For exercise. □

Lemma 0.1.6. *If $\omega_1, \ldots, \omega_n$ is a basis for V^* then a basis for $\bigwedge^p V^*$ is given by*

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid i_1 < \ldots < i_p\}.$$

Proof. Let $\{v_1, \ldots, v_n\}$ be the dual basis to $\{\omega_1, \ldots, \omega_n\}$, i.e. $\omega_i(v_j) = \delta_{ij}$ for all i, j. Given $\omega \in \bigwedge^p V^*$, define the real number

$$\omega_{i_1, \ldots, i_p} = \omega(v_{i_1}, \ldots, v_{i_p})$$

for all increasing sequences $i_1 < \ldots < i_p$. We claim that

$$\omega(v_{j_1}, \ldots, v_{j_p}) = \sum_{i_1 < \ldots < i_p} (\omega_{i_1, \ldots, i_p} \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p})(v_{j_1}, \ldots, v_{j_p}).$$

Indeed, $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})(v_{j_1}, \ldots, v_{j_p}) = 0$ unless $\{i_1, \ldots, i_p\} = \{j_1, \ldots, j_p\}$, in which case the result is 1. This proves the claimed formula, and in particular the formula holds for all size p subsets of $\{v_1, \ldots, v_n\}$ so

$$\omega = \sum_{i_1 < \ldots < i_p} \omega_{i_1, \ldots, i_p} \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Hence the given set spans $\bigwedge^p V^*$. Now suppose $\sum a_{i_1, \ldots, i_p} \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = 0$. Evaluating this on $(v_1, \ldots, v_p)$ shows that all $a_{i_1, \ldots, i_p} = 0$. This proves linear independence. □

Corollary 0.1.7. *If* $\dim V = n$ *then* $\dim \bigwedge^p V^* = \binom{n}{p}$. *In particular,* $\bigwedge^p V^* = 0$ *whenever* $p > \dim V$.

Lemma 0.1.8. *If* $\omega \in \bigwedge^p V^*$ *and* $\eta \in \bigwedge^q V^*$ *then* $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$.

Example 0.1.9. On $\mathbb{R}^3$, exterior p-forms have the following forms:

- $p = 0$: $\omega = a \in \mathbb{R}$.
- $p = 1$: $\omega = a\,dx + b\,dy + c\,dz$ where $a, b, c \in \mathbb{R}$.
- $p = 2$: $\omega = a\,dy \wedge dz + b\,dx \wedge dz + c\,dx \wedge dy$ for $a, b, c \in \mathbb{R}$. Applying this to a vector may be evaluated using a determinant:

$$\omega(v, w) = \det \begin{pmatrix} a & b & c \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

- $p = 3$: $\omega = a\,dx \wedge dy \wedge dz$ for $a \in \mathbb{R}$.
- $p \geq 4$: all further exterior forms are 0.

The main use of exterior forms is in defining the following generalization of multivariable calculus. Let M be a smooth manifold and view $V = T_xM$ as the tangent space at some point $x \in M$.

Definition. *A* **differential** p**-form** *on* M *is a smooth assignment of an exterior* p*-form* $\omega_x \in \bigwedge^p T_x^*M$ *to each* $x \in M$.

When $M = \mathbb{R}^n$, there is a natural identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$. A differential p-form on $\mathbb{R}^n$ can be written

$$\omega = \sum_{i_1 < \dots i_p} \omega_{i_1,\dots,i_p} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where each $\omega_{i_1,\dots,i_p}$ is a smooth function $\mathbb{R}^n \to \mathbb{R}$. An example of a differential 1-form on $\mathbb{R}^3$ is $x^3 y\,dx + \sin z\,dy + \left(\tan\left(\frac{x}{y}\right)\right)^2 dz$.

For a general n-manifold M, choose local coordinates $\{x_1, \dots, x_n\}$ which determine a basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ for each tangent space T_xM. This also determines a dual basis for T_x^*M which we will write as $\{dx_1, \dots, dx_n\}$. Any differential p-form may be expressed in these local coordinates as

$$\omega = \sum_{i_1 < \dots < i_p} \omega_{i_1,\dots,i_p} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for locally defined, smooth functions $\omega_{i_1,\dots,i_p}$.

Definition. *The vector space of all differential* p*-forms on* M *is denoted* $\Omega^p(M)$.

Note that $\Omega^0(M) = C^\infty(M)$, the space of smooth functions on M, which is infinite dimensional. In general, $\Omega^p(M)$ will be infinite dimensional unless M is a finite set.

For each $x \in M$, the wedge product determines a map

$$\bigwedge^p T_x^*M \times \bigwedge^q T_x^*M \longrightarrow \bigwedge^{p+q} T_x^*M.$$

This further determines a map on the space of differential forms, which we also denote with the wedge symbol $\wedge$:

$$\begin{aligned}\Omega^p(M) \times \Omega^q(M) &\longrightarrow \Omega^{p+q}(M)\\ (\omega, \eta) &\longmapsto \omega \wedge \eta.\end{aligned}$$

This wedge product satisfies $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$.

Example 0.1.10. On $\mathbb{R}^3$, consider two 1-forms $\omega = x\,dx - 3y\,dy$ and $\eta = z\,dx + 4e^y\,dz$. Their wedge product is computed to be

$$\begin{aligned}\omega \wedge \eta &= (x\,dx - 3y\,dy) \wedge (z\,dx + 4e^y\,dz)\\ &= xz\,dx \wedge dx + 4xe^y\,dx \wedge dz - 3yz\,dy \wedge dx - 12ye^y\,dy \wedge dz\\ &= 3yz\,dx \wedge dy + 4xe^y\,dx \wedge dz - 12ye^y\,dy \wedge dz.\end{aligned}$$

Remark. If $\omega \in \Omega^p(M)$ and $X_1, \ldots, X_p$ are vector fields on M, we can evaluate ω on the X_i to get a smooth function on M:

$$\omega(X_1, \ldots, X_p) \in \Omega^0(M) = C^\infty(M).$$

Suppose $f : M \to \mathbb{R}$ is smooth. The differential of f is a linear map $D_xf : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}$, so the differential gives us a 1-form on M. Locally, i.e. in coordinates $x_1, \ldots, x_n$ in a coordinate chart of M, Df is a $1 \times n$ matrix of partial derivatives:

$$Df = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

Traditionally, we write df when thinking of the differential as a 1-form. Using these local coordinates, we can write

$$df = \sum_{i=1}^n a_i dx_i$$

for coefficients $a_i \in \mathbb{R}$. To evaluate these coefficients, notice that for each $1 \le j \le n$,

$$\frac{\partial f}{\partial x_j} = df\left(\frac{\partial}{\partial x_j}\right) = \left(\sum_{i=1}^n a_i dx_i\right)\left(\frac{\partial}{\partial x_j}\right) = \sum_{i=1}^n a_i dx_i\left(\frac{\partial}{\partial x_j}\right) = \sum_{i=1}^n a_i\delta_{ij} = a_j.$$

Therefore we can write the differential 1-form df in local coordinates as

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$

Example 0.1.11. For $M = \mathbb{R}^3$, consider the function $f(x, y, z) = x^2 \sin(y + z)$. Then

$$df = 2x\sin(y + z)\,dx + x^2\cos(y + z)\,dy + x^2\cos(y + z)\,dz.$$

This differential 1-form df generalizes the gradient from multivariable calculus, as one can see in the above formula. We will see that other familiar objects from multivariable calculus show up as differential forms.

0.2 The Exterior Derivative

Definition. *For a smooth manifold M, the* **exterior derivative** *on p-forms is the linear map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ defined as follows: given $\omega \in \Omega^p(M)$ and a point in M, choose local coordinates $x_1, \ldots, x_n \in M$ and write $\omega = \sum_I \omega_I \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ where the sum is over all sequences $I : i_1 < \ldots < i_p$. Then we define*

$$d\omega = \sum_I d\omega_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Lemma 0.2.1. *The definition of the exterior derivative is independent of the local coordinates on M.*

Proof. We will prove that on an open set $U \subset \mathbb{R}^n$, the exterior derivative d is the only operator $\Omega^p(U) \to \Omega^{p+1}(U)$ satisfying:

(a) $d(\omega + \eta) = d\omega + d\eta$;

(b) If $\omega \in \Omega^p(U)$ and $\eta \in \Omega^q(U)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$;

(c) If $f \in \Omega^0(U)$ then $df(X) = X(f)$ (directional derivative);

(d) If $f \in \Omega^0(U)$ then $d(df) = 0$.

Take a p-form ω on U, which can be written

$$\omega = \sum_I \omega_I \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \text{ for some smooth functions } \omega_I \in \Omega^0(U) = C^\infty(U).$$

Suppose $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is an operator satisfying (a) – (d). We will show that it must coincide with our definition of the exterior derivative; that is, we will show:

$$d\omega = \sum_I d\omega_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

For vector fields $X = (X_1, \ldots, X_{p+1})$ defined on U, consider

$$\begin{aligned}
d\omega(X) &= d\left(\sum_I \omega_I \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right)(X) \\
&= \left(\sum_I d(\omega_I \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p})\right)(X) \quad \text{by (a)} \\
&= \sum_I \left[(d\omega_I)(X) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} + (-1)^0 \omega_I(X) \cdot d(dx_{i_1} \wedge \cdots \wedge dx_{i_p})\right] \text{ by (b)} \\
&= \sum_I (d\omega_I)(X) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad \text{by applying (b) and (d) several times} \\
&= \sum_I X(\omega_I) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad \text{by (c).}
\end{aligned}$$

Since the directional derivative property is satisfied for all smooth functions ω_I, we see that d agrees with our definition of the exterior derivative.

Finally, suppose d_U and d_V are defined on overlapping charts $U \cap V$ in an atlas for $\mathbb{R}^n$ such that d_U and d_V each satisfy (a) – (d). Then on the open set $U \cap V$, we have

$$(d_U\omega)|_{U\cap V} = d_{U\cap V}\omega = (d_V\omega)|_{U\cap V}$$

by uniqueness on each neighborhood. Therefore $d : \Omega^p(\mathbb{R}^n) \to \Omega^{p+1}(\mathbb{R}^n)$ is well-defined for each $p \geq 0$. One can extend this to manifolds by composing with coordinate charts. $\square$

Example 0.2.2. In Example 0.1.11, we saw that the gradient arose as a special case of a differential 1-form on $\mathbb{R}^3$. In this example, we will generalize two more quantities associated to vector fields: curl and divergence. If $f \in \Omega^0(\mathbb{R}^3)$ is a smooth function, its exterior derivative is just the differential:

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \frac{\partial f}{\partial x_3}\,dx_3.$$

Again, this looks like the gradient ∇f.

For a 1-form $\omega = f\,dx_1 + g\,dx_2 + h\,dx_3 \in \Omega^1(\mathbb{R}^3)$, the exterior derivative yields a 2-form:

$$\begin{aligned}
d\omega &= df \wedge dx_1 + dg \wedge dx_2 + dh \wedge dx_3 \\
&= \left(\frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \frac{\partial f}{\partial x_3}\,dx_3\right) \wedge dx_1 \\
&\quad + \left(\frac{\partial g}{\partial x_1}\,dx_1 + \frac{\partial g}{\partial x_2}\,dx_2 + \frac{\partial g}{\partial x_3}\,dx_3\right) \wedge dx_2 \\
&\quad + \left(\frac{\partial h}{\partial x_1}\,dx_1 + \frac{\partial h}{\partial x_2}\,dx_2 + \frac{\partial h}{\partial x_3}\,dx_3\right) \wedge dx_3 \\
&= \left(\frac{\partial h}{\partial x_2} - \frac{\partial g}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f}{\partial x_3} - \frac{\partial h}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) dx_1 \wedge dx_2.
\end{aligned}$$

Viewing ω as a vector field, we can clearly see this as the formula for curl $\nabla \times \omega$.

Going further, a 2-form $\eta = f\,dx_2 \wedge dx_3 + g\,dx_3 \wedge dx_1 + h\,dx_1 \wedge dx_2$ yields a 3-form:

$$\begin{aligned}
d\eta &= df \wedge dx_2 \wedge dx_3 + dg \wedge dx_3 \wedge dx_1 + dh \wedge dx_1 \wedge dx_2 \\
&= \left(\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2} + \frac{\partial h}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3.
\end{aligned}$$

In the same manner as above, this generalizes the divergence $\nabla \cdot \eta$ of a vector field.

Theorem 0.2.3. *For any $\omega \in \Omega^p(M)$, $d(d\omega) = 0$. In other words, the sequence*

$$\cdots \to \Omega^p(M) \to \Omega^{p+1}(M) \to \Omega^{p+2}(M) \to \cdots$$

is a chain complex.

Proof. Let $\omega \in \Omega^p(M)$. In local coordinates, $\omega = f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for some smooth function f. Then

$$d\omega = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Applying the exterior derivative again yields

$$d^2\omega = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i}\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

By equality of mixed partials, all symmetric terms will cancel, and all other terms contain at least two repeated dx_{i_k} terms, so we are left with 0. □

The next result states that the exterior derivative obeys a type of 'signed Leibniz rule' with respect to the wedge product of differential forms.

Proposition 0.2.4. *If $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

0.3 de Rham Cohomology

In this section we define de Rham cohomology for a smooth manifold M. Going forward, this will be our case study of the connections between algebraic and topological information; such connections are conveyed by homology.

Definition. *A p-form $\omega \in \Omega^p(M)$ is* **closed** *if $d\omega = 0$. Also, ω is* **exact** *if $\omega = d\eta$ for some $(p-1)$-form $\eta \in \Omega^{p-1}(M)$. In particular, every exact form is closed.*

Consider the sequence of vector spaces induced by the exterior derivative on differential forms of M:

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \cdots \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \cdots$$

Here we are letting d_p denote the exterior derivative on p-forms; we will often just write d to represent the exterior derivative in any degree. Notice that for each $p \geq 0$, the closed p-forms are exactly $\ker d_p$ and the exact p-forms are $\operatorname{im} d_{p-1}$. These are, in particular, vector spaces of differential forms. Moreover, $\operatorname{im} d_{p-1} \subseteq \ker d_p$ which allows us to define:

Definition. *For a smooth manifold M, the pth* **de Rham cohomology** *of M is defined to be the quotient space*

$$H^p_{dR}(M) = \frac{\ker d_p}{\operatorname{im} d_{p-1}}.$$

Since $\ker d_p$ and $\operatorname{im} d_{p-1}$ are subspaces of infinite dimensional vector spaces of differential forms, it is likely that they too are infinite dimensional. However, in some cases the quotient may be finite. This is the case when M is compact, for example:

Theorem 0.3.1. *If M is compact, then $H^p_{dR}(M)$ is finite dimensional for all $p \geq 0$.*

Remark. Let $n = \dim M$. Since $\Omega^p(M) = 0$ for all $p > n$, $H^p_{dR}(M) = 0$ for all $p > n$ as well. This means the chain complex for differential forms on M has finite length:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M).$$

It is possible to calculate de Rham cohomology in low degrees. One sees that significant topological information is conveyed by these cohomologies:

Proposition 0.3.2. *For any smooth manifold M, $H^0_{dR}(M) \cong \mathbb{R}^k$, where k is the number of connected components of M.*

Proof. For $p = 0$, $H^0_{dR}(M)$ is just equal to $\ker d_0$, the closed 0-forms on M. If $f \in \Omega^0(M) = C^\infty(M)$, then f is simply a smooth function, whose exterior derivative can be written in local coordinates. Thus we have

$$\begin{aligned} df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i = 0 &\iff \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n \\ &\iff f \text{ is locally constant.} \end{aligned}$$

It is known that the vector space of locally constant functions on M has dimension k, where k is the number of connected components of M. Therefore $H^0_{dR}(M) \cong \mathbb{R}^k$ as desired. $\square$

The takeaway here is that this algebraic object $H^p_{dR}(M)$ contains topological information about the manifold M – in the $p = 0$ case, the number of connected components. It turns out that de Rham cohomology is a *topological invariant*, that is, if M and N are homeomorphic then they have isomorphic de Rham cohomologies in every degree.

Example 0.3.3. Suppose $\omega \in \Omega^1(\mathbb{R}) = \mathrm{Hom}(\mathbb{R}, \mathbb{R})$. Then ω has the form $\omega = f\, dx$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$. It is a fact from real analysis that every continuous (and therefore every smooth) function is integrable, so let F be defined by

$$F(x) = \int_0^x f(t)\, dt \in \Omega^0(\mathbb{R}).$$

Then by the definition of exterior derivative of F and the fundamental theorem of calculus, we have

$$dF = \left(\frac{d}{dx} \int_0^x f(t)\, dt \right) dx = f(x)\, dx = \omega.$$

Therefore ω is exact, so it follows that $H^1_{dR}(\mathbb{R}) = 0$. Thus the complete de Rham cohomology of $\mathbb{R}$ is

$$H^p_{dR}(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Remark. For each $p \geq 0$, $H^p_{dR}(-)$ is a functor from the category $\mathtt{SmMfld}$ of smooth manifolds to the category $\mathtt{Vec}_{\mathbb{R}}$ of real vector spaces. It is a *contravariant* functor, meaning if $f : M \to N$ is a smooth map then there is a corresponding linear map $f^* : H^p_{dR}(N) \to H^p_{dR}(M)$. We describe this map next.

Definition. *Let $f : M \to N$ be a smooth map and take $\omega \in \Omega^p(N)$. The* **pullback** *of ω along f is a p-form $f^*\omega \in \Omega^p(M)$ defined by*

$$(f^*\omega)(X_1, \ldots, X_p) = \omega(Df(X_1), \ldots, Df(X_p)).$$

Proposition 0.3.4. *For two smooth maps $L \xrightarrow{g} M \xrightarrow{f} N$ and any two differential forms ω, η on N, we have*

(1) $(f \circ g)^* = g^* \circ f^*$.

(2) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

(3) In local coordinates, $f^*(\omega_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = (\omega_I \circ f) d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_p} \circ f)$.

(4) $f^*(d\omega) = d(f^*\omega)$, *that is, pullback is natural for the exterior derivative.*

Proof. (1) Take $\omega \in \Omega^p(N)$ and let $X_1, \ldots, X_p$ be vector fields on N. Then

$$\begin{aligned}
(f \circ g)^*(\omega)(X_1, \ldots, X_p) &= \omega(D(f \circ g)(X_1), \ldots, D(f \circ g)(X_p)) \quad \text{by def. of pullback} \\
&= \omega((Df \circ Dg)(X_1), \ldots, (Df \circ Dg)(X_p)) \quad \text{by chain rule} \\
&= (f^*\omega)(Dg(X_1), \ldots, Dg(X_p)) \\
&= (g^*(f^*\omega))(X_1, \ldots, X_p) \\
&= (g^* \circ f^*)(\omega)(X_1, \ldots, X_p).
\end{aligned}$$

Since they agree on all vector fields $X_1, \ldots, X_p$ for any arbitrary p-form ω, we conclude that $(f \circ g)^*$ and $g^* \circ f^*$ are equal.

(2) Take $\omega \in \Omega^p(N), \eta \in \Omega^q(N)$ and $X_1, \ldots, X_{p+q}$ vector fields on N. Then

$$\begin{aligned}
f^*(\omega \wedge \eta) &= (\omega \wedge \eta)(Df(X_1), \ldots, Df(X_{p+q})) \\
&= \sum_\sigma (-1)^{|\sigma|} \omega(Df(X_{\sigma(1)}), \ldots, Df(X_{\sigma(p)})) \eta(Df(X_{\sigma(p+1)}), \ldots, Df(X_{\sigma(p+q)})) \\
&= \sum_\sigma (-1)^{|\sigma|} (f^*\omega)(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) (f^*\eta)(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}) \\
&= (f^*\omega \wedge f^*\eta)(X_1, \ldots, X_{p+q}),
\end{aligned}$$

where all sums are over permutations σ satisfying $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$. Since ω, η and $X_1, \ldots, X_p$ were all arbitrary, $f^*(\omega \wedge \eta) = f^* \wedge f^*\eta$.

(3) and (4) exercise. □

Corollary 0.3.5. *For a smooth map $f : M \to N$, there is a well-defined linear map $f^* : H^p_{dR}(N) \to H^p_{dR}(M)$ for all $p \geq 0$.*

Proof. By (4) of Proposition 0.3.4, ω is closed $\implies$ $f^*\omega$ is closed, and ω is exact $\implies$ $f^*\omega$ is exact. This gives us a well-defined linear map on the cohomologies as described. □

Corollary 0.3.6. *If $f : M \to N$ is a diffeomorphism then $f^* : H^p_{dR}(N) \to H^p_{dR}(M)$ is an isomorphism for each $p \geq 0$.*

Proof. If f is a diffeomorphism then there is a smooth inverse $f^{-1} : N \to M$ such that $f \circ f^{-1} = id_N$ and $f^{-1} \circ f = id_M$. Taking pullbacks, we get $id_{H^p_{dR}(N)} = (f \circ f^{-1})^* = (f^{-1})^* \circ f^*$ by (1) of Proposition 0.3.4. Likewise, $id_{H^p_{dR}(M)} = (f^{-1} \circ f)^* = f^* \circ (f^{-1})^*$. This shows f^* is invertible, and hence an isomorphism. $\square$

Example 0.3.7. Take a 0-form $g \in \Omega^0(M)$ and a smooth map $f : M \to N$. Then $(f^*g)(x) = g(f(x))$ for all $x \in M$. Thus $f^*g = g \circ f$ so the pullback of a 0-form coincides with composition. Applying the exterior derivative gives a 1-form $dg \in \Omega^1(M)$. Then

$$\begin{aligned}(f^*dg)(X) &= (dg)(Df(X)) = Df(X)(g)\\ &= X(g \circ f) \quad \text{by the chain rule}\\ &= X(f^*g) \quad \text{by the above}\\ &= d(f^*g)(X).\end{aligned}$$

This verifies (4) of Proposition 0.3.4 that $f^*(dg) = d(f^*g)$ when g is a 1-form.

Example 0.3.8. Let $f : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ be defined by $f(r, \theta) = (r \cos\theta, r \sin\theta)$. That is, f is a parametrization of $\mathbb{R}^2$ in polar coordinates. We calculate the pullbacks of the 1-forms dx and dy, and the 2-form $dx \wedge dy$ below:

$$\begin{aligned}f^*dx &= d(x \circ f) = d(r\cos\theta) = \cos\theta\, dr - r\sin\theta\, d\theta\\ f^*dy &= d(y \circ f) = d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta\\ f^*(dx \wedge dy) &= f^*(dx) \wedge f^*(dy)\\ &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)\\ &= (r\cos^2\theta + r\sin^2\theta)\, dr \wedge d\theta = r\, dr \wedge d\theta.\end{aligned}$$

One can think of this 2-form $f^*(dx \wedge dy)$ as a generalization of the change-of-coordinates coefficient for $(x, y) \to (r, \theta)$.

Example 0.3.9. The *standard volume form* on Euclidean space $\mathbb{R}^{n+1}$ is the $(n+1)$-form $v = dx_1 \wedge \cdots \wedge dx_n \wedge dx_{n+1}$. Set $\omega = \displaystyle\sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{n+1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$. Then applying the exterior derivative yields

$$\begin{aligned}d\omega &= \sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{n+1} dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}\\ &= \sum_{i=1}^{n+1} \frac{1}{n+1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_i \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}\\ &= dx_1 \wedge \cdots \wedge dx_{n+1} = v.\end{aligned}$$

Therefore $d\omega = v$ so v is an exact form on $\mathbb{R}^{n+1}$.

0.4 Integration of Differential Forms

In this section we investigate the following question.

Question. *How does one decide if $H^p_{dR}(M)$ is nontrivial?*

In order to answer this, we develop a theory of integration for differential forms on a smooth manifold M. To start, let $M = \mathbb{R}^n$ and suppose an n-form $\omega \in \Omega^n(\mathbb{R}^n)$ has compact support in an open set $U \subset \mathbb{R}^n$. Then $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ where f is a function with compact support in U. We define the *integral* of ω on $\mathbb{R}^n$ as

$$\int_{\mathbb{R}^n} \omega = \int_U f\,dx_1 \cdots dx_n.$$

Now suppose $U, V \subset \mathbb{R}^n$ are open and $f : U \to V$ is a diffeomorphism between them. Suppose μ_U are local coordinates on U and μ_V are local coordinates on V. From multivariable calculus, we know that one can compute the integral of any smooth function $g : V \to \mathbb{R}$ by a change of variables:

$$\int_V g\,d\mu_V = \int_U (g \circ f)|\deg J(f)|\,d\mu_U$$

where $J(f)$ is the Jacobian matrix of f. On the other hand, if $x_1, \ldots, x_n$ are the local coordinates on U and $y_1, \ldots, y_n$ are the local coordinates on V, then

$$f^*(dy_1 \wedge \cdots \wedge dy_n) = \det J(f) \cdot dx_1 \wedge \cdots \wedge dx_n.$$

Assume that f is orientation-preserving, i.e. that $\det J(f) > 0$. Then the above allows us to write

$$\int_V \omega = \int_U f^*\omega.$$

To generalize this to an arbitrary n-manifold M, suppose $\omega \in \Omega^n(M)$. We then define

Definition. *For $\omega \in \Omega^n(M)$ supported compactly in an open set $U \subset M$ that is the domain of some coordinate chart $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, the* **integral** *of ω over M is*

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

Lemma 0.4.1. *This definition of the integral of ω is independent of the choice of coordinate chart (U, φ).*

Proof. Use the change-of-variables formula from above. □

Assume M is oriented, i.e. that we are given a (maximal) atlas such that all transition functions are orientation-preserving.

Definition. *A* **partition of unity** *on M is a collection of smooth functions $\{f_i : M \to \mathbb{R}\}_{i \in I}$ such that*

(a) Each f_i is identically 0 outside a chart U_i.

(b) *$f_i \geq 0$ for all i.*

(c) *The charts $\{U_i\}_{i\in I}$ cover M.*

(d) *The atlas $\{U_i\}_{i\in I}$ is locally finite: each point $x \in M$ lies in a neighborhood intersecting a finite number of the U_i.*

(e) *For all $x \in M$, $\sum_{i\in I} f_i(x) = 1$.*

Lemma 0.4.2. *Partitions of unity exist on every smooth manifold M.*

Notice that if $\{f_i\}$ is a partition of unity on M, then we can write any differential form $\omega \in \Omega^n(M)$ as

$$\omega = \sum_{i\in I} f_i\omega.$$

Since integration is linear, we can now define the integral for any differential n-form on M.

Definition. *Let $\{f_i\}_{i\in I}$ be a partition of unity on M. For an n-form $\omega \in \Omega^n(M)$, its* **integral** *is defined to be*

$$\int_M \omega = \sum_{i\in I} \int_M f_i\omega$$

if the sum converges.

Lemma 0.4.3. *The definition of $\int_M \omega$ is independent of the choice of partition of unity.*

Example 0.4.4. Let $M = S^1 \subset \mathbb{R}^2$ be the unit circle. Consider the 1-form

$$d\theta = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

A parametrization for S^1 is $\psi : [0, 2\pi) \to S^1$ defined by $\psi(t) = (\cos t, \sin t)$ (here we are regarding $\psi = \varphi^{-1}$ for some coordinate chart φ). Then the pullback of $d\theta$ along this parametrization is computed to be

$$\psi^*(d\theta) = -\sin t\, d(\cos t) + \cos t\, d(\sin t) = \sin^2 t\, dt + \cos^2 t\, dt = dt.$$

Thus the integral of $d\theta$ over the circle is

$$\int_{S^1} d\theta = \int_0^{2\pi} dt = 2\pi.$$

The central result that unites the theory of differential forms is known as Stokes' Theorem.

Theorem 0.4.5 (Stokes)**.** *Suppose M is an oriented n-manifold with boundary and $\omega \in \Omega^{n-1}(M)$ is a differential form with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Note that on the right, ω is really standing in for the restriction $\omega|_{\partial M}$. Also keep in mind that ∂M is oriented with orientation induced from that of M. When $\partial M = \varnothing$, we interpret the integral over ∂M to be 0.

Example 0.4.6. Let $\omega = f\,dx + g\,dy$ be a 1-form on $\mathbb{R}^2$, where f, g are smooth functions on $\mathbb{R}^2$. If ω is exact, with $\omega = dF$ for a smooth function F, then for any smooth curve $\gamma \subset \mathbb{R}^2$, Stokes' Theorem says

$$\int_\gamma (f\,dx + g\,dy) = F(x_1, y_1) - F(x_0, y_0)$$

where (x_0, y_0) and (x_1, y_1) are the starting and ending points, respectively, of γ. For instance, if we take γ to be the straight line segment from $(0, 0)$ to (x, y), then the function

$$F(x, y) = \int_\gamma (f\,dx + g\,dy) = \int_0^1 (xf(tx, ty) + yg(tx, ty))\,dt$$

is a 0-form. We use this information below.

Let $\omega = f\,dx + g\,dy$ be a 1-form on $\mathbb{R}^2$, where f, g are smooth functions on $\mathbb{R}^2$. If ω is closed, then $0 = d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$ which happens if and only if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. Define the function

$$F(x, y) = \int_0^1 (xf(tx, ty) + yg(tx, ty))\,dt \in \Omega^0(\mathbb{R}^2).$$

We claim $dF = \omega$. By the chain rule and Leibniz's rule, we have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \int_0^1 f(tx, ty)\,dt + \int_0^1 \left[tx\frac{\partial f}{\partial x}(tx, ty) + ty\frac{\partial g}{\partial x}(tx, ty)\right] dt \\
&= \int_0^1 f(tx, ty)\,dt + \int_0^1 \left[tx\frac{\partial f}{\partial x}(tx, ty) + ty\frac{\partial f}{\partial y}(tx, ty)\right] dt \\
&= \int_0^1 f(tx, ty)\,dt + t\int_0^1 \frac{d}{dt} f(tx, ty)\,dt = f(x, y); \\
\frac{\partial F}{\partial y} &= g(x, y) \text{ similarly.}
\end{aligned}$$

Then the exterior derivative of F is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = f\,dx + g\,dy.$$

So $dF = \omega$, showing ω is exact. Therefore $H^1_{dR}(\mathbb{R}^2) = 0$.

Now suppose $\eta \in \Omega^2(\mathbb{R}^2)$ is a 2-form, written $\eta = f\,dx \wedge dy$ for a smooth function f. Since $\Omega^3(\mathbb{R}^2) = 0$, every 2-form is closed. Using the polar substitution formula (Example 0.3.8), write $\omega = f\,dx \wedge dy = f(r, \theta) r\,dr \wedge d\theta$. Fixing θ, set

$$F(r, \theta) = \int_0^r tf(t, \theta)\,dt.$$

Then $F\,d\theta$ is a 1-form and its exterior derivative is

$$\begin{aligned} d(F\,d\theta) &= dF \wedge d\theta + (-1)^0 F \cdot d(d\theta) = dF \wedge d\theta \quad \text{since } d^2 = 0 \\ &= \left[\left(\frac{\partial}{\partial r}\int_0^r tf(t,\theta)\,dt\right) dr + \left(\frac{\partial}{\partial \theta}\int_0^r tf(t,\theta)\,dt\right) d\theta\right] \wedge d\theta \\ &= \left(\frac{\partial}{\partial r}\int_0^r tf(t,\theta)\,dt\right) dr \wedge d\theta \quad \text{by the alternating property} \\ &= rf(r,\theta)\,dr \wedge d\theta = \omega. \end{aligned}$$

Hence ω is exact, so $H^2_{dR}(\mathbb{R}^2) = 0$. As in Example 0.3.3, we can write down the complete cohomology of $\mathbb{R}^2$:

$$H^p_{dR}(\mathbb{R}^2) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Remark. Suppose $N \subset M$ is a p-submanifold without boundary. Then for an exact p-form $\omega \in \Omega^p(N)$ such that $\omega = d\eta$ for a $(p-1)$-form η, Stokes' Theorem (0.4.5) shows that

$$\int_N \omega = \int_N d\eta = \int_{\partial N} \eta = 0.$$

This induces a linear map $\Omega^p(M) \to \mathbb{R}$ and thus a well-defined map on de Rham cohomology:

$$\begin{aligned} H^p_{dR}(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_N \omega. \end{aligned}$$

Corollary 0.4.7. *Suppose $N \subset M$ is a p-submanifold without boundary such that there exists a closed p-form $\omega \in \Omega^p(M)$ such that $\int_N \omega \neq 0$. Then N is not the boundary of some $(p+1)$-submanifold of M.*

Proof. If $\int_N \omega \neq 0$ then by the contrapositive to the remark above, ω is not exact, so in particular $H^p_{dR}(N) \neq 0$. Moreover, Stokes' Theorem (0.4.5) shows that N cannot be the boundary of another submanifold W of M, since otherwise $d\omega = 0$ would integrate to 0 over W, contradicting the hypothesis $\int_{\partial W} \omega = \int_N \omega \neq 0$. □

Example 0.4.8. Let $T = S^1 \times S^1$ be a torus. Then by Corollary 0.4.7, $H^1_{dR}(T) \neq 0$ and no longitude or meridian of T bounds a submanifold of T.

Stokes' Theorem (0.4.5) is recognizable in various disguises in calculus, including the following famous theorems.

Corollary 0.4.9 (Fundamental Theorem of Calculus)**.** *For a continuous function $f : [a,b] \to \mathbb{R}$ which is differentiable on (a,b),*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Proof. The interval $[a, b]$ is a compact, oriented manifold with boundary $\{a, b\}$ and the expression $f'(x)\,dx$ is the exterior derivative of f as a 0-form. Therefore the fundamental theorem of calculus follows directly from Stokes' Theorem (0.4.5). □

Corollary 0.4.10 (Fundamental Theorem of Line Integrals)**.** *Suppose C is a smooth curve parametrized by $\gamma : [a, b] \to C$. Then for any function f which is smooth on C, we have*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\gamma(b)) - f(\gamma(a))$$

where ∇f is the gradient of f.

Proof. As we saw in Example 0.2.2, ∇f is a special case of the exterior derivative df for any 0-form f. □

Corollary 0.4.11 (Green's Theorem)**.** *Let $D \subset \mathbb{R}^2$ be a compact region bounded by C a piecewise smooth, positively oriented, simple closed curve, and suppose $P(x, y)$ and $Q(x, y)$ are continuously differentiable on D. Then*

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Proof. Here D is a 2-manifold with boundary C and $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ is the exterior derivative of the 1-form $\omega = P\,dx + Q\,dy$. □

Corollary 0.4.12 (Divergence Theorem)**.** *Suppose V is a compact subset of $\mathbb{R}^3$ with piecewise smooth boundary $S = \partial V$ oriented with outward-pointing normal vectors. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V then*

$$\iiint_V \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS,$$

where $\nabla \cdot \mathbf{F}$ is the divergence and $\mathbf{n}$ is the vector field of (outward-pointing) normal vectors to S.

Proof. This is Stokes' Theorem (0.4.5) applied to the 2-form $\mathbf{F} \cdot \mathbf{n}\,dS$. □

Corollary 0.4.13 (Classical Stokes' Theorem)**.** *Let S be an oriented smooth surface with boundary $C = \partial S$ which is a smooth, simple closed curve with positive orientation. If $\mathbf{F}$ is a vector field defined on S then*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

where $\nabla \times \mathbf{F}$ is the curl.

Proof. As we showed in Example 0.2.2, the exterior derivative of the 1-form $\mathbf{F} \cdot d\mathbf{r}$ is precisely curl: $d(\mathbf{F} \cdot d\mathbf{r}) = \nabla \times \mathbf{F} \cdot d\mathbf{S}$. □

Definition. *An n-dimensional manifold M is* **orientable** *if there exists a differential n-form $\omega \in \Omega^n(M)$ with the property that $\omega_x \neq 0$ for every $x \in M$.*

Corollary 0.4.14. *If M is a compact, orientable n-manifold without boundary, $H^n_{dR}(M) \neq 0$.*

Proof. If M then any n-form $\omega \in \Omega^n(M)$ has compact support over all of M so we can integrate over all M. Since M has no boundary, Stokes' Theorem (0.4.5) implies

$$\int_M d\omega = \int_{\partial M} \omega = 0.$$

Thus exact forms integrate to 0. This induces a well-defined linear map

$$\begin{aligned} \Phi : H^n_{dR}(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega. \end{aligned}$$

If M is orientable, then by definition there is an n-form $\omega \in \Omega^n(M)$ such that $\omega_x \neq 0$ for all $x \in M$. Then $\int_M \omega \neq 0$ so because Φ is linear, $[\omega] \neq 0$ in $H^n_{dR}(M)$. □

Recall that a *homotopy* of a continuous map $f : X \to Y$ between general topological spaces is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ for all $x \in X$. It is more common to refer to a homotopy between maps $f, g : X \to Y$, which satisfies the conditions that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. Alternatively, one can think of F as a one-parameter family of maps $f_t : X \to Y$. Furthermore, we say f and g are *smoothly homotopic* if there is a homotopy F between them such that F is smooth.

Proposition 0.4.15. *If $f, g : N^n \to M^m$ are smoothly homotopic maps between compact manifolds N and M without boundary, and $\omega \in \Omega^n(M)$ then*

$$\int_N f^*\omega = \int_N g^*\omega.$$

Proof. Exercise. □

Corollary 0.4.16. *The integral map $\int_N : H^p_{dR}(M) \to \mathbb{R}$ depends on the submanifold N only up to homotopy.*

Our plan in the next chapter is to generalize some of this theory to arbitrary topological spaces and homotopy classes of maps.

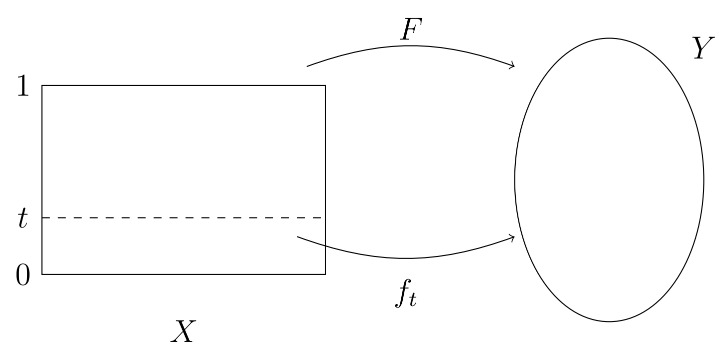
1 Homotopy Theory

1.1 Homotopy

Let X and Y be topological spaces with subspaces $A \subset X$ and $B \subset Y$. Write $\mathrm{Map}(X, Y)$ for the space of continuous maps $X \to Y$, or $\mathrm{Map}((X, A), (Y, B))$ for the space of continuous functions $f : X \to Y$ such that $f(A) \subseteq B$.

Example 1.1.1. If $A = \{x_0\}$ and $B = \{y_0\}$, the points x_0 and y_0 are called *base points*, the spaces $(X, \{x_0\})$ and $(Y, \{y_0\})$ are called *pointed spaces* and we write $\mathrm{Map}_*(X, Y) = \mathrm{Map}((X, \{x_0\}), (Y, \{y_0\}))$ for the space of *pointed* (or *based*) *maps* between $(X, \{x_0\})$ and $(Y, \{y_0\})$.

Definition. *A* **homotopy** *between two maps* $f, g \in \mathrm{Map}(X, Y)$ *is a continuous map* $F : X \times [0, 1] \to Y$ *such that* $F(x, 0) = f(x)$ *and* $F(x, 1) = g(x)$ *for all* $x \in X$*. If such a map exists, we say* f *and* g *are* **homotopic**, *denoted* $f \simeq g$*. We can alternatively think of a homotopy* F *as a one-parameter family of continuous maps* $f_t : X \to Y$



Definition. *If* $A \subset X$ *and* $B \subset Y$ *are subspaces and* $f, g : X \to Y$ *are homotopic with homotopy* $F = f_t$*, we say* F *is a* **homotopy relative to** A **and** B *if* $f_t(A) \subseteq B$ *for all* $t \in [0, 1]$*. In the case that* $X = Y, A = B$ *and* $f_t(A) = A$ *for all* $t \in [0, 1]$*,* f_t *is called a* **homotopy rel** A*.*

Lemma 1.1.2. *Homotopy is an equivalence relation on* $\mathrm{Map}(X, Y)$ *and* $\mathrm{Map}((X, A), (Y, B))$*.*

Proof. Letting $A = B = \varnothing$ shows that the first statement follows from the second, so it will suffice to prove the equivalence relation for $\mathrm{Map}((X, A), (Y, B))$ in general. Let $f, g, h : (X, A) \to (Y, B)$ be continuous maps such that $f(A) \subseteq B$ and likewise for g, h.

Note that the map $F : X \times [0, 1] \to Y, F(x, t) = f(x)$ for all $x \in X, t \in [0, 1]$, is continuous, preserves $A \to B$ and $F(x, 0) = f(x) = F(x, 1)$ for all $x \in X$. Therefore $f \simeq f$, so $\simeq$ is reflexive.

Next, suppose $f \simeq g$. If $F : X \times [0, 1] \to Y$ is a homotopy between them respecting $A \to B$, then define $G : X \times [0, 1] \to Y$ by $G(x, t) = F(x, 1 - t)$. Then $G(A, t) = F(A, 1 - t) \subseteq B$, $G(x, 0) = F(x, 1) = g(x)$ and $G(x, 1) = F(x, 0) = f(x)$. Moreover, G inherits continuity from F (more precisely, G is the composition of F with the homeomorphism $t \mapsto 1 - t$ on $[0, 1]$) so G is a homotopy. Therefore $g \simeq f$.

Finally, suppose $f \simeq g$ and $g \simeq h$. Then there are continuous maps $F, G : X \times [0,1] \to Y$ such that: $F(A \times [0,1]), G(A \times [0,1]) \subseteq B$, $F(x,0) = f(x), F(x,1) = g(x) = G(x,0)$ and $G(x,1) = h(x)$ for all $x \in X$. Define $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

The properties of F, G give us $H(A \times [0,1]) \subseteq B$ as required. Moreover, H is continuous away from the line $t = \frac{1}{2}$, and at this line we have

$$\lim_{t \to 1/2^-} H(x,t) = \lim_{t \to 1^-} F(x,t) = g(x)$$
$$\text{and} \quad \lim_{t \to 1/2^+} H(x,t) = \lim_{t \to 0^+} G(x,t) = g(x).$$

So H is continuous on $X \times [0,1]$. Also, from the way we defined the function, we have $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$ for all $x \in X$. So H is a homotopy, proving $f \simeq h$. We conclude that $\simeq$ is an equivalence relation. □

Definition. *The set of equivalence classes of maps $f : X \to Y$ under homotopy is denoted $[X, Y]$, called the set of* **homotopy classes** *from X to Y. If X and Y are pointed spaces, this is written $[X, Y]_*$.*

In some situations, we can turn $[X, Y]$ into a group. Given X, one can define a product on a certain subset of $[X \times [0,1], Y]$ as follows. If $f, g : X \times [0,1] \to Y$ are continuous and $f(x,1) = g(x,0)$ for all $x \in X$, then the product $f * g : X \times [0,1] \to Y$ is defined by

$$(f * g)(x,t) = \begin{cases} f(x,2t) & 0 \le t \le \frac{1}{2} \\ g(x,2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

However, it's not even clear if this operation is well-defined on homotopy classes yet. To remedy this, we specify base points $x_0 \in X$ and $y_0 \in Y$ and restrict our attention to maps $f : X \times [0,1] \to Y$ such that $f(x,0) = y_0$ is constant on X.

Definition. *The (reduced)* **suspension** *of a topological space X at a base point x_0 is the quotient space*

$$\Sigma X := \frac{X \times [0,1]}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [0,1])}.$$

Remark. Given based maps $f, g : \Sigma X \to Y$, or equivalently $f, g : X \times [0,1] \to Y$ such that $f(x,0) = f(x,1) = f(x_0,t) = y_0$ for all $x \in X, t \in [0,1]$, then the product of f and g is well-defined as a map $f * g : \Sigma X \to Y$. Rigorously, the composition $f * g$ gives a map $\Sigma X \to \Sigma X \vee \Sigma X \to Y$, where $\Sigma X \vee \Sigma X$ is the wedge of two copies of the suspension of X.

Lemma 1.1.3. *The composition $[f] \cdot [g] = [f * g]$ makes $[\Sigma X, Y]_*$ into a group.*

Proof. We need to prove four things:

(i) Concatenation is well-defined on homotopy classes, i.e. $f * g$ does not depend (up to homotopy) on the homotopy classes of f or g.

(ii) Concatenation is associative.

(iii) The pointed map $e : \Sigma X \to \{y_0\} \subset Y$ taking all points in the suspension of X to the base point in Y is an identity element.

(iv) For a pointed map $f : \Sigma X \to Y$, $[h] = [f]^{-1}$, where $h = f(x, 1-t)$.

Once we have checked all four, we will have proven that $[\Sigma X, Y]_*$ is a group.

(i) Suppose $f_0 \simeq f$ and $g_0 \simeq g$. Then there are (based) homotopies $F, G : \Sigma X \times [0,1] \to Y$ with $F(x,t,0) = f_0(x,t), F(x,t,1) = f(x,t), G(x,t,0) = g_0(x,t)$ and $G(x,t,1) = g(x,t)$ for all $(x,t) \in \Sigma X$. Consider the map $H = F * G : \Sigma X \times [0,1] \to Y$,

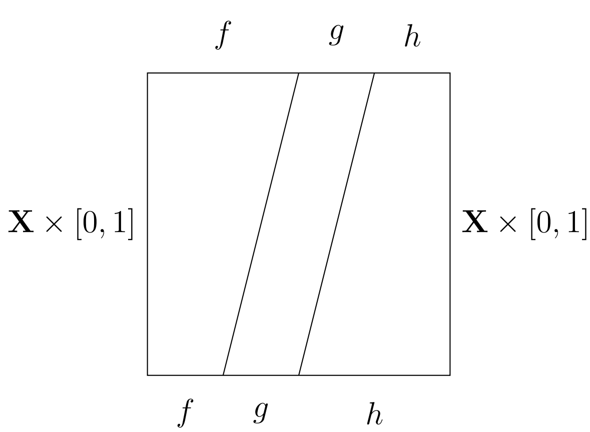
$$H(x,t,s) = \begin{cases} F(x,2t,2s) & 0 \le t \le \frac{1}{2}, 0 \le s \le \frac{1}{2} \\ G(x,2t-1,2s) & \frac{1}{2} < t \le 1, 0 \le s \le \frac{1}{2} \\ F(x,2t,2s-1) & 0 \le t \le \frac{1}{2}, \frac{1}{2} < s \le 1 \\ G(x,2t-1,2s-1) & \frac{1}{2} < t \le 1, \frac{1}{2} < s \le 1 \end{cases} \quad \text{for all } x \in \Sigma X.$$

Note that $H(x,t,0) = (f_0 * g_0)(x,t)$ and $H(x,t,1) = (f * g)(x,t)$ for all $(x,t) \in \Sigma X$. Moreover H is continuous since all the maps start and end at the base point of ΣX. Hence H is a homotopy from $f_0 * g_0$ to $f * g$, i.e. $[f_0 * g_0] = [f * g]$, so the group law is well-defined on homotopy classes.

(ii) Let $f, g, h : \Sigma X \to Y$ be based maps. To verify associativity, we must show $f * (g * h) \simeq (f * g) * h$. Define $F : \Sigma X \times [0,1] \to Y$ by

$$F(x,t,s) = \begin{cases} f(x, 2ts + 4t(1-s)) & 0 \le t \le \frac{s+1}{4} \\ g(x, 4t-2) & \frac{s+1}{4} < t \le \frac{s+2}{4} \\ h(x, (4t-3)s + (2t-1)(1-s)) & \frac{s+2}{4} < t \le 1. \end{cases}$$

The figure below is a good guide.



(Here $\mathbf{X}$ denotes the base point $X \times \{0,1\} \cup \{x_0\} \times [0,1]$.) Then F has the following special values:

$$F(x,t,0) = \begin{cases} f(x,4t) & 0 \le t \le \frac{1}{4} \\ g(x,4t-2) & \frac{1}{4} < t \le \frac{1}{2} \\ h(x,2t-1) & \frac{1}{2} < t \le 1 \end{cases} \quad = ((f * g) * h)(x,t)$$

$$F(x,t,1) = \begin{cases} f(x,2t) & 0 \le t \le \frac{1}{2} \\ g(x,4t-2) & \frac{1}{2} < t \le \frac{3}{4} \\ h(x,4t-3) & \frac{3}{4} < t \le 1 \end{cases} \quad = (f * (g * h))(x,t)$$

for all $(x,t) \in \Sigma X$. Clearly F preserves the base points since it does so piecewise; and F is continuous since all functions agree on the seams. Therefore F is a homotopy $(f * g) * h \simeq f * (g * h)$.

(iii) Let $f : \Sigma X \to Y$ be continuous. Define $F : \Sigma X \times [0,1] \to Y$ by

$$F(x,t,s) = \begin{cases} f(x,(2-s)t) & 0 \le t \le \frac{s+1}{2} \\ y_0 & \frac{s+1}{2} < t \le 1. \end{cases}$$

Then F preserves base points since it does piecewise, F is continuous since all maps meet at the base point on the boundary, and

$$F(x,t,0) = \begin{cases} f(x,0) & 0 \le t \le \frac{1}{2} \\ y_0 & \frac{1}{2} < t \le 1 \end{cases} \quad = f * e$$
$$F(x,t,1) = f(x,t).$$

Therefore F is a homotopy showing $f * e \simeq f$. The proof that $e * f \simeq f$ is similar. Therefore e is an identity element.

(iv) Define

$$F(x,t,s) = \begin{cases} f(x,t) & 0 \le t < \frac{1-s}{2} \\ f\left(\frac{1-s}{2}\right) & \frac{1-s}{2} \le t \le \frac{1+s}{2} \\ h(t) & \frac{1+s}{2} < t \le 1. \end{cases}$$

Then as before, F is continuous everywhere, F preserves base points piecewise, $F(x,t,0) = (f * h)(x,t)$ and $F(x,t,1) = f(0) = y_0$ for all x, t. Hence $f * h \simeq e$, and the proof that $h * f \simeq e$ is similar. We conclude that $[\Sigma X, Y]_*$ is indeed a group under the given laws. ☐

Proposition 1.1.4. *Suspension defines a functor* $\mathtt{Top}_* \to \mathtt{Top}_*$ *from the category of pointed spaces to itself. In particular, if* $f : X \to Y$ *is a pointed map then there is a suspended map* $\Sigma f : \Sigma X \to \Sigma Y$ *such that* $\Sigma(f \circ g) = \Sigma f \circ \Sigma g$. *Moreover, if* $f, g \in \mathrm{Map}_*(X,Y)$ *are homotopic, then so are* Σf *and* Σg.

Proof. Define $\Sigma : \mathtt{Top}_* \to \mathtt{Top}_*$ on objects by $X \mapsto \Sigma X$. For a morphism $f : X \to Y$ which is a based (continuous) map, define $\Sigma f : \Sigma X \to \Sigma Y$ for all $(x,t) \in \Sigma X$ by

$$\Sigma f(x,t) = (f(x),t).$$

Then Σf is continuous since it's continuous componentwise. Also, Σf is a based map since for all $t \in [0,1]$, $\Sigma f(x_0, t) = f(x_0) = y_0$, and for any $x \in X$, $\Sigma f(x, 0) = (f(x), 0) = (f(x), 1) = \Sigma f(x, 1)$ in ΣY; so in particular, $\Sigma f(X \times \{0,1\} \cup \{x_0\} \times [0,1]) = (Y \times \{0,1\} \cup \{y_0\} \times [0,1])$. This argument in addition shows Σf is well-defined.

To prove $\Sigma(\cdot)$ is a functor, we must show $\Sigma(f \circ g) = \Sigma f \circ \Sigma g$, where $X \xrightarrow{g} Y \xrightarrow{f} Z$ are based. But this is immediate from the definition: for all $x \in X, t \in [0,1]$, we have

$$\Sigma(f \circ g)(x,t) = ((f \circ g)(x), t) = (f(g(x)), t) = \Sigma f(g(x), t) = \Sigma f \circ \Sigma g(x,t).$$

Finally, if $f, g : X \to Y$ are based homotopic, let $F : X \times [0,1] \to Y$ be a based homotopy taking f to g. Define $H : \Sigma X \times [0,1] \to \Sigma Y$ by

$$H(x,t,s) = (F(x,s), t).$$

Then H is continuous and based by construction, $H(x,t,0) = (F(x,0),t) = (f(x),t) = \Sigma f(x,t)$ and $H(x,t,1) = (F(x,1),t) = (g(x),t) = \Sigma g(x,t)$. Hence $\Sigma f \simeq \Sigma g$. $\square$

Example 1.1.5. Let $X = S^n$ be the n-sphere with base point x_0, the north pole for example. We claim that $\Sigma S^n = S^{n+1}$. Indeed, if one removes the base point from S^n, this yields Euclidean space: $S^n \smallsetminus \{x_0\} \cong \mathbb{R}^{n+1}$. Therefore ΣS^n is the one-point compactification of $\mathbb{R}^{n+1}$, that is, $\Sigma S^n \cong S^{n+1}$.

Example 1.1.6. If $X = S^0 = \{x_0, x_1\}$ is the 0-sphere then from the previous example, $\Sigma S^0 = S^1$. The multiplication

$$[(S^1, x_0), (Y, y_0)]_* \times [(S^1, x_0), (Y, y_0)]_* \longrightarrow [(S^1, x_0), (Y, y_0)]_*$$

is given by concatenation:

$$(f * g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

This says that the set $\mathrm{Map}(S^1, Y)_*$ consists of all paths in Y starting and ending at y_0.

Definition. *Let (Y, y_0) be a pointed space. For $n \ge 1$, n***th homotopy group** *of (Y, y_0) is $\pi_n(Y, y_0) := [S^n, Y]_*$, where we view the n-sphere as $S^n = \Sigma S^{n-1}$ for $n \ge 1$. For $n = 1$, the group $\pi_1(Y, y_0)$ is called the* **fundamental group** *of Y with base point y_0.*

Remark. For $n = 0$, the 0th homotopy $\pi_0(Y, y_0)$ is just a set which is naturally identified with the collection of path components of Y.

We will prove:

Theorem. *For any pointed space (Y, y_0), $\pi_n(Y, y_0)$ is an abelian group for $n \ge 2$.*

The fundamental group is not abelian in general. However, it is much better understood than higher homotopy groups. For example, we have the following result for compact manifolds:

Proposition 1.1.7. *If Y is a compact manifold, then $\pi_1(Y, y_0)$ is finitely presented.*

In contrast with cohomology, $\pi_n(Y, y_0)$ may be hard to compute, and in fact can be nontrivial for infinitely many n. What's surprising is how little we really know about higher homotopy groups. For example, $\pi_k(S^n)$ is not known for every value of n, k.

Definition. *If $g : (X, A) \to (Y, B)$ is continuous, for any space/subspace pair (W, C), the* **induced map** *on homotopy is*

$$\begin{aligned} g_* : [(W, C), (X, A)] &\longrightarrow [(W, C), (Y, B)] \\ [f] &\longmapsto [g \circ f]. \end{aligned}$$

Lemma 1.1.8. *If $g : (X, x_0) \to (Y, y_0)$ is a based map, the induced map on homotopy $g_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a homomorphism for all n.*

Lemma 1.1.9. *For any maps $g, h : (X, A) \to (Y, B)$ which are homotopic and for any (W, C), the induced maps*

$$g_*, h_* : [(W, C), (X, A)] \longrightarrow [(W, C), (Y, B)]$$

are equal.

Proof. If $g \simeq h$ then $g \circ f \simeq h \circ f$. □

Corollary 1.1.10. *If $f, g : (X, x_0) \to (Y, y_0)$ are based maps which are based homotopic, the induced homomorphisms $f_*, g_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are equal for every $n \in \mathbb{N}$.*

Definition. *Two spaces X and Y are* **homotopy equivalent** *if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Similarly, two pointed spaces (X, x_0) and (Y, y_0) are* **based homotopy equivalent** *if the homotopies above are based homotopies.*

Lemma 1.1.11. *The induced map commutes with composition; that is, for any spaces $(X, A), (Y, B), (Z, C)$ and continuous maps $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$, we have*

$$(g \circ f)_* = g_* \circ f_*.$$

Corollary 1.1.12. *If (X, x_0) and (Y, y_0) are (based) homotopy equivalent, then $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$ for all $n \in \mathbb{N}$.*

Proof. Let f and g be as in the definition of homotopy equivalence. Then the homomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ and $g_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$ are inverses, since

$$g_* \circ f_* = (g \circ f)_* = (id_X)_* \quad \text{and} \quad f_* \circ g_* = (f \circ g)_* = (id_Y)_*$$

by Lemma 1.1.11. □

Definition. *A space X is* **contractible** *if it is homotopy equivalent to a point set $\{x_0\}$.*

Corollary 1.1.13. *For any contractible space X for which the homotopy equivalence $X \simeq \{x_0\}$ is based, every homotopy group is trivial: $\pi_n(X, x_0) = \{1\}$ for all $n \geq 0$.*

To make some computations of homotopy groups, we first need to take a detour and discuss a notion from differential topology called *smooth approximation*.

Theorem 1.1.14. *Let $A, B \subset M^n$ be closed subsets of a smooth manifold M and suppose $f : M \to \mathbb{R}^k$ is continuous on M and smooth on A. Then for all $\varepsilon > 0$ there is a continuous map $g : M \to \mathbb{R}^k$ such that*

(1) g is smooth on $M \smallsetminus B$ and $g|_A = f|_A$ and $g|_B = g|_A$.

(2) $|g(x) - f(x)| < \varepsilon$ for all $x \in M$.

(3) $f \simeq g$ and the homotopy $F : M \times [0,1] \to \mathbb{R}^k$ is ε-small, meaning $|F(x,t) - f(x)| < \varepsilon$ for all $x \in M$ and $t \in [0,1]$.

Proof. (1) Choose a metric d on M (a Riemannian metric or the metric induced from the standard metric on $\mathbb{R}^\ell$, where $M \subseteq \mathbb{R}^\ell$). For $x \in M$, let $\varepsilon(x) = \min\{\varepsilon, d(x,B)\}$. Then for $x \in M \smallsetminus B$, choose a neighborhood $V_x \subset M \smallsetminus B$ of x and a function $h_x : V_x \to \mathbb{R}^k$ such that

(a) If $x \in A$ then h_x is smooth and $h_x = f$ on $A \cap V_x$.

(b) If $x \notin A$ then $A \cap V_x = \varnothing$ and $h_x = f(x)$ is constant on V_x.

(c) Each V_x is small enough so that for all $x \in V_x$,

$$|f(y) - f(x)| < \frac{\varepsilon(x)}{2}, \qquad |h_x(y) - f(x)| < \frac{\varepsilon(x)}{2} \quad \text{and} \quad d(x,y) < \frac{\varepsilon(x)}{2}.$$

Then $\{V_x\}_{x\in M}$ is a cover of M, so take a locally finite refinement $\{U_\alpha\}_{\alpha\in I}$ such that the U_α cover M and for all $\alpha \in I$, there is an $x = x(\alpha)$ such that $U_\alpha \subset V_{x(\alpha)}$. Take a partition of unity $\{\lambda_\alpha\}$ such that λ_α is supported in U_α. Define

$$g(y) = \begin{cases} \sum_{\alpha\in I} \lambda_\alpha(y) h_{x(\alpha)}(y) & \text{if } y \notin B \\ f(y) & \text{if } y \in B. \end{cases}$$

Since the cover $\{U_\alpha\}$ is locally finite, it follows that g is smooth on $M \smallsetminus B$. If $y \in A \smallsetminus B$, the only α with $\lambda_\alpha(y) \neq 0$ are those with $x(\alpha) \in A \smallsetminus B$. For these y, $h_{x(\alpha)}(y) = f(y)$ so

$$g(y) = \sum_\alpha \lambda_\alpha(y) f(y) = f(y).$$

So $g|_A = f|_A$, and $g|_B = f|_B$ is by construction.

(2) Next, we have

$$\begin{aligned} |g(y) - f(y)| &= \left| \sum_\alpha \lambda_\alpha(y) h_{x(\alpha)}(y) - f(y) \right| \\ &\leq \left| \sum_\alpha \lambda_\alpha(y)(h_{x(\alpha)}(y) - f(x(\alpha))) \right| + \left| \sum_\alpha \lambda_\alpha(y) f(x(\alpha)) - f(y) \right| \\ &\leq \sum_\alpha \lambda_\alpha(y) \Big(|h_{x(\alpha)}(y) - f(x(\alpha))| + |f(x(\alpha)) - f(y)| \Big) \\ &< \sum_\alpha \lambda_\alpha(y) \left(\frac{\varepsilon(x(\alpha))}{2} + \frac{\varepsilon(x(\alpha))}{2} \right) < \varepsilon. \end{aligned}$$

(3) Next, g is continuous on M. To see this, consider

$$\begin{aligned}\varepsilon(x(\alpha)) \leq d(x(\alpha), B) \leq d(x(\alpha), y) + d(y, B) \leq \frac{\varepsilon(x(\alpha))}{2} + d(y, B) \\ \implies \varepsilon(x(\alpha)) \leq 2d(y, B).\end{aligned}$$

In particular, as $y \to B$, $|g(y) - f(y)| \to 0$. Finally, the straight line homotopy

$$F(x, t) = tg(x) + (1 - t)f(x)$$

is the desired ε-small homotopy between f and g. □

Corollary 1.1.15 (Smooth Approximation Theorem)**.** *Let M^m and N^n be smooth manifolds that are metric spaces, where N is compact, and let $A \subset M$ be a closed subset. Suppose $f : M \to N$ is continuous such that $f|_A$ is smooth. Then there exists a function $h : M \to N$ such that*

(1) h is smooth on M.

(2) $h|_A = f|_A$.

(3) $d(h(x), f(x)) < \varepsilon$ for all $x \in M$.

(4) f is ε-homotopic to h.

Proof. Let $N \subseteq \mathbb{R}^k$ for some k. Take a tubular neighborhood V of N in $\mathbb{R}^k$, i.e. a 'thickening' of N by ε in every normal direction to N such that V retracts onto N. Since $N \subset \mathbb{R}^k$, Theorem 1.1.14 allows us to approximate f by a smooth map $g : M \to \mathbb{R}^k$ such that $g(M) \subseteq V$. Composing g with the retraction onto N gives the desired map $h : M \to N$. □

Remark. For sufficiently small ε, any two such approximations g and h are smoothly homotopic and the homotopy between them is ε-small. Therefore any $f : M \to N$ is homotopic to a smooth map (hence the name smooth approximation) and if $f, g : M \to N$ are both smooth and homotopic, then f and g are smoothly homotopic.

Definition. *A space X is* **simply connected** *if X is path-connected and $\pi_1(X, x_0) = 0$ for any $x_0 \in X$.*

Proposition 1.1.16. *The circle S^1 is not simply connected.*

Proof. Consider the identity map $\gamma : S^1 \to S^1$ as a loop with base point 1 (viewing S^1 as a subset of $\mathbb{C}$). We will show $[\gamma]$ is not the trivial class in $\pi_1(S^1, 1)$. In Example 0.4.4, we proved that

$$\int_{S^1} \gamma^* \, d\theta = \int_{S^1} d\theta = 2\pi.$$

On the other hand, if $c : S^1 \to S^1$ is the constant map $c(x) = 1$ for all $x \in S^1$, then $c^* d\theta = 0$ since the differential of c is 0. Thus $\int_{S^1} c^* \, d\theta = 0 \neq 2\pi$ so by Proposition 0.4.15, γ and c are not smoothly homotopic. Therefore by the remark, they cannot be homotopic at all. Thus $[\gamma] \neq e$ in $\pi_1(S^1, 1)$. □

Theorem 1.1.17. *For $n > 1$, $\pi_k(S^n) = 0$ for all $0 \le k < n$ and $\pi_n(S^n) \neq 0$.*

Proof. For the first statement, suppose $f : (S^k, x_0) \to (S^n, x_0)$ is continuous. Take $A = \{x_0\}$. Then f is based homotopic to a smooth map $(S^k, x_0) \to (S^n, x_0)$. By Sard's Theorem, such a smooth map cannot be surjective so we really have a smooth map $g : (S^k, x_0) \to (S^n \smallsetminus \{p\}, x_0)$ for some $p \neq x_0$. We claim there exists a based homotopy from $id : S^n \smallsetminus \{p\} \to S^n \smallsetminus \{p\}$ to the constant map $c : S^n \smallsetminus \{p\} \to \{x_0\} \subset S^n \smallsetminus \{p\}$. This results from the fact that $S^n \smallsetminus \{p\} \cong \mathbb{R}^n$ and we can choose the homeomorphism such that x_0 corresponds to $0 \in \mathbb{R}^n$. Then $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n, F(x, t) = (1 - t)x$ is a homotopy from id to c. Finally, compose g with this homotopy to obtain a homotopy from g to the constant map $(S^n \smallsetminus \{p\}, \{x_0\}) \to \{x_0\}$. This proves $\pi_k(S^n) = 0$ for all $k < n$.

For the second statement, let $v \in \Omega^{n+1}(\mathbb{R}^{n+1})$ be the standard volume form from Example 0.3.9. In that example, we proved v is exact, so let $\omega \in \Omega^n(\mathbb{R}^{n+1})$ such that $d\omega = v$. By Stokes' Theorem (0.4.5),

$$\int_{S^n} \omega = \int_D v$$

where D is the open unit $(n + 1)$-ball in $\mathbb{R}^{n+1}$. Integrating the volume form over D^{n+1} is just integrating over an open set in $\mathbb{R}^{n+1}$, which is our original formulation of the integral of differential forms:

$$\int_D v = \int_D dx_1 \cdots dx_{n+1} = \mathrm{vol}(D) \neq 0.$$

On the other hand, if $c : S^n \to S^n$ is the (based) constant map $c(x) = x_0$ for all $x \in S^n$, where x_0 is the north pole, then $c^*\omega = 0$ since the differential of a constant map is 0. Thus

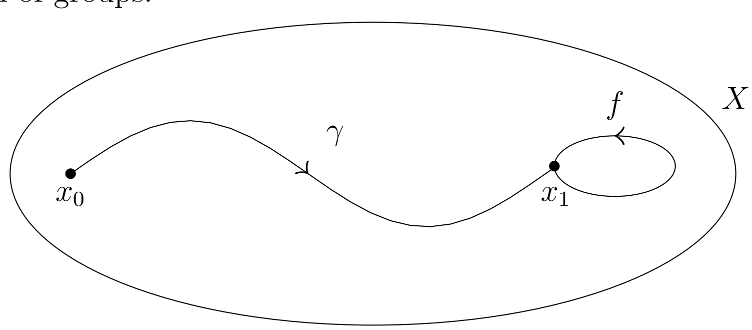
$$\int_{S^n} id^*\omega = \int_{S^n} \omega \neq 0 = \int_{S^n} 0 = \int_{S^n} c^*\omega.$$

By Proposition 0.4.15, this means id and c are not smoothly homotopic, but by the remarks following the smooth approximation theorem, this implies they cannot be homotopic at all. Therefore $[id]$ is a nontrivial class in $\pi_n(S^n)$. □

For a pointed space (X, x_0), the fundamental group $\pi_1(X, x_0)$ only contains information about the path component of the base point x_0. In fact, if $x_0, x_1 \in X$ lie in the same path component of X (or if X is path-connected), let $\gamma : [0, 1] \to X$ be a path from x_0 to x_1. Then the map

$$\begin{aligned} \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [\gamma * f * \gamma^{-1}] \end{aligned}$$

is an isomorphism of groups.



In the case that X is path-connected, we will write $\pi_1(X) = \pi_1(X, x_0)$ for any base point $x_0 \in X$. In many instances we will just assume X is path-connected to begin with. To summarize:

Theorem 1.1.18. *If there exists a path γ in X from x_0 to x_1 then there is an isomorphism of groups*

$$\begin{aligned} h_\gamma : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [\gamma * f * \gamma^{-1}] \end{aligned}$$

with inverse $h_{\gamma^{-1}}$. Moreover, if γ is a loop based at x_0 then h_γ is the inner automorphism $\alpha \mapsto \beta\alpha\beta^{-1}$ where $\beta = [\gamma] \in \pi_1(X, x_0)$.

Definition. *Two loops $f_0, f_1 : [0,1] \to X$ are* **freely homotopic** *if there is a homotopy $F : [0,1]^2 \to X$ from f_0 to f_1 such that $F(0,s) = F(1,s)$ for all $s \in [0,1]$.*

Note that a free homotopy need not based. Visually, the intermediate maps $F(t,s)$ "float" from f_0 to f_1 without any restrictions on their endpoints. Let p be the path in X defined by $p(s) = F(0,s)$ for $s \in [0,1]$. Then we write $f_0 \simeq_p f_1$ to denote that f_0 and f_1 are freely homotopic via the path p.

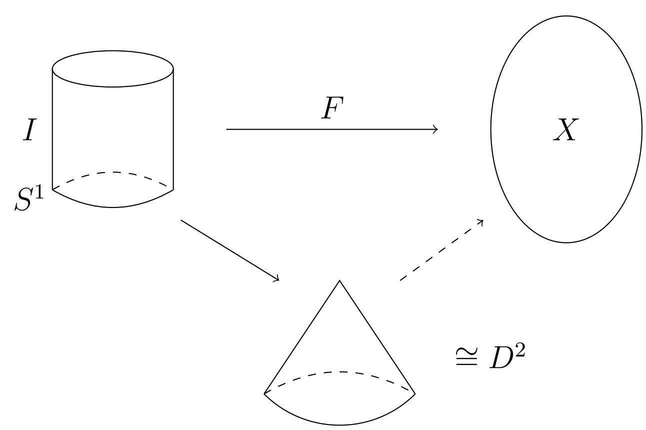
Lemma 1.1.19. *If $f_0, f_1 : [0,1] \to X$ are loops, $f_0 \simeq_p f_1$ if and only if $h_p[f_1] = [f_0]$.*

Theorem 1.1.20. *If X and Y are path-connected and $\varphi : X \to Y$ is a homotopy equivalence, then the induced map $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups.*

Proof. Set $y_0 = \varphi(x_0)$. Since φ is a homotopy equivalence, there exists a homotopy inverse $\psi : Y \to X$ such that $\psi \circ \varphi \simeq id_X$ by a homotopy $F : X \times I \to X$. During F, $\psi(\varphi(x_0))$ traces a path p from $\psi(\varphi(x_0))$ to x_0. Then for a loop f with base point x_0, $(\psi \circ \varphi) \circ f \simeq_p f$. Therefore by Lemma 1.1.19, $(\psi \circ \varphi)_*[f] = h_p[f]$. This holds for all homotopy classes $[f] \in \pi_1(X, x_0)$ so $(\psi \circ \varphi)_* = h_p$ which is an isomorphism. In particular, φ_* is an injection and ψ_* is a surjection. Reversing the roles of φ and ψ, we get that ψ_* is injective and hence an isomorphism Finally, $\varphi_* = \psi_*^{-1}(\psi \circ \varphi)_*$ which shows φ_* is also an isomorphism. $\square$

Proposition 1.1.21. *A (based) loop $f : S^1 \to (X, x_0)$ lies in the trivial class $e \in \pi_1(X, x_0)$ if and only if f extends continuously to a map $D^2 \to X$.*

Proof. ($\implies$) Suppose $[f] = e \in \pi_1(X, x_0)$. Then there exists a homotopy $F : S^1 \times [0,1] \to X$ from f to the constant map $c(t) = x_0$ for all $t \in S^1$. The homotopy factors through the quotient $S^1 \times [0,1]/S^1 \times \{1\} \cong D^2$:



Therefore f extends to a map $D^2 \to X$.

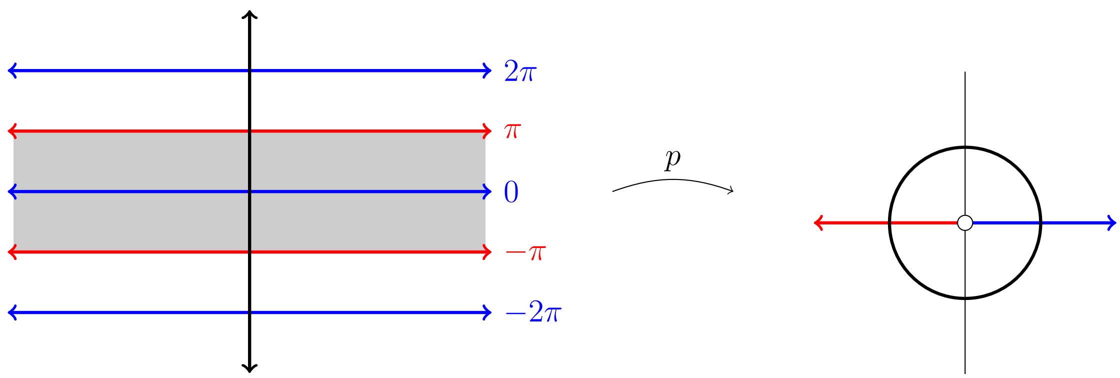
($\Longleftarrow$) In the other direction, if f extends to D^2 then composition with the quotient map $S^1 \times I \to D^2$ from above gives a homotopy $S^1 \times I \to X$. This shows f is freely homotopic to a constant map, $f \simeq_p c_{x_1}$, $x_1 \in X$. By Lemma 1.1.19, $h_p[c_{x_1}] = [f]$ but $[c_{x_1}] = e \in \pi_1(X, x_1)$ so we must have $[f] = e \in \pi_1(X, x_0)$. $\square$

The takeaway from Proposition 1.1.21 is that triviality in $\pi_1(X, x_0)$ is equivalent to bounding a disk in X.

1.2 Covering Spaces

To motivate the study of covering spaces, consider the following extended example.

Let $p : \mathbb{R}^2 \to \mathbb{R}^2 \setminus (0,0)$ be the map $p(z) = e^z$, i.e. p is like complex exponentiation: $p(x, y) = (e^x \cos y, e^x \sin y)$.



Any strip between a pair of red lines maps (locally) homeomorphically (in fact, diffeomorphically) onto all of $\mathbb{R}^2 \setminus (0,0)$. Set $X = \mathbb{R}^2 \setminus (0,0)$ and fix a base point, say $x_0 = (1,0)$. We will use p to study the fundamental group $\pi_1(X, x_0)$. Suppose $\gamma : [0,1] \to X$ is a loop in X with $\gamma(0) = \gamma(1) = x_0$. If γ is contained in an appropriate neighborhood of $(1,0)$ then p is a local homeomorphism so there exists an inverse map p^{-1} on the defined neighborhood. We know from complex analysis that for $p(z) = e^z$, this inverse map is a branch of the complex logarithm, $p^{-1}(z) = \log(z)$. Then $\tilde{\gamma} = p^{-1} \circ \gamma$ is a loop in the designated strip in $\mathbb{R}^2$. Such a $\tilde{\gamma}$ is called a *lift* of γ to $\mathbb{R}^2$.

Claim (Path Lifting)**.** *For any path $\gamma : [0,1] \to \mathbb{R}^2 \setminus (0,0)$, there is a lift $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ such that $\gamma = p \circ \tilde{\gamma}$. Moreover, $\tilde{\gamma}$ is unique once a starting point $\tilde{\gamma}(0)$ is specified.*

Idea: At each point z on the trace of γ, the preimage of a neighborhood of z under p is a disjoint union of neighborhoods in $\mathbb{R}^2$. The claim follows from a more general notion of 'path lifting' which will be proven later.

Claim (Homotopy Lifting)**.** *If $F : [0,1] \times [0,1] \to \mathbb{R}^2 \setminus (0,0)$ is a homotopy rel endpoints between f_0 and f_1, and $\tilde{f}_0$ is a lift of f_0, then there is a unique lift $\widetilde{F} : [0,1] \times [0,1] \to \mathbb{R}^2$ of F such that $\widetilde{F}(t, 0) = \tilde{f}_0(t)$ for all $t \in [0,1]$ and $\widetilde{F}$ is a homotopy rel endpoints.*

One method to find $\widetilde{F}$ is to subdivide $[0,1]^2$ so that each sub-rectangle is mapped into the domain of some p^{-1}. Another method is to define $\widetilde{F}$ using the path lifting property on each vertical line segment in $[0,1]^2$. This claim follows from the more general 'homotopy lifting' property which will also be proven later.

Corollary 1.2.1. *If $f : [0,1] \to \mathbb{R}^2 \smallsetminus (0,0)$ is a loop with $[f] = e$ in $\pi_1(\mathbb{R}^2 \smallsetminus (0,0), x_0)$ then any lift $\tilde{f} : [0,1] \to \mathbb{R}^2$ along p is a loop in $\mathbb{R}^2$.*

Proof. By hypothesis, there is a homotopy F from f to a constant map, so we can lift rel endpoints to a homotopy $\widetilde{F}$ from $\tilde{f}$ to the lift of the constant map, which is itself constant. Since a constant map ony has one endpoint and $\widetilde{F}$ respects endpoints, $\tilde{f}$ must be a loop. □

Let $X = \mathbb{R}^2 \smallsetminus (0,0)$ and $x_0 = (1,0)$ once again. If $[f] \in \pi_1(X, x_0)$ is a loop then in general $\tilde{f}$ is a path, say starting at $(0,0)$. The endpoint of $\tilde{f}$ must be $(0, 2\pi n)$ for some $n \in \mathbb{Z}$. Call this n the *degree* of f. This induces a map $\deg : \pi_1(\mathbb{R}^2 \smallsetminus (0,0), x_0) \to \mathbb{Z}$ defined by $\deg[f] = n$.

Claim. *The induced map $\deg : \pi_1(\mathbb{R}^2 \smallsetminus (0,0), x_0) \to \mathbb{Z}$ is a well-defined isomorphism of groups.*

Proof. By the homotopy lifting property, n is invariant under homotopy, so deg is well-defined. To show deg is a homomorphism, take two classes $[f], [g] \in \pi_1(\mathbb{R}^2 \smallsetminus (0,0))$ with representatives f and g. Lift f to $\tilde{f}$, then lift g to a path starting at the endpoint of $\tilde{f}$. Then $f * g$ lifts to a path with degree $\deg f + \deg g$. For surjectivity, map a segment $(0,0) \to (0, 2\pi n)$ through p; by construction this is a loop in $\mathbb{R}^2 \smallsetminus (0,0)$ at x_0 with degree n. For injectivity, if $\deg f = 0$ then f lifts to a loop $\tilde{f}$ in $\mathbb{R}^2$, but $\mathbb{R}^2$ is contractible so by Corollary cor:contractiblehtpy, $\tilde{f}$ is trivial. Therefore deg is an isomorphism. □

Corollary 1.2.2. *$\pi_1(S^1) \cong \mathbb{Z}$.*

Proof. Since S^1 is path-connected, $\pi_1(S^1)$ does not depend on a base point (up to isomorphism). By Theorem 1.1.20, the homotopy equivalence $\mathbb{R}^2 \smallsetminus (0,0) \to S^1$ induces an isomorphism of groups $\pi_1(\mathbb{R}^2 \smallsetminus (0,0)) \xrightarrow{\cong} \pi_1(S^1)$. Then by the previous calculation, $\pi_1(S^1) \cong \mathbb{Z}$. □

Definition. *A map $p : X \to Y$ between connected, Hausdorff spaces is a* **covering map** *if each point $y \in Y$ has a neighborhood U such that $p^{-1}(U) \subseteq X$ is a nonempty disjoint union $p^{-1}(U) = \coprod U_\alpha$ such that the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for each U_α. Such a neighborhood U is called an* **evenly covered** *neighborhood of y, and the U_α are called the* **sheets** *of the cover over y. The domain space X is called a* **covering space** *of Y.*

Examples.

1. The exponential map $p : \mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus (0,0)$, $p(z) = e^z$, is a covering map.

2. For any space Y, the identity map $Y \to Y$ is called the *trivial covering map.*

We want to generalize the path lifting and homotopy lifting properties described above.

Lemma 1.2.3 (Lebesgue)**.** *If X is a compact metric space and $\{U_\alpha\}$ is an open cover of X, then there exists a number $\delta > 0$, called a* **Lebesgue number** *for the cover, such that for all $x \in X$, the ball $B(x, \delta)$ is contained in some U_α.*

Lemma 1.2.4. *Let W be a topological space and let $\{U_\alpha\}$ be an open cover of $W \times [0,1]$. Then for all $w \in W$, there is a neighborhood N of w and an integer $n \in \mathbb{N}$ such that $N \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$ lies in some U_α for each $0 \le i < n$.*

Proof. We may cover $\{w\} \times [0,1]$ by finitely many open sets $N_1 \times V_1, \ldots, N_k \times V_k$ since $\{w\} \times [0,1] \cong [0,1]$ is compact. By Lebesgue's lemma, there is an $n \in \mathbb{N}$ such that each $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ lies in some V_j. The set $N = \bigcap_{j=1}^k N_j$ is open and satisfies the required property. $\square$

Theorem 1.2.5 (Path Lifting)**.** *Suppose $p : X \to Y$ is a covering map and $f : [0,1] \to Y$ is a path in Y with $y_0 = f(0)$. Then for each $x_0 \in p^{-1}(y_0)$, there is a unique path $g : [0,1] \to X$ such that $g(0) = x_0$ and g lifts f, i.e. $p \circ g = f$.*

$$\begin{tikzcd}
\{0\} \arrow[r, "x_0"] \arrow[d, hook] & X \arrow[d, "p"] \\
{[0,1]} \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

Proof. Each $f(t)$ lies in an evenly covered neighborhood $V_t \subset Y$ so $\{f^{-1}(V_t)\}$ is an open cover of $[0,1]$. By Lebesgue's lemma, there exists an $n \in \mathbb{N}$ such that $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ is contained in some member of the cover for each $0 \le i \le n-1$. Namely, $f\left(\left[\frac{i}{n}, \frac{i+1}{n}\right]\right)$ lies in an evenly covered neighborhood V_i. Define $g : [0,1] \to X$ inductively by:

- For $t \in \left[0, \frac{1}{n}\right]$, let U_0 be the component of $p^{-1}(V_0)$ containing x_0. Write $p_0 = p|_{U_0}$ and define g on $\left[0, \frac{1}{n}\right]$ by $g = p_0^{-1} \circ f$.
- If g has been defined on $\left[0, \frac{i}{n}\right]$, let U_i be the component of $p^{-1}(V_i)$ containing $g\left(\frac{i}{n}\right)$. Set $p_i = p|_{U_i}$ and define g on $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ by $g = p_i^{-1} \circ f$.

Then g is unique because U_0 is the unique component of $p^{-1}(V_0)$ containing x_0. By construction, g makes the diagram commute, so we are done. $\square$

Theorem 1.2.6 (Homotopy Lifting)**.** *Suppose W is a locally connected space and $p : X \to Y$ is a covering space. Let $F : W \times [0,1] \to Y$ be a homotopy and $f : W \times \{0\} \to X$ be a lift of $F|_{W \times \{0\}}$, that is, $p \circ f = F$. Then there is a unique homotopy $G; W \times [0,1] \to X$ lifting F such that $p \circ G = F$. Moreover, if F is a homotopy rel W' for some subset $W' \subset W$ then so is G.*

$$\begin{tikzcd}
W \times \{0\} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "p"] \\
W \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "G"] & Y
\end{tikzcd}$$

Proof. Define G using path lifting on each $\{w\} \times [0,1]$. Then the diagram commutes and G is unique as long as it is continuous. (Note that G has to be constant on $\{w'\} \times [0,1]$ for all $w' \in W'$ if $W' \subset W$ is a fixed subset, since each constant map $\{w'\} \times [0,1] \xrightarrow{F} Y$ lifts uniquely to a constant map.) To show G is continuous, take $w \in W$. By Lemma 1.2.4, there is a neighborhood N of w and an integer $n \in \mathbb{N}$ such that $F\left(N \times \left[\frac{i}{n}, \frac{i+1}{n}\right]\right)$ lies in an evenly covered neighborhood in Y for all $0 \leq i \leq n-1$. Since W is locally connected, we may assume N is connected. Now taking G to already be continuous on $N \times \left[0, \frac{i}{n}\right]$, we must have that $G\left(N \times \left\{\frac{i}{n}\right\}\right)$ is connected, so in particular $G\left(N \times \left\{\frac{i}{n}\right\}\right)$ lies in a unique component $U_i \subset p^{-1}(V_i$, where V_i is an evenly covered neighborhood of $F\left(N \times \left[\frac{i}{n}, \frac{i+1}{n}\right]\right)$. By uniqueness of path lifting (1.2.5) we have $G = p_i^{-1} \circ F$ on $N \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$, where $p_i = p|_{U_i}$. Then $p_i^{-1} \circ F$ is continuous on $N \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$ so G is continuous. □

Corollary 1.2.7. *Suppose $p : X \to Y$ is a covering space and $f_0, f_1 : [0,1] \to Y$ are paths in Y that are homotopic rel endpoints. Take $\tilde{f}_0$ and $\tilde{f}_1$ to be lifts of f_0 and f_1, respectively, such that $\tilde{f}_0(0) = \tilde{f}_1(0)$. Then $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic rel endpoints, and in particular, $\tilde{f}_0(1) = \tilde{f}_1(1)$.*

Proof. Apply Theorem 1.2.6 to $W = [0,1]$ and $W' = \{0,1\}$. □

Corollary 1.2.8. *Suppose $p : X \to Y$ is a covering space and $f : [0,1] \to Y$ is a loop which is homotopic rel endpoints to a constant map. Then any lift $\tilde{f} : [0,1] \to Y$ of f is a loop and is homotopic rel endpoints to a constant map in X.*

Corollary 1.2.9. *If $p : (X, x_0) \to (Y, y_0)$ is a (based) covering map then the induced homomorphism*

$$p_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is injective with image consisting of $[f]$ such that f lifts to a loop in X.

Proof. If f is a loop in Y based at y_0 such that $[f] = e \in \pi_1(Y, y_0)$, then by Corollary 1.2.8, the only lift of $[f]$ in X is the constant class $[c] = e \in \pi_1(X, x_0)$. □

The key result of this sequence of theorems and corollaries is that every covering space $p : (X, x_0) \to (Y, y_0)$ is associated to a subgroup of $\pi_1(Y, y_0)$.

Example 1.2.10. In Corollary 1.2.2, we showed that the fundamental group of the circle is $\pi_1(S^1, 1) \cong \mathbb{Z}$. The subgroup/covering space correspondence in this case is easy to describe:

subgroups of $\mathbb{Z}$	covering spaces of S^1
$\mathbb{Z}$	$id : S^1 \to S^1, z \mapsto z$
$\{0\}$	$p_\infty : \mathbb{R} \to S^1, z \mapsto e^z$
$n\mathbb{Z}, n \geq 2$	$p_n : S^1 \to S^1, z \mapsto z^n$

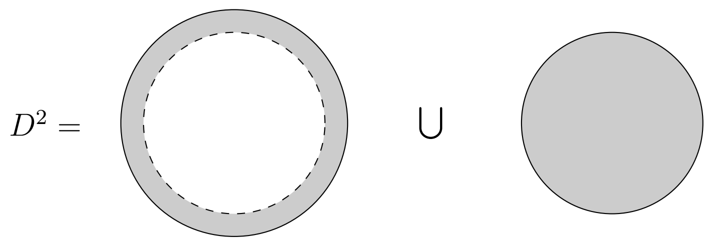
Corollary 1.2.11. *If Y is Hausdorff, path-connected and locally path-connected and $p : X \to Y$ is a nontrivial covering map, then $\pi_1(Y, y_0) \neq 1$.*

Proof. Such a cover X must be path-connected since it inherits connectedness and local path-connectedness from Y. So given a fixed $y_0 \in Y$, choose distinct $x_0, x_1 \in p^{-1}(y_0)$ and a path $\tilde{f}$ between them. Then $p_*(\tilde{f}) = p \circ \tilde{f}$ is a loop in Y since $p(x_0) = p(x_1)$, but $p \circ \tilde{f}$ lifts to a path that is not a loop. Therefore $[p_*(\tilde{f})] \neq e$ in $\pi_1(Y, y_0)$. □

Example 1.2.12. (Real) projective space $\mathbb{R}P^n$ has several equivalent definitions. We will focus on two: $\mathbb{R}P^n$ is equal to the set of one-dimensional linear subspaces of $\mathbb{R}^{n+1}$ (i.e. lines through the origin in $(n+1)$-space); or $\mathbb{R}P^n$ is the quotient space $S^n/\sim$ where $\sim$ is the antipodal action $x \sim -x$ on the n-sphere. In view of the latter description, there is a continuous quotient map $p : S^n \to \mathbb{R}P^n$ which is everywhere 2-to-1. In fact, p is a covering map. We can alternatively identify $\mathbb{R}P^n$ with the closed n-disk D^n with antipodal points on the boundary glued together.

For $n = 1$, $\mathbb{R}P^1 \cong S^1$ but the covering map $p : S^1 \to S^1$ is a double cover – this corresponds to the subgroup $2\mathbb{Z}$ in Example 1.2.10.

For $n = 2$, to understand $\mathbb{R}P^2$, cut the disk D^2 into an annulus and a smaller disk:



Identifying the edges of the annulus with the antipodal action produces a Möbius band, so we can also view $\mathbb{R}P^2$ as the space obtained by gluing a disk to a Möbius band along the boundary.

In the same way, $\mathbb{R}P^3 = \mathbb{R}P^2 \cup D^3$. We claim that $\mathbb{R}P^3 \cong SO_3$, the space of 3×3 orthogonal, orientation-preserving matrices, which is equivalently the space of rotations of $\mathbb{R}^3$. Geometrically, each rotation ρ is uniquely defined by an axis (a line through the origin) and an angle $0 \leq \pi < 2\pi$. Under this identification, a point $v \in D^3$ (where D^3 now has radius π) corresponds to the matrix A_v which represents rotation about the axis defined by v by $|v| = \pi$ radians, counterclockwise. Since π and $-\pi$ define the same rotation, this gives us a bijection $D^3/\sim \leftrightarrow SO_3$.

Given a diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

we want to define conditions for when there exists a lift $g : W \to X$ of f. This situation is completely described in the next theorem.

Theorem 1.2.13 (Lifting)**.** *Given a based covering map $p : (X, x_0) \to (Y, y_0)$ and a map $f : (W, w_0) \to (Y, y_0)$, where W is locally path-connected, a lift $g : (W, w_0) \to (X, x_0)$ of f exists if and only if $f_*(\pi_1(W, w_0)) \subseteq p_*(\pi_1(X, x_0))$. Moreover, if g exists, it is unique for the given choices of base points.*

Proof. ($\implies$) Given a lift $g : (W, w_0) \to (X, x_0)$, the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, so $f_*(\pi_1(W, w_0)) = (p \circ g)_*(\pi_1(W, w_0)) = p_*(g_*(\pi_1(W, w_0)) \subseteq p_*(\pi_1(X, x_0))$.

($\Longleftarrow$) Define the lift g as follows. Given $w \in W$, choose a path $\tilde{\gamma}$ from w_0 to w. Then $f \circ \gamma$ is a path in Y from y_0 to $y = f(w)$. Lift $f \circ \gamma$ to a path $\tilde{\gamma}$ in X starting at x_0, which exists and is unique by Theorem 1.2.5. Then define $g(w) = \tilde{\gamma}(1)$, the endpoint of the lifted path. Once we prove g is well-defined and continuous on W, uniqueness will follow from uniqueness of path-lifting.

To show g is well-defined, supose λ is another path in W from w_0 to w. Then we must show that for the unique lift $\tilde{\lambda}$ of $f \circ \lambda$ at x_0, we have $\tilde{\lambda}(1) = \tilde{\gamma}(1)$. The product path $\gamma * \lambda^{-1}$ is a loop in W, so it defines a class $[\gamma * \lambda^{-1}] \in \pi_1(W, w_0)$. Then $(f \circ \gamma) * (f \circ \lambda^{-1})$ is a loop in Y at y_0 such that

$$f_*[\gamma * \lambda^{-1}] = [(f \circ \gamma) * (f \circ \lambda^{-1})] \in f_*(\pi_1(W, w_0)).$$

By assumption, $(f \circ \gamma) * (f \circ \lambda^{-1})$ lifts to a loop in X at x_0, but by uniqueness of lifting, this is $\tilde{\gamma} * \tilde{\lambda}^{-1}$. Hence $\tilde{\gamma}$ and $\tilde{\lambda}$ start at x_0 and end at the same point, so g is well-defined.

Finally, given $w \in W$, take an evenly covered neighborhood $U \subset Y$ of $f(w)$. Then there exists a path-connected neighborhood $V \subset W$ of w such that $f(V) \subseteq U$, which is possible by continuity of f and local path-connectedness of W. Fix a path λ from w_0 to w. Then for $w' \in V$, choose a path λ' from w to w'. Lift $f \circ \lambda'$ starting at $g(w)$ to a path $\tilde{\lambda}'$ at $x_0 \in X$. Then $\tilde{\lambda} * \tilde{\lambda}'$ is the (unique) path from x_0 to $g(w')$. But now $\tilde{\lambda}' = p_0^{-1}(f \circ \lambda')$, where p_0 is the restriction of p to the component of $p^{-1}(U)$ containing $g(w)$. On V we then have $g = p_0^{-1} \circ f$ which is continuous because it is the composition of continuous functions. Since w was arbitrary, g is continuous everywhere. $\square$

Example 1.2.14. Recall the covering space $p : \mathbb{R} \to S^1$. Then the identity $id : S^1 \to S^1$ *does not* lift to $S^1 \to \mathbb{R}$ because $id_*(\pi_1(S^1)) = id_*(\mathbb{Z}) \cong \mathbb{Z}$ is not contained in $p_*(\pi_1(\mathbb{R})) = p_*(1) = 1$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[ur, dashed, "\nexists g"] \arrow[r, "id"'] & S^1 \end{tikzcd}$$

Example 1.2.15. Any map from the projective plane $\mathbb{R}P^2$ to the circle is nullhomotopic. Indeed, let $f : \mathbb{R}P^2 \to S^1$ be continuous and consider the universal cover $p : \mathbb{R} \to S^1$. We will prove (Corollary 1.3.10) that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and we already have seen (Corollary 1.2.2) that $\pi_1(S^1) \cong \mathbb{Z}$ so the induced map on homotopy becomes $f^* : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Since the only finite subgroup of $\mathbb{Z}$ is $\{0\}$, f^* must be the zero map. Therefore $f_*(\pi_1(\mathbb{R}P^2)) = 0$ so by the lifting theorem, f lifts uniquely to a map $g : \mathbb{R}P^2 \to \mathbb{R}$ such that $f = p \circ g$. However, $\mathbb{R}$ is contractible so g is homotopic to a constant map c_{x_0} by a homotopy $G : \mathbb{R}P^2 \times [0,1] \to \mathbb{R}$. Then $F = p \circ G : \mathbb{R}P^2 \times [0,1] \to S^1$ is continuous, $F(x,0) = p \circ G(x,0) = p \circ g(x) = f(x)$ and $F(x,1) = p \circ G(x,1) = p \circ c_{x_0} = c_{p(x_0)}$, so F is a homotopy from f to a constant map in S^1.

Example 1.2.16. Suppose $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ is continuous and nontrivial on the fundamental group. By Corollary 1.3.10, $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ so f_* must be an isomorphism. Let $p : S^2 \to \mathbb{R}P^2$ be the canonical cover and consider the map $g = f \circ p : S^2 \to \mathbb{R}P^2$. Then the induced map $g_* : \pi_1(S^2) \to \pi_1(\mathbb{R}P^2)$ is trivial since S^2 is simply connected, so by the lifting theorem, there is a unique lift $\tilde{g} : S^2 \to S^2$ such that $g = p \circ \tilde{g}$. Set $T = \tilde{g}$.

$$\begin{tikzcd}
S^2 \arrow[r, dashed, "T"] \arrow[d, "p"'] \arrow[dr, "g"] & S^2 \arrow[d, "p"] \\
\mathbb{R}P^2 \arrow[r, "f"'] & \mathbb{R}P^2
\end{tikzcd}$$

Fix $x \in S^2$. Since S^2 is path-connected, choose a path γ in S^2 from x to $-x$. Then $p \circ \gamma$ is a loop at $y = p(x)$, so $[p \circ \gamma]$ is a homotopy class in $\pi_1(\mathbb{R}P^2, y)$. Moreover, $[p \circ \gamma]$ is nontrivial in $\pi_1(\mathbb{R}P^2, y)$ since any loop in the trivial class lifts uniquely to a nullhomotopic loop in S^2, whereas $p \circ \gamma$ lifts to the non-closed path γ. Since f_* is an isomorphism, $f_*[p \circ \gamma]$ is nontrivial in $\pi_1(\mathbb{R}P^2, f(y))$. By definition of g and T, $f_*[p \circ \gamma] = [f \circ p \circ \gamma] = [g \circ \gamma] = [p \circ T \circ \gamma] = p_*[T \circ \gamma]$. This shows that $p_*[T \circ \gamma]$ is nontrivial in $\pi_1(\mathbb{R}P^2, f(y))$. Note that we cannot have $T(-x) = T(x)$, else $T \circ \gamma$ is a loop in S^2 based at $T(x)$, which is nullhomotopic, and thus $p_*[T \circ \gamma]$ is trivial. So $T(-x) \neq T(x)$ but $p \circ T(-x) = f \circ p(-x) = f \circ p(x) = p \circ T(x)$ so we must have $T(-x) = -T(x)$ since p is a degree 2 covering of $\mathbb{R}P^2$. This shows that any $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ such that $f_* \neq 0$ induces a map $T : S^2 \to S^2$ that commutes with the antipodal action: $T(-x) = -T(x)$ for all $x \in S^2$.

Corollary 1.2.17. *If W is simply connected, path-connected and locally path-connected, then for any covering $p : X \to Y$, every map $f : (W, w_0) \to (Y, y_0)$ lifts to a map $g : (W, w_0) \to (X, x_0)$ which is unique once $x_1 \in p^{-1}(y_0)$ is specified.*

Corollary 1.2.18. *All higher homotopy groups of the circle are trivial: $\pi_n(S^1) = 0$ for $n \geq 2$.*

Proof. By definition, $\pi_n(S^1) = [S^n, S^1]$ so if $f : S^n \to S^1$ is given, by Theorem 1.2.13 there is a unique lift $\tilde{f} : S^n \to \mathbb{R}$ making the diagram commute:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
S^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1
\end{tikzcd}$$

However, every path in $\mathbb{R}$ is homotopic to a constant map, so in particular $\tilde{f} \simeq c$ by some homotopy $\widetilde{F}$. Then $F = p \circ \widetilde{F}$ is a homotopy from f to a constant map in S^1, demonstrating $[f] = e \in \pi_n(S^1)$. Therefore $\pi_n(S^1) = 0$. □

Corollary 1.2.19. *If Y has a contractible covering space then $\pi_n(Y, y_0) = 0$ for all $n \geq 2$.*

Proof. Same as the proof for Corollary 1.2.18. □

Example 1.2.20. Fix $m \geq 3$. Then $\pi_n(\mathbb{R}P^m) = 0$ for all $1 < n < m$.

Proof. Let $p : S^m \to \mathbb{R}P^m$ be the canonical covering space and suppose $f : S^n \to \mathbb{R}P^m$ is a continuous map. Then $\pi_1(S^n) = 0$ so $f_*(\pi_1(S^1)) \subseteq p_*(\pi_1(S^m))$ and therefore by the lifting theorem (1.2.13), there exists a unique lift $g : S^n \to S^m$ such that $f = p \circ g$. However, by Theorem 1.1.17, $\pi_n(S^m) = 0$ since $n < m$, so the map $g : S^n \to S^m$ is homotopic to a constant map: $g \simeq c$ by some homotopy $\widetilde{F}$. Then by uniqueness of homotopy lifting, $F = p \circ \widetilde{F}$ must be a homotopy from f to a constant map in $\mathbb{R}P^m$. Therefore $[f]$ is trivial in $\pi_n(\mathbb{R}P^m)$, but because f was arbitrary, we conclude that $\pi_n(\mathbb{R}P^m) = 0$. □

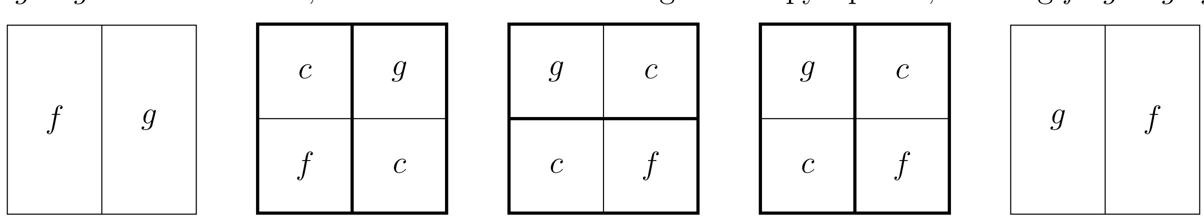
Theorem 1.2.21. *If $p : (X, x_0) \to (Y, y_0)$ is a based covering map, then the induced map $p_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for all $n \geq 2$.*

Proof. First, let $f : S^n \to (Y, y_0)$ be continuous. Then since $\pi_1(S_n) = 0$, the lifting theorem (1.2.13) provides a unique lift $g : S^n \to (X, x_0)$ such that $f = p \circ g$. By definition of the induced map, we have $p_*[g] = [p \circ g] = [f]$ so in particular p_* is surjective.

Now if $[f] = e \in \pi_n(Y, y_0)$, with $F : S_n \times [0, 1] \to (Y, y_0)$ a based homotopy from f to the constant map c_{y_0} in Y, then by homotopy lifting (1.2.6), there is a unique homotopy $G : S^n \times [0, 1] \to (X, x_0)$ such that $F = p \circ G$. Notice that for any $s \in S^n$, $p \circ G(s, 0) = F(s, 0) = f(s)$ and $p \circ G(s, 1) = F(s, 1) = y_0$, so by uniqueness of the lift g, we must have $G(s, 0) = g(s)$ and $G(s, 1) = x_0$ for all s. Therefore G is a homotopy from g to the constant map c_{x_0} in X, so $[g] = e \in \pi_n(X, x_0)$. This proves p_* is injective, and hence an isomorphism. □

Theorem 1.2.22. *For any space Y, $\pi_n(Y, y_0)$ is abelian for $n \geq 2$.*

Proof. Take $f, g : S^n \to (Y, y_0)$. We must show $[f * g] = [g * f]$. Let $c = c_{y_0}$ be the constant map at the base point y_0. Then $[c]$ is the identity class in $\pi_n(Y, y_0)$, so we have that $[f * c] = [f]$ and $[c * g] = [g]$. Since S^n can be viewed as a suspension, $S^n = \Sigma S^{n-1}$, we can view f, g and c as based maps $f, g, c : S^{n-1} \times [0, 1] \to (Y, y_0)$ which are all constant on the base point $(S^{n-1} \times \{0, 1\} \cup \{x_0\} \times [0, 1])$ of the suspension. Therefore c represents the identity homotopy class of based maps on S^{n-1}, so $f * c \simeq c * f$ and $c * g \simeq g * c$. Passing back to homotopy classes in $\pi_n(Y, y_0)$, we have the same homotopy equivalences $f * c \simeq f$ and $c * g \simeq g$. In other words, we can fill in the following homotopy squares, showing $f * g \simeq g * f$.



□

Theorem 1.2.13 gives a characterization of when a (based) map lifts to a cover. It will be interesting to note then that every finite covering has loops that do not lift to paths in the covering space. To see this, we need the following fact from group theory.

Theorem 1.2.23 (Jordan)**.** *Assume G is a group acting transitively on a set X, where $|X| = n$ and $2 \leq n < \infty$. Then there exists a $g \in G$ such that $gx \neq x$ for all $x \in X$.*

Corollary 1.2.24. *Let $p : T \to S$ be a finite covering space of degree $n \geq 2$, where T is path-connected. Then there exists a loop $f : S^1 \to S$ which does not lift to a path $\tilde{f} : S^1 \to T$.*

Proof. Fix $s \in S$ and set $X = p^{-1}(s)$ and $G = \pi_1(S, s)$. Then G acts transitively on X and $|X| = n$ by hypothesis. Since $|X| \geq 2$, by Theorem 1.2.23 there is a class $g \in G$ such that $gx \neq x$ for all $x \in X$. Fixing some base point $a \in S^1$, g is represented by a loop $f : (S^1, a) \to (S, s)$. If $\tilde{f} : S^1 \to T$ were a lift of f along p, then we would have $(p \circ \tilde{f})(a) = f(a)$ but then $g \cdot \tilde{f}(a) = \tilde{f}(a)$ by uniqueness of path lifting (Theorem 1.2.13). Thus $x = \tilde{f}(a)$ is a fixed point of g, a contradiction. $\square$

1.3 Classifying Covering Spaces

The goal in this section is to classify the covering spaces of a base space (Y, y_0) by relating them to the subgroups of $\pi_1(Y, y_0)$. The starting point of this theory is Theorem 1.2.13, which establishes the correspondence of subgroups of the fundamental group and covering spaces. To classify covers, we must have a notion of 'equivalent covers' to work with.

Definition. *Let $p_1 : X_1 \to Y$ and $p_2 : X_2 \to Y$ be covering spaces. An* **equivalence of coverings** *is a homeomorphism $g : X_1 \to X_2$ making the diagram commute:*

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

Lemma 1.3.1. *Assume W is connected, $p : X \to Y$ is a covering and $f : W \to Y$ is continuous. Suppose $g_1, g_2 : W \to X$ are lifts of f such that $g_1(w) = g_2(w)$ for some $w \in W$. Then $g_1 = g_2$ on all of W.*

Proof. Consider the set $A = \{w \in W \mid g_1(w) = g_2(w)\}$. Then for a fixed $w \in W$, there is a neighborhood $U_0 \subset W$ of w such that $f(U_0)$ lies in an evenly covered neighborhood $U \subset X$ of $f(w)$. Let V be the component of $p^{-1}(U)$ containing $g_1(w) = g_2(w)$. Since g_1 and g_2 are continuous, $\widetilde{V} = g_1^{-1}(V) \cap g_2^{-1}(V)$ is an open neighborhood in W, and since $p|_V$ is a homeomorphism, we must have $g_1|_{\widetilde{V}} = (p|_V)^{-1} \circ f = g_2|_{\widetilde{V}}$. Thus A is open. On the other hand, let $g_1 \times g_2 : W \to X \times X$ be the map $(g_1 \times g_2)(w) = (g_1(w), g_2(w))$. Then $A = (g_1 \times g_2)^{-1}(\Delta)$ where $\Delta \subseteq X \times X$ is the diagonal. Since X is Hausdorff, Δ is closed so A is closed in W. By assumption, A is nonempty so we must have $A = W$. $\square$

Theorem 1.3.2. *Given two covering spaces $p_1 : (W_1, w_1) \to (Y, y_0)$ and $p_2 : (W_2, w_2) \to (Y, y_0)$ where W_1 is simply connected, there is a unique covering map $g : W_1 \to W_2$ making the diagram commute:*

$$\begin{tikzcd} W_1 \arrow[dr, dashed, "g"] \arrow[dd, "p_1"'] & \\ & W_2 \arrow[dl, "p_2"] \\ Y & \end{tikzcd}$$

Proof. If W_1 is simply connected, $\pi_1(W_1, w_1) = 1$. Then there is a unique lift g of p_2 such that g makes the above diagram commute. It remains to show g is a covering map of W_2. Let $x \in W_2$ and set $y = p_2(x) \in Y$. Then there are neighborhoods U_1 and U_2 of y in Y such that $p_1^{-1}(U_1) = \coprod_\alpha U_\alpha^1$ in W_1, with $p_1|_{U_\alpha^1} : U_\alpha^1 \to U_1$ a homeomorphism for each α; and $p_2^{-1}(U_2) = \coprod_\beta U_\beta^2$ in W_2, with $p_2|_{U_\beta^2} : U_\beta^2 \to U_2$ a homeomorphism for each β. Intersecting $U_1 \cap U_2$ if necessary, we may assume $U_1 = U_2 =: U$. Set $V = U_\beta^2$ for the component U_β^2 of $p_2^{-1}(U)$ containing x. For each U_α^1 in $p_1^{-1}(U)$, $g(U_\alpha^1)$ lies in some U_β^2 in $p_2^{-1}(U)$; let $\{V_\gamma\}$ be the collection of these landing in V Then since $p_1|_{V_\alpha} : V_\alpha \to U$ and $p_2|_V : V \to U$ are homeomorphisms and the diagram commutes, we must have $g|_{V_\alpha} : V_\alpha \to V$ a homeomorphism for each V_α and $g^{-1}(V) = \coprod V_\alpha$. Therefore V is an evenly-covered neighborhood of x in W_2 with respect to the map g, so g is a covering map $W_1 \to W_2$. $\square$

Corollary 1.3.3. *Simply connected covering spaces are unique up to equivalence of coverings.*

Proof. Apply Theorem 1.3.2 to lift p_1 and p_2 to unique covering maps $g : W_1 \to W_2$ and $h : W_2 \to W_1$. The lifting property says that $p_1 = p_2 \circ g$ and $p_2 = p_1 \circ h$, so these together give $p_1 = p_1 \circ h \circ g$. Since id_{W_1} and $h \circ g$ agree at the base point w_1 and both are lifts of p_1 along itself, we must have $h \circ g = id_{W_1}$. Similarly, $g \circ h = id_{W_2}$ so g is an equivalence of covers with inverse h. $\square$

Definition. *If $X \to Y$ is a simply connected covering space of Y, then X is called the* **universal cover** *of Y.*

Example 1.3.4. Recall the cover $p : \mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus (0, 0)$ from Section 1.2. Fixing the base point $y_0 = (1, 0)$, we "see" the fundamental group $\pi_1(\mathbb{R}^2 \smallsetminus (0, 0), y_0)$ as the preimage $p^{-1}(y_0) \subset \mathbb{R}^2$, that is, $\pi_1(\mathbb{R}^2 \smallsetminus (0, 0), y_0)$ is in bijection with the fibre $p^{-1}(y_0)$ as sets. This identification depends on the base point chosen, so a priori the set $p^{-1}(y_0)$ does not have a group structure. However, the fundamental group acts on the fibre in a certain fashion so as to define a group structure.

Given a covering space $p : X \to Y$ and a base point $y_0 \in Y$, the fundamental group $\pi_1(Y, y_0)$ acts on the fibre $p^{-1}(y_0)$ as follows: for $x_0 \in p^{-1}(y_0)$ and $[\gamma] \in \pi_1(Y, y_0)$, define

$$x_0 \cdot [\gamma] = \tilde{\gamma}(1)$$

where $\tilde{\gamma}$ is the unique lift of γ starting at x_0.

Proposition 1.3.5. *$x_0 \cdot [\gamma]$ defines a group action $p^{-1}(y_0) \times \pi_1(Y, y_0) \to p^{-1}(y_0)$.*

Proof. First, the given action is well-defined on homotopy classes by the lifting theorem (1.2.13). If $e \in \pi_1(Y, y_0)$ then e lifts uniquely to the constant map at x_0. Therefore $x_0 \cdot e = x_0$. Finally, for any $[\gamma], [\eta] \in \pi_1(Y, y_0)$, we have

$$(x_0) \cdot [\gamma]) \cdot [\eta] = \tilde{\gamma}(1) \cdot \eta = \tilde{\eta}(1)$$

where $\tilde{\eta}$ is the unique lift of η starting at $\tilde{\gamma}(1)$. On the other hand,

$$x_0 \cdot [\gamma * \eta] = \widetilde{\gamma * \eta}(1)$$

where $\widetilde{\gamma * \eta}$ is the unique lift of $\gamma * \eta$ starting at x_0. By definition of path products, these must agree since $\tilde{\eta}$ starts at $\tilde{\gamma}(1)$. □

Lemma 1.3.6. *For a cover $p : X \to Y$, the action of $\pi_1(Y, y_0)$ on $p^{-1}(y_0)$ is transitive.*

Proof. X is path-connected, so given $x_0, x_1 \in p^{-1}(y_0)$, there exists a path f in X from x_0 to x_1. Then $\gamma = p \circ f$ is a loop in Y with base point y_0 that lifts to f, so $x_0 \cdot [\gamma] = x_1$. □

Lemma 1.3.7. *For any $x_0 \in p^{-1}(y_0)$, the stabilizer G_{x_0} is equal to $p_*(\pi_1(X, x_0))$.*

Proof. For any $\alpha \in \pi_1(Y, y_0)$,

$$\begin{aligned} \alpha \in G_{x_0} &\iff \alpha = [f] \text{ for some } f : S^1 \to (Y, y_0) \text{ lifting to a loop } g \text{ at } x_0 \\ &\iff [g] \in \pi_1(X, x_0) \\ &\iff [f] \in p_*(\pi_1(X, x_0)) \text{ by uniqueness of path lifting.} \end{aligned}$$

Therefore $G_{x_0} = p_*(\pi_1(X, x_0))$ as claimed. □

Theorem 1.3.8. *Let $p : X \to Y$ be a covering space and $y_0 \in Y$ be a point. For each $x_0 \in p^{-1}(y_0)$, the map*

$$\begin{aligned} \pi_1(Y, y_0) &\longrightarrow p^{-1}(y_0) \\ [\gamma] &\longmapsto x_0 \cdot [\gamma] \end{aligned}$$

induces a bijection $\pi_1(Y, y_0)/p_*(\pi_1(X, x_0)) \cong p^{-1}(y_0)$, where the quotient $\pi_1(Y, y_0)/p_*(X, x_0)$ is the collection of right *cosets.*

Proof. This is simply the orbit-stabilizer theorem. □

Corollary 1.3.9. *The number of sheets in a cover $p : X \to Y$ equals $[\pi_1(Y, y_0) : p_*(\pi_1(X, x_0))]$ for any $x_0 \in p^{-1}(y_0)$.*

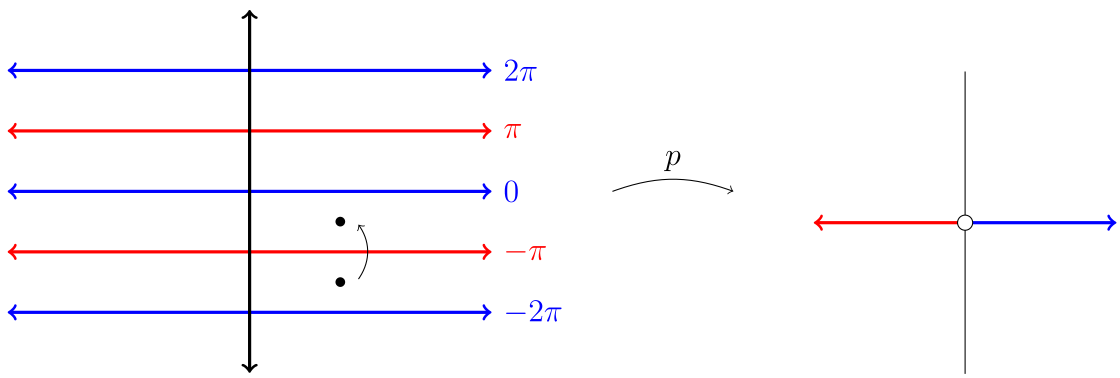
In particular, if X is simply connected, the number of sheets of $X \to Y$ is equal to $|\pi_1(Y, y_0)|$.

Corollary 1.3.10. *For all $n \geq 2$, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$.*

Remark. If X is the universal cover of Y and $x_0, x_1 \in p^{-1}(y_0)$, there is a unique equivalence of covers $g : X \to X$ with $g(x_0) = x_1$. This is generalized by the notion of deck transformations.

Definition. *For a covering space $p : X \to Y$, a* **deck transformation** *of X is an equivalence of covers $g : X \to X$.*

Example 1.3.11. Let $p : \mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus (0,0), p(z) = e^z$ be the universal cover of the punctured plane.



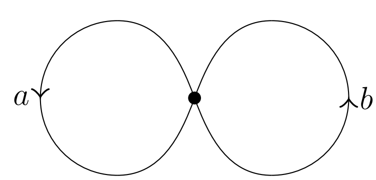
The equivalences $g_n : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x, y + 2\pi n)$ are deck transformations. By Lemma 1.3.1, if $g(x) = x$ for some $x \in X$ then $g(x) = x$ for all $x \in X$ so these are all the deck transformations of the cover $\mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus (0,0)$.

Proposition 1.3.12. *The set $\Delta(p)$ of deck transformations of a cover $p : X \to Y$ is a group under composition.*

We saw that $\pi_1(Y, y_0)$ acts on the fibre $p^{-1}(y_0)$. The group of deck transformations also has a natural action on $p^{-1}(y_0)$: If $g : X \to X$ is a deck transformation and $x_0 \in p^{-1}(y_0)$ then $g \cdot x_0 = g(x_0)$. This is well-defined since g is a homeomorphism that commutes with the covering map p. Moreover, the action of $\Delta(p)$ commutes with the fundamental group action:

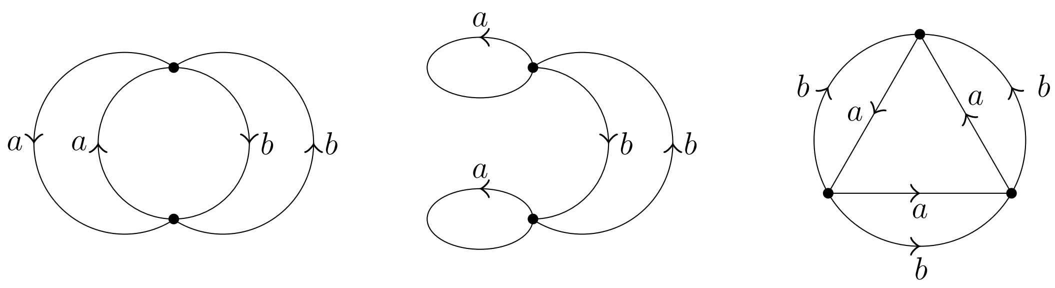
Proposition 1.3.13. *For any $g \in \Delta(p), x \in p^{-1}(y_0)$ and $\alpha \in \pi_1(Y, y_0)$, we have $g(x \cdot \alpha) = g(x) \cdot \alpha$.*

Example 1.3.14. Consider the figure-eight space, which topologically is the wedge of two circles:



$S^1 \vee S^1$

The easiest way to view some of the covering spaces of $S^1 \vee S^1$ is by drawing them as connected graphs:



Notice that the graphs A and B are each a 2-fold cover of $X = S^1 \vee S^1$, while C is a 3-fold cover. The group of deck transformations Δ_A for A consists of the trivial equivalence and the symmetry consisting of flipping the graph along its horizontal axis and then swapping the inner and outer strands (of course swapping a for a and b for b). So $\Delta_A \cong \mathbb{Z}/2\mathbb{Z}$. Similarly, $\Delta_B \cong \mathbb{Z}/2\mathbb{Z}$ consists of the trivial equivalence and then the transformation that flips B along its horizontal axis, swaps the b's and flips over the a's.

In the case of graph C, we know Δ_C is a subgroup of S_3, the group of permutations of 3 elements. But it's not the whole symmetric group, since a deck transformation that fixes any element must be the identity. So the only nontrivial deck transformations of this cover are the rotations, and hence $\Delta_C \cong \mathbb{Z}/3\mathbb{Z}$.

For a cover $p : X \to Y$ and a point $x \in p^{-1}(y_0)$, we will abbreviate $p_*(\pi_1(X, x))$ by J_x.

Lemma 1.3.15. *For any $x_0 \in p^{-1}(y_0)$ and $\alpha \in \pi_1(Y, y_0)$, $J_{x_0 \cdot \alpha} = \alpha^{-1} J_{x_0} \alpha$.*

Proof. For any $x_0 \in p^{-1}(y_0)$ and $\alpha \in \pi_1(Y, y_0)$,

$$\begin{aligned} \beta \in J_{x_0 \cdot \alpha} &\iff (x_0 \cdot \alpha) \cdot \beta = x_0 \cdot \alpha \quad \text{by Lemma 1.3.7} \\ &\iff x_0 \cdot \alpha\beta\alpha^{-1} = x_0 \\ &\iff \alpha\beta\alpha^{-1} \in J_{x_0}. \end{aligned}$$

Hence $J_{x_0 \cdot \alpha} = \alpha^{-1} J_{x_0} \alpha$. □

Let G be a group. Recall for a subgroup $H \leq G$, the *normalizer* of H in G is the subgroup $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. This is the stabilizer of H under the conjugation action of G on itself. If $H \lhd G$ is a normal subgroup, then $N_G(H) = G$.

Theorem 1.3.16. *Let $p : X \to Y$ be a covering space, $y_0 \in Y$ a point and $x_0, x_1 \in p^{-1}(y_0)$. Then the following are equivalent:*

(1) There exists a deck transformation $g : X \to X$ satisfying $g(x_0) = x_1$.

(2) There exists a loop $\alpha \in \pi_1(Y, y_0)$ lying in the normalizer $N(p_*(\pi_1(X, x_0)))$ such that $x_0 \cdot \alpha = x_1$.

(3) $p_*(\pi_1(X, x_0)) = p_*(\pi_1(X, x_1))$.

Proof. (1) $\iff$ (3) By Theorem 1.2.13 there is some $g : X \to X$ lifting p with $g(x_0) = x_1$ if and only if $J_{x_0} \subseteq J_{x_1}$. Then there is an inverse $h : X \to X$ of g if and only if $J_{x_1} \subseteq J_{x_0}$. In particular, g is a deck transformation if and only if $J_{x_0} = J_{x_1}$.

(2) $\iff$ (3) If $\alpha \in N(J_{x_0})$ with $x_0 \cdot \alpha = x_1$, then by Lemma 1.3.15,

$$J_{x_1} = J_{x_0 \cdot \alpha} = \alpha^{-1} J_{x_0} \alpha = J_{x_0}.$$

On the other hand, if $J_{x_0} = J_{x_1}$ then by Lemma 1.3.6, there is some $\alpha \in \pi_1(Y, y_0)$ such that $x_0 \cdot \alpha = x_1$. Then $\alpha^{-1} J_{x_0} \alpha = J_{x_0 \cdot \alpha} = J_{x_1} = J_{x_0}$, so we see that $\alpha \in N(J_{x_0})$. □

Corollary 1.3.17. *$J_{x_0} = p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$ if and only if $\Delta(p)$ acts transitively on $p^{-1}(y_0)$.*

Proof. Exercise. □

Corollary 1.3.18. *Fix $x_0 \in p^{-1}(y_0)$. Then as x_1 ranges over $p^{-1}(y_0)$, the groups J_{x_1} vary over all conjugates of J_{x_0} in $\pi_1(Y, y_0)$.*

Definition. *A covering space $p : X \to Y$ is* **regular** *(or* **normal***) if $\Delta(p)$ acts transitively on $p^{-1}(y_0)$.*

Lemma 1.3.19. *A covering space $p : X \to Y$ is regular if and only if $J_{x_0} = p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$ for any $x_0 \in p^{-1}(y_0)$.*

Theorem 1.3.20. *Let $p : X \to Y$ be a cover and take $x_0 \in p^{-1}(y_0)$. Then there is a short exact sequence*

$$1 \to p_*(\pi_1(X, x_0)) \to N(p_*(\pi_1(X, x_0))) \to \Delta(p) \to 1.$$

Proof. Define a function

$$\begin{aligned} \Theta : N(J_{x_0}) &\longrightarrow \Delta(p) \\ \alpha &\longmapsto g_\alpha \end{aligned}$$

where g_α is the unique deck transformation of X such that $g_\alpha(x_0) = x_0 \cdot \alpha$. Then for any $\alpha, \beta \in N(J_{x_0})$,

$$\begin{aligned} g_{\alpha\beta}(x_0) &= x_0 \cdot \alpha\beta = (x_0 \cdot \alpha) \cdot \beta = g_\alpha(x_0) \cdot \beta \\ &= g_\alpha(x_0 \cdot \beta) \quad \text{by Prop. 1.3.13} \\ &= g_\alpha g_\beta(x_0). \end{aligned}$$

Therefore by Lemma 1.3.1, $g_{\alpha\beta} = g_\alpha g_\beta$ everywhere since they agree at x_0. So Θ is a homomorphism, and it is surjective by Theorem 1.3.16. Finally, by the same theorem $\ker \Theta = \{\alpha \in \pi_1(Y, y_0) \mid g_\alpha(x_0) = x_0\} = J_{x_0}$, so we have the desired exact sequence. □

Corollary 1.3.21. *If $p : X \to Y$ is a regular cover, then $\Delta(p) \cong \pi_1(Y, y_0)/p^*(\pi_1(X, x_0))$.*

Corollary 1.3.22. *If X is a universal cover of Y then $\Delta(p) \cong \pi_1(Y, y_0)$.*

If $p : X \to Y$ is a universal cover of Y, a useful perspective is to think of Y as the quotient space $Y \cong X/\Delta$, where Δ is the action of the deck transformations on X.

Example 1.3.23. The universal cover $\mathbb{R} \to S^1$ gives rise to an action of $\mathbb{Z}$ on $\mathbb{R}$:

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R},\ (n, x) \mapsto n + x.$$

The quotient space $\mathbb{R}/\mathbb{Z}$ is therefore canonically identified with S^1.

Question. *Given an action of a group G on a space X, when is the quotient map $X \to X/G$ a covering map?*

If the question has an affirmative answer, G will end up being be the group of deck transformations of $X \to X/G$. The following properties are certainly required for one to have a covering map in such a situation:

- The action is free: for any $x \in X$, $gx = x$ if and only if $g = e$.
- There is some condition guaranteeing the evenly covered property of $X \to X/G$.

Definition. *An action of G on X is* **properly discontinuous** *if every point $x \in X$ has a neighborhood U such that for all $g \neq e$ in G, $gU \cap U = \varnothing$.*

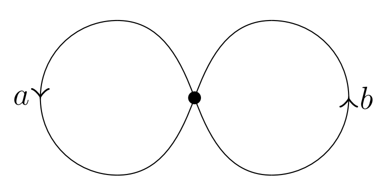
Lemma 1.3.24. *Every properly discontinuous action is also a free action.*

Proposition 1.3.25. *If G acts properly discontinuously on a path-connected, locally path-connected, Hausdorff space X, then the quotient map $p : X \to X/G$ is a covering map with $\Delta(p) = G$.*

Proof. The neighborhoods in the definition of a properly discontinuous action descend via p to evenly covered neighborhoods of the points in X/G. If $\overline{U}$ is such a neighborhood in X/G, then $p^{-1}(\overline{U}) = \bigcup_{g\in G} gU$ is a disjoint union and $gU \to \overline{U}$ is a homeomorphism for each g. Clearly G acts by deck transformations in general, but by definition of the quotient action, $G = \Delta(p)$ where p is the quotient map. $\square$

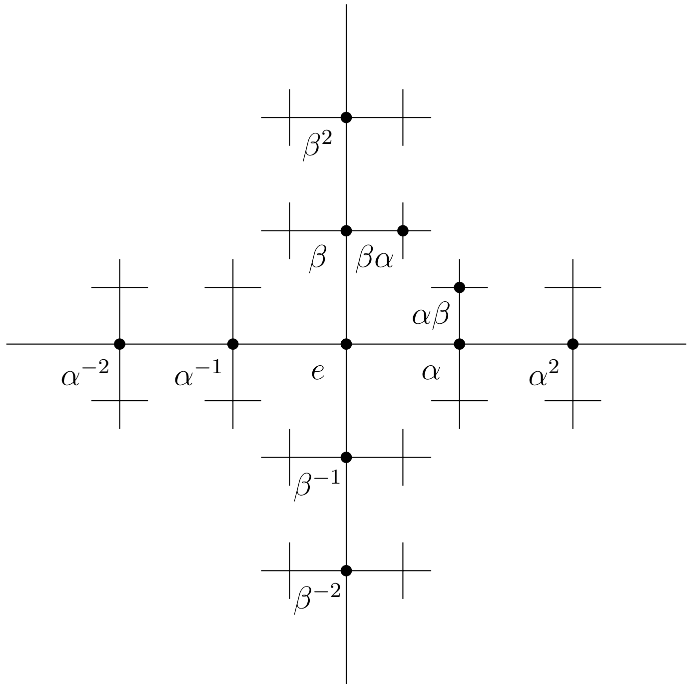
Corollary 1.3.26. *If X is simply connected and G acts properly discontinuously on X, then $\pi_1(X/G) = G$.*

Example 1.3.27. Picking up where we left off in Example 1.3.14, consider once again the figure eight space:



$S^1 \vee S^1$

Let $G = \langle \alpha, \beta \rangle$ be the free group on two generators and let Γ be the Cayley graph of G. Explicitly, the vertices of Γ are the elements of G and the edges of Γ are pairs of vertices of the form $(g, g\alpha)$ or $(g, g\beta)$.



The group G acts on Γ by the following:

- On vertices, by left multiplication: $h \cdot g = hg$.
- On edges, by translation: $h \cdot (g, g\alpha) = (hg, hg\alpha)$ and $h \cdot (g, g\beta) = (hg, hg\beta)$.

Clearly this action is free, and in fact it's properly discontinuous. The quotient space Γ/G is homeomorphic to the figure eight space: $S^1 \vee S^1 \cong \Gamma/G$. We claim that Γ is simply connected, and therefore a universal cover of $S^1 \vee S^1$. This follows from the facts in graph theory that (1) any tree is contractible, and (2) the Cayley graph of a free group is a tree. Hence by Corollary 1.3.26, we conclude that $\pi_1(S^1 \vee S^1) = \langle \alpha, \beta \rangle$, the free group on two generators.

Remark. Any finite graph Γ is homotopy equivalent to an n-fold wedge of circles $S^1 \vee \cdots \vee S^1 = \bigvee^n S^1$. Then by the above argument, $\pi_1(\bigvee^n S^1) = \langle \alpha_1, \ldots, \alpha_n \rangle = F_n$, the free group on n generators. In fact, $n = 1 - \chi(\Gamma)$, where χ denotes the Euler characteristic of the finite graph.

1.4 The Fundamental Theorem of Covering Spaces

We now prove the fundamental theorem of covering spaces.

Theorem 1.4.1. *Let Y be a path-connected, locally path-connected space and suppose $\tilde{p} : \widetilde{X} \to Y$ is a simply connected covering space. Then there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{equivalence classes of covering} \\ \text{spaces } p : X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of subgroups} \\ H \leq \pi_1(Y, y_0) \end{array} \right\}.$$

Moreover, restricting one's attention to based coverings, there is a bijective correspondence

$$\left\{\begin{matrix}\text{equivalence classes of based covering}\\ \text{spaces } p:(X,x_0)\to(Y,y_0)\end{matrix}\right\}\longleftrightarrow\left\{\begin{matrix}\text{subgroups}\\ H\le\pi_1(Y,y_0)\end{matrix}\right\}.$$

Proof. The first correspondence on conjugacy classes of subgroups follows from the second correspondence and Lemma 1.3.15. To prove the main correspondence on subgroups and based covers, define a map

$$\Phi:(p:(X,x_0)\to(Y,y_0))\longmapsto H=p_*(\pi_1(X,x_0)).$$

By the lifting theorem (1.2.13), Φ is injective, so it remains to show it is surjective.

Take a subgroup $H\le\pi_1(Y,y_0)$. We know $\Delta(\tilde{p})\cong\pi_1(Y,y_0)$ so view H as a subgroup of the deck transformation group $\Delta(\tilde{p})$. Since $\Delta(\tilde{p})$ acts properly discontinuously on Y, H also acts properly discontinuously on Y so by Proposition 1.3.25, there is a quotient space $X=\widetilde{X}/H$ such that the quotient map $p:X=\widetilde{X}/H\to Y$ is a covering map. It then suffices to show $H=p_*(\pi_1(X,x_0))$. Let $\gamma\in\pi_1(X,x_0)$. Under p_*, this becomes a class $p_*(\gamma)\in\pi_1(Y,y_0)$, which one can think of as a based loop in Y. The lift of $p_*(\gamma)$ along $\tilde{p}$ is then a path $\tilde{\gamma}$ in $\widetilde{X}$ starting at $\tilde{x}_0$ and ending at the point $\tilde{x}_0\cdot p_*(\gamma)$. Now passing to the quotient space $X=\widetilde{X}/H$, $\tilde{\gamma}$ becomes the loop γ, meaning the endpoints of $\tilde{\gamma}$ lie in the same orbit of H in $\widetilde{X}$. Hence by Theorem 1.3.16, there exists a deck transformation $g_{p_*(\gamma)}\in\Delta(\tilde{p})$ such that $\tilde{x}_0\cdot p_*(\gamma)=g_{p_*(\gamma)}(\tilde{x}_0)$. We have shown that $g_{p_*(\gamma)}$ is determined uniquely either by its action on $\tilde{x}_0$ or by the loop $p_*(\gamma)$. This is possible if and only if $p_*(\gamma)\in H$, in which case $H=p_*(\pi_1(X,x_0))$. Thus Φ is surjective. □

What's more, Lemma 1.3.19 says that the correspondence is bijective on regular covering spaces and normal subgroups of $\pi_1(Y,y_0)$. This is remarkably similar to the bijective correspondence in the fundamental theorem of Galois theory.

Corollary 1.4.2. *If F_n is the free group on n generators and $H\le F_n$ is a subgroup of finite index, then H is free. In fact, if $p=[F_n:H]$ then the rank of H is $pn-p+1$.*

Proof. By Theorem 1.4.1, every subgroup $H\le F_n$ corresponds (up to an isomorphism of covers) to a covering space of $\bigvee^n S^1$ having p sheets. Let $X\to\bigvee^n S^1$ be this cover. Then X is a finite graph, which means $H=\pi_1(X,x_0)$ is free on $1-\chi(X)$ generators, by the final remark in Section 1.3. Since $X\to\bigvee^n S^1$ is a p-to-1 cover, $\chi(X)=p\chi(\bigvee^n S^1)$ but one can count vertices and edges to see that $\chi(\bigvee^n S^1)=n-1$. Therefore the rank of H is

$$1-\chi(X)=1-p(1-n)=pn-p+1.$$

□

Remark. From Corollary 1.4.2, one obtains the surprising fact that there is a free subgroup of infinite rank inside the free group F_2 on two generators.

To wrap up the theory of covering spaces and truly have a complete characterization via the fundamental theorem of covering spaces, we must find conditions for when a space (Y,y_0) has a universal cover. The characterizing condition, stated below, is known to hold in almost all 'natural' settings, but as a technical requirement it is involved.

Definition. *A space Y is* **semilocally simply connected** *if every point $y \in Y$ has a neighborhood U such that $\pi_1(U, y) \to \pi_1(Y, y)$ is trivial.*

Theorem 1.4.3. *Suppose Y is path-connected and locally path-connected. Then Y has a universal cover if and only if Y is semilocally simply connected.*

Proof. ($\Longrightarrow$) Suppose $\tilde{p} : \widetilde{X} \to Y$ is a universal cover. Take $y \in Y$ and let U be an evenly covered neighborhood of y. If γ is a loop at y contained in U, then γ lifts to a loop $\tilde{\gamma} \subset \widetilde{X}$, so $\tilde{\gamma}$ contracts to a point in $\widetilde{X}$ since $\widetilde{X}$ is simply connected. Composing with $\tilde{p}$, we see that γ is nullhomotopic in Y. Therefore $\pi_1(U, y) \to \pi_1(Y, y)$ is trivial.

($\Longleftarrow$, Sketch) We construct a universal cover $\widetilde{X}$ as follows. Fix $y_0 \in Y$ and set

$$\widetilde{X} = \{(y, [f]) : y \in Y \text{ and } f \text{ is a path from } y_0 \text{ to } y\}.$$

Define $\tilde{p} : \widetilde{X} \to Y$ by $\tilde{p}(y, [f]) = y$. Since Y is locally path-connected, such a pair $(y, [f])$ exists for every $y \in Y$, and hence $\tilde{p}$ is surjective. Suppose $U \subset Y$ is a neighborhood of y such that $\pi_1(U, y) \to \pi_1(Y, y)$ is trivial. Then for any $(y, [f]) \in \widetilde{X}$, we have a set $U_{[f]} \subset \widetilde{X}$ defined by

$$U_{[f]} = \{(y', [f']) : y' \in U, f' = f * \alpha \text{ for some path } \alpha \subset U \text{ from } y \text{ to } y'\}.$$

One can show that the collection of $U_{[f]}$ ranging over all $[f]$ and $U \subset Y$ generate a topology on $\widetilde{X}$ with respect to which $\tilde{p}$ is continuous. Finally, one finishes by showing that $U = \coprod_{[f]} U_{[f]}$ and $\tilde{p}|_{U_{[f]}}$ is a homeomorphism for each open set $U_{[f]}$. We also need to verify that $\widetilde{X}$ is path-connected, locally path-connected and simply connected, but these details follow from the definitions of $\widetilde{X}$ and $\tilde{p}$. □

1.5 The Seifert-van Kampen Theorem

A theorem of van Kampen's, later generalized by Seifert, makes it possible to compute a presentation of the fundamental group $\pi_1(Y, y_0)$ when Y is sufficiently nice union of path-connected spaces. Before stating the theorem, we review the notion of free products of groups.

Definition. *If G and H are groups, their* **free product** *is the group $G * H$ whose elements are reduced words $g_1h_1g_2h_2 \cdots g_nh_n$, where $g_i \in G, h_i \in H$ and none are the identity other than possibly g_1 or h_n. The group law on $G * H$ is concatenation (followed by reduction).*

Examples.

1. If $G = \langle a \rangle$ and $H = \langle b \rangle$ are free groups of rank 1, their free product is $G * H = \langle a, b \rangle$. This generalizes to free groups of any finite rank.

2. If $G = \langle a_i \mid r_j \rangle$ and $H = \langle b_k \mid s_\ell \rangle$ are groups with generators a_i and b_k, and relations r_j and s_ℓ, respectively, then their free product is simply the union of the generators together with all the relations from both groups:

$$G * H = \langle a_i, b_k \mid r_j, s_\ell \rangle.$$

Definition. *If G, H and K are groups and $g : K \to G$ and $h : K \to H$ are homomorphisms, the* **amalgamated free product** *$G *_K H$ is defined as the quotient group of $G * H$ by the normal subgroup consisting of all elements of the form $g(k)h(k)^{-1}$ for $g, h, k \in K$.*

Remark. The amalgamated free product is an example of a *pushout* of groups.

Example.

3. Let $G = \langle a \rangle, H = \langle b \rangle$ and $K = \langle t \rangle$ all be free groups. Then any maps from K to the others are of the form

$$\begin{aligned} g : K &\longrightarrow G,\ t \mapsto a^n \\ \text{and} \quad h : K &\longrightarrow H,\ t \mapsto b^m \quad \text{for } n, m \in \mathbb{N}_0. \end{aligned}$$

In this case $G *_K H$ is easy to present:

$$G *_K H = \langle a, b \mid a^n = b^m \rangle.$$

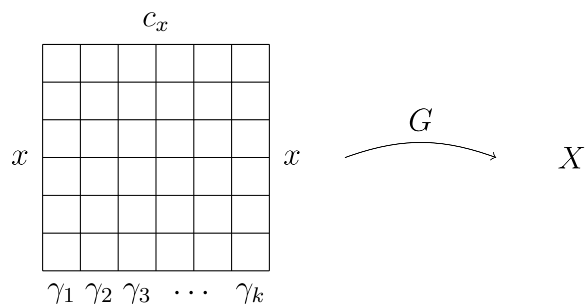
Define a map $\varphi : G *_K H \to \mathbb{Z}$ by sending $a \mapsto m$ and $b \mapsto n$. This map is well-defined since $\varphi(a^n) = mn = \varphi(b^m)$. Moreover, φ is surjective, and nontrivial when $n, m \geq 1$. Thus $G *_K H$ is infinite when $n, m \geq 1$.

Theorem 1.5.1 (Seifert-van Kampen)**.** *Suppose X is a space and $U, V, U \cap V \subset X$ are open, path-connected subsets such that $U \cup V = X$. For any $x \in U \cap V$, set $K = \pi_1(U \cap V, x)$. Then $\pi_1(X, x) \cong \pi_1(U, x) *_K \pi_1(V, x)$, where the maps $K \to \pi_1(U, x), \pi_1(V, x)$ are the natural inclusions.*

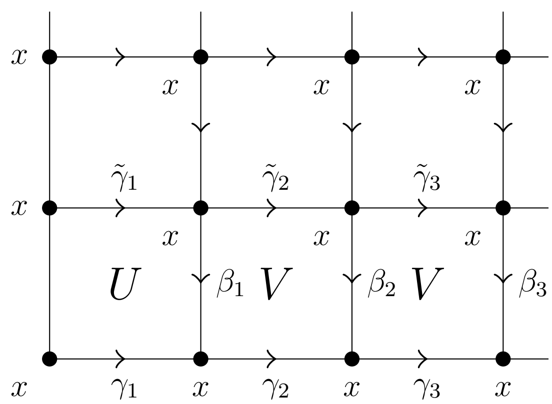
Proof. (Sketch) The inclusions $\pi_1(U, x) \hookrightarrow \pi_1(X, x)$ and $\pi_1(V, x) \hookrightarrow \pi_1(X, x)$ induce a map $\pi_1(U, x) * \pi_1(V, x) \to \pi_1(X, x)$ by the universal property of the free product. Now K is a quotient of this free product, so there is a homomorphism

$$\Phi : \pi_1(U, x) *_K \pi_1(V, x) \longrightarrow \pi_1(X, x).$$

It remains to show Φ is an isomorphism. First let γ be a loop in X based at x. Then $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ is an open cover of $[0, 1]$, so by Lebesgue's lemma (1.2.3), for sufficiently large n all intervals of length $\frac{1}{n}$ in $[0, 1]$ lie in one of $\gamma^{-1}(U), \gamma^{-1}(V)$. For each $\frac{i}{n}$ such that $\gamma\left(\frac{i}{n}\right) \in U \cap V$, take a path β_i from $\gamma\left(\frac{i}{n}\right)$ to x. Then $\gamma \simeq (\gamma_1 * \beta_i) * (\beta_i^{-1} * \gamma_2)$. Doing this for all i shows that γ can be written as a product of loops in U or V; hence Φ is surjective. On the other hand, suppose $\gamma = \gamma_1 * \cdots * \gamma_k \in \pi_1(U, x) *_K \pi_1(V, x)$ is trivial in $\pi_1(X, x)$ via Φ. Then there is a homotopy rel endpoints $F : [0, 1]^2 \to X$ between γ and the constant map at x. By Lebesgue's lemma (1.2.3), every square of edge length $\frac{1}{n}$ in $[0, 1]^2$ must map into either U or V for large enough n. Without loss of generality we may assume $k = n$. We will use this to construct a homotopy $G : [0, 1]^2 \to X$ between γ and a constant map in $\pi_1(U, x) *_K \pi_1(V, x)$.



We may arrange so that F is constant in the horizontal direction in neighborhoods of segments of the form $\left\{\frac{i}{n}\right\} \times [0,1]$; likewise, we can arrange F to be constant in the vertical direction in neighborhoods of segments $[0,1] \times \left\{\frac{i}{n}\right\}$. By interposing paths to x in either U or V at each vertex of the homotopy square for F, we may assume each vertex maps to x and each square maps in U or V:



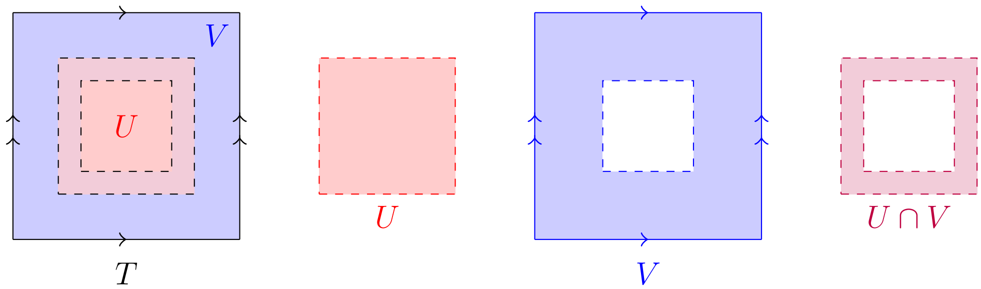
By the amalgamation relations, we compute:

$$\begin{aligned}
\gamma_1 * \gamma_2 * \gamma_3 * \cdots * \gamma_k &= (\tilde{\gamma}_1 * \beta_1)_U * (\beta_1^{-1} * \tilde{\gamma}_2 * \beta_2)_V * (\beta_2^{-1} * \tilde{\gamma}_3 * \beta_3)_V * \cdots \\
&= (\tilde{\gamma}_1 * \beta_1)_U * (\beta_1^{-1} * \tilde{\gamma}_2 * \tilde{\gamma}_3 * \beta_3)_V * \cdots \\
&= \tilde{\gamma}_1 \tilde{\gamma}_2 * \tilde{\gamma}_3 * \cdots * \tilde{\gamma}_k.
\end{aligned}$$

Continuing this process constructs the homotopy square for G, which again is a homotopy from γ to a constant map c_x in $\pi_1(U,x) *_K \pi_1(V,x)$. Hence Φ is injective, so $\pi_1(X,x) \cong \pi_1(U,x) *_K \pi_1(V,x)$. $\square$

Corollary 1.5.2. *If $X = U \cup V$ for open, path-connected sets $U, V \subset X$ and $U \cap V$ is contractible, then for any $x \in U \cap V$, $\pi_1(X,x) \cong \pi_1(U,x) * \pi_1(V,x)$, the free product of $\pi_1(U,x)$ and $\pi_1(V,x)$.*

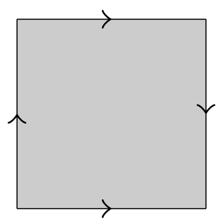
Example 1.5.3. We show that the torus $T = S^1 \times S^1$ has fundamental group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Write T as quotient space of the square $[0,1]^2$ with opposite sides identified. Then $T = U \cup V$, where U and V are the open sets shown below:



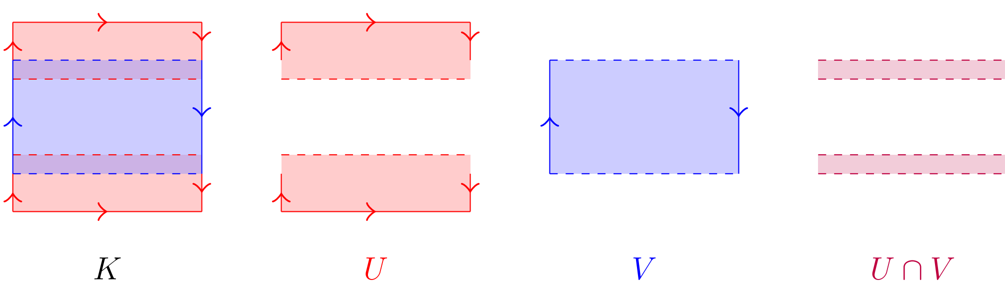
Then $\pi_1(U) = 1$ since U is contractible and $\pi_1(V) = \langle a, b\rangle$, the free group on two generators, since B retracts to the wedge of two circles. Further, $U \cap V$ retracts to a circle, so $\pi_1(U \cap V) = \langle c\rangle \cong \mathbb{Z}$ by Corollary 1.2.2. As a loop in U, $[c]$ is trivial since $\pi_1(U) = 1$, while in V, $[c] = aba^{-1}b^{-1}$. Hence by the Seifert-van Kampen theorem,

$$\pi_1(T, x) \cong \pi_1(U, x) * \pi_1(V, x) = \langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

Example 1.5.4. Let K be the Klein bottle, identified as the following quotient of the unit square:

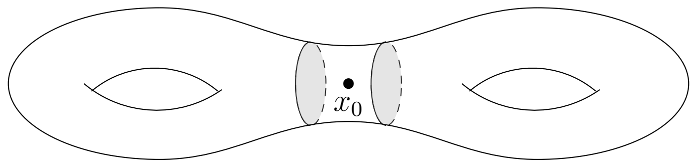


Decompose K into the following two open sets:

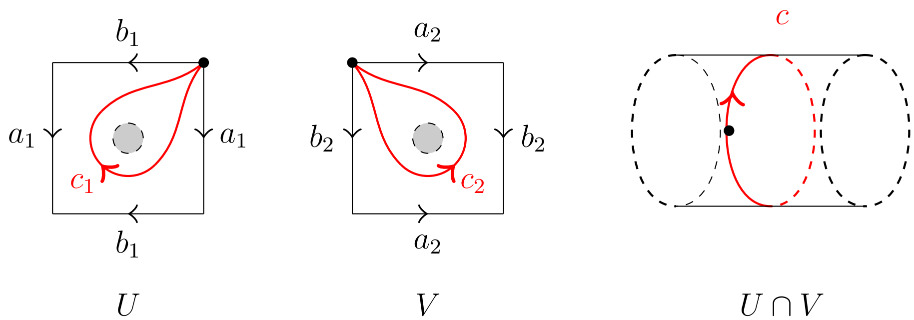


Then U glues together to form a Möbius band, which retracts onto its center circle and thus $\pi_1(U) = \langle a\rangle \cong \mathbb{Z}$ for a generating loop traveling from left to right in the diagram above. Likewise, V is homeomorphic to a Möbius band, so $\pi_1(V) = \langle b\rangle \cong \mathbb{Z}$, where b travels from left to right. Finally, $U \cap V$ glues together into a strip which retracts to a circle, so $\pi_1(U \cap V) = \langle c\rangle \cong \mathbb{Z}$ for a loop traveling left to right. By the Seifert-van Kampen theorem, $\pi_1(K) \cong \langle a\rangle *_{\langle c\rangle} \langle b\rangle$. The generators of this amalgamated product are just the free generators a and b. Under the inclusion $U \cap V \hookrightarrow U$, c maps to a^2, while under the inclusion $U \cap V \hookrightarrow V$, c maps to b^2. Hence the presentation for the fundamental group of the Klein bottle is $\pi_1(K) = \langle a, b \mid a^2b^{-2}\rangle$. Another presentation, perhaps more useful in some circumstances, is $\pi_1(K) = \langle a, b \mid aba^{-1}b\rangle$.

Example 1.5.5. Consider $\Sigma_2 = T\#T$, the connect sum of two tori.



Decompose Σ_2 into open sets U and V which overlap in the neighborhood between the two disks in the figure above. View U and V as quotients of squares, and $U \cap V$ as a cylinder:



Then U and V are each a torus with one puncture, but each of these retracts onto the wedge of two circles. Hence $\pi_1(U, x_0) \cong F_2 \cong \pi_1(V, x_0)$, the free group on two generators. On the other hand, a cylinder retracts onto a circle, so $\pi_1(U \cap V, x_0) \cong \mathbb{Z}$. Let $\pi_1(U, x_0) = \langle a_1, b_1 \rangle$, $\pi_1(U, x_0) = \langle a_2, b_2 \rangle$ and $\pi_1(U \cap V, x_0) = \langle c \rangle$ as pictured above. By the Seifert-van Kampen theorem, $\pi_1(\Sigma_2, x_0) \cong \langle a_1, b_1 \rangle *_{\langle c \rangle} \langle a_2, b_2 \rangle$. Under the inclusion $U \cap V \hookrightarrow U$, we can see that c maps to c_1 which is homotopic to $a_1 b_1 a_1^{-1} b_1^{-1}$. On the other hand, along $U \cap V \hookrightarrow V$, c maps to $c_2 \simeq b_2 a_2 b_2^{-1} a_2^{-1}$. Putting these together, we obtain the relation $c_1 c_2^{-1} = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = [a_1, b_1][a_2, b_2]$. Hence

$$\pi_1(\Sigma_2, x_0) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle.$$

The genus g surface is a g-holed torus, defined recursively by Σ_g recursively by $\Sigma_g = \Sigma_{g-1} \# T$. By induction, the Seifert-van Kampen shows that the fundamental group of Σ_g is

$$\pi_1(\Sigma_g, x_0) = \left\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \right\rangle \quad \text{for every } g \geq 1.$$

The group $\pi_1(\Sigma_g)$ is called the *gth surface group*. These groups are quite large. In the case of Σ_2, let $F_2 = \langle x, y \rangle$ be the free group on two generators and define a homomorphism

$$\begin{aligned} \varphi : \pi_1(\Sigma_2) &\longrightarrow F_2 \\ a_1 &\longmapsto x \\ a_2 &\longmapsto y \\ b_1, b_2 &\longmapsto 1. \end{aligned}$$

Then φ is clearly surjective, so the surface group for a 2-holed torus is at least as 'big' as the free group F_2. Moreover, $\pi_1(\Sigma_2)/\ker\varphi \cong F_2$ by the first isomorphism theorem, and since F_2 is the deck transformations of the universal cover X of the figure eight space (Example 1.3.27), we get a cover $X \to X/\Sigma_2$. Explicitly, X/Σ_2 is obtained by 'enlarging' the universal covering graph $X \to S^1 \vee S^1$.

This is generalized by the following proposition.

Proposition 1.5.6. *Suppose M and N are connected n-manifolds, for $n \geq 3$. Then the fundamental group of their connect sum, $M\#N$, is the free product of their fundamental groups: $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(N)$.*

Proof. To obtain the connect sum $M\#N$, we remove two coordinate neighborhoods $\widetilde{U}$ and $\widetilde{V}$, one from each manifold, and glue the boundary of each (a boundary which is homeomorphic to S^{n-1}) to a copy of S^{n-1} just inside the neighborhood removed from the other. (This is precisely what is depicted in the first figure in Example 1.5.5.) Define open sets U_1, V_1, U_2, V_2 so that $M = U_1 \cup V_1$ and $N = U_2 \cup V_2$ as follows. Let U_1 be M minus some point in $\widetilde{U} \cap M$; likewise set U_2 to be N minus some point in the corresponding neighborhood $\widetilde{V} \cap N$. Let $V_1 = \widetilde{U} \cap M$ and $V_2 = \widetilde{V} \cap N$. Then $M\#N$ is homotopy equivalent to $U_1 \cup U_2$ so we can use Seifert-van Kampen to obtain a description of $\pi_1(M\#N) \cong \pi_1(U_1 \cup U_2)$.

First we consider the decompositions $M = U_1 \cup V_1$ and $N = U_2 \cup V_2$. By construction V_1 is homeomorphic to an n-ball B^n, so it is contractible and $\pi_1(V_1) = \{1\}$. Assuming $n \geq 3$, the intersection $U_1 \cap V_1$ is homeomorphic to $B^n \setminus \{p\}$, which is contractible. Thus $\pi_1(U_1 \cap V_1) = \{1\}$ as well, and so by Seifert-van Kampen we must have $\pi_1(M) \cong \pi_1(U_1)$. A similar proof shows $\pi_1(N) \cong \pi_1(U_2)$.

Now consider the decomposition $M\#N = U_1 \cup U_2$. In our situation $U_1 \cap U_2$ is contractible, so $\pi_1(U_1 \cap U_2) = \{1\}$. By Seifert-van Kampen, we get that $\pi_1(U_1 \cup U_2)$ is isomorphic to the free product $\pi_1(U_1) * \pi_1(U_2)$. By the above calculations, this shows $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(N)$ as desired. $\square$

2 Homology

One of the most important areas in topology is homology. Homology was originally conceived of as a way to distinguish between two topological spaces by studying their 'holes'. For example, we have seen that the genus g of a surface is a topological invariant; that is, two compact, orientable surfaces are homeomorphic if and only if they have the same number of holes. This idea can be captured by a homology theory such as singular homology, but more exotic homology theories abound. These capture many different types of information that may be useful to a topologist.

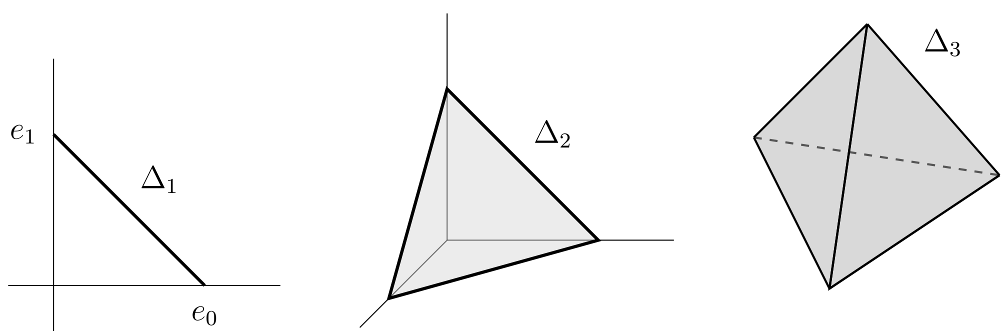
2.1 Singular Homology

Our first approach to a homology theory is singular homology. Loosely, the philosophy of singular homology is to view a space as a linear combination of formal 'triangles', or images of n-dimensional triangular structures under continuous maps.

Definition. *For each $p \geq 1$, the* **standard p-simplex** *is the compact subset $\Delta_p \subset \mathbb{R}^{p+1}$ defined by*

$$\Delta_p = \left\{ \sum_{i=0}^{p} \lambda_i e_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{p} \lambda_i = 1 \right\}.$$

Example 2.1.1. The first three standard simplices are depicted below:



line segment triangle tetrahedron

Definition. *Let $p \geq 1$. A* **singular p-simplex** *in a topological space X is a continuous map $\sigma : \Delta_p \to X$.*

Example 2.1.2. If $v_0, v_1, \ldots, v_p \in \mathbb{R}$ are any vectors, there is a corresponding p-simplex given by the map

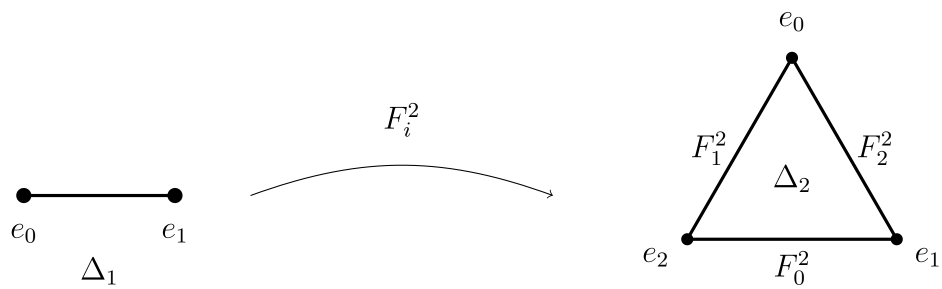
$$\sigma : \sum_{i=0}^{p} \lambda_i e_i \longmapsto \sum_{i=0}^{p} \lambda_i v_i.$$

This is an example of an *affine p-simplex*, denoted $\sigma = [v_0, v_1, \ldots, v_p]$.

Example 2.1.3. For each $p \geq 2$, there are natural affine simplices $\Delta_{p-1} \to \Delta_p$, called *face maps*, defined by

$$\begin{aligned} F_i^p = [e_0, \ldots, \hat{e}_i, \ldots, e_p] : \Delta_{p-1} &\longrightarrow \Delta_p \\ \sum_{j=0}^{p-1} \lambda_j e_j &\longmapsto \lambda_0 e_0 + \ldots + \lambda_{i-1} e_{i-1} + \lambda_i e_{i+1} + \ldots + \lambda_{p-1} e_p. \end{aligned}$$

For example, the face maps from Δ_1 to Δ_2 are



Definition. *For a topological space X, the* p**th singular chain group** *of X is the free abelian group generated by all p-simplices in X, written $\Delta_p(X)$. An element of $\Delta_p(X)$ is a finite $\mathbb{Z}$-linear combination of simplices $\sigma : \Delta_p \to X$, called a* p**-chain**.

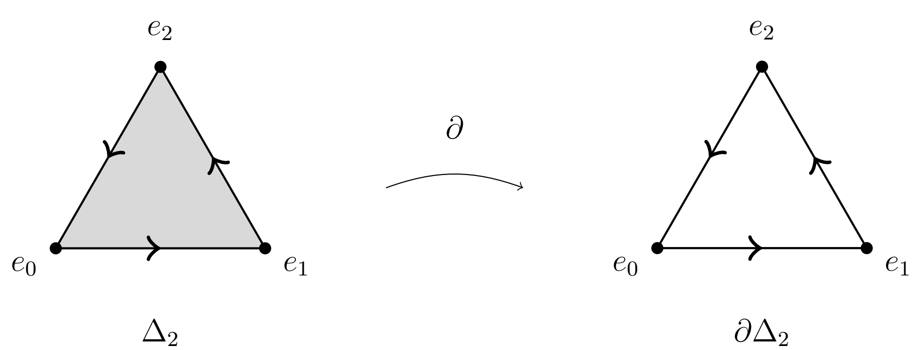
Example 2.1.4. For any space X, $\Delta_0(X)$ is formally defined to be the free abelian group generated by the points in X, and $\Delta_1(X)$ is the abelian group consisting of $\mathbb{Z}$-linear combinations of paths in X.

Definition. *Let $\sigma : \Delta_p \to X$ be a p-simplex. Then the* i**th face** *of σ is the $(p-1)$-simplex $\sigma^{(i)} : \Delta_{p-1} \to X$ given by $\sigma^{(i)} = \sigma \circ F_i^p : \Delta_{p-1} \xrightarrow{F_i^p} \Delta_p \xrightarrow{\sigma} X$. The* **boundary** *of σ is the $(p-1)$-chain $\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$.*

To each simplex we associate its boundary as described above. This defines a homomorphism of abelian groups, called the *boundary operator*:

$$\begin{aligned} \partial : \Delta_p(X) &\longrightarrow \Delta_{p-1}(X) \\ \sum n_\sigma \sigma &\longmapsto \sum n_\sigma \partial\sigma. \end{aligned}$$

Example 2.1.5. In $\mathbb{R}^2$, the triangle Δ_2 is an affine simplex with boundary $[e_1, e_2] - [e_0, e_2] + [e_0, e_1]$. Visually, the boundary operator takes an affine simplex to its frame:



Lemma 2.1.6. *For any space X, $\partial^2 = \partial \circ \partial : \Delta_p(X) \to \Delta_{p-2}(X)$ is trivial for all $p \geq 2$.*

Proof. For the affine simplex $\sigma = [v_0, \ldots, v_p]$, observe that for all $0 \leq j \leq p$, $\sigma^{(j)} = \sigma \circ F_j^p = [v_0, \ldots, \hat{v}_j, \ldots, v_p]$. Further, for all $0 \leq i \leq p-1$,

$$(\sigma^{(j)})^{(i)} = \sigma \circ F_j^p \circ F_i^{p-1} = \begin{cases} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_p], & i < j \\ [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_{i+1}, \ldots, v_p], & i \geq j. \end{cases} \qquad (*)$$

Using these computations, we see that for any p-simplex $\sigma \in \Delta_p(X)$,

$$\begin{aligned}
\partial^2 \sigma &= \partial \left(\sum_{j=0}^{p} (-1)^j \sigma \circ F_j^p \right) \\
&= \sum_{j=0}^{p} (-1)^j \sum_{i=0}^{p-1} (-1)^i \sigma \circ F_j^p \circ F_i^{p-1} \\
&= \sum_{0 \leq i < j \leq p} (-1)^{i+j} \sigma \circ F_j \circ F_i^{p-1} + \sum_{0 \leq j \leq i \leq p-1} (-1)^{i+j} \sigma \circ F_j^p \circ F_i^{p-1}.
\end{aligned}$$

Now by the formula $(*)$, we have

$$\sigma \circ F_j^p \circ F_i^{p-1} = \begin{cases} \sigma \circ F_i^p \circ F_j^{p-1}, & i < j \\ \sigma \circ F_{i+1}^p \circ F_j^{p-1}, & i \geq j. \end{cases}$$

Therefore in the sum, we get

$$\partial^2 \sigma = \sum_{0 \leq i < j \leq p} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} + \sum_{0 \leq j \leq i \leq p-1} (-1)^{i+j} \sigma \circ F_{i+1}^p \circ F_j^{p-1}.$$

Notice that these are the same up to a single multiple of -1, so the sum vanishes. Hence $\partial^2 = 0$. $\square$

We formally set $\Delta_{-1}(X) = 0$. The boundary operator forms a sequence of abelian groups

$$\Delta_\bullet(X) : \cdots \to \Delta_p(X) \to \Delta_{p-1}(X) \to \cdots \to \Delta_1(X) \to \Delta_0(X) \to 0.$$

Then Lemma 2.1.6 says that $\Delta_\bullet(X)$ is a *chain complex*.

Definition. *Let X be a topological space. Then the* p**-cycles** *are the elements of the pth cycle group*

$$Z_p(X) = \ker(\partial : \Delta_p(X) \to \Delta_{p-1}(X))$$

while the p**-boundaries** *are the elements of the pth boundary group*

$$B_p(X) = \mathrm{im}(\partial : \Delta_{p+1}(X) \to \Delta_p(X)).$$

By Lemma 2.1.6, $B_p(X) \subseteq Z_p(X)$ for every $p \geq 0$; that is, every p-boundary is a p-cycle as well. Thus we can form a quotient of abelian groups.

Definition. *For a topological space X and $p \geq 0$, the* p**th singular homology group** *of X is the quotient group*

$$H_p(X) := Z_p(X)/B_p(X) = \frac{\ker(\partial : \Delta_p(X) \to \Delta_{p-1}(X))}{\mathrm{im}(\partial : \Delta_{p+1}(X) \to \Delta_p(X))}.$$

Example 2.1.7. Let X be a topological space. Since $\Delta_{-1}(X) = 0$, $\partial : \Delta_0(X) \to \Delta_{-1}(X)$ must be the zero map, so $Z_0(X) = \Delta_0(X)$ is the free abelian group on the points of X. Next, $\partial : \Delta_1(X) \to \Delta_0(X)$ has image consisting of 0-simplices $\partial\sigma = \sigma(1) - \sigma(0)$ for $\sigma : [0,1] \to X$. Thus two points in X are equivalent in $H_0(X)$ if and only if there exists a path between them. This shows that $H_0(X)$ is the free abelian group on the path components of X. Compare this to the characterizations of $H^0_{dR}(X)$ in Proposition 0.3.2 and of $\pi_0(X)$ in Section 1.1.

Definition. *The* **augmentation map** *is the linear map $\varepsilon : \Delta_0(X) \to \mathbb{Z}$ defined by*

$$\varepsilon\left(\sum_{x\in X} n_x x\right) = \sum_{x\in X} n_x.$$

Observe that if $\sigma : [0,1] \to X$ is a 1-simplex in X, then $\partial\sigma = \sigma(1) - \sigma(0)$, so $\varepsilon(\partial\sigma) = 1 - 1 = 0$. Thus for any 1-chain $c \in \Delta_1(X)$, $\varepsilon(\partial c) = 0$ and so $\varepsilon|_{B_0(X)} = 0$. Since $Z_0(X) = \Delta_0(X)$, augmentation induces a homomorphism

$$\varepsilon_* : H_0(X) \longrightarrow \mathbb{Z}.$$

Proposition 2.1.8. *If X is nonempty and path-connected, then $\varepsilon_* : H_0(X) \to \mathbb{Z}$ is an isomorphism.*

Proof. Since X is nonempty and $\varepsilon|_{B_0(X)} = 0$, ε_* is clearly surjective. To show it is injective, fix a point $x_0 \in X$. For each $x \in X$, choose a path λ_x from x_0 to x. Suppose $c = \sum_{x\in X} n_x x$ is a 0-chain in $\ker\varepsilon$. Then $\sum_{x\in X} n_x = 0$. We must show $c \in B_0(X)$, that is, we must find a 1-chain whose boundary is c. Set $\Lambda = \sum_{x\in X} n_x\lambda_x \in \Delta_1(X)$. Then

$$\begin{aligned}\partial\Lambda = \partial\left(\sum_{x\in X} n_x\lambda_x\right) &= \sum_{x\in X} n_x\partial\lambda_x = \sum_{x\in X} n_x(x - x_0)\\ &= \sum_{x\in X} n_x x - \sum_{x\in X} n_x x_0 = c - x_0\sum_{x\in X} n_x = c - 0 = c.\end{aligned}$$

Hence $c \in B_0(X)$ so ε_* is an isomorphism on 0th homology. □

Lemma 2.1.9. *If $X = \coprod_\alpha X_\alpha$ is a disjoint union of path components, then for each $p \geq 0$,*

$$H_p(X) = \bigoplus_\alpha H_p(X_\alpha).$$

Proof. A simplex $\sigma : \Delta_p \to X$ is a continuous map and Δ_p is path-connected, so its image lies in a unique path component of X. It follows that all chains, cycles and boundaries of degree p lie in a single path component of X, so $H_p(X)$ splits as a direct product of the $H_p(X_\alpha)$. □

Corollary 2.1.10. *For any space X, $H_0(X) = \bigoplus_{\alpha} \mathbb{Z}$ where $\{X_\alpha\}$ is the set of path components of X.*

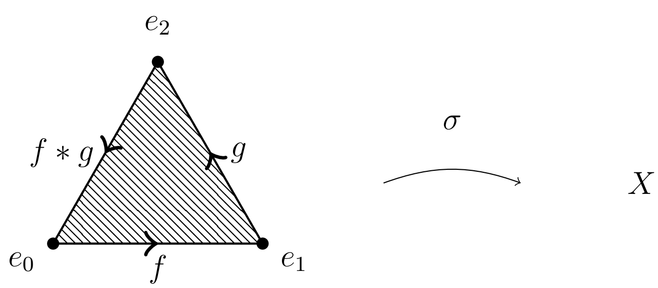
As we saw above, there is a similarity between the 0th homotopy group $\pi_0(X)$ and the 0th homology group $H_0(X)$. We next explore the relations between the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ of a space. Suppose $[\gamma] \in \pi_1(X, x_0)$ is a class of loops based at $x_0 \in X$. Then there is a representative $\gamma : [0,1] \to X$ which is a continuous map, and therefore a 1-simplex: $\gamma \in \Delta_1(X)$. Since γ is a loop, $\partial\gamma = \gamma(1) - \gamma(0) = 0$, so in fact $\gamma \in Z_1(X)$. This allows us to define a map

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow H_1(X) \\ [\gamma] &\longmapsto [\gamma] \end{aligned}$$

called the *Hurewicz homomorphism*. We will prove this is a well-defined group homomorphism. First we need the following lemmas.

Lemma 2.1.11. *If $f, g : [0,1] \to X$ are paths with $f(1) = g(0)$, then $f * g - f - g \in B_1(X)$.*

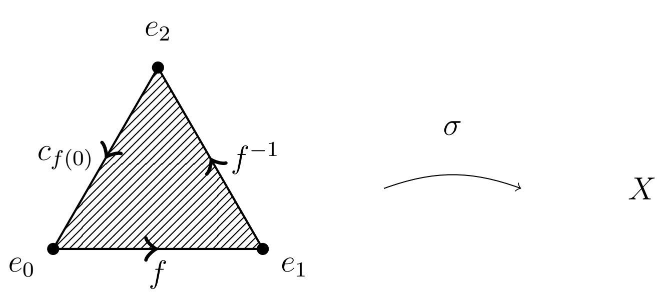
Proof. Define a 2-simplex $\sigma : \Delta_2 \to X$ by



where σ is constant along the diagonal lines. Then $\partial\sigma = g - f * g + f \in B_1(X)$. $\square$

Lemma 2.1.12. *A constant path is a boundary. Moreover, for any path $f : [0,1] \to X$, the 1-chain $f * f^{-1}$ is a boundary.*

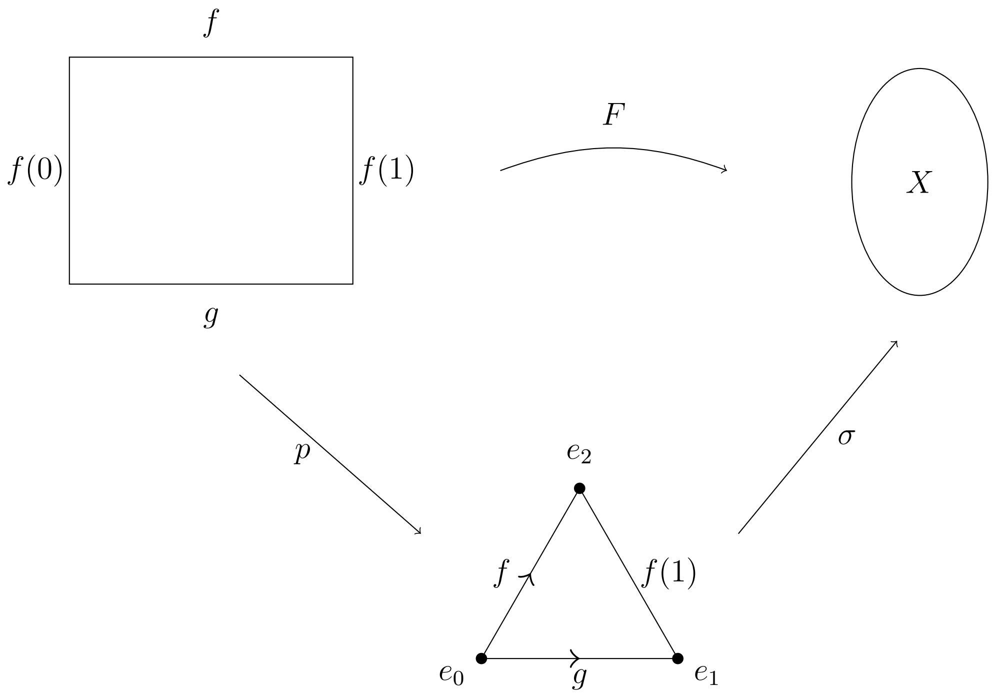
Proof. Let $\sigma : \Delta_2 \to X$ be constant, say at $x_0 \in X$. Then $\partial\sigma = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)} = x_0 - x_0 + x_0 = x_0$. Thus the constant map at x_0 is a boundary. Now let $f : [0,1] \to X$ be any path and define a 2-simplex by



where $c_{f(0)}$ is the constant map at $f(0)$ and σ is constant on the diagonal lines. Then $\partial\sigma = f^{-1} - c_{f(0)} + f = f^{-1} - \partial\sigma_{f(0)} + f$, where $\sigma_{f(0)}$ is the constant 2-simplex at $f(0)$ from above. Thus $f + f^{-1} = \partial(\sigma - \sigma_{f(0)}) \in B_1(X)$. It follows by Lemma 2.1.11 that $f * f^{-1}$ and $f^{-1} * f$ are both boundaries. $\square$

Lemma 2.1.13. *Suppose $f, g : [0,1] \to X$ are two paths that are homotopic rel endpoints. Then $f - g \in B_1(X)$.*

Proof. Let $F : [0,1]^2 \to X$ be a homotopy rel endpoints from g to f. We may then factor the homotopy square $[0,1]^2$ through a 2-simplex by $F = \sigma \circ p$:



Then σ is a 2-simplex and $\partial\sigma = f + c_{f(1)} - g$, which implies $\partial(\sigma - \sigma_{f(1)}) = f - g$. Hence $f - g \in B_1(X)$. ☐

Theorem 2.1.14 (Hurewicz)**.** *Let X be a path-connected space and fix $x_0 \in X$. Then there is an isomorphism $\varphi : \pi_1(X, x_0)^{ab} \to H_1(X)$, where $\pi_1(X, x_0)^{ab} = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$ is the abelianization of the fundamental group of X.*

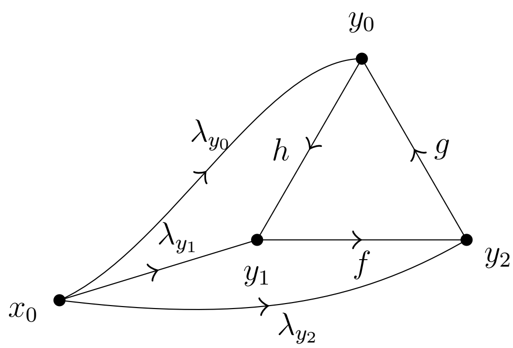
Proof. The Hurewicz map $\pi_1(X, x_0) \to H_1(X)$ is well-defined by Lemma 2.1.13 and a group homomorphism by Lemmas 2.1.11 and 2.1.12. Further, since $H_1(X)$ is an abelian group, the Hurewicz map factors through the abelianization of $\pi_1(X, x_0)$:

$$\pi_1(X, x_0) \to \pi_1(X, x_0)^{ab} \xrightarrow{\varphi} H_1(X).$$

To define an inverse to φ, fix a path $\lambda_x : x_0 \mapsto x$ for each $x \in X$. For a 1-simplex $f : [0,1] \to X$, notice that $[\lambda_{f(0)} * f * \lambda_{f(1)}^{-1}]$ is a well-defined class in $\pi_1(X, x_0)^{ab}$. Extending linearly, this gives a homomorphism

$$\begin{aligned} \psi : \Delta_1(X) &\longrightarrow \pi_1(X, x_0)^{ab} \\ f &\longmapsto [\lambda_{f(0)} * f * \lambda_{f(1)}^{-1}]. \end{aligned}$$

We next show ψ is trivial on $B_1(X)$. For a 2-simplex $\sigma \in \Delta_2(X)$, write $\sigma = f - g + h$, where $f = [y_1, y_2], g = [y_0, y_2], h = [y_0, y_1]$ and $y_i = \sigma(e_i)$ for each $i = 1, 2, 3$.



Then

$$\begin{aligned}\psi(\partial\sigma) &= \psi(f-g+h) = \psi(f)\psi(h)\psi(g)^{-1} \quad \text{in } \pi_1(X,x_0)^{ab}\\ &= [\lambda_{y_0} * h * \lambda_{y_1}^{-1}][\lambda_{y_1} * f * \lambda_{y_2}^{-1}][\lambda_{y_2} * g * \lambda_{y_0}^{-1}]\\ &= [\lambda_{y_0} * h * \lambda_{y_1}^{-1} * \lambda_{y_1} * f * \lambda_{y_2}^{-1} * \lambda_{y_2} * g * \lambda_{y_0}^{-1}]\\ &= [\lambda_{y_0} * h * f * g * \lambda_{y_0}^{-1}].\end{aligned}$$

Clearly this is nullhomotopic in X, so $\psi(\partial\sigma) = 1$ in $\pi_1(X,x_0)^{ab}$. Therefore ψ induces a map

$$\psi_* : H_1(X) \longrightarrow \pi_1(X,x_0)^{ab}.$$

Finally, we show φ and ψ are inverses. If $f \in \pi_1(X,x_0)$ is a loop at x_0 then

$$(\psi\circ\varphi)(f) = \psi(f) = [\lambda_{x_0} * f * \lambda_{x_0}^{-1}] = [f]$$

since λ_{x_0} is a constant path. So $\psi_* \circ \varphi_*$ is the identity on $\pi_1(X,x_0)^{ab}$. On the other hand, for a 1-simplex $\sigma : [0,1] \to X$,

$$\begin{aligned}(\varphi\circ\psi)(\sigma) &= \varphi(\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1})\\ &= [\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}]\\ &= [\lambda_{\sigma(0)} + \sigma + \lambda_{\sigma(1)}^{-1}] \quad \text{by Lemma 2.1.11}\\ &= [\lambda_{\sigma(0)} + \sigma - \lambda_{\sigma(1)}] \quad \text{by Lemma 2.1.12}\\ &= [\sigma - \lambda_{\partial\sigma}]\end{aligned}$$

where $\lambda_{\partial\sigma} = \lambda_{\sigma(1)} - \lambda_{\sigma(0)}$. This means for a 1-chain $c = \sum n_\sigma \sigma$ in $\Delta_1(X)$,

$$(\varphi_* \circ \psi_*)(c) = \left[\sum n_\sigma(\sigma - \lambda_{\partial\sigma})\right] = [c - \lambda_{\partial c}].$$

Thus if $\partial c = 0$, i.e. c is a 1-cycle, $[c - \lambda_{\partial\sigma}] = [c] \in H_1(X)$. Therefore $\varphi_* \circ \psi_*$ is the identity on $H_1(X)$ so φ is an isomorphism. □

Examples. Hurewicz's theorem is a powerful tool for calculating homology. Using some results from Chapter 1, we get the following homology groups for familiar spaces.

1. $H_1(S^1) = \mathbb{Z}$ (Corollary 1.2.2) and for $n \geq 2$, $H_1(S^n) = 0$ (Theorem 1.1.17).

(2) For all $n \geq 2$, $H_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ (Corollary 1.3.10).

(3) For the torus $T = S^1 \times S^1$, $H_1(T) = \mathbb{Z}^2$ (Example 1.5.3). More generally, if Σ_g is the (orientable) surface of genus g then $H_1(\Sigma_g) = \mathbb{Z}^{2g}$ (Example 1.5.5).

(4) By Example 1.3.27, $\pi_1(S^1 \vee S^1) = F_2$, the free group on two generators. Thus by Hurewicz's theorem, $H_1(S^1 \vee S^1) = \mathbb{Z}^2$, the free abelian group of rank 2. Similarly, $H_1\left(\bigvee^n S^1\right) = \mathbb{Z}^n$. In particular, the first homology group cannot distinguish between the torus and the figure eight space.

(5) For the Klein bottle K, $\pi_1(K) = \langle a, b \mid a^2b^{-2}\rangle$ by Example 1.5.4. We claim $\pi_1(X, x_0)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. To see this, consider the homomorphism

$$\begin{aligned} \langle a, b \mid a^2b^{-2}\rangle &\longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ a &\longmapsto (1, 0) \\ b &\longmapsto (1, 1). \end{aligned}$$

Then $\varphi(a^2b^{-2}) = 2\varphi(a) - 2\varphi(b) = (2, 0) - (2, 2) = (2, 0) - (2, 0) = (0, 0)$ so φ is well-defined. Clearly φ is surjective, e.g. by linear algebra. Also, the commutator $H = [\pi_1(K), \pi_1(K)]$ must be contained in $\ker\varphi$ since $\operatorname{im}\varphi = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is abelian. On the other hand, if $x = a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k}$ is an arbitrary element of the kernel, we can commute even powers of b past any powers of a, which allows us to write $x = a^m b^i a^n b^j$, where $m, n \in \mathbb{Z}$ and $i, j \in \{\pm 1, 0\}$. Then $\varphi(x) = (m + n + i + j, i + j)$ so if $x \in \ker\varphi$ it must be that $i+j$ is even and $m+n = -(i+j)$. Write $i+j = 2k$. Then $x = a^{-2k-n}b^i a^n b^j$. If $i = j = 0$, this is just a power of a, which lies in H. The remaining possibilities are:

$$\begin{aligned} i = j = -1 &\implies k = -1 \implies x = a^{2-n}b^{-1}a^n b^{-1} = a^{-n}ba^n b^{-1}; \\ i = j = 1 &\implies k = 1 \implies x = a^{-2-n}ba^n b = a^{-n}b^{-1}a^n b; \\ i = 1, j = -1 &\implies k = 0 \implies x = a^{-n}ba^n b^{-1}; \\ i = -1, j = 1 &\implies k = 0 \implies x = a^{-n}b^{-1}a^n b. \end{aligned}$$

In all cases $x \in H$, so we have proven that $\ker\varphi$ is the commutator. Therefore $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as claimed.

(6) One can view the 3-sphere $S^3 \subset \mathbb{R}^4$ as the union of two solid tori:

$$S^3 = \partial B^4 \cong \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2) = (S^1 \times D^2) \cup (D^2 \times S^1).$$

This can be generalized by gluing two solid tori along their boundary tori via the function

$$\begin{aligned} f : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ [(x, y)] &\longmapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] \end{aligned}$$

for $a, b, c, d \in \mathbb{Z}$. (This is viewing $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$.) Call the resulting quotient space X. The special case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ gives the above decomposition of S^4. However, other choices of a, b, c, d give more exotic spaces – called *lens spaces* if $\gcd(a, c) = 1$ and $ad - bc = 1$, i.e. the gluing map is represented by multiplication by a matrix in $SL_2(\mathbb{R})$.

Viewing $S^1 \times S^1 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, we can write f by $f(x, y) = (ax + by, cx + dy) \in (\mathbb{R}/\mathbb{Z})^2$. Of course since $a, b, c, d \in \mathbb{Z}$ this is well-defined. Assume $ad - bc \neq 0$, i.e. the matrix must be invertible, so that f is a homeomorphism. To determine $H_1(X)$, we calculate $\pi_1(X)$ using the Seifert-van Kampen theorem (1.5.1) and then apply Hurewicz's theorem (2.1.14).

Let U equal T_1 together with a small neighborhood of ∂T_2 and let V equal T_2 together with a small neighborhood of ∂T_1, so that $X = U \cup V$. Then U and V each retract to their 'core', a circle, so $\pi_1(U) = \langle \alpha \rangle$ and $\pi_1(V) = \langle \beta \rangle$; both are isomorphic to $\mathbb{Z}$. On the other hand, $U \cap V$ retracts onto the torus $S^1 \times S^1$ that sits inside X after gluing. So $\pi_1(U \cap V) = \langle \gamma, \delta \mid \gamma\delta\gamma^{-1}\delta^{-1} \rangle \cong \mathbb{Z}^2$, where γ is represented by a meridian and δ by a longitude. Now the amalgamation maps are (up to homotopy equivalence) the canonical inclusion $i : U \cap V \hookrightarrow U$ and the homeomorphism $f : U \cap V \hookrightarrow V$. Then $i_*(\gamma)$ is trivial in the solid torus U, while $i_*(\delta) = \alpha$ since any longitude is homotopic to the core in a solid torus. On the other hand, γ is represented by the map $\gamma_0 : [0, 1] \to S^1 \times S^1, \gamma_0(t) = (t, 0)$ and δ is represented by $\delta_0 : [0, 1] \to S^1 \times S^1, \delta_0(t) = (0, t)$. So we have $f_*\gamma = f_*[\gamma_0] = [f \circ \gamma_0]$ and $f_*\delta = f_*[\delta_0] = [f \circ \delta_0]$. Viewing $f \circ \gamma_0 : [0, 1] \to S^1 \times S^1$ as a loop, we see that for any $t \in [0, 1]$, $(f \circ \gamma_0)(t) = f(t, 0) = (at, ct)$. Likewise, $(f \circ \delta_0)(t) = f(0, t) = (bt, dt)$. The map $F(t, u) = (at, (1 - u)ct)$ is a homotopy from $f \circ \gamma_0 = (at, ct)$ to $\gamma_0^a = (at, 0)$. Likewise, the map $G(t, u) = ((1-u)bt, dt)$ is a homotopy from $f \circ \delta_0 = (bt, dt)$ to $\delta_0^d = (0, dt)$. Hence $f_*\gamma = \gamma^a \simeq \beta^a$ and $f_*\delta = \delta^d \simeq \beta^d$. By Seifert-van Kampen, we get

$$\begin{aligned}
\pi_1(X) = \pi_1(U \cup V) &= \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \\
&= \langle \alpha, \beta \mid i_*(\gamma) f_*(\gamma)^{-1}, i_*(\delta) f_*(\delta)^{-1} \rangle \\
&= \langle \alpha, \beta \mid e\beta^{-a}, \alpha\beta^{-d} \rangle \\
&= \langle \alpha, \beta \mid \beta^a, \alpha\beta^{-d} \rangle.
\end{aligned}$$

Now using the relation $\alpha = \beta^d$, we can remove α from our list of generators to get

$$\pi_1(X) = \langle \beta \mid \beta^a \rangle \cong \mathbb{Z}/a\mathbb{Z}.$$

Since the fundamental group is already abelian, $H_1(X) = \mathbb{Z}/a\mathbb{Z}$ as well. The amazing part is that the first homology of X only depends on a. It turns out that different choices of b give distinct homeomorphism classes of spaces, although H_1 cannot detect this.

Lemma 2.1.15. *For two groups G and H, the abelianization of their free product $G * H$ is the direct sum of their abelianizations, $G^{ab} \oplus H^{ab}$.*

Proposition 2.1.16. *Suppose M and N are connected n-manifolds, for $n \geq 3$. Then $H_1(M \# N) \cong H_1(M) \oplus H_1(N)$.*

Proof. Apply Proposition 1.5.6. □

Proposition 2.1.17. *The Hurewicz homomorphism is natural; that is, given a continuous map $f : X \to Y$ with $f(x_0) = y_0$, the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\varphi_X"'] & \pi_1(Y, y_0) \arrow[d, "\varphi_Y"] \\
H_1(X) \arrow[r, "f_*"] & H_1(Y)
\end{tikzcd}$$

where φ_X and φ_Y are the Hurewicz homomorphisms.

Proof. For any $\alpha \in \pi_1(X, x_0)$, choose a loop $\gamma : [0, 1] \to (X, x_0)$ so that $[\gamma] = \alpha$. Notice that since $\Delta_1 = [0, 1]$, γ is also a 1-simplex. Then $f_*\alpha = [f \circ \gamma]$ by definition, and $\varphi_Y(f_*\alpha) = \varphi_Y(f \circ \gamma) = [f \circ \gamma] \in H_1(Y)$. On the other hand, $\varphi_X(\alpha) = \varphi_X(\gamma) = [\gamma] \in H_1(X)$ and $f_*(\varphi_X(\gamma)) = f_*[\gamma] = [f \circ \gamma]$ since γ is a 1-simplex. Therefore the diagram commutes. $\square$

Theorem 2.1.18. *For each $p \geq 0$, the association $X \mapsto H_p(X)$ is a functor $H_p(-) : \mathtt{Top} \to \mathtt{Ab}$ from the category of topological spaces to the category of abelian groups.*

Proof. For a map $f : X \to Y$, there is an induced homomorphism $f_* : H_p(X) \to H_p(Y)$ defined as follows. For a p-simplex $\sigma : \Delta_p \to X$, we get a p-simplex $f_*\sigma : \Delta_p \to Y$ by composing with f: $f_*\sigma = f \circ \sigma$. Since $\Delta_p(Y)$ is generated freely by p-simplices, the universal property of free groups gives us a map $f_* : \Delta_p(X) \to \Delta_p(Y)$. We claim f_* is a chain map, i.e. $f_*\partial = \partial f_*$. To show this, we must prove the following diagram commutes:

$$\begin{tikzcd}
\Delta_p(X) \arrow[r, "f_*"] \arrow[d, "\partial"'] & \Delta_p(Y) \arrow[d, "\partial"] \\
\Delta_{p-1}(X) \arrow[r, "f_*"] & \Delta_{p-1}(Y)
\end{tikzcd}$$

If σ is a p-simplex in X, then we have

$$\begin{aligned}
f_*(\partial\sigma) &= f_*\left(\sum_{i=0}^{p}(-1)^i\sigma^{(i)}\right) = f_*\left(\sum_{i=0}^{p}(-1)^i\sigma \circ F_i\right) \\
&= \sum_{i=0}^{p}(-1)^i f_*(\sigma \circ F_i) = \sum_{i=0}^{p}(-1)^i(f \circ \sigma \circ F_i) \\
&= \sum_{i=0}^{p}(-1)^i(f \circ \sigma) \circ F_i = \partial(f \circ \sigma) = \partial(f_*\sigma).
\end{aligned}$$

Extending by linearity to all of $\Delta_p(X)$ gives the result.

Now suppose $\alpha \in Z_p(X)$ is a cycle in X. Then by the above, $\partial f_*(\alpha) = f_*(\partial\alpha) = f_*(0) = 0$ so $f_*(\alpha) \in Z_p(Y)$. Thus f_* restricts to a well-defined map on p-cycles, which we also denote by $f_* : Z_p(X) \to Z_p(Y)$. If $\beta \in B_p(X)$ is a boundary, let $\gamma \in \Delta_{p+1}(X)$ be such that $\partial\gamma = \beta$.

Then $f_*(\beta) = f_*(\partial\gamma) = \partial f_*(\gamma)$ which implies $f_*(\beta) \in B_p(Y)$. Thus $f_*(B_p(X)) \subseteq B_p(Y)$, so we get a well-defined induced map on homology, $f_* : H_p(X) \to H_p(Y)$. This completes the proof. □

We are finally able to prove that homology is a topological property; that is, the homology groups $H_p(-)$ are the same up to isomorphism for homeomorphic spaces.

Corollary 2.1.19. *If $f : X \to Y$ is a homeomorphism then the induced map on homology $f_* : H_p(X) \to H_p(Y)$ is an isomorphism for all $p \geq 0$.*

2.2 Some Homological Algebra

To further develop a theory of homology, we need some tools from homological algebra. Recall the following terms from abelian group theory.

Definition. *A sequence of abelian groups $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be* **exact** *at B if $\ker g = \operatorname{im} f$. An exact sequence of the form*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a **short exact sequence***. In particular, this means that f is injective, g is surjective and $\ker g = \operatorname{im} f$.*

Definition. *A* **chain complex** *is an abelian group C with an endomorphism $\partial : C \to C$ such that $\partial^2 = 0$.*

Definition. *A* **graded abelian group** *is a direct sum $C_\bullet = \bigoplus_{n\in\mathbb{Z}} C_n$. An element (x_n) of a graded group $C_\bullet = \bigoplus_{n\in\mathbb{Z}}$ is* **homogeneous** *of degree i if $x_n = 0$ for all $n \neq i$. A homomorphism $\varphi : C_\bullet \to D_\bullet$ between graded groups is a* **homogeneous map** *of degree d if whenever $x \in C_n$ is homogeneous of degree n, $\varphi(x) \in D_{n+d}$ is homogeneous of degree $n + d$.*

Definition. *A* **graded chain complex** *is a graded abelian group $C_\bullet$ equipped with a* **boundary operator** *$\partial : C_\bullet \to C_\bullet$ such that $\partial^2 = 0$ and $\partial(C_n) \subseteq C_{n-1}$ for all $p \in \mathbb{Z}$. That is, $C_\bullet$ is a chain complex and its boundary operator is a homogeneous map of degree -1.*

Example 2.2.1. Let X be a topological space with singular chain groups $\Delta_n(X)$ for $p \in \mathbb{Z}$, where $\Delta_n(X) = 0$ if $p < 0$. Then $\Delta_\bullet(X) = \bigoplus_{n\in\mathbb{Z}} \Delta_n(X)$ is a graded chain complex with boundary operator $\delta : \Delta_n(X) \to \Delta_{n-1}(X)$ as described in Section 2.1, which is a homogeneous map of degree -1. For any map $f : X \to Y$, the induced map $f_* : \Delta_\bullet(X) \to \Delta_\bullet(Y)$ is a homogeneous map of degree 0, by Theorem 2.1.18.

Definition. *The* **homology of a graded chain complex** *$(C_\bullet, \partial)$ is the graded group $H_\bullet(C_\bullet) = \bigoplus_{n\in\mathbb{Z}} H_n(C_\bullet)$, where*

$$H_n(C_\bullet) = \frac{\ker(\partial : C_n \to C_{n-1})}{\operatorname{im}(\partial : C_{n+1} \to C_n)}.$$

Definition. *Given two chain complexes $A_\bullet$ and $B_\bullet$, a* **chain map** *is a homogeneous map $f : A_\bullet \to B_\bullet$ satisfying $f\partial = \partial f$.*

If $f : A_\bullet \to B_\bullet$ is a chain map, there is an induced map on homology, $f_\bullet : H_\bullet(A) \to H_\bullet(B)$. One of the most important theoretical results is the long exact sequence in homology. In the language of functors, the next results shows what happens to a short exact sequence when the homology functors are applied.

Theorem 2.2.2 (Long Exact Sequence in Homology)**.** *Given a short exact sequence of graded chain complexes*

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0,$$

there is a long exact sequence

$$\cdots \to H_{n+1}(C_\bullet) \xrightarrow{\partial_*} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial_*} H_{n-1}(A_\bullet) \to \cdots$$

Proof. Fix $n \in \mathbb{Z}$. Then since i and j are chain maps, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"'] & B_n \arrow[r, "j"] \arrow[d, "\partial"'] & C_n \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] & B_{n-1} \arrow[r, "j"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

For a homology class $[c] \in H_n(C)$, there exists $b \in B_n$ such that $j(b) = c$ by exactness of the top row. Since the diagram above commutes, $j\partial(b) = \partial j(b) = \partial(c) = 0$ because c is a cycle. Hence by exactness of the bottom row, there is some $a \in A_{n-1}$ such that $i(a) = \partial(b)$ in B_{n-1}. Extending the commutative diagram further, we have

$$\begin{tikzcd}
 & A_{n-1} \arrow[r] \arrow[dd] & B_{n-1} \arrow[dd] \\
a \arrow[r, mapsto] \arrow[dd, mapsto] & \partial b \arrow[dd, mapsto] & \\
 & A_{n-2} \arrow[r] & B_{n-2} \\
\partial a \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

This commutes and i is injective, so $\partial a = 0$. Thus $[a]$ is a well-defined homology class in $H_{n-1}(A)$. Define the *connecting homomorphism* $\partial_* : H_n(C) \to H_{n-1}(A)$ by $\partial_*[c] = [a]$. To see that ∂_* is well-defined, suppose $b' \in B_n$ such that $j(b') = c$ as well. Let $a' \in A_{n-1}$ be the element for which $i(a') = \partial(b')$. Then $j(b' - b) = c - c = 0$ so by exactness, there is some $a'' \in A_n$ such that $i(a'') = b' - b$. Now by commutativity,

$$i(a + \partial(a'')) = i(a) + i\partial(a'') = \partial(b) + \partial i(a'') = \partial(b) + \partial(b') - \partial(b) = \partial(b') = i(a').$$

Since i is injective, $a' = a + \partial(a'')$ so $[a] = [a']$ in $H_{n-1}(A)$. On the other hand, another choice of representatives of $[c]$ would be of the form $c' = c + \partial(c'')$ for $c'' \in C_{n+1}$. Let $b'' \in B_{n+1}$ be such that $j(b'') = c''$. Then

$$j(b + \partial(b'')) = j(b) + j\partial(b'') = c + \partial(c'') = c'$$

but $\partial(b+\partial(b''))=\partial(b)+\partial^2(b'')=\partial(b)$. Since i is injective, different choices of representatives of $[c]$ determine the same $a\in A_{n-1}$ for which $i(a)=\partial(b)$. Therefore ∂_* is a well-defined map. It is obvious that it is a homomorphism.

Now we show the sequence

$$\cdots\to H_{n+1}(C_\bullet)\xrightarrow{\partial_*}H_n(A_\bullet)\xrightarrow{i_*}H_n(B_\bullet)\xrightarrow{j_*}H_n(C_\bullet)\xrightarrow{\partial_*}H_{n-1}(A_\bullet)\to\cdots$$

is exact. We first show the homology sequence is a complex. For $[c]\in H_n(C_\bullet)$, let $b\in B_n$ and $a\in A_{n-1}$ be as in the definition of ∂_*. Then $i_*\partial_*[c]=i_*[a]$, but by definition $i(a)=\partial(b)$ which is a boundary and hence trivial in $H_n(B_\bullet)$. This shows $i_*[a]=0$, so $i_*\partial_*=0$. Next, $j_*i_*=(j\circ i)_*=0_*=0$ because $0\to A_n\xrightarrow{i}B_n\xrightarrow{j}C_n\to 0$ is exact for all n. Finally, for any $[b]\in H_n(B_\bullet)$ we have $\partial(b)=0$. Set $c=j(b)$ so that $j_*[b]=[c]$. Then as above, there is some $a\in A_{n-1}$ satisfying $i(a)=\partial(b)=0$ and $\partial_*[c]=[a]$. Since i is injective, $i(a)=0$ implies $a=0$ and so $\partial_*j_*[b]=\partial_*[c]=[0]=0$. Therefore the sequence in homology is a complex.

It remains to show exactness at $H_n(A_\bullet),H_n(B_\bullet)$ and $H_n(C_\bullet)$. Suppose $i_*[a]=0$ for a cycle $a\in A_n$. Then $i(a)$ is a boundary, so $i(a)=\partial(b)$ for some $b\in B_n$. Setting $j(b)=c\in C_n$, naturality of ∂ gives us $\partial(c)=\partial(j(b))=j(\partial(b))=j(i(a))=0$ by exactness. This shows c defines a homology class $[c]\in H_n(C_\bullet)$, and moreover $\partial_*[c]=[a]$ by definition of the connecting homomorphism. Hence $\ker i_*\subseteq\operatorname{im}\partial_*$, so the sequence is exact $H_n(A_\bullet)$.

Now suppose $j_*[b]=0$ for a cycle $b\in B_n$. Then $j(b)$ is a boundary, so there is some $c\in C_n$ such that $\partial(c)=j(b)$. By surjectivity of j, there is some $b'\in B_n$ so that $j(b')=c$, and we have

$$j(b-\partial(b'))=j(b)-j(\partial(b'))=\partial(c)-\partial(j(b'))=\partial(c)-\partial(c)=0$$

by naturality of ∂. As $b-\partial(b')$ only differs from b by a boundary, $[b]=[b-\partial(b')]$ so replacing b with $b-\partial(b')$, we may assume $j(b)=0$. By exactness, there exists an $a\in A_n$ with $i(a)=b$. Notice that $i(\partial(a))=\partial(i(a))=\partial(b)=0$ but because i is injective, this implies $\partial(a)=0$. Hence $[a]$ is defined in $H_n(A_\bullet)$, so we have $i_*[a]=[b]$. This proves exactness at $H_n(B_\bullet)$.

Lastly, take a cycle $c\in C_n$ and suppose $\partial_*[c]=0$. Then as above, there are $b\in B_n$ and $a\in A_{n-1}$ satisfying $j(b)=c,i(a)=\partial(b)$ and $\partial_*[c]=[a]$. Now $[a]=0$, so a is a boundary, meaning $a=\partial(a')$ for some $a'\in A_n$. So by naturality of ∂, $\partial(i(a'))=i(\partial(a'))=i(a)=\partial(b)$, and thus $\partial(b-i(a'))=\partial(b)-\partial(b)=0$ so $b-i(a')$ defines a homology class $[b-i(a')]\in H_n(B_\bullet)$. Also, exactness of $0\to A_n\xrightarrow{i}B_n\xrightarrow{j}C_n\to 0$ implies $j(i(a'))=0$, so $j(b-i(a'))=j(b)-j(i(a'))=c-0=c$ which shows $j_*[b-i(a')]=[c]$. Hence $\ker\partial_*\subseteq\operatorname{im}j_*$, i.e. the sequence is exact at $H_n(C_\bullet)$. $\square$

The above proof is an example of a 'diagram chase': one sets up a commutative diagram with certain conditions, and then 'chases' elements around the diagram to verify other properties. Another useful result in homological algebra is the Five Lemma, which is also proved with a diagram chase:

Lemma 2.2.3 (Five Lemma). *Consider a commutative diagram with exact rows*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5
\end{tikzcd}$$

If f_2 and f_4 are isomorphisms, f_1 is surjective, and f_5 is injective, then f_3 is an isomorphism.

Proof. Suppose $a_3 \in A_3$ such that $f_3(a_3) = 0$. Set $a_4 = \alpha_3(a_3) \in A_4$. Then $\beta_3 f_3(a_3) = \beta_3(0) = 0$ so by commutativity of the third square, $f_4\alpha_3(a_3) = f_4(a_4) = 0$ as well. Since f_4 is an isomorphism, this makes $a_4 = 0$, that is, $\alpha_3(a_3) = 0$. By exactness of the top row, there is some $a_2 \in A_2$ with $\alpha_2(a_2) = a_3$. Now $f_3\alpha_2(a_2) = f_3(a_3) = 0$ so by commutativity of the second square, $\beta_2 f_2(a_2) = 0$. By exactness of the bottom row, there is a $b_1 \in B_1$ such that $\beta_1(b_1) = f_2(a_2)$. By surjectivity of f_1, we may choose $a_1 \in A_1$ such that $f_1(a_1) = b_1$. By commutativity of the first square, we have $f_2\alpha_1(a_1) = \beta_1 f_1(a_1) = \beta_1(b_1) = f_2(a_2)$. This implies $\alpha_1(a_1) = a_2$ since f_2 is an isomorphism, so in particular $a_3 = \alpha_2(a_2) = \alpha_2\alpha_1(a_1) = 0$ because the top row is a complex. This proves f_3 is injective.

On the other hand, suppose $b_3 \in B_3$. We must construct an $a_3 \in A_3$ mapping along f_3 to b_3. Set $b_4 = \beta_3(b_3)$ and lift along the isomorphism f_4 to a unique $a_4 \in A_4$ with $f_4(a_4) = b_4$. Notice that $\beta_4 f_4(a_4) = \beta_4(b_4) = \beta_4\beta_3(b_3) = 0$ because the bottom row is a complex, so by commutativity of the fourth square, $f_5\alpha_4(a_4) = 0$ as well. Now f_5 is injective, so this implies $\alpha_4(a_4) = 0$. Since the top row is exact, there is some $a_3 \in A_3$ with $\alpha_3(a_3) = a_4$. By commutativity of the third square, $b_4 = f_4(a_4) = f_4\alpha_3(a_3) = \beta_3 f_3(a_3)$. In particular if we set $b_3' = f_3(a_3)$, we have $\beta_3(b_3 - b_3') = \beta_3(b_3) - \beta_3 f_3(a_3) = b_4 - b_4 = 0$. By exactness of the bottom row, there is a $b_2 \in B_2$ with $\beta_2(b_2) = b_3 - b_3'$. Since f_2 is an isomorphism, take the unique lift $a_2 \in A_2$ satisfying $f_2(a_2) = b_2$. Now we have $f_3\alpha_2(a_2) = \beta_2 f_2(a_2) = \beta_2(b_2) = b_3 - b_3'$. Finally, observe that $f_3(\alpha_2(a_2) + a_3) = f_3\alpha_2(a_2) + f_3(a_3) = b_3 - b_3' + b_3' = b_3$. Therefore f_3 is surjective, and hence an isomorphism. □

We use the homology of a chain complex to construct three other versions of singular homology. In the next section, we will see that all of these versions of homology satisfy certain underlying axioms of a generalized homology theory.

Let X be a topological space and $A \subset X$ a subset. We denote this by a pair (X, A). The inclusion $i : A \hookrightarrow X$ induces a subgroup inclusion $i_* : \Delta_n(A) \hookrightarrow \Delta_n(X)$ for each $n \in \mathbb{Z}$, and therefore a chain map $i_* : \Delta_\bullet(A) \to \Delta_n(X)$. Define the *relative chain group* $\Delta_\bullet(X, A) = \bigoplus_{n\in\mathbb{Z}} \Delta_n(X, A)$, where $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$ is a quotient of abelian groups. This determines a short exact sequence of chain complexes

$$0 \to \Delta_\bullet(A) \xrightarrow{i} \Delta_\bullet(X) \xrightarrow{j} \Delta_\bullet(X, A) \to 0.$$

Definition. *The* n**th relative homology group** *of a pair (X, A) is the nth homology of this chain complex:*

$$H_n(X, A) := H_n(\Delta_\bullet(X, A)).$$

Theorem 2.2.4 (Long Exact Sequence in Relative Homology)**.** *For a pair (X, A), there is a long exact sequence*

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

Proof. Apply Theorem 2.2.2 to the short exact sequence

$$0 \to \Delta_\bullet(A) \xrightarrow{i} \Delta_\bullet(X) \xrightarrow{j} \Delta_\bullet(X, A) \to 0.$$

□

Next, let G be an abelian group with singular homology complex $\Delta_\bullet(X)$. Tensoring with G gives a new chain complex $\Delta_\bullet(X) \otimes G = \bigoplus_{n\in\mathbb{Z}}(\Delta_n(X) \otimes G)$.

Definition. *For a space X and an abelian group G, we define the* n**th homology of** X **with coefficients in** G *by*

$$H_n(X; G) := H_n(\Delta_\bullet(X) \otimes G).$$

Lemma 2.2.5. *If R is a commutative ring, then any homology group $H_n(X; R)$ for a space X is an R-module.*

If (X, A) is a pair, the short exact sequence $0 \to \Delta_n(A) \to \Delta_n(X) \to \Delta_n(X, A) \to 0$ is split for every $n \in \mathbb{Z}$, so tensoring with G preserves exactness:

$$0 \to \Delta_n(A) \otimes G \to \Delta_n(X) \otimes G \to \Delta_n(X, A) \otimes G \to 0.$$

Definition. *The* n**th relative homology with coefficients** *in G for the pair (X, A) is defined as the homology of this exact sequence:*

$$H_n(X, A; G) := H_n(\Delta_\bullet(X, A) \otimes G).$$

Theorem 2.2.6 (Long Exact Sequence for Homology with Coefficients)**.** *Let (X, A) be a pair of topological spaces and G an abelian group. Then there is a long exact sequence*

$$\cdots \to H_{n+1}(X, A; G) \to H_n(A; G) \to H_n(X; G) \to H_n(X, A; G) \to H_{n-1}(A; G) \to \cdots$$

Proof. Apply Theorem 2.2.2 to the exact sequence

$$0 \to \Delta_\bullet(A) \otimes G \to \Delta_\bullet(X) \otimes G \to \Delta_\bullet(X, A) \otimes G \to 0.$$

□

Let X be a nonempty space and let $P = \{x_0\}$ be a point. Then there is a unique map $\varepsilon : X \to P$ sending everything to x_0, and this induces a map on homology, $\varepsilon_* : H_\bullet(X) \to H_\bullet(P)$. For a fixed $x \in X$, the inclusion $i : P \to X, x_0 \mapsto x$, is a right inverse to ε, i.e. the composition $P \xrightarrow{i} X \xrightarrow{\varepsilon} P$ is the identity. Hence $\varepsilon_* \circ i_* = id$ so ε_* is surjective.

Definition. *For a space X, the* n**th reduced homology** *is defined by $\widetilde{H}_n(X) = \ker \varepsilon_*$.*

Clearly $\widetilde{H}_n(X) = H_n(X)$ when $n > 0$. Further, $\varepsilon_* : H_0(X) \to \mathbb{Z}$ coincides with the map induced by the augmentation map. In particular, Proposition 2.1.8 implies that if X is path-connected, then $\widetilde{H}_0(X) = 0$.

Definition. *A space X is* **acyclic** *if $\widetilde{H}_n(X) = 0$ for all n.*

2.3 The Eilenberg-Steenrod Axioms

In this section we introduce five axioms for a general homology theory. We will show that singular homology satisfies these axioms, and derive important homology calculations from the axioms themselves.

Definition (Eilenberg-Steenrod Axioms)**.** *A* **homology theory** *is a functor H that assigns to each pair (X, A) of topological spaces a graded abelian group $(H_\bullet(X, A), \partial)$ such that H is natural with respect to maps of pairs $f : (X, A) \to (Y, B)$, i.e. there exists an* **induced map** *$f_* : H_\bullet(X, A) \to H_\bullet(Y, B)$ satisfying $Hf_* = f_*H$, and the following axioms hold:*

(1) (Homotopy) If $f, g : (X, A) \to (Y, B)$ are homotopic maps then $f_* = g_*$.

(2) (Exactness) For each pair of inclusions $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$, there is a long exact sequence

$$\cdots \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

(3) (Excision) For any pair (X, A) and any open set $U \subseteq X$ such that $\overline{U} \subseteq A$, the inclusion $(X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism on homology.

(4) (Dimension) If $P = \{x\}$ is a point space, then $H_n(P) \neq 0$ only if $n = 0$. The group $H_0(P)$ is called the **coefficient group** *of the homology theory H.*

(5) (Additivity) If $(X_\alpha)_{\alpha\in I}$ is an arbitrary collection of topological spaces then

$$H_\bullet\left(\coprod_{\alpha\in I} X_\alpha\right) = \bigoplus_{\alpha\in I} H_\bullet(X_\alpha).$$

As one may notice in the statement of the axioms, we typically replace the notation $H_n(X, \varnothing)$ with $H_n(X)$, thereby simplifying many expressions.

An important result is that any homology theory satisfying the Eilenberg-Steenrod axioms is a homotopy invariant.

Proposition 2.3.1. *If (X, A) and (Y, B) are homotopy equivalent pairs of topological spaces, then $H_\bullet(X, A) \cong H_\bullet(Y, B)$ as abelian groups.*

Proof. Let H be a homology theory and suppose $f : (X, A) \to (Y, B)$ is a homotopy equivalent, with $g : (Y, B) \to (X, A)$ such that $G \circ F \simeq id_{(X,A)}$ and $F \circ G \simeq id_{(Y,B)}$. Then since H is a functor, $g_* \circ f_* = (g \circ f)_* = (id_{(X,A)})_* = 1$ and $f_* \circ g_* = (f \circ g)_* = (id_{(Y,B)})_* = 1$ so $f_* : H_\bullet(X, A) \to H_\bullet(Y, B)$ is an isomorphism. □

As with singular homology, we can define a reduced homology from a general homology theory as follows. Let $P = \{x_0\}$ be a point and let X be any nonempty space, with the unique map $\varepsilon : X \to P$ sending all elements of X to x_0.

Definition. *Let H be a homology theory. The* **reduced homology** *of X is the graded abelian group $\widetilde{H}_\bullet(X) = \ker \varepsilon_*$, where $\varepsilon_* : H_\bullet(X) \to H_\bullet(P)$ is the induced map on homology.*

Remark. For $n > 0$, $\widetilde{H}_n(X) = H_n(X)$ as in the singular theory. For $n = 0$, we have $\widetilde{H}_0(X) \oplus G \cong H_0(X)$, where $G = H_0(P)$ is the coefficient group of the homology theory. However, this isomorphism is not canonical.

Theorem 2.3.2. *Given a pair of spaces (X, A), the following diagram commutes:*

$$\begin{tikzcd}
 & & \widetilde{H}_0(A) \arrow[r] \arrow[d] & \widetilde{H}_0(X) \arrow[d] \arrow[dr] & & \\
\cdots \arrow[r] & H_1(X,A) \arrow[r] & H_0(A) \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & H_0(X,A) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & H_1(P,P) \arrow[r] & H_0(P) \arrow[r] & H_0(P) \arrow[r] & H_0(P,P) = 0 &
\end{tikzcd}$$

Proof. Diagram chase. □

Corollary 2.3.3. *If X is a contractible space, $\widetilde{H}_n(X) = 0$ for all n and hence $\widetilde{H}_\bullet(X) \cong H_\bullet(X)$.*

Theorem 2.3.4. *Let X be a Hausdorff space and $x_0 \in X$ a point having a closed neighborhood N of which $\{x_0\}$ is a strong deformation retract. Let Y be a Hausdorff space and $y_0 \in Y$. If $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is the wedge, or one-point union, of X and Y (as a subspace of $X \times Y$), then the inclusion maps induce isomorphisms*

$$\widetilde{H}_i(X) \oplus \widetilde{H}_i(Y) \longrightarrow \widetilde{H}_i(X \vee Y)$$

for each i, whose inverse is induced by the projections of $X \vee Y$ to each of X and Y.

Proof. Set $Z = X \vee Y$ and consider the pair of spaces (Z, X). The long exact sequence in reduced homology is

$$\cdots \to \widetilde{H}_{n+1}(Z, X) \to \widetilde{H}_n(X) \to \widetilde{H}_n(Z) \to \widetilde{H}_n(Z, X) \to \cdots$$

By the excision axiom, the inclusion $(Z \smallsetminus (X \smallsetminus N), N) \subset (Z, X)$ induces an isomorphism on homology: $\widetilde{H}_*(Z \smallsetminus (X \smallsetminus N), N) \cong \widetilde{H}_*(Z, X)$. Since N deformation retracts onto $\{x_0\}$, $(Z \smallsetminus (X \smallsetminus N), N) = (N \vee Y, N)$ is homotopy equivalent to $(Y, \{x_0\})$ so the homotopy axiom gives an isomorphism on homology: $\widetilde{H}_*(N \vee Y, N) \cong \widetilde{H}_*(Y, \{x_0\})$. These facts allow us to replace $\widetilde{H}_n(Z, X)$ with $\widetilde{H}_n(Y)$.

Now consider $i_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Z)$. This is the map induced from inclusion $i : X \hookrightarrow X \vee Y$ which has a natural left inverse $p : X \vee Y \to X$. Then $p \circ i = id_X$ so $p_* \circ i_* = (p \circ i)_* = (id_X)_* = 1$ so i_* is injective. Similarly, $\widetilde{H}_n(Z) \to \widetilde{H}_n(Y)$ is induced by $q : X \vee Y \to Y$, and

$p \circ j = id_Y$ where $j : Y \hookrightarrow X \vee Y$ is the natural inclusion. So $q_* \circ j_* = (q \circ j)_* = (id_Y)_* = 1$, showing q_* is surjective. Thus we have a short exact sequence

$$0 \to \widetilde{H}_n(X) \xrightarrow{i_*} \widetilde{H}_n(Z) \xrightarrow{q_*} \widetilde{H}_n(Y) \to 0.$$

The inclusion $j : Y \hookrightarrow Z$ provides a section $j_* : \widetilde{H}_n(Y) \to \widetilde{H}_n(Z)$ of q_*, so the sequence is split. Therefore $\widetilde{H}_n(X \vee Y) = \widetilde{H}_n(Z) = \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$ for all n. □

We now proceed to give some elementary homology calculations of spheres and disks in a general homology theory, which will become useful for more involved computations later. Note that once we prove singular homology satisfies the Eilenberg-Steenrod axioms, these computations will hold for singular homology.

Theorem 2.3.5. *Let H be a homology theory and $G = H_0(P)$ its coefficient group. For any $n \geq 0$,*

$$\widetilde{H}_i(S^n) = \begin{cases} G, & i = n \\ 0, & i \neq n \end{cases} \tag{S_n}$$

$$H_i(D^n, S^{n-1}) = \begin{cases} G, & i = n \\ 0, & i \neq n \end{cases} \tag{D_n}$$

$$H_i(S^n, D^n_+) = \begin{cases} G, & i = n \\ 0, & i \neq n \end{cases} \tag{R_n}$$

where D^n_+ is the closed upper hemisphere of D^n.

Proof. We prove the statements $(S_n), (D_n)$ and (R_n) recursively for all $n \geq 0$. First, (R_0) follows for the dimension and excision axioms, using $H_i(S^0, D^0_+) \cong H_i(P)$. Consider the inclusion $D^n_+ \hookrightarrow S^n$. Since D^n_+ is contractible, Corollary 2.3.3 shows that the long exact sequence in the pair (S_n, D^n_+) becomes

$$0 = \widetilde{H}_i(D^n_+) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow H_i(S^n, D^n_+) \longrightarrow \widetilde{H}_{i-1}(D^n_+) = 0.$$

So $\widetilde{H}_i(S^n) \cong H_i(S^n, D^n_+)$ for all i by exactness. This proves $(R_n) \iff (S_n)$. Next, by the excision axiom and Proposition 2.3.1, we have

$$H_i(S^n, D^n_+) \cong H_i(S^n \setminus U, D^n_+ \setminus U) \cong H_i(D^n_-, S_{n-1})$$

where U is a small neighborhood of the north pole in S^n and D^n_- is the closed lower hemisphere of D^n. Hence $(D_n) \iff (R_n)$. Finally, the long exact sequence in the pair (D^n, S^{n-1}) is

$$0 = \widetilde{H}_i(D^n) \longrightarrow H_i(D^n, S^{n-1}) \longrightarrow \widetilde{H}_{i-1}(S^{n-1}) \longrightarrow \widetilde{H}_{i-1}(D^n) = 0,$$

since D^n is contractible. By exactness, $H_i(D^n, S^{n-1}) \cong \widetilde{H}_{i-1}(S^{n-1})$ and so we have $(D_n) \iff (S_{n-1})$. Since the base (R_0) was established, all statements $(S_n), (D_n), (R_n)$ now follow by induction. □

Corollary 2.3.6. *D^n does not retract onto S^{n-1} for any $n \geq 1$.*

Proof. If such a retraction exists, it is a map $r : D^n \to S^{n-1}$ such that $r \circ i \simeq id_{S^{n-1}}$ where $i : S^{n-1} \hookrightarrow D^n$ is the inclusion. Then by the homotopy axiom, $r_* \circ i_* = (r \circ i)_* = (id_{S^{n-1}})_* = 1$, that is, the composition $H_\bullet(S^{n-1}) \xrightarrow{i_*} H_\bullet(D^n) \xrightarrow{r_*} H_\bullet(S^{n-1})$ is the identity. However, by Corollary 2.3.3 $H_\bullet(D^n) = 0$ since D^n is contractible, and by Theorem 2.3.5, $H_\bullet(S^{n-1}) = G \neq 0$, so we obtain a contradiction. Hence no such retraction exists. □

Corollary 2.3.7 (Brouwer's Fixed Point Theorem)**.** *Any smooth map $f : D^n \to D^n$ has a fixed point.*

Proof. Suppose there is a smooth map $f : D^n \to D^n$ without a fixed point. Then for every $x \in D^n$, $x \neq f(x)$ so define L_x to be the line in $\mathbb{R}^n$ containing the distinct points $x, f(x)$. Define the map $g : D^n \to S^{n-1}$ by setting $g(x)$ to be the intersection of L_x and S^{n-1} in the closest point to x. Then g is smooth and for any boundary point $x \in S^{n-1}$, $g(x)$ is precisely x. Thus g is a retract of D^n onto S^{n-1}, contradicting Corollary 2.3.6. Hence f must have a fixed point. □

Definition. *For a map $f : S^n \to S^n$, the* **degree** *of f is defined to be the induced map $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$.*

Example 2.3.8. For singular homology, or any homology theory where $H_n(S^n) = \mathbb{Z}$, the degree of a map $f : S^n \to S^n$ is a homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$ and is therefore uniquely determined by an integer $k \in \mathbb{Z}$ such that $f(1) = k$. This is often what is referred to as degree. We will see that this notion agrees with the definition of degree from differential topology.

Lemma 2.3.9. *Reflections of the sphere S^n have degree -1.*

Proof. Viewing $S^n \subseteq \mathbb{R}^{n+1}$, a reflection is of the form $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, (x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$. This reduces to a self-map of S^n, say $r = f|_{S^n} : S^n \to S^n$. We prove that $\deg r = -1$ by induction. For $n = 0$, $H_0(S^0) = \mathbb{Z}^2$ and $\widetilde{H}_0(S^0) = (t, -t)\mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}$. The induced map $r_* : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)$ switches t and $-t$, so r_* must be multiplication by -1. To induct, assume the statement holds for reflections of S^k for $k < n$. Define the hemispheres

$$\begin{aligned} D^n_+ &= \{(x_0, \ldots, x_n) \in D^n \mid x_n \geq 0\} \\ \text{and} \quad D^n_- &= \{(x_0, \ldots, x_n) \in D^n \mid x_n \leq 0\}. \end{aligned}$$

These are invariant under $r : (x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$ so we have a commutative diagram

$$\begin{tikzcd} \widetilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "r_*"'] & H_n(S^n, D^n_+) \arrow[d, "f_*"'] & H_n(D^n_-, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "f_*"'] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[d, "r_*"'] \\ \widetilde{H}_n(S^n) \arrow[r, "\cong"'] & H_n(S^n, D^n_+) & H_n(D^n_-, S^{n-1}) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \widetilde{H}_{n-1}(S^{n-1}) \end{tikzcd}$$

where the horizontal arrows are isomorphisms from Theorem 2.3.5. Since the diagram commutes, the left map $r_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ is equal to the right map $\widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ so by induction, $\deg r = -1$. $\square$

Corollary 2.3.10. *The antipodal map $a : S^n \to S^n, x \mapsto -x$ has degree $(-1)^{n+1}$. In particular, if n is even, a is not homotopic to the identity.*

Proof. By definition $a = r_0 \circ r_1 \circ \cdots \circ r_n$, where $r_i : (x_0, \ldots, x_i, \ldots, x_n) \mapsto (x_0, \ldots, -x_i, \ldots, x_n)$ is the ith rotation. Thus by functoriality of the induced map, $a_* = (r_0)_* \circ (r_1)_* \circ \cdots \circ (r_n)_*$. The result then follows from Lemma 2.3.9. Finally, if a were homotopic to the identity on S^n then by the homotopy axiom, $a_* = id_* = 1$. However, the identity has degree 1 so this is impossible when n is even. $\square$

Corollary 2.3.11. *If n is even, then any map $f : S^n \to S^n$ has a point $x \in S^n$ such that $f(x) = \pm x$.*

Proof. Suppose $f(x) \neq x$ for all $x \in S^n$. Then the straight line between $f(x)$ and x does not pass through the origin in $\mathbb{R}^{n+1}$ for any $x \in S^n$, so we can construct a well-defined homotopy

$$F(x,t) = \frac{tf(x) + (1-t)x}{||tf(x) + (1-t)x||} : S^n \times [0,1] \longrightarrow S^n$$

from the identity on S^n to f. Similarly, if $f(x) \neq -x$ for all $x \in S^n$, we can construct a homotopy

$$G(x,t) = \frac{tf(x) + (t-1)x}{||tf(x) + (t-1)x||} : S^n \times [0,1] \longrightarrow S^n$$

from the antipodal map to f. Hence the antipodal map and the identity have the same degree by the homotopy axiom, but since n is even, Corollary 2.3.10 gives us $\deg a = (-1)^{n+1} = -1$, a contradiction. Hence there must exist a point $x \in S^n$ such that $f(x) = \pm x$. $\square$

From this, we obtain the famous 'hairy ball theorem'.

Corollary 2.3.12 (Hairy Ball Theorem)**.** *If n is even, S^n has no nonvanishing tangent vector field.*

Proof. If ξ_x is such a tangent vector field, the map $f(x) = \frac{\xi_x}{||\xi_x||}$ is a well-defined function $S^n \to S^n$ but has no point such that $f(x) = \pm x$, contradicting Corollary 2.3.11. $\square$

Next, we connect the degree of a map as defined above to the differential notion of degree.

Lemma 2.3.13. *Let $A \in GL_n(\mathbb{R})$ be a matrix representing a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Then A induces a map $f : S^n \to S^n$ of the unit n-sphere such that* $\deg f = \operatorname{sign}(\det A)$.

Proof. View $S^n \cong \mathbb{R}^n/\mathbb{Z}^n$. Since A represents a homeomorphism, we get an induced map $f : S^n \to S^n$. Further, det is multiplicative so it's enough to check the result when A is an elementary matrix. If A is a row replacement matrix, $\det A = 1$ and there is a homotopy from f to the identity map id_{S^n}, so $\deg f = 1$. Likewise, if A is a scaling matrix, $\det A = c$ and $\deg f = 1$ or -1 depending on the sign of c. Finally, if A is a row-swap matrix, $\det A = -1$ but f is homotopic to a reflection, so $\deg f = -1$ by Lemma 2.3.9. $\square$

Proposition 2.3.14. *If $f : S^n \to S^n$ is a smooth map and $p \in S^n$ is any regular value of f such that $f^{-1}(p) = \{q\}$, then $\deg f = \operatorname{sign}(\det D_q f)$ where the determinant of the differential $D_q f$ is taken with respect to bases of $T_q S^n$ and $T_p S^n$ possibly differing by a rotation.*

Proof. By composing with a rotation of S^n, we may assume $p = q$. Further, composing with the linear map $(D_q f)^{-1}$, we may assume $D_q f = id_{T_q S^n}$, so $\det D_q f = 1$. By differential topology, f is the identity on a neighborhood of q, which is a disk D. On the complementary disk $D' = S^n \smallsetminus D$, f is the identity on $\partial D'$ and so is homotopic to the identity on all of D'. Hence $\deg f = \deg id_{S^n} = 1 = \det D_q f$ as required. $\square$

Proposition 2.3.15. *Let $X = \bigvee^k S^n$ be the wedge of k copies of the n-sphere and consider the map $f : \bigoplus_{i=1}^k H_n(S^n) \to H_n(X)$ induced by the sum of the inclusions $\{f_i : S^n \hookrightarrow X\}_{i=1}^k$. Then f is an isomorphism with inverse given by the sum $\sum_{i=1}^k (p_i)_*$, where $p_i : X \to S^n$ are the projection maps.*

Proof. Follows from Theorem 2.3.4 and induction on k. $\square$

Theorem 2.3.16. *Let $f : S^n \to (Y, y_0)$ be a pointed map and suppose there are open sets $E_1, \ldots, E_k \subseteq S^n$, each homeomorphic to a disk, such that $f(S^n \smallsetminus (E_1 \cup \cdots \cup E_k)) = y_0$, that is, f is constant off of $E_1, \ldots, E_k$. Then f factors through the quotient space $S^n / S^n \smallsetminus (E_1 \cup \cdots \cup E_k) \cong \bigvee^k S^n$. Further, if $f_j : S^n \to Y, 1 \le j \le k$, is the map equal to f on E_j and constant on $S^n \smallsetminus E_j$, then $f_* = \sum_{j=1}^k (f_j)_* : H_n(S^n) \to H_n(Y)$.*

Proof. Factor f through the wedge product using the universal property of quotient maps:

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & \bigvee^k S^n \arrow[ur, "h"'] & \end{tikzcd}$$

Let $p_j : \bigvee^k S^n \to S^n$ be the jth projection map and $i_j : S^n \hookrightarrow \bigvee^k S^n$ be the jth inclusion map. Then by Proposition 2.3.15, $\sum_{j=1}^k (i_j)_* \circ (p_j)_* = 1$ on $H_n(S^n)$. Note that f_j is the composition $f_j : S^n \xrightarrow{g} \bigvee^k S^n \xrightarrow{p_j} S^n \xrightarrow{i_j} \bigvee^k S^n \xrightarrow{h} Y$, so by naturality of the induced maps,

$$\sum_{j=1}^k (f_j)_* = \sum_{j=1}^k h_* \circ (i_j)_* \circ (p_j)_* \circ g_* = \sum_{j=1}^k h_* \circ g_* = h_* \circ g_* = f_*.$$

$\square$

Corollary 2.3.17. *Let $f : S^n \to S^n$ be smooth and take $p \in S^n$ to be any regular value of f. Then $\deg f$ equals the sum of signs of Jacobian determinants over the fibre $f^{-1}(p) = \{q_1, \ldots, q_k\}$.*

Proof. Let D be a small disk around p such that $f^{-1}(D) = \coprod_{j=1}^k D_j$, where $D_1, \ldots, D_k$ are a collection of disjoint disks with $q_j \in D_j$ for each $1 \le j \le k$. Compose f with the map which collapses $S^n \setminus D$ to a point to obtain a map $g : S^n \to S^n$. Since the collapsing map is homotopic to the identity on S^n, it has degree 1 and therefore $\deg g = \deg f$. Finally, f maps $S^n \setminus D$ to a point, so Theorem 2.3.16 applies. The result then follows from Proposition 2.3.14. □

This result shows that the algebraic notion of degree agrees with the differential topological one. Furthermore, the algebraic definition of degree does not depend on the homology theory chosen, since we only appealed to the Eilenberg-Steenrod axioms in the sequence of proofs.

Example 2.3.18. Let X be the manifold with boundary obtained by removing a small open disk from the 2-torus. The inclusions $i : \partial X \hookrightarrow X$ and $j : (X, \varnothing) \hookrightarrow (X, \partial X)$ induce the following long exact sequence in homology:

$$\begin{aligned} \cdots \to H_n(\partial X) \to H_n(X) \to H_n(X, \partial X) \to \cdots \\ \to H_2(\partial X) \to H_2(X) \to H_2(X, \partial X) \to \\ \to H_1(\partial X) \to H_1(X) \to H_1(X, \partial X) \to \\ \to H_0(\partial X) \to H_0(X) \to H_0(X, \partial X) \to 0. \end{aligned}$$

We will compute the relative homology groups $H_n(X, \partial X)$ for all $n \geq 0$. Note that X retracts onto the wedge of two circles and ∂X is itself a circle, so by Theorems 2.3.5 and 2.3.4, $H_n(X) = H_n(\partial X) = 0$ for $n \geq 2$. It follows by exactness that $H_n(X, \partial X) = 0$ for $n \geq 3$, so it suffices to consider the bottom three rows in the sequence above.

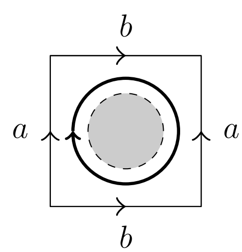
Now, H_0 is free abelian on path components, so $H_0(X) = H_0(\partial X) = \mathbb{Z}$. In general, we have $H_0(X, \partial X) = 0$, e.g. by considering reduced homology: $\widetilde{H}_0(X, \partial X) \cong H_0(X, \partial X)$ and the last row of the long exact sequence in reduced homology is $\to 0 \to 0 \to \widetilde{H}_0(X, \partial X) \to 0$ so we have that $H_0(X, \partial X) = 0$ as well. Since X retracts onto the wedge of two circles and ∂X is a circle, we get $H_1(X) = \mathbb{Z}^2$ and $H_1(\partial X) = \mathbb{Z}$, by Examples 4 and 1 in Section 2.1. Filling in these terms, we get an exact sequence

$$0 \to H_2(X, \partial X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}^2 \to H_1(X, \partial X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to 0 \to 0.$$

The map i_* is the map induced from $i : \partial X \hookrightarrow X$, which is an isomorphism $H_0(\partial X) \to H_0(X)$, so by exactness the image of the map out of $H_1(X, \partial X)$ is 0. We therefore have only two terms left to compute:

$$0 \to H_2(X, \partial X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}^2 \to H_1(X, \partial X) \to 0. \qquad (*)$$

We need to understand how the generator of $\mathbb{Z} = H_1(\partial X)$ maps into $H_1(X)$. Explicitly, i takes the circle ∂X to the boundary of X, so on the level of fundamental groups, a loop α generating $\pi_1(\partial X) = \langle \alpha \rangle \cong \mathbb{Z}$ maps into $\pi_1(X)$ by the following:



Recall that the wedge of two circles has fundamental group F_2, the free group on two generators. If $\pi_1(X) = \langle a, b \mid \rangle \cong F^2$ then $i_*\alpha = aba^{-1}b^{-1}$ by the figure above. By Proposition 2.1.17, i_* commutes with the Hurewicz homomorphism, so on the level of homology, $i_*\alpha = [a, b] = 0 \in H_1(X)$. Hence i_* is the zero map, so we can divide the exact sequence $(*)$ into two sequences:

$$0 \to H_2(X, \partial X) \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z}^2 \to H_1(X, \partial X) \to 0.$$

These give isomorphisms $H_2(X, \partial X) \cong \mathbb{Z}$ and $H_1(X, \partial X) \cong \mathbb{Z}^2$. To summarize,

$$H_n(X, \partial X) = \begin{cases} 0, & n = 0 \\ \mathbb{Z}^2, & n = 1 \\ \mathbb{Z}, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

Example 2.3.19. Let M is the Möbius band. We will compute the relative homology groups $H_n(M, \partial M)$ for all n. Consider the long exact sequence in relative homology:

$$\begin{aligned} \cdots \to H_n(\partial M) \to H_n(M) \to H_n(M, \partial M) \to \cdots \\ \to H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to \\ \to H_1(\partial M) \to H_1(M) \to H_1(M, \partial M) \to \\ \to H_0(\partial M) \to H_0(M) \to H_0(M, \partial M) \to 0. \end{aligned}$$

Here M retracts onto a circle and ∂M is itself a circle, so $H_n(M) = H_n(\partial M) = 0$ for $n \geq 2$, $H_1(M) = H_1(\partial M) = \mathbb{Z}$ and $H_0(M) = H_0(\partial M) = \mathbb{Z}$. As in Example 2.3.18, $H_n(M, \partial M) = 0$ for $n \geq 3$ by exactness and $H_0(M, \partial M) = 0$ by considering reduced homology in the bottom row of the sequence. Also, $i_* : H_0(\partial M) \to H_0(M)$ is the zero map as before. Therefore we are left with the following terms to compute:

$$0 \to H_2(M, \partial M) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to H_1(M, \partial M) \to 0.$$

Here the inclusion $i : \partial M \hookrightarrow M$ may be composed with the deformation retract $r : M \to \partial M$ to obtain the two-fold cover of S^1 by itself: it is obvious that every point in S^1 has a two-sheeted fibre, and the fact that this is a cover follows from the definition of a deformation retract. So that $(r \circ i)_* = r_* \circ i_*$ represents a degree 2 map, and since r_* is an isomorphism on homology (by the homotopy axiom), this means i_* is multiplication by 2. In particular, i_* is injective, so the sequence above becomes

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

To summarize the calculations, we have

$$H_n(M, \partial M) = \begin{cases} 0, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & n \geq 2. \end{cases}$$

Example 2.3.20. Our goal is to compute ordinary homology $H_n(\mathbb{R}P^2)$ and homology with coefficients $H_n(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$ for all n for the projective plane. We may construct $\mathbb{R}P^2$ by gluing a disk D^2 along its boundary $\partial D^2 \cong S^1$ to the Möbius band M along its boundary $\partial M \cong S^1$. Then the pair $(\mathbb{R}P^2, D^2)$ induces a long exact sequence

$$\begin{aligned}
\cdots \to H_n(D^2) \to H_n(\mathbb{R}P^2) \to H_n(\mathbb{R}P^2, D^2) \to \cdots\\
\to H_2(D^2) \to H_2(\mathbb{R}P^2) \to H_2(\mathbb{R}P^2, D^2) \to\\
\to H_1(D^2) \to H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2, D^2) \to\\
\to H_0(D^2) \to H_0(\mathbb{R}P^2) \to H_0(\mathbb{R}P^2, D^2) \to 0.
\end{aligned}$$

Let U be the interior of D^2. Then the excision axiom gives us an isomorphism $H_n(\mathbb{R}P^2, D^2) \cong H_n(\mathbb{R}P^2 \smallsetminus U, D^2 \smallsetminus U)$ for each n. Now $\mathbb{R}P^2 \smallsetminus U$ deformation retracts to the copy of the Möbius band sitting inside $\mathbb{R}P^2$ and $D^2 \smallsetminus U = \partial D^2 = S^1 \cong \partial M$, so by Example 2.3.19 and the homotopy axiom,

$$H_n(\mathbb{R}P^2 \smallsetminus U, D^2 \smallsetminus U) \cong H_n(M, \partial M) = \begin{cases} 0, & n = 0\\ \mathbb{Z}/2\mathbb{Z}, & n = 1\\ 0, & n \geq 2.\end{cases}$$

We also know that since D^2 is contractible, $H_n(D^2)$ is 0 for $n \geq 1$ and is $\mathbb{Z}$ for $n = 0$. In particular, we have $0 \to H_n(\mathbb{R}P^2) \to 0$ in the nth row of the long exact sequence for every $n \geq 2$, so $H_n(\mathbb{R}P^2) = 0$ for $n \geq 2$. Filling in the rest of the terms we know, we are left with the following:

$$0 \to H_1(\mathbb{R}P^2) \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \to H_0(\mathbb{R}P^2) \to 0 \to 0.$$

Since the only finite subgroup of $\mathbb{Z}$ is $\{0\}$, the map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be the zero map. Therefore our exact sequence becomes two exact sequences:

$$0 \to H_1(\mathbb{R}P^2) \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to H_0(\mathbb{R}P^2) \to 0.$$

These tell us $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $H_0(\mathbb{R}P^2) = \mathbb{Z}$. Altogether, we have determined

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n = 0\\ \mathbb{Z}/2\mathbb{Z}, & n = 1\\ 0 & n \geq 2.\end{cases}$$

Now for homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$, the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ induces the following long exact sequence:

$$\cdots \to H_n(\mathbb{R}P^2;\mathbb{Z}) \xrightarrow{2} H_n(\mathbb{R}P^2;\mathbb{Z}) \to H_n(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \to \cdots$$

Plugging in the calculations from above, we only have a few nonzero terms:

$$0 \to H_2(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \to H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to H_0(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \to 0.$$

Since multiplication by 2 is the zero map on $\mathbb{Z}/2\mathbb{Z}$ and is injective on $\mathbb{Z}$, we get the following exact sequences:

$$0 \to H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$
$$0 \to \mathbb{Z}/2\mathbb{Z} \to H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \to 0$$
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to H_0(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \to 0.$$

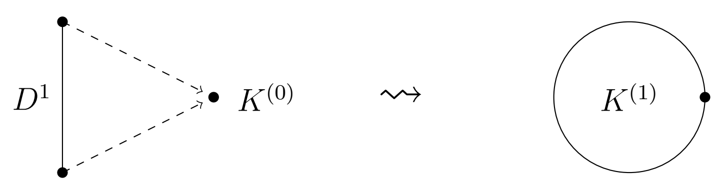
This shows that $H_0(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}), H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ and $H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In all, we have

$$H_n(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0, 1, 2 \\ 0, & n \geq 3. \end{cases}$$

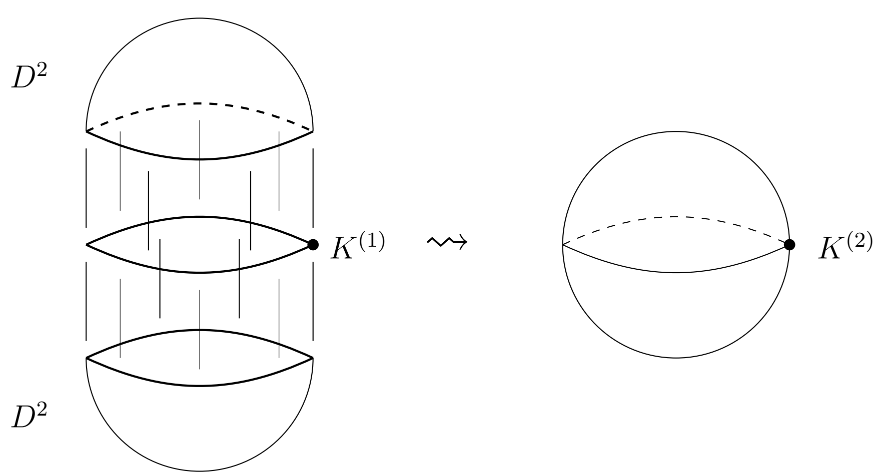
2.4 CW-Complexes

One of the most useful tools for computing homology is called cellular homology. To define this, we imitate the homology calculations for spheres and disks from Section 2.3 for a more general class of spaces called CW-complexes.

To motivate the definition of a CW-complex, consider two different constructions of the 2-sphere. In version 1, start with a point x_0 and call this structure $K^{(0)}$. Then glue a copy of the unit interval $D^1 = [0, 1]$ to $K^{(0)}$ by attaching both endpoints to x_0:



The result is a circle, which we denote by $K^{(1)}$. Finally, we glue a pair of disks D^2 to the existing space by identifying the circle with each of their boundaries $\partial D^2 = S^1$; call the result $K^{(2)}$:



The process described above is called a *CW-decomposition*, or CW-structure, for S^2. The various disks D^n comprising the structure of S^2 are called *CW-cells* and the total space

$\bigcup_{n=0}^2 K^{(n)}$ a *CW-complex.* An alternative CW-structure for the 2-sphere consists of two 0-cells, two 1-cells attached to each of the two 0-cells at their endpoints and two 2-cells attached to the resulting circle as above. This example is made formal in the definition below.

Definition. *A* **CW-complex** *is a space K that is an increasing union of subspaces $K = \bigcup_{n=0}^\infty K^{(n)}$, called* **skeleta***, which inductively satisfy the following construction:*

- *$K^{(0)}$ is a discrete set of points.*
- *For each $n \geq 1$, $K^{(n)} = K^{(n-1)} \bigcup_f \coprod_\sigma D_\sigma^n$, where $\{D_\sigma^n\}_{\sigma\in S}$ is an indexed collection of n-disks, called n-***cells***, $f_{\partial\sigma} : \partial D_\sigma^n \to K^{(n-1)}$ are some prescribed attaching maps and $f = \coprod_\sigma f_{\partial\sigma}$.*

K is made into a topological space by equipping the **weak topology***: a subset $U \subset K$ is open if and only if $U \cap K^{(n)}$ is open in $K^{(n)}$ for all $n \geq 0$.*

Definition. *A CW-complex K has* **dimension** *n if the largest dimension of any cell of K is n. If no such n exists, K is said to be* **infinite dimensional***.*

We will primarily concern ourselves with finite dimensional CW-complexes.

Remark. As a consequence of the definition of the weak topology, a map $g : K \to X$ on a CW-complex K is continuous if and only if $g|_{D_\sigma^n} : D_\sigma^n \to X$ is continuous for each cell D_σ^n.

Definition. *A* **subcomplex** *of a CW-complex K is a union of cells of K that is itself a CW-complex with the same attaching maps.*

Example 2.4.1. As described at the start of the section, S^2 can be made into a CW-complex with one 0-cell, one 1-cell and two 2-cells; or alternatively, with two 0-cells, two 1-cells and two 2-cells. Notice that in both cases, the circle S^1 is a subcomplex of S^2 equal to the union of the 0- and 1-cells. The pattern can be extended to describe a CW-structure on S^n for any n. A simpler CW-structure for S^n is to take one 0-cell x_0 and attach one n-cell by the unique map $\partial D^n \to x_0$.

CW-structures are incredibly useful for computing the homology groups of a wide array of spaces. Let H be a homology theory with coefficients in $\mathbb{Z}$ and fix a CW-complex K. In general, the quotient space $K^{(n)}/K^{(n-1)}$ is homeomorphic to a wedge sum of spheres $\bigvee_\sigma S^n$, one for each n-cell. This suggests using $H_n(K^{(n)}, K^{(n-1)})$ as a chain group in the construction of a chain complex for K. By Theorem 2.3.5 and the additivity axiom,

$$H_n\left(\coprod_\sigma D_\sigma^n, \coprod_\sigma \partial D_\sigma^n\right) = \bigoplus_\sigma H_n(D_\sigma^n, \partial D_\sigma^n) \cong \bigoplus_\sigma \mathbb{Z},$$

so the nth relative homology for the pair $(K^{(n)}, K^{(n-1)})$ is the free abelian group on the n-cells of K. Each n-cell comes equipped with a natural inclusion map $f_\sigma : (D_\sigma^n, \partial D_\sigma^n) \hookrightarrow (K^{(n)}, K^{(n-1)})$.

Lemma 2.4.2. *For all $n \geq 0$, $\bigoplus_\sigma (f_\sigma)_* : \bigoplus_\sigma H_n(D^n_\sigma, \partial D^n_\sigma) \to H_n(K^{(n)}, K^{(n-1)})$ is an isomorphism. Moreover, for all $i \neq n$, $H_i(K^{(n)}, K^{(n-1)}) = 0$.*

Proof. Let $f = \bigoplus_\sigma (f_\sigma)_* : H_\bullet(\coprod_\sigma D^n_\sigma, \coprod_\sigma \partial D^n_\sigma) \to H_\bullet(K^{(n)}, K^{(n-1)})$. Then we have a diagram:

$$\begin{tikzcd}
H_\bullet\left(\coprod_\sigma D^n_\sigma, \coprod_\sigma \partial D^n_\sigma\right) \arrow[r, "f"] \arrow[d, "\cong"'] & H_\bullet(K^{(n)}, K^{(n-1)}) \arrow[d, "\cong"] \\
H_\bullet\left(\coprod_\sigma D^n_\sigma, \coprod_\sigma D^n_\sigma \setminus \{0\}\right) \arrow[r] & H_\bullet\left(K^{(n)}, K^{(n)} \setminus \bigcup_\sigma f_\sigma(0)\right) \\
H_\bullet\left(\coprod_\sigma \operatorname{Int}(D^n_\sigma), \coprod_\sigma \operatorname{Int}(D^n_\sigma \setminus \{0\})\right) \arrow[u, "\cong"] \arrow[r, "\cong"'] & H_\bullet(K^{(n)} \setminus K^{(n-1)}, K^{(n)} \setminus (K^{(n-1)} \cup f_\sigma(0))) \arrow[u, "\cong"']
\end{tikzcd}$$

The diagram commutes by naturality of the induced maps. The top (downward) vertical arrows are isomorphisms by the homotopy axiom, while the bottom (upward) vertical arrows are isomorphisms by the excision axiom. Finally, the bottom arrow is an isomorphism because it is induced from a homeomorphism of the pairs of spaces. This implies f is an isomorphism. The statement for $i \neq n$ then follows immediately. $\square$

We will construct a chain complex whose nth chain group is $H_n(K^{(n)}, K^{(n-1)})$. To do so, we have:

Lemma 2.4.3. *There is a commutative diagram*

$$\begin{tikzcd}
H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_*"] & H_{n-1}(K^{(n-1)}) \\
\bigoplus_\sigma H_n(D^n_\sigma, \partial D^n_\sigma) \arrow[u, "\bigoplus f_\sigma"] \arrow[r, "\bigoplus \partial_*"] & \bigoplus_\sigma H_{n-1}(\partial D^{n-1}_\sigma) \arrow[u, "\bigoplus f_{\partial\sigma}"']
\end{tikzcd}$$

Proof. This comes from naturality and the exact sequences in homology for the pairs $(K^{(n)}, K^{(n-1)})$ and $(D^n_\sigma, \partial D^n_\sigma)$. $\square$

Definition. *Given a CW-complex K, its* **cellular chain complex** *is the chain complex $(C_\bullet(K), \partial^{cell})$, where $C_n(K) = H_n(K^{(n)}, K^{(n-1)})$ and*

$$\partial^{cell} : H_n(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_{n-1}(K^{(n-1)}) \xrightarrow{j_*} H_{n-1}(K^{(n-1)}, K^{(n-2)}).$$

Proposition 2.4.4. *For any CW-complex K, $(C_\bullet(K), \partial^{cell})$ is a chain complex.*

Proof. The long exact sequences for the pairs $(K^{(n)}, K^{(n-1)})$ and $(K^{(n-1)}, K^{(n-2)})$ give us a commutative diagram

$$\begin{tikzcd}
 & & H_{n-1}(K^{(n-2)}) \arrow[d] & \\
H_n(K^{(n)}) \arrow[r] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_*"] \arrow[dr, dashed, "\partial^{cell}"'] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[d, "j_{n-1}"] & H_{n-1}(K^{(n)}) \\
 & & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[d, "\partial_*"] \arrow[dr, "\partial^{cell}"] & \\
 & & H_{n-2}(K^{(n-2)}) \arrow[r, "j_{n-2}"] & H_{n-2}(K^{(n-2)}, K^{(n-3)})
\end{tikzcd}$$

Then $\partial^{cell} \circ \partial^{cell} = j_{n-2} \circ \partial_* \circ j_{n-1} \circ \partial_*$ but the column is exact, so $\partial_* \circ j_{n-1} = 0$. Hence $\partial^{cell} \circ \partial^{cell} = 0$. $\square$

Definition. *The* n**th cellular homology group** *of a CW-complex* K *is* $H_n(C_\bullet(K))$.

Remark. CW-complexes are universal in topology:

- Every topological manifold is homotopy equivalent to a CW-complex.
- More generally, every topological space is *weakly* homotopy equivalent to a CW-complex, i.e. it has the same homology theory. This means that every possible homology theory is realizable through a CW-complex.

As the next theorem shows, the utility of CW-complexes is that the cellular homology coincides with the ordinary homology of any space with a CW-structure.

Theorem 2.4.5. *Let* K *be a CW-complex. Then there is an isomorphism* $H_n(C_\bullet(K)) \cong H_n(K)$ *for each* n *and any homology theory* H.

Proof. By Lemma 2.4.2, $H_i(K^{(n)}, K^{(n-1)}) = 0$ whenever $i \neq n$. Fix $i \geq 1$ and consider $n = 0$. Then by the additivity and dimension axioms, $H_i(K^{(0)}) = 0$. Next assume $1 \leq n < i$ and assume the result holds for $n - 1$. Then the long exact sequence for the pair $(K^{(n)}, K^{(n-1)})$ is

$$0 = H_{i+1}(K^{(n)}, K^{(n-1)}) \to H_i(K^{(n-1)}) \to H_i(K^{(n)}) \to H_i(K^{(n)}, K^{(n-1)}) = 0$$

By the inductive hypothesis, $H_i(K^{(n-1)}) = 0$ so we get $H_i(K^{(n)}) = 0$. We now extend the commutative diagram from Proposition 2.4.4:

$$\begin{tikzcd}
 & & & H_{n-1}(K^{(n-2)}) = 0 \arrow[d] & \\
0 = H_n(K^{(n-1)}) \arrow[r] & H_n(K^{(n)}) \arrow[r, "j_n"] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_*"] \arrow[dr, dashed, "\partial_n^{cell}"'] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[d, "j_{n-1}"] & H_{n-1}(K^{(n)}) \\
 & & & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[d, "\partial_*"] \arrow[dr, "\partial_{n-1}^{cell}"] & \\
 & & & H_{n-2}(K^{(n-2)}) \arrow[r, "j_{n-2}"] & H_{n-2}(K^{(n-2)}, K^{(n-3)})
\end{tikzcd}$$

Filling in 0's and using exactness of the column and row, we get that j_n and j_{n-1} are injective, and $\ker \partial_{n-1}^{cell} = \ker \partial_* = \operatorname{im} j_{n-1}$. On the other hand, $\operatorname{im} \partial_n^{cell} = j_{n-1}(\operatorname{im} \partial_*) = j_{n-1}(H_n(K^{(n)}, K^{(n-1)}))/j_n(H_n(K^{(n)}))$. From this, we get a commutative square

$$\begin{tikzcd}
\operatorname{im} \partial_* \arrow[r, hookrightarrow] \arrow[d] & H_{n-1}(K^{(n-1)}) \arrow[d, hookrightarrow, "j_{n-1}"] \\
\operatorname{im} \partial_n^{cell} \arrow[r, hookrightarrow] & H_{n-1}(K^{(n-1)}, K^{(n-2)})
\end{tikzcd}$$

Then by commutativity of the big diagram,

$$\ker \partial_{n-1}^{cell} / \operatorname{im} \partial_n^{cell} = \operatorname{im} j_{n-1} / j_{n-1}(\operatorname{im} \partial_*) = H_{n-1}(K^{(n-1)}) / \operatorname{im} \partial_* \cong H_{n-1}(K^{(n)}).$$

To complete the proof, we show $H_{n-1}(K^{(n)}) = H_{n-1}(K)$. For technical reasons, we assume K is a finite dimensional CW-complex (the general result follows from taking direct limits of the homology groups). We have shown that $H_i(K^{(n-1)}) \cong H_i(K^{(n)})$ whenever $n \neq i, i+1$. Equivalently, $H_i(K^{(n)}) \cong H_i(K^{(n+1)})$ when $i \neq n, n+1$. This implies

$$H_{n-1}(K^{(n)}) \cong H_{n-1}(K^{(n+1)}) \cong \cdots H_{n-1}(K)$$

which terminates since K is finite dimensional. Hence by induction we are done. $\square$

To make this useful in practice, we must further understand the cellular complex:

$$C_n(K) = H_n(K^{(n)}, K^{(n-1)}) = \bigoplus_\sigma H_n(D^n_\sigma, \partial D^n_\sigma).$$

Recall that $C_n(K)$ is the free abelian group on the n-cells of K. Then an element of $C_n(K)$ is a formal sum $c = \sum_\sigma n_\sigma \sigma$, where σ are the n-cells of K and $n_\sigma \in \mathbb{Z}$. For such a chain c, its cellular boundary is $\partial^{cell}(c) = \sum_\sigma n_\sigma \partial^{cell}(\sigma)$. Moreover, for a given cell σ, its boundary is of the form $\partial^{cell}(\sigma) = \sum_\tau [\tau : \sigma]\tau$, where the sum is over all $(n-1)$-cells τ and $[\tau : \sigma] \in \mathbb{Z}$. We have a commutative diagram

$$
\begin{tikzcd}
\partial^{cell}: & H_n(K^{(n)},K^{(n-1)}) \arrow[r, "\partial_*"] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_n"] & H_{n-1}(K^{(n-1)},K^{(n-2)}) \arrow[d, dashed, bend left, "\Phi"] \\
& H_n(D^n,\partial D^n) \arrow[u, "(f_\sigma)_*"] \arrow[r, "\partial_*"] & H_{n-1}(\partial D^{n-1}) \arrow[u, "(f_{\partial\sigma})_*"] \arrow[ur, dashed] & \bigoplus_\tau H_{n-1}(D^{n-1}_\tau,\partial D^{n-1}_\tau) \arrow[u, "\Psi"] \\
& \mathbb{Z} = \langle [D^n,\partial D^n]\rangle \arrow[r] & \mathbb{Z} = \langle[\partial D^{n-1}]\rangle &
\end{tikzcd}
$$

where Ψ is the isomorphism $\sum n_\tau \tau \mapsto \sum n_\tau (f_\tau)_*([D^{n-1},\partial D^{n-1}])$. The inverse of Ψ is the map

$$
\begin{aligned}
\Phi : H_n(K^{(n)},K^{(n-1)}) &\longrightarrow \bigoplus_\sigma \mathbb{Z}\langle\sigma\rangle \\
\alpha &\longmapsto \sum_\sigma \phi((p_\sigma)_*(\alpha))\sigma,
\end{aligned}
$$

where $p_\sigma : (K^{(n)},K^{(n-1)}) \to (S^n,x_0)$ collapses $K^{(n-1)}$ and all n-cells except σ to a point $x_0 \in S^n$, and $\phi : H_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism. To see explicitly that Φ is the inverse of Ψ, it's enough to show one composition is the identity, since Ψ is already known to be an isomorphism. In this case, for any n-cell σ, we have

$$\Phi(\Psi(\sigma)) = \Phi((f_\sigma)_*([D^n,D^{n-1}])) = \sum_\tau \phi((p_\tau)_*((f_\sigma)_*([D^n,D^{n-1}])))\tau$$

and for any $(n-1)$-cell τ,

$$p_\tau \circ f_\sigma = \begin{cases} c_{x_0}, & \tau \neq \sigma \\ id, & \tau = \sigma. \end{cases}$$

This implies $\Phi(\Psi(\sigma)) = 1\cdot\sigma = \sigma$ so Φ and Ψ are inverses. These computations show that $[\tau:\sigma] = \phi((p_\tau\circ f_{\partial\sigma})_*[\partial D^n]) = \deg(p_\tau \circ f_{\partial\sigma})$ for any n-cell σ and $(n-1)$-cell τ. This describes the cellular boundary map completely.

Example 2.4.6. The circle S^1 has an obvious CW-structure consisting of a single 1-cell attached at its endpoints to a single 0-cell. Thus the cellular complex is

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

Since the gluing map has degree 0 (it identifies the endpoints of the 1-cell), the cellular homology for S^1 may be computed as

$$H_0(S^1) = \mathbb{Z}, \quad H_1(S^1) = \mathbb{Z}, \quad H_n(S^1) = 0 \text{ for } n \geq 2.$$

Example 2.4.7. Using the CW-structure for S^2 from Example 2.4.1, we get the following cellular complex:

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial^{cell}} \mathbb{Z} \xrightarrow{\partial^{cell}} \mathbb{Z} \to 0.$$

The gluing map $f : D^1 \to K^{(0)} = \{x_0\}$ identifies the endpoints of D^1 to x_0, so its induced map is zero: $f_*([D^1]) = x_0 - x_0 = 0$. Further, the gluing map $g : D^2_1 \cup D^2_2 \to K^{(1)} \cong S^1$ attaches each 2-disk to the 1-skeleton by a degree 1 map, so the induced map is surjective. Since the groups in the exact sequence above are free abelian, there is only one option for ∂_2^{cell}. We have the following homology calculations:

$$H_0(S^2) = \mathbb{Z}, \quad H_1(S^2) = 0, \quad H_2(S^2) = \mathbb{Z}, \quad H_n(S^2) = 0 \text{ for } n \geq 3.$$

Compare this to Theorem 2.3.5.

Example 2.4.8. Let X be the space obtained from $S^1 \times S^2$ by attaching a 2-cell along a map $\partial D^2 \to S^1 \times \{\text{pt}\}$ of degree k. A CW-structure consists of the 0-cell $\{x\} \times \{y\}$; 1-cell $I \times \{y\}$ attached to $K^{(0)}$ at the endpoints of I; 2-cell $D^2 \times \{y\}$ attached to $K^{(1)} = S^1 \times \{y\}$ by a map of degree k; and a 3-cell $\{x\} \times D^3$ attached along its boundary S^2 to the point $\{x\} \times \{y\}$. This gives us the following cellular complex:

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0.$$

To compute homology, we must describe the cellular boundary maps ∂_i. First, ∂_1 is induced by a constant map $I \times \{y\} \to \{x\} \times \{y\}$ so it is zero (with kernel $\mathbb{Z}$), and therefore $H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}/0 = \mathbb{Z}$. Next, ∂_2 is induced by a degree k map $D^2 \times \{y\} \to K^{(1)}$ so $\partial_2 a = ka$ for all $a \in C_2(X)$. In particular, $\ker \partial_2 = 0$ and $\operatorname{im} \partial_2 = k\mathbb{Z}$. This gives us $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}/k\mathbb{Z}$. Finally, ∂_3 is induced by another constant map $\{x\} \times D^3 \to \{x\} \times \{y\}$ so $\partial_3 = 0$, in which case $\ker \partial_3 = \mathbb{Z}$ and $\operatorname{im} \partial_3 = 0$. This allows us to compute the last two homology groups: $H_2(X) = \ker \partial_2 / \operatorname{im} \partial_3 = 0$ and $H_3(X) = \ker \partial_3 = \mathbb{Z}$. To summarize, we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k\mathbb{Z}, & i = 1 \\ 0, & i = 2 \\ \mathbb{Z}, & i = 3 \\ 0, & i \geq 4. \end{cases}$$

Example 2.4.9. For $0 < k < \infty$ and $n > 0$, let X_k be the union of k n-disks along their boundary. A CW-structure for this space is: one 0-cell $K^{(0)} = \{x\}$; one 1-cell I glued to x at its endpoints to yield a circle $K^{(1)} = S^1$; and k n-cells $D^n_1, \ldots, D^n_k$, each glued to $K^{(1)}$ along the boundary. We get the following cellular chain complex:

$$0 \to \mathbb{Z}^k \xrightarrow{\partial_n} 0 \to \cdots \to 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0.$$

Then $\ker \partial_0 = \mathbb{Z}$ and ∂_1 is induced from the constant map $I \to \{x\}$, so it is the zero map, and we get $\ker \partial_1 = \mathbb{Z}$ and $\operatorname{im} \partial_1 = 0$. This gives us $H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}/0 = \mathbb{Z}$. Next, ∂_2 must be the zero map, so $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}/0 = \mathbb{Z}$. All the homology groups are zero for $i = 2$ up to $n-1$, but $\partial_n = 0$ implies $\ker \partial_n = \mathbb{Z}^k$. Therefore $H_n(X) = \ker \partial_n / 0 = \mathbb{Z}^k$. To summarize, we have the following homology groups for X_k:

$$H_i(X_k) = \begin{cases} \mathbb{Z}, & i = 0, 1 \\ 0, & 2 \leq i \leq n-1 \\ \mathbb{Z}^k, & i = n \\ 0, & i > n. \end{cases}$$

Example 2.4.10. Let $n, m > 0$. We will describe a CW-structure on the product $S^n \times S^m$, and use it to compute the homology groups $H_i(S^n \times S^m)$ for all $i \geq 0$. Set $X = S^n \times S^m$ and suppose without loss of generality that $m \geq n > 0$. The individual spheres have cellular structures consisting of a point and a single cell in the top dimension: $S^n = \{x\} \cup D^n$ and $S^m = \{y\} \cup D^m$. A CW-structure on $S^n \times S^m$ consists of a single 0-cell $\{x\} \times \{y\}$; a single n-cell $D^n \times \{y\}$; a single m-cell $\{x\} \times D^m$; and a single $(n+m)$-cell $D^n \times D^m$. There are two cases. First, if $n = m$, we really have two n-cells $D^n \times \{y\}, \{x\} \times D^n$ and one $2n$-cell $D^n \times D^n$, so the cellular homology groups look like $H_0(C_\bullet(X)) = \mathbb{Z}, H_n(C_\bullet(X)) = \mathbb{Z}^2, H_{2n}(C_\bullet(X)) = \mathbb{Z}$ and $H_i(C_\bullet(X)) = 0$ otherwise. By the isomorphisms $H_i(C_\bullet(X)) \cong H_i(X)$, we get

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^2, & i = n \\ \mathbb{Z}, & i = 2n \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $m > n$. Then the cellular homology is $H_0(C_\bullet(X)) = \mathbb{Z}, H_n(C_\bullet(X)) = \mathbb{Z}, H_m(C_\bullet(X)) = \mathbb{Z}, H_{n+m}(C_\bullet(X)) = \mathbb{Z}$ and $H_i(X) = 0$ otherwise. Therefore the isomorphisms $H_i(C_\bullet(X)) \cong H_i(X)$ yield

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}, & i = n, m, n+m \\ 0, & \text{otherwise.} \end{cases}$$

This generalizes to arbitrary products of CW-complexes. Suppose K and L are CW-complexes. For each p-cell σ of K and each q-cell τ of L, we get a $(p+q)$-cell of $K \times L$, which we may denote by $\sigma \times \tau$, with the attaching map $f_\sigma \times f_\tau : \partial D^{p+q} \to K \times L$. This describes the entire n-skeleton of $K \times L$:

$$(K \times L)^{(n)} = \bigcup_{0 \leq p \leq n} K^{(p)} \times L^{(n-p)}.$$

Definition. *A map $f : K \to L$ between CW-complexes is a* **cellular map** *if $f(K^{(n)}) \subseteq L^{(n)}$ for each $n \geq 0$.*

We will show that every map between CW-complexes can be approximated by a cellular map.

Lemma 2.4.11. *Let K be a CW-complex. For any map $\varphi : D^n \to K$ such that $\varphi(S^{n-1}) \subseteq K^{(n-1)}$, φ is homotopic rel S^{n-1} to a map $D^n \to K^{(n)}$.*

Proof. Since D^n is compact, $\varphi(D^n) \subseteq K'$ for some finite subcomplex $K' \subseteq K$. Thus we may replace K by a space $X = Y \cup D^m$ for $m > n$ and assume $\varphi : (D^n, \partial D^n) \to (X, Y)$. Take an open set $U = X \smallsetminus Y = \text{Int}(D^m)$. Then there is some open set $E \subseteq U$ with compact closure $\overline{E}$. By smooth approximation (Corollary 1.1.15), φ is homotopic rel $D^n \smallsetminus \varphi^{-1}(E)$ to a map g that is smooth on $\varphi^{-1}(E)$. Since $m > n$, Sard's theorem guarantees there is a point $p \in E$ outside the image of φ. Now $Y \smallsetminus \{p\}$ deformation retracts to X so there is a homotopy rel S^{n-1} of g to a map with codomain Y. Since homotopy is an equivalence relation, we get the desired homotopy rel S^{n-1} of φ. $\square$

Corollary 2.4.12. *Any map $f : D^n \times \{0\} \cup S^{n-1} \times I \to K$ with $f(S^{n-1} \times \{1\}) \subseteq K^{(n-1)}$ extends to a map $g : D^n \times I \to K$ such that $g(D^n \times \{1\}) \subseteq K^{(n)}$.*

Proof. The pair $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ is homeomorphic to $(D^n \times I, D^n \times \{0\})$ so apply Lemma 2.4.11. □

Theorem 2.4.13 (Cellular Approximation)**.** *If $\varphi : K \to Y$ is a map between CW-complexes and $L \subseteq K$ is a subcomplex such that $\varphi|_L$ is a cellular map, then φ is homotopic to a cellular map on K.*

Proof. By hypothesis, we have a map $K \times \{x_0\} \cup L \times I \to Y$. We extend this inductively over the increasing n-skeleta of K using Corollary 2.4.14. □

Corollary 2.4.14. *If $f, g : K \to Y$ are cellular maps that are homotopic, then they are homotopic via a cellular homotopy.*

Proof. Apply cellular approximation to such a homotopy, $F : K \times I \to Y$. □

Next, we describe the maps in homology induced by cellular maps between CW-complexes. Given a cellular map $g : K \to L$, there is an induced map on the cellular complexes $C_n(K) \to C_n(L)$ given by $g_* : H_n(K^{(n)}, K^{(n-1)}) \to H_n(L^{(n)}, L^{(n-1)})$. This is a chain map, i.e. the following diagram commutes:

$$\begin{tikzcd}
H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_*"] \arrow[d, "g_*"] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_*"] \arrow[d, "g_*"] & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[d, "g_*"] \\
H_n(L^{(n)}, L^{(n-1)}) \arrow[r, "\partial_*"] & H_{n-1}(L^{(n-1)}) \arrow[r, "j_*"] & H_{n-1}(L^{(n-1)}, L^{(n-2)})
\end{tikzcd}$$

Lemma 2.4.15. *Under the isomorphism $H_n(C_\bullet(K), \partial^{cell}) \cong H_n(K)$, the induced map $g_* : C_n(K) \to C_n(L)$ coincides with the induced map $H_n(K) \to H_n(L)$.*

Proof. From Theorem 2.4.5, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} \partial_n \arrow[r] \arrow[d, "j"] & H_n(K^{(n)}) \arrow[r] \arrow[d, "j"] & H_n(K) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \operatorname{im} \partial^{cell} \arrow[r] & \ker \partial^{cell} \arrow[r] & H_n(C_\bullet(K), \partial^{cell}) \arrow[r] & 0
\end{tikzcd}$$

Applying g_* gives us a commutative diagram

$$\begin{tikzcd}
H_n(K) \arrow[r, "g_*"] \arrow[d] & H_n(L) \arrow[d] \\
H_n(C_\bullet(K), \partial^{cell}) \arrow[r, "g_*"] & H_n(C_\bullet(L), \partial^{cell})
\end{tikzcd}$$

which commutes by the preceding discussion. Thus the two induced maps coincide. $\square$

Next, take two cells $\sigma \in C_n(K)$ and $\tau \in C_n(L)$. Then the diagram

$$\begin{tikzcd} D^n \arrow[r, "f_\sigma"] \arrow[d] & K^{(n)} \arrow[r, "g"] \arrow[d] & L^{(n)} \arrow[r, "p_\tau"] \arrow[d] & S^n \arrow[d, equal] \\ S^n \arrow[r] & K^{(n)}/K^{(n-1)} \arrow[r] & L^{(n)}/L^{(n-1)} \arrow[r] & S^n \end{tikzcd}$$

induces a map $g_{\tau,\sigma} : S^n \to S^n$.

Lemma 2.4.16. *For any cells $\sigma \in C_n(K)$ and $\tau \in C_n(L)$, $g_*(\sigma) = \sum_{\tau \in C_n(L)} (\deg g_{\tau,\sigma})\tau$.*

Proof. Consider the diagram

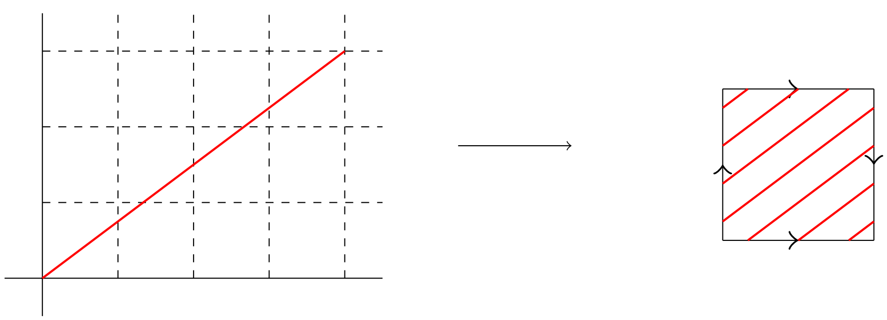
$$\begin{tikzcd} C_n(K) \arrow[r, "g"] \arrow[d, "\Phi_K"'] & C_n(L) \arrow[d, "\Phi_L"] \\ H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "g_*"] & H_n(L^{(n)}, L^{(n-1)}) \end{tikzcd}$$

By previous work,

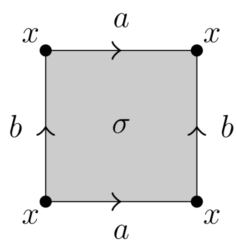
$$g_*(\sigma) = \sum_{\tau \in C_n(L)} \phi((p_\tau)_* g_* (f_\sigma)_* [D^n, \partial D^n])\tau = \sum_{\tau \in C_n(L)} (\deg g_{\tau,\sigma})\tau.$$

$\square$

Example 2.4.17. Take a (p, q) curve on the torus T: for relatively prime integers p and q, the map $t \mapsto t(p, q) \in \mathbb{R}^2$ induces a closed curve in $T \cong \mathbb{R}^2/\mathbb{Z}^2$:



This induces a map in homology $H_1(S^1) \to H_1(T)$. The torus may be given the following CW-structure:



1	0-cell	x
2	1-cells	a, b
1	2-cell	σ

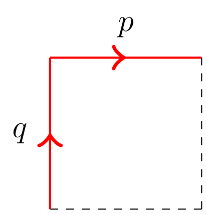
Thus the cellular complex for T is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0.$$

In particular, the complex is already exact, so it is its own homology:

$$H_0(T) = \mathbb{Z}, \quad H_1(T) = \mathbb{Z}^2, \quad H_2(T) = \mathbb{Z}, \quad H_n(T) = 0 \text{ for } n \geq 3.$$

The map $\gamma : S^1 \to T$ which realizes the (p, q) curve is homotopic to a map along the frame of the quotient square:



Hence the induced map on homology is $\gamma_*(c) = pa+qb$, where c is a generator of $H_1(S^1) \cong \mathbb{Z}$.

2.5 Euler Characteristic

Recall that if G is a finitely generated abelian group, then $G \cong F \oplus T$ where F is a free abelian group of finite rank and T is a finite torsion group, $T \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$ for integers $n_1 \mid \cdots \mid n_k$. The *rank* of G is defined to be $\operatorname{rank} G = r$, where r is the rank of the free part of G. It is a well-known fact from algebra that rank is *additive* on short exact sequences, meaning if

$$0 \to G' \to G \to G'' \to 0$$

is a short exact sequence of abelian groups, then $\operatorname{rank} G = \operatorname{rank} G' + \operatorname{rank} G''$.

In this section, let X be a topological space such that $\operatorname{rank} H_i(X)$ is finite for all i and the set $\{i \in \mathbb{N}_0 : \operatorname{rank} H_i(X) > 0\}$ is finite.

Definition. *The rank* $\beta_i := \operatorname{rank} H_i(X)$ *is called the ith* **Betti number** *of* X*. We call the alternating sum*

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i \beta_i = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

the **Euler characteristic** *of* X*.*

Examples.

(1) By Theorem 2.3.5, we can compute the Euler characteristic of any sphere:

$$\chi(S^n) = \begin{cases} 0, & n \text{ is odd} \\ 2, & \text{ is even.} \end{cases}$$

2. By Example 2.3.20, the Euler characteristic of the projective plane is $\chi(\mathbb{R}P^2) = 1$. Moreover, the Klein bottle is a connect sum of projective planes, $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$, so by the additivity axiom, $\chi(K) = 0$.

3. Our computations of the homology of a torus in Example 2.4.17 show that $\chi(T) = 0$ as well. However, the torus and Klein bottle are not homeomorphic (the Klein bottle is non-orientable, for example), so this shows that Euler characteristic cannot detect topological equivalence completely.

Theorem 2.5.1 (Euler-Poincaré)**.** *Let $(C_\bullet, \partial)$ be a finite dimensional chain complex, i.e. a chain complex such that* $\operatorname{rank}\left(\bigoplus_{i=0}^\infty C_i\right) < \infty$. *Then*

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{rank} H_i(C_\bullet) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} C_i.$$

Proof. Consider the exact sequences

$$\begin{aligned} & 0 \to \ker \partial_i \to C_i \to \operatorname{im} \partial_i \to 0 \\ \text{and} \quad & 0 \to \operatorname{im} \partial_{i+1} \to \ker \partial_i \to H_i(C_\bullet) \to 0. \end{aligned}$$

Since rank is additive,

$$\begin{aligned} \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} C_i &= \sum_{i=0}^{\infty}(\operatorname{rank} \ker \partial_i + \operatorname{rank} \operatorname{im} \partial_i) \\ &= \sum_{i=0}^{\infty}(-1)^i(\operatorname{rank} \operatorname{im} \partial_{i+1} + \operatorname{rank} H_i(C_\bullet) + \operatorname{rank} \operatorname{im} \partial_i) \\ &= \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} H_i(C_\bullet) \end{aligned}$$

since the first and third terms in each summand telescope. □

In particular, the Euler characteristic for a (finite) CW-complex is easy to calculate if only its CW-structure is known.

Corollary 2.5.2. *For a CW-complex X, its Euler characteristic may be computed by*

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i n_i$$

where n_i is the number of i-cells of X.

Corollary 2.5.3. *For any complex polyhedron X, $\chi(X) = V - E + F = 2$, where V, E and F are, respectively, the number of vertices, edges and faces in the polyhedron.*

Proposition 2.5.4. *Given a covering space $p : X \to Y$ with k sheets, where Y is a finite CW-complex, then X is also a finite CW-complex with Euler characteristic $\chi(X) = k\chi(Y)$.*

Proof. By covering space theory, each map $f_\sigma : D^i_\sigma \to Y$ lifts to X in exactly k ways. This gives a CW-structure on X consisting of exactly k times as many i-cells as Y for each $i \geq 0$. The Euler characteristic equation follows immediately. □

Example 2.5.5. If $p : S^{2k} \to Y$ is a covering space of a finite CW-complex Y by an even-dimensional sphere S^{2k}, then the number of sheets of the cover is 1 or 2. In particular, every even-dimensional projective space $\mathbb{R}P^{2k}$ has Euler characteristic 1 since $p : S^{2k} \to \mathbb{R}P^{2k}$ is a nontrivial cover (see Example 1.2.12).

Example 2.5.6. Let $\Sigma_g = \underbrace{T\#\cdots\#T}_{g \text{ times}}$ be the closed, orientable surface obtained as the connected sum of $g \geq 1$ copies of T. Using a CW-structure, one can show that

$$H_i(\Sigma_g) = \begin{cases} \mathbb{Z}, & i = 0 \quad (\Sigma_g \text{ is path-connected}) \\ \mathbb{Z}^{2g}, & i = 1 \\ \mathbb{Z}, & i = 2 \quad (\text{e.g. there is one 2-cell in a CW structure}) \\ 0, & i \geq 3. \end{cases}$$

Then the Euler characteristic is $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$. To complete the picture, we may view S^2 as the 'connected sum of zero tori', by which $\chi(S^2) = 2$. Thus the Euler characteristic completely determines a closed, orientable surface.

2.6 More Singular Homology

In this section we finally prove the existence of a homology theory by verifying that singular homology satisfies the Eilenberg-Steenrod axioms of Section 2.3. Note that we already have proven three of the axioms, namely exactness (Theorem 2.2.4), dimension (this is obvious from the definition of singular homology; see Theorem 2.6.3 below) and additivity (Lemma 2.1.9). This leaves the homotopy and excision axioms. To prove the homotopy axiom, we introduce another useful concept from homological algebra.

Definition. *Let $A_\bullet$ and $B_\bullet$ be chain complexes and $\varphi, \psi : A_\bullet \to B_\bullet$ be chain maps. We say φ and ψ are* **chain homotopic** *if there exists a homomorphism $H : A_\bullet \to B_\bullet$ such that $H(A_n) \subseteq B_{n+1}$ for each $n \in \mathbb{Z}$, and $\partial \circ H + H \circ \partial = \varphi - \psi$.*

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\psi"] & A_n \arrow[r, "\partial"] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\psi"] \arrow[dl, "H"'] & A_{n-1} \arrow[r] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\psi"] \arrow[dl, "H"'] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"'] & B_n \arrow[r, "\partial"'] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 2.6.1. *If $\varphi, \psi : A_\bullet \to B_\bullet$ are chain homotopic then $\varphi_* = \psi_*$ on $H_n(A_\bullet)$ for all $n \in \mathbb{Z}$.*

Proof. For any n-cycle $a \in A_n$, we have

$$(\varphi_* - \psi_*)[a] = [\varphi(a) - \psi(a)] = [\partial H(a) - H\partial(a)] = [\partial H(a) - H(0)] = [\partial H(a)] = 0.$$

Hence $\varphi_* = \psi_*$ as claimed. □

Lemma 2.6.2. *Chain homotopy is an equivalence relation.*

Proof. Take $\varphi, \psi, \chi : A_\bullet \to B_\bullet$ and suppose there exist degree -1 maps $H, G : A_\bullet \to B_\bullet$ such that $\partial H + H\partial = \varphi - \psi$ and $\partial G + G\partial = \psi - \chi$. Then

$$\varphi - \chi = (\varphi - \psi) + (\psi - \chi) = (\partial H + H\partial) + (\partial G + G\partial) = \partial(H + G) + (H + G)\partial.$$

Therefore φ and χ are chain homotopic by the map $H + G$. □

Definition. *A chain map $\varphi : A_\bullet \to B_\bullet$ is a* **(chain) homotopy equivalence** *if there exists some chain map $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are chain homotopic to id_B and id_A, respectively.*

Theorem 2.6.3 (Dimension)**.** *If X is contractible then its nth singular homology group $H_n(X)$ is zero for all $n > 0$.*

Proof. Define a map $D : \Delta_n(X) \to \Delta_{n+1}(X)$ as follows. Let $F : X \times I \to X$ be the contraction to a point $x_0 \in X$. For an n-simplex $\sigma \in \Delta_n(X)$, define an $(n+1)$-simplex

$$\begin{aligned} D(\sigma) : \Delta_{n+1} &\longrightarrow X \\ \sum_{j=0}^{n+1} \lambda_j e_j &\longmapsto F\left(\sigma\left(\sum_{j=0}^{n+1} \frac{\lambda_j}{\lambda} e_j\right), \lambda_0\right) \end{aligned}$$

where $\lambda = \sum_{j=0}^{n+1} \lambda_j$. This defines our map $D : \Delta_n(X) \to \Delta_{n+1}(X)$. Notice that when $n > 0$,

$$\begin{aligned} \partial D(\sigma) &= \sigma - \sum_{j=1}^{n+1} (-1)^{j-1} D(\sigma)^{(j)} = \sigma - \sum_{j=1}^{n+1} (-1)^{j-1} D(\sigma^{(j-1)}) \\ &= \sigma - D\left(\sum_{j=0}^{n} (-1)^j \sigma^{(j)}\right) = \sigma - D(\partial\sigma). \end{aligned}$$

If $n = 0$, then $\partial D(\sigma) = \sigma - \sigma_0$, where σ_0 is the unique 0-simplex at x_0. Also, $D(\partial\sigma) = D(0) = 0$, so we have

$$(\partial D + D\partial)(\sigma) = \begin{cases} \sigma, & n > 0 \\ \sigma - \sigma_0, & n = 0. \end{cases}$$

We thus see that $\partial D + D\partial = 1 - \varepsilon$, where $\varepsilon : X \to \{x_0\}$ is the augmentation map. Since $\varepsilon_* = 0$ on homology for all $n > 0$, we have constructed a chain homotopy from 1 to 0 on $H_n(X)$ for all $n > 0$. Therefore $H_n(X) = 0$ for all $n > 0$. □

This establishes the dimension axiom for singular homology. To prove the homotopy axiom, we introduce a bilinear map on products of homology groups, called the *cross product*. First, we define a bilinear map on the product of the singular chain groups.

Theorem 2.6.4. *Let X and Y be spaces and fix $p, q \geq 0$. Then there is a bilinear map*

$$\begin{aligned} \Delta_p(X) \times \Delta_q(Y) &\longrightarrow \Delta_{p+q}(X \times Y) \\ (a, b) &\longmapsto a \times b \end{aligned}$$

which satisfies

(a) If $q = 0$ then $\sigma \times \{y\} = (\sigma(\cdot), y) : \Delta_p \to X \times Y$. Similarly, if $p = 0$ then $\{x\} \times \tau = (x, \tau(\cdot)) : \Delta_q \to X \times Y$.

(b) If $f : X \to X'$ and $g : Y \to Y'$, then $(f \times g)_*(a \times b) = f_*(a) \times g_*(b)$, i.e. we have a commutative diagram

$$\begin{tikzcd} \Delta_\bullet(X) \times \Delta_\bullet(Y) \arrow[r] \arrow[d, "f_* \times g_*"'] & \Delta_\bullet(X \times Y) \arrow[d, "(f \times g)_*"] \\ \Delta_\bullet(X') \times \Delta_\bullet(Y') \arrow[r] & \Delta_\bullet(X' \times Y') \end{tikzcd}$$

(c) $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$.

Proof. We prove all three statements by induction on $p+q$. If $p = 0$ or $q = 0$, the statements are clear by the definition in (a). Now fix $p + q$ where $p > 0$ and $q > 0$. View the identity maps $i_p : \Delta_p \to \Delta_p$ and $i_q : \Delta_q \to \Delta_q$ as a p-simplex and a q-simplex, respectively. Then ∂i_p is a $(p-1)$-simplex so by induction, $\partial i_p \times i_q + (-1)^p i_p \times \partial i_q$ is defined and satisfies

$$\begin{aligned} \partial(\partial i_p \times i_q + (-1)^p i_p \times \partial i_q) &= \partial^2 i_p \times i_q + (-1)^{p-1} \partial i_p \times \partial i_q + (-1)^p \partial i_p \times \partial i_q + (-1)^{p^2} i_p \times \partial^2 i_q \\ &= 0 + (-1)^{p-1}(\partial i_p \times \partial i_q - \partial i_p \times \partial i_q) + 0 = 0. \end{aligned}$$

So $\partial i_p \times i_q + (-1)^p \times \partial i_q$ is a cycle in $Z_{p+q-1}(\Delta_p \times \Delta_q)$. However, $\Delta_p \times \Delta_q$ is a contractible space, so by Theorem 2.6.3, $H_{p+q-1}(\Delta_p \times \Delta_q) = 0$. This implies $\partial i_p \times i_q + (-1)^p i_p \times \partial i_q$ is a boundary, that is, there is some $c \in \Delta_{p+q}(\Delta_p \times \Delta_q)$ for which

$$\partial(c) = \partial i_p \times i_q + (-1)^p i_p \times \partial i_q.$$

Define the cross product on these 'standard' simplices to be $i_p \times i_q := c$, where c is chosen as above. Now for any simplices $\sigma : \Delta_p \to X$ and $\tau : \Delta_q \to Y$, we have $\sigma = \sigma_*(i_p)$ and $\tau = \tau_*(i_q)$ so it makes sense to define $\sigma \times \tau := (\sigma, \tau)_*(i_p \times i_q)$. Extend this definition linearly to all of $\Delta_p(X) \times \Delta_q(Y)$. In the base case, we already proved (a) and (b) holds by our definition of the cross product on simplices. We finish by showing (c) holds: for any simplices $\sigma \in \Delta_p(X)$ and $\tau \in \Delta_q(Y)$, we have

$$\begin{aligned} \partial(\sigma \times \tau) &= \partial((\sigma, \tau)_*(i_p \times i_q)) \\ &= (\sigma, \tau)_* \partial(i_p \times i_q) \quad \text{by naturality of chain maps} \\ &= (\sigma, \tau)_*(\partial i_p \times i_q + (-1)^p i_p \times \partial i_q) \\ &= (\sigma, \tau)_*(\partial i_p \times i_q) + (\sigma, \tau)_*((-1)^p i_p \times \partial i_q) \\ &= \sigma_*(\partial i_p) \times \tau_*(i_q) + (-1)^p \sigma_*(i_p) \times \tau_*(\partial i_q) \quad \text{by induction} \\ &= \partial \sigma_*(i_p) \times \tau_*(i_q) + (-1)^p \sigma_*(i_p) \times \partial \tau_*(i_q) \quad \text{by naturality} \\ &= \partial \sigma \times \tau + (-1)^p \sigma \times \partial \tau. \end{aligned}$$

Finally, extend by linearity to all of $\Delta_p \times \Delta_q$ to obtain the result. □

Corollary 2.6.5. *There is a bilinear map, called the cross product, defined on homology by*

$$\begin{aligned} H_p(X) \times H_q(Y) &\longrightarrow H_{p+q}(X \times Y) \\ ([a],[b]) &\longmapsto [a \times b]. \end{aligned}$$

Proof. It's enough to check that the bilinear map in Theorem 2.6.4 is well-defined on homology classes. For any cycles $a \in Z_p(X), b \in Z_q(Y)$ and boundary $\partial a' \in B_p(X)$, we have

$$\begin{aligned} (a + \partial a') \times b &= a \times b + \partial a' \times b \\ &= a \times b + \partial(a' \times b) - (-1)^p a' \times \partial b \quad \text{by Theorem 2.6.4} \\ &= a \times b + \partial(a' \times b) - 0. \end{aligned}$$

So $[(a + \partial a') \times b] = [a \times b]$. □

Theorem 2.6.6 (Homotopy)**.** *If $f, g : X \to Y$ are homotopic maps then the induced maps on homology $f_*, g_* : H_\bullet(X) \to H_\bullet(Y)$ are equal.*

Proof. We will show that the inclusions

$$\eta_0, \eta_1 : X \hookrightarrow X \times I,\ \eta_0(X) = X \times \{0\},\ \eta_1(X) = X \times \{1\},$$

induce chain homotopic maps $\Delta_\bullet(X) \to \Delta_\bullet(X \times I)$. This implies the homotopy axiomm, since if $F : X \times I \to Y$ is a homotopy from f to g then we will have $F \circ \eta_0 = f, F \circ \eta_1 = g$ and

$$f_* = (F \circ \eta_0)_* = F_* \circ (\eta_0)_* = F_* \circ (\eta_1)_* = (F \circ \eta_1)_* = g_*.$$

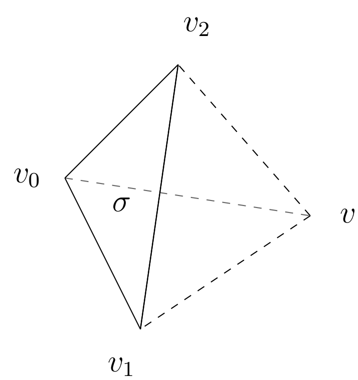
View the interval I as a 1-simplex with endpoints e_0, e_1 so that $\partial I = e_1 - e_0$. Define $D : \Delta_p(X) \to \Delta_{p+1}(X)$ by $D(\sigma) = \sigma \times I$, using the cross product from Theorem 2.6.4. Then for any chain $c \in \Delta_p(X)$,

$$\begin{aligned} (\partial D - D\partial)(c) &= \partial(c \times I) - \partial c \times I \\ &= \partial c \times I + (-1)^p c \times \partial I - \partial c \times I \quad \text{by Theorem 2.6.4} \\ &= (-1)^p c \times (e_1 - e_0) \\ &= (-1)^p (c \times e_1 - c \times e_0) \\ &= (-1)^p ((\eta_1)_*(c) - (\eta_0)_*(c)). \end{aligned}$$

This shows that $(\eta_0)_*$ is chain homotopic to $(\eta_1)_*$ as claimed. □

To prove the remaining axiom of excision, we need to understand how to subdivide open sets U which are to be excised. View $\Delta_n \subseteq \mathbb{R}^{n+1}$ as a subspace of Euclidean space. For $0 \le p \le n$, let $L_p(\Delta_n) \subset \Delta_p(\Delta_n)$ be the subcomplex generated by affine simplices $\sigma = [v_0, \ldots, v_p]$.

Definition. *For a point $v \in \Delta_n$, the* **cone** *of v on an affine p-simplex $\sigma = [v_0, \ldots, v_p]$ is the affine $(p+1)$-simplex $v\sigma = [v, v_0, \ldots, v_p]$.*



the cone $v\sigma$ on an affine 2-simplex σ

Extending the cone definition by linearity, we get a map $v : L_p(\Delta_n) \to L_{p+1}(\Delta_n)$ which satisfies the following properties:

- For $p > 0$, $\partial(v\sigma) = \partial[v, v_0, \dots, v_p] = [v_0, \dots, v_p] - v\partial[v_0, \dots, v_p] = \sigma - v\partial\sigma$. As a result, for a general p-chain c, $\partial(vc) = \sigma - v\partial c$.
- For $p = 0$, $\partial(vc) = c - \varepsilon(c)[v]$.

In order to subdivide, we need to be able to turn a simplex into multiple smaller simplices. To this end, we have:

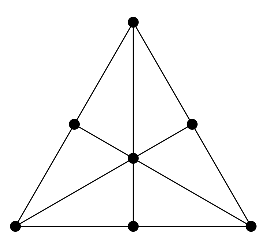
Definition. *Let $\sigma = [v_0, \dots, v_p]$ be a p-simplex and define its* **barycenter** *by*

$$\underline{\sigma} = \frac{1}{p+1} \sum_{i=0}^{p} v_p.$$

This defines a map, called **barycentric subdivision**, *$\Upsilon : L_p(\Delta_n) \to L_p(\Delta_n)$ defined on affine p-simplices by*

$$\Upsilon(\sigma) = \begin{cases} \sigma, & p = 0 \\ \underline{\sigma}(\Upsilon(\partial\sigma)), & p > 0 \end{cases}$$

and extended by linearity.



barycentric subdivision of a 2-simplex

Lemma 2.6.7. *Υ is a chain map.*

Proof. Let σ be a p-simplex. For $p = 0$, $\Upsilon(\partial\sigma) = 0 = \partial\Upsilon(\sigma)$. For $p = 1$, $\Upsilon(\partial\sigma) = \partial\sigma$, while on the other hand,

$$\partial\Upsilon(\sigma) = \partial\underset{\sim}{\sigma}(\Upsilon(\partial\sigma)) = \partial\underset{\sim}{\sigma}(\partial\sigma) = \partial\sigma - \varepsilon(\partial\sigma)[\underset{\sim}{\sigma}] = \partial\sigma$$

so $\Upsilon\partial$ and $\partial\Upsilon$ agree on 1-simplices. Finally, if $p > 1$,

$$\begin{aligned}\partial\Upsilon(\sigma) &= \partial\underset{\sim}{\sigma}(\Upsilon(\partial\sigma)) = \Upsilon(\partial\sigma) - \underset{\sim}{\sigma}(\partial\Upsilon(\partial\sigma)) \\ &= \Upsilon(\partial\sigma) - \underset{\sim}{\sigma}(\partial^2\Upsilon(\sigma)) \quad \text{by induction} \\ &= \Upsilon(\partial\sigma) - 0 = \Upsilon(\partial\sigma).\end{aligned}$$

So $\Upsilon\partial = \partial\Upsilon$. □

Lemma 2.6.8. *Υ is chain homotopic to the identity on $L_p(\Delta_n)$.*

Proof. Inductively define the map

$$\begin{aligned}T : L_p(\Delta_n) &\longrightarrow L_{p+1}(\Delta_n) \\ \sigma &\longmapsto \underset{\sim}{\sigma}(\Upsilon(\sigma) - \sigma - T(\partial\sigma))\end{aligned}$$

for p-simplices σ and extend linearly. For any 0-simplex σ, we have

$$(T\partial + \partial T)(\sigma) = 0 = (\Upsilon - 1)(\sigma).$$

Further, for $p > 0$ and any p-simplex σ,

$$\begin{aligned}\partial T\sigma &= \partial\underset{\sim}{\sigma}(\Upsilon(\sigma) - \sigma - T\partial\sigma) \\ &= \Upsilon(\sigma) - \sigma - T\partial\sigma - \underset{\sim}{\sigma}(\partial\Upsilon(\sigma) - \partial\sigma - \partial(T\partial\sigma)) \\ &= \Upsilon(\sigma) - \sigma - T\partial\sigma - \underset{\sim}{\sigma}(\partial\Upsilon(\sigma) - \partial\sigma - (T\partial(\partial\sigma) + \Upsilon(\partial\sigma) - \partial\sigma)) \\ &= \Upsilon(\sigma) - \sigma - T\partial\sigma - \underset{\sim}{\sigma}(\partial\Upsilon(\sigma) - \Upsilon(\partial\sigma)) \\ &= \Upsilon(\sigma) - \sigma - T\partial\sigma - 0 \quad \text{since } \Upsilon \text{ is a chain map} \\ &= (\Upsilon - 1)(\sigma) - T\partial\sigma.\end{aligned}$$

Hence $\partial T + T\partial = \Upsilon - 1$ so Υ is chain homotopic to the identity. □

Theorem 2.6.9. *There exists a chain map $\Upsilon : \Delta_p(X) \to \Delta_p(X)$ and a chain homotopy $T : \Delta_p(X) \to \Delta_{p+1}(X)$ from Υ to the identity such that*

(1) Υ and T agree with their definitions on $L_p(\Delta_n)$.

(2) $\Upsilon(f_*c) = f_*\Upsilon(c)$ and $T(f_*c) = f_*T(c)$ for any map $f : X \to Y$.

(3) $\Upsilon(\sigma)$ and $T(\sigma)$ are chains in the image of σ.

Proof. As above, a p-simplex $\sigma : \Delta_p \to X$ can be written $\sigma = \sigma_*(i_p)$. Define the maps Υ and T on p-simplices by $\Upsilon(\sigma) = \sigma_*(\Upsilon(i_p))$ and $T(\sigma) = \sigma_*(T(i_p))$ and extend to all of $\Delta_p(X)$

by linearity. Then (1) holds by construction, (2) follows from naturality of Υ and (3) follows from (2). Now, Υ is a chain map:

$$\begin{aligned}\Upsilon(\partial\sigma) &= \Upsilon(\partial\sigma_*(i_p)) = \Upsilon(\sigma_*(\partial i_p)) \\ &= \sigma_*(\Upsilon\partial i_p) = \sigma_*(\partial\Upsilon(i_p)) \\ &= \partial\sigma_*(\Upsilon(i_p)) = \partial\Upsilon(\sigma_*(i_p)) = \partial\Upsilon(\sigma).\end{aligned}$$

Also, T is a chain map:

$$\begin{aligned}T\partial\sigma &= T(\sigma_*(\partial i_p)) = \sigma_*(T\partial i_p) \\ &= \sigma_*(\partial T(i_p)) = \partial\sigma_*(T(i_p)) = \partial T(\sigma).\end{aligned}$$

Finally,

$$\begin{aligned}(T\partial + \partial T)(\sigma) &= \sigma_*((T\partial + \partial T)i_p) = \sigma_*((\Upsilon - 1)i_p) \\ &= (\Upsilon - 1)\sigma_*(i_p) = (\Upsilon - 1)(\sigma).\end{aligned}$$

Therefore $T\partial + \partial T = \Upsilon - 1$ so T is a chain homotopy as desired. □

Corollary 2.6.10. *For any $k \geq 1$, the map $\Upsilon^k : \Delta_p(X) \to \Delta_p(X)$ is chain homotopic to the identity on $\Delta_p(X)$.*

Proof. Follows inductively from the fact that $\Upsilon^2 = \Upsilon \circ \Upsilon \simeq \Upsilon \circ 1 = \Upsilon \simeq 1$. □

The key idea with barycentric subdivision is that for any affine simplex $\sigma \in L_p(\Delta_n)$, each $\Upsilon^k(\sigma)$ is a linear combination of simplices whose diameters approach 0 as $k \to \infty$. (This is made precise in IV.17 of Bredon.) As a result, we have:

Corollary 2.6.11. *If X is a space with an open cover $\mathcal{U} = \{U_\alpha\}$, then for any simplex $\sigma \in \Delta_p(X)$, there exists a $k \geq 0$ such that each simplex in $\Upsilon^k(\sigma)$ has image in some U_σ.*

Let X be a space and suppose $\mathcal{U}$ is any collection of sets in X whose interiors cover X. Let $\Delta_\bullet^{\mathcal{U}}(X) \subseteq \Delta_\bullet(X)$ be the subcomplex generated by simplices with image lying in the interior of some $U \in \mathcal{U}$. We denote the homology of this subcomplex by $H_\bullet^{\mathcal{U}}(X) := H_\bullet(\Delta_\bullet^{\mathcal{U}}(X))$.

Theorem 2.6.12. *The inclusion $I : \Delta_\bullet^{\mathcal{U}}(X) \hookrightarrow \Delta_\bullet(X)$ induces an isomorphism $I_* : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$.*

Proof. Suppose $\partial c = 0$ in $\Delta_\bullet^{\mathcal{U}}(X)$ and $I_*[c] = 0$ in $H_\bullet(X)$. Then there exists $d \in \Delta_\bullet(X)$ such that $\partial d = c$. By Corollary 2.6.11, one can find $k \geq 0$ such that $\Upsilon^k(d) \in \Delta_\bullet^{\mathcal{U}}(X)$. Since Υ^k is homotopic to the identity by Theorem 2.6.9, there is some chain homotopy T_k such that

$$\begin{aligned}\Upsilon^k(d) - d &= T_k\partial d + \partial T_k d = T_k c + \partial T_k d \\ \implies \partial\Upsilon^k(d) - \partial d &= \partial T_k c + 0 \\ \implies c &= \partial d = \partial(\Upsilon^k(d) + T_k c).\end{aligned}$$

Since $\Upsilon^k(d)$ and $T_k c$ both lie in $\Delta_\bullet^{\mathcal{U}}(X)$, $c \in \partial(\Delta_\bullet^{\mathcal{U}}(X))$, i.e. $[c] = 0$ in $H_\bullet^{\mathcal{U}}(X)$ as desired. This shows injectivity. For surjectivity, take $c \in \Delta_\bullet(X)$ with $\partial c = 0$. Let $k \geq 0$ such that $\Upsilon^k(c) \in \Delta_\bullet^{\mathcal{U}}(X)$ using Corollary 2.6.11. Then

$$\Upsilon^k(c) - c = T_k\partial c + \partial T_k c = \partial T_k c$$

so $[c] = [\Upsilon^k(c)] \in H_\bullet^{\mathcal{U}}(X)$. Hence I_* is an isomorphism. □

For a pair (X, A) and a cover $\mathcal{U}$ as above, let $\Delta_\bullet^{\mathcal{U}}(A)$ denote the subcomplex of $\Delta_\bullet(A)$ generated by all simplices having image in the interior of a set of the form $U \cap A$ for $U \in \mathcal{U}$. Set

$$\Delta_\bullet^{\mathcal{U}}(X, A) := \Delta_\bullet^{\mathcal{U}}(X)/\Delta_\bullet^{\mathcal{U}}(A).$$

Then there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Delta_\bullet^{\mathcal{U}}(A) \arrow[r] \arrow[d, "\cong"'] & \Delta_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\cong"'] & \Delta_\bullet^{\mathcal{U}}(X, A) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \Delta_\bullet(A) \arrow[r] & \Delta_\bullet(X) \arrow[r] & \Delta_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma (2.2.3), the dashed arrow is an isomorphism: $H_\bullet^{\mathcal{U}}(X, A) \cong H_\bullet(X, A)$.

We are now prepared to prove the excision axiom for singular homology.

Theorem 2.6.13 (Excision)**.** *Let $B \subseteq A \subseteq X$ be sets such that $\overline{B} \subseteq Int(A)$. Then the inclusion $(X \smallsetminus B, A \smallsetminus B) \hookrightarrow (X, A)$ induces an isomorphism on relative homology:*

$$H_\bullet(X \smallsetminus B, A \smallsetminus B) \cong H_\bullet(X, A).$$

Proof. The collection $\mathcal{U} = \{A, X \smallsetminus B\}$ covers X so by the above work, $H_\bullet^{\mathcal{U}}(X, A) \cong H_\bullet(X, A)$. Every simplex $\sigma \in \Delta_\bullet^{\mathcal{U}}(X)$ has image lying in A or $X \smallsetminus B$, so we can write $\Delta_\bullet^{\mathcal{U}}(X) = \Delta_\bullet(A) + \Delta_\bullet(X \smallsetminus B)$. Moreover, $\Delta_\bullet(A \smallsetminus B) = \Delta_\bullet(A) \cap \Delta_\bullet(X \smallsetminus B)$ so there exists an isomorphism

$$\Delta_\bullet(X \smallsetminus B, A \smallsetminus B) = \frac{\Delta_\bullet(X \smallsetminus B)}{\Delta_\bullet(A \smallsetminus B)} = \frac{\Delta_\bullet(X \smallsetminus B)}{\Delta_\bullet(A) \cap \Delta_\bullet(X \smallsetminus B)} \cong \frac{\Delta_\bullet(X \smallsetminus B) + \Delta_\bullet(A)}{\Delta_\bullet(A)} = \frac{\Delta_\bullet^{\mathcal{U}}(X)}{\Delta_\bullet(A)}$$

by the isomorphism theorems. This gives us a commutative diagram

$$\begin{tikzcd}
\Delta_\bullet(X \smallsetminus B, A \smallsetminus B) \arrow[rr, "\cong"] \arrow[dr, dashed] & & \Delta_\bullet^{\mathcal{U}}(X)/\Delta_\bullet(A) \arrow[dl, "I"] \\
& \Delta_\bullet(X)/\Delta_\bullet(A) &
\end{tikzcd}$$

By Theorem 2.6.12, I_* is an isomorphism on homology so $H_\bullet(X \smallsetminus B, A \smallsetminus B) \cong H_\bullet(X, A)$ as desired. $\square$

Corollary 2.6.14. *Singular homology is a homology theory.*

2.7 The Mayer-Vietoris Sequence

In this section we construct a long exact sequence called the Mayer-Vietoris sequence which is useful for computing homology groups. This can be seen as an analog to the Seifert-van Kampen theorem (1.5.1).

Theorem 2.7.1 (Mayer-Vietoris Sequence)**.** *Suppose $A, B \subseteq X$ are sets whose interiors cover X; set $\mathcal{U} = \{A, B\}$. Then there is a short exact sequence of chain complexes*

$$0 \to \Delta_\bullet(A \cap B) \to \Delta_\bullet(A) \oplus \Delta_\bullet(B) \to \Delta_\bullet^{\mathcal{U}}(A \cup B) \to 0$$

inducing a long exact sequence in homology

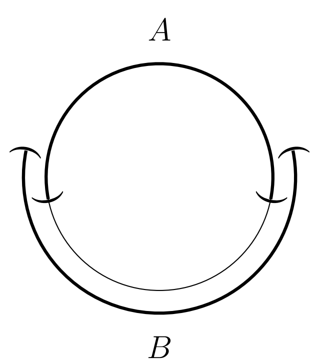
$$\cdots \to H_{n+1}(X) \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots$$

Proof. Define the inclusions $i_A : A \cap B \hookrightarrow A, i_B : A \cap B \hookrightarrow B, j_A : A \hookrightarrow A \cup B$ and $j_B : B \hookrightarrow A \cup B$. Then the maps determining the short exact sequence are:

$$0 \to \Delta_\bullet(A \cap B) \xrightarrow{i_A \oplus i_B} \Delta_\bullet(A) \oplus \Delta_\bullet(B) \xrightarrow{j_A - j_B} \Delta_\bullet^{\mathcal{U}}(A \cup B) \to 0.$$

All that remains is to define the boundary map $\partial_* : H_{n+1}(X) \to H_n(A \cap B)$. To do this, take a cycle $c \in Z_{n+1}(X)$ and use barycentric subdivision (e.g. Theorem 2.6.12) to write $c = a + b$ for chains $a \in \Delta_{n+1}(A)$ and $b \in \Delta_{n+1}(B)$. Then $\partial a + \partial b = \partial(a + b) = \partial c = 0$ so we can take $\partial_*[c] = [\partial a] = [-\partial b]$. Exactness of the long sequence is now a straightforward diagram chase. □

Example 2.7.2. The Mayer-Vietoris sequence allows us to calculate the homology groups for all spheres in a straightforward fashion (compare this to Theorem 2.3.5 and the cellular homology calculations in Examples 2.4.6 and 2.4.7). For $n = 1$, write $S^1 = A \cup B$ where A and B are the illustrated arcs:

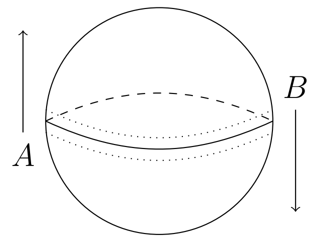


Then A and B are acyclic and $A \cap B$ is homotopy equivalent to S^0. The Mayer-Vietoris sequence for $S^1 = A \cup B$ then reduces to the following terms:

$$0 = H_1(A) \oplus H_1(B) \to H_1(S^1) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(S^1) \to 0.$$

By Corollary 2.1.10, $H_0(A \cap B) = H_0(S^0) = \mathbb{Z}^2$, and also $H_0(A) = H_0(B) = \mathbb{Z}$, so it's enough to describe the map $\mathbb{Z}^2 \xrightarrow{i_A \oplus i_B} \mathbb{Z}^2$ between these groups. One can check that this is represented by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which has null space of rank 1, and thus $H_1(S^1) = H_0(S^1) = \mathbb{Z}$. This agrees with earlier computations.

Now for any $n \geq 2$, decompose S^n into 'enlarged' hemispheres A and B:



Then $A \cap B$ is homotopy equivalent to S^{n-1} so by induction and the same Mayer-Vietoris sequence argument as above, we get

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.7.3. In this example, we use the Mayer-Vietoris sequence to compute the homology of $\mathbb{R}P^2$. (Compare to Example 2.3.20.) View $\mathbb{R}P^2$ as the union of a disk $A = D^2$ and a Möbius band B, allowing these two sets A, B to overlap a bit so that there interiors cover $\mathbb{R}P^2$. The Mayer-Vietoris sequence for $X = A \cup B$ is

$$\begin{aligned} \cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(\mathbb{R}P^2) \to \cdots \\ \cdots \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(\mathbb{R}P^2) \to \\ \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(\mathbb{R}P^2) \to \\ \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(\mathbb{R}P^2) \to 0. \end{aligned}$$

Now A is a disk and B is homotopy equivalent to a circle, so we know their homology groups:

$$H_n(A) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0 \end{cases} \quad \text{and} \quad H_n(B) = \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & n > 1. \end{cases}$$

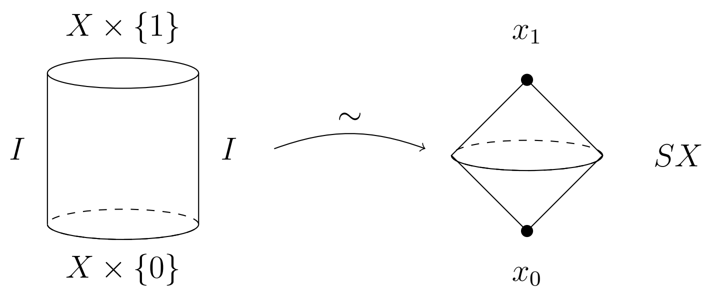
Also, $A \cap B$ has the homotopy type of a circle, so $H_0(A \cap B) = H_1(A \cap B) = \mathbb{Z}$ and $H_n(A \cap B) = 0$ otherwise. We are therefore left with the following terms to compute:

$$\begin{aligned} 0 \to 0 \to H_2(\mathbb{R}P^2) \to \\ \to \mathbb{Z} \to \mathbb{Z} \to H_1(\mathbb{R}P^2) \to \\ \to \mathbb{Z} \to \mathbb{Z}^2 \to H_0(\mathbb{R}P^2) \to 0. \end{aligned}$$

The inclusions $A \cap B \subset A$ and $A \cap B \subset B$ induce an injection $H_0(A \cap B) \hookrightarrow H_0(A) \oplus H_0(B)$, so the arrow $\mathbb{Z} \to \mathbb{Z}^2$ is injective in the diagram above. It follows from exactness that $H_0(\mathbb{R}P^2) = \mathbb{Z}$. Finally, $H_1(A \cap B) \to H_1(A) \oplus H_1(B) = 0 \oplus \mathbb{Z}$ is the zero map in the first component plus a degree two map (see Example 2.3.20), so $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2. In particular, this is injective, so $H_2(\mathbb{R}P^2) = 0$, and the cokernel is $\mathbb{Z}/2\mathbb{Z}$, proving $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$. In total, we have the expected homology of the projective plane:

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & n \geq 2. \end{cases}$$

Definition. *For a space X, define its* **(unreduced) suspension** SX *as the quotient of $X \times I$ by the equivalence relation $\sim$ identifying $X \times \{0\}$ to a point x_0 and $X \times \{1\}$ to another point x_1.*



Proposition 2.7.4. *For a space X, there are isomorphisms $\widetilde{H}_i(SX) \cong \widetilde{H}_{i-1}(X)$ for all i.*

Proof. Consider the open sets $A = SX \smallsetminus \{x_0\}$ and $B = SX \smallsetminus \{x_1\}$. Then $SX = A \cup B$ so the Mayer-Vietoris sequence for this covering is

$$\cdots \to H_{i+1}(SX) \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(SX) \to \cdots$$

Notice that $A \cap B = SX \smallsetminus \{x_0, x_1\}$ is homotopy equivalent to X; in fact, there is a deformation retract of $A \cap B$ to the image of $X \times \{1/2\}$ in SX obtained by composing the deformation retract $X \times (0,1) \to X \times \{1/2\}$, $f_s(x,t) = (x, |t - 1/2|)$, with the restriction of the quotient map $X \times (0,1) \to SX \smallsetminus \{x_0, x_1\}$. Thus $H_i(A \cap B) \cong H_i(X)$ for all $i \geq 0$. By a similar argument, A contracts to the point x_1 and B contracts to x_1, so we get $H_i(A) = H_i(B) = 0$ for $i > 0$ and $H_0(A) \cong H_0(B) \cong \mathbb{Z}$. Thus the Mayer-Vietoris sequence becomes

$$\cdots \to H_{i+1}(SX) \to H_i(X) \to 0 \to H_i(SX) \to \cdots H_1(SX) \to H_0(X) \to \mathbb{Z}^2 \to H_0(SX) \to 0.$$

For all $i > 0$, we immediately get isomorphisms $H_{i+1}(SX) \cong H_i(X)$ and since reduced homology is isomorphic to ordinary homology for $i > 0$, we get $\widetilde{H}_{i+1}(SX) \cong \widetilde{H}_i(X)$ for $i > 0$. The case $i = 0$ is easy using the definitions of the suspension and reduced homology. $\square$

There is an analogue of the Mayer-Vietoris sequence for relative homology:

Theorem 2.7.5 (Relative Mayer-Vietoris)**.** *For any open sets $U, V \subseteq X$, there is a short exact sequence*

$$0 \to \frac{\Delta_\bullet(X)}{\Delta_\bullet(U \cap V)} \to \frac{\Delta_\bullet(X)}{\Delta_\bullet(U)} \oplus \frac{\Delta_\bullet(X)}{\Delta_\bullet(V)} \to \frac{\Delta_\bullet(X)}{\Delta_\bullet^{\mathcal{U}}(U \cup V)} \to 0,$$

where $\mathcal{U} = \{U, V\}$, inducing a long exact sequence in relative homology

$$\cdots \to H_{n+1}(X, U \cup V) \to H_n(X, U \cap V) \to H_n(X, U) \oplus H_n(X, V) \to H_n(X, U \cup V) \to \cdots$$

2.8 Jordan-Brouwer Separation Theorem

Recall that the classic Jordan Curve Theorem states that any embedding of S^1 in the plane $\mathbb{R}^2$ separates the plane into two regions, an 'inside' and an 'outside'. In this section we generalize this property using homology.

Lemma 2.8.1. *Fix $n \in \mathbb{N}$ and suppose a compact space Y has the property that for any embedding $f : Y \hookrightarrow S^n$, $\widetilde{H}_i(S^n \smallsetminus f(Y)) = 0$ for all i. Then $Y \times I$ has the same property.*

Proof. Fix an embedding $f : Y \times I \hookrightarrow S^n$ and suppose $\widetilde{H}_j(S^n \smallsetminus f(Y \times I)) \neq 0$ for some j. Then there is a cover $A \cup B = S^n \smallsetminus f(Y \times \{1/2\})$ consisting of

$$\begin{aligned} A &= S^n \smallsetminus f(Y \times [0, 1/2]) \\ B &= S^n \smallsetminus f(Y \times [1/2, 1]) \\ A \cap B &= S^n \smallsetminus f(Y \times I) \\ A \cup B &= S^n \smallsetminus f(Y \times \{1/2\}). \end{aligned}$$

Viewing $Y \times \{1/2\} = Y$, the hypothesis gives $\widetilde{H}_i(A \cup B) = 0$ for all i. Then the Mayer-Vietoris sequence for $A \cup B$ is

$$\cdots \to \widetilde{H}_{i+1}(A \cup B) = 0 \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B) = 0 \to \cdots$$

so we have isomorphisms $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$ for all i. Therefore at least one of $\widetilde{H}_j(A), \widetilde{H}_j(B)$ is nonzero. Now repeat the Mayer-Vietoris argument using $[0, 1/2]$ in place of $[0, 1]$. A fact from homological algebra says that direct limits commute with homology, so if $I_0 = [0, 1], I_1 = [0, 1/2]$, etc. then we have

$$0 \neq \varinjlim \widetilde{H}_i \left(\bigcup_{r=0}^{\infty} S^n \smallsetminus f(Y \times I_r) \right) \cong \widetilde{H}_i(S^n \smallsetminus f(Y \times \{x_0\})) = 0,$$

a contradiction. □

Lemma 2.8.2. *Let $f : D^m \to S^n$ be an embedding. Then $\widetilde{H}_i(S^n \smallsetminus f(D^m)) = 0$ for all $i \geq 0$.*

Proof. Induct on m, using $D^m \cong D^{m-1} \times I$ and Lemma 2.8.1. □

In particular, this shows that $S^n \smallsetminus f(D^m)$ is a connected space.

Theorem 2.8.3 (Generalized Jordan Curve Theorem)**.** *For any embedding $f : S^m \hookrightarrow S^n$,*

$$\widetilde{H}_i(S^n \smallsetminus f(S^m)) = \begin{cases} \mathbb{Z}, & \text{if } i = n - m - 1 \\ 0, & \text{if } i \neq n - m - 1. \end{cases}$$

In particular, $\widetilde{H}_i(S^n - f(S^m)) \cong \widetilde{H}_i(S^{n-m-1})$ for all i.

Proof. We induct on m. For the base case, $m = 0$ and $S^n \smallsetminus f(S^0) \cong \mathbb{R}^n \smallsetminus \{0\}$ which is homotopy equivalent to S^{n-1}. The result then follows from Theorem 2.3.5. Now suppose that for any embedding $g : S^{m-1} \hookrightarrow S^n$, $\widetilde{H}_i(S^n \smallsetminus g(S^{m-1})) \cong \mathbb{Z}$ for $i = n - m$ and is 0 otherwise. Let $f : S^m \hookrightarrow S^n$ be an embedding and define

$$\begin{aligned} A &= S^n \smallsetminus f(D^m_+) \\ \text{and} \quad B &= S^n \smallsetminus f(D^m_-), \end{aligned}$$

where D^m_+ and D^m_- are the closed upper and lower hemispheres of D^m, respectively. Then $A \cap B = S^n \smallsetminus f(S^m)$ and $A \cup B = S^n \smallsetminus f(S^{m-1})$ so the Mayer-Vietoris sequence for $A \cup B$ is

$$\cdots \to \widetilde{H}_{i+1}(S^n \smallsetminus f(S^{m-1})) \to \widetilde{H}_i(S^n \smallsetminus f(S^m)) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \cdots$$

Now by Lemma 2.8.2, $\widetilde{H}_i(A) \oplus \widetilde{H}_i(B) = 0$ for all i, so we get isomorphisms $\widetilde{H}_i(S^n \smallsetminus f(S^m)) \cong \widetilde{H}_{i+1}(S^n \smallsetminus f(S^{m-1}))$. The result follows by the inductive hypothesis. □

Corollary 2.8.4 (Jordan-Brouwer Separation Theorem)**.** *Suppose $f : S^{n-1} \hookrightarrow S^n$ is an embedding. Then $S^n \smallsetminus f(S^{n-1})$ consists of two connected components, U and V, such that $\widetilde{H}_i(U) = \widetilde{H}_i(V) = 0$ for all i, and $f(S^{n-1})$ is the boundary of each of U and V.*

Proof. By the generalized Jordan curve theorem, $\widetilde{H}_0(S^n \smallsetminus f(S^{n-1})) = \mathbb{Z}$ so $H_0(S^n \smallsetminus f(S^{n-1})) = \mathbb{Z} \oplus \mathbb{Z}$. Then Corollary 2.1.10 says that $S^n \smallsetminus f(S^{n-1})$ has two (path) components, call them U and V. Theorem 2.8.3 also says that $H_i(S^n \smallsetminus f(S^{n-1})) = 0$ for all $i > 0$, so by the additivity axiom, $H_i(U) = H_i(V) = 0$ for all $i > 0$.

Finally, since $f(S^{n-1})$ is compact, it is closed in S^n so any point outside $f(S^{n-1})$ lies in an open set completely contained in one of U or V. This proves $\partial U \subseteq f(S^{n-1})$ and $\partial V \subseteq f(S^{n-1})$. If $f(x) \in f(S^{n-1})$ but $f(x) \notin \partial U$ for some $x \in S^{n-1}$, then $f(x)$ lies in a neighborhood N that does not intersect U. Thus there exists an open $(n-1)$-disk $\widetilde{N} \subset S^{n-1}$ such that $f(\widetilde{N}) \subseteq N$. Since $S^{n-1} \smallsetminus \widetilde{N} \cong D^{n-1}$, Lemma 2.8.2 shows that $W = S^n \smallsetminus f(S^{n-1} \smallsetminus \widetilde{N})$ is acyclic. In particular, it is open and connected. On the other hand, we can write

$$W = S^n \smallsetminus f(S^{n-1} \smallsetminus \widetilde{N}) = (U \cap W) \cup ((V \cup N) \cap W)$$

which is a disjoint union of nonempty, open sets, a contradiction. Hence $\partial U = f(S^{n-1}) = \partial V$ as claimed. □

Corollary 2.8.5. *Suppose $n \geq 2$ and $f : S^{n-1} \hookrightarrow \mathbb{R}^n$ is an embedding. Then $\mathbb{R}^n \smallsetminus f(S^{n-1})$ has exactly two connected components, U and V, such that V is bounded and acyclic and $H_i(U) \cong H_i(S^{n-1})$ for all i.*

Proof. By the Jordan-Brouwer separation theorem, $S^n \smallsetminus f(S^{n-1})$ has two components, call one of them V. Consider the long exact sequence for the pair $(V, V \smallsetminus \{x\})$:

$$\cdots \to \widetilde{H}_{i+1}(V) \to H_{i+1}(V, V \smallsetminus \{x\}) \to \widetilde{H}_i(V \smallsetminus \{x\}) \to \widetilde{H}_i(V) \to \cdots$$

Then since V is acyclic, we get isomorphisms

$$\begin{aligned} \widetilde{H}_i(V \smallsetminus \{x\}) &\cong H_{i+1}(V, V \smallsetminus \{x\}) \\ &\cong H_{i+1}(D^n, D^n \smallsetminus \{0\}) \quad \text{by excision} \\ &\cong \widetilde{H}_i(D^n \smallsetminus \{0\}) \quad \text{by the exact sequence for the pair } (D^n, D^n \smallsetminus \{0\}) \\ &\cong \widetilde{H}_i(S^{n-1}) \quad \text{by the homotopy axiom.} \end{aligned}$$

Since $\mathbb{R}^n \smallsetminus (V \smallsetminus \{x\})$ is homotopy equivalent to $U := \mathbb{R}^n \smallsetminus V$, the result follows. $\square$

2.9 Borsuk-Ulam Theorem

In this section we combine the theory of covering spaces with homology to prove several important results in topology, including the Borsuk-Ulam Theorem. Suppose $\pi : X \to Y$ is a double covering and fix a deck transformation $g : X \to X$ such that $g(x) \neq x$ for any $x \in X$ and $g^2 = 1$. For any p-simplex $\tau \in \Delta_p(Y)$, there are precisely two lifts of τ to $\Delta_p(X)$, σ and $g \circ \sigma$.

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ \Delta_p \arrow[ur, bend left, "g \circ \sigma"] \arrow[ur, "\sigma"'] \arrow[r, "\tau"'] & Y \end{tikzcd}$$

Definition. *For the double covering $\pi : X \to Y$, the* **transfer map** *is define to be the chain map*

$$\begin{aligned} t : \Delta_\bullet(Y; \mathbb{Z}/2\mathbb{Z}) &\longrightarrow \Delta_\bullet(X; \mathbb{Z}/2\mathbb{Z}) \\ \tau &\longmapsto \sigma + g \circ \sigma \end{aligned}$$

where σ is any lift of τ to X.

There is another important chain map induced by the cover:

$$\pi_* : \Delta_\bullet(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \Delta_\bullet(Y; \mathbb{Z}/2\mathbb{Z}).$$

This fits into an exact sequence with the transfer map, as the following lemma proves.

Lemma 2.9.1. *There is a short exact sequence of chain complexes*

$$0 \to \Delta_\bullet(Y; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{t} \Delta_\bullet(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi_*} \Delta_\bullet(Y; \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Proof. First, t is injective by the lifting property (Theorem 1.2.13) and π_* is surjective by the lifting property applied to simplices in Y, as above. Next, for any simplex $\tau \in \Delta_\bullet(Y; \mathbb{Z}/2\mathbb{Z})$,

$$\pi_*(t(\tau)) = \pi_*(\sigma + g \circ \tau) = \tau + \tau = 0 \pmod 2$$

by definition of lifts, so $\pi_* \circ t = 0$. Finally, if $c \in \ker_*$, we can write $c = \sum a_i \sigma_i$ such that if σ_i appears in c, so does $g \circ \sigma_i$. Then $c \in \operatorname{im} t$ so we have shown the sequence is exact. $\square$

Theorem 2.9.2. *For any double covering $\pi : X \to Y$, there is a long exact sequence in homology*

$$\to \cdots H_{i+1}(Y; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_*} H_i(Y; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{t_*} H_i(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi_*} H_i(Y; \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

Proof. This is just Theorem 2.2.2 applied to the short exact sequence in Lemma 2.9.1. ☐

Example 2.9.3. An important application of these mod 2 homology calculations is to the double cover $\pi : S^n \to \mathbb{R}P^n$. Since the homology groups for S^n and $\mathbb{R}P^n$ vanish in dimension greater than n, the long exact sequence from Theorem 2.9.2 is

$$0 \to H_n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{t_*} H_n(S^n; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi_*} H_n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

By Theorem 2.3.5, $H_n(S^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. For any $[\sigma] \in H_n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$, set $\tau = \pi]circ\sigma$. Then we have $t_*(\pi_*(\sigma)) = t_*(\tau) = \sigma + g \circ \sigma$, but since g is a deck transformation, hence a homeomorphism, Corollary 2.1.19 says that g_* is an isomorphism on homology. Therefore

$$t_* \circ \pi_* = 1 + g_* = 1 + 1 = 0 \pmod 2.$$

Since t_* is injective, we must have $\pi_* = 0$ as well. Thus the long exact sequence becomes

i	$H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$		$H_i(S^n; \mathbb{Z}/2\mathbb{Z})$		$H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$	
n	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{0}$	$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{\cong}$
$n-1$	$\mathbb{Z}/2\mathbb{Z}$		0		$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{\cong}$
$n-2$	$\mathbb{Z}/2\mathbb{Z}$		0		$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{\cong}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	
1	$\mathbb{Z}/2\mathbb{Z}$		0		$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{\cong}$
0	$\mathbb{Z}/2\mathbb{Z}$	$\xrightarrow{0}$	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	

In particular, all the connecting homomorphisms $\partial_* : H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \to H_{i-1}(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ are isomorphisms. These computations of transfer and induced maps allow us to prove important results for sphere homology.

Definition. *Let $a : S^n \to S^n$ be the antipodal map. We say a map $\varphi : S^n \to S^m$ is* **equivariant** *if $\varphi \circ a = a \circ \varphi$.*

Theorem 2.9.4. *For any map $\varphi : S^n \to S^m$, if φ is equivariant, then $n \le m$.*

Proof. An equivariant map induces a map on projective spaces $\psi : \mathbb{R}P^n \to \mathbb{R}P^m$. By the long exact sequences in mod 2 homology for each space (Theorem 2.2.6), we get a commutative diagram for each $i \ge 0$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \arrow[r, "\partial_*"] \arrow[d, "\psi_*"'] & H_{i-1}(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\psi_*"] & \cdots \\
\cdots \arrow[r] & H_i(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \arrow[r, "\partial_*"] & H_{i-1}(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

For $i = 0$, the homology groups on the right are 0 so ψ_* on the right is an isomorphism. Moreover, by Example 2.9.3, the connecting homomorphisms ∂_* are isomorphisms in every degree. Thus by commutativity, ψ_* on the left is an isomorphism. By the same argument, ψ_* is an isomorphism $H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \to H_i(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z})$ whenever $i \le \min(m, n)$. Suppose $n > m$. Then ψ_* is an isomorphism when $i \le m$, but one of the squares of the above commutative diagram is

$$\begin{array}{ccc} H_m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{t_*} & H_m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = 0 \\ \psi_* \big\downarrow & & \big\downarrow \varphi_* \\ H_m(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{t_*} & H_m(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \end{array}$$

The above discussion shows ψ_* are isomorphisms, but $H_m(S^m;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, a contradiction. Therefore $n \leq m$. □

Theorem 2.9.5 (Borsuk-Ulam)**.** *For any map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(-x) = f(x)$.*

Proof. Suppose to the contrary that $f(-x) \neq f(x)$ for any $x \in S^n$. Then the map $\varphi : S^n \to S^{n-1}$ given by

$$\varphi(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

is well-defined and continuous. Notice that $\varphi(x) = \varphi(-x)$, so φ is equivariant. However, $n > n-1$ so this contradicts Theorem 2.9.4. □

Although the Borsuk-Ulam theorem has a simple statement, it truly requires the full power of homology theory to prove. We provide two corollaries to Borsuk-Ulam which are important theorems in their own right.

Corollary 2.9.6 (Ham Sandwich Theorem)**.** *Suppose $A_1, \ldots, A_m \subset \mathbb{R}^m$ are bounded, measurable subsets. Then there is an $(m-1)$-dimensional plane cutting each A_j into two pieces of equal measure.*

Proof. View $\mathbb{R}^m \subset \mathbb{R}^{m+1}$ as the set of points with $x_{m+1} = 1$. Then every affine plane in $\mathbb{R}^m$ is uniquely determined by a subspace in $\mathbb{R}^{m-1}$ and therefore by a unit vector $x \in \mathbb{R}^{m+1}$. For such a vector x, set

$$V_x = \{y \in \mathbb{R}^m \times \{1\} \mid \langle x, y\rangle \geq 0\}$$
$$H_x = \{y \in \mathbb{R}^m \times \{1\} \mid \langle x, y\rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ is the standard vector inner product on $\mathbb{R}^{m+1}$. Let μ be Lebesgue measure on $\mathbb{R}^{m+1}$ and define $f_j(x) = \mu(A_j \cap V_x)$ for each $1 \leq j \leq m$. Set $f = (f_1, \ldots, f_m) : S^m \to \mathbb{R}^m$. Since each A_j is bounded, f is a continuous map by standard measure theory. Now by the Borsuk-Ulam theorem, there exists $x_0 \in S^m \subseteq \mathbb{R}^{m+1}$ such that $f(x_0) = f(-x_0)$. The plane H_{x_0} is the desired affine plane. □

Corollary 2.9.7 (Lusternik-Schnirelmann)**.** *If $A_1, \ldots, A_{n+1}$ are closed sets covering S^n, then at least one of the A_j contains a pair of antipodal points of the sphere.*

Proof. Suppose none of the A_j contains a pair of antipodal points. Then for all $1 \leq j \leq n$, A_j and $-A_j$ are disjoint closed subsets of S^n. By Urysohn's Lemma (see Bredon), one can define continuous maps $f_j : S^n \to [0,1]$ such that $f_j = 0$ on A_j and $f_j = 1$ on $-A_j$, for each $1 \leq j \leq n$. Then the map $f = (f_1, \ldots, f_n) : S^n \to \mathbb{R}^n$ is continuous, so by the Borsuk-Ulam theorem, there exists a point $x \in S^n$ with $f(x) = f(-x)$. However, by construction of f, none of $A_1, \ldots, A_n$ can contain either x or $-x$. Therefore since the A_j, $1 \leq j \leq n+1$, cover S^n, we must have $x, -x \in A_{n+1}$. □

2.10 Simplicial Homology

Definition. *A* **simplicial complex** *is a subset K of $\mathbb{R}^\infty$ that is a union of affine simplices satisfying:*

(1) For each simplex σ in K, every face $\sigma^{(i)}$ is in K as well.

(2) If σ, τ are simplices, either $\sigma \cap \tau = \varnothing$ or $\sigma \cap \tau = \sigma^{(i)} = \tau^{(j)}$ is a face of each simplex.

Definition. *A space X is said to be* **triangulable** *if there exists a homeomorphism $f : K \to X$ for some simplicial complex K. The pair (K, f) is called a* **triangulation** *of X.*

If K is a simplicial complex, then K has a natural CW-structure formed by identifying simplices as cells. The advantage of the simplicial approach is that subdivision of a simplicial complex is much easier than with CW-complexes.

Let K be a finite simplicial complex, i.e. one that has bounded dimension n on its simplices. We denote a simplex σ of K by $\sigma = \langle v_1, \dots, v_n]$. The boundary formula

$$\partial\langle v_1, \dots, v_n\rangle = \sum_{i=1}^{n} (-1)^i \langle v_1, \dots, v_{i-1}, v_i, \dots, v_n\rangle$$

describes a chain complex $C_\bullet^{sing}(K)$ from which we can define the *simplicial homology groups* $H_i(C_\bullet^{sing}(K))$. By Theorem 2.4.5, and considering K as a finite CW-complex, we get isomorphisms

$$H_i(K) \cong H_i(C_\bullet^{sing}(K))$$

for all $i \geq 0$.

Definition. *Let K and L be simplicial complexes. A map $f : K \to L$ is called a* **simplicial map** *if f maps the set vertices of any simplex of K into the vertices of a simplex of L, and for any simplex $\sigma = \sum_{i=1}^\infty \lambda_i v_i$, $f(\sigma) = f\left(\sum_{i=1}^\infty \lambda_i v_i\right) = \sum_{i=1}^\infty \lambda_i f(v_i)$.*

Definition. *For any map $f : K \to L$ between simplicial complexes, a* **simplicial approximation** *to f is a map $\tilde{f} : K \to L$ satisfying*

(1) $\tilde{f}$ is a simplicial map that is homotopic to f.

(2) For all $x \in K$, $\tilde{f}(x)$ lies in the smallest simplex containing $f(x)$.

As with cellular approximation (Theorem 2.4.13), we have:

Theorem 2.10.1 (Simplicial Approximation)**.** *For every map $f : K \to L$ between simplicial complexes, there is some $n \in \mathbb{N}$ and a map $\tilde{f} : K^{[n]} \to L$ such that $K^{[n]}$ is the nth subdivision of K and $\tilde{f}$ is a simplicial approximation to f.*

Proof. Omitted; see Bredon, IV.22. □

2.11 Lefschetz's Fixed Point Theorem

Let X be a topological space and consider a map $f : X \to X$. We are interested in finding criteria for when f has a fixed point, i.e. a point $x \in X$ such that $f(x) = x$. Fix a homology theory H with coefficients in $\mathbb{Z}$ and suppose the homology $H_\bullet(X)$ has finite rank, or equivalently, that $H_\bullet(X; \mathbb{Q})$ is a finite dimensional vector space. For simplicity, we also suppose $H_i(X) = 0$ for $i < 0$.

Definition. *The* **Lefschetz number** *of X is the alternating sum*

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q}))$$

where tr *is the trace of a linear map.*

Remarks.

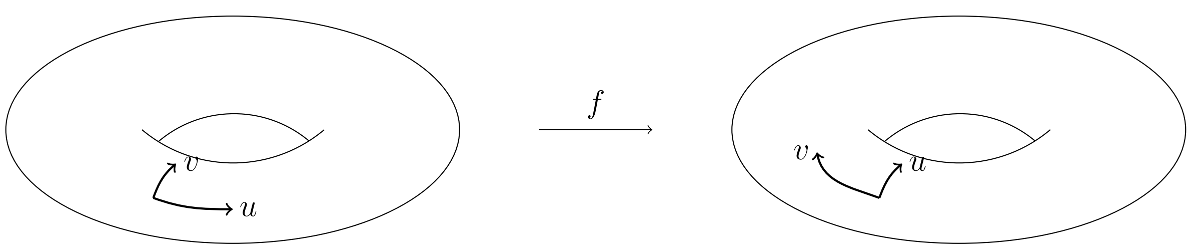
1. One can define the Lefschetz number over an arbitrary field K by
$$L(f, K) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_* : H_i(X; K) \to H_i(X; K)).$$

2. For each $i \geq 0$, $H_i(X; \mathbb{Q}) \cong H_i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ so each trace $\operatorname{tr}(f_*)$ can be computed over $\mathbb{Z}$ by restricting to the free part of $H_i(X; \mathbb{Z})$. Therefore each $\operatorname{tr}(f_*)$ lies in $\mathbb{Z}$.

3. If f is the identity on X, $L(f) = \chi(X)$, so our results about the Lefschetz number will apply to the Euler characteristic (Section 2.5).

4. If f and g are homotopic maps on X, then $L(f) = L(g)$ since f and g induce the same map on homology by the homotopy axiom.

Example 2.11.1. For the torus $T \cong \mathbb{R}^2/\mathbb{Z}^2$, consider the map $f : T \to T$ induced by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.



Then we compute

$$\begin{aligned} L(f) &= \operatorname{tr}(f_* : H_0(T) \to H_0(T)) - \operatorname{tr}(f_* : H_1(T) \to H_1(T)) + \operatorname{tr}(f_* : H_2(T) \to H_2(T)) \\ &= 1 - 0 + 1 = 2. \end{aligned}$$

Notice that $L(f) \neq \chi(T)$ (the Euler characteristic is 0 by Example 3 in Section 2.5) so f is not homotopic to the identity on T.

The following lemma shows that one may compute the Lefschetz number of a map using a chain complex or its homology. This is a direct generalization of the Euler-Poincaré theorem (2.5.1) for Euler characteristic.

Lemma 2.11.2 (Hopf Trace Formula). *For a finite dimensional chain complex $C_\bullet$ that is a vector space over some field K, and a chain map $f_\bullet : C_\bullet \to C_\bullet$,*

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{tr}(f_i) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr}((f_i)_* : H_i(C_\bullet) \to H_\bullet(C_\bullet)).$$

Proof. Given a short exact sequence of vector spaces $0 \to W \to V/W \to 0$ and a map $f : V \to V$ such that $f(W) \subseteq W$, by linear algebra we have

$$\operatorname{tr}(f) = \operatorname{tr}(f|_W) + \operatorname{tr}(\bar{f})$$

where $\tilde{f}$ is the induced map $V/W \to V/W$. Applying this to the chain map $f_\bullet : C_\bullet \to C_\bullet$ and the short exact sequence $0 \to \ker \partial_i \to C_i \xrightarrow{\partial_i} \operatorname{im} \partial_i \to 0$, we get

$$\operatorname{tr}(f_i) = \operatorname{tr}(f|_{\ker \partial_i}) + \operatorname{tr}(f|_{\operatorname{im} \partial_i}).$$

Likewise, the short exact sequence $0 \to \operatorname{im} \partial_{i+1} \to \ker \partial_i \to H_i(C_\bullet) \to 0$ gives us

$$\operatorname{tr}(f|_{\ker \partial_i}) = \operatorname{tr}(f|_{\operatorname{im} \partial_{i+1}}) + \operatorname{tr}(f_* : H_i(C_\bullet) \to H_i(C_\bullet)).$$

Combining these and cancelling terms gives the result. ☐

Theorem 2.11.3 (Lefschetz Fixed Point Theorem). *Suppose X is a finite simplicial complex or a compact manifold. Then for any map $f : X \to X$ with $L(f) \neq 0$, f has a fixed point.*

Proof. We show the contrapositive, that if $f : X \to X$ has no fixed points, then $L(f) = 0$. By hypothesis, we can view X as a compact subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. By compactness, there exists some $\delta > 0$ such that $d(x, f(x)) \geq \delta$ for all $x \in X$, where d is the Euclidean distance function. We may subdivide X so that each new simplex has diameter less than $\frac{\delta}{2}$ using barycentric subdivision (see Section 2.6). Write $X^{[n]}$ for this subdivided complex. By the simplicial approximation theorem (2.10.1), there exists a large enough n so that $\tilde{f} : X^{[n]} \to X$ is a simplicial approximation to f. Viewing only the underlying topological space, $X = X^{[n]}$ so we have a cellular map $\tilde{f} : X^{[n]} \to X^{[n]}$ – this may not be simplicial. Since $\tilde{f}$ moves cells of $X^{[n]}$ off themselves (there are no fixed points), $\operatorname{tr}(f_*) = 0$ on the chain complex $(C_\bullet, \partial^{cell})$. However, this is enough by Lemma 2.11.2 to deduce that $L(f) = 0$. ☐

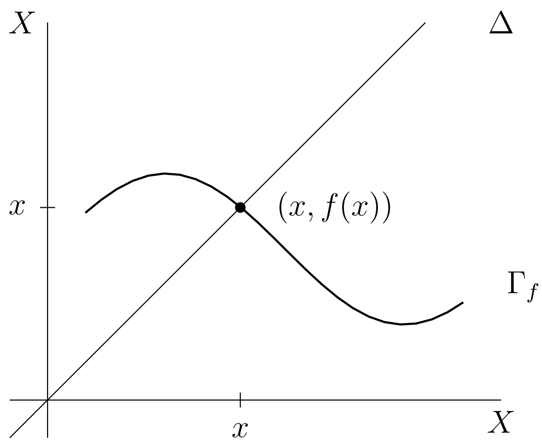
The Lefschetz fixed point theorem gives yet another proof of Brouwer's fixed point theorem: by Theorem 2.3.5, $H_i(D^n; \mathbb{Q}) = 0$ for all $i \neq 0$ and $H_0(D^n; \mathbb{Q}) = \mathbb{Q}$, so every map $f : D^n \to D^n$ induces the identity on $H_0(D^n; \mathbb{Q})$, and thus has Lefschetz number 1.

Corollary 2.11.4. *For a compact smooth manifold M without boundary, if $\chi(M) \neq 0$ then any vector field on M must have a zero.*

Corollary 2.11.5. *For any map $f : S^n \to S^n$ such that $\deg f \neq (-1)^{n+1}$, f has a fixed point.*

Note that this shows once again that the antipodal map on S^n has degree $(-1)^{n+1}$ (as in Corollary 2.3.10).

In differential topology, the Lefschetz number has a different interpretation. Suppose X is a compact manifold and $f : X \to X$ is a smooth map. Then fixed points of X coincide with intersections between the graph Γ_f of f and the diagonal Δ of X within $X \times X$.



Definition. *A fixed point of $f : X \to X$ is* **nondegenerate** *if the intersection of Δ_f and Δ is transverse at $(x, f(x))$ in $X \times X$. Equivalently, x is a nondegenerate fixed point of f if* $\det(id_X - D_x f) \neq 0$. *Otherwise, x is* **degenerate**.

Assuming $f : X \to X$ has no degenerate fixed points, the Lefschetz number can be written

$$L(f) = \sum_{\substack{x \in X \\ f(x)=x}} \operatorname{sign}(\det(id_X - D_x f)).$$

This is really the jumping off point for the study of intersection theory in differential topology.

3 Cohomology

3.1 Singular Cohomology

Recall from Section 0.4 that if M is a smooth manifold and $V \subseteq M$ is a closed, oriented submanifold of dimension p, then integration over V induces a well-defined linear map

$$\begin{aligned} H^p_{dR}(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_V \omega. \end{aligned}$$

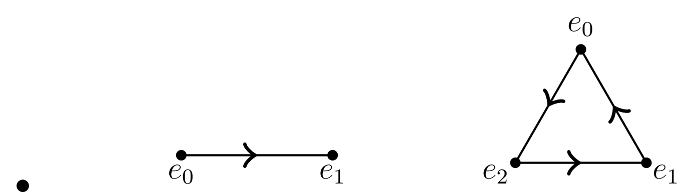
Moreover, if $W \subseteq M$ is a $(p+1)$-submanifold such that $\partial W = V$, then $\int_V \equiv 0$ on $H^p_{dR}(M)$ since by Stokes' theorem (0.4.5),

$$\int_V \omega = \int_{\partial W} \omega = \int_W d\omega = 0.$$

Thus boundaries of submanifolds correspond to the trivial class in $H^p_{dR}(M)$. This is similar to the trivial class in singular homology $H_p(M)$ defined by boundaries of singular simplices in M.

To compare de Rham cohomology and singular homology, we instead look at smooth singular homology, that is, the homology of the chain complex generated by *smooth* simplices $\Delta_p \to M$. This poses only technical problem, as the homology groups coming from these two chain complexes are the same.

In addition, we must orient the standard p-simplices in a canonical way. Given an orientation of a point (say, positive), inductively choose an orientation on Δ_p such that the 0th face map $F_0 : \Delta_{p-1} \to \Delta_p$ is orientation-preserving.



$$\begin{array}{ccccc} \Delta_0 & \xrightarrow{F_0} & \Delta_1 & \xrightarrow{F_0} & \Delta_2 \\ \bullet & \longmapsto & e_1 & & \\ & & [e_0, e_1] & \longmapsto & [e_1, e_2]. \end{array}$$

Thus, since $F_i = [e_i, e_0, \dots, \hat{e}_i, \dots, e_n] \circ F_0$, the ith face map F_i is orientation-preserving if and only if i is even.

Now if $\omega \in \Omega^p(M)$ is a p-form and $\sigma : \Delta_p \to M$ is a smooth p-simplex, define the integral of ω over σ by

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega.$$

Extend to all chains $c = \sum n_\sigma \sigma \in \Delta_p(M)$ by $\int_c \omega = \sum n_\sigma \int_\sigma \omega$. Thus each p-form ω induces a linear map $\Delta_p(M) \to \mathbb{R}$, so we have a map

$$\Psi : \Omega^p(M) \longrightarrow \mathrm{Hom}(\Delta_p(M), \mathbb{R}).$$

Proposition 3.1.1. *The map $\Psi : \Omega^\bullet(M) \to \mathrm{Hom}(\Delta_\bullet(M), \mathbb{R})$ is a chain map.*

Proof. For every p-form ω and p-simplex σ,

$$\begin{aligned}
\Psi(d\omega)(\sigma) &= \int_\sigma d\omega = \int_{\Delta_p} \sigma^*(d\omega) \\
&= \int_{\Delta_p} d(\sigma^*\omega) \quad \text{by Proposition 0.3.4} \\
&= \int_{\partial\Delta_p} \sigma^*\omega \quad \text{by Stokes' theorem (0.4.5)} \\
&= \sum_{i=0}^{p} (-1)^i \int_{\Delta_{p-1}} F_i^*\sigma^*\omega = \sum_{i=0}^{p} (-1)^i \int_{\sigma\circ F_i} \omega \\
&= \int_{\partial\sigma} \omega = \Psi(\omega)(\partial\sigma).
\end{aligned}$$

This shows that the following diagram is commutative:

$$\begin{tikzcd}
\Omega^p(M) \arrow[r, "\Psi"] \arrow[d, "d"'] & \mathrm{Hom}(\Delta_p(M), \mathbb{R}) \arrow[d, "\delta"] \\
\Omega^{p+1}(M) \arrow[r, "\Psi"] & \mathrm{Hom}(\Delta_{p+1}(M), \mathbb{R})
\end{tikzcd}$$

where $\delta : \alpha \mapsto \alpha \circ \partial$. Therefore Ψ is a chain map. □

In the diagram above, the left column is the familiar chain complex defining de Rham cohomology $H^p_{dR}(M)$. The right column is a different kind of complex, called a *cochain complex.*

Definition. *For a topological space X, the* **singular cochain complex** *of X with coefficients in an abelian group G is the chain $\mathrm{Hom}(\Delta_\bullet(X), G)$ with differential $\delta : \alpha \mapsto \alpha \circ \partial$, where ∂ is the singular boundary operator, written*

$$\Delta^\bullet(X; G) := \mathrm{Hom}(\Delta_\bullet(X), G).$$

The prefix "co" (e.g. cochain, cocycle, coboundary) signifies that the boundary operator *raises* the degree of the chain: $\delta : \Delta^p(X; G) \to \Delta^{p+1}(X; G)$.

Definition. *The* **pth singular cohomology** *of a topological space X is the homology of the cochain complex $\Delta^\bullet(X; G)$, that is,*

$$H^p(X; G) := H_p(\Delta^\bullet(X; G)).$$

We will prove in Section 3.6 that the map Ψ induces an isomorphism on cohomology $\Psi^* : H^\bullet_{dR}(M) \to H^\bullet(M; \mathbb{R})$ for any smooth manifold M.

3.2 Exact Sequences and Functors

The discussion of singular cohomology in the previous section motivates the following question: What effect do functors (such as $\mathrm{Hom}(-, G)$) have on complexes and their homology? For example, one may ask if the following diagram of functors commutes:

$$\begin{tikzcd} \texttt{ChainCx} \arrow[r, "- \otimes G"] \arrow[d, "H_\bullet(-)"'] & \texttt{ChainCx} \arrow[d, "H_\bullet(-)"] \\ \texttt{Groups} \arrow[r, "- \otimes G"] & \texttt{Groups} \end{tikzcd}$$

where `ChainCx` is the category of chain complexes. It turns out that such a diagram does not commute in general.

Lemma 3.2.1 (Right Exactness of Tensor)**.** *Given a short exact sequence of abelian groups $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ and an abelian group G, the following sequence is exact:*

$$A' \otimes G \xrightarrow{i \otimes 1} A \otimes G \xrightarrow{j \otimes 1} A'' \otimes G \to 0.$$

In other words, $- \otimes G$ is a right exact functor.

Proof. Observe that $(j \otimes 1) \circ (i \otimes 1) = j \circ i \otimes 1 \circ 1 = 0 \otimes 1 = 0$ because the original sequence is exact. So the tensored sequence is a complex. Next, $j \otimes 1$ is still surjective: any element of $A'' \otimes G$ is a linear combination of elementary tensors $a'' \otimes g$ for $a'' \in A''$ and $g \in G$, and since j is surjective, $a'' = j(a)$ for some $a \in A$. Then $a'' \otimes g = (j \otimes 1)(a \otimes g)$. This shows $j \otimes 1$ is surjective on a generating set of $A'' \otimes G$, so $j \otimes 1$ is surjective. Finally, the original short exact sequence gives $A'' = A/i(A')$. Then $A/i(A') \otimes G = A \otimes G/i(A' \otimes G)$ implies exactness at $A \otimes G$. Hence the tensor functor is right exact. □

Example 3.2.2. For an integer $n > 0$,

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is a short exact sequence. Tensoring with the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ gives a sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to 0.$$

However, multiplication by n is the zero map on $\mathbb{Z}/n\mathbb{Z}$, so this shows that $- \otimes \mathbb{Z}/n\mathbb{Z}$ is not left exact. As a bonus, the tensored exact sequence splits into

$$0 \to \mathbb{Z}/n\mathbb{Z} \to Z/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to 0.$$

So $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$ as abelian groups.

Lemma 3.2.3 (Left Exactness of Hom)**.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of left R-modules and G a left R-module. Then the following sequences are exact:*

$$0 \to \mathrm{Hom}_R(G, A) \xrightarrow{f_*} \mathrm{Hom}_R(G, B) \xrightarrow{g_*} \mathrm{Hom}_R(G, C)$$
$$\text{and} \quad 0 \to \mathrm{Hom}_R(C, G) \xrightarrow{g^*} \mathrm{Hom}_R(B, G) \xrightarrow{f^*} \mathrm{Hom}_R(A, G).$$

In other words, $\mathrm{Hom}_R(G, -)$ *and* $\mathrm{Hom}_R(-, G)$ *are left exact functors.*

Proof. The proof for $\mathrm{Hom}_R(G, -)$ is given; the proof for $\mathrm{Hom}_R(-, G)$ is similar.

We need to show that f_* is one-to-one and $\ker g_* = \operatorname{im} f_*$. First, take $h \in \mathrm{Hom}_R(G, A)$ such that $fh = 0$. Note that f is one-to-one if and only if there exists a morphism $\tilde{f} : \operatorname{im} f \to A$ such that $\tilde{f} f = id_A$. In this case

$$h = id_A h = \tilde{f} f h = \tilde{f} 0 = 0$$

which shows that $h = 0$, i.e. f_* is one-to-one.

Now let $j \in \mathrm{Hom}_R(G, B)$ such that $gj = 0$, i.e. $j \in \ker g_*$. We need to construct a function $k : G \to A$ such that $fk = j$. Let $x \in G$. Then $g(j(x)) = 0$ so $j(x) \in \ker g$. But $\ker g = \operatorname{im} f$ so there exists a (unique) element $a \in A$ such that $f(a) = j(x)$. Now we can define $k : G \to A$ by mapping x to this unique a. Then $j(x) = f(a) = f(k(x))$ so $j = fk$ as desired. Hence $\mathrm{Hom}_R(G, -)$ is left exact. □

Note that $\mathrm{Hom}_R(X, -)$ may not always be right exact. In fact, even in the category of abelian groups, $\mathrm{Hom}_{\mathbb{Z}}(X, -)$ is not right exact, as the following example shows.

Example 3.2.4. Consider the exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Let $X = \mathbb{Z}/2\mathbb{Z}$. Then Lemma 3.2.3 tells us that

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, 2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

is exact. But up to isomorphism, this sequence is

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z}.$$

Clearly adding a zero on the right makes the sequence not exact, since the kernel of $\mathbb{Z}/2\mathbb{Z} \to 0$ is necessarily $\mathbb{Z}/2\mathbb{Z}$.

Definition. *An R-module P is* **projective** *if the following diagram with exact bottom row can always be made to commute:*

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

In other words, projectives allow you to *lift along surjections*. There is a dual notion obtained by reversing the arrows:

Definition. *An R-module E is* **injective** *if the following diagram with exact bottom row can always be made to commute:*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dashed] \end{tikzcd}$$

Proposition 3.2.5. *An R-module P is projective if and only if* $\operatorname{Hom}_R(P, -)$ *is exact.*

Proof. By Lemma 3.2.3, $\operatorname{Hom}_R(P, -)$ is always left exact. That is, for a sequence

$$0 \to M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \to 0$$

applying $\operatorname{Hom}_R(P, -)$ induces an exact sequence with solid arrows:

$$0 \to \operatorname{Hom}(P, M') \to \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, M'') \dashrightarrow 0.$$

To get the 0 on the right, note that by definition of projective, every $f : P \to M''$ factors through $\alpha : M \to M''$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{f}"'] \arrow[d, "f"] & \\ M'' \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

Notice that is equivalent to showing $\operatorname{Hom}(P, M) \to \operatorname{Hom}(P, M'')$ is surjective. The converse follows by reversing the entire argument. $\square$

The dual statement holds for injective modules:

Proposition 3.2.6. *A left R-module E is injective if and only if* $\operatorname{Hom}_R(-, E)$ *is exact.*

Proof. Suppose we have an exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

We must show that the sequence

$$0 \to \operatorname{Hom}_R(C, E) \xrightarrow{p^*} \operatorname{Hom}_R(B, E) \xrightarrow{i^*} \operatorname{Hom}_R(A, E) \to 0$$

is exact. By Lemma 3.2.3 that $\operatorname{Hom}_R(-, E)$ is always left exact, so it remains to show exactness at $\operatorname{Hom}_R(A, E)$. In other words we must prove i^* is surjective if and only if i is injective. On one hand, if $f \in \operatorname{Hom}(A, E)$ there exists $g \in \operatorname{Hom}(B, E)$ with $f = i^*(g) = gi$; that is,

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, "g"'] \end{tikzcd}$$

commutes, showing E is injective. Conversely, if E is injective then for any $f : A \to E$ there exists $g : B \to E$ making the above diagram commute. Then we see that $f = gi = i^*(g) \in \operatorname{im} i^*$ so i^* is surjective. This proves $\operatorname{Hom}(-, E)$ is an exact functor. □

Proposition 3.2.7. *Every free module is projective.*

Proof. Suppose F is a free module with basis $\{f_i\}_{i \in I}$. Let $\alpha : M \to N$ be a surjection, i.e. the row of the following diagram with solid arrows is exact.

$$\begin{tikzcd} & F \arrow[dl, dashed, "\tilde{\alpha}"'] \arrow[d, "\varphi"] & \\ M \arrow[r, "\alpha"] & N \arrow[r] & 0 \end{tikzcd}$$

If $\varphi : F \to N$ is an R-linear map, denote $\varphi(f_i) = n_i \in N$. Since α is surjective, there exists an $m_i \in M$ such that $\alpha(m_i) = n_i$. Then we will define $\tilde{\alpha}(f_i) = m_i$ and extend by linearity to all of F. It is clear that $\alpha\tilde{\alpha}$ and φ agree on $\{f_i\}$ which is a basis for F; therefore the diagram commutes. □

Lemma 3.2.8. *An R-module P is projective if and only if P is a direct summand of a free R-module.*

Proof. ($\Longrightarrow$) Observe that if P is projective and $0 \to A \to B \to C \to 0$ is short exact, then the sequence splits via the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "id"] & \\ B \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

Next, every module is the image of a surjective morphism out of a free module, so let F be such a free module for P. Then there is an exact sequence

$$0 \to Q \to F \xrightarrow{\pi} P \to 0$$

where $Q = \ker \pi$. By the above, this sequence splits, so $F = P \oplus Q$.

($\Longleftarrow$) Suppose $F = P \oplus Q$ is free and consider a diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Let $\pi : P \oplus Q \to P$ and $i : P \to P \oplus Q$ be the canonical projection and inclusion. Since F is free, it is projective by Proposition 3.2.7, so the composition $f \circ \pi$ lifts to $j : F \to N$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, shift left, "\pi"] \arrow[d, dashed, "j"'] \arrow[dr] & P \arrow[l, shift left, "i"] \arrow[d, "f"] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Then $g = j \circ i : P \to N$ lifts f as required. □

Corollary 3.2.9. *(a) Every module over a field is projective.*

(b) If R is a PID, then every projective R-module is also free.

Proof. (a) A module over a field is a vector space which is free, hence projective by Proposition 3.2.7.

(b) For modules over a PID, every submodule of a free module is free. Apply Lemma 3.2.8. □

We have similar results as those above for injective modules, though we will not prove them here.

Proposition 3.2.10. *Direct sums and direct summands of injectives are injective.*

Definition. *A module M over a domain R is* **divisible** *if for all $m \in M$ and nonzero $r \in R$, there is some $m' \in M$ such that $m = rm'$.*

Informally, this says that in a divisible R-module, you can 'divide' by R.

Example 3.2.11. $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module. In fact, this holds for the field of fractions of any domain.

Theorem 3.2.12. *Every injective left R-module is divisible.*

The converse holds when R is a PID:

Theorem 3.2.13. *Let R be a PID. Then*

(1) Every divisible R-module is injective.

(2) Quotients of injectives are injective.

Example 3.2.14. By Example 3.2.11 we see that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Our next goal is to show that every left R-module can be realized as a submodule of an injective left R-module. We begin by proving this for $\mathbb{Z}$-modules (abelian groups). Let M be a $\mathbb{Z}$-module. Then $M \cong F/S$ where F is some free abelian group and S is the module of relations. By the fundamental theorem of abelian groups, $M \cong \bigoplus_{i \in I} \mathbb{Z}/S$ and $\bigoplus \mathbb{Z}$ can be embedded in $\bigoplus \mathbb{Q}$ so we have a composition

$$M \cong \bigoplus_{i \in I} \mathbb{Z}/S \hookrightarrow \bigoplus_{i \in I} \mathbb{Q}/S.$$

Now $\mathbb{Q}$ is divisible so $\bigoplus \mathbb{Q}$ is also divisible. Then by Theorem 3.2.13, $\bigoplus \mathbb{Q}/S$ is injective.

Before proving the property for R-modules in general, we will need

Proposition 3.2.15. *Let $\varphi : R \to S$ be a ring homomorphism and E an injective left R-module. Then $\mathrm{Hom}_R(S, E)$ is an injective left S-module.*

Proof. Note that by Proposition 3.2.6, $\mathrm{Hom}_R(S, E)$ is an injective left S-module if and only if $\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, E))$ is an exact functor. By Hom-Tensor adjointness,

$$\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, E)) \cong \mathrm{Hom}_R(S \otimes_S -, E) \cong \mathrm{Hom}_R(-, E)$$

and $\mathrm{Hom}_R(-, E)$ is exact precisely when E is injective. (In fact we have proven the converse of the proposition as well.) $\square$

Corollary 3.2.16. *For any divisible abelian group D, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective left R-module.*

Theorem 3.2.17. *For any left R-module M, there is an embedding $M \hookrightarrow E$ where E is an injective left R-module.*

Proof. (Sketch) Every module is an abelian group so embed $M \hookrightarrow D$ where D is a divisible abelian group. Apply $\mathrm{Hom}_{\mathbb{Z}}(R, -)$, which preserves injectivity by left exactness:

$$\mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, D).$$

To put M inside $\mathrm{Hom}_{\mathbb{Z}}(R, M)$, consider the map

$$\begin{aligned} M &\hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \\ m &\mapsto (r \mapsto rm). \end{aligned}$$

We can verify that the composite $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, D)$ is R-linear. Then by Corollary 3.2.16, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective which proves we can embed M into an injective left R-module. $\square$

Proposition 3.2.18. *If the short exact sequence $0 \to M' \to M \to M'' \to 0$ is split, and N is another R-module, then the sequences*

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0 \tag{1}$$

$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to 0 \tag{2}$$

$$0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \to 0 \tag{3}$$

are all exact.

Proof. (1) Tensor is right exact so it suffices to show $i \otimes 1 : M' \otimes N \to M \otimes N$ is injective. Notice that $(\pi \otimes 1)(i \otimes 1) = \pi i \otimes 1 = 0 \otimes 1 = 0$ so $i \otimes 1$ has a left inverse and is therefore injective.

(2) By Lemma 3.2.3, we only need to check that $i^* : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$ is surjective. But the splitting $M' \xleftarrow{\pi} M$ gives $i^*\pi^* = (\pi i)^* = (id_{M'})^* = 1$ so i^* has a right inverse and is therefore surjective.

(3) Similarly, covariant Hom is always left exact so it's enough to show $j_* : \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'')$ is surjective. The other splitting $M \xleftarrow{f} M''$ gives us $j_* f_* = (jf)_* = (id_{M''})_* = 1$, so j_* has a right inverse and is surjective. □

Definition. *A* **projective resolution** *of an R-module M is an exact sequence*

$$P_\bullet = \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_j is projective. If there exists a smallest n such that $P_j = 0$ for all $j > n$, then the resolution is said to have **length** *n; otherwise it has infinite length.*

As always, there is a dual notion for injectives:

Definition. *An* **injective resolution** *of an R-module M is an exact sequence*

$$E_\bullet = 0 \to M \to E^0 \to E^1 \to \cdots \to E_{n-1} \to E_n \to \cdots$$

such that each E^j is injective. The **length** *of an injective resolution is defined as above.*

Lemma 3.2.19. *(a) Every module over a field has a projective resolution of length 0.*

(b) Every module over a PID has a projective resolution of length at most 1.

Proof. (a) If V is a module over a field, it is a vector space and therefore free and projective. Hence $0 \to V \xrightarrow{id} V \to 0$ is a projective resolution.

(b) Let P be a projective module with a surjection $P \to M$, so that P is also free by Corollary 3.2.9. Then the kernel K of this map is a submodule of P, so it's free and therefore projective by Corollary 3.2.9 again. Hence $0 \to K \to P \to M \to 0$ is a projective resolution of M. □

Definition. *A* **deleted projective resolution** *of an R-module M is a complex*

$$P_\bullet = \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

such that $P_\bullet \to M$ is a projective resolution. Likewise, a **deleted injective resolution** *of M is a complex*

$$E_\bullet = 0 \to E^0 \to E^1 \to \cdots \to E_{n-1} \to E_n \to \cdots$$

such that $E_\bullet \to M$ is an injective resolution. Notice that deleted resolutions need not be exact.

3.3 Tor and Ext

A fundamental concept in homological algebra is the derived functors of Hom and tensor. We define derived functors in their most general form, but we will only concern ourselves with Tor and Ext, whose definitions follow.

Definition. *Suppose $T : \mathcal{A} \to \mathcal{C}$ is a covariant additive functor between categories of modules. The* n**th left derived functor** *of T is*

$$\begin{aligned} L_nT : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto H_n(T(P_\bullet)) \end{aligned}$$

where $P_\bullet$ is a fixed deleted projective resolution of A.

Derived functors in some sense measure the failure of exactness in the nth homological dimension of the functor T.

Definition. *In the category R-*`Mod` *of left R-modules, if $T_M = - \otimes_R M$ then its left derived functors are called* **Tor***:*

$$\mathrm{Tor}_n^R(M, N) := (L_nT_M)(N) = H_n(P_\bullet \otimes_R N).$$

Likewise for a right R-module M, the left derived functors of $T'_M = M \otimes_R -$ are called **tor***:*

$$\mathrm{tor}_n^R(N, M) := (L_nT'_M)(N) = H_n(M \otimes_R P_\bullet).$$

The notation tor is only temporary, as the following theorem shows (for a proof, see Rotmatn's *Introduction to Homological Algebra*).

Theorem 3.3.1. *For all left R-modules A and right R-modules B, and all $n \geq 0$,*

$$\mathrm{Tor}_n^R(A, B) \cong \mathrm{tor}_n^R(A, B).$$

The dual of left derived functors is right derived functors, which we define now.

Definition. *For a covariant additive functor $S : \mathcal{A} \to \mathcal{C}$ between categories of modules, the* n**th right derived functor** *of S is*

$$\begin{aligned} R^nS : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto H^n(S(E_\bullet)) \end{aligned}$$

where $E_\bullet$ is a fixed injective resolution of A.

The counterpart to Tor is a right derived functor called Ext.

Definition. *In R-*`Mod`*, the right derived functors of $S_M = \mathrm{Hom}_R(M, -)$ are called* **Ext***:*

$$\mathrm{Ext}_R^n(M, N) := (R^nS_M)(N) = H^n(\mathrm{Hom}_R(M, E_\bullet)).$$

Example 3.3.2. For all right R-modules A and left R-modules B, $\mathrm{Tor}_0^R(A,B) \cong A \otimes_R B$. Likewise, if A and B are both left R-modules then $\mathrm{Hom}_R(A,B) \cong \mathrm{Ext}_R^0(A,B)$.

Example 3.3.3. Fix $n \geq 0, m \geq 1$ and consider $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$. A simple projective resolution of $\mathbb{Z}/m\mathbb{Z}$ is the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

Applying $\mathrm{Hom}(-,\mathbb{Z})$ and deleting $\mathbb{Z}/m\mathbb{Z}$, we obtain a complex

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z} \xrightarrow{m} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z} \to 0.$$

The homology of this sequence is now easy to calculate:

$$\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}) = 0 \quad \text{and} \quad \mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}.$$

In defining left and right derived functors, we have neglected the fact that we are choosing a particular projective (or injective) resolution of A. The comparison theorem resolves this issue using chain homotopy (see Section 2.6).

Theorem 3.3.4 (Comparison Theorem). *Let $P_\bullet : \cdots \to P_2 \to P_1 \to P_0$ be a projective chain complex and suppose $C_\bullet : \cdots C_2 \to C_1 \to C_0$ is an acyclic chain complex. Then for any homomorphism $\varphi : H_0(P_\bullet) \to H_0(C_\bullet)$, there is a chain map $f : P_\bullet \to C_\bullet$ whose induced map on H_0 is φ, and φ is unique up to chain homotopy.*

Proof. Consider the diagram

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r, "\partial_2"] \arrow[d, dashed, "f_2"'] & P_1 \arrow[r, "\partial_1"] \arrow[d, dashed, "f_1"'] & P_0 \arrow[r, "\partial_0"] \arrow[d, dashed, "f_0"'] & H_0(P_\bullet) \arrow[r] \arrow[d, "\varphi"] & 0 \\
{} \arrow[r] & C_2 \arrow[r, "\partial_2'"] & C_1 \arrow[r, "\partial_1'"] & C_0 \arrow[r, "\partial_0'"] & H_0(C_\bullet) \arrow[r] & 0
\end{tikzcd}$$

Since P_0 is projective, there exists an f_0 lifting φ to $P_0 \to C_0$. Inductively, given f_{n-1} we have a diagram

$$\begin{tikzcd}
P_n \arrow[r, "\partial_n"] \arrow[d, dashed, "f_n"'] \arrow[dr] & P_{n-1} \arrow[d, "f_{n-1}"] \\
C_n \arrow[r, "\partial_n'"] & C_{n-1}
\end{tikzcd}$$

Note that $f_{n-1}\partial$ has image lying in $\ker \partial_n' \subseteq C_{n-1}$, but $C_\bullet$ is acyclic, so $\partial_n'(C_n) = \ker \partial_{n-1}'$. Since P_n is projective, we can lift $f_{n-1}\partial_n$ to the desired map $f_n : P_n \to C_n$. By construction, $f = \{f_n\}_{n=0}^\infty$ satisfies the desired properties.

For uniqueness, suppose $g : P_\bullet \to C_\bullet$ is another chain map restricting to φ on H_0. Since $f_0 - g_0 = 0$ on H_0, it must be that $(f_0 - g_0)(P_0) \subseteq \ker \partial_0' = \mathrm{im}\, \partial_1'$, so by projectivity of P_0 there exists $s_0 : P_0 \to C_1$ making the following diagram commute:

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "s_0"'] \arrow[d, "f_0 - g_0"] & \\ C_1 \arrow[r] & \operatorname{im} \partial_1' \arrow[r] & 0 \end{tikzcd}$$

Inductively, given $s_0, \dots, s_{n-1}$ satisfying $f_k - g_k = \partial_{k+1}' s_k + s_{k+1}\partial_k$ for all $0 \le k \le n-1$, we have

$$\begin{aligned} \partial_n'(f_n - g_n - s_{n-1}\partial_n) &= (f_{n-1} - g_{n-1})\partial_n - \partial_n' s_{n-1}\partial_n \quad \text{since } f, g \text{ are chain maps} \\ &= (\partial_n' s_{n-1} - s_{n-2}\partial_{n-1})\partial_n - \partial_n' s_{n-1}\partial_n = 0. \end{aligned}$$

Hence there is a commutative diagram

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "s_n"'] \arrow[d, "f_n - g_n - s_{n-1}\partial_n"] & \\ C_{n+1} \arrow[r, "\partial_{n+1}'"'] & \ker \partial_n' \arrow[r] & 0 \end{tikzcd}$$

This establishes the chain homotopy $s : P_\bullet \to C_\bullet$ such that $f_n - g_n = \partial_{n+1}' s_n + s_{n+1}\partial_n$ for all $n \ge 0$. Hence f is unique up to chain homotopy. □

Corollary 3.3.5. *Let $g : M \to N$ be R-linear and pick projective resolutions $P_\bullet$ and $Q_\bullet$ of M and N, respectively. Then there exists a chain map $f : P_\bullet \to Q_\bullet$ such that $H_0(f) = g$ and f is unique up to chain homotopy.*

Proof. Given projective resolutions $P_\bullet, Q_\bullet \to M$, we have $M = H_0(P_\bullet) = H_0(Q_\bullet)$ so let $\varphi = id_M$. Since projective resolutions are acyclic, the comparison theorem gives us a chain map $f : P_\bullet \to Q_\bullet$. Reversing the roles of $P_\bullet$ and $Q_\bullet$ gives a chain map in the opposite direction, and uniqueness forces the composition of these maps to be the identity in either direction. □

Corollary 3.3.6. *For R-modules M and N, the assignments $(M, N) \mapsto H_n(P_\bullet \otimes N)$ and $(M, N) \mapsto H^n(\operatorname{Hom}(P_\bullet, N))$ are independent of the deleted projective resolution $P_\bullet \to M$ and are functorial in both M and N. In particular,* Ext *and* Tor *are well-defined functors.*

Proof. If $P_\bullet, Q_\bullet \to M$ are two projective resolutions, then by Corollary 3.3.5, $P_\bullet$ and $Q_\bullet$ are chain homotopy equivalent, and it easily follows that $P_\bullet \otimes N$ and $Q_\bullet \otimes N$, as well as $\operatorname{Hom}(P_\bullet, N)$ and $\operatorname{Hom}(Q_\bullet, N)$, are chain homotopy equivalent as well.

Next, a map $N \to N'$ induces a map $P_\bullet \otimes N \to P_\bullet \otimes N'$ which is functorial since $\otimes$ and $H_\bullet$ are functors to begin with. On the other hand, suppose $P_\bullet \to M$ and $P_\bullet' \to M'$ are projective resolutions. Then given a map $M \to M'$, the uniqueness portion of Corollary 3.3.5 says that the map is functorial.

The proof for Hom is analagous. □

Corollary 3.3.7. *If $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules, then for any R-module M, there are long exact sequences in* Tor *and* Ext*:*

$$\begin{aligned} &\cdots \to \operatorname{Tor}_n(M, N') \to \operatorname{Tor}_n(M, N) \to \operatorname{Tor}_n(M, N'') \to \operatorname{Tor}_{n-1}(M, N') \to \cdots \\ \textit{and} \quad &\cdots \to \operatorname{Ext}^n(M, N') \to \operatorname{Ext}^n(M, N) \to \operatorname{Ext}^n(M, N'') \to \operatorname{Ext}^{n+1}(M, N') \to \cdots \end{aligned}$$

Proof. Both Tor and Ext are defined in terms of the homology of a chain complex, so this is a direct application of Theorem 2.2.2. □

Proposition 3.3.8. *Let R be a ring and let M and N be R-modules.*

(a) If M is a free R-module, then $\operatorname{Tor}_n^R(M, N) = \operatorname{Ext}_R^n(M, N) = 0$ for $n > 0$.

(b) If R is a PID, then $\operatorname{Tor}_n^R(M, N) = \operatorname{Ext}_R^n(M, N) = 0$ for $n > 1$.

Proof. Apply Lemma 3.2.19. □

Lemma 3.3.9 (Horseshoe Lemma)**.** *Given a short exact sequence of R-modules $0 \to A \to B \to C \to 0$ and deleted projective resolutions $P_\bullet \to A$ and $R_\bullet \to C$, there is a deleted projective resolution $Q_\bullet \to B$ and a short exact sequence of chain complexes $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$ inducing the maps of the original short exact sequence on H_0.*

Proof. We must have $Q_\bullet = P_\bullet \oplus R_\bullet$ so that the short exact sequence $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$ splits. Consider the following diagram with exact columns and exact top and bottom row:

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[r] \arrow[d, "i"] & 0 \\
 & Q_n \arrow[d] & \cdots & Q_1 \arrow[d] & Q_0 \arrow[r, "\delta_0"] \arrow[d] & B \arrow[r, dashed] \arrow[d, "j"] & 0 \\
\cdots \arrow[r] & R_n \arrow[r] & \cdots \arrow[r] & R_1 \arrow[r] & R_0 \arrow[r] \arrow[ur, dashed, "\Phi"'] & C \arrow[r] & 0
\end{tikzcd}$$

Our goal is to complete the middle row such that the diagram commutes. We begin by constructing $\delta_0 : Q_0 \to B$. Since R_0 is projective, there exists a map $\Phi : R_0 \to B$ lifting j, given by the dashed arrow above. Then define δ_0 on $Q_0 = P_0 \oplus R_0$ by $\delta_0 = (i \circ \varepsilon) \oplus (-\Phi)$. Having constructed $\delta_0, \ldots, \delta_{n-1}$, we have the following portion of the above diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Q_n \arrow[r, dashed] \arrow[d] & Q_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d] & Q_{n-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & R_n \arrow[r] & R_{n-1} \arrow[r] & R_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Setting $K = \ker(P_{n-1} \to P_{n-2}), L = \ker(Q_{n-1} \to Q_{n-2})$ and $M = \ker(R_{n-1} \to R_{n-2})$, the above diagram induces a smaller diagram of the same form as the base case:

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
& Q_n \arrow[r, "\delta_n"] \arrow[d] & L \arrow[d] & \\
\cdots \arrow[r] & R_n \arrow[r] \arrow[ur, dashed] & M \arrow[r] & 0
\end{tikzcd}$$

Complete the diagram as in the base case to construct δ_n. $\square$

Corollary 3.3.10. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules and N is any R-module, then there are long exact sequences*

$$\to \mathrm{Tor}_n(M', N) \to \mathrm{Tor}_n(M, N) \to \mathrm{Tor}_n(M'', N) \to \mathrm{Tor}_{n-1}(M', N) \to$$
$$\to \mathrm{Ext}^n(M'', N) \to \mathrm{Ext}^n(M, N) \to \mathrm{Ext}^n(M', N) \to \mathrm{Ext}^{n+1}(M'', N) \to$$

Proof. Take projective resolutions $P'_\bullet \to M'$ and $P''_\bullet \to M''$; then Lemma 3.3.9 provides a projective resolution $P_\bullet$ for M and a short exact sequence of complexes

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0.$$

Since each term in the sequence is a complex of projective modules, applying $- \otimes N$ and $\mathrm{Hom}(-, N)$ yield short exact sequences again. Then the desired long exact sequences are merely the long exact sequences in homology for these induced short exact sequences. $\square$

Theorem 3.3.11. *For each $n \geq 0$, there exists a functor $\mathrm{Tor}_n^R : R\text{–}\mathtt{Mod} \times R\text{–}\mathtt{Mod} \to R\text{–}\mathtt{Mod}$ which satisfies*

(1) $\mathrm{Tor}_0^R(M, N) = M \otimes N$.

(2) For any short exact sequence $0 \to M' \to M \to M'' \to 0$ and any R-module N, there is a long exact sequence

$$\to \mathrm{Tor}_n(M', N) \to \mathrm{Tor}_n(M, N) \to \mathrm{Tor}_n(M'', N) \to \mathrm{Tor}_{n-1}(M', N) \to$$

which is natural in N.

(3) For any free module F, $\mathrm{Tor}_n^R(F, N) = 0$ for all $n > 0$.

Moreover, any functor satisfying these three properties is naturally isomorphic to Tor_n^R.

Proof. We have proven that Tor_n satisfies the stated properties so it remains to show that these in fact characterize Tor_n. We prove this inductively on n. For $n = 0$, uniqueness follows from the universal property of the tensor product. For $n \geq 1$, take modules M and

N and a free module F such that there is an exact sequence $0 \to K \to F \to M \to 0$. Then by (2), there is a long exact sequence

$$0 = \mathrm{Tor}_n(F,N) \to \mathrm{Tor}_n(M,N) \to \mathrm{Tor}_{n-1}(K,N) \to \mathrm{Tor}_{n-1}(F,N) \to \cdots$$

When $n > 1$, $\mathrm{Tor}_{n-1}(F,N) = 0$ as well so $\mathrm{Tor}_n(M,N) \cong \mathrm{Tor}_{n-1}(K,N)$. When $n = 1$, $\mathrm{Tor}_1(M,N) \cong \ker(K \otimes N \to F \otimes N)$. In all cases, induction implies that Tor_n is determined as a functor by Tor_{n-1} so uniqueness holds. □

Corollary 3.3.12. *For any $n \geq 0$ and any modules M and N, there is a natural isomorphism*

$$\mathrm{Tor}_n(M,N) \cong \mathrm{Tor}_n(N,M).$$

Proof. Consider the assignment $(M,N) \mapsto \mathrm{tor}_n(M,N) := \mathrm{Tor}_n(N,M)$. Then

(1) $\mathrm{tor}_0(M,N) = \mathrm{Tor}_0(N,M) = N \otimes M \cong M \otimes N$.

(2) tor_n has a long exact sequence in the first variable because Tor_n has a long exact sequence in the second variable by Corollary 3.3.7.

(3) For a free module F, $\mathrm{tor}_n(F,N) = \mathrm{Tor}_n(N,F) = 0$ for each $n > 0$ since $- \otimes F$ preserves exactness.

Hence by Theorem 3.3.11, Tor_n and tor_n are naturally isomorphic. □

There is an analagous theorem for Ext:

Theorem 3.3.13. *For each $n \geq 0$, there exists a functor $\mathrm{Ext}_R^n : R\text{–}\mathtt{Mod} \times R\text{–}\mathtt{Mod} \to R\text{–}\mathtt{Mod}$ which satisfies*

(1) $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$.

(2) For any short exact sequence $0 \to M' \to M \to M'' \to 0$ and any R-module N, there is a long exact sequence

$$\to \mathrm{Ext}^n(M'',N) \to \mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(M',N) \to \mathrm{Ext}^{n+1}(M'',N) \to$$

which is natural in N.

(3) For any free module F, $\mathrm{Ext}_R^n(F,N) = 0$ for all $n > 0$.

Moreover, any functor satisfying these three properties is naturally isomorphic to Ext_R^n.

Proof. Reverse the arrows in the proof of Theorem 3.3.11. □

Proposition 3.3.14. *Let R be a commutative ring, $a \in R$ a non-zero-divisor and M an R-module. Write ${}_aM = \{m \in M \mid am = 0\}$ for the a-torsion part of M. Then*

(a) $R/aR \otimes M \cong M/aM$.

(b) $\mathrm{Tor}_1(R/aR, M) \cong {}_aM$.

(c) $\mathrm{Hom}(R/aR, M) \cong {}_aM$.

(d) $\mathrm{Ext}^1(R/aR, M) \cong M/aM$.

Proof. (a) Define a map $\phi : R/aR \times M \to M/aM$ by $(r, m) \mapsto rm$. If $r - r' \in aR$, then $r - r' = as$ for some $s \in R$. Then $(r - r')m = (as)m = a(sm) \in aM$ so the map is well-defined on the quotient R/aR. It is also clearly bilinear. By the universal property of tensor products, this determines a linear map $\Phi : R/aR \otimes M \to M/aM$. If $\Phi(r \otimes m) = 0$ in M/aM then $\Phi(r \otimes m) = rm \in aM$. Thus $rm = am'$ for some $m' \in M$, and since the tensor is over R, we can pass elements of R across the tensor:

$$r \otimes m = 1 \otimes rm = 1 \otimes am' = a \otimes m' = 0 \otimes m' = 0 \text{ in } M/aM.$$

So Φ is one-to-one. Clearly Φ is also surjective: $1 \otimes m \mapsto m$ for any $m \in M$. Hence we have the desired isomorphism.

(b) Consider the exact sequence

$$0 \to R \xrightarrow{a} R \to R/aR \to 0.$$

Tensoring with M gives a long exact sequence in Tor:

$$0 = \mathrm{Tor}_1(R, M) \to \mathrm{Tor}_1(R/aR, M) \to R \otimes M \to R \otimes M \to R/aR \otimes M \to 0.$$

(By Proposition 3.3.8, since R is a free R-module, $\mathrm{Tor}_1(R, M) = 0$.) Using the facts that $R \otimes M \cong M$ and, from part (a), $R/aR \otimes M \cong M/aM$, we get an isomorphic exact sequence:

$$0 \to \mathrm{Tor}_1(R/aR, M) \to M \xrightarrow{a} M \to M/aM \to 0.$$

Since the map out of $\mathrm{Tor}_1(R/aR, M)$ is injective and the sequence is exact, we see that $\mathrm{Tor}_1(R/aR, M)$ is isomorphic to the kernel of $M \xrightarrow{a} M$, which is clearly ${}_aM$. Hence $\mathrm{Tor}_1(R/aR, M) \cong {}_aM$.

(c) For each $m \in M$, there is an R-map $f_m : R \to M, r \mapsto rm$. Moreover, if $m \in {}_aM$, then f_m factors through the quotient R/aR, giving an element $\bar{f}_m$ of $\mathrm{Hom}(R/aR, M)$. This determines a map $\varphi : {}_aM \to \mathrm{Hom}(R/aR, M), m \mapsto \bar{f}_m$. Conversely, each $g \in \mathrm{Hom}(R/aR, M)$ determines an element $g(1) \in {}_aM$, since $ag(1) = g(a1) = g(a) = 0$. This gives $\psi : \mathrm{Hom}(R/aR, M) \to {}_aM$, and we have

$$\begin{aligned} \varphi\psi(g)(r) &= \varphi(g(1)) = \bar{f}_{g(1)}(r) = rg(1) = g(r) \\ \text{and} \quad \psi\varphi(m) &= \psi(\bar{f}_m) = \bar{f}_m(1) = 1m = m. \end{aligned}$$

So we see that φ and ψ are inverses. Hence $\mathrm{Hom}(R/aR, M) \cong {}_aM$.

(d) Using the same exact sequence

$$0 \to R \xrightarrow{a} R \to R/aR \to 0$$

and applying $\mathrm{Hom}(-, M)$, we get a long exact sequence in Ext:

$$0 \to \mathrm{Hom}(R/aR, M) \to \mathrm{Hom}(R, M) \to \mathrm{Hom}(R, M) \to \mathrm{Ext}^1(R/aR, M) \to \mathrm{Ext}^1(R, M) = 0.$$

(Again since R is free, $\mathrm{Ext}^1(R, M) = 0$.) Replacing $\mathrm{Hom}(R/aR, M)$ with ${}_aM$ and $\mathrm{Hom}(R, M)$ with M, we get

$$0 \to {}_aM \to M \xrightarrow{a} M \to \mathrm{Ext}^1(R/aR, M) \to 0.$$

By exactness, $\mathrm{Ext}^1(R/aR, M)$ is isomorphic to the cokernel of the map $M \xrightarrow{a} M$, which is M/aM by definition. □

Example 3.3.15. Let G be an abelian group, with torsion part $T(G)$. For another abelian group B, applying $- \otimes B$ to the short exact sequence $0 \to T(G) \to G \to G/T(G) \to 0$ gives a long exact sequence in Tor (coefficients in $\mathbb{Z}$):

$$0 \to \mathrm{Tor}_1(T(G), B) \to \mathrm{Tor}_1(G, B) \to \mathrm{Tor}_1(G/T(G), B) = 0,$$

with the final 0 coming from Proposition 3.3.8 (using that $G/T(G)$ is torsion-free). Thus one may replace G by $T(G)$ when computing Tor_1. Further, consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Applying $T(G) \otimes -$ gives another long exact sequence

$$\mathrm{Tor}_1(T(G), \mathbb{Z}) \to \mathrm{Tor}_1(T(G), \mathbb{Q}) \to \mathrm{Tor}_1(T(G), \mathbb{Q}/\mathbb{Z}) \to T(G) \otimes \mathbb{Z} \to T(G) \otimes \mathbb{Q} = 0.$$

Further, $\mathrm{Tor}_1(T(G), \mathbb{Q}) = 0$, so we get $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, G) \cong \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, T(G)) \cong T(G)$.

Example 3.3.16. Let $m \geq 2$ and $n \geq 0$ be integers. Then using the above results, we can compute

$$\mathrm{Tor}_n(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Tor}_n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & n = 0 \\ 0, & n > 0 \end{cases}$$

$$\mathrm{Ext}^n(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0, & n = 0 \\ \mathbb{Z}/m\mathbb{Z}, & n = 1 \\ 0, & n > 1 \end{cases}$$

$$\mathrm{Ext}^n(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & n = 0 \\ 0, & n > 0. \end{cases}$$

3.4 Universal Coefficient Theorems

Now that we have sufficiently described the functors $\otimes$ and Hom and their derived functors, it is natural to ask: what is the relation between $H_\bullet(C_\bullet \otimes M)$ and $H_\bullet(C_\bullet) \otimes M$ for a chain complex $C_\bullet$ and module M? Likewise, what is the relation between $H_\bullet(\mathrm{Hom}(C_\bullet, M))$ and $\mathrm{Hom}(H_\bullet(C_\bullet), M)$? The universal coefficient theorems give answers to these questions in terms of Tor and Ext.

Theorem 3.4.1 (Universal Coefficient Theorem for Homology with Coefficients)**.** *For a free chain complex $C_\bullet$ over a PID and a coefficient module M, for each $n \geq 0$ there is a short exact sequence*

$$0 \to H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M) \to \mathrm{Tor}_1(H_{n-1}(C_\bullet), M) \to 0.$$

Moreover, this short exact sequence splits and is natural in $C_\bullet$ and M.

Proof. The map $g : H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M)$ may be defined by $[a] \otimes m \mapsto [a \otimes m]$. Write $Z_n = \ker(\partial : C_n \to C_{n-1})$, $B_n = \operatorname{im}(\partial : C_{n+1} \to C_n)$ and $H_n = H_n(C_\bullet)$. Then for each $n \geq 0$, there are exact sequences

$$0 \to B_n \xrightarrow{i} Z_n \xrightarrow{q} H_n \to 0 \tag{1}$$

$$0 \to Z_n \xrightarrow{j} C_n \xrightarrow{\partial} B_{n-1} \to 0 \tag{2}$$

where (2) is split. Tensoring with M, we get a diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 \arrow[d] & & 0 \arrow[dl, dashed] \\
 & B_n \otimes M \arrow[u] \arrow[r, "i"] & Z_n \otimes M \arrow[r, "q"] \arrow[d, "j"] & H_n \otimes M \arrow[r] \arrow[dl, dashed, "g"] & 0 \\
 & C_{n+1} \otimes M \arrow[u] \arrow[r, "\partial"] & C_n \otimes M \arrow[r, "\partial"] \arrow[d, "\partial"] \arrow[dl, dashed, "f"] & C_{n-1} \otimes M & \\
0 \arrow[r] & \operatorname{Tor}_1(H_{n-1}, M) \arrow[r] \arrow[dl, dashed] & B_{n-1} \otimes M \arrow[r] \arrow[d] & Z_{n-1} \otimes M \arrow[u] & \\
0 & & 0 & 0 \arrow[u] &
\end{tikzcd}$$

The bottom row is the long exact sequence in Tor coming from the short exact sequence (1), but it terminates after Tor_1 because Z_{n-1} is free. Also, since (2) is split we get the zeroes at the top of the middle column and bottom of the right column. By diagram chasing, one can define f and g and show that the sequence is exact. Moreover, the sequence is split due to the fact that sequence (2) is split. Finally, naturality follows from naturality of tensor, H_n and Tor_1. □

Theorem 3.4.2 (Universal Coefficient Theorem for Cohomology)**.** *For a free chain complex $C_\bullet$ over a PID and a coefficient module M, for each $n \geq 0$ there is a short exact sequence*

$$0 \to \operatorname{Ext}^1(H_{n-1}(C_\bullet), M) \to H^n(\operatorname{Hom}(C_\bullet, M)) \to \operatorname{Hom}(H_n(C_\bullet), M) \to 0$$

which splits and is natural in $C_\bullet$ and M.

Proof. Replacing $- \otimes M$ with $\operatorname{Hom}(-, M)$, Tor_1 with Ext^1 and reversing the arrows in the proof of Theorem 3.4.1 gives a diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 & & 0 \\
 & \mathrm{Hom}(B_n, M) \arrow[d] & \mathrm{Hom}(Z_n, M) \arrow[l, "i"'] \arrow[u] & \mathrm{Hom}(H_n, M) \arrow[l, "q"'] \arrow[ur, dashed] & 0 \arrow[l] \\
 & \mathrm{Hom}(C_{n+1}, M) & \mathrm{Hom}(C_n, M) \arrow[l, "\partial"'] \arrow[u, "j"] \arrow[ur, dashed, "g"] & \mathrm{Hom}(C_{n-1}, M) \arrow[l, "\partial"'] \arrow[d] & \\
0 & \mathrm{Ext}^1(H_{n-1}, M) \arrow[l] \arrow[ur, dashed, "f"] & \mathrm{Hom}(B_{n-1}, M) \arrow[l] \arrow[u, "\partial"'] & \mathrm{Hom}(Z_{n-1}, M) \arrow[l] \arrow[d] & \\
0 \arrow[ur, dashed] & & 0 \arrow[u] & 0 &
\end{tikzcd}$$

The proof is dual to the proof of Theorem 3.4.1. □

Example 3.4.3. In this example we compute the cohomology groups of S^n. Fix $k \geq 0$. By the universal coefficient theorem for cohomology, $H^k(S^n) = \mathrm{Hom}(H_k(S^n), \mathbb{Z}) \oplus \mathrm{Ext}^1(H_{k-1}(S^n), \mathbb{Z})$. Recall from Theorem 2.3.5 that

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & k = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Since $H_k(S^n)$ is always free, we see that $\mathrm{Ext}^1(H_k(S^n), \mathbb{Z}) = 0$ for all k. Thus

$$H^k(S^n) = \mathrm{Hom}(H_k(S^n), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $H^k(S^n) = H_k(S^n)^*$, i.e. cohomology and homology are dual.

Example 3.4.4. Recall from Example 6 in Section 2.1 the definition of Lens space $L(p, q)$ as a quotient of S^3 by the $\mathbb{Z}/p\mathbb{Z}$ action. Using a CW-structure on $L(p, q)$, one can compute its singular homology groups:

$$H_k(L(p, q)) = \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}/p\mathbb{Z}, & k = 1 \\ 0, & k = 2, k \geq 4. \end{cases}$$

Then by the universal coefficient theorem,

$$\begin{aligned}
H^0(L(p, q)) &= \mathbb{Z} \quad \text{by duality} \\
H^1(L(p, q)) &= \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0 \oplus 0 = 0 \\
H^2(L(p, q)) &= \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \quad \text{by Example 3.3.16} \\
H^3(L(p, q)) &= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1(0, \mathbb{Z}) = \mathbb{Z} \oplus 0 = \mathbb{Z}.
\end{aligned}$$

Lemma 3.4.5. *Let $G = F(G) \oplus T(G)$ be a finitely generated abelian group, where $F(G)$ is the free part of G and $T(G)$ is the torsion part of G. Then* $\mathrm{Hom}(G, \mathbb{Z}) \cong F(G)$ *and* $\mathrm{Ext}^1(G, \mathbb{Z}) \cong T(G)$.

Proof. Use the universal coefficient theorem, Example 3.3.16 and additivity. □

Corollary 3.4.6. *Suppose X is a topological space and $A \subseteq X$ is a subspace. If $H_n(X, A)$ and $H_{n-1}(X, A)$ are finitely generated, then so is $H^n(X, A)$ and there is an isomorphism*

$$H^n(X, A) \cong F_n \oplus T_{n-1},$$

where $F_n = F(H_n(X, A))$ and $T_{n-1} = T(H_{n-1}(X, A))$.

Note that the isomorphism in Corollary 3.4.6 is *not* canonical.

3.5 Properties of Cohomology

Lemma 3.5.1. *Let $A_\bullet$ and $B_\bullet$ be free chain complexes over a PID and take a chain map $\varphi : A_\bullet \to B_\bullet$ such that $\varphi^* : H_n(A_\bullet) \to H_n(B_\bullet)$ is an isomorphism for all $n \geq 0$. Then*

(a) For any M, the map $\varphi^M : A_\bullet \otimes M \to B_\bullet \otimes M$ induces isomorphisms $H_n(A_\bullet; M) \to H_n(B_\bullet; M)$ for all $n \geq 0$.

(b) For any M, the map $\varphi_M : \mathrm{Hom}(B_\bullet, M) \to \mathrm{Hom}(A_\bullet, M)$ induces isomorphisms $H^n(A_\bullet; M) \to H^n(B_\bullet; M)$ for all $n \geq 0$.

Proof. (a) From the universal coefficient theorem for homology with coefficients, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_n(A_\bullet) \otimes M \arrow[r] \arrow[d, "\varphi \otimes 1"] & H_n(A_\bullet \otimes M) \arrow[r] \arrow[d, "\varphi^M"] & \mathrm{Tor}_1(H_{n+1}(A_\bullet), M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_n(B_\bullet) \otimes M \arrow[r] & H_n(B_\bullet \otimes M) \arrow[r] & \mathrm{Tor}_1(H_{n+1}(B_\bullet), M) \arrow[r] & 0
\end{tikzcd}$$

By naturality of each row in the universal coefficient theorem, the third column is an isomorphism and by hypothesis, $\varphi \otimes 1$ is an isomorphism. Hence by the Five Lemma (2.2.3), φ^M is an isomorphism.

(b) is similar, using the universal coefficient theorem for cohomology. □

This implies the excision axiom for homology with coefficients and cohomology.

Corollary 3.5.2 (Excision)**.** *If $B \subseteq A \subseteq X$ are sets such that $\overline{B} \subseteq Int(A)$, then the inclusion $(X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_\bullet(X \setminus B, A \setminus B; M) \longrightarrow H_\bullet(X, A; M)$$
$$H^\bullet(X \setminus B, A \setminus B; M) \longrightarrow H^\bullet(X, A; M)$$

for any coefficient module M.

We also get an analogue of the Mayer-Vietoris sequence (Theorem 2.7.1) for these homology theories.

Theorem 3.5.3 (Mayer-Vietoris)**.** *If $A, B \subseteq X$ are subspaces such that Int(A) and Int(B) cover X, then for any coefficient module M, there are long exact sequences*

$$\cdots \to H_{k-1}(A \cap B; M) \to H_k(X; M) \to H_k(A; M) \oplus H_k(B; M) \to H_k(A \cap B; M) \to \cdots$$
$$\cdots \to H^{k-1}(A \cap B) \to H^k(X) \to H^k(A) \oplus H^k(B) \to H^k(A \cap B) \to \cdots$$

Proof. The ordinary Mayer-Vietoris sequence arises from a short exact sequence

$$0 \to \Delta_\bullet(A \cap B) \to \Delta_\bullet(A) \oplus \Delta_\bullet(B) \to \Delta_\bullet^{\mathcal{U}}(X) \to 0$$

for $\mathcal{U} = \{A, B\}$. Recall that $\Delta_\bullet^{\mathcal{U}}(X) \to \Delta_\bullet(X)$ induces an isomorphism on homology. Then by Lemma 3.5.1, there are induced isomorphisms on homology with coefficients and cohomology. □

Theorem 3.5.4 (Additivity)**.** *If $X = \coprod X_\bullet$ is a disjoint union of subspaces, then the inclusions $X_\alpha \hookrightarrow X$ and projections $X \to X_\alpha$ induce isomorphisms*

$$H^\bullet(X; M) \cong \prod_\alpha H^\bullet(X_\alpha; M)$$

for any coefficient module M.

Proof. We have $\Delta_\bullet\left(\coprod X_\bullet\right) \cong \bigoplus \Delta_\bullet(X_\bullet)$ and applying Hom to a direct sum gives a direct product, so we have

$$\operatorname{Hom}\left(\Delta_\bullet\left(\coprod_\alpha X_\alpha\right), M\right) \cong \operatorname{Hom}\left(\bigoplus_\alpha \Delta_\bullet(X_\alpha), M\right) \cong \prod_\alpha \operatorname{Hom}(\Delta_\bullet(X_\alpha), M).$$

□

Theorem 3.5.5 (Homotopy)**.** *If $f, g : (X, A) \to (Y, B)$ are homotopic maps then they induce the same map on homology with coefficients and cohomology.*

Proof. Given the notation in Theorem 2.6.6, η_0 and η_1 are chain homotopic maps on $\Delta_\bullet(X)$. It follows that $\eta_0 \otimes 1$ and $\eta_1 \otimes 1$ are chain homotopic, so the same proof goes through. Likewise, $\operatorname{Hom}(\eta_0, M)$ and $\operatorname{Hom}(\eta_1, M)$ are chain homotopic, so we get the same result for cohomology. □

Corollary 3.5.6. *For any coefficient module M, the functors $H_n(-; M)$ and $H^n(-; M)$ are homology theories.*

3.6 de Rham's Theorem

Recall from Section 3.1 that integration of differential forms determines a map

$$\Psi : \Omega^p(M) \longrightarrow \mathrm{Hom}(\Delta_p(M), \mathbb{R})$$

We know that Ψ is linear (since integration is linear) and Stokes' theorem (0.4.5) shows that Ψ is a chain map (see Proposition 3.1.1). Thus we get an induced map on cohomology:

$$\Psi^* : H^p_{dR}(M) \longrightarrow H^p(M; \mathbb{R}).$$

De Rham's theorem says that this map is an isomorphism for any smooth manifold M. To prove this, we need a Mayer-Vietoris sequence for de Rham cohomology.

Theorem 3.6.1 (Mayer-Vietoris)**.** *Suppose $M = U \cup V$ for open subsets $U, V \subseteq M$. Then the inclusions $i_U : U \hookrightarrow M, i_V : V \hookrightarrow M$ and $j_U : U \cap V \hookrightarrow U, j_V : U \cap V \hookrightarrow V$ induce a short exact sequence for each $p \geq 0$:*

$$0 \to \Omega^p(M) \xrightarrow{i_U^* \oplus i_V^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{j_U^* - j_V^*} \Omega^p(U \cap V) \to 0.$$

Proof. For exactness, suppose $\omega \in \Omega^p(M)$ such that $i_U^*\omega = 0$ and $i_V^*\omega = 0$. Then $\omega = 0$ since $M = U \cup V$ so $i_U^* \oplus i_V^*$ is injective. Next, if $(\omega, \eta) \mapsto j_U^*\omega - j_V^*\omega = 0$ then $\omega = \eta$ on $U \cap V$. One can define $\alpha \in \Omega^p(M)$ by $\alpha|_U = \omega$ and $\alpha|_V = \eta$, so that $(\omega, \eta) = (i_U^* \oplus i_V^*)(\alpha)$. This shows exactness at the middle term. Finally, if $\tau \in \Omega^p(U \cap V)$, take a partition of unity (Section 0.4) to extend τ to ω on U and η on V, such that $\omega + \eta \equiv 1$ on M. Then $(j_U^* - j_V^*)(\omega, -\eta) = (\omega + \eta)|_{U \cap V} = \tau$ so the last arrow is surjective. $\square$

Corollary 3.6.2. *If the de Rham map $\Psi : H^\bullet_{dR}(-) \to H^\bullet(-; \mathbb{R})$ is an isomorphism on the cohomologies of open sets U, V and $U \cap V \subseteq M$, then it is also an isomorphism for $U \cup V$.*

Proof. The Mayer-Vietoris sequence in Theorem 3.6.1 is natural and Ψ is a chain map, so we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet(U \cup V) \arrow[r] \arrow[d] & \Omega^\bullet(U) \oplus \Omega^p(V) \arrow[r] \arrow[d] & \Omega^\bullet(U \cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta^\bullet(U \cup V) \arrow[r] & \Delta^\bullet(U) \oplus \Delta^p(V) \arrow[r] & \Delta^\bullet(U \cap V) \arrow[r] & 0
\end{tikzcd}$$

Taking cohomology, we get a similar diagram

$$\begin{tikzcd}
0 \arrow[r] & H^\bullet_{dR}(U \cup V) \arrow[r] \arrow[d] & H^\bullet_{dR}(U) \oplus H^p(V) \arrow[r] \arrow[d] & H^\bullet_{dR}(U \cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^\bullet(U \cup V; \mathbb{R}) \arrow[r] & H^\bullet(U; \mathbb{R}) \oplus H^p(V; \mathbb{R}) \arrow[r] & H^\bullet(U \cap V; \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

By hypothesis, the middle and right arrows are isomorphisms so by the Five Lemma (2.2.3), $H^\bullet_{dR}(U \cup V) \cong H^\bullet(U \cup V; \mathbb{R})$. $\square$

Lemma 3.6.3 (Poincaré). *The de Rham map is an isomorphism for all convex, open subsets of $\mathbb{R}^n$. In particular, for such a set $U \subseteq \mathbb{R}^n$,*

$$H^k_{dR}(U) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0. \end{cases}$$

Proof. For $k = 0$, $H^0_{dR}(U) = \ker(d : \Omega^0(U) \to \Omega^1(U))$. A function $f \in \Omega^0(U)$ is in $\ker d$ if and only if f is locally constant, which is equivalent to f being constant since U is a convex set. Say $f \equiv r$. Then Ψ assigns to f the cochain that maps a 0-simplex x to $\int_x f = f(x) = r$. Thus $\Psi(f) : \Delta_0(\mathcal{U}) \to \mathbb{R}$ is the constant map on $\mathbb{R}$. It follows that $H^0(U;\mathbb{R}) \cong \mathbb{R}$, so the Poincaré Lemma holds for $k = 0$.

Now suppose $k > 0$. We must show that $H^k_{dR}(U) = 0$, i.e. every closed k-form on U is exact. Consider $dx_I = dx_{i_0} \wedge \cdots \wedge dx_{i_{k-1}}$ for $I = (i_0, \ldots, i_{k-1})$. Define $\eta_I \in \Omega^{k-1}(U)$ by

$$\eta_I = \sum_{j=0}^{k-1} (-1)^j x_{i_j} dx_{i_0} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_{k-1}}.$$

Then $d\eta_I = k dx_I$, which determines a map $\varphi : \Omega^k(U) \to \Omega^{k-1}(U)$, where if $\omega = f(x_1, \ldots, x_n) dx_I$,

$$\varphi(\omega) = \left(\int_0^1 t^{k-1} f(tx)\, dt \right) \eta_I.$$

(Such a $(k-1)$-form is well-defined since U is convex. Moreover, after translation we may assume without loss of generality that U contains the origin in $\mathbb{R}^n$.) Now by the chain rule for the exterior derivative, we have

$$\begin{aligned} d\varphi(\omega) &= d\left(\int_0^1 t^{k-1} f(tx)\, dt \right) \wedge \eta_I + \left(\int_0^1 t^{k-1} f(tx)\, dt \right) d\eta_I \\ &= \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I + \left(\int_0^1 t^{k-1} f(tx)\, dt \right) d\eta_I. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi(d\omega) = \varphi\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I \right) &= \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) \eta_{\{j\} \cup I} \\ &= \sum_{j=1}^n x_j \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_I - \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I \\ &= \left(\int_0^1 t^k \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}(tx) \right) dx_I - \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I \\ &= \left(\int_0^1 t^k \frac{\partial f}{\partial t}(tx)\, dt \right) dx_I - \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I \end{aligned}$$

$$= \left[t^k f(tx)|_0^1 - \int_0^1 k t^{k-1} f(tx)\, dt \right] dx_I - \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I$$

by integration by parts

$$= f(x) dx_I - \left(\int_0^1 t^{k-1} f(tx)\, dt \right) d\eta_I - \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial f}{\partial x_j}(tx)\, dt \right) dx_j \wedge \eta_I$$

$$= \omega - d\varphi(\omega).$$

So $\varphi(d\omega) + d\varphi(\omega) = \omega$ for all ω, and in particular $\omega = d\varphi(\omega)$ if ω is closed. Therefore closed forms are exact. □

Lemma 3.6.4. *If the de Rham map Ψ is an isomorphism for any collection of disjoint open sets $\{U_\alpha\}$ then it is an isomorphism for $\coprod U_\alpha$.*

Proof. By additivity, we have natural isomorphisms

$$H^\bullet_{dR}\left(\coprod_\alpha U_\alpha \right) \cong \prod_\alpha H^\bullet_{dR}(U_\alpha) \quad \text{and} \quad H^\bullet\left(\coprod_\alpha U_\alpha; \mathbb{R} \right) \cong \prod_\alpha H^\bullet(U_\alpha; \mathbb{R}).$$

□

Lemma 3.6.5. *Let $P(U)$ be a statement about open sets U of a smooth manifold M which satisfies:*

(1) $P(U)$ holds for all U diffeomorphic to a convex, open subset of $\mathbb{R}^n$.

(2) If $P(U), P(V)$ and $P(U \cap V)$ are valid then so is $P(U \cup V)$.

(3) If $\{U_\alpha\}$ are disjoint, open sets and each $P(U_\alpha)$ is valid, then $P\left(\coprod U_\alpha\right)$ is valid.

Then $P(M)$ is true.

Proof. We construct a smooth, proper function $f : M \to [0, \infty)$ as follows. When $M \subseteq \mathbb{R}^n$, this is easy: $f(x) = |x|^2$ for all $x \in M$ will suffice. In general, take a locally finite open cover $\{U_j\}$ of M with each $\overline{U}_j$ compact, and a partition of unity $\{f_j\}$ subordinate to $\{U_j\}$. Then $f(x) = \sum_{j \geq 1} j f_j(x)$ will work.

To prove the lemma, first suppose $M \subseteq \mathbb{R}^n$ is itself an open subset of Euclidean space. If M is convex, the proof is trivial so suppose otherwise. Then $P(U_1 \cup \cdots \cup U_m)$ is true for any finite collection of convex, open subsets $U_1, \ldots, U_m \subseteq M$. Let $f : M \to [0, \infty)$ be the proper map constructed in the first paragraph. For each $m \geq 0$, let $A_m = f^{-1}([m, m+1]) \subseteq M$ which is compact by properness of f. Then A_m may be covered by finitely many open balls U_m^j which we may assume are contained in $f^{-1}\left(\left(m - \frac{1}{2}, m + \frac{3}{2}\right)\right)$. Set $U_m = \bigcup_j U_m^j$. By construction, the U_m for m even are all disjoint, so by condition (3), $P(U_{even}) = P\left(\coprod U_{2k}\right)$ holds. Likewise, the U_m for m odd are disjoint so $P(U_{odd}) = P\left(\coprod U_{2k+1}\right)$ holds. Moreover,

$$U_{even} \cap U_{odd} = \coprod_{k \geq 0} (U_{2k} \cap U_{2k+1})$$

is a disjoint union and each $U_{2k} \cap U_{2k+1}$ is convex and open, so by conditions (1) and (3), $P(U_{even} \cap U_{odd})$ holds. Since M is covered by the A_m, we are done by condition (2).

Now if M is any smooth manifold, we can cover each $A_m = f^{-1}([m, m+1])$ by finitely many open sets which are diffeomorphic to open sets of $\mathbb{R}^n$. By the above, P holds for all elements of this cover, hence for each A_m and M by condition (2). $\square$

Theorem 3.6.6 (De Rham)**.** *The de Rham map* $\Psi : H^\bullet_{dR}(M) \to H^\bullet(M;\mathbb{R})$ *is an isomorphism for all smooth manifolds* M*.*

Proof. Let $P(U)$ be the statement that $\Psi : H^\bullet_{dR}(U) \to H^\bullet(U;\mathbb{R})$ is an isomorphism. To prove de Rham's theorem, we need only check that the conditions of Lemma 3.6.5 are satisfied. (1) follows from the Poincaré Lemma (3.6.3); (2) was proven in Corollary 3.6.2; and (3) is Lemma 3.6.4. Therefore $P(M)$ holds. $\square$

Example 3.6.7. Consider complex projective n-space $\mathbb{C}P^n$, the set of lines through the origin in $\mathbb{C}^{n+1}$, given by

$$\mathbb{C}P^n = \{[z_0, \ldots, z_n] : [\lambda z_0, \ldots, \lambda z_n] = [z_0, \ldots, z_n] \text{ for any } \lambda \in \mathbb{C} \smallsetminus \{0\}\}.$$

As with real projective space, $\mathbb{C}P^n$ can be covered by coordinate charts

$$U_j = \{[z_0, \ldots, z_n] : z_j \neq 0\} = \left\{\left[\tfrac{z_0}{z_j}, \ldots, 1, \ldots, \tfrac{z_n}{z_j}\right]\right\} \cong \mathbb{C}^n.$$

For instance, in the complex projective plane $\mathbb{C}P^2$ we have a chart

$$U_0 = \{[1, u, v]\} \cong \mathbb{C}^2_{u,v}.$$

This can be parametrized by polar coordinates: set $u = \frac{z_1}{z_0} = re^{2\pi i\theta}$ and $v = \frac{z_2}{z_0} = se^{2\pi i\phi}$. Define a 1-form $\eta \in \Omega^1(U_0)$ by

$$\eta = \frac{r^2\, d\theta + s^2\, d\phi}{1 + r^2 + s^2}.$$

Note that since $\theta = \arctan\left(\frac{y}{x}\right)$, the differential $d\theta$ is given by

$$d\theta = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}dx + \frac{1}{x}dy\right) = \frac{1}{x^2 + y^2}(-y\, dx + x\, dy).$$

Thus $d\theta$ is defined everywhere except the origin. Moreover, $r^2\, d\theta = -y\, dx + x\, dy$ is smooth everywhere on U_0. Since $r^2 = x^2 + y^2$ is also smooth and $r\, dr = x\, dx + y\, dy$, we see that $r\, dr \wedge d\theta = dx \wedge dy$ is smooth everywhere.

Unfortunately, η does not define a 1-form on all of $\mathbb{C}P^2$. Indeed, for one of the other coordinate charts of $\mathbb{C}P^2$,

$$U_1 = \{[u', 1, v']\} \cong \mathbb{C}^2_{u',v'},$$

we have $u' = \frac{z_0}{z_1} = r'e^{2\pi i\theta'}$ and $v' = \frac{z_2}{z_1} = s'e^{2\pi i\phi'}$. Thus $u = \frac{1}{u'}$ and $v = \frac{v'}{u'}$, so $r = \frac{1}{r'}, \theta = -\theta', s = \frac{s'}{r'}$ and $\phi = \phi' - \theta'$. Thus θ can be written in the coordinates u', v' as

$$\theta' = \frac{1}{1 + (r')^2 + (s')^2}(-((s')^2 + 1)\, d\theta' + (s')^2\, d\phi').$$

But $-((s')^2+1)\,d\theta'$ is singular at $r'=0$. However, the 2-form $d\eta \in \Omega^2(U_0)$ *does* extend to a 2-form on all of $\mathbb{C}P^2$, since for example on U_1, we have

$$\begin{aligned}(d\eta)' &= d\eta' \quad \text{since } d \text{ is natural with change of coordinates}\\ &= \frac{1}{(1+(r')^2+(s')^2)^2}[((s')^2+1)2r'\,dr'\wedge d\theta' - 2r'\,dr'\wedge (s')^2\,d\phi' - 2(s')^2\,ds'\wedge d\phi' \\ &\qquad - 2s'\,ds'\wedge (r')^2\,d\theta' + (1+(r')^2+(s')^2)2s'\,ds'\wedge d\phi'].\end{aligned}$$

By the calculations above, each piece is smooth on all of $\mathbb{C}P^2$. The form $\omega \in \Omega^2(\mathbb{C}P^2)$ defined by extending $d\eta$ or $d\eta'$ to $\mathbb{C}P^2$ is called the *Fubini-Study form* on $\mathbb{C}P^2$. A similar construction on $\mathbb{C}P^n$ defines a Fubini-Study form $\omega \in \Omega^2(\mathbb{C}P^n)$.

Note that $d\omega = 0$ everywhere since $\omega = d\eta$ is closed on each affine patch of $\mathbb{C}P^2$. We claim that ω is not exact. Notice that it suffices to show $\omega\wedge\omega$ is not exact: by Proposition 0.2.4, $d(\omega\wedge\omega) = d\omega\wedge\omega + (-1)^2\omega\wedge d\omega = 0+0=0$, so $\omega\wedge\omega$ is closed. Then if $\omega = d\alpha$ for some $\alpha \in \Omega^1(\mathbb{C}P^2)$, we have $d(\alpha\wedge\omega) = d\alpha\wedge\omega + (-1)^2\alpha\wedge d\omega = d\alpha\wedge\omega = \omega\wedge\omega$.

By Stokes' theorem (0.4.5), it's enough to check that $\int_{\mathbb{C}P^2}\omega\wedge\omega \neq 0$. Since U_0 only misses a measure zero subset of $\mathbb{C}P^2$, we can integrate over this coordinate patch:

$$\int_{\mathbb{C}P^2}\omega\wedge\omega = \int_{U_0}\frac{8rs}{(1+r^2+s^2)^3}dr\wedge d\theta\wedge ds\wedge d\phi = 1.$$

Likewise, the Fubini-Study form $\omega \in \Omega^2(\mathbb{C}P^n)$ determines a top form $\underbrace{\omega\wedge\cdots\wedge\omega}_{n}$ which is closed but not exact. This proves:

Theorem 3.6.8. *Let $n \geq 1$. Then*

$$H^{2k}_{dR}(\mathbb{C}P^n) = \begin{cases}\mathbb{R}, & 0 \leq k \leq n\\ 0, & k > n.\end{cases}$$

Moreover, $H^2_{dR}(\mathbb{C}P^n)$ is generated by $[\omega]$ and for each $2 \leq k \leq n$, $H^{2k}_{dR}(\mathbb{C}P^n)$ is generated by $[\omega^k]$, where $\omega^k = \underbrace{\omega\wedge\cdots\wedge\omega}_{k}$.

Recall that the wedge product induces a multiplication on de Rham cohomology:

$$\begin{aligned}H^p_{dR}(M)\times H^q_{dR}(M) &\longrightarrow H^{p+q}_{dR}(M)\\ ([\omega],[\eta]) &\longmapsto [\omega\wedge\eta].\end{aligned}$$

This gives $H^\bullet_{dR}(M)$ the structure of a *differential graded algebra*.

Corollary 3.6.9. *As an algebra, $H^\bullet_{dR}(\mathbb{C}P^n) = \mathbb{R}[c]/(c^{n+1})$, where $c = [\omega]$.*

This additional structure on cohomology allows us to deduce more topological information than homology. For example, Lefschetz's fixed point theorem (2.11.3) and the work above imply:

Corollary 3.6.10. *Any smooth map $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ has a fixed point.*

Proof. By Lefschetz's fixed point theorem, it suffices to show $L(f) \neq 0$. Using the universal coefficient theorem (3.4.2) and de Rham's theorem, one can show that the Lefschetz number may be computed by taking traces on de Rham cohomology instead of real homology:

$$\begin{aligned} L(f) &= \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_* : H_i(\mathbb{C}P^2;\mathbb{Q}) \to H_i(\mathbb{C}P^2;\mathbb{Q})) \\ &= \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f^* : H^i_{dR}(\mathbb{C}P^2) \to H^i_{dR}(\mathbb{C}P^2)). \end{aligned}$$

Since $H^2_{dR}(\mathbb{C}P^2) = \mathbb{R}\langle c\rangle$ by Theorem 3.6.8, we know that $f^*(c) = rc$ for some $c \in \mathbb{R}$. Let f_i denote the restriction of f^* to $H^i_{dR}(\mathbb{C}P^2)$. Then

$$\begin{aligned} \operatorname{tr}(f_0) &= \operatorname{tr}(id) = 1 \\ \operatorname{tr}(f_1) &= \operatorname{tr}(f_3) = 0 \\ \operatorname{tr}(f_2) &= \operatorname{tr}([r]) = r \\ \operatorname{tr}(f_4) &= \operatorname{tr}([r^2]) = r^2. \end{aligned}$$

Thus $L(f) = 1 + r + r^2$ for some $r \in \mathbb{R}$, but $1 + x + x^2$ is irreducible over $\mathbb{R}$, i.e. never vanishes. Hence $L(f) \neq 0$. □

Example 3.6.11. $\mathbb{C}P^2$ is obtained as a CW-complex by attaching a 4-cell to $\mathbb{C}P^1$ via the Hopf map $\partial D^4 = S^3 \to S^2 = \mathbb{C}P^1$.

Corollary 3.6.12. *The Hopf map $h : S^3 \to S^2$ is not homotopic to a constant. In particular, $\pi_3(S^2) \neq 0$.*

Proof. If $h : S^3 \to S^2$ is homotopic to a constant, Example 3.6.11 shows that $\mathbb{C}P^2$ is homotopy equivalent to $S^2 \vee S^4$. Under this identification, we get a map

$$g : S^2 \vee S^4 \to S^2 \xrightarrow{a} S^2 \hookrightarrow S^2 \vee S^4$$

which has no fixed points, where a is the antipodal map. This contradicts Corollary 3.6.10. Hence $h : \Sigma S^2 = S^3 \to S^2$ defines a nontrivial class in $\pi_3(S^2)$. □

It turns out that $\pi_3(S^2) \cong \mathbb{Z}$. We saw in Chapter 1 that the higher homotopy groups of spheres are difficult to compute, and in most cases are not fully understood yet, so Corollary 3.6.12 is a highly nontrivial result.

Example 3.6.13. In this example we prove that $\mathbb{C}P^3$ and $S^2 \times S^4$ have the same cohomology groups, but that they are not homotopy equivalent. All cohomology groups are assumed to have coefficients in $\mathbb{R}$ but notation will be suppressed. $\mathbb{C}P^3$ and $S^2 \times S^4$ each have a CW-structure consisting of one cell in dimensions 0, 2, 4 and 6. In each case the cell complex is

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0.$$

Each arrow is 0, so the complex is its own homology:

$$H_k(\mathbb{C}P^3) = H_k(S^2 \times S^4) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4, 6 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\mathbb{C}P^3$ and $S^2 \times S^4$ are homotopy equivalent. This would determine an isomorphism on cohomology $\psi : H^\bullet(\mathbb{C}P^3) \to H^\bullet(S^2 \times S^4)$ (e.g. by the universal coefficient theorem for cohomology). We will prove that $H^\bullet(\mathbb{C}P^3)$ is not isomorphic as a graded ring to $H^\bullet(S^2 \times S^4)$.

We know $H^2(\mathbb{C}P^3) \cong H^2_{dR}(\mathbb{C}P^3)$ is generated by $c = [\omega]$, the class of the Fubini-Study form $\omega \in \Omega^2(\mathbb{C}P^3)$. On the other hand, $H^n(S^n) = \mathbb{R}$ for each $n \geq 1$, and by the universal coefficient theorem, $H^2(S^2 \times S^4) = \mathbb{R}$. Consider the projection $f : S^2 \times S^4 \to S^2$. This induces a contravariant map on cohomology,

$$f^* : H^2(S^2) \longrightarrow H^2(S^2 \times S^4).$$

Explicitly, for a 2-form $\eta \in \Omega_2(S^2)$, a point $(x, y) \in S^2 \times S^4$ and vector fields $v = v_1 + v_2$ and $w = w_1 + w_2 \in T_{(x,y)}(S^2 \times S^4) = T_x S^2 \oplus T_y S^4$, we have

$$f^*\eta_{(x,y)}(v, w) = \eta_{f(x,y)}(df(v), df(w)) = \eta_{f(x,y)}(v_1, w_1).$$

Since $H^2(S^2 \times S^4) = \mathbb{R} \neq 0$, this cannot be the zero map on $H^2(S^2)$, so f^* is an isomorphism $\mathbb{R} \to \mathbb{R}$. If $r \in H^2(S^2)$ is a generator (i.e. any nonzero element), then f^*r is a generator of $H^2(S^2 \times S^4)$. But $r \wedge r \in H^4(S^2) = 0$, so we must have $(f^*r)^2 = 0$. Therefore such a map $\psi : H^\bullet(\mathbb{C}P^3) \to H^\bullet(S^2 \times S^4)$ would take $c = [\omega]$ to a generator $r \in H^2(S^2 \times S^4)$, but $\psi(c^2) = (\psi c)^2 = r^2 = 0$, so the map is not injective. Hence $H^\bullet(\mathbb{C}P^3)$ and $H^\bullet(S^2 \times S^4)$ are not isomorphic. We conclude that $\mathbb{C}P^3$ and $S^2 \times S^4$ cannot be homotopy equivalent.

4 Products in Homology and Cohomology

One of the easiest products on homology and cohomology rings is the *Kronecker pairing*, which is established via the following proposition.

Proposition 4.0.1. *For a chain complex (C_*, ∂) of R-modules, where R is a commutative ring, with corresponding cochain complex $C^j = \operatorname{Hom}_R(C_j, R)$, there is a bilinear map*

$$C^j \times C_j \to R$$

given by the "application" $(\alpha, \sigma) \mapsto \alpha(\sigma)$. This induces a bilinear pairing of R-modules,

$$\begin{aligned} H^j(C^*) \times H_j(C_*) &\longrightarrow R \\ ([\alpha], [\sigma]) &\longmapsto \langle \alpha, \sigma \rangle = \alpha(\sigma). \end{aligned}$$

Proof. Take a cocycle $\alpha \in C^j$ and a cycle $\sigma \in C_j$. If $\sigma = \partial(\tau)$ for $\tau \in C_{j+1}$ then $\alpha(\sigma) = \alpha(\partial\tau) = (\alpha\partial)\tau = \delta(\alpha)\tau = 0$ since α is a cocycle. Likewise, if $\alpha = \delta(\beta) = \beta\partial$ for some $\beta \in C^{j-1}$ then $\alpha(\sigma) = \beta\partial(\sigma) = \beta(0) = 0$ since σ is a cycle. Hence if α' is a coboundary and σ' is a boundary, then

$$\begin{aligned} &\langle \alpha + \alpha', \sigma \rangle = \langle \alpha, \sigma \rangle + \langle \alpha', \sigma \rangle = \langle \alpha, \sigma \rangle + 0 = \langle \alpha, \sigma \rangle \\ \text{and} \quad &\langle \alpha, \sigma + \sigma' \rangle = \langle \alpha, \sigma \rangle + \langle \alpha, \sigma' \rangle = \langle \alpha, \sigma \rangle + 0 = \langle \alpha, \sigma \rangle. \end{aligned}$$

So $\langle \cdot, \cdot \rangle : H^j(C^*) \times H_j(C_*) \to R$ is well-defined. □

Corollary 4.0.2. *There is a map*

$$H^j(C^*) \to \operatorname{Hom}(H_j(C_*), R).$$

Proof. This is just given by $[\alpha] \mapsto \langle \alpha, \cdot \rangle$, which is a well-defined functional by Proposition 4.0.1. □

The Kronecker pairing is a special case of the cap product, which will be described in Section 4.4.

4.1 Acyclic Models

Suppose $(\mathcal{A}, \mathcal{M})$ is a *category with models*, that is, a category $\mathcal{A}$ with a collection of *models* $\mathcal{M} \subseteq \operatorname{obj}(\mathcal{A})$. Let $\mathcal{C}$ is the category of chain complexes over a ring R.

Definition. *We say a functor $F : \mathcal{A} \to \mathcal{C}$ is* **acyclic** *if $F(M)$ is an acyclic chain complex for all models $M \in \mathcal{M}$.*

Definition. *A functor $F : \mathcal{A} \to \mathcal{C}$ is* **free** *if for any $X \in \mathcal{A}$, the chain group $F(A)_q$ is free with basis some subset of*

$$\{F(u)(F(M)_q) \mid M \in \mathcal{M}, u \in \operatorname{Hom}_{\mathcal{A}}(M, X)\}.$$

Example 4.1.1. Singular homology is a free, acyclic functor on the category of topological spaces with models $\mathcal{M} = \{\text{simplices}\}$.

Theorem 4.1.2. *Let $F, G : \mathcal{A} \to \mathcal{C}$ be functors such that F is free, G is acyclic and $\varphi_0 : H_0(F) \to H_0(G)$ is a natural transformation. Then there is a natural transformation $\Phi : F \to G$ inducing φ_0 on H_0 which is unique up to chain homotopy.*

Proof. For an object $X \in \mathcal{A}$, we must define a map $\Phi_X : F(X) \to G(X)$ which is natural in X. Start with a model $M \in \mathcal{M}$. Then $F(M)$ is a free chain complex and $G(M)$ is acyclic. Since we have a homomorphism $\varphi_0 : H_0(F(M)) \to H_0(G(M))$, the comparison theorem (3.3.4) provides a chain map $\Phi_M : F(M) \to G(M)$ which is unique up to chain homotopy. Fix such a Φ_M for each model M.

Now for $X \in \mathcal{A}$, $F(X)$ is free with basis in $\{F(u)(F(M)) \mid M \in \mathcal{M}, u \in \mathrm{Hom}(M, X)\}$. Define $\Phi_X : F(X) \to G(X)$ by specifying the images of the generators, using Φ_M:

$$\Phi_X : F(u)(F(M)) \longrightarrow G(u)(\Phi_M(F(M))).$$

(Note that Φ_X must be defined this way for it to be natural.) It follows that since the Φ_M are unique up to chain homotopy, the same is true of the Φ_X. This completes the proof. $\square$

Corollary 4.1.3. *If $F, G : \mathcal{A} \to \mathcal{C}$ are free, acyclic functors on a category with models $(\mathcal{A}, \mathcal{M})$ such that $H_0(F)$ and $H_0(G)$ are naturally isomorphic, then F and G are naturally isomorphic.*

4.2 The Künneth Theorem

We saw in Section 3.6 that the homology $H^\bullet(\mathbb{C}P^n)$ inherits a product structure from the wedge product on $H^\bullet_{dR}(\mathbb{C}P^n)$ via de Rham's theorem. In the next few sections, we generalize this structure to the cohomology ring of an arbitrary space in the form of the cup product. To define this, we first must understand the tensor product of chain complexes.

Definition. *For chain complexes $A_\bullet$ and $B_\bullet$, their* **tensor product** *is the complex*

$$(A_\bullet \otimes B_\bullet)_k = \bigoplus_{i+j=k} A_i \otimes B_j$$

with differential $\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b$.

Theorem 4.2.1. *For a pair of spaces X, Y, there exists a cross product on singular chain groups*

$$\begin{aligned} \Delta_\bullet(X) \otimes \Delta_\bullet(Y) &\longrightarrow \Delta_\bullet(X \times Y) \\ \sigma \otimes \tau &\longmapsto \sigma \times \tau \end{aligned}$$

which is unique up to chain homotopy, such that for $x \in \Delta_0(X)$ and $y \in \Delta_0(Y)$, $x \otimes y \mapsto (x, y)$.

Proof. Let $\mathcal{A}$ be the category of pairs of spaces with pairs of morphisms, and define functors $F, G : \mathcal{A} \to \mathcal{C}$ by $F(X,Y) = \Delta_\bullet(X) \otimes \Delta_\bullet(Y)$ and $G(X,Y) = \Delta_\bullet(X,Y)$. On H_0, the map $\varphi_0 : x \otimes y \mapsto (x,y)$ induces the natural transformation

$$\begin{aligned} H_0(\Delta_\bullet(X) \otimes \Delta_\bullet(Y)) &\longrightarrow H_0(\Delta_\bullet(X \times Y)) \\ [x \otimes y] &\longmapsto [(x,y)]. \end{aligned}$$

The models in $\mathcal{A}$ are pairs of simplices (Δ_p, Δ_q), so clearly F is a free functor (by definition of singular chain complexes) and G is acyclic (since $\Delta_p \times \Delta_q$ is contractible). Apply acyclic models (Theorem 4.1.2) to define the cross product on all of $\Delta_\bullet(X) \otimes \Delta_\bullet(Y)$. □

Theorem 4.2.2 (Eilenberg-Zilber)**.** *For any spaces X, Y, there is a natural chain homotopy equivalence $\Delta_\bullet(X) \otimes \Delta_\bullet(Y) \to \Delta_\bullet(X \times Y)$ inducing a natural isomorphism*

$$H_\bullet(X \times Y) \cong H_\bullet(\Delta_\bullet(X) \otimes \Delta_\bullet(Y)).$$

Definition. *A chain homotopy equivalence $\theta : \Delta_\bullet(X \times Y) \to \Delta_\bullet(X) \otimes \Delta_\bullet(Y)$ is called an* **Eilenberg-Zilber map**.

The Künneth theorem gives a formula in terms of Tor_1 for computing the homology of a tensor product of chain complexes.

Theorem 4.2.3 (Künneth)**.** *Let $K_\bullet$ and $L_\bullet$ be free chain complexes over a PID R. Then for each $n \geq 0$, there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(K_\bullet) \otimes H_q(L_\bullet) \to H_n(K_\bullet \otimes L_\bullet) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(K_\bullet), H_q(L_\bullet)) \to 0$$

which is split and natural in $K_\bullet$ and $L_\bullet$.

Proof. The map $\bigoplus_{p+q=n} H_p(K_\bullet) \otimes H_q(L_\bullet) \xrightarrow{\times} H_n(K_\bullet \otimes L_\bullet)$ is given by $[k] \otimes [\ell] \mapsto [k \otimes \ell]$. Fix $n \geq 0$ and set $Z_n = \ker(\partial : K_n \to K_{n-1})$ and $B_n = \mathrm{im}(\partial : K_{n+1} \to K_n)$. Then there is a short exact sequence

$$0 \to Z_n \to K_n \xrightarrow{\partial} B_{n-1} \to 0$$

which splits since $K_\bullet$ is free. Thus tensoring with $L_\bullet$ preserves exactness, so we get another short exact sequence

$$0 \to Z_\bullet \otimes L_\bullet \to K_\bullet \otimes L_\bullet \to B_{\bullet-1} \otimes L_\bullet \to 0. \tag{1}$$

Then $H_\bullet(Z_\bullet \otimes L_\bullet) = Z_\bullet \otimes H_\bullet(L_\bullet)$. In particular,

$$H_n(Z_\bullet \otimes L_\bullet) = \bigoplus_{p+q=n} Z_p \otimes H_q(L_\bullet).$$

Likewise, $H_\bullet(B_\bullet \otimes L_\bullet) = B_\bullet \otimes H_\bullet(L_\bullet)$. The long exact sequence in homology coming from (1) is

$$\begin{tikzcd}[column sep=small]
& H_n(Z_\bullet \otimes L_\bullet) \arrow[r] \arrow[d, equal] & H_n(K_\bullet \otimes L_\bullet) \arrow[r] \arrow[d, equal] & H_n(B_{\bullet-1} \otimes L_\bullet) \arrow[r] \arrow[d, equal] & H_{n-1}(Z_\bullet \otimes L_\bullet) \arrow[d, equal] \\
\arrow[r, "\Delta_n"] & \bigoplus_{p+q=n} Z_p \otimes H_n(L_\bullet) \arrow[r] & H_n(K_\bullet \otimes L_\bullet) \arrow[r] & \bigoplus_{p+q=n-1} B_p \otimes H_q(L_\bullet) \arrow[r, "\Delta_{n-1}"] & \bigoplus_{p+q=n-1} Z_p \otimes H_q(L_\bullet)
\end{tikzcd}$$

This induces a pair of short exact sequences

$$0 \to \operatorname{coker} \Delta_n \to H_n(K_\bullet \otimes L_\bullet) \to \ker \Delta_{n-1} \to 0 \tag{2}$$
$$0 \to B_p \to Z_p \to H_p(K_\bullet) \to 0. \tag{3}$$

Tensoring (3) with $H_q(L_\bullet)$ gives a long exact sequence in Tor,

$$0 = \operatorname{Tor}_1(Z_p H_q(L_\bullet)) \to \operatorname{Tor}_1(H_p(K_\bullet), H_q(L_\bullet)) \to B_p \otimes H_q(L_\bullet) \xrightarrow{\delta} Z_p \otimes H_q(L_\bullet) \to H_p(K_\bullet) \otimes H_q(L_\bullet) \to 0.$$

(Here, $\delta : b \otimes [\ell] \mapsto b \otimes [\ell]$.) One can show that the map

$$\Delta_n : \bigoplus_{p+q=n} B_{p-1} \otimes H_q(L_\bullet) \to \bigoplus_{p+q=n-1} Z_p \otimes H_q(L_\bullet) = \bigoplus_{p+q=n} Z_p \otimes H_q(L_\bullet)$$

is just the diagonal sum of the δ from the long exact sequence above. Hence

$$\operatorname{coker} \Delta_n = \bigoplus_{p+q=n} H_p(K_\bullet) \otimes H_q(L_\bullet)$$
$$\text{and} \quad \ker \Delta_{n-1} = \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(K_\bullet), H_q(L_\bullet))$$

so sequence (3) becomes the desired short exact sequence. The proofs of splitting and naturality of this sequence are routine. □

Corollary 4.2.4. *For two spaces X and Y and any integer $n \geq 0$, there is an isomorphism*

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(X), H_q(Y)).$$

Example 4.2.5. If $X = Y = \mathbb{R}P^2$, then by Example 2.3.20, we have

$$H_p(X) = H_p(Y) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/2\mathbb{Z}, & p = 1 \\ 0, & p \geq 2 \end{cases}$$

Applying the Künneth theorem and relevant calculations in Tor_1 from Example 3.3.16, we obtain

$$H_n(\mathbb{R}P^2 \times \mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2, 3 \\ 0, & n \geq 4. \end{cases}$$

Example 4.2.6. For a pair of spheres $X = S^n$ and $Y = S^m$, all singular homology groups are free $\mathbb{Z}$-modules by Theorem 2.3.5 so Tor_1 always vanishes. Hence the Künneth theorem gives

$$H_k(S^n \times S^m) = \bigoplus_{p+q=k} H_p(S^n) \otimes H_q(S^m) = \begin{cases} \mathbb{Z}, & k = 0, n, m, n+m \\ 0, & \text{otherwise.} \end{cases}$$

4.3 The Cup Product

We have defined a cross product $\Delta_\bullet(X) \otimes \Delta_\bullet(Y) \xrightarrow{\times} \Delta_\bullet(X \times Y)$ which is a chain homotopy equivalence by the Eilenberg-Zilber theorem. We would like to dualize this to get a product on cochain complexes, but this requires that we first understand the relation between the dual $(\Delta_\bullet(X) \otimes \Delta_\bullet(Y))^*$ and $\Delta^\bullet(X) \otimes \Delta^\bullet(Y)$.

Definition. *The map defined by*

$$\begin{aligned} \times_{alg} : \Delta^\bullet(X) \otimes \Delta^\bullet(Y) &\longrightarrow (\Delta_\bullet(X) \otimes \Delta_\bullet(Y))^* \\ \alpha \otimes \beta &\longmapsto \Big(\alpha \times_{alg} \beta : \sum z_i \otimes w_j \mapsto \sum (-1)^{\deg z_i \deg \beta} \alpha(z_i)\beta(w_j)\Big) \end{aligned}$$

is called the **algebraic cross product** *on cochains.*

Lemma 4.3.1. *The algebraic cross product is a chain map.*

The algebraic cross product determines a map on the tensor product of homologies:

$$\begin{aligned} H^p(X) \otimes H^q &\longrightarrow H^{p+q}((\Delta_\bullet(X) \otimes \Delta_\bullet(Y))^*) \\ [\alpha] \otimes [\beta] &\longmapsto [\alpha \times_{alg} \beta]. \end{aligned}$$

Definition. *The* **cohomology cross product** *is the composition*

$$\times : H^p(X) \otimes H^q(Y) \xrightarrow{\times_{alg}} H^{p+q}((\Delta_\bullet(X) \otimes \Delta_\bullet(Y))^*) \xrightarrow{\theta^*} H^{p+q}(X \times Y)$$

where $\theta : \Delta_\bullet(X \times Y) \to \Delta_\bullet(X) \otimes \Delta_\bullet(Y)$ is an Eilenberg-Zilber map.

Note that since any Eilenberg-Zilber map is unique up to chain homotopy, the induced map θ^* is well-defined.

Definition. *For a space X, the* **cup product** *on cohomology is the product map*

$$\cup : H^p(X) \otimes H^q(X) \longrightarrow H^{p+q}(X)$$

defined on cochains by

$$\Delta^p(X) \otimes \Delta^q(X) \xrightarrow{\times} \Delta^{p+q}(X \times X) \xrightarrow{\Delta^*} \Delta^{p+q}(X),$$

where $\Delta : X \to X \times X$ is the diagonal map $x \mapsto (x, x)$. That is, for $\alpha \in H^p(X)$ and $\beta \in H^q(X)$, $\alpha \cup \beta = \Delta^*(\alpha \times \beta)$.

Proposition 4.3.2. *Let X and Y be spaces and $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the natural projections. Then*

(1) For any maps $f : X' \to X$ and $g : Y' \to Y$, $(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$.

(2) The cup product is natural, i.e. $f^*(\alpha_1 \cup \alpha_2) = f^*\alpha_1 \cup f^*\alpha_2$ for any $\alpha_1, \alpha_2 \in H^\bullet(X)$ and map $f : Y \to X$.

(3) $\alpha \times \beta = p_X^*\alpha \cup p_Y^*\beta$ for any $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$.

Proof. (1) An Eilenberg-Zilber map θ is a chain map and so is its transpose θ^*, so the cross product is natural in both X and Y.

(2) We have

$$\begin{aligned} f^*\alpha_1 \cup f^*\alpha_2 &= \Delta^*(f^*\alpha_1 \times f^*\alpha_2) \\ &= \Delta^*(f \times f)^*(\alpha_1 \times \alpha_2) \quad \text{by (1)} \\ &= ((f \times f)\Delta)^*(\alpha_1 \times \alpha_2) \\ &= (\Delta f)^*(\alpha_1 \times \alpha_2) \\ &= f^*\Delta^*(\alpha_1 \times \alpha_2) \\ &= f^*(\alpha_1 \cup \alpha_2). \end{aligned}$$

(3) Let $\Delta_{X\times Y} : X \times Y \to (X \times Y) \times (X \times Y)$ be the diagonal map. For the projection maps p_X, p_Y and any $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$,

$$\begin{aligned} p_X^*\alpha \cup p_Y^*\beta &= \Delta_{X\times Y}^*(p_X^*\alpha \times p_Y^*\beta) \\ &= \Delta_{X\times Y}^*(p_X \times p_Y)^*(\alpha \times \beta) \\ &= ((p_X \times p_Y)\Delta)^*(\alpha \times \beta) \\ &= id_{X\times Y}^*(\alpha \times \beta) = \alpha \times \beta. \end{aligned}$$

□

Definition. *A* **diagonal approximation** *is a chain map $\tau : \Delta_\bullet(X) \to \Delta_\bullet(X) \otimes \Delta_\bullet(X)$ for each space X such that τ is natural in X and $\tau(x) = x \otimes x$ on 0-simplices $x \in \Delta_0(X)$.*

By acyclic models (Theorem 4.1.2), such a τ exists and is unique up to chain homotopy. Observe that if $\theta : \Delta_\bullet(X \times X) \to \Delta_\bullet(X) \otimes \Delta_\bullet(X)$ is an Eilenberg-Zilber map, then the composition

$$\tau : \Delta(X) \xrightarrow{\Delta_*} \Delta_\bullet(X \times X) \xrightarrow{\theta} \Delta_\bullet(X) \otimes \Delta_\bullet(X)$$

is a diagonal approximation. Then the cup product can be written in terms of a diagonal approximation: $\alpha \cup \beta = \tau^*(\alpha \otimes \beta)$.

Lemma 4.3.3. *Any Eilenberg-Zilber map $\theta : \Delta_\bullet(X \times X) \to \Delta_\bullet(X) \otimes \Delta_\bullet(X)$ uniquely determines a diagonal approximation $\tau : \Delta_\bullet(X) \to \Delta_\bullet(X) \otimes \Delta_\bullet(X)$.*

Proof. We saw above that $\tau = \theta \circ \Delta_*$ is a diagonal approximation whenever θ is an Eilenberg-Zilber map. Conversely, a diagonal approximation τ determines θ by the composition

$$\Delta_\bullet(X \times Y) \xrightarrow{\tau} \Delta_\bullet(X \times Y) \otimes \Delta_\bullet(X \times Y) \xrightarrow{p_X \otimes p_Y} \Delta_\bullet(X) \otimes \Delta_\bullet(Y).$$

□

For any space X, there is a "distinguished" cochain $\varepsilon \in \Delta^0(X)$ satisfying $\varepsilon(x) = 1$ for every 0-simplex x. As a map, this is the just the augmentation map $\varepsilon : \Delta_0(X) \to \mathbb{Z}$. As a cohomology class, we write $[\varepsilon] = 1 \in H^0(X)$. We can now show that the cup product endows $H^\bullet(X)$ with the structure of a graded commutative ring with unity $1 = [\varepsilon]$.

Proposition 4.3.4. *For any $\alpha \in H^p(X)$,*

(a) $\alpha \times 1_Y = p_X^*(\alpha) \in H^p(X \times Y)$ for any space Y.

(b) $\alpha \cup 1 = 1 \cup \alpha = \alpha$.

Proof. (a) Let P be a point space and consider the functors

$$F : X \longmapsto \Delta_\bullet(X \times P) \quad \text{and} \quad G : X \longmapsto \Delta_\bullet(X).$$

Then there is a natural transformation $F \to G$ given by

$$\Delta_\bullet(X \times P) \xrightarrow{\theta} \Delta_\bullet(X) \otimes \Delta_\bullet(P) \xrightarrow{1\otimes\varepsilon} \Delta_\bullet(X) \otimes \mathbb{Z} = \Delta_\bullet(X). \tag{$*$}$$

On the other hand, $(p_X)_* : \Delta_\bullet(X\times P) \to \Delta_\bullet(X)$ gives another natural transformation $F \to G$ that determines the same map on 0-simplices as $(*)$. Thus by acyclic models (Theorem 4.1.2), F and G are naturally isomorphic. Taking the dual of $(*)$, we get $\alpha \mapsto \alpha \times 1_P$, while the dual of $(p_X)_*$ is $(p_X)^*$. Hence (a) holds for a point space P.

Now for any Y, there is a unique map $f : Y \to P$ to a point space, with $f^*\varepsilon = \varepsilon$, and thus $f^*(1_P) = 1_Y$. Then under the composition $H^p(X) \to H^p(X \times P) \xrightarrow{(id\times f)^*} H^p(X \times Y)$, any $\alpha \in H^p(X)$ maps to

$$(id \times f)^*(\alpha \times 1_P) = \alpha \times f^*(1_P) = \alpha \times 1_Y$$

but by the above case for P, $\alpha \times 1_P = p_X^*(\alpha)$, so we also have

$$(id \times f)^*(\alpha \times 1_P) = (id \times f)^* p_X^*(\alpha) = (p_X(id \times f))^*(\alpha) = p_X^*(\alpha).$$

Hence $p_X^*(\alpha) = \alpha \times 1_Y$ as claimed.

(b) Now for $\alpha \in H^p(X)$,

$$\alpha \cup 1 = \Delta^*(\alpha \times 1) = \Delta^* p_X^*(\alpha) = (p_X\Delta)^*(\alpha) = id^*\alpha = \alpha.$$

The proof is similar for $1 \cup \alpha$. □

Theorem 4.3.5. *The cup product is associative and graded-commutative, that is, $\alpha \cup \beta = (-1)^{|\alpha|\,|\beta|}\beta \cup \alpha$ for any $\alpha, \beta \in \Delta^\bullet(X)$.*

Proof. First, there are Eilenberg-Zilber maps θ and ϕ that fit into the following diagram:

$$\begin{tikzcd}
\Delta_\bullet(X \times Y \times Z) \arrow[r, "\theta"] \arrow[d, "\phi"'] & \Delta_\bullet(X \times Y) \otimes \Delta_\bullet(Z) \arrow[d] \\
\Delta_\bullet(X) \otimes \Delta_\bullet(Y \times Z) \arrow[r] & \Delta_\bullet(X) \otimes \Delta_\bullet(Y) \otimes \Delta_\bullet(Z)
\end{tikzcd}$$

Both are natural transformations of functors $(X, Y, Z) \mapsto \Delta_\bullet(X \times Y \times Z)$, so by acyclic models (Theorem 4.1.2), it's enough to show that θ and ϕ induce the same map on H_0, which clearly they do since tensor products are associative. Hence θ and ϕ are naturally chain homotopic. Taking the transpose, we see that $\times_{alg}$ is associative. Finally, associativity of the cup product follows from this observation, and the commutativity of the diagram:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "\Delta \times id"] \\
X \times X \arrow[r, "id \times \Delta"'] & X \times X \times X
\end{tikzcd}$$

where Δ is the diagonal map.

Next, we show $\cup$ is graded-commutative. Consider the "geometric" and "algebraic" order-reversing maps:

$$\begin{array}{rccc} S: & X \times Y & \longrightarrow & Y \times X \\ & (x, y) & \longmapsto & (y, x) \end{array} \quad \text{and} \quad \begin{array}{rccc} T: & \Delta_\bullet(X) \otimes \Delta_\bullet(Y) & \longrightarrow & \Delta_\bullet(Y) \otimes \Delta_\bullet(X) \\ & a \otimes b & \longmapsto & (-1)^{|a|\,|b|} b \otimes a. \end{array}$$

Note that S is a homeomorphism and T is a chain homotopy equivalence. Then the maps $T\theta$ and θS are both natural transformations giving the same map on H_0, so by acyclic models (Theorem 4.1.2), they are naturally isomorphic; that is, the following diagram commutes:

$$\begin{tikzcd}
\Delta_\bullet(X \times Y) \arrow[r, "\theta"] \arrow[d, "S"'] & \Delta_\bullet(X) \otimes \Delta_\bullet(Y) \arrow[d, "T"] \\
\Delta_\bullet(Y \times X) \arrow[r, "\theta"'] & \Delta_\bullet(Y) \otimes \Delta_\bullet(X)
\end{tikzcd}$$

Further, $S\Delta = \Delta$, so we have

$$\begin{aligned}
\alpha \cup \beta &= \Delta^*(\alpha \times \beta) = \Delta^* S^*(\alpha \times \beta) \\
&= \Delta^* S^* \theta^*(\alpha \otimes \beta) \quad \text{interpreting cross product using diagonal approximation} \\
&= \Delta^* \theta^* T^*(\alpha \otimes \beta) \quad \text{up to a boundary} \\
&= \Delta^* \theta^*((-1)^{|\alpha|\,|\beta|} \beta \otimes \alpha) = (-1)^{|\alpha|\,|\beta|} \beta \cup \alpha.
\end{aligned}$$

Therefore the formulas agree on cohomology. □

How do we compute cup products in practice?

Definition. *Suppose $\sigma : \Delta_n \to X$ is an n-simplex and $p, q \geq n$. The* **front p-face** *of σ is the p-simplex $\sigma\rfloor_p : \Delta_p \to X$ given by $\sigma\rfloor_p(t_0, \dots, t_p) = \sigma(t_0, \dots, t_p, 0, \dots, 0)$. Similarly, the* **back q-face** *of σ is the q-simplex ${}_q\lfloor\sigma : \Delta_q \to X$, ${}_q\lfloor\sigma(t_0, \dots, t_q) = \sigma(0, \dots, 0, t_0, \dots, t_q)$.*

Definition. *For a pair (X, Y), the* **Alexander-Whitney map** *is the map $\theta : \Delta_\bullet(X \times Y) \to \Delta_\bullet(X) \otimes \Delta_\bullet(Y)$ defined on an n-simplex $\sigma \in \Delta_n(X \times Y)$ by*

$$\theta(\sigma) = \sum_{p+q=n} (p_X \otimes \sigma)\rfloor_p \otimes {}_q\lfloor(p_Y \otimes \sigma).$$

Lemma 4.3.6. *The Alexander-Whitney map is an Eilenberg-Zilber map.*

Proof. One sees that θ induces $(x, y) \mapsto x \otimes y$ on H_0 and is chain map. □

By Lemma 4.3.3, the Alexander-Whitney map determines a diagonal approximation τ.

Definition. *The diagonal approximation corresponding to θ is called the* **Alexander-Whitney diagonal approximation***; explicitly,*

$$\begin{aligned} \tau : \Delta_\bullet(X) &\longrightarrow \Delta_\bullet(X) \otimes \Delta_\bullet(X) \\ \sigma &\longmapsto \sum_{p+q=n} \sigma\rfloor_p \otimes {}_q\lfloor\sigma. \end{aligned}$$

Corollary 4.3.7. *The cup product may be computed on the chain level for $\alpha \in \Delta^p(X), \beta \in \Delta^q(X)$ and $\sigma \in \Delta_{p+q}(X)$ by*

$$(\alpha \cup \beta)(\sigma) = (-1)^{pq}\alpha(\sigma\rfloor_p)\beta({}_q\lfloor\sigma).$$

Proof. Let τ be the Alexander-Whitney diagonal approximation. Then we have

$$\begin{aligned} (\alpha \cup \beta)(\sigma) &= \tau^*(\alpha \times \beta)(\sigma) = (\alpha \times \beta)\tau(\sigma) \\ &= (\alpha \times \beta) \sum_{r+s=p+q} \sigma\rfloor_r \otimes {}_s\lfloor\sigma \\ &= \sum_{r+s=p+q} (-1)^{|\beta|r} \quad \text{by definition of the cross product} \\ &= (-1)^{pq}\alpha(\sigma\rfloor_p)\beta({}_q\lfloor\sigma). \end{aligned}$$

□

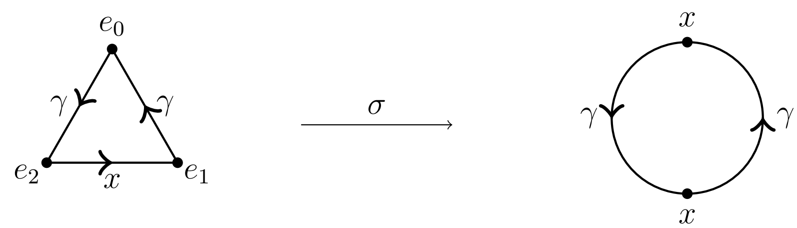
Example 4.3.8. In this example, we compute the cup products of $X = \mathbb{R}P^2$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Recall from Example 2.3.20 that the projective plane has the following mod 2 homology groups:

$$H_k(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0, 1, 2 \\ 0, & k \geq 3. \end{cases}$$

Since $\mathbb{Z}/2\mathbb{Z}$ is a field, Proposition 3.3.8 ensures the Ext groups in the universal coefficient theorem vanish, so over $\mathbb{Z}/2\mathbb{Z}$, the cohomology and homology of $\mathbb{R}P^2$ are isomorphic:

$$H^k(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0, 1, 2 \\ 0, & k \geq 3. \end{cases}$$

Let α be a 1-cochain such that $[\alpha]$ generates $H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ and take γ to be a 1-chain such that $[\gamma] \in H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ is a generator; then $\alpha(\gamma) = 1$. We would like to compute $[\alpha] \cup [\alpha]$. Consider the following 2-simplex σ:



Notice that $\partial\sigma = \gamma - c_1 + \gamma = 2\gamma - c_1$, where c_1 is a constant 1-simplex. On the other hand, if c_2 is a constant 2-simplex then $\partial c_2 = c_1$, so we have $\partial(\sigma + c_1) = 2\gamma$ but this is 0 in $\Delta_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. Thus $[\sigma + c_1]$ generates $H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. Now using the Alexander-Whitney approximation, we have

$$\begin{aligned} &(\alpha \cup \alpha)(\sigma) = \alpha(\sigma\rfloor_1)\alpha({}_1\lfloor\sigma) = \alpha(\gamma)\alpha(\gamma) = 1 \cdot 1 = 1 \\ \text{and} \quad &(\alpha \cup \alpha)(c_1) = \alpha(c_1\rfloor_1)\alpha({}_1\lfloor c_1) = \alpha(\partial c_2\rfloor_2)\alpha({}_1\lfloor\partial c_2) = 0 \cdot 0 = 0, \end{aligned}$$

since α is a cocycle. Hence $(\alpha \cup \alpha)(\sigma + c_1) = 1$ so $[\alpha \cup \alpha]$ must be a generator of $H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. The cohomology of $\mathbb{R}P^2$ with mod 2 coefficients therefore has the following ring structure: $H^\bullet(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[\alpha]/(\alpha^3)$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and $[\alpha]$ generates $H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. This generalizes to projective n-space:

Theorem 4.3.9. *For any $n \geq 2$, $H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[\alpha]/(\alpha^{n+1})$, where $[\alpha]$ is a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.*

4.4 The Cap Product

In this section we define the cap product between homology and cohomology, that is, a map

$$H^p(X) \otimes H_n(X) \longrightarrow H_{n-p}(X)$$

which generalizes the Kronecker pairing (the case $n = p$) of Proposition 4.0.1.

Definition. *For a space X, the* **cap product** *is the product map*

$$\cap : H^p(X) \otimes H_n(X) \longrightarrow H_{n-p}(X)$$

defined on cochains and chains by

$$\Delta^p(X) \otimes \Delta_n(X) \xrightarrow{1\otimes\tau} \Delta^p(X) \otimes \bigoplus_{p'+q'=n} (\Delta_{p'}(X) \otimes \Delta_{q'}(X)) \xrightarrow{\kappa} \mathbb{Z} \otimes \Delta_q = \Delta_q$$

where τ is a diagonal approximation and κ is the Kronecker pairing of $\Delta^p(X)$ with each $\Delta_{q'}(X)$, which is zero when $q' \neq n - p$.

Using the Alexander-Whitney diagonal approximation, the cap product can be written

$$\alpha \cap \sigma = (-1)^{pq}\alpha(\sigma\rfloor_p)\,{}_q\lfloor\sigma$$

for any cochain $\alpha \in \Delta^p(X)$ and chain $\sigma \in \Delta_n(X)$, with $p + q = n$.

Theorem 4.4.1. *The cap product is a well-defined product $H^p(X) \otimes H_n(X) \to H_{n-p}(X)$ satisfying the following properties:*

(a) When $n = p$, the cap product equals the Kronecker pairing $H^p(X) \otimes H_p(X) \to \mathbb{Z}$.

(b) For any $\alpha \in \Delta^p(X)$ and $\sigma \in \Delta_n(X)$, $\partial(\alpha \cap \sigma) = \delta\alpha \cap \sigma + (-1)^p \alpha \cap \partial\sigma$.

(c) $\cap$ is natural and is the unique product satisfying (a) – (c).

(d) For any chain σ, $1 \cap \sigma = \sigma$ where $1 = [\varepsilon]$ is the cohomology class of the augmentation $\varepsilon : \Delta_0(X) \to \mathbb{Z}$.

(e) $\alpha \cap (\beta \cap \sigma) = (\alpha \cup \beta) \cap \sigma$ for any $\alpha, \beta \in \Delta^\bullet(X)$ and $\sigma \in \Delta_\bullet(X)$.

(f) For spaces X and Y with a continuous map $f : X \to Y$, and for classes $\alpha \in H^p(Y)$ and $c \in H_n(X)$, $f_*(f^*(\alpha) \cap c) = \alpha \cap f_*(c)$.

Proof. (a) is clear using the Alexander-Whitney diagonal approximation: for $\alpha \in \Delta^p(X)$ and $\sigma \in \Delta_p(X)$,

$$\alpha \cap \sigma = (-1)^{p^2} \alpha(\sigma\rfloor_p)\, {}_q\lfloor\sigma = \alpha(\sigma) = \langle \alpha, \sigma \rangle.$$

(b) Consider the diagram

$$\begin{tikzcd} H^\bullet(X) \otimes H_\bullet(X) \arrow[r, "\delta \otimes 1 + 1 \otimes \partial"] \arrow[d, "\cap"'] & H^\bullet(X) \otimes H_\bullet(X) \arrow[d, "\cap"] \\ H_\bullet(X) \arrow[r, "\partial"'] & H_\bullet(X) \end{tikzcd}$$

This represents two functors $H^\bullet(X) \otimes H_\bullet(X) \to H_\bullet(X)$, so to show the diagram commutes, it's enough by Theorem 4.1.2 to show the diagram commutes on the level of $H_0(X)$. But by (a), for any 0-chain x we have

$$\partial(\alpha \cap x) = \partial \circ \alpha(x) = (\delta\alpha)(x) = (\delta\alpha)(x) + (-1)^p \alpha \cap 0 = \delta\alpha \cap x + (-1)^p \alpha \cap \partial x.$$

We now use (b) to prove that cap product is well-defined on (co)homology. It is clear that $\cap$ is bilinear. If $\alpha = \delta\alpha'$ for some cochain α', then by (b),

$$\partial(\alpha \cap \sigma) = \partial(\delta\alpha' \cap \sigma) = \delta^2\alpha' \cap \sigma + (-1)^p \delta\alpha' \cap \partial\sigma = (-1)^p \delta\alpha' \cap \partial\sigma = 0.$$

Likewise, if $\sigma = \partial\sigma'$ then

$$\partial(\alpha \cap \sigma) = \partial(\alpha \cap \partial\sigma') = \delta\alpha \cap \partial\sigma' + (-1)^p \alpha \cap \partial^2\sigma' = \delta\alpha \cap \partial\sigma' = 0.$$

Therefore $\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X), [\alpha] \otimes [\sigma] \mapsto [\alpha \cap \sigma]$ is well-defined.

(c) Naturality follows from naturality of diagonal approximations (by definition) and the Kronecker pairing (Proposition 4.0.1). Uniqueness follows from the fact that diagonal approximations are up to chain homotopy.

(d) is trivial from the Alexander-Whitney diagonal approximation definition of $\cap$.

(e) Consider the diagram

$$\begin{tikzcd} H^\bullet(X) \otimes H^\bullet(X) \otimes H_\bullet(X) \arrow[r, "\cup \otimes 1"] \arrow[d, "1 \otimes \cap"'] & H^\bullet(X) \otimes H_\bullet(X) \arrow[d, "\cap"] \\ H^\bullet(X) \otimes H_\bullet(X) \arrow[r, "\cap"'] & H_\bullet(X) \end{tikzcd}$$

As in part (b), it suffices to show the diagram commutes on 0-cochains and 0-chains, by Theorem 4.1.2. If $x, y \in H^0(X)$ and $z \in H_0(X)$, we have

$$(x \cup y) \cap z = (xy)(z) = x(y(z)) = x \cap y(z) = x \cap (y \cap z)$$

so the diagram commutes.

(f) Set $q = n - p$. Using the Alexander-Whitney diagonal approximation, we have

$$\begin{aligned} f_*(f^*\alpha \cap c) &= f_*[(-1)^{pq}(f^*\alpha)({}_p\lfloor c)c\rfloor_q] = (-1)^{pq}(f^*\alpha)({}_p\lfloor c)f_*(c\rfloor_q) \\ &= (-1)^{pq}(\alpha \circ f)({}_p\lfloor c)f_*(c\rfloor_q); \\ \text{while} \quad \alpha \cap f_*(c) &= (-1)^{pq}\alpha({}_p\lfloor (f_*c))(f_*c)\rfloor_q = (-1)^{pq}\alpha(f_*({}_p\lfloor c))f_*(c\rfloor_q) \\ &= (-1)^{pq}(\alpha \circ f)({}_p\lfloor c)f_*(c\rfloor_q). \end{aligned}$$

Thus they are equal. □

Property (e) shows that $H_\bullet(X)$ can be made into a module over the cohomology ring $H^\bullet(X)$. Cap product can be defined analogously for homology and cohomology with coefficients in an abelian group G. There is also a relative cap product which we define as follows. Let $A \subseteq X$. Define the relative cap product on chains by:

$$\begin{aligned} \Delta^p(X, A) \otimes \Delta_{p+q}(X, A) &\longrightarrow \Delta_q(X) \\ \alpha \otimes c &\longmapsto \alpha \cap c. \end{aligned}$$

Write $\Delta^p(X, A) = \operatorname{Hom}\left(\dfrac{\Delta_p(X)}{\Delta_p(A)}, \mathbb{Z}\right)$ and $\Delta_{p+q}(X, A) = \dfrac{\Delta_{p+q}(X)}{\Delta_{p+q}(A)}$. Then α lifts uniquely to a cochain $\bar{\alpha} : \Delta_p(X) \to \mathbb{Z}$ that vanishes on $\Delta_p(A)$. So if $c \in \Delta_{p+q}(A)$, then

$$\alpha \cap c = \bar{\alpha} \cap c = (-1)^{pq}\bar{\alpha}({}_p\lfloor c)c\rfloor_q = (-1)^{pq} \cdot 0 \cdot c\rfloor_q = 0$$

since ${}_p\lfloor c \in \Delta_p(A)$. Thus the relative cap product is well-defined on chains. Now to show this defines a cap product on relative (co)homology, take a cocycle $\alpha \in \Delta^\bullet(X, A)$ and a cycle $c \in \Delta_\bullet(X, A)$. Then by Theorem 4.4.1(b), we have

$$\partial(\alpha \cap c) = \delta\alpha \cap c + (-1)^p \alpha \cap \partial c = 0 + 0$$

since α is a cocycle and $\partial c \in \Delta_\bullet(A)$. Moreover, if $\alpha = \delta\alpha'$ then

$$\alpha \cap c = \delta\alpha' \cap c = \partial(\alpha' \cap c) - (-1)^p \alpha' \cap \partial c = \partial(\alpha' \cap c)$$

since $\partial c \in \Delta_\bullet(A)$, so $[\alpha \cap c] = 0$ in homology. Similarly, if $c = \partial c'$, then

$$\alpha \cap c = \alpha \cap \partial c' = (-1)^p \partial(\alpha \cap c') - (-1)^p \delta\alpha \cap c' = (-1)^p \partial(\alpha \cap c')$$

which is 0 in homology. Thus the relative cap product is well-defined:

$$\begin{aligned} \cap : H^p(X, A) \otimes H_n(X, A) &\longrightarrow H_{n-p}(X) \\ [\alpha] \cap [c] &\longmapsto [\alpha \cap c]. \end{aligned}$$

More generally, if $A, B \subseteq X$ are subspaces for which $\Delta_\bullet(A) + \Delta_\bullet(B) \to \Delta_\bullet(A \cup B)$ is an isomorphism on homology, then there is a relative cap product

$$\cap : H^p(X, A) \otimes H_n(X, A \cup B) \longrightarrow H_{n-p}(X, B).$$

All of the above definitions generalize to arbitrary coefficients.

The formulas in Theorem 4.4.1 specialize to the Kronecker pairing:

Corollary 4.4.2. *Let X and Y be spaces, $\alpha, \beta \in H^\bullet(X)$, $\gamma \in H^\bullet(Y)$ and $\sigma \in H_\bullet(X)$. Then*

(a) $\langle \alpha \cup \beta, \sigma \rangle = \langle \alpha, \beta \cap \sigma \rangle$.

(b) $\langle f^*(\alpha), \sigma \rangle = \langle \alpha, f_*(\sigma) \rangle$.

In particular, (b) demonstrates the 'duality' of the cup and cap products. This will become more evident in Chapter 5.

5 Duality

In the previous chapter, we saw how the cap and cup products exhibited a kind of duality between the homology and cohomology groups of a space. In this chapter we make that connection explicit for manifolds. We will prove that if M is a compact, oriented n-manifold then there is an isomorphism
$$H_p(M) \cong H^{n-p}(M).$$
More generally, for a compact manifold with boundary we will also obtain isomorphisms $H_p(M, \partial M) \cong H^{n-p}(M)$ and $H_p(M) \cong H^{n-p}(M, \partial M)$. From this we will deduce many beautiful results about manifolds.

5.1 Direct Limits

Definition. *A* **direct system** *is a family of groups $(A_\alpha)_{\alpha\in I}$, for some directed index set I, together with homomorphisms $f_{\beta\alpha} : A_\alpha \to A_\beta$ for all $\beta \geq \alpha$ such that when $\gamma \geq \beta \geq \alpha$, we have $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}$.*

Definition. *Given a direct system $(A_\alpha, f_{\beta\alpha})$ over a direct set I, the* **direct limit** *is defined as the quotient group*

$$\varinjlim A_\alpha = \left(\bigoplus_{\alpha\in I} A_\alpha\right) / \langle f_{\beta\alpha}(a) - a \mid a \in A_\alpha, \beta \geq \alpha\rangle.$$

There are natural inclusions $A_\alpha \hookrightarrow \bigoplus A_\alpha$ which induce homomorphisms $i_\alpha : A_\alpha \to \varinjlim A_\alpha$ making the following diagrams commute:

$$\begin{tikzcd}
A_\alpha \arrow[r, "f_{\beta\alpha}"] \arrow[dr, "i_\alpha"'] & A_\beta \arrow[d, "i_\beta"] \\
& \varinjlim A_\alpha
\end{tikzcd}$$

The next result says that direct limits are a solution to a universal mapping property for direct systems.

Proposition 5.1.1. *Suppose $(A_\alpha, f_{\beta\alpha})$ is a direct system of abelian groups with direct limit $A = \varinjlim A_\alpha$. Then for any abelian group G and any collection of homomorphisms $g_\alpha : A_\alpha \to G$ satisfying $g_\beta \circ f_{\beta\alpha} = g_\alpha$, there exists a unique homomorphism $g : \varinjlim A_\alpha \to G$ such that*

(1) $g \circ i_\alpha = g_\alpha$ *for all* $\alpha \in I$.

(2) $\operatorname{im} g = \{x \in G \mid x = g_\alpha(x_\alpha)$ *for some* α *and* $x_\alpha \in A_\alpha\}$.

(3) $\ker g = \{a \in A \mid a = i_\alpha(x_\alpha)$ *and* $g_\alpha(x_\alpha) = 0$ *for some* $\alpha\}$.

Corollary 5.1.2. *Let $(A_\alpha, f_{\beta\alpha})$ be a direct system, with $A = \varinjlim A_\alpha$. Then a collection of homomorphisms $g_\alpha : A_\alpha \to G$ for which $g_\beta \circ f_{\beta\alpha} = g_\alpha$ defines an isomorphism $g : A \to G$ if and only if the following conditions hold:*

(1) For all $x \in G$, there exists $\alpha \in I$ and $x_\alpha \in A_\alpha$ such that $g_\alpha(x_\alpha) = x$.

(2) If $g_\alpha(x_\alpha) = 0$ then there exists $\beta \geq \alpha$ for which $f_{\beta\alpha}(x_\alpha) = 0$.

Theorem 5.1.3. *Suppose $(A_\alpha), (A'_\alpha)$ and (A''_α) are direct systems over I. Then if for each $\alpha \in I$ there is a short exact sequence $0 \to A'_\alpha \to A_\alpha \to A''_\alpha \to 0$ which is natural with respect to the homomorphisms in each direct system, then the induced sequence*

$$0 \to \varinjlim A'_\alpha \to \varinjlim A_\alpha \to \varinjlim A''_\alpha \to 0$$

is exact.

5.2 The Orientation Bundle

To 'orient' our path to proving Poincaré duality so to speak, we begin with the following observation.

Lemma 5.2.1. *Let M be an n-dimensional manifold and G any coefficient group. Then for each $x \in M$, there is an isomorphism $H_n(M, M \smallsetminus \{x\}; G) \cong G$.*

Proof. Let $x \in A \subseteq M$ be any bounded, convex subset with respect to some coordinate chart of M. Then A is contained in a closed disk $D^n \subset \mathbb{R}^n \subseteq M$. Consider the commutative diagram

$$\begin{tikzcd}
H_n(M, M \smallsetminus A; G) \arrow[r] & H_n(M, M \smallsetminus \{x\}; G) \\
H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus A; G) \arrow[r] \arrow[u, "\cong"] & H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x\}; G) \arrow[u, "\cong"'] \\
H_n(D^n, \partial D^n; G) \arrow[r, "\mathrm{id}"'] \arrow[u, "\cong"] & H_n(D^n, \partial D^n; G) \arrow[u, "\cong"']
\end{tikzcd}$$

Here, the horizontal arrows are induced by inclusion, the top pair of vertical arrows are by excision and the bottom pair of vertical arrows are by homotopy. Moreover, by Theorem 2.3.5, $H_n(D^n, \partial D^n; G) \cong G$ so the diagram implies $H_n(M, M \smallsetminus A; G)$ and $H_n(M, M \smallsetminus \{x\}; G)$ are each isomorphic to G. $\square$

Definition. *When $G = \mathbb{Z}$, a* **local orientation** *of M at $x \in M$ is a choice of generator $\theta_x \in H_n(M, M \smallsetminus \{x\}) \cong \mathbb{Z}$.*

Definition. *An* **orientation** *of M is a choice of local orientation $x \mapsto \theta_x \in H_n(M, M \smallsetminus \{x\})$ for each $x \in M$ that is locally constant. That is, for each $x \in M$ there is a neighborhood $U \subseteq M$ of x and a class $\theta_U \in H_n(M, M \smallsetminus \overline{U})$ such that for each $y \in \overline{U}$, the restriction map $j_{y,U} : H_n(M, M \smallsetminus \overline{U}) \to H_n(M, M \smallsetminus \{y\})$ satisfies $j_{y,U}(\theta_U) = \theta_y$.*

Definition. *The assignment $U \mapsto \langle \theta_U \rangle$ defines the structure of a sheaf on M with stalks $\langle \theta_x \rangle = \mathbb{Z}$, called the* **orientation sheaf** *of M.*

Definition. *The* **orientation bundle** *of an n-manifold M with coefficients in G is the set*

$$\Theta_M(G) = \coprod_{x \in M} H_n(M, M \smallsetminus \{x\}; G)$$

with the following topology. Let $p : \Theta_M(G) \to M$ be the map sending $H_n(M, M \smallsetminus \{x\}; G)$ to x. For $U \subseteq M$ open and $\alpha \in H_n(M, M \smallsetminus \overline{U}; G)$, define the subset

$$U_\alpha = \{j_{y,U}(\alpha) \in H_n(M, M \smallsetminus \{y\}; G) \mid y \in U\}.$$

Then the collection $\{U_\alpha \mid U \subseteq M \text{ is open and } \alpha \in H_n(M, M \smallsetminus \overline{U}; G)\}$ is a basis for the topology on $\Theta_M(G)$.

Lemma 5.2.2. *The orientation bundle $\Theta_M(G)$ is a topological space with the described topology, with respect to which $p : \Theta_M(G) \to M$ and fibrewise addition are continuous maps.*

Proof. If $g \in H_n(M, M \smallsetminus \{x\}; G) \subseteq \Theta_M(G)$ then for any convex neighborhood U of x, Lemma 5.2.1 says that $g = j_{y,U}(\beta)$ for some $\beta \in H_n(M, M \smallsetminus \overline{U}; G)$ and $y \in U$. Hence $\{U_\alpha\}$ is a cover of $\Theta_M(G)$. Next, if $g \in U_\alpha \cap V_\beta$ for open sets $U, V \subseteq M$ and elements α, β of the corresponding homology groups, we must produce an open set $W \subseteq M$ and a class $\gamma \in H_n(M, M \smallsetminus \overline{W}; G)$ such that $g \in W_\gamma \subseteq U_\alpha \cap V_\beta$. Note that if $g = p^{-1}(x) = H_n(M, M \smallsetminus \{x\}; G)$, then $g = j_{x,U}(\alpha) = j_{x,V}(\beta)$. Take $W \subseteq U \cap V$, a convex neighborhood of x. By Lemma 5.2.1, the map $j_{y,W} : H_n(M, M \smallsetminus W; G) \to H_n(M, M \smallsetminus \{y\}; G)$ is an isomorphism for all $y \in W$. Let $\gamma = j_{x,W}^{-1}(g) \in H_n(M, M \smallsetminus \overline{W}; G)$. Then $g \in W_\gamma$ and by definition of the basis sets, $W_\gamma = \{j_{y,W}(\gamma) \mid y \in W\}$. If $g' = j_{y,W}(\gamma) \in W_\gamma$, we must show $g' \in U_\alpha \cap V_\beta$. Consider the maps

$$\begin{tikzcd}
H_n(M, M \smallsetminus \overline{U}; G) \arrow[dr, "j_{W,U}"] & & \\
& H_n(M, M \smallsetminus \overline{W}; G) \arrow[r, "j_{y,W}"] & H_n(M, M \smallsetminus \{y\}; G) \\
H_n(M, M \smallsetminus \overline{V}; G) \arrow[ur, "j_{W,V}"'] & &
\end{tikzcd}$$

Since $j_{y,W} \circ j_{W,U} = j_{y,U}$, we have $j_{y,W} \circ j_{W,U}(\alpha) = j_{y,U}(\alpha)$, but since $j_{y,W}$ is an isomorphism, we get $j_{W,U}(\alpha) = j_{y,W}^{-1} \circ j_{y,U}(\alpha)$. This holds for any $y \in W$. In particular, for $y = x$ we know $j_{x,U}(\alpha) = g$ and $j_{x,W}^{-1}(g) = \gamma$, so we must have $j_{W,U}(\alpha) = \gamma$. Hence $g' = j_{y,W}(\gamma) =$

$j_{y,W} \circ j_{W,U}(\alpha) = j_{y,U}(\alpha) \in U_\alpha$. A similar proof shows that $g' \in V_\beta$. Hence $\{U_\alpha\}$ forms a basis for a topology on $\Theta_M(G)$.

Next, take an open, convex set $U \subseteq M$ with $\overline{U}$ compact (within some coordinate chart in M). Then for $\alpha \in H_n(M, M \smallsetminus \overline{U}; G) \cong G$, the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a bijection. In particular, $p^{-1}(U) = \bigcup_{x \in U} H_n(M, M \smallsetminus \{x\}; G)$ is the union of open sets in the collection $\{j_{x,U}(H_n(M, M \smallsetminus \overline{U}; G))\}$ so $p^{-1}(U)$ itself is open.

Finally, for any convex $U \subseteq M$, the map $\varphi_U : U \times H_n(M, M \smallsetminus \overline{U}; G) \to H_n(M, M \smallsetminus \overline{U}; G)$ given by $(x, \alpha) \mapsto j_{x,U}(\alpha)$ is a bijection, since each $j_{x,U}$ is an isomorphism. Here, we view $H_n(M, M \smallsetminus \overline{U}; G) \cong G$ with the discrete topology. If $V \subseteq U$ is open and $\alpha \in H_n(M, M \smallsetminus \overline{U}; G)$, then $\varphi_U(V \times \{\alpha\}) = V_\alpha$ which is open by definition of the topology on $\Theta_M(G)$. Hence φ_U is open. Now $\{V_\alpha \mid V \subseteq U\}$ is a basis for $p^{-1}(U)$, so we also see that φ_U is continuous. Hence φ_U is a homeomorphism. As a consequence, we see that $p^{-1}(U)$ has the discrete topology, so it follows that $H_n(M, M \smallsetminus \{x\}; G) \times H_n(M, M \smallsetminus \{x\}; G) \to H_n(M, M \smallsetminus \{x\}; G)$ is continuous with respect to this topology. □

This proves something more:

Corollary 5.2.3. *$p : \Theta_M(G) \to M$ is a topological cover.*

Proof. For each convex, open neighborhood $U \subseteq M$ and for each $\alpha \in H_n(M, M \smallsetminus \overline{U}; G)$, the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism. It follows that p is a local homeomorphism, hence a cover. □

Definition. *A* **section** *of the orientation bundle over a subset $A \subseteq M$ is a continuous map $\sigma : A \to \Theta_M(G)$ such that $p \circ \sigma = id_A$. The collection of all sections over A is denoted $\Gamma(A, \Theta_M(G))$.*

Concretely, a section $\sigma \in \Gamma(A, \Theta_M(G))$ is an assignment of an element of $H_n(M, M \smallsetminus \{x\}; G)$ to each $x \in A$ that varies continuously with x. By the proof of Lemma 5.2.2, this is equivalent to $\sigma : x \mapsto \sigma(x)$ being locally constant.

Definition. *Let M be a manifold and $A \subseteq M$ a subset. An* **orientation along** *A is a section $\theta_A \in \Gamma(A, \Theta_M(\mathbb{Z}))$ such that $\theta_A(x)$ is a generator of $H_n(M, M \smallsetminus \{x\}) \cong \mathbb{Z}$ for each $x \in A$. If such a section exists, we say M is* **orientable along** *A. For $A = M$, we say an orientation along M is a* **global orientation** *and M is* **orientable**.

Proposition 5.2.4. *For an n-manifold M, the following are equivalent:*

(1) M is orientable.

(2) M is orientable along every compact subset of M.

(3) M is orientable along every loop in M.

(4) The collection of units in each fibre $p^{-1}(x) \subseteq \Theta_M(\mathbb{Z})$ forms a trivial double cover of M, i.e. one homeomorphic to $M \times \mathbb{Z}/2\mathbb{Z}$.

(5) There is a bundle isomorphism $\Theta_M(\mathbb{Z}) \cong M \times \mathbb{Z}$.

Proof. (1) $\implies$ (2) $\implies$ (3) are easy.

(3) $\implies$ (4) Assume M is connected. Then the units in $\Theta_M(\mathbb{Z})$ form a double cover with either one or two components; if two components, we are done. Suppose instead that the cover is nontrivial, so that there is a path in $\Theta_M(\mathbb{Z})$ connecting the two elements of $p^{-1}(x)$ for some $x \in M$. Project this path to M to get a loop based at x. Since the units form a double cover, there can be no section of $\Theta_M(\mathbb{Z})$ along this loop, so (3) fails. Thus the implication holds by contrapositive.

(4) $\implies$ (5) Fix a component of the double cover, U_1. Then any $g \in \Theta_M(\mathbb{Z})$ can be written uniquely as $g = nu_x$ for some $n \in \mathbb{Z}$ and $u_x \in U_1$. The map $g \mapsto (x, n)$ gives the desired bundle isomorphism.

(5) $\implies$ (1) $x \mapsto (x, 1)$ defines a global section, making M orientable. □

The key idea contained in (3) of Proposition 5.2.4 is the orientability is a "one-dimensional question", decidable by orientability along loops.

Sections of the orientation bundle can arise as follows. Let $A \subseteq M$. For a class $\alpha \in H_n(M, M \setminus A; G)$, define a section over A by $x \mapsto j_{x,A}(\alpha)$. This determines a function

$$\begin{aligned} J_A : H_n(M, M \setminus A; G) &\longrightarrow \Gamma(A, \Theta_M(G)) \\ \alpha &\longmapsto (x \mapsto j_{x,A}(\alpha)). \end{aligned}$$

Lemma 5.2.5. *For any $A \subseteq M$, J_A is continuous, and for any $\alpha \in H_n(M, M \setminus A; G)$, the section $J_A(\alpha)$ has compact support, i.e. $J_A(\alpha)(y) = 0$ for all y outside some compact set.*

Proof. Represent α by a chain a, so that a lies in some compact set B. If $x \notin B$, then $a \in \Delta_n(M, M \setminus A; G)$ maps to 0 in $\Delta_n(M, M \setminus \{x\}; G)$ via $j_{x,A}$. Thus $J_A(\alpha) = 0$ on $M \setminus B$. □

Definition. *For a set $A \subseteq M$, define the set of* **compactly supported sections** *of the orientation bundle over A by*

$$\Gamma_c(A, \Theta_M(G)) := \{\sigma \in \Gamma(A, \Theta_M(G)) \mid \sigma = 0 \text{ on } M \setminus B \text{ for some compact } B\}.$$

Then Lemma 5.2.5 says that the image of J_A lies in $\Gamma_c(A, \Theta_M(G))$. We will prove that whenever A is a closed subset of M, J_A determines an isomorphism $H_n(M, M \setminus A; G) \cong \Gamma_c(A, \Theta_M(G))$. To prove this, we need:

Lemma 5.2.6. *The map $J_A : H_n(M, M \setminus A; G) \to \Gamma(A, \Theta_M(G))$ is natural with respect to inclusions of open sets $B \hookrightarrow A$.*

Proof. We must show that for any closed sets $B \subseteq A$, the following diagram commutes:

$$\begin{tikzcd} H_n(M, M \setminus A; G) \arrow[r, "i^*"] \arrow[d, "J_A"'] & H_n(M, M \setminus B; G) \arrow[d, "J_B"] \\ \Gamma_c(A, \Theta_M(G)) \arrow[r] & \Gamma_c(B, \Theta_M(G)) \end{tikzcd}$$

But this follows from the fact that $j_{x,A}$ and $j_{x,B}$ agree for all $x \in B$. □

Lemma 5.2.7. *Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of an n-manifold M, and let $A = \bigcap A_i$. Then for any p, inclusion induces an isomorphism*

$$\varinjlim H_p(M, M - A_i) \cong H_p(M, M - A).$$

Proof. Let $H_p(M, M-A_1) \xrightarrow{f_1} H_p(M, M-A_2) \xrightarrow{f_2} \cdots$ be the maps induced by the inclusions $(M, M-A_1) \hookrightarrow (M, M-A_2) \hookrightarrow \cdots$ and set $f_{ij} = f_j f_i$. Let $g_i : H_p(M, M-A_i) \to \varinjlim H_p(M, M-A_i)$ be the canonical maps. Since $A \subseteq A_i$ for each i, we also have inclusions $(M, M-A_i) \hookrightarrow (M, M-A)$. These induce maps on homology $\varphi_i : H_p(M, M-A_i) \to H_p(M, M-A_i)$ which commute with the f_{ij}, so by the universal property of direct limits (Proposition 5.1.1), we get a map

$$\Phi : \varinjlim H_p(M, M-A_i) \longrightarrow H_p(M, M-A)$$

which commutes with the g_i, that is, $\Phi g_i = \varphi_i$.

For any p-chain $\sigma \in \Delta_p(M, M-A) = \dfrac{\Delta_p(M)}{\Delta_p(M-A)}$, lift σ to $\tau \in \Delta_p(M)$ and set $\sigma_i = \tau + \Delta_p(M - A_i) \in \dfrac{\Delta_p(M)}{\Delta_p(M-A_i)} = \Delta_p(M, M-A_i)$. Then $\alpha = [H_p(M, M-A_i), \sigma_i]$ defines an element of the direct limit, and

$$\begin{aligned}\Phi(\alpha) &= \Phi[H_p(M, M-A_i), \sigma_i] \\ &= \Phi(g_i(\sigma_i)) \quad \text{by Proposition 5.1.1} \\ &= \varphi_i(\sigma_i) = \sigma.\end{aligned}$$

Therefore Φ is surjective on the level of chains.

Next, suppose $\Phi(\alpha) = \sigma = 0$ in $\Delta_p(M, M-A)$. This means $\tau \in \Delta_p(M-A)$ so τ must lie in $\Delta_p(M-A_j)$ for some j. This says that $\sigma_j = \tau + \Delta_p(M-A_j) = 0$ in $\Delta_p(M, M-A_j)$, so $\alpha = [H_p(M, M-A_j), \sigma_j] = g_j(\sigma_j) = g_j(0) = 0$. Hence Φ is injective. $\square$

Lemma 5.2.8. *Let $P_M(A)$ be a statement about closed subsets A of a manifold M and suppose that the following conditions are met:*

(i) $P_M(A)$ is true when A is a convex, compact subset of a coordinate chart of M.

(ii) If $P_M(A), P_M(B)$ and $P_M(A \cap B)$ are true then $P_M(A \cup B)$ is true.

(iii) If $A_1 \supseteq A_2 \supseteq \cdots$ is a descending chain of compact subsets and $P_M(A_j)$ holds for all $j \geq 1$, then $P_M\left(\bigcap_{j=1}^\infty A_j\right)$ holds.

Then $P_M(A)$ is true for all compact $A \subseteq M$. If in addition,

(iv) If $\{A_j\}$ are compact subsets of M that have disjoint neighborhoods separating them and $P_M(A_j)$ holds for all j, then $P_M\left(\bigcup_{j=1}^\infty A_j\right)$ holds,

then $P_M(A)$ is true for all closed subsets $A \subseteq M$. Further, if (i) – (iv) hold for all closed $A \subseteq M$ and all manifolds M, and the following condition also holds:

(v) If $A \subseteq M$ is closed and $P_W(W \cap A)$ holds for all open sets $W \subseteq M$ with $\overline{W}$ compact, then $P_M(A)$ also holds,

then $P_M(A)$ is true for all closed subsets A of all manifolds M.

Proof. Similar to the proof of Lemma 3.6.5; see Bredon for the full proof. □

Theorem 5.2.9. *For an n-manifold M, a closed subset $A \subseteq M$ and any coefficient group G, we have*

(a) $H_i(M, M \smallsetminus A; G) = 0$ for all $i > n$.

(b) J_A gives an isomorphism $H_n(M, M \smallsetminus A; G) \to \Gamma_c(A, \Theta_M(G))$.

Proof. For a closed set $A \subseteq M$, let $P_M(A)$ be the statement that $H_i(M, M \smallsetminus A; G) = 0$ for $i > n$ and J_A is an isomorphism. By Lemma 5.2.8, it suffices to prove that conditions (i) – (v) are met for this statement $P_M(A)$.

(i) This follows immediately from Lemma 5.2.1 – for $i > n$, the diagram in the proof of said lemma is identical and has $H_n(D^n, \partial D^n; G) = 0$ by Theorem 2.3.5.

(ii) Set $X = M, U = M \smallsetminus A$ and $V = M \smallsetminus B$, so that $U \cap V = M \smallsetminus (A \cup B)$ and $U \cup V = M \smallsetminus (A \cap B)$. Applying the relative Mayer-Vietoris sequence (Theorem 2.7.5), we get a diagram with exact rows:

$$\begin{tikzcd}
H^{loc}_{n+1}(A\cap B) \arrow[r] \arrow[d, equal] & H^{loc}_n(A\cup B) \arrow[r] \arrow[d, "J_{A\cup B}"] & H^{loc}_n(A)\oplus H^{loc}_n(B) \arrow[r] \arrow[d, "J_A \oplus J_B"] & H^{loc}_n(A\cap B) \arrow[d, "J_{A\cap B}"] \\
0 \arrow[r] & \Gamma_c(A\cup B,\Theta) \arrow[r] & \Gamma_c(A,\Theta)\oplus\Gamma_c(B,\Theta) \arrow[r] & \Gamma_c(A\cap B,\Theta)
\end{tikzcd}$$

where $H^{loc}_i(Y) = H_i(M, M \smallsetminus Y; G)$ and $\Theta = \Theta_M(G)$. The diagram commutes by Lemma 5.2.6. Moreover, by hypothesis, $J_A \oplus J_B$ and $J_{A\cap B}$ are isomorphisms so the Five Lemma (2.2.3), $J_{A\cup B}$ is also an isomorphism. Moreover, the top row of the diagram makes it clear that $H_i(M, M \smallsetminus (A \cup B); G) = 0$ whenever $i > n$.

(iii) Consider a sequence of compact sets $A_1 \supseteq A_2 \supseteq \cdots$ for which the hypotheses $P_M(A_j)$ hold. Then if $A = \bigcap_{j=1}^{\infty} A_j$, the restriction maps induce an isomorphism

$$\begin{aligned}
\Phi : \varinjlim \Gamma(A_i, \Theta_M(G)) &\longrightarrow \Gamma(A, \Theta_M(G)) \\
[\Gamma(A_i, \Theta_M(G)), \sigma_i] &\longmapsto \sigma_i|_A
\end{aligned}$$

where the direct limit is ordered by restriction of sections. Indeed, if $[\Gamma(A_i, \Theta_M(G)), \sigma_i]$ maps to 0 under this assignment, then for all $x \in A$, there exists a neighborhood U_x of x such that $\sigma_i|_{U_x} = 0$. Thus $\sigma_i|_U = 0$, where $U = \bigcup_{x\in A} U_x$, so for some $j \geq 1$, $A_j \subseteq U$ and $\sigma_i|_{A_j} = 0$. By Proposition 5.1.1, this shows Φ is one-to-one. In order to show it is onto, for $\sigma \in \Gamma(A, \Theta_M(G))$, we want to extend σ to a neighborhood of A. For each $x \in A$, there is a neighborhood U_x and a section $\sigma_x \in \Gamma(U_x, \Theta_M(G))$ such that $\sigma_x|_{A\cap U_x} = \sigma|_{A\cap U_x}$. Since A is compact, we may cover A by finitely many of these $U_{x_1}, \ldots, U_{x_k}$. Then

$$U = \{y \in M \mid \sigma_{x_i}(y) = \sigma_{x_j}(y) \text{ if } y \in U_{x_i} \cap U_{x_j}\}$$

is an open set containing A and the σ_{x_j} give an extension of σ to U. This proves Φ is an isomorphism. Now we have a commutative diagram

$$\begin{tikzcd}
\varinjlim H_j(M, M \setminus A_i; G) \arrow[r] \arrow[d] & H_j(M, M \setminus A; G) \arrow[d, "J_A"] \\
\varinjlim \Gamma(A_i, \Theta_M(G)) \arrow[r] & \Gamma(A, \Theta_M(G))
\end{tikzcd}$$

where the top row is an isomorphism by Lemma 5.2.7, the left column is an isomorphism by hypothesis and the bottom row is the isomorphism Φ. When $j = n$, the diagram shows J_A is an isomorphism, and when $j > n$, the top row implies $H_j(M, M \setminus A; G) = 0$.

(iv) Say $\{A_i\}$ are compact sets with disjoint neighborhoods $\{N_i\}$ in M. Then for $A = \bigcup A_i$ and any $j \in \mathbb{N}$, we have isomorphisms

$$\begin{aligned}
H_j(M, M \setminus A; G) &\cong H_j\left(\bigcup N_i, \bigcup N_i \setminus \bigcup A_i; G\right) \quad \text{by excision} \\
&\cong \bigoplus_{i=1}^{\infty} H_j(N_i, N_i \setminus A_i; G) \\
&\cong \bigoplus_{i=1}^{\infty} H_j(M, M \setminus A_i; G) \quad \text{by excision.}
\end{aligned}$$

Therefore when $j > n$, property (a) of $P_M(A)$ follows immediately. Moreover, since the A_i are compact, we have

$$\Gamma_c(A, \Theta_M(G)) = \bigoplus_i \Gamma_c(A_i, \Theta_M(G))$$

so property (b) also follows.

(v) Consider the system of neighborhoods $W \subseteq M$ such that $\overline{W}$ is compact. Then $\{H_j(W, W \setminus (W \cap A); G)\}$ and $\{\Gamma_c(W \cap A, \Theta_M(G))\}$ are direct systems ordered by inclusion $W \subseteq W'$. So for each j we have a diagram

$$\begin{tikzcd}
\varinjlim H_j(W, W \setminus (W \cap A); G) \arrow[r] \arrow[d] & H_j(M, M \setminus A; G) \arrow[d, "J_A"] \\
\varinjlim \Gamma_c(W \cap A, \Theta_M(G)) \arrow[r] & \Gamma_c(A, \Theta_M(G))
\end{tikzcd}$$

Here, the top row is an isomorphism by Lemma 5.2.7, the left column by hypothesis and the bottom row by a similar argument as in (iii). Hence $H_j(M, M \setminus A; G) = 0$ for $j > n$ and J_A is an isomorphism for $j = n$.

We have verified statements (i) – (v) of Lemma 5.2.8, so the entirety of the theorem is proved. $\square$

Proposition 5.2.10. *If $A \subseteq M$ is closed and connected, then*

$$\Gamma(A, \Theta_M(G)) = \begin{cases} G, & \text{if } M \text{ is orientable along } A \\ {}_2G, & \text{if } M \text{ is not orientable along } A \end{cases}$$

where ${}_2G$ denotes the 2-torsion part of G.

Proof. Define the map

$$\begin{aligned} \Phi : \Gamma(A, \Theta_M(G)) &\longrightarrow H_n(M, M \smallsetminus \{x\}; G) \cong G \\ \sigma &\longmapsto \sigma(x). \end{aligned}$$

Since $\Theta_M(G)$ is locally constant and A is connected, we see that if any two sections agree at x then they agree locally and therefore on a set which is both open and closed. This implies Φ is one-to-one. On the other hand, for each $g \in G$, there is a map $\Theta_M(\mathbb{Z}) \to \Theta_M(G)$ induced by

$$\begin{aligned} H_n(M, M \smallsetminus \{x\}) \otimes \mathbb{Z} &\longrightarrow H_n(M, M \smallsetminus \{x\}) \otimes G = H_n(M, M \smallsetminus \{x\}; G) \\ \alpha \otimes 1 &\longmapsto \alpha \otimes g \end{aligned}$$

for each $x \in M$ and extended to $\Theta_M(\mathbb{Z}) = \bigcup_{x \in M} H_n(M, M \smallsetminus \{x\}; \mathbb{Z})$. Applying the section functor $\Gamma(A, -)$ for a fixed closed subset A gives a map of bundles $\psi_g : \Gamma(A, \Theta_M(\mathbb{Z})) \to \Gamma(A, \Theta_M(G))$. Supposing M is orientable along A, define

$$\begin{aligned} \Psi : G &\longrightarrow \Gamma(A, \Theta_M(G)) \\ g &\longmapsto \psi_g(1), \end{aligned}$$

where $1 \in \Gamma(A, \Theta_M(A))$ is an orientation of M along A. We claim Φ and Ψ are inverses. On one hand, Ψ is injective: if $\Psi(g)$ is the zero section of $\Theta_M(G)$, by definition $\psi_g(1) = 0$ but this implies $1 \otimes g = 0$ and hence $g = 0$. On the other hand, for $\sigma \in \Gamma(A, \Theta_M(G))$,

$$\Psi \circ \Phi(\sigma)(x) = \Psi(\sigma(x)) = \psi_{\sigma(x)}(1) = \sigma(x)$$

so $\Psi \circ \Phi = id$. In particular, Φ is an isomorphism $\Gamma(A, \Theta_M(G)) \cong G$.

Now suppose M is not orientable along A. Fix a section $\sigma \in \Gamma(A, \Theta_M(G))$ and set $g = \Phi(\sigma) \in G$. Consider the preimage $\psi_g^{-1}(\sigma) \subseteq \Theta_M(\mathbb{Z})$. Over x, this consists precisely of $\alpha \in H_n(M, M \smallsetminus \{x\})$ such that $\alpha \otimes g \mapsto g \in \Theta_M(G)_x$, so that we may view $\psi^{-1}(\sigma(x)) = \{a \in \mathbb{Z} \mid ag = g\}$. Then $1 \in \psi^{-1}(\sigma(x))$ and for each $y \in A$, $\psi^{-1}(\sigma(y))$ must contain at least one of the two generators of $H_n(M, M \smallsetminus \{y\})$. Since the orientation bundle is locally constant and A is connected, this means the number of generators in $\psi^{-1}(\sigma(y))$ is independent of the choice of $y \in A$. Supposing this number is 1, this determines an orientation of M over A by Proposition 5.2.4, contradicting the assumption. Therefore $\{\pm 1\} \subseteq \psi^{-1}(\sigma(x))$. Of course, this now implies that $-g = g$, that is, $g \in {}_2G$. It follows that Φ has image lying in ${}_2G$, but for any $g \in {}_2G$, the assignment $y \mapsto \psi_g(1)$ determines a section of $\Theta_M(G)$ over A mapping to g under Φ. As above, Φ is injective so we have proven $\Gamma(A, \Theta_M(G)) \cong {}_2G$ as required. $\square$

We now derive some important results about manifold homology and cohomology.

Corollary 5.2.11. *If $A \subseteq M$ is any closed, connected set of an n-manifold, then for any coefficient group G,*

$$H_n(M, M \smallsetminus A; G) = \begin{cases} G, & A \text{ is compact and } M \text{ is orientable along } A \\ {}_2G, & A \text{ is compact and } M \text{ is not orientable along } A \\ 0, & A \text{ is not compact.} \end{cases}$$

Proof. Apply Theorem 5.2.9. □

Corollary 5.2.12. *For a connected n-manifold M without boundary and a coefficient group G, $H_i(M; G) = 0$ for all $i > n$, and*

$$H_n(M; G) = \begin{cases} G, & M \text{ is compact, orientable} \\ {}_2G, & M \text{ is compact, not orientable} \\ 0, & M \text{ is not compact.} \end{cases}$$

Corollary 5.2.13. *For a connected n-manifold M, the torsion part of $H_{n-1}(M)$ is $\mathbb{Z}/2\mathbb{Z}$ if M is compact and non-orientable, and 0 otherwise.*

Proof. First, apply Example 3.3.15 to $G = H_{n-1}(M)$ to get that the torsion part is $TH_{n-1}(M) = \text{Tor}_1(H_{n-1}(M), \mathbb{Q}/\mathbb{Z})$. Then the universal coefficient theorem (3.4.1) gives a short exact sequence

$$0 \to H_n(M) \otimes \mathbb{Q}/\mathbb{Z} \to H_n(M; \mathbb{Q}/\mathbb{Z}) \to \text{Tor}_1(H_{n-1}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

When M is non-compact, $H_n(M) \otimes \mathbb{Q}/\mathbb{Z}$ and $H_n(M; \mathbb{Q}/\mathbb{Z})$ are 0 by Corollary 5.2.12, so the third term is 0 as well. When M is compact and orientable, both $H_n(M) \otimes \mathbb{Q}/\mathbb{Z}$ and $H_n(M; \mathbb{Q}/\mathbb{Z})$ are $\mathbb{Q}/\mathbb{Z}$, and the map between them is the identity, so by exactness, the third term is 0. Finally, when M is compact and non-orientable, $H_n(M) \otimes \mathbb{Q}/\mathbb{Z} = 0$ but $H_n(M; \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ by Corollary 5.2.12. This implies $TH_{n-1}(M) = \mathbb{Z}/2\mathbb{Z}$. □

We can now deduce the top cohomology of any compact manifold.

Corollary 5.2.14. *If M is a compact, connected n-manifold without boundary, then for any coefficient group G,*

$$H^n(M; G) = \begin{cases} G, & M \text{ is orientable} \\ G/2G, & M \text{ is not orientable.} \end{cases}$$

Proof. By the universal coefficient theorem (3.4.2),

$$H^n(M; G) \cong \text{Hom}(H_n(M), G) \oplus \text{Ext}^1(H_{n-1}(M), G).$$

If M is orientable, then $\text{Hom}(H_n(M), G) \cong \text{Hom}(\mathbb{Z}, G) \cong G$, while $H_{n-1}(M)$ is torsion-free by Corollary 5.2.13. We will show that the homology groups of a compact manifold are finitely generated (cite), but taking this to be true, the above implies $H_{n-1}(M)$ is in fact free, and therefore $\text{Ext}^1(H_{n-1}(M), G) = 0$ by Proposition 3.3.8. Hence $H^n(M; G) \cong \text{Hom}(H_n(M), G) \cong G$.

On the other hand, if M is not orientable, then $\text{Hom}(H_n(M), G) \cong \text{Hom}(0, G) = 0$ and $\text{Ext}^1(H_{n-1}(M), G) \cong \text{Ext}^1(\mathbb{Z}/2\mathbb{Z}, G) = G/2G$ by Proposition 3.3.14. □

Finally, we connect this chapter's definition of orientability (in terms of a global section of the orientation bundle) with the notion of orientability for smooth manifolds.

Theorem 5.2.15. *If M is a smooth manifold, then it is chartwise orientable if and only if there exists a global section of the orientation bundle $\Theta_M(\mathbb{Z})$.*

5.3 Čech Cohomology

In this section we introduce an important cohomology theory, called Čech cohomology, which is vital for proving the duality theorems in Section 5.4. Assume M is an oriented n-manifold and $L \subseteq K \subseteq M$ are compact subsets. Suppose $U \supset K$ and $V \supset L$ are open sets in M containing each compact set, with $V \subseteq U$. A key idea in the proof of duality is that $H^p(U,V)$ approximates $H^p(K,L)$ in some fashion. Indeed, if $U' \subseteq U$ and $V' \subseteq V$ are smaller open subsets containing the respective compact sets, then inclusion of pairs induces a map $H^p(U,V) \to H^p(U',V')$. In fact, the pairs of open sets $\{(U,V) : U \supset K, V \supset L\}$ form a direct system under inclusion.

Definition. *The pth* **Čech cohomology** *of the pair (K,L) is the direct limit*

$$\check{H}^p(K,L) := \varinjlim H^p(U,V),$$

where the direct limit is over all pairs $U \supset K, V \supset L$.

Lemma 5.3.1. *If there exists a pair of neighborhoods $U \supset K, V \supset L$ such that the pair (U,V) deformation retracts to (K,L), then $\check{H}^p(K,L) = H^p(K,L)$ for all $p \geq 0$.*

We will show in this section that:

(1) When M is a manifold or a finite CW-complex, $\check{H}^\bullet(M) \cong H^\bullet(M)$.

(2) The definition of $\check{H}^p(A,B)$ does not depend on the embeddings $A, B \hookrightarrow M$.

(3) In general, $\check{H}^\bullet(X) \ncong H^\bullet(X)$.

Definition. *A set $X \subseteq \mathbb{R}^n$ is called a* **Euclidean neighborhood retract** *(ENR) if X is the retract of some neighborhood $U \subseteq \mathbb{R}^n$ of X.*

A key observation is that an ENR $X \hookrightarrow U \subseteq \mathbb{R}^n$ is closed in $\mathbb{R}^n$ and therefore locally compact.

Lemma 5.3.2. *Let $Y \subseteq \mathbb{R}^n$ be locally compact. Then there is an embedding $Y \hookrightarrow \mathbb{R}^{n+1}$ as a closed subset.*

Proof. By local compactness, $Y = U \cap \overline{Y}$ for some open set $U \subseteq \mathbb{R}^n$. Then $C := \overline{Y} \smallsetminus U$ is closed and the map

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \operatorname{dist}(x, C) \end{aligned}$$

is continuous. Define

$$\begin{aligned}\varphi : Y &\longrightarrow \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \\ y &\longmapsto \left(y, \frac{1}{f(y)}\right).\end{aligned}$$

Then it is easy to check that φ is an embedding and $\varphi(Y)$ is closed in $\mathbb{R}^{n+1}$. □

Theorem 5.3.3 (Borsuk)**.** *If X is locally compact and locally contractible, then any embedding $X \hookrightarrow \mathbb{R}^n$ is an ENR.*

Proof. By Lemma 5.3.2, we may assume $X \subseteq \mathbb{R}^n$ is closed. The idea now is to divide $\mathbb{R}^n \setminus X$ into cells which get "small" close to X, and define a retraction by defining a map on enough of these cells around X. Explicitly, divide $\mathbb{R}^n$ into hypercubes with vertices having integer coordinates. Let C_1 be the union of all cubes not intersecting X. For $p \in \mathbb{R}^n$, if $\mathrm{dist}(p, X) > \sqrt{n}$, where $\sqrt{n}$ is viewed as the diameter of a cube with side-length 1, then $p \in C_1$ and so $X \subseteq \mathbb{R}^n \setminus C_1$.

Next, divide $\mathbb{R}^n \setminus \mathrm{Int}(C_1)$ into cubes of side-length $\frac{1}{2}$ and set C_2 as the union of all $\frac{1}{2}$-cubes not intersecting X. Continue in this manner to define a chain of subsets

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$$

and set $C = \bigcup_{i=1}^{\infty} C_i$. By the first paragraph, any $p \in \mathbb{R}^n$ satisfying $\mathrm{dist}(p, X) > \frac{\sqrt{n}}{2^i}$ lies in C_{i-1}, so every points $p \in \mathbb{R}^n \setminus X$ lies in C. Further, each of these points lies in a *finite number* of the C_i. Thus $C = \mathbb{R}^n \setminus X$ so $\mathbb{R}^n \setminus X$ is a locally finite CW-complex.

Now let a be a 0-cell of C and choose $r_0(a) \in X$ such that

$$\mathrm{dist}(a, r_0(a)) < 2 \inf\{\mathrm{dist}(a, x) \mid x \in X\}.$$

For any cell $\sigma \subset C$ and any map $f : \sigma \to X$, define the number

$$\rho(f) := \max\{\mathrm{dist}(z, f(z)) \mid z \in \sigma\}.$$

Given a map $r_i : A_i \to X$ where $A_i \subseteq C$ is a union of some i-cells, we define r_{i+1} as follows. Let σ be an $(i+1)$-cell of C such that $\partial\sigma \in A_i$. Let A_{i+1} be the cells in C such that $r_i|_{\partial\sigma} : \partial\sigma \to X$ extends to $\sigma \to X$ and let $f_\sigma : \sigma \to X$ be some extension such that

$$\rho(f_\sigma) < 2 \inf\{\rho(f) \mid f : \sigma \to X \text{ and } f|_{\partial\sigma} = r_i|_{\partial\sigma}\}.$$

Then the f_σ together define an extension $r_{i+1} : A_{i+1} \to X$.

Now set $A = \bigcup_{i=1}^{\infty} A_i \cup X$ and let $r : A \to X$ be defined by $r|_{A_i} = r_i$ and $r|_X = id_X$. Then it follows from the above construction that r is a continuous retraction and A is a neighborhood of X. Hence X is an ENR. □

Corollary 5.3.4. *Every topological manifold and every CW-complex is an ENR.*

We get the following important consequence for manifold homology and cohomology.

Corollary 5.3.5. *For every manifold M, $H_\bullet(M)$ and $H^\bullet(M)$ are finitely generated.*

Proof. By the proof of Theorem 5.3.3, there is an embedding $i : M \hookrightarrow K \subseteq \mathbb{R}^n$ where K is a finite CW-complex with some retraction $r : K \to M$. Then $1 = r_* i_* : H_\bullet(M) \to H_\bullet(K) \to H_\bullet(M)$ so $H_\bullet(M)$ must be finitely generated. For cohomology, apply the universal coefficient theorem (3.4.2). □

Lemma 5.3.6. *If $X \hookrightarrow \mathbb{R}^n$ is an ENR and $U \subseteq \mathbb{R}^n$ is any neighborhood of X, then there is a smaller neighborhood $V \subseteq U$ such that V deformation retracts onto X through U.*

Corollary 5.3.7. *If X is an ENR, then the neighborhoods $X \hookrightarrow \mathbb{R}^n$ of a fixed embedding form a direct system and the resulting map*

$$\check{H}^p(X) = \varinjlim H^p(U) \to H^p(X)$$

is an isomorphism for all $p \geq 0$.

Proof. Choose a neighborhood $U \subseteq \mathbb{R}^n$ of X small enough so that $r : U \to X$ is a retract. Then the induced map $i^* : H^\bullet(U) \to H^\bullet(X)$ has right inverse r^* and so is surjective. By Lemma 5.3.6, choose a neighborhood $X \subseteq V \subseteq U$ and a deformation $F : V \times I \to U$. Then $s = F(\cdot, 1)|_{V \times \{1\}} : V \times \{1\} \to X \subseteq U$ is a retract. Consider the commutative diagram

$$\begin{tikzcd}
H^\bullet(X) \arrow[r, "s^*"] & H^\bullet(V \times \{1\}) \\
H^\bullet(U) \arrow[u, "i^*"] \arrow[r, "F^*"] \arrow[dr, "j^*"'] & H^\bullet(V \times I) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
& H^\bullet(V \times \{0\})
\end{tikzcd}$$

Here, the isomorphisms are by homotopy and j^* is induced by inclusion. Then for any $\alpha \in H^\bullet(U)$ such that $i^*\alpha = 0$, the diagram implies $j^*\alpha = 0$ in $H^\bullet(V) = H^\bullet(V \times \{0\})$ as well, so there is a triangle

$$\begin{tikzcd}
H^\bullet(U) \arrow[rr] \arrow[dr] & & H^\bullet(V) \arrow[dl] \\
& X &
\end{tikzcd}$$

which commutes with $\alpha \mapsto 0$ for every inclusion of open sets $V \subseteq U$. Hence we get an isomorphism

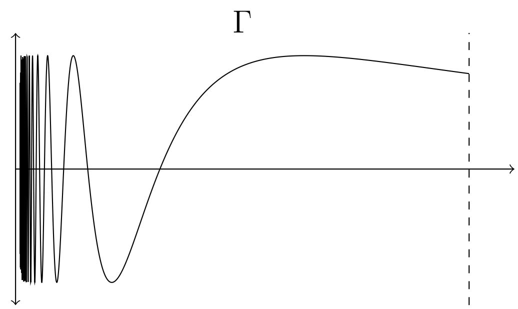
$$\check{H}^\bullet(X) = \varinjlim H^\bullet(U) \xrightarrow{\cong} H^\bullet(X)$$

as claimed. □

Corollary 5.3.8. *For an ENR X, the Čech cohomology groups $\check{H}^p(X)$ are independent of the embedding $X \hookrightarrow \mathbb{R}^n$.*

Remark. The property that $\check{H}^\bullet(X)$ is independent of the embedding of X in $\mathbb{R}^n$ holds for arbitrary subsets X of $\mathbb{R}^n$, but it is harder to prove. Spanier (*Algebraic Topology*, Ch. 6) does this by generalizing to sheaf cohomology, where the coefficient group is allowed to vary across X in some fashion. For our purposes, Corollary 5.3.8 will suffice.

Example 5.3.9. Let $\Gamma \subseteq \mathbb{R}^2$ be the topologist's sine curve:



A standard exercise in general topology shows that Γ is connected but has two path components. Hence $H^\bullet(\Gamma;\mathbb{Z}) = \mathbb{Z}^2$ but for any neighborhood $U \subseteq \mathbb{R}^2$ of Γ, U is locally path-connected, thus locally connected, so Γ must lie in a single component of U. Therefore

$$\check{H}^0(\Gamma;\mathbb{Z}) = \varinjlim H^0(C;\mathbb{Z})$$

where the limit is over all *connected* neighborhoods $\Gamma \subseteq C \subseteq \mathbb{R}^2$. In particular, $\check{H}^0(\Gamma;\mathbb{Z}) = \mathbb{Z} \neq H^0(\Gamma;\mathbb{Z})$.

The moral of this story is the singular cohomology detects path-components, while Čech cohomology only detects connected components.

5.4 Poincaré Duality

Poincaré duality is a classic result in algebraic topology that relates the homology groups of a manifold with the cohomology groups of complementary dimension. Let M be an orientable n-dimensional manifold and fix an orientation $\theta_M \in \Gamma(M, \Theta_M(\mathbb{Z}))$. We first prove a duality theorem for so-called compactly supported cohomology, using the cap product with the orientation class and the bootstrap method of Lemma 3.6.5.

Lemma 5.4.1. *If $K \subset M$ is a compact subset, then any choice of orientation θ_M determines a generator θ_K for $H_n(M, M\setminus K)$, and this choice is natural with respect to inclusions $K \subset L$ of compact subsets.*

Proof. Restriction of sections $\Gamma(M,\Theta(M)) \to \Gamma(K,\Theta(M))$, $\sigma \mapsto \sigma|_K$ defines the desired generator: θ_M restricts to $\theta_K \in H_n(M, M \setminus K)$. Moreover, for an inclusion $K \hookrightarrow L$ of compact subsets of M, we have maps

$$\begin{array}{ccc} H_n(M) & \xrightarrow{J_M} & \Gamma_c(M,\Theta(M)) \\ \downarrow & & \downarrow \\ H_n(M, M \smallsetminus L) & \xrightarrow{J_L} & \Gamma(L,\Theta(M)) \\ \downarrow & & \downarrow r \\ H_n(M, M \smallsetminus K) & \xrightarrow{J_K} & \Gamma(K,\Theta(M)) \end{array}$$

Here, r is restriction of sections. Since the J_K maps are natural by Lemma 5.2.6, the diagram commutes so we must have $r(\theta_L) = \theta_K$. ☐

Lemma 5.4.2. *Cap product with the orientation class θ_K gives a homomorphism*

$$\begin{aligned} H^p(M, M \smallsetminus K; G) &\longrightarrow H_{n-p}(M;G) \\ \alpha &\longmapsto \alpha \cap \theta_K \end{aligned}$$

that is natural with respect to inclusions $K \subset L$, and therefore this operation gives rise to a well-defined homomorphism

$$\varinjlim_K H^p(M, M \smallsetminus K; G) \to H_{n-p}(M;G)$$

where the direct limit is over the collection of compact subsets $K \subset M$, directed by inclusion.

Proof. That this defines a homomorphism is obvious from the construction of the relative cap product. Thus it suffices to show this map is natural with respect to inclusions. By the universal property of the direct limit, this will define a homomorphism

$$\varinjlim_K H^p(M, M \smallsetminus K; G) \to H_{n-p}(M;G).$$

If $K \hookrightarrow L$ is an inclusion of compact sets, we get an inclusion of pairs $i : (M, M \smallsetminus L) \hookrightarrow (M, M \smallsetminus K)$. Then we must show the following diagram commutes:

$$\begin{tikzcd} H^p(M, M \smallsetminus K) \arrow[d, "i^*"'] \arrow[dr] & \\ H^p(M, M \smallsetminus L) \arrow[r] & H_{n-p}(M) \end{tikzcd}$$

For $\alpha \in H^p(M, M \smallsetminus K)$, Theorem 4.4.1(f) gives us

$$i^*\alpha \cap \theta_L = i_*(i^*\alpha \cap \theta_L) = \alpha \cap i_*\theta_L = \alpha \cap \theta_K.$$

Thus the diagram commutes. ☐

Definition. *For a manifold M, a coefficient group G and an integer $p \geq 0$, the pth* **compactly supported cohomology** *of M is defined to be*

$$H^p_{cpt}(M;G) := \varinjlim_K H^p(M, M \smallsetminus K; G).$$

Theorem 5.4.3 (Poincaré)**.** *Let M be an oriented n-manifold. Then cap product with the orientation class gives an isomorphism for each p and any coefficient group G:*

$$\begin{aligned} H^p_{cpt}(M;G) &\longrightarrow H_{n-p}(M;G) \\ \alpha &\longmapsto \alpha \cap \theta_M. \end{aligned}$$

Proof. Let $P_M(U)$ be the statement that cap product with the orientation class induces an isomorphism $H^p_{cpt}(U;G) \to H_{n-p}(U;G)$ for each p. We prove the duality theorem by verifying (1) – (3) of Lemma 3.6.5.

(1) Consider an "exhausting sequence" of compact subsets $K_1 \subset K_2 \subset \cdots \subset U$, with the property that each K_i is homeomorphic to a ball, and any compact subset $K \subset U$ lies in K_i. We may assume $K_1 = \{x\}$ for a point $x \in U$. Then each pair $(U, U \smallsetminus K_i)$ retracts onto $(U, U \smallsetminus K_{i+1})$, so we get isomorphisms

$$\varphi_i : H^p(U, U \smallsetminus K_i) \cong H^p(U, U \smallsetminus K_{i+1}).$$

Taking $\psi_i = \varphi_i^{-1}$ and $f_i = \psi_i \circ \cdots \circ \psi_2$ gives compatible maps

$$\begin{tikzcd} H^p(U, U \smallsetminus K_i) \arrow[d, "\psi_i"'] \arrow[dr, "f_i"] & \\ & H^p(U, U \smallsetminus K_1) \\ H^p(U, U \smallsetminus K_{i-1}) \arrow[ur, "f_{i-1}"'] & \end{tikzcd}$$

Thus we get an isomorphism $\varinjlim H^p(U, U \smallsetminus K_i) \cong H^p(U, U \smallsetminus K_1)$, which is *unique* since the direct limit is a universal object. Finally, the duality homomorphism

$$\varinjlim H^p(U, U \smallsetminus K_i) \cong H^p(U, U \smallsetminus K_1) \to H_{n-p}(U)$$

is given by $\alpha \mapsto \alpha \cap \theta_{K_1}$ where θ_{K_1} is a generator of $H_n(U, U \smallsetminus K_1) \cong G$ (recalling that $K_1 = \{x\}$ is a point). This is clearly an isomorphism when $p \neq n$, since $H^i(U, U \smallsetminus K_1)$ and $H_i(U)$ are 0 when $i \neq n$. When $p = n$, the cap product reduces to the Kronecker pairing: $H^n(U, U \smallsetminus K_1) \to H_0(U) \cong G$, $\alpha \mapsto \alpha \cap \theta_{K_1} = \alpha(\theta_{K_1})$. So for a generator $1 \in H^n(U, U \smallsetminus K_1)$, $1(\theta_{K_1}) = 1$. Hence the duality cap product is an isomorphism for every p.

(2) First, if $A \hookrightarrow B$ is an inclusion and $K \subseteq A$ is any compact set, then excision gives us an isomorphism $\varphi_{A,B}$ making the diagram commute:

$$\begin{tikzcd} H^p(A, A \smallsetminus K) \arrow[r, "\varphi_{A,B}"] \arrow[d, equal] & H^p(B, B \smallsetminus K) \arrow[dl, "\text{excision}"] \\ H^p(B \smallsetminus (B \smallsetminus A), (A \smallsetminus K) \smallsetminus (B \smallsetminus A)) & \end{tikzcd}$$

Using the inclusions $U \cap V \hookrightarrow U, U \cap V \hookrightarrow V, U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$, we get maps $i_U = \varphi_{U\cap V,U}, i_V \varphi_{U\cap V,V}, j_U = \varphi_{U,U\cup V}$ and $j_V = \varphi_{V,U\cup V}$. This determines the top row of the following diagram:

$$\begin{tikzcd}
H^p(U\cap V, U\cap V \setminus K) \arrow[r, "i_U + i_V"] \arrow[d] & H^p(U, U\setminus K) \oplus H^p(V, V\setminus K) \arrow[r, "j_U - j_V"] \arrow[d] & H^p(U\cup V, U\cup V\setminus K) \arrow[d] \\
H_{n-p}(U\cap V) \arrow[r] & H_{n-p}(U) \oplus H_{n-p}(V) \arrow[r] & H_{n-p}(U\cup V)
\end{tikzcd}$$

where the bottom row is the Mayer-Vietoris sequence in ordinary homology for (U, V). Since the top row is induced from inclusions, it is clear that it is exact, with connecting homomorphism $H^p(U \cup V, U \cup V \setminus K) \to H^{p+1}(U \cap V, U \cap V \setminus K)$ given by $\alpha \mapsto \delta\alpha|_{U\cap V}$. Moreover, by Lemma 5.4.2 the vertical arrows commute with maps induced from inclusions, so the whole diagram commutes. Since the above construction holds for all compact $K \subseteq U \cap V$, we may apply the exact (by Theorem 5.1.3) functor $\varinjlim$ over the set of all such K, directed by inclusion, to obtain a similar commutative diagram with exact rows:

$$\begin{tikzcd}
H^p_{cpt}(U\cap V) \arrow[r] \arrow[d] & H^p_{cpt}(U) \oplus H^p_{cpt}(V) \arrow[r] \arrow[d] & H^p_{cpt}(U\cup V) \arrow[d] \\
H_{n-p}(U\cap V) \arrow[r] & H_{n-p}(U) \oplus H_{n-p}(V) \arrow[r] & H_{n-p}(U\cup V)
\end{tikzcd}$$

By the hypotheses $P_M(U), P_M(V)$ and $P_M(U \cap V)$, two out of every three of these vertical arrows is an isomorphism so by the Five Lemma (2.2.3), we get $P_M(U \cup V)$.

(3) Set $U = \bigcup U_\alpha$. Then this follows from additivity for compactly-supported cohomology, which in turn is a consequence of direct limits commuting with direct sums:

$$H_{n-p}(U) \cong \bigoplus H_{n-p}(U_\alpha) \cong \bigoplus H^p_{cpt}(U_\alpha) \cong \varinjlim \bigoplus H^p(U_\alpha, U_\alpha \setminus K_\alpha) \cong \varinjlim H^p(U, U \setminus K),$$

where the first direct limit is over compact $K_\alpha \subset U_\alpha$, and in the second direct limit, any compact $K \subset U = \bigcup U_\alpha$ can be written as a disjoint sum of compact sets $K = \bigcup K_\alpha$. This concludes the proof. $\square$

The version of Poincaré duality we have just proved relies on three things: the trivial local structure of manifolds, compactness (in some form) and orientability. In proving more modern versions of duality, we will see how these conditions can be stretched.

Now, as in Section 5.3, assume M is an oriented n-manifold, $L \subseteq K \subseteq M$ are compact subsets and $U \supset K, V \supset L$ are open neighborhoods with $V \subseteq U$.

Lemma 5.4.4. *If M is an oriented manifold, then cap product determines a homomorphism*

$$\begin{aligned} \check{H}^p(K, L) &\longrightarrow H_{n-p}(M \setminus L, M \setminus K) \\ \alpha &\longmapsto \alpha \cap \theta_K, \end{aligned}$$

where $\theta_K \in \Gamma(K, \Theta_M(\mathbb{Z}))$ is a fixed orientation.

Proof. For each pair of neighborhoods (U,V) of (K,L), we have isomorphisms $H_\bullet(M \smallsetminus L, M \smallsetminus K) \cong H_\bullet(U \smallsetminus L, U \smallsetminus K)$ and $H_\bullet(M, M \smallsetminus K) \cong H_\bullet(U, U \smallsetminus K)$. Consider the cap product

$$\begin{aligned} H^p(U,V) \otimes H_{n-p}(U, U \smallsetminus K) &\longrightarrow H_{n-p}(U \smallsetminus L, U \smallsetminus K) \\ [\alpha] \otimes [c] &\longmapsto [\alpha \cap c]. \end{aligned}$$

Explicitly, take $[\alpha] \in H^p(U,V)$, where $\alpha \in \Delta^p(U)$ such that $\delta\alpha = 0$ and $\alpha|_V = 0$; and take $[c] \in H_n(U, U \smallsetminus K)$ with $c = c_V + c_{U \smallsetminus L} \in \Delta_\bullet(V) + \Delta_\bullet(U \smallsetminus L)$, with $\partial c \in \Delta_\bullet(U \smallsetminus K)$. Then

$$\alpha \cap c = \alpha \cap (c_V + c_{U\smallsetminus L}) = \alpha \cap c_V + \alpha \cap c_{U \smallsetminus L} = \alpha \cap c_{U \smallsetminus L}$$

since $\alpha|_V = 0$, so we see that $\alpha \cap c \in \Delta_\bullet(U \smallsetminus L)$. Further,

$$\partial(\alpha \cap c) = \delta\alpha \cap c + (-1)^p \alpha \cap \partial c = 0 + (-1)^p \alpha \cap \partial c,$$

and since $\partial c \in \Delta_\bullet(U \smallsetminus K)$, this shows $\partial(\alpha \cap c) \in \Delta_\bullet(U \smallsetminus K)$. As a result, we have $[\alpha \cap c] \in H_{n-p}(U \smallsetminus L, U \smallsetminus K)$. It is routine to check that this cap product is well-defined on classes and that it is compatible with inclusions of pairs $(U', V') \hookrightarrow (U,V)$. Thus we get a diagram

$$\begin{tikzcd} H^p(U,V) \otimes H_n(U, U \smallsetminus K) \arrow[r] \arrow[d] & H_{n-p}(U \smallsetminus L, U \smallsetminus K) \arrow[d] \\ H^p(U',V') \otimes H_n(U', U' \smallsetminus K) \arrow[r] & H_{n-p}(U' \smallsetminus L, U' \smallsetminus K) \end{tikzcd}$$

where the left vertical arrow is the induced map $i^* : H^p(U,V) \to H^p(U',V')$ tensored with an excision isomorphism and the right vertical arrow is an excision isomorphism. Further, it follows from Theorem 4.4.1 that the diagram commutes. Extending to the direct limit, we may define a cap product

$$\check{H}^p(K,L) \otimes H_n(M, M \smallsetminus K) \longrightarrow H_{n-p}(M \smallsetminus L, M \smallsetminus K).$$

Since K is compact, θ_M defines a generator $\theta_K \in H_n(M, M \smallsetminus K)$, so we get the desired homomorphism $\check{H}^p(K,L) \to H_{n-p}(M \smallsetminus L, M \smallsetminus K)$. $\square$

Lemma 5.4.5. *For an oriented n-manifold M and compact subsets $L \subseteq K \subseteq M$, there is a commutative diagram with exact rows:*

$$\begin{tikzcd} \check{H}^p(K,L) \arrow[r] \arrow[d] & \check{H}^p(K) \arrow[r] \arrow[d] & \check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^{p+1}(K,L) \arrow[d] \\ H_{n-p}(M \smallsetminus L, M \smallsetminus K) \arrow[r] & H_{n-p}(M, M \smallsetminus K) \arrow[r] & H_{n-p}(M, M \smallsetminus L) \arrow[r] & H_{n-p-1}(M \smallsetminus L, M \smallsetminus K) \end{tikzcd}$$

Proof. The top row comes from the long exact sequence in cohomology for each pair (U, V); moreover, taking the direct limit in each term preserves exactness (Theorem 5.1.3), so we get the desired long exact sequence in Čech cohomology. The left and middle squares in the diagram are all induced by inclusion of pairs, so commutativity is ensured. Finally, for the right square, take a class $[\theta] \in H_n(M, M \smallsetminus K)$ and represent it by a chain $\theta \in \Delta_n(M)$ such that $\partial\theta \in \Delta_\bullet(M \smallsetminus K)$ – by naturality, θ also represents the orientation class in $H_n(M, M \smallsetminus L)$. We have the following diagram for any pair (U, V) of neighborhoods of (K, L):

$$\begin{array}{ccc} H^p(V) & \longrightarrow & H^{p+1}(U, V) \\ \downarrow & & \downarrow \\ H_{n-p}(M, M \smallsetminus L) & \longrightarrow & H_{n-p-1}(M \smallsetminus L, M \smallsetminus K) \end{array}$$

For $\alpha \in H^p(V)$, we have

$$\partial(\alpha \cap \theta) = \delta\alpha \cap \theta + (-1)^p \alpha \cap \partial\theta,$$

but $\alpha \cap \partial\theta \in \Delta_{n-p-1}(M \smallsetminus K)$, so $[\partial(\alpha \cap \theta)] = [\delta\alpha \cap \theta]$ in $H_{n-p-1}(M \smallsetminus L, M \smallsetminus K)$. Hence the diagram commutes. □

Theorem 5.4.6 (Poincaré-Alexander-Lefschetz Duality)**.** *Let M be an oriented n-manifold, $L \subseteq K \subseteq M$ compact subsets and G any coefficient group. Then for each $p \geq 0$, cap product with an orientation class $\theta_K \in H_n(M, M \smallsetminus K)$ gives an isomorphism*

$$\begin{aligned} \check{H}^p(K, L; G) &\longrightarrow H_{n-p}(M \smallsetminus L, M \smallsetminus K; G) \\ \alpha &\longmapsto \alpha \cap \theta_K. \end{aligned}$$

Proof. The coefficient group G has no bearing on the proof, so we will suppress it in the notation for homology/cohomology groups. Since Lemma 5.4.5 and the Five Lemma (2.2.3) allow us to reduce to the case $L = \varnothing$, let K be a compact subset of M and let $P_M(K)$ be the statement that the duality homomorphism $\check{H}^p(K) \to H_{n-p}(M, M \smallsetminus K)$ is an isomorphism. We prove duality via (i) – (iii) of Lemma 5.2.8 as follows.

(i) Suppose K is a compact, convex subset of a coordinate chart of M. When $K = \{x\}$, we have

$$\check{H}^p(K) = H^p(K) = \begin{cases} 0, & p > 0 \\ G, & p = 0 \end{cases}$$

by the dimension axiom, while Lemma 5.2.1 gives us

$$H_{n-p}(M, M \smallsetminus \{x\}) = \begin{cases} 0, & p > 0 \\ G, & p = 0. \end{cases}$$

Over $\mathbb{Z}$, the map $H^0(\{x\}) \to H_n(M, M \smallsetminus \{x\})$ is given by $1 \mapsto 1 \cap \theta = \theta$, and θ is a generator of the local cohomology group at x, so this must be an isomorphism. By the universal coefficient theorem (3.4.1), this holds for any G. Now for an arbitrary compact, convex set K, we have a commutative diagram for any point $x \in K$:

$$\begin{array}{ccc}
\check{H}^p(K) & \xrightarrow{\cong} & \check{H}^p(\{x\}) \\
\downarrow & & \downarrow \cong \\
H_{n-p}(M, M \smallsetminus K) & \xrightarrow{\cong} & H_{n-p}(M, M \smallsetminus \{x\})
\end{array}$$

Since three out of the four arrows are isomorphisms, the fourth is as well, which establishes $P_M(K)$.

(ii) The following diagram has exact rows and commutes:

$$\begin{array}{ccccccc}
\check{H}^p(K \cup L) & \longrightarrow & \check{H}^p(K) \oplus \check{H}^p(L) & \longrightarrow & \check{H}^p(K \cap L) & \longrightarrow & \check{H}^{p+1} \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n-p}(M, M \smallsetminus (K \cup L)) & \longrightarrow & \begin{array}{c} H_{n-p}(M, M \smallsetminus K) \\ \oplus \\ H_{n-p}(M, M \smallsetminus L) \end{array} & \longrightarrow & H_{n-p}(M, M \smallsetminus (K \cap L)) & \longrightarrow & H_{n-p-1} \cdots
\end{array}$$

Therefore by the Five Lemma (2.2.3), $P_M(K), P_M(L)$ and $P_M(K \cap L)$ imply that $P_M(K \cup L)$ holds as well.

(iii) Suppose $A_1 \supseteq A_2 \supseteq \cdots$ are compact sets and put $A = \bigcap_{i=1}^{\infty} A_i$. The restriction maps $\check{H}^p(A_i) \to \check{H}^p(A_{i+1})$ form a direct system, and one can show that

$$\varinjlim \check{H}^p(A_i) \cong \check{H}^p(A).$$

Thus there is a commutative diagram

$$\begin{array}{ccc}
\varinjlim \check{H}^p(A_i) & \xrightarrow{\cong} & \check{H}^p(A) \\
\cong \downarrow & & \downarrow \\
\varinjlim H_{n-p}(M, M \smallsetminus A_i) & \xrightarrow{\cong} & H_{n-p}(M, M \smallsetminus A)
\end{array}$$

Here, the top row isomorphism comes from above, the left isomorphism is by each hypothesis $P_M(A_i)$ and the bottom row isomorphism comes from Lemma 5.2.7. Hence the right arrow is also an isomorphism, that is, $P_M(A)$ holds. Now the duality theorem follows from Lemma 5.2.8 as desired. $\square$

Let M be a compact, oriented n-manifold and fix an orientation $\theta \in \Gamma(M, \Theta_M(\mathbb{Z}))$.

Definition. *The generator of $H_n(M) \cong \mathbb{Z}$ corresponding to θ is called the* **fundamental class** *of M, denoted $[M]$.*

In this language, Poincaré duality (either Theorem 5.4.3 or 5.4.6) says that

$$\begin{aligned} H^p(M; G) &\longrightarrow H_{n-p}(M; G) \\ \alpha &\longmapsto \alpha \cap [M] \end{aligned}$$

is an isomorphism for each $p \geq 0$. Note that if M is non-orientable, the proof of duality still works as long as one can find a class $[M]$ generating $H_n(M;G)$ – this is possible when $G = {}_2G$, e.g. if $G = \mathbb{Z}/2\mathbb{Z}$.

Example 5.4.7. For the torus T^2, we have the following homology and cohomology groups:

$$H_i(T^2;\mathbb{Z}) = H^i(T^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0,2 \\ \mathbb{Z}^2, & i = 1 \\ 0, & i > 2. \end{cases}$$

Example 5.4.8. Let $L(p,q)$ be the (p,q) Lens space. Then duality implies that

$$H_i(L(p,q);\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0,3 \\ \mathbb{Z}/p\mathbb{Z}, & i = 1 \\ 0, & i = 2, i > 3 \end{cases} \quad \text{and} \quad H^i(L(p,q);\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0,3 \\ 0, & i = 1, i > 3 \\ \mathbb{Z}/p\mathbb{Z}, & i = 2. \end{cases}$$

(Compare this to Example 3.4.4.)

Example 5.4.9. $\mathbb{R}P^2$ is non-orientable, so we cannot use duality with $\mathbb{Z}$ coefficients. However, with $\mathbb{Z}/2\mathbb{Z}$ coefficients, we have

$$H_i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i = 0,1,2 \\ 0, & i > 2 \end{cases} \quad \text{and} \quad H^i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i = 0,1,2 \\ 0, & i > 2. \end{cases}$$

(Compare this to Example 4.3.8.)

Corollary 5.4.10 (Lefschetz Duality)**.** *For a compact, oriented n-manifold M and any compact subset $L \subseteq M$, there is a commutative diagram*

$$\begin{tikzcd} \check{H}^p(M,L) \arrow[r] \arrow[d] & \check{H}^p(M) \arrow[r] \arrow[d] & \check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^{p+1}(M,L) \arrow[d] \\ H_{n-p}(M \smallsetminus L) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M, M \smallsetminus L) \arrow[r] & H_{n-p-1}(M \smallsetminus L) \end{tikzcd}$$

In particular, there is an isomorphism $\check{H}^p(M,L) \to H_{n-p}(M \smallsetminus L)$ for every $p \geq 0$.

Corollary 5.4.11. *Suppose M is a connected, compact, oriented n-manifold such that $H_1(M;\mathbb{Z}) = 0$ and A is any closed set, then*

(1) $\check{H}^{n-1}(A)$ is free.

(2) The number of components of $M \smallsetminus A$ is equal to $1 + \operatorname{rank} \check{H}^{n-1}(A)$.

Proof. Consider the sequence in reduced homology for the pair (M,A):

$$0 = H_1(M) \to H_1(M, M \smallsetminus A) \to \widetilde{H}_0(M \smallsetminus A) \to \widetilde{H}_0(M) = 0.$$

Then by duality, $\check{H}^{n-1}(A) \cong H_1(M, M \smallsetminus A) \cong \widetilde{H}_0(M \smallsetminus A)$ which is always free. Finally, the number of components of the compliment is given by

$$\operatorname{rank} H_0(M \smallsetminus A) = 1 + \operatorname{rank} \widetilde{H}_0(M \smallsetminus A) = 1 + \operatorname{rank} \check{H}^{n-1}(A)$$

as claimed. □

This provides another proof of the Jordan-Brouwer separation theorem (Corollary 2.8.4).

Corollary 5.4.12. *Any topological embedding $S^n \hookrightarrow S^{n+1}$ separates S^{n+1} into two components.*

Proof. Let A be the image of S^n in S^{n+1}. Then $\check{H}^n(A) = \check{H}^n(S^n) = \mathbb{Z}$, so by (2) of Corollary 5.4.11, $S^{n+1} \smallsetminus A$ has two components. □

5.5 Duality of Manifolds with Boundary

If M is a compact, oriented manifold with nonempty boundary ∂M, we can still apply the results of the previous section to the manifold $\mathring{M} = \operatorname{Int}(M) = M \smallsetminus \partial M$, which is now a manifold without boundary, though not necessarily compact.

Lemma 5.5.1. *For any manifold M with boundary and any $p \geq 0$, there is an isomorphism*

$$H^p_{cpt}(\mathring{M}) \cong H^p(M, \partial M).$$

Theorem 5.5.2. *Let M be a compact, oriented n-manifold with boundary ∂M. Then there are isomorphisms $H^p(M, \partial M) \cong H_{n-p}(M)$ and $H^p(M) \cong H_{n-p}(M, \partial M)$ for every $p \geq 0$.*

Proof. Taking a "collared neighborhood" $\partial M \times [0, 1)$ of the boundary, we have $H_{n-p}(M, \partial M) = H_{n-p}(M, \partial M \times [0, 1))$. Consider the isomorphisms

$$\begin{aligned} H_{n-p}(M, \partial M \times [0,1)) &\cong H_{n-p}(\mathring{M}, \partial M \times (0,1)) \quad \text{by excision} \\ &\cong H_{n-p}(\mathring{M}, \mathring{M} \smallsetminus K) \quad \text{where } K = M \smallsetminus \partial M \times [0,1) \\ &\cong H^p(M) \quad \text{by duality.} \end{aligned}$$

Likewise,

$$\begin{aligned} H^p(M) &= H^p(M \smallsetminus \partial M \times [0,1)) \\ &\cong H_{n-p}(\mathring{M}, \partial M \times (0,1)) \quad \text{by duality for } \mathring{M} \\ &\cong H_{n-p}(M, \partial M \times [0,1)) \\ &\cong H_{n-p}(M, \partial M). \end{aligned}$$

□

Note the special case $H_n(M, \partial M) \cong H^0(M) = \mathbb{Z}$.

Definition. *A generator of $H_n(M, \partial M) \cong \mathbb{Z}$ is called a* **relative orientation class** *of M, denoted $[M]$.*

Given a relative orientation class $[M] \in H_n(M, \partial M)$, cap product gives us two duality maps

$$H^p(M, \partial M) \xrightarrow{\cap[M]} H_{n-p}(M)$$
$$\text{and} \quad H^p(M) \xrightarrow{\cap[M]} H_{n-p}(M, \partial M).$$

Lemma 5.5.3. *If M is compact and orientable, then ∂M is orientable and $[\partial M] := \partial_*[M]$ is an orientation class for ∂M.*

Proof. First, suppose ∂M and consider the long exact sequence

$$\cdots \to H_n(M) \to H_n(M, \partial M) \to H_{n-1}(\partial M) \to H_{n-1}(M) \to \cdots$$

Then it's enough to show that $H_n(M) = 0$, since then we will have $\mathbb{Z}H_n(M, \partial M) \hookrightarrow H_{n-1}(\partial M)$, so it will follow that $\partial_*[M]$ is a generator and thus ∂M is orientable. By excision and Corollary 5.2.12, $H_n(M) = H_n(\mathring{M}) = 0$ so the result follows. Further, Corollary 5.2.12 says that $H_n(M; G) = 0$ holds for any coefficient group G, so by the universal coefficient theorem,

$$0 = H_n(M; \mathbb{Q}/\mathbb{Z}) \cong H_n(M) \otimes \mathbb{Q}/\mathbb{Z} \oplus \mathrm{Tor}_1(H_{n-1}(M), \mathbb{Q}/\mathbb{Z}) = 0 \oplus TH_{n-1}(M)$$

(with the last equality coming from Example 3.3.15). Hence $H_{n-1}(M)$ has no torsion, so $[M]$ maps to a generator in $H_{n-1}(\partial M)$. For the disconnected case, see Bredon. $\square$

Theorem 5.5.4 (Duality for Manifolds with Boundary)**.** *If M is a connected, compact, oriented n-manifold with boundary, then there is a commutative diagram with exact rows*

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(M) \arrow[r] \arrow[d, "\cap[M]"] & H^p(\partial M) \arrow[r] \arrow[d, "\cap[\partial M]"] & H^{p+1}(M, \partial M) \arrow[r] \arrow[d, "\cap[M]"] & H^{p+1}(M) \arrow[r] \arrow[d, "\cap[M]"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M, \partial M) \arrow[r] & H_{n-p-1}(\partial M) \arrow[r] & H_{n-p-1}(M) \arrow[r] & H_{n-p-1}(M, \partial M) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, all the columns are isomorphisms.

Proof. The first column is an isomorphism by Theorem 5.5.2 and the second is an isomorphism by Poincaré duality for ∂M, so it's enough to show the diagram commutes and apply the Five Lemma.

Consider the diagram

$$\begin{tikzcd}
H^p(M) \arrow[r, "i^*"] \arrow[d] & H^p(\partial M) \arrow[d] \\
H_{n-p}(M, \partial M) \arrow[r, "\partial"] & H_{n-p-1}(\partial M)
\end{tikzcd}$$

For $\alpha \in H^p(M)$, we have

$$\begin{aligned}\partial[\alpha \cap M] &= \delta\alpha \cap [M] + (-1)^p \alpha \cap \partial[M] \\ &= 0 + (-1)^p \alpha \cap [\partial M] \quad \text{by Lemma 5.5.3} \\ &= (-1)^p i^* \alpha \cap [\partial M].\end{aligned}$$

So the diagram commutes. For the next square,

$$\begin{tikzcd} H^p(\partial M) \arrow[r, "\delta"] \arrow[d] & H^{p+1}(M, \partial M) \arrow[d] \\ H_{n-p-1}(\partial M) \arrow[r] & H_{n-p-1}(M) \end{tikzcd}$$

we have for any $\alpha \in H^p(\partial M)$ that $\delta\alpha \cap [M] = \partial(\alpha \cap [M]) - (-1)^p(\alpha \cap [\partial M])$, but $\partial(\alpha \cap [M]) = 0$ in homology, so $\delta\alpha \cap [M]$ and $(-1)^{p-1}(\alpha \cap [\partial M])$ define the same homology class. Hence this diagram commutes as well. Finally, the proof that the third square commutes is similar. □

We now investigate some applications of duality for manifolds with boundary.

Definition. *For a compact, oriented n-manifold M, there is a pairing*

$$\begin{aligned} H^p(M) \otimes H^{n-p}(M, \partial M) &\longrightarrow H_0(M) = \mathbb{Z} \\ \alpha \otimes \beta &\longmapsto \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

called the **cup product pairing**.

Theorem 5.5.5. *The cup product pairing is nondegenerate on free parts, i.e. for each $p \geq 0$ there is an induced isomorphism*

$$\begin{aligned} FH^p(M) &\longrightarrow \mathrm{Hom}(FH^{n-p}(M, \partial M), \mathbb{Z}) \\ \beta &\longmapsto (\alpha \mapsto \langle \alpha \cup \beta, [M] \rangle). \end{aligned}$$

Proof. By the universal coefficient theorem and duality, there are isomorphisms

$$FH^p(M) \cong \mathrm{Hom}(FH_p(M), \mathbb{Z}) \cong \mathrm{Hom}(FH^{n-p}(M, \partial M), \mathbb{Z}).$$

Explicitly, these maps are given by $\alpha \mapsto \langle \alpha, - \rangle$ and $\beta \mapsto \langle \beta, - \cap [M] \rangle$. However,

$$\begin{aligned} \langle \alpha, \beta \cap [M] \rangle &= \alpha \cap (\beta \cap [M]) \\ &= (\alpha \cup \beta) \cap [M] \quad \text{by Theorem 4.4.1(e)} \\ &= \langle \alpha \cup \beta, [M] \rangle. \end{aligned}$$

So the isomorphism is as described. □

A classic question in topology is: Given a manifold M^n, how can one determine if M is the boundary of a compact manifold V^{n+1}?

Example 5.5.6. For $n = 1, 2$, all compact, orientable n-manifolds M are the boundary of a compact manifold V^{n+1}. In fact, for $n = 3$ this is true as well, although it is highly nontrivial to prove.

Theorem 5.5.7. *Let Λ be a field, M^{2n} a compact, connected manifold and V^{2n+1} a compact, oriented manifold such that $\partial V = M$. Then*

(a) The cohomology $H^n(M; \Lambda)$ has even dimension over Λ.

(b) Let $K = \ker(i_* : H_n(M; \Lambda) \to H_n(V; \Lambda))$. Then

$$\dim K = \frac{1}{2} \dim H^n(M; \Lambda) = \dim \operatorname{im}(i^* : H^n(V; \Lambda) \to H^n(M; \Lambda)).$$

(c) If $\alpha, \beta \in \operatorname{im}(i^* : H^n(V; \Lambda) \to H^n(M; \Lambda))$ then $\alpha \cup \beta = 0$.

Proof. (a) – (b) We will suppress the coefficients Λ. By duality for manifolds with boundary (Theorem 5.5.4), there is a commutative diagram with vertical isomorphisms and exact rows:

$$\begin{tikzcd}
H^n(V) \arrow[r, "i^*"] & H^n(M) \arrow[r, "\delta"] \arrow[d, "\cap[M]"] & H^{n+1}(V, M) \arrow[d, "\cap[V]"] \\
& H_n(M) \arrow[r] & H_n(V)
\end{tikzcd}$$

Then $\ker \delta = \operatorname{im} i^* \subseteq H^n(M)$ maps isomorphically onto $K = \ker i_* \subseteq H_n(M)$. Thus

$$\begin{aligned}
\operatorname{rank} i^* &= \dim \operatorname{im} i^* = \dim \ker i_* = \dim K \\
&= \dim H_n(M) - \operatorname{rank} i_*.
\end{aligned}$$

But by duality, $i_* : H_n(M) \to H_n(V)$ and $i^* : H^n(V) \to H^n(M)$ may be viewed as each other's transpose, so in particular they have the same rank. Hence $\dim H_n(M) = 2 \operatorname{rank} i^* = 2 \dim \operatorname{im} i^* = 2 \dim K$.

(c) For $a, b \in H^n(V)$, let $\alpha = i^*a$ and $\beta = i^*b$. Then by (2) of Proposition 4.3.2,

$$\alpha \cup \beta = i^*a \cup i^*b = i^*(a \cup b).$$

Thus by exactness, $\delta(\alpha \cup \beta) = \delta i^*(a \cup b) = 0$. Now, the diagram

$$\begin{tikzcd}
H^{2n}(M) \arrow[r, "\delta"] \arrow[d, "\cong"'] & H^{2n+1}(V, M) \arrow[d, "\cong"] \\
H_0(M) \arrow[r] & H_0(V)
\end{tikzcd}$$

commutes and the bottom row is injective, so we must have $\alpha \cup \beta = 0$. $\square$

Corollary 5.5.8. *For n even, $\mathbb{C}P^n$ is not the boundary of a compact, orientable $(2n+1)$-manifold.*

Proof. By Theorem 3.6.8, $\mathbb{C}P^n$ has real cohomology isomorphic to $\mathbb{R}$ in even dimensions, so when n is even, $\dim H^n(\mathbb{C}P^n;\mathbb{R})$ is odd. Apply Theorem 5.5.7 with $\Lambda = \mathbb{R}$. $\square$

Corollary 5.5.9. *For n even, $\mathbb{R}P^n$ is not the boundary of a compact, orientable $(n+1)$-manifold.*

Proof. Apply Theorem 5.5.7 with $\Lambda = \mathbb{Z}/2\mathbb{Z}$. $\square$

Let M be a $4n$-dimensional closed, orientable manifold. Then the cup product pairing

$$\begin{aligned} b_M : H^{2n}(M;\mathbb{R}) \otimes H^{2n}(M;\mathbb{R}) &\longrightarrow H^{4n}(M;\mathbb{R}) \cong \mathbb{R} \\ \alpha \otimes \beta &\longmapsto \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

is bilinear, symmetric and nondegenerate and has, with respect to an appropriate basis, a diagonal matrix form Q_M. Let p be the number of positive eigenvalues and m be the number of negative eigenvalues along the diagonal of Q_M.

Definition. *The* **signature** *of M^{4n} is $\sigma(M) = p - m$, where p and m are defined for the diagonal matrix Q_M above.*

Since the sign of each eigenvalue of Q_M is preserved under basis change, we see that $\sigma(M)$ is well-defined. The signature gives a way of determining when a $4n$-manifold is the boundary of another manifold.

Corollary 5.5.10. *If M is a closed, orientable $4n$-manifold and $M = \partial V$ for some compact, orientable $(4n+1)$-manifold V, then $\sigma(M) = 0$.*

Proof. Write $H = H^{2n}(M;\mathbb{R})$ so that by Theorem 5.5.7(a), $\dim H = 2k$. Diagonalizing Q_M allows us to decompose H as a sum of subspaces $H = H^+ \oplus H^-$, where Q_M is positive-definite on H^+ and negative-definite on H^-. Then $\dim H^+ = p$, $\dim H^- = m$ and we have the following equations:

$$\sigma(M) = p - m \quad \text{and} \quad 2k = p + m.$$

By Theorem 5.5.7(c), there exists a subspace $K = \operatorname{im}(i^* : H^{2n}(V;\mathbb{R}) \to H)$ of dimension k on which the cup product is trivial. Hence $K \cap H^+ = 0$ and $K \cap H^- = 0$. The first implies $k + p \leq 2k$, so $p \leq k$; while the second gives $k + m \leq 2k$, or $m \leq k$. Putting these together with the equations above, we must have $p = m = k$. Hence $\sigma(M) = p - m = k - k = 0$. $\square$

Example 5.5.11. By Corollary 5.5.8, we know $\mathbb{C}P^2$ is not the boundary of an orientable 5-manifold, but $M = \mathbb{C}P^2 \# \mathbb{C}P^2$ has $H^2(M) \cong \mathbb{R}^2$, which is even dimensional, so the same trick will not work for M. However, one can compute the matrix for the cup product pairing b_M to be

$$Q_M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\sigma(M) = 2$, so by Corollary 5.5.10, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not the boundary of a 5-manifold either.

6 Intersection Theory

The objective in intersection theory is to give a geometric way of computing cup products using Poincaré duality. In particular, if M^n is a compact, connected, oriented manifold, we will define an "intersection product"

$$H_i(M) \otimes H_j(M) \longrightarrow H_{i+j-n}(M)$$

that is dual to the cup product. This will be given a geometric interpretation in terms of submanifolds of M, which will make it easier to compute cup products of M in general.

6.1 The Thom Isomorphism Theorem

Let N be a compact manifold without boundary and let $\pi : W \to N$ be a disk bundle over N, i.e. a surjection in which each preimage $\pi^{-1}(U)$ is homeomorphic to $U \times D^k$ and the choices of homeomorphisms are compatible with overlaps of charts $U \cap V \subseteq N$. If $\dim N = n$ and $\dim \pi^{-1}(U) = k$ (i.e. the disks are k-disks), then $\dim W = n + k$. Assume N and W are both oriented.

Definition. *The* **Thom class** *of the bundle $\pi : W \to N$ is the Poincaré dual τ of the 0-section of the bundle π. That is, if $i : N \to W$ is the section $i(n) = 0 \in \pi^{-1}(n) \cong D^k$, then $\tau \in H^k(W, \partial W)$ is the unique class such that $\tau \cap [W] = i_*[N]$.*

We will prove that for a disk bundle $\pi : W \to N$, the composition $H^p(N) \xrightarrow{\pi^*} H^p(W) \xrightarrow{\cup\tau} H^{p+k}(W, \partial W)$ is an isomorphism for any $p \geq 0$. This is known as the Thom isomorphism theorem.

Definition. *Let $D_M : H_i(M, \partial M) \to H^{n-i}(M)$, be the inverse of the Poincaré duality isomorphism. We define the* **intersection product** *on M to be the map*

$$\begin{aligned} H_i(M, \partial M) \otimes H_j(M) &\longrightarrow H_{i+j-n}(M) \\ a \otimes b &\longmapsto D_M^{-1}(D_M(b) \cup D_M(a)), \end{aligned}$$

where $n = \dim M$.

Alternatively for the dual isomorphism $D_M : H_i(M) \to H^{n-i}(M, \partial M)$, we can define the intersection product by

$$\begin{aligned} H_i(M, \partial M) \otimes H_j(M, \partial M) &\longrightarrow H_{i+j-n}(M, \partial M) \\ a \otimes b &\longmapsto D_M^{-1}(D_M(b) \cup D_M(a)). \end{aligned}$$

Remark. By the definition of D_M, we have

$$D_M^{-1}(D_M(b) \cup D_M(a)) = (D_M(b) \cup D_M(a)) \cap [M] = D_M(b) \cap (D_M(a) \cap [M]) = D_M(b) \cap a.$$

Equivalently, the intersection product of a and b is the unique class $a \bullet b \in H_{i+j-n}(M)$ satisfying $D_M(a \bullet b) = D_M(b) \cup D_M(a)$.

Lemma 6.1.1. *For any $a \in H_i(M), b \in H_j(M)$ and $c \in H_k(M)$, the intersection product satisfies:*

(a) (Graded commutativity) $a \bullet b = (-1)^{(n-i)(n-j)} b \bullet a$.

(b) (Associativity) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

Proof. (1) follows from the Theorem 4.3.5.

(2) Since D_M is an isomorphism, it suffices to show $D_M(a \bullet (b \bullet c)) = D_M((a \bullet b) \bullet c)$. Indeed, we have

$$\begin{aligned} D_M(a \bullet (b \bullet c)) &= D_M(b \bullet c) \cup D_M(a) \\ &= (D_M(c) \cup D_M(b)) \cup D_M(a) \\ &= D_M(c) \cup (D_M(b) \cup D_M(a)) \quad \text{by Theorem 4.3.5} \\ &= D_M(c) \cup (a \bullet b) \\ &= D_M((a \bullet b) \bullet c). \end{aligned}$$

Therefore $\bullet$ is associative. □

Definition. *Let M be an m-manifold, N be an n-manifold and $f : (N, \partial N) \to (M, \partial M)$ be a map of manifolds with boundary. The* **transfer maps** *of f are defined on cohomology by*

$$\begin{aligned} f^! : H^{n-p}(N) &\longrightarrow H^{m-p}(M) \\ \alpha &\longmapsto f^!(\alpha) := D_M f_* D_N^{-1}(\alpha) \end{aligned}$$

and on homology by

$$\begin{aligned} f_! : H_{m-p}(M) &\longrightarrow H_{n-p}(N) \\ \sigma &\longmapsto f_!(\sigma) := D_N^{-1} f^* D_M(\sigma). \end{aligned}$$

Let N be a connected, oriented, closed n-manifold and let $\pi : W \to N$ be a k-disk bundle over N. This means that $\dim W = n + k$. Then the 0-section $i : N \hookrightarrow W$, i.e. the section taking $x \in N$ to the center of the corresponding disk D_x^k, is an embedding. Also, ∂W is a $(k-1)$-sphere bundle over N. Here, the transfer map $i_! : H_n(N) \to H_n(W)$ determines the Thom class:

$$\tau = D_W i_! [N] \in H^k(W, \partial W).$$

Explicitly, $\tau \cap [W] = i_! [N]$.

Theorem 6.1.2 (Thom Isomorphism Theorem)**.** *Let $\pi : W \to N$ be a k-disk bundle and $i : N \hookrightarrow W$ the 0-section. Then for any $p \geq 0$,*

$$H^p(N) \xrightarrow{\pi^*} H^p(W) \xrightarrow{- \cup \tau} H^{p+k}(W, \partial W)$$

is an isomorphism which is equal to the transfer map $i^!$.

Proof. By definition, $i^! = D_W i_* D_N^{-1}$, so because the duality maps are isomorphisms, it's enough to show that i_* is an isomorphism. Note that since i is a section, $\pi i = id_N$. On the other hand, W deformation retracts onto $i(N)$, so $i\pi$ is homotopic to the identity on W. Thus π and i are homotopy inverses, which implies i_* is an isomorphism as desired. Now we show the Thom isomorphism $\pi^*(-) \cup \tau$ is equal to $i^!$. For $\beta \in H^p(N)$, let $\alpha = \pi^*(\beta)$ so that $\beta = i^*(\alpha)$. Then

$$\begin{aligned} i^!(\beta) &= D_W i_* D_N^{-1}(\beta) \\ &= D_W i_*(\beta \cap [N]) \\ &= D_W i_*(i^*(\alpha) \cap [N]) \\ &= D_W(\alpha \cap i_*[N]) \quad \text{by Theorem 4.4.1(f)} \\ &= D_W(\alpha \cap (\tau \cap [W])) \quad \text{by the above} \\ &= D_W((\alpha \cup \tau) \cap [W]) \quad \text{by Theorem 4.4.1(e)} \\ &= \alpha \cup \tau = \pi^*(\beta) \cup \tau. \end{aligned}$$

Therefore the Thom map is an isomorphism. □

Lemma 6.1.3. *Let $\pi : W \to N$ be a k-disk bundle. For any closed subset $A \subseteq N$, let $\widetilde{A} = \pi^{-1}(A) \subseteq W$ and $\partial\widetilde{A} = \widetilde{A} \cap \partial W$. Then*

(a) For all $i < k$, $\check{H}^i(\widetilde{A}, \partial\widetilde{A}) = 0$.

(b) The restriction $\tau_x \in \check{H}^k(\widetilde{\{x\}}, \partial\widetilde{\{x\}})$ for any $x \in N$ is a generator.

We now give a geometric interpretation of the intersection product and Thom class.

Definition. *Let $i : N \hookrightarrow W$ be a smooth embedding of smooth manifolds such that ∂N meets ∂W transversely. We define the* **Thom class of the embedding** *by $\tau_N^W = D_W[N]_W$, where $[N]_W = i_*[N]$ is the restriction of the fundamental class of N to W. In other words, τ_N^W is the unique class satisfying $\tau_N^W \cap [W] = [N]_W$.*

Theorem 6.1.4. *If $K, N \hookrightarrow W$ are two embedded submanifolds of W whose boundaries intersect ∂W transversely, then $[K]_W \bullet [N]_W = (\tau_N^W \cup \tau_K^W) \cap [W]$. Moreover, if K and N intersect transversely in W, then their normal bundles satisfy $\mathcal{V}_{N\cap K}^W = \mathcal{V}_K^W|_{N\cap K} = \mathcal{V}_N^W|_{N\cap K}$.*

Corollary 6.1.5. *If $K, N \hookrightarrow W$ are embedded submanifolds intersecting transversely in W, then $[N \cap K]_W = [N]_W \bullet [K]_W$.*

Proof. Let $i : K \hookrightarrow W$ and $j : N \hookrightarrow W$ be the embeddings. Then $\tau_{N\cap K}^N = j^*\tau_K^W$ by definition of the Thom class, so by Theorem 6.1.4,

$$\begin{aligned} [N]_W \bullet [K]_W &= (\tau_K^W \cup \tau_N^W) \cap [W] \\ &= \tau_K^W \cap (\tau_N^W \cap [W]) \quad \text{by Theorem 4.4.1(e)} \\ &= \tau_K W \cap [N]_W \\ &= [N \cap K]_W. \end{aligned}$$

□

Corollary 6.1.6. *If $K, N \hookrightarrow W$ are submanifolds such that $N \cap K = \varnothing$, then the intersection product on N and K is degenerate, i.e. for all $\sigma \in H_i(N)$ and $\eta \in H_j(K)$, $\sigma \bullet \eta = 0$.*

Example 6.1.7. Let $W = S^n \times S^m$, fix a point $(x, y) \in S^n \times S^m$ and set $N = S^n \times \{y\}$ and $K = \{x\} \times S^m$. Then $N \cap K = \{(x, y)\}$ so N and K intersect transversely. By Corollary 6.1.5, $[N]_W \bullet [K]_W = [N \cap K]_W = \pm[(x, y)]$. Thus if α is the Poincaré dual of N, i.e. $[N] = \alpha \cap [W]$, and β is the Poincaré dual of K, i.e. $[K] = \beta \cap [W]$, then $\alpha \cup \beta$ generates $H^{n+m}(S^n \times S^m)$. Also, taking two copies of N intersecting transversely in W gives $[N] \bullet [N] = 0$ which implies $\alpha^2 = \alpha \cup \alpha = 0$. Likewise, $\beta^2 = \beta \cup \beta = 0$. This information completely determines the cohomology ring of $S^n \times S^m$.

Let $\pi : E \to N$ be a k-disk bundle over an oriented, closed, compact n-manifold N. Then the Thom class of the bundle is $\tau \in H^k(E, \partial E) = H^k(E, E \smallsetminus N)$. There is a cup product

$$H^j(E) \otimes H^\ell(E, \partial E) \longrightarrow H^{j+\ell}(E, \partial E)$$

and $\pi^* : H^\bullet(N) \to H^\bullet(E)$ is an isomorphism, so this turns $H^\bullet(E, \partial E)$ into a module over $H^\bullet(N)$ via

$$\begin{aligned} H^\bullet(N) \times H^\bullet(E, \partial E) &\longrightarrow H^\bullet(E, \partial E) \\ (\alpha, \beta) &\longrightarrow \pi^*(\alpha) \cup \beta. \end{aligned}$$

By the Thom isomorphism theorem (6.1.2), there is an isomorphism $H^\bullet(N) \cong H^\bullet(E, \partial E)$, so it follows that $H^\bullet(E, \partial E)$ is a *free* module of rank 1 over $H^\bullet(N)$ generated by the Thom class τ.

Next, suppose $i : N^n \hookrightarrow W^w$ is an embedding of closed, oriented manifolds. The normal bundle of N in W, $\mathcal{V}_N^W$, has a Thom class τ and N has a Poincaré dual $\alpha_N := [N]_W$. What is the relation between these classes τ and α_N? By definition, $\tau \in H^{w-n}(\mathcal{V}_N^W, \partial \mathcal{V}_N^W)$ and $\alpha_N = D_W i_*[N] \in H^{w-n}(W)$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{w-n}(W, W \smallsetminus \mathcal{V}_N^W) \arrow[r] \arrow[d, "\cong"'] & H^{w-n}(W) \arrow[r] \arrow[dd, "-\cap [W]"] & H^{w-n}(W \smallsetminus \mathcal{V}_N^W) \arrow[r] & \cdots \\
 & & H^{w-n}(\mathcal{V}_N^W, \partial \mathcal{V}_N^W) \arrow[d, "-\cap [N]"'] & & & \\
\cdots \arrow[r] & H_n(N) \arrow[r] & H_n(\mathcal{V}_N^W) \arrow[r] & H_n(W) \arrow[r] & \cdots &
\end{tikzcd}$$

where the vertical arrows are isomorphisms by excision (top left) and duality (bottom left and right). This shows that α_N is the image of τ under the map $H^{w-n}(\mathcal{V}_N^W, \partial \mathcal{V}_N^W) \to H^{w-n}(W)$. Hence the Thom class coincides with the Poincaré dual of N in W.

There is a smooth manifold version of the Thom isomorphism theorem which is compatible with de Rham cohomology. Suppose $M = \mathbb{R}^k$ is Euclidean k-space. Then by de Rham's theorem (3.6.6) and compact duality (5.4.3),

$$H_\bullet(D^k, \partial D^k) = H_\bullet(\mathbb{R}^k, \mathbb{R}^k \smallsetminus \{0\}) \cong H^\bullet_{cpt}(\mathbb{R}^k) \cong H^\bullet_{dR,cpt}(\mathbb{R}^k).$$

Moreover, we have

$$H^{\ell}_{cpt} = \begin{cases} \mathbb{R}^k, & \ell = k \\ 0, & \ell \neq k. \end{cases}$$

On the other hand, $H^k_{dR,cpt}(\mathbb{R}^k)$ is generated by $\varphi\, dx_1 \wedge \cdots \wedge dx_k$ for some compactly supported function φ satisfying $\int_{\mathbb{R}^k} \varphi = 1$.

Theorem 6.1.8 (Thom)**.** *Let τ be the Thom class of a vector bundle over $\mathbb{R}^k$. Then $\tau_x = \varphi\, dx_1 \wedge \cdots \wedge dx_k$, where τ_x is the restriction of τ to the fibre $\pi^{-1}(x)$ and φ is a compactly supported function satisfying $\int_{\mathbb{R}^k} \varphi = 1$.*

This is in fact a characterization of the Thom class in the smooth case: τ is the unique cohomology class in $H^k_{dR,cpt}(\mathbb{R}^k)$ which integrates to 1 on each fibre. Therefore the following holds:

Corollary 6.1.9. *For any vector bundle $\pi : E \to \mathbb{R}^k$, there is an isomorphism*

$$H^i_{dR}(\mathbb{R}^k) \longrightarrow H^{i+k}_{dR,cpt}(E), \quad \alpha \longmapsto \pi^*\alpha \wedge \tau$$

where τ is the Thom class.

6.2 Euler Class

Let $\pi : E \to N$ be a k-disk bundle over a closed n-manifold N. Consider the diagram

$$\begin{array}{ccccccccc} H_n(N) & \xrightarrow{i_*} & H_n(E) & \xrightarrow{D} & H^k(E, \partial E) & \xrightarrow{j^*} & H^k(E) & \xrightarrow{i^*} & H^k(N) \\ [N] & \longmapsto & & & Di_*[N] = \tau & \longmapsto & & & i^*j^*\tau = e(E) \end{array}$$

where τ is the Thom class of π and D is duality.

Definition. *The class $e(E) = i^*j^*\tau = i^*j^*Di_*[N]$ is called the* **Euler class** *of N for the bundle $\pi : E \to N$.*

Lemma 6.2.1. *Suppose a bundle $\pi : E \to N$ has a nonzero section $\sigma : N \to E$ such that $\sigma(N) \subseteq E \setminus N \subseteq \partial N$. Then $e(E) = 0$.*

Proof. Such a section σ determines maps

$$H^\bullet(N) \xrightarrow{\pi^*} H^\bullet(E) \to H^\bullet(\partial E) \xrightarrow{\sigma^*} H^\bullet(N)$$

which is the identity on $H^\bullet(N)$. In particular, $H^\bullet(E) \to H^\bullet(\partial E)$ is injective, so in the exact sequence

$$H^\bullet(E, \partial E) \xrightarrow{j^*} H^\bullet(E) \to H^\bullet(\partial E)$$

we get that j^* is 0. Therefore $e(E) = i^*j^*\tau = 0$. $\square$

Suppose $N \hookrightarrow W$ is an embedding of smooth manifolds. Then TN is a submanifold of TW, so the normal bundle $\mathcal{V} = \mathcal{V}_N^W = (TN)^\perp \subseteq TW$ is defined. Consider the composition

$$\begin{array}{ccccccccc} H_n(W) & \xrightarrow{D^{-1}} & H^k(W, W \smallsetminus \mathcal{V}) & \longrightarrow & H^k(\mathcal{V}, \partial\mathcal{V}) & \longrightarrow & H^k(\mathcal{V}) & \longrightarrow & H^k(N) \\ \alpha_N & & \longmapsto & & \tau(\mathcal{V}) & & \longmapsto & & e(\mathcal{V}) \end{array}$$

Definition. *The* **Euler class** *of the embedding $N \hookrightarrow W$ is $\chi_N^W := e(\mathcal{V}) \in H^k(N)$.*

A special case of interest is when $W = N \times N$ and $d : N \to \Delta \subseteq N \times N$ is the diagonal embedding. On the level of tangent spaces, $TN \cong T\Delta \subseteq T(N \times N) = TN \oplus TN$. Observe that $T\Delta \cap TN \times \{0\} = 0$ and $T\Delta \cap \{0\} \times TN = 0$. Thus the quotient map

$$\pi : T(N \times N) \to T(N \times N)/T\Delta$$

induces a homeomorphism $TN \times \{0\} \cong \pi(TN \times \{0\}) = T(N \times N)/T\Delta$. Identifying N with its diagonal via d, we can write the Thom class of the embedding $\tau = \alpha_\Delta \in H^n(N \times N)$, i.e. $\tau = d_*[N]$.

Definition. *The* **Euler class** *of any n-manifold N is $\chi := d^*\tau \in H^n(N)$.*

By definition, we have that $\chi = d^*\tau = d^*\chi_\Delta^{N\times N} = d^*e(\mathcal{V}(\Delta)) = e(TN)$, so this shows that χ is equivalent to the Euler class of the tangent bundle to the diagonal $\Delta \subseteq N \times N$.

Let Λ be a field. Then by the universal coefficient theorem (3.4.2) and Poincaré duality, we get isomorphisms

$$\begin{array}{ccccc} H^i(N;\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(H_i(N;\Lambda),\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(H^{n-i}(N;\Lambda),\Lambda) \\ \alpha & \longmapsto & \langle \alpha, - \rangle & \longmapsto & \langle \alpha, - \cap [N] \rangle \end{array}$$

By Theorem 4.4.1(a) and (e), we can interpret $\langle \alpha, - \cap [N]\rangle$ as $\langle \alpha \cup -, [N]\rangle$. For any basis B of $H^\bullet(N)$, this determines a dual basis $\{\alpha^0\}_{\alpha\in B}$ such that $\langle \alpha^0 \cup \beta, [N]\rangle = \delta_{\alpha,\beta}$.

Theorem 6.2.2. *For any n-manifold N, the Thom class $\tau = d_*[N]$ satisfies*

$$\tau = \sum_{\alpha\in B} (-1)^{|\alpha|}\alpha^0 \times \alpha \in H^n(N \times N;\Lambda).$$

Proof. By the Künneth formula (Theorem 4.2.3), τ can be written

$$\tau = \sum_{\alpha,\beta\in B} A_{\alpha,\beta}\, \alpha^0 \times \beta$$

for some $A_{\alpha,\beta} \in \Lambda$, but all of these coefficients are 0 except when $|\alpha| = |\beta|$. Let $|\alpha| = |\beta| = p$. Then

$$\begin{aligned} \langle \alpha \times \beta^0 \cup \tau, [N \times N]\rangle &= \langle \alpha \times \beta^0, \tau \cap [N \times N]\rangle \\ &= \langle \alpha \times \beta^0, d_*[N]\rangle \\ &= \langle d^*(\alpha \times \beta^0), [N]\rangle \quad \text{by Theorem 4.4.1(a) and (f)} \\ &= \langle \alpha \cup \beta^0, [N]\rangle \quad \text{by definition of } \cup \\ &= (-1)^{p(n-p)}\langle \beta^0 \cup \alpha, [N]\rangle \quad \text{by Theorem 4.3.5} \\ &= (-1)^{p(n-p)}\delta_{\alpha,\beta}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \alpha \times \beta^0 \cup \tau, [N \times N] \rangle &= \left\langle \alpha \times \beta^0 \cup \sum_{\alpha', \beta' \in B} A_{\alpha', \beta'} (\alpha')^0 \times \beta', [N \times N] \right\rangle \\
&= (-1)^{(n-p)^2} A_{\alpha,\beta} \langle (\alpha \cup \alpha^0) \times (\beta^0 \cup \beta), [N] \times [N] \rangle \\
&= (-1)^{n-p} A_{\alpha,\beta} ((\alpha \cup \alpha^0) \cap [N]) \times ((\beta^0 \cup \beta) \cap [N]) (-1)^n \\
&= (-1)^p A_{\alpha,\beta} \langle \alpha^0 \cup \alpha, [N] \rangle \langle \beta^0 \cup \beta, [N] \rangle \quad \text{in } H^0 \\
&= (-1)^{p(n-p)-p} A_{\alpha,\beta}.
\end{aligned}$$

Setting these expressions equal gives $A_{\alpha,\beta} = (-1)^p \delta_{\alpha,\beta}$. The result follows. □

Corollary 6.2.3. *For a manifold N, the Euler class can be written*

$$\chi = \sum_{\alpha \in B} (-1)^{|\alpha|} \alpha^0 \cup \alpha$$

and moreover, $\langle \chi, [N] \rangle = \chi(N)$ is the Euler characteristic of N.

Proof. For $\Lambda = \mathbb{Q}$, we have $\chi = d^* \tau = \sum_\alpha (-1)^{|\alpha|} \alpha^0 \cup \alpha$ by definition of the cup product (see Section 4.3). Thus

$$\langle \chi, [N] \rangle = \sum_\alpha (-1)^{|\alpha|} \langle \alpha^0 \cup \alpha, [N] \rangle = \sum_\alpha (-1)^{|\alpha|} = \chi(N).$$

□

For a map $f : N \to N$, let $G_f = (1 \times f) \circ d : N \to N \times N$ be the graph of f, i.e. $G_f(N) = \{(x, f(x)) \mid x \in N\} =: \Gamma$. Then with appropriate orientation, we have $[\Gamma] = (G_f)_*[N]$.

Definition. *The* **intersection number** *of Γ with Δ is the augmentation of the intersection product $[\Gamma][\Delta] = \varepsilon_*([\Gamma] \bullet [\Delta])$.*

The intersection number gives us an interpretation of Lefschetz theory (Section 2.11) in terms of intersection products.

Theorem 6.2.4. *The Lefschetz number of $f : N \to N$ is equal to $[\Gamma][\Delta]$.*

Proof. Let B be a basis of $H^\bullet(N; \mathbb{Q})$. Then for any $\alpha \in H^\bullet(N; \mathbb{Q})$, we may write

$$f^* \alpha = \sum_{\beta \in B} f_{\alpha,\beta} \beta$$

for $f_{\alpha,\beta} \in \mathbb{Q}$. Let γ be the Poincaré dual of $[\Gamma]$, i.e. so that $\gamma \cap [N \times N] = [\Gamma]$. Then

$$\begin{aligned}
[\Gamma][\Delta] &= \varepsilon_*((\tau \cup \gamma) \cap [N \times N]) \\
&= \langle \tau \cup \gamma, [N \times N] \rangle \\
&= \langle \tau, \gamma \cap [N \times N] \rangle \quad \text{by Theorem 4.4.1(e)} \\
&= \langle \tau, [\Gamma] \rangle = \langle \tau, (G_f)_*[N] \rangle \quad \text{by the above} \\
&= \langle (G_f)^* \tau, [N] \rangle \quad \text{by Theorem 4.4.1(f)} \\
&= \sum_{\alpha \in B} (-1)^{|\alpha|} \langle (G_f)^*(\alpha^0 \times \alpha), [N] \rangle \\
&= \sum_{\alpha \in B} (-1)^{|\alpha|} \langle \alpha^0 \cup f^* \alpha, [N] \rangle \quad \text{by (2) of Prop. 4.3.2} \\
&= \sum_{\alpha \in B} (-1)^{|\alpha|} \left\langle \alpha^0 \cup \sum_{\beta \in B} f_{\alpha,\beta} \beta, [N] \right\rangle \\
&= \sum_{\alpha \in B} (-1)^{|\alpha|} f_{\alpha,\alpha} \quad \text{by definition of the } \alpha^0 \\
&= \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f^* : H^n(N;\mathbb{Q}) \to H^n(N;\mathbb{Q})) \quad \text{by Corollary 6.2.3} \\
&= L(f).
\end{aligned}$$

□

Corollary 6.2.5. *If f_* does not have 1 as an eigenvalue for any fixed point $x \in N$, then*

$$L(f) = \sum_{f(x)=x} \operatorname{sign}(\det(I - f_*)).$$

This gives another proof of Lefschetz's fixed point theorem (2.11.3):

Corollary 6.2.6. *If f has no fixed points, then $L(f) = 0$.*

Example 6.2.7. Let M be a smooth, oriented n-manifold. Then TM is an oriented vector bundle of rank n. This defines an Euler class $e(M) := e(TM) \in H^n(M)$ which satisfies

$$\langle e(M), [M] \rangle = \chi(M).$$

Proposition 6.2.8. *For an oriented disk bundle $\pi : E \to N$, the Euler class $e(N)$ is dual in N to the self-intersection class of N in E.*

Proof. The self-intersection class of N in E is $i_*[N] \bullet i_*[N] = (D_E[N] \cup D_E[N]) \cup [E]$, where $i : N \hookrightarrow E$ is the 0-section of the bundle (sending each $x \in N$ to the zero vector in $\pi^{-1}(x)$) and $D_E : H_{n-k}(E) \to H^k(E, \partial E)$ is Poincaré duality. On the other hand, by definition $D_E[N] = \tau_E$ is the Thom class of the bundle, so we have

$$\begin{aligned}
i_*[N] \bullet i_*[N] &= (\tau_E \cup \tau_E) \cap [E] \\
&= \tau_E \cap (\tau_E \cup [E]) \quad \text{by Theorem 4.4.1(e)} \\
&= \tau_E \cap i_*[N] \quad \text{by definition of } \tau_E \\
&= i_*(i^* \tau_E \cap [N]) \quad \text{by Theorem 4.4.1(f)} \\
&= i_*(e(E) \cap [N]) \quad \text{by definition of the Euler class.}
\end{aligned}$$

Hence $[N] \cdot [N]$ and $e(E)$ are Poincaré dual in N. □

6.3 The Gysin Sequence

Lemma 6.3.1. *Suppose $\pi : E \to N$ is a k-disk bundle. Then there is a commutative diagram of $H^\bullet(N)$-modules*

$$\begin{tikzcd} H^p(N) \arrow[r, "- \cup e"] \arrow[d, "\pi^*(-) \cup \tau"'] & H^{p+k}(N) \arrow[d, "\pi^*"] \\ H^{p+k}(E, \partial E) \arrow[r, "j^*"'] & H^{p+k}(E) \end{tikzcd}$$

Proof. Notice that $\pi^* = (i^*)^{-1}$ is an isomorphism, where i is the 0-section, and $\pi^*(-) \cup \tau$ is an isomorphism by the Thom isomorphism theorem (6.1.2). Then for $\beta \in H^p(N)$, we have

$$i^* j^*(\pi^*\beta \cup \tau) = i^*(\pi^*\beta \cup j^*\tau) = i^*\pi^*\beta \cup i^*j^*\tau = \beta \cup e.$$

Therefore the diagram commutes. Moreover, the maps are $H^\bullet(N)$-modules because cup product is natural. □

Theorem 6.3.2 (Gysin Sequence)**.** *Suppose $\pi : X \to N$ is an oriented sphere bundle with fibres S^{k-1}. Then there is a long exact sequence of $H^\bullet(N)$-modules*

$$\cdots \to H^p(N) \xrightarrow{-\cup e} H^{p+k}(N) \xrightarrow{\pi^*} H^{p+k}(X) \xrightarrow{\sigma^*} H^{p+1}(N) \to \cdots$$

where e is the Euler class of the disk bundle $E \to N$ obtained by the mapping cone of π.

Proof. The long exact sequence for the disk bundle $\pi_E : E \to N$ is the top row in the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & H^{p+k}(E, \partial E) \arrow[r, "j^*"] & H^{p+k}(E) \arrow[r, "k^*"] & H^{p+1}(\partial E) \arrow[r, "\delta^*"] & H^{p+k+1}(E, \partial E) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^p(N) \arrow[r, "- \cup e"'] \arrow[u, "\pi^*(-) \cup \tau"'] & H^{p+k}(N) \arrow[r, "\pi^*"'] \arrow[u, "\pi_E^*"'] & H^{p+k}(X) \arrow[r, "\sigma^*"'] \arrow[u, "\mathrm{id}"'] & H^{p+1}(N) \arrow[r] \arrow[u, "\pi^*(-) \cup \tau"'] & \cdots \end{tikzcd}$$

where the first and fourth vertical arrows are the Thom isomorphisms. The left square commutes by Lemma 6.3.1, while the middle and right squares commute since the third vertical arrow is just the identity.

To check that the module structure is compatible, take $\alpha, \beta \in H^\bullet(N)$. Then $\pi^*(\alpha \cup \beta) = \pi^*\alpha \cup \pi^*\beta$ by Proposition 4.3.2, so π^* is a module map. Likewise, $(\alpha \cup \beta) \cup e = \alpha \cup (\beta \cup e)$ by

Theorem 4.3.5, so $-\cup e$ is also a module map. Finally, $\sigma^* = (\pi^*(-)\cup\tau)^{-1}\delta^*$ by definition. Write $h = (\pi^*(-)\cup\tau)^{-1}$. Then for any $\gamma \in H^\bullet(X)$, we have

$$\begin{aligned}
\sigma^*(\pi^*\beta\cup\gamma) &= h\circ\delta^*(\pi^*\beta\cup\gamma)\\
&= (-1)^{|\beta|\,|\gamma|}h\circ\delta^*(\gamma\cup k^*\pi_E^*\beta)\\
&= (-1)^{|\beta|\,|\gamma|}h(\delta^*\gamma\cup\pi_E^*\beta) \quad \text{by naturality of } \cup\\
&= (-1)^{|\beta|\,|\gamma|+|\beta|(|\gamma|+1)}h(\pi_E^*\beta\cup\delta^*\gamma) \quad \text{by Theorem 4.3.5}\\
&= (-1)^{|\beta|\,|\gamma|+|\beta|(|\gamma|+1)}\beta\cup h\delta^*\gamma \quad \text{since } h \text{ is a module map}\\
&= (-1)^{|\beta|}\beta\cup\sigma^*\gamma.
\end{aligned}$$

Hence σ^* is a module map. □

Corollary 6.3.3. *Let X be a $(k-1)$-sphere bundle over M such that $X = S^{n+k-1}$ is also a sphere. Then $n = kr$ for some $r \in \mathbb{Z}$ and $H^\bullet(M) = \mathbb{Z}[e]/e^{r+1}$ where $e \in H^k(M)$ is the Euler class of the disk bundle $E \to M$ having X as its boundary.*

Proof. In this situation, the Gysin sequence is

$$\cdots \to H^{p+k-1}(S^{n+k-1}) \to H^p(M) \xrightarrow{-\cup e} H^{p+k}(M) \to H^{p+k}(S^{n+k-1}) \to \cdots$$

Suppose $\alpha \in H^{p+k}(M)$ is nonzero for $p+k > 0$. Since $\dim M = n$, $0 < p+k \le n < n+k-1$ so α maps to 0 in $H^{p+k}(S^{n+k-1})$. Thus $\alpha = \beta\cup e$ for some $\beta \in H^p(M)$ by exactness. Replacing p with $p' = p-k$, the argument may be repeated as long as $p'+k > 0$. This terminates when $p' = 0$. Then M has nontrivial cohomology only in dimensions which are multiples of k, and moreover $H^\bullet(M)$ is spanned as a ring by $\{1, e, \ldots, e^r\}$ where r satisfies $kr = n$. □

6.4 Stiefel Manifolds

Definition. *For any integers $0 < k \le n$, the* **Stiefel manifold** *$V_{n,k}$ is defined to be the space of orthonormal k-frames in $\mathbb{R}^n$, i.e. the set*

$$V_{n,k} := \{(\vec{v}_1,\ldots,\vec{v}_k) \in \mathbb{R}^{nk} : \langle\vec{v}_i,\vec{v}_j\rangle = \delta_{ij} \text{ for all } 1 \le i,j \le k\}.$$

Remark. Complex Stiefel manifolds $V_{n,k}(\mathbb{C})$ can be defined similarly using the Hermitian inner product. As a manifold, $V_{n,k}(\mathbb{C})$ is *not* a complex manifold. To distinguish between the real and complex case, we denote the real Stiefel manifolds by $V_{n,k}(\mathbb{R})$ when necessary.

Proposition 6.4.1. *For all $0 < k \le n$, $V_{n,k}(\mathbb{R})$ and $V_{n,k}(\mathbb{C})$ are smooth, compact, orientable manifolds of finite (real) dimension.*

Example 6.4.2. For any n, $V_{n,1}(\mathbb{R}) \cong S^{n-1}$ and $V_{n,1}(\mathbb{C}) \cong S^{2n-1}$ are homeomorphic to unit spheres.

Example 6.4.3. For any n, $V_{n,n}(\mathbb{R}) = O_n(\mathbb{R})$, the set of $n\times n$ orthogonal matrices, and $V_{n,n}(\mathbb{C}) = U_n(\mathbb{R})$, the set of $n\times n$ unitary matrices.

Example 6.4.4. There is a natural map $V_{n,k} \to \mathrm{Gr}(k,n)$ into the Grassmannian manifold $\mathrm{Gr}(k,n)$, defined as the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, which takes the k-tuple $(\vec{v}_1, \ldots, \vec{v}_k)$ to the k-dimensional subspace they span in $\mathbb{R}^n$. In fact, this map is a fibre bundle over the Grassmannian with fibres $O_k(\mathbb{R})$ in the real case and $U_k(\mathbb{R})$ in the complex case.

There is a transitive action of $O_n(\mathbb{R})$ on $V_{n,k}(\mathbb{R})$ given by

$$A \cdot (\vec{v}_1, \ldots, \vec{v}_k) = (A\vec{v}_1, \ldots, A\vec{v}_k).$$

Under this action, the stabilizer of a point is isomorphic to $O_{n-k}(\mathbb{R})$. In particular, $V_{n,k}(\mathbb{R}) \cong O_n(\mathbb{R})/O_{n-k}(\mathbb{R})$ as a coset space.

If $\ell < k$, there is a natural map $V_{n,k} \to V_{n,\ell}$ sending $(\vec{v}_1, \ldots, \vec{v}_k) \mapsto (\vec{v}_1, \ldots, \vec{v}_\ell)$. This coincides with the induced map

$$V_{n,k} \cong O_n/O_{n-k} \longrightarrow O_n/O_{n-\ell} \cong V_{n,\ell}.$$

Hence $V_{n,k} \to V_{n,\ell}$ can be viewed as a bundle projection with fibres $O_{n-\ell}/O_{n-k}$. Specifically, projecting one dimension at a time, we get fibre bundles $V_{n,k} \to V_{n,k-1}$ with fibres S^{n-k}. Similarly, $V_{n,k}(\mathbb{C}) \to V_{n,k-1}(\mathbb{C})$ may be viewed as a fibre bundle with fibres $S^{2(n-k)+1}$.

Our goal is to compute the cohomology ring of $V_{n,k}(\mathbb{C})$. If $k = 1$, $V_{n,1}(\mathbb{C}) = S^{2n-1}$ by Example 6.4.2, so we know that $H^\bullet(V_{n,1}(\mathbb{C})) = \bigwedge\langle x_{2n-1}\rangle$, the exterior algebra on a generator $x_{2n-1} \in H^{2n-1}(V_{n,1}(\mathbb{C}))$. Next, if $k = 2$, there is a bundle projection $V_{n,2}(\mathbb{C}) \to V_{n,1}(\mathbb{C}) = S^{2n-1}$ with fibres S^{2n-3}, by the above. The Euler class for this bundle lies in $H^{2n-2}(V_{n,1}(\mathbb{C}))$, but since $V_{n,1}(\mathbb{C}) = S^{2n-1}$, we have $H^{2n-2}(V_{n,1}(\mathbb{C})) = 0$. Thus the Euler class is 0, so the Gysin sequence (Theorem 6.3.2) for this bundle breaks into short exact sequences:

$$0 \to H^p(S^{2n-1}) \xrightarrow{\pi^*} H^p(V_{n,2}(\mathbb{C})) \xrightarrow{\sigma^*} H^{p-2n+3}(S^{2n-1}) \to 0.$$

Since $H^p(S^{2n-1})$ is free abelian, this sequence splits and therefore $H^p(V_{n,2})$ is also free. We then can compute:

p	$H^p(S^{2n-1})$	$H^p(V_{n,2}(\mathbb{C}))$	$H^{p-2n+3}(S^{2n-1})$
0	$\mathbb{Z}$	$\mathbb{Z}$	0
1	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n-3$	0	$\mathbb{Z}$	$\mathbb{Z}$
$2n-2$	0	0	0
$2n-1$	$\mathbb{Z}$	$\mathbb{Z}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4n-4$	0	$\mathbb{Z}$	$\mathbb{Z}$

This implies that

$$H^p(V_{n,2}(\mathbb{C})) = \begin{cases} \mathbb{Z}, & p = 0, 2n-3, 2n-1, 4n-4 \\ 0, & \text{otherwise.} \end{cases}$$

To determine the ring structure, take a generator $x \in H^{2n-3}(V_{n,2}(\mathbb{C}))$ such that $\sigma^* x = 1$. Define

$$\begin{aligned}\Psi : H^\bullet(S^{2n-1}) \otimes \bigwedge(x_{2n-3}) &\longrightarrow H^\bullet(V_{n,2}(\mathbb{C}))\\ \alpha \otimes 1 &\longmapsto \pi^*\alpha\\ \alpha \otimes x_{2n-3} &\longmapsto \pi^*\alpha \cup x_{2n-3}.\end{aligned}$$

Since the degree of x_{2n-3} is odd, $2x_{2n-3}^2 = 2(x_{2n-3} \cup x_{2n-3}) = 0$ but since $H^\bullet(V_{n,2}(\mathbb{C}))$ is free, we must have $x_{2n-3}^2 = 0$. Hence Ψ is a well-defined ring map. Further, by the naturality statement in Theorem 6.3.2, we get

$$\sigma^*(\pi^*\alpha \cup x_{2n-3}) = (-1)^{|\alpha|}\alpha \cup \sigma^*(x_{2n-3}) = (-1)^{|\alpha|}\alpha \cup 1 = \pm\alpha.$$

So Ψ is a group isomorphism and thus a ring isomorphism. This generalizes as follows.

Theorem 6.4.5. *For any $0 < k \leq n$, the cohomology ring of the Stiefel manifold $V_{n,k}$ is the exterior algebra*

$$H^\bullet(V_{n,k}(\mathbb{C})) = \bigwedge\langle x_{2n-1}, x_{2n-3}, \ldots, x_{2(n-k)+1}\rangle$$

where x_j is a generator of $H^j(V_{n,k}(\mathbb{C}))$.

Proof. Induct on k; the base cases were shown above. Using the bundle projection $V_{n,k} \to V_{n,k-1}$ with fibres $S^{2(n-k)+1}$, we know its Euler class lies in $H^{2(n-k)+2}(V_{n,k-1})$, but since

$$2(n-k)+2 < 2(n-k)+3 = 2(n-(k-1))+1$$

is the dimension of the smallest nonzero cohomology in $H^{2(n-k)+2}(V_{n,k-1})$ by induction, we get that $e = 0$. Now the inductive argument from above goes through. $\square$

Corollary 6.4.6. *The unitary group $U_n(\mathbb{R})$ has cohomology*

$$H^\bullet(U_n(\mathbb{R})) = \bigwedge\langle x_{2n-1}, x_{2n-3}, \ldots, x_1\rangle.$$

Remark. Notice that as rings, $H^\bullet(V_{n,k}(\mathbb{C})) \cong H^\bullet(S^{2n-1} \times S^{2n-3} \times \cdots \times S^{2(n-k)+1})$. But when $n > 2$, $U_n(\mathbb{R})$ is not even homotopy equivalent to a product of spheres. This shows that cohomology is not a complete homotopy invariant.

6.5 Steenrod Squares

The cup product "square" is a cohomology operation

$$H^i(X) \longrightarrow H^{2i}(X)$$

which maps $\alpha \mapsto \alpha^2 = \alpha \cup \alpha$. The Steenrod operations generalize the cup product square for cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Definition. *A set of* **Steenrod operations**, *or* **Steenrod squares**, *are cohomology operations*

$$\mathrm{Sq}^i : H^n(X, A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{n+i}(X, A; \mathbb{Z}/2\mathbb{Z})$$

which are natural and satisfy

(i) Sq^0 *is the identity on* $H^n(X, A; \mathbb{Z}/2\mathbb{Z})$.

(ii) *If* $x \in H^i(X, A; \mathbb{Z}/2\mathbb{Z})$, *then* $\mathrm{Sq}^i(x) = x^2 = x \cup x$.

(iii) *If* $x \in H^n(X, A; \mathbb{Z}/2\mathbb{Z})$, *then for all* $i > n$, $\mathrm{Sq}^i(x) = 0$.

(iv) *(Cartan formula)* $\mathrm{Sq}^k(x \cup y) = \displaystyle\sum_{i=0}^{k} \mathrm{Sq}^i(x) \cup \mathrm{Sq}^{k-i}(y)$.

(v) *(Adém relation)* $\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j \pmod 2$.

In fact, Axiom (v) can be deduced from the other axioms, so we will not deal with it for now.

Theorem 6.5.1. *Steenrod squares exist and are unique.*

Definition. *The* **total Steenrod square** *is the ring homomorphism*

$$\begin{aligned} \mathrm{Sq} : H^\bullet(X, A; \mathbb{Z}/2\mathbb{Z}) &\longrightarrow H^\bullet(X, A; \mathbb{Z}/2\mathbb{Z}) \\ x &\longmapsto \mathrm{Sq}(x) = \sum_{i=0}^{\infty} \mathrm{Sq}^i(x). \end{aligned}$$

By Axiom (iii), the sum is finite so these are well-defined.

Notice that Axiom (iv) implies that $\mathrm{Sq}(x \cup y) = \mathrm{Sq}(x) \cup \mathrm{Sq}(y)$ for any $x, y \in H^\bullet(X, A; \mathbb{Z}/2\mathbb{Z})$.

Proposition 6.5.2. *If* $x \in H^1(X, A; \mathbb{Z}/2\mathbb{Z})$ *then for any* $i, k \geq 1$, $\mathrm{Sq}^i(x^k) = \binom{k}{i} x^{k+i}$.

Proof. By the comment preceding the proposition, $\mathrm{Sq}(x^k) = \mathrm{Sq}(x)^k$, and we have

$$\begin{aligned} \mathrm{Sq}(x)^k &= (x + x^2)^k \quad \text{by Axioms (i) – (iii)} \\ &= x^k (x + 1)^k \\ &= x^k \sum_{i=0}^{k} \binom{k}{i} x^{k+i}. \end{aligned}$$

Therefore the ith Steenrod square of x^k is $\binom{k}{i} x^{k+i}$. $\square$

Lemma 6.5.3. *Let* $x, y \in H^\bullet(X)$ *so that* $x \times y \in H^\bullet(X \times Y)$, *where* $\times$ *is the cohomology cross product. Then for any* $n \geq 0$,

$$\mathrm{Sq}^n(x \times y) = \sum_{i=0}^{n} \mathrm{Sq}^i(x) \times \mathrm{Sq}^{n-i}(y).$$

Proof. By (3) of Proposition 4.3.2, $x \times y = (x \times 1) \cup (1 \times y)$. Now apply Cartan's formula. □

Proposition 6.5.4. *The Steenrod operations commute with the coboundary map, i.e. for any pair (X, A) there is a commutative diagram*

$$\begin{tikzcd} H^n(A) \arrow[r, "\delta"] \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(X,A) \arrow[d, "\mathrm{Sq}^i"] \\ H^{n+i}(A) \arrow[r, "\delta"'] & H^{n+i+1}(X,A) \end{tikzcd}$$

Corollary 6.5.5. *The Steenrod squares are stable, i.e. they commute with suspension.*

Proof. Suspension may be realized as a connecting homomorphism:

$$\Sigma : \widetilde{H}^n(X) \xrightarrow{\delta} \widetilde{H}^{n+1}(CX, X) \cong H^{n+1}(\Sigma X),$$

where CX is the cone on X and the last isomorphism is by the homotopy axiom. Thus Proposition 6.5.4 applies. □

We have the following deep result on homotopy groups of spheres, generalizing Corollary 3.6.12.

Corollary 6.5.6. *For all $n \geq 2$, $\pi_{n+1}(S^n) \neq 0$.*

Proof. Consider $\mathbb{C}P^2 = S^2 \cup_h D^4$ where $h : S^3 \to S^2$ is the Hopf map. Taking suspensions on each side, we get $\Sigma\mathbb{C}P^2 = S^3 \cup_{\Sigma h} D^5$. If $\Sigma h = 0$, then $\Sigma\mathbb{C}P^2 = S^3 \wedge S^5$ is just a wedge of spheres. Since Steenrod squares are natural, we have a commutative diagram

$$\begin{tikzcd} H^3(S^3) \arrow[r, "\mathrm{Sq}^2"] \arrow[d] & H^5(S^3) \arrow[d] \\ H^3(S^3 \wedge S^5) \arrow[r, "\mathrm{Sq}^2"'] & H^5(S^3 \wedge S^5) \end{tikzcd}$$

Since $H^5(S^3) = 0$ by Theorem 2.3.5, this implies $\mathrm{Sq}^2 = 0$ on $\Sigma\mathbb{C}P^2$. However, the cohomology ring of $\mathbb{C}P^2$ is generated by $1 \in H^0(\mathbb{C}P^2)$, $x \in H^2(\mathbb{C}P^2)$ and $x^2 \in H^4(\mathbb{C}P^2)$. By the Steenrod axioms, we must have $\mathrm{Sq}^2(x) = x^2$ and in particular $\mathrm{Sq}^2 \neq 0$. Thus $\mathrm{Sq}^2 \neq 0$ on $\Sigma\mathbb{C}P^2$ since Steenrod squares commute with suspension by Corollary 6.5.5. This implies that Σh cannot be trivial. Now induct to see that the nth suspension of the Hopf map, $\Sigma^n h : S^{n+3} \to S^{n+2}$, cannot be trivial. By definition $\Sigma^n h$ represents a class in $[S^{n+1}, S^n] = \pi_{n+1}(S^n)$ so $\Sigma h \neq 0$ implies the result. □

The following formula helps us compute the coefficients of Steenrod squares.

Lemma 6.5.7. *Let $a = \sum a_j 2^j$ and $b = \sum b_j 2^j$ be formal sums with coefficients $a_j, b_j \in \mathbb{Z}/2\mathbb{Z}$. Then*

$$\binom{a}{b} = \prod \binom{a_j}{b_j} \pmod 2.$$

Proof. Over the polynomial ring $\mathbb{Z}/2\mathbb{Z}[x]$, we have

$$(1+x)^a = \prod (1+x)^{a_j 2^j} = \prod (1+x^{2^j})^{a_j} = \prod \sum_k \binom{a_j}{k} x^{2^j k} \pmod 2.$$

Now let $k = b$ to obtain the desired formula. □

Corollary 6.5.8. *If $x \in H^1(X, A; \mathbb{Z}/2\mathbb{Z})$, then*

$$\mathrm{Sq}^i(x^{2^k}) = \begin{cases} x^{2^k}, & i = 0 \\ x^{2^k+i}, & i = 2^k \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 6.5.9. *If i is not a power of 2, then Sq^i can be written as a sum of compositions of Sq^k for $k < i$.*

Proof. The Adém relations can be written

$$\binom{b-1}{a} \mathrm{Sq}^{a+b} = \mathrm{Sq}^a \mathrm{Sq}^b + \sum_{j=1}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

whenever $0 < a < 2b$. Thus Sq^{a+b} can be written as a sum of smaller degree Steenrod squares whenever $\binom{b-1}{a} = 1 \pmod 2$. Supposing i is not a power of 2, write $i = a + 2^k$ for $0 < a < 2^k$. Then $2^k - 1 = 1 + 2 + 2^2 + \ldots + 2^{k-1}$ and this implies by Lemma 6.5.7 that $\binom{2^k-1}{a} = 1 \pmod 2$. Hence Sq^i can be decomposed. □

Example 6.5.10. Some of the following decompositions are useful:

$$\begin{aligned} \mathrm{Sq}^3 &= \mathrm{Sq}^1 \mathrm{Sq}^2, \\ \mathrm{Sq}^5 &= \mathrm{Sq}^1 \mathrm{Sq}^4, \\ \mathrm{Sq}^6 &= \mathrm{Sq}^2 \mathrm{Sq}^4 + \mathrm{Sq}^1 \mathrm{Sq}^4 \mathrm{Sq}^1. \end{aligned}$$

Corollary 6.5.11. *If $x \in H^n(X, A; \mathbb{Z}/2\mathbb{Z})$ such that $x^2 \neq 0$, then $\mathrm{Sq}^i(x) \neq 0$ for some i which satisfies $0 \leq 2^i \leq n$.*

Corollary 6.5.12. *If M is a closed $2n$-manifold with $H_i(M; \mathbb{Z}/2\mathbb{Z}) = 0$ for all $0 < i < n$ and $H_n(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, then n is a power of 2.*

In fact, Adams has shown that the only possibilities for the dimension of such a manifold are $n = 1, 2, 4, 8$.

For any map $f : S^{2n-1} \to S^n$, $n \geq 2$, let $X = C_f = S^n \cup_f D^n$ be the mapping cone of f. Then X has a natural CW-structure with a single cell in dimensions 0, n and $2n$. The inclusion $S^n \hookrightarrow X$ and collapsing map $X \to S^{2n}$ induce isomorphisms on cohomology:

$$H^n(S^n) \cong H^n(X) \quad \text{and} \quad H^{2n}(X) \cong H^{2n}(S^{2n}).$$

Therefore $x = [S^n] \in H^n(X)$ and $y = [S^{2n}] \in H^{2n}(X)$ are well-defined. Since y generates $H^{2n}(X)$ by definition, we must have $x^2 = h_f y$ for some integer h_f.

Definition. *The integer h_f is called the* **Hopf invariant** *of $f : S^{2n-1} \to S^n$.*

Corollary 6.5.13. *If $f : S^{2n-1} \to S^n$ is a map such that h_f is odd, then n is a power of 2 equal to one of $n = 1, 2, 4$ or 8.*

7 Higher Homotopy Theory

The contents of this chapter follow Chapter 6 in Davis and Kirk's *Lectures Notes in Algebraic Topology*. We first discuss compactly generated spaces. For the entire chapter, assume all spaces are Hausdorff.

Definition. *A Hausdorff topological space X is* **compactly generated** *if for every set $A \subseteq X$, A is closed if and only if $A \cap K$ is closed for all compact $K \subseteq X$.*

Example 7.0.1. Every locally compact Hausdorff space is clearly compactly generated. This includes all manifolds.

Example 7.0.2. More generally, any metric space is compactly generated.

Example 7.0.3. An important class of compactly generated spaces is the CW-complexes having finitely many cells in each dimension.

Definition. *Let X be any Hausdorff space. Define $k(X)$ to be X with the topology such that A is closed in $k(X)$ if and only if $A \cap K$ is closed in X for all compact $K \subseteq X$.*

Lemma 7.0.4. *Let X be a Hausdorff space. Then*

(a) $k(X)$ is compactly generated.

(b) If X is compactly generated, then $k(X) = X$.

(c) For any function $f : X \to Y$, the corresponding map $k(f) : k(X) \to k(Y)$ is continuous if and only if $f|_K$ is continuous for all compact $K \subseteq X$.

(d) If X is compactly generated and $C(X, Y)$ denotes the space of continuous maps $X \to Y$, then the assignment $k : f \mapsto k(f)$ is a bijection $k : C(X, Y) \leftrightarrow C(X, k(y))$.

(e) X and $k(X)$ have the same singular chain complexes and the same homotopy groups.

Let $\mathcal{K}$ be the category of compactly generated topological spaces. Then the above shows that $k : X \mapsto k(X)$ is a functor $\texttt{HTop} \to \mathcal{K}$ on the category $\texttt{HTop}$ of Hausdorff topological spaces.

Definition. *The* **compact-open topology** *on $C(X, Y)$ is the topology generated by sets of the form*

$$U(K, W) = \{f : X \to Y \mid f(K) \subseteq W\}$$

where $K \subseteq X$ is compact and $W \subseteq Y$ is open.

Let $\mathrm{Map}(X, Y) = k(C(X, Y))$. For two compactly generated spaces $X, Y \in \mathcal{K}$, we define their product in this category to be $X \times Y := k(X \times Y)$ – where the product on the right is taken in the category of topological spaces.

Theorem 7.0.5. *For $X, Y, Z \in \mathcal{K}$, there is a homeomorphism*

$$\begin{aligned} \operatorname{Map}(X \times Y, Z) &\longrightarrow \operatorname{Map}(X, \operatorname{Map}(Y, Z)) \\ f &\longmapsto (x \mapsto f_x) \end{aligned}$$

where f_x is the map $y \mapsto f(x, y)$.

Proposition 7.0.6. *Let $X, Y, Z \in \mathcal{K}$. Then*

(1) The evaluation map

$$\begin{aligned} e : \operatorname{Map}(X, Y) \times X &\longrightarrow Y \\ (f, x) &\longmapsto f(x) \end{aligned}$$

is continuous.

(2) $\operatorname{Map}(X, Y \times Z) = \operatorname{Map}(X, Y) \times \operatorname{Map}(X, Z)$.

(3) Composition gives a continuous map $\operatorname{Map}(X, Y) \times \operatorname{Map}(Y, Z) \to \operatorname{Map}(X, Z)$.

7.1 Fibration

Definition. *A map $p : E \to B$ is called a* **fibration** *if it has the* **homotopy lifting property***, i.e. for any space Y, the following diagram can be completed:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

Remark. The property of being a fibration is a 'local condition', i.e. $p : E \to B$ is a fibration if and only if every point $b \in B$ has a neighborhood U such that $p : p^{-1}(U) \to U$ is a fibration. (This follows from a theorem of Hurewicz.)

Example 7.1.1. Theorem 1.2.6 says precisely that every covering map is a fibration, and that the $\widetilde{G}$ completing the diagram in the definition above is always *unique*. This uniqueness property is not true of every fibration.

Example 7.1.2. Let $p : E \to B$ be a fibre bundle, i.e. a surjection in which each preimage $p^{-1}(U)$ of an open set $U \subseteq B$ in a covering of B is homeomorphic to $U \times F$ for some fixed space F. Then p is a fibration.

Theorem 7.1.3. *Suppose B is path-connected and $p : E \to B$ is a fibration. Then*

(1) All of the fibres $E_x := p^{-1}(x)$ are homotopy equivalent.

(2) Every choice of path α in B from x to y determines a homotopy class of homotopy equivalences $\alpha_* : E_x \to E_y$ depending only on the homotopy class of α rel endpoints.

(3) Under the above, concatenation of paths corresponds to composition of homotopy equivalences. In other words, there is a well-defined homomorphism of groups

$$\begin{aligned} \pi_1(B, x) &\longrightarrow \{\textit{homotopy classes of self homotopy equivalences of } E_x\} \\ [\alpha] &\longmapsto (\alpha^{-1})_*. \end{aligned}$$

Proof. Take a path α from x to y in B. Then the inclusion $E_x \hookrightarrow E$ induces the following diagram:

$$\begin{tikzcd} E_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_x \times [0,1] \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

where $G(e, t) = \alpha(t)$ for all $t \in [0, 1]$. Since p is a fibration, we get a lift $\widetilde{G}$. At $t = 0$, $\widetilde{G}_0 : E_x \times \{0\} \to E$ is just the inclusion of the fibre $E_x \hookrightarrow E$. On the other hand, for any t, $p \circ \widetilde{G}_t$ is the constant map at $\alpha(t)$ so in particular at $t = 1$, $\widetilde{G}_1$ gives a map $E_x \to E_{\alpha(1)} = E_y$. Set $\alpha_* = [\widetilde{G}_1]$. To check α_* is well-defined, suppose $\alpha' : [0, 1] \to B$ is another path homotopic rel endpoints to α. Set $H = \alpha' \circ \mathrm{proj}_{[0,1]}$ where $\mathrm{proj}_{[0,1]} : E_x \times [0, 1] \to [0, 1]$ is the second coordinate projection. Then using the homotopy lifting property on the diagram

$$\begin{tikzcd} E_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_x \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

we get a map $\widetilde{H} : E_x \times [0, 1] \to E$ and, as above, a map $\widetilde{H}_1 : E_x \to E_y$. One then constructs a homotopy from $\widetilde{G} \to \widetilde{H}$ using that $\alpha \simeq \alpha'$ rel endpoints; this then induces a homotopy $\widetilde{G}_1 \to \widetilde{H}_1$. Hence α_* is well-defined and (2) is proved.

It is clear that for paths α, β in B such that $\beta(0) = \alpha(1)$, we have $(\alpha * \beta)_* = \beta_* \circ \alpha_*$. Thus when $\beta = \alpha^{-1}$, $\beta_* \circ \alpha_* = (c_x)_*$, where c_x is the constant path at x. Since $(c_x)_* = [id_{E_x}]$, we have that $\beta_* = (\alpha^{-1})_*$ is a homotopy inverse of α_*. Hence α_* is a homotopy equivalence, so using path-connectedness we see that all fibres are homotopy equivalent, proving (1).

Finally, it is routine to prove the homotopy classes of homotopy equivalences $E_x \to E_x$ form a group under composition. Then for (3), the above shows that $(\alpha * \beta)_* = \beta_* \circ \alpha_*$ and the trivial class goes to the homotopy class of the identity map $E_x \to E_x$, so $[\alpha] \mapsto (\alpha^{-1})_*$ is a homomorphism. □

Definition. *For a space Y, the* **free path space** *on Y is*

$$Y^I = \mathrm{Map}([0, 1], Y).$$

For a point $y_0 \in Y$, the **based path space** *on Y with basepoint y_0 is*

$$P_{y_0}Y = \{\alpha \in Y^I \mid \alpha(0) = y_0\}.$$

Further, the **based loop space** *at y_0 is*

$$\Omega_{y_0}Y = \{\alpha \in P_{y_0}Y \mid \alpha(1) = y_0\}.$$

For Y^I and $P_{y_0}Y$, define the *endpoint map*,

$$\begin{aligned} p : Y^I &\longrightarrow Y \\ \alpha &\longmapsto \alpha(1) \end{aligned}$$

(and restricting to $P_{y_0}Y$ for the based version), which is continuous on both Y_I and $P_{y_0}Y$. The fibre of $p : P_{y_0}Y \to Y$ is, up to homotopy, exactly the loop space $\Omega_{y_0}Y$.

Theorem 7.1.4. *For any Y, the endpoint map $p : Y^I \to Y$ is a fibration with fibre over y_0 homeomorphic to $P_{y_0}Y$.*

Proof. To prove p satisfies the homotopy lifting property, we need to complete the following diagram for any space A:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g"] \arrow[d, hook] & Y^I \arrow[d, "p"] \\ A \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y \end{tikzcd}$$

For $a \in A$, $g(a) \in Y^I$ is a map such that $p \circ g(a) = H(a, 0)$. That is, $g(a)$ is a path ending at the starting point of the homotopy $H(a, -)$. To lift, just continue this path by defining

$$\widetilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t), & 0 \le t \le \frac{1}{1+s} \\ H(a, (1+s)t - 1), & \frac{1}{1+s} < t \le 1. \end{cases}$$

Then $\widetilde{H}$ is continuous, $\widetilde{H}(a, 0) = g(a)$ and $p \circ \widetilde{H}(a,s)(-) = \widetilde{H}(a,s)(1) = H(a,s)$. Hence $p : Y^I \to Y$ is a fibration. □

Corollary 7.1.5. *For any $y_0 \in Y$, $p : P_{y_0}Y \to Y$ is a fibration with fibres $\Omega_{y_0}Y$.*

Proof. The same proof goes through. □

Lemma 7.1.6. *The map $p : Y^I \to Y$ is a homotopy equivalence.*

Proof. Define $i : Y \to Y^I$ by $i(y) = c_y$, the constant path at $y \in Y$. Then $p \circ i(y) = p(c_y) = c_y(1) = y$, while for $\alpha \in Y^I$, $i \circ p(\alpha) = i(\alpha(1)) = c_{\alpha(1)}$, but of course α is homotopy equivalent to the constant path at its endpoint (since $[0,1]$ is contractible). Hence $i \circ p \simeq id$ so i and p are homotopy inverses. □

Lemma 7.1.7. *For any $y_0 \in Y$, $P_{y_0}Y$ is contractible.*

Proof. Similar. □

One also defines the *starting point fibration* $q : Y^I \to Y$ (and its restriction to $P_{y_0}Y$) by $\alpha \mapsto \alpha(0)$. Then analogues to the preceding results hold for q by identical proofs.

Definition. *If $p : E \to B$ is a fibration and $f : A \to B$ is a continuous map, then the* **pullback fibration** *of f along p is the space*

$$f^*E = \{(a, e) \in A \times E \mid f(a) = p(e)\}$$

along with the map $f^*p : f^*E \to A, (a, e) \mapsto a$. That is, f^*p is the pullback in the category of topological spaces.

Proposition 7.1.8. *For any fibration $p : E \to B$ and map $f : A \to B$, $f^*p : f^*E \to A$ is a fibration.*

Proof. Routine. □

Definition. *A* **map of fibrations** *between $p : E \to B$ and $p' : E' \to B'$ is a pair of maps $f : B \to B'$ and $\tilde{f} : E \to E'$ making the following diagram commute:*

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Definition. *A* **fibre homotopy** *between maps of fibrations $(f_0, \tilde{f}_0)$ and $(f_1, \tilde{f}_1)$ is a pair of homotopies $(H, \widetilde{H})$ such that H is a homotopy $f_0 \to f_1$, $\widetilde{H}$ is a homotopy $\tilde{f}_0 \to \tilde{f}_1$ and the following diagram commutes:*

$$\begin{tikzcd} E \times I \arrow[r, "\widetilde{H}"] \arrow[d, "p \times id"'] & E' \arrow[d, "p'"] \\ B \times I \arrow[r, "H"'] & B' \end{tikzcd}$$

Two fibrations $p : E \to B$ and $q : E' \to B$ over the same base are said to be **fibre homotopy equivalent** *if there exist maps of fibrations*

$$\begin{tikzcd} E \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & E' \arrow[ll, shift left, "g"] \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

such that $f \circ g$ and $g \circ f$ are fibre homotopic to the identity.

We next make rigorous the idea that 'every continuous map is a fibration'. Suppose $f : X \to Y$ is continuous.

Definition. *The* **mapping path space** *of f is the pullback fibration $P_f := f^*(Y^I)$ along the starting point fibration $q : Y^I \to Y, \alpha \mapsto \alpha(0)$. That is, $P_f = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x)\}$ and there is a commutative diagram*

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "f^*q"'] & Y^I \arrow[d, "q"] \\ X \arrow[r] & Y \end{tikzcd}$$

Definition. *The* **mapping path fibration** *of $f : X \to Y$ is the map $p_f : P_f \to Y$ given by $p(x, \alpha) = \alpha(1)$, that is, the restriction of the endpoint fibration.*

Theorem 7.1.9. *For any continuous $f : X \to Y$,*

(1) There is a homotopy equivalence $h : X \to P_f$ such that the diagram

$$\begin{tikzcd} & P_f \arrow[dr, "p_f"] & \\ X \arrow[ur, "h"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

commutes.

(2) $p_f : P_f \to Y$ is a fibration.

(3) If f is a fibration, then h is a fibre homotopy equivalence.

Proof. (1) Define $h(x) = (x, c_{f(x)})$ where $c_{f(x)}$ is the constant path (in Y) at $f(x)$. Then the projection $\pi : P_f \to X$ is obviously a homotopy inverse to h, since $\pi \circ h(x) = \pi(x, c_{f(x)}) = x$; and $h \circ \pi \simeq id$ via $F((x, \alpha), s) = (x, \alpha_s)$, where $\alpha_s(t) = \alpha(st)$.

(2) We must complete the following diagram for any space A:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g"] \arrow[d, hook] & P_f \arrow[d, "p_f"] \\ A \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y \end{tikzcd}$$

For $a \in A$, we have $g(a) = (g_1(a), g_2(a))$ where $g_1(a) \in X$ and $g_2(a)$ is a path in Y starting at $f(g_1(a))$ and ending at $H(a, 0)$. As in the proof of Theorem 7.1.4, continue this path to get the desired lift by setting $\widetilde{H}(a, s)(t) = (g_1(a), \widetilde{H}_2(a, s)(t))$, where

$$\widetilde{H}_2(a, s)(t) = \begin{cases} g_2(a)((1+s)t), & 0 \leq t \leq \frac{1}{1+s} \\ H(a, (1+s)t - 1), & \frac{1}{1+s} < t \leq 1. \end{cases}$$

(3) Note that $\pi : P_f \to X$ is not a fibration map a priori. To fix this, define $\gamma : P_f \times I \to Y$ by $\gamma(x, \alpha, t) = \alpha(t)$. Then we have a diagram

$$\begin{tikzcd} P_f \times \{0\} \arrow[r, "\pi"] \arrow[d, hook] & X \arrow[d, "f"] \\ P_f \times I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & Y \end{tikzcd}$$

which commutes by definition of P_f, so there exists a lift $\tilde{\gamma}$ since f is a fibration. Define $g : P_f \to X$ by $g(x, \alpha) = \tilde{\gamma}(x, \alpha, 1)$. Then the diagram

$$\begin{tikzcd} P_f \arrow[rr, shift left, "g"] \arrow[dr, "p_f"'] & & X \arrow[ll, shift left, "h"] \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

commutes by construction and g is a fibre homotopy inverse of f. $\square$

7.2 Fibration Sequences

Suppose $f : E \to B$ is a fibration with fibres F (up to homotopy). Then the inclusion of fibres $i : F \to E$ is, up to homotopy, a fibration by Theorem 7.1.9. This process can be iterated to produce a sequence of fibrations.

Theorem 7.2.1. *Let $f : E \to B$ be a fibration with fibre $F \to E$ and Z equal to the homotopy fibre of $i : F \to E$, i.e. the fibre of the fibration $P_i \to E$. Then Z is homotopy equivalent to the loop space $\Omega_{y_0} B$, when all maps are based at $y_0 \in B$.*

Proof. By Theorem 7.1.9, we may replace E with P_f, with path space fibration $p : P_f \to B, (e, \alpha) \mapsto \alpha(1)$. The fibre of p over $y_0 \in B$ is

$$P_{f,y_0} = \{(e, \alpha) \mid f(e) = \alpha(0), y_0 = \alpha(1)\}.$$

Define $\pi : P_{f,y_0} \to E$ by $\pi(e, \alpha) = e$. Then it suffices to show π is a fibration with fibre $\Omega_{y_0} B$. Indeed, since f is a fibration, $E \to P_f, e \mapsto (e, c_{f(e)})$ is a fibration homotopy equivalence, so $f^{-1}(y_0) \hookrightarrow P_{f,y_0}$. Now to prove the new statement, take $e_0 \in f^{-1}(y_0) \subseteq E$. Then

$$\pi^{-1}(e_0) = \{(e_0, \alpha) \mid y_0 = f(e_0) = \alpha(0), y_0 = \alpha(1)\} = \Omega_{y_0} B,$$

so we need only check that π is a fibration. For a space A, consider the diagram

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "g"] \arrow[d, hook] & P_{f,y_0} \arrow[d, "\pi"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & E
\end{tikzcd}$$

If $a \in A = A \times \{0\}$, then $g(a) = (g_1(a), g_2(a))$ where $g_1(a) = H(a, 0) \in E$ is the start of the path $H(a,t)$, while $g_2(a)$ is the path starting at $g_2(a)(0) = f(g_1(a))$ and ending at $g_2(a)(1) = y_0$. Extend H to P_{f,y_0} by:

$$\widetilde{H}(a,s)(t) = \begin{cases} f(H(a, -(1+s)t+s)), & 0 \le t \le \frac{s}{1+s} \\ g_2(a)((1+s)t - s), & \frac{s}{1+s} < t \le 1. \end{cases}$$

As in previous proofs, this $\widetilde{H}$ completes the diagram. □

This allows us to construct a so-called long exact sequence of fibrations.

Corollary 7.2.2. *If $E \to B$ is a fibration with fibre $F \hookrightarrow E$, there is a sequence of fibrations*

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

where each pair of consecutive terms is the inclusion of a fibre in a fibration.

Definition. *A sequence of set maps $A \xrightarrow{f} B \xrightarrow{g} C$ with fixed basepoint $c_0 \in C$ is* **exact** *at B if $f(A) = g^{-1}(c_0)$.*

Theorem 7.2.3. *Let $p : E \to B$ be a fibration, $b_0 \in B$ a basepoint and $F = p^{-1}(b_0)$ its fibre, with some point $e_0 \in F$ specified. Then for any space Y, the sequence of maps*

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

is exact at $[Y, E]$.

Proof. Clearly $p_* \circ i_*$ is equal to c_{b_0}, the constant map at b_0, which is the basepoint of the set $[Y, B]$. Suppose $f : Y \to E$ has $p_*[f] = [c_{b_0}]$. Then there exists a null-homotopy of $p \circ f$, call it H, making the diagram commute:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

Then by the homotopy lifting property, we get a lift $\widetilde{H}$ such that $\widetilde{H}(-,1) : Y \to E$ lifts c_{b_0}. In other words, $[f] = i_*[\widetilde{H}(-,1)]$. Hence the sequence is exact. □

Corollary 7.2.4. *For any fibration $E \to B$ with fibre $F \hookrightarrow E$ and any space Y, there is a long exact sequence*

$$\cdots \to [Y, \Omega^{n+1} B]_0 \to [Y, \Omega^n F]_0 \to [Y, \Omega^n E]_0 \to [Y, \Omega^n B]_0 \to \cdots$$

Proof. The proof of Theorem 7.2.3 can be directly adapted to based maps and based homotopy spaces. This implies the result. □

Let $X \wedge Y = X \times Y/(\{x_0\} \times Y \cup X \times \{y_0\})$ be the *smash product* of X and Y at a pair of points $x_0 \in X, y_0 \in Y$. Then there are bijections

$$\mathrm{Map}(X \times Y, Z) \longleftrightarrow \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$
$$\mathrm{Map}_0(X \wedge Y, Z) \longleftrightarrow \mathrm{Map}_0(X, \mathrm{Map}_0(Y, Z)),$$

where Map_0 denotes the space of *based* maps (with the compactly generated topology). Let $\Sigma X = X \wedge S^1$ be the *reduced suspension* of X. Then we have:

Corollary 7.2.5. *For spaces X and Y and any integer $n \geq 0$,*

(1) $\mathrm{Map}_0(\Sigma X, Y) \cong \mathrm{Map}_0(X, \Omega Y)$.

(2) $[S^n, \Omega Y]_0 \cong [S^{n-1}, Y]_0$.

(3) $\pi_n(\Omega Y) \cong \pi_{n+1}(Y)$.

Theorem 7.2.6 (Homotopy Long Exact Sequence)**.** *For any fibration $E \to B$ with fibre $F \hookrightarrow E$, there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \cdots$$

Proof. Apply Corollary 7.2.4. □

Corollary 7.2.7. *If $p : Y \to X$ is a covering map, then $p_* : \pi_n(Y) \to \pi_n(X)$ is an isomorphism for all $n > 1$.*

Proof. For any covering space, the fibre $F \hookrightarrow Y$ is a discrete set, so $\pi_n(F) = 0$ for all $n > 0$. Thus in the long exact sequence of homotopy groups,

$$\cdots \to \pi_n(F) \to \pi_n(Y) \xrightarrow{p_*} \pi_n(X) \to \pi_{n-1}(F) \to \cdots$$

we get a 0 in every third term, so p_* is an isomorphism as claimed. □

Example 7.2.8. Consider the standard circle bundle $S^\infty \to \mathbb{C}P^\infty$ (i.e. the fibre is, up to homotopy, S^1). Then Theorem 7.2.6 gives a long exact sequence

$$\cdots \to \pi_{n+1}(S^\infty) \to \pi_{n+1}(\mathbb{C}P^\infty) \to \pi_n(S^1) \to \pi_n(S^\infty) \to \cdots$$

It is a standard fact that S^∞ is contractible, so $\pi_n(S^\infty) = 0$ for $n \geq 1$ by Corollary 1.1.13. Hence the long exact sequence tells us that

$$\pi_n(S^1) = \begin{cases} \mathbb{Z}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

thus confirming Corollary 1.2.18. In fact, this shows that $\mathbb{C}P^\infty$ is an *Eilenberg-Maclane space* of type $K(\mathbb{Z}, 2)$ – see Section 7.6. Further, the fibration long exact sequence (Corollary 7.2.2) shows j is an isomorphism on all homotopy groups:

$$\cdots \to \Omega S^\infty \to \Omega\mathbb{C}P^\infty \xrightarrow{j} S^1 \to S^\infty \to \mathbb{C}P^\infty.$$

By definition, $j : \Omega\mathbb{C}P^\infty \to S^1$ is a weak homotopy equivalence, so we see that S^1 is the loop space of $\mathbb{C}P^\infty$.

7.3 Hurewicz Homomorphisms

Recall from Theorem 2.1.14 that the homomorphism

$$\begin{aligned} \pi_1(X) &\longrightarrow H_1(X) \\ [\alpha] &\longmapsto [\alpha] \end{aligned}$$

is surjective with kernel $[\pi_1(X), \pi_1(X)]$, the commutator of the fundamental group of X, so that $H_1(X) \cong \pi_1(X)^{ab}$. In this section, we generalize this result to higher homotopy groups.

Definition. *For each $n \geq 1$, the* **Hurewicz homomorphism** *is defined on $\pi_n(X) \to H_n(X)$ by sending a class $[f] \in \pi_n(X)$, represented by a map $f : S^n \to X$, to the pushforward $f_*[S^n]$, where $[S^n]$ is the fundamental class of the n-sphere.*

Example 7.3.1. Note that $\pi_n(\mathbb{C}P^\infty) = \mathbb{Z}$ for $n = 2$ and is zero otherwise, whereas $H_n(\mathbb{C}P^\infty) = \mathbb{Z}$ for all even n and is zero otherwise. Thus Hurewicz homomorphisms are not all isomorphisms. However, Hurewicz's theorem for higher homotopy groups gives the correct generalization:

Theorem 7.3.2 (Hurewicz)**.** *Let X be path-connected, $n > 1$ and assume $\pi_k(X) = 0$ for all $k < n$. Then the Hurewicz map $\pi_k(X) \to \widetilde{H}_k(X)$ is an isomorphism for all $k \leq n$.*

Example 7.3.3. By Theorem 1.1.17, we know that for $n > 1$ and for all $k < n$, $\pi_k(S^n) = 0$. Moreover, Theorem 2.3.5 gives $H_n(S^n) = \mathbb{Z}$, so by Hurewicz's theorem, $\pi_n(S^n) = \mathbb{Z}$, a fact that is very difficult to prove otherwise.

Example 7.3.4. Let $n \geq 1$. A space X is said to be n-**connected** if $\pi_k(X) = 0$ for all $k \leq n$. A map $f : X \to Y$ is n-connected if $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k < n$ and $f_* : \pi_n(X) \to \pi_n(Y)$.

Note that 1-connected is just a new way to say simply connected.

Definition. *For a pair (X, A) and $n \geq 1$, the nth* **relative homotopy group** *is defined as the based homotopy space*

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

Proposition 7.3.5. *For all pairs (X, A), there is a long exact sequence in relative homotopy*

$$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$$

Given a map $f : X \to Y$, one defines the *mapping cylinder* of f by

$$M_f := X \times [0,1] \cup Y/(x,1) \sim f(x).$$

Note that M_f is homotopy equivalent to Y, since $[0,1]$ is contractible, and the inclusion $X \hookrightarrow X \times \{0\} \subset M_f$ makes the following diagram commute:

$$\begin{tikzcd} & M_f \arrow[dr, "\cong"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

(In terminology we have not introduced, these facts prove that every map is a *cofibration*, just as Theorem 7.1.9 showed that every map is a fibration.)

Proposition 7.3.6. *Let $n \geq 1$. A map $f : X \to Y$ is n-connected if and only if the pair (M_f, X) is n-connected, i.e. $\pi_k(M_f, X) = 0$ for all $k \leq n$.*

Proof. Apply the long exact sequence in Proposition 7.3.5. □

There is a relative version of Hurewicz's theorem as well.

Theorem 7.3.7. *If (X, A) is a CW-pair which is n-connected, i.e. $\pi_k(X, A) = 0$ for all $k < n$, then $\pi_k(X, A) \to H_k(X, A)$ is an isomorphism for all $k < n$ and is surjective for $k = n$.*

From this, one can deduce several very important results in homotopy theory.

Corollary 7.3.8 (Whitehead Theorem)**.** *If $f : X \to Y$ is n-connected, then $f_* : H_k(X) \to H_k(Y)$ is an isomorphism for $k < n$ and is surjective for $k = n$.*

Corollary 7.3.9. *If X and Y are simply connected spaces and $f : X \to Y$ induces an isomorphism $f_* : H_k(X) \to H_k(Y)$ for all $k < n$ and is surjective for $k = n$, then f is an n-connected map.*

Corollary 7.3.10. *If X and Y are simply connected and $f : X \to Y$ induces an isomorphism on all homology groups, then f is a weak homotopy equivalence.*

Corollary 7.3.11. *If X and Y are simply connected CW-complexes and $f : X \to Y$ induces an isomorphism on all homology groups, then f is a homotopy equivalence.*

7.4 Obstruction Theory

A main principle in homotopy theory is the following. A map $S^n \to Y$ extends to $D^{n+1} \to Y$ if and only if the map is nullhomotopic. This fact is used to study the *extension problem*: suppose (X, A) is a CW-pair and $f : A \to Y$ is any map. Then does f extend to a map $X \to Y$? With CW-complexes, we can work cell-by-cell. First, let $(X, A)^{(n)} = A \cup X^{(n)}$ be the n-skeleton of the pair. It is trivial to extend $f : A \to Y$ over the 0-skeleton $(X, A)^{(0)}$. Next, for $(X, A)^{(1)}$, we can extend as long as Y is connected (i.e. $\pi_0(Y) = 0$). In this way, we can work inductively.

Definition. *A space Y is called n-**simple** if $[S^n, Y] = [S^n, Y]_*$.*

For the remainder of the section, assume Y is n-simple.

Definition. *Let $g : (X, A)^{(n)} \to Y$. The **obstruction cochain** of g is the element $\theta^{n+1}(g) \in C^{n+1}(X, A; \pi_n(Y))$ whose value on an $(n+1)$-cell e_{n+1} is the class $[\theta^{n+1}(g)]$ represented by the composition $\theta^{n+1}(g) = g \circ \phi_{\partial e_{n+1}} : S^n \xrightarrow{\phi} X^{(n)} \xrightarrow{g} Y$, where ϕ is the characteristic map.*

Lemma 7.4.1. *A map $g : (X, A)^{(n)} \to Y$ extends to $(X, A)^{(n+1)}$ if and only if $\theta^{n+1}(g) = 0$.*

Proposition 7.4.2. *$\theta^{n+1}(g)$ is a cocycle.*

Proof. Recall from Section 2.4 that the cellular chain groups are the free abelian groups $C_n(X) = H_n(X^{(n)}, X^{(n-1)})$, for a fixed homology theory $(H_\bullet, \partial)$, together with the cellular boundary maps

$$\partial^{cell} : H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{\delta} H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

Consider the diagram

$$\begin{tikzcd}
\pi_{n+2}(X^{(n+2)}, X^{(n+1)}) \arrow[r] \arrow[d, "a"'] & \pi_{n+1}(X^{(n+1)}) \arrow[r] \arrow[d, "b"'] & \pi_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r] \arrow[d, "c"'] & \pi_n(X^{(n)}) \arrow[d, "g_*"'] \\
H_{n+2}(X^{(n+2)}, X^{(n+1)}) \arrow[r, "\delta"] & H_{n+1}(X^{(n+1)}) \arrow[r] & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\theta^{n+1}(g)"] & H_n(X^{(n)})
\end{tikzcd}$$

where the vertical arrows are the Hurewicz homomorphisms, the top row is the long exact sequence from Proposition 7.3.5, the bottom row is $\theta^{n+1} \circ \partial^{cell} = \delta\theta^{n+1}$. Note that $\pi_k(X^{(n+2)}, X^{(n+1)}) = 0$ for all $k \le n+1$, so it follows from Hurewicz's theorem (7.3.2) that a is surjective. Further, the Hurewicz maps are natural, so each square commutes. This means that $\delta\theta^{n+1}(g)$ is equal to the composition along the top row which is 0 by exactness. Hence $\theta^{n+1}(g)$ is a cocycle. □

Theorem 7.4.3. *If $g : (X, A)^{(n)} \to Y$ and Y is n-simple, then the cohomology class $[\theta^{n+1}(g)] \in H^{n+1}(X, A; \pi_n(Y))$ is trivial if and only if $g|_{(X,A)^{(n-1)}}$ extends to $(X, A)^{(n+1)}$.*

Proof. Suppose $f_0, f_1 : X^{(n)} \to Y$ are homotopic on $X^{(n-1)}$. Then any homotopy F between them determines a *difference cochain* $d = d(f_0, f_1, F) \in C^n(X, A; \pi_n(Y))$ with the property that $\delta d = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$. Consider the CW-pair $(X \times I, A \times I)$; note that the k-skeleton of this pair is $(X, A)^{(k)} \times \partial I \cup (X, A)^{(k-1)} \times I$. So a map $(X \times I, A \times I)^{(n)} \to Y$ is a pair of maps $f_0, f_1 : (X, A)^{(n)} \to Y$ and a homotopy between their restrictions on the $(n-1)$-skeleton, $f_0|_{(X,A)^{(n-1)}}$ and $f_1|_{(X,A)^{(n-1)}}$. This is an extension problem which can be described by the obstruction cocycle

$$\theta^{n+1}(f_0, f_1, F) \in C^{n+1}(X \times I, A \times I; \pi_n(Y)).$$

Define $d(f_0, f_1, F)$ by restricting this cocycle to $(X, A)^{(n)} \times I$ and defining

$$d(f_0, f_1, F)(e_n) = (-1)^{n+1}\theta^{n+1}(f_0, f_1, F)(e_n \times I).$$

Then we have

$$\begin{aligned} 0 = \delta\theta^{n+1}(e_{n+1} \times I) &= \theta^{n+1}\partial(e_{n+1} \times I) \\ &= \theta^{n+1}(\partial e_{n+1} \times I) + (-1)^{n+1}(\theta^{n+1}(e_{n+1} \times \{1\}) - \theta^{n+1}(e_{n+1} \times \{0\})) \\ &= (-1)^n[\delta(e_{n+1}) + (\theta^{n+1}(f_1) - \theta^{n+1}(f_0))(e_{n+1})]. \end{aligned}$$

This proves that if f_0, f_1 are homotopic on $(X, A)^{(n-1)}$ then the corresponding obstruction cocycles are cohomologous.

We need the converse as well, namely that if $[\theta^{n+1}(f_0)] = [\theta^{n+1}(f_1)]$ in $H^{n+1}(X, A; \pi_n(Y))$, then f_0 and f_1 are homotopic on $(X, A)^{(n-1)}$. The details of this can be found in Davis and Kirk. But given the converse, we finally need to realize any $d \in C^n(X, A; \pi_n(Y))$ as a difference cochain for some f_0, f_1, F – this is also in Davis and Kirk. Then if $[\theta^{n+1}(g)] = 0$, we would have

$$\theta^{n+1}(g) = \delta d = \delta d(f_0, f_1, F) = \theta^{n+1}(f_0) - \theta^{n+1}(f_1).$$

So if $\theta^{n+1}(f_1) = 0$, then f_1 will extend to $(X, A)^{(n+1)}$. □

Suppose two maps $f_0, f_1 : X \to Y$ are homotopic on $A \subseteq X$. What are the obstructions to them being homotopic on all of X? This question can be stated as an extension problem via the diagram

$$\begin{tikzcd} X \times \{0,1\} \cup A \times [0,1] \arrow[r, "F"] \arrow[d, hook] & Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

There is an obstruction

$$[\theta^{n+1}(F)] \in H^{n+1}(X \times I, X \times \{0, 1\} \cup A \times I; \pi_n(Y)) \cong H^n(X, A; \pi_n(Y))$$

explicitly given by $[\theta^{n+1}(F)] = d(f_0, f_1, F)$ using the notation of Theorem 7.4.3, with the result that $[\theta^{n+1}(F)] = 0$ in $H^n(X, A; \pi_n(Y))$ if and only if $F\_X^{(n-2)} \times I$ extends to a homotopy between $f_0|_{X^{(n)}}$ and $f_1|_{X^{(n)}}$.

Corollary 7.4.4. *A continuous map from an n-dimensional CW-complex into an n-connected space is nullhomotopic.*

Proof. Let $f_0 : X \to Y$ be such a map and let $f_1 : X \to Y$ be a constant map. Then obstructions to a homotopy $f_0 \simeq f_1$ lie in $H^k(X; \pi_k(Y))$, but by hypothesis $\pi_k(Y) = 0$ for $k \le n$, so $H^k(X; \pi_k(Y)) = 0$ for these k. On the other hand, for $k > n$, $H^k(X; \pi_k(Y)) = 0$ since X is n-dimensional. Hence there are actually no obstructions to a homotopy. □

Theorem 7.4.5. *If Y is $(n-1)$-connected, $A \subseteq X$ and $g : A \to Y$ is any map, then g extends to a map $(X, A)^{(n)} \to Y$ and any two such extensions are homotopic.*

Proof. Obstructions to an extension to $(X, A)^{(k+1)}$ lie in $H^{k+1}(X, A; \pi_k(Y))$ which is 0 for $k \le n - 1$, so we can extend to the n-skeleton by Theorem 7.4.3. On the other hand, obstructions to homotopy between these extensions lie in $H^k(X, A; \pi_k(Y))$ which is 0 for $k \le n - 1$, but note that for any two extensions g_0, g_1, $[\theta^{n+1}(g_0)] = [\theta^{n+1}(g_1)]$ since these classes are determined by $g_0|_{X^{(n-1)}} = g = g_1|_{X^{(n-1)}}$. □

7.5 Hopf's Theorem

The goal in this section is, for an n-dimensional CW-complex K, to completely describe $[K, S^n]$ in terms of cohomology. Recall the notation of Section 2.4, where σ denotes an n-cell of K, $f_\sigma : D^n \to K$ the characteristic map, $f_{\partial\sigma} : S^{n-1} \to K$ the attaching map along the disk's boundary and $p_\sigma : K^{(n)} \to S^n$ the map which collapses the n-skeleton around σ.

Definition. *For cells $\sigma \in K^{(n)}$ and $\tau \in K^{(n-1)}$, define the* **incidence number** *of σ and τ to be*

$$[\tau, \sigma] := \det(p_\tau \circ f_{\partial\sigma}).$$

Then the cellular chain complex $(C_\bullet, \partial^{cell})$ consists of $C_n = C_n(K) = \mathbb{Z}\{\sigma \mid \sigma \text{ is an } n\text{-cell}\}$ with boundary map $\partial^{cell}\sigma = \sum_\tau [\tau, \sigma]\tau$.

One defines a *cellular cohomology* for K (with coefficents in any coefficient module G) by setting

$$C^n(K; G) = \operatorname{Hom}(C_n(K), G)$$
$$\delta : c \longmapsto c \circ \partial^{cell}.$$

This makes $(C^\bullet, \delta)$ into a cochain complex. Further, it follows from the universal coefficient theorem (3.4.2) that $H^n(K; G) \cong H^n(C^\bullet(K; G))$, so cellular cohomology coincides with ordinary cohomology for K.

We treat S^n and D^{n+1} as nested CW-complexes, with a 0-cell $*$, an n-cell e_n and an $(n+1)$-cell e_{n+1} such that $\partial e_{n+1} = e_n$. To study $[K, S^n]$, we will use cellular approximation quite freely, which says:

(1) (Theorem 2.4.13) Any map $\varphi : K \to S^n$ is homotopic to a cellular map, i.e. a map that restricts to $K^{(n-1)} \to *$.

(2) (Corollary 2.4.14) If $\varphi, \psi : K \to S^n$ are homotopic maps, then they are homotopic via a cellular homotopy $K \times I \to S^n$, i.e. such that $(K \times I)^{(n-1)} \to *$.

Let $e = e_n$. There is a commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "f_\sigma"] \arrow[d, "p_\tau \circ f_\tau"'] & K^{(n)} \arrow[d, "\varphi"] \\ S^n \arrow[r, dashed, "\varphi_{e,\tau}"'] & S^n \end{tikzcd}$$

where $\varphi_{e,\tau}$ is the map in Lemma 2.4.15. We know that $\varphi : K^{(n)} \to S^n$ induces a linear map

$$\varphi_* : C_n(K) \longrightarrow C_n(S^n)$$
$$\tau \longmapsto (\deg \varphi_{e,\tau})e.$$

Define $c_\varphi \in C^n(K; \mathbb{Z})$ by $c_\varphi(\tau) = \deg \varphi_{e,\tau}$, so that $\varphi_*(\tau) = c_\varphi(\tau)e$.

Proposition 7.5.1. *For any map $\varphi : K \to S^n$, c_φ is a cocycle.*

Definition. *c_φ is called the* **obstruction cocycle** *for $\varphi : K \to S^n$.*

This definition agrees with the definition of obstruction cocycle given in the previous section (see Davis and Kirk for details).

Proposition 7.5.2. *If σ is an $(n+1)$-cell of K and $\varphi : K^{(n)} \to S^n$, then*

$$(\delta c_\varphi)(\sigma) = c_\varphi(\partial\sigma) = \deg(\varphi \circ f_{\partial\sigma}).$$

Proof. This follows from a diagram chase of the commutative diagram:

$$\begin{array}{ccccc} C_{n+1}(D^{n+1}) & \xrightarrow{(f_\sigma)_*} & C_{n+1}(K) & & \\ \downarrow \partial & & \downarrow \partial & & \\ C_n(S^{n+1}) & \xrightarrow[(f_{\partial\sigma})_*]{} & C_n(K) & \xrightarrow[\partial_*]{} & C_n(S^n) \end{array}$$

□

Now suppose $F : (K \times I)^{(n)} \to S^n$ is a continuous map, with $\varphi_0(x) = F(x, 0)$ and $\varphi_1(x) = F(x, 1)$ for certain maps $\varphi_i : K^{(n)} \to S^n$. We next compare c_{φ_0} and c_{φ_1}. Define $d_F \in C^{n-1}(K; \mathbb{Z})$ by $d_F(\tau) = c_F(\tau \times I) = \deg F_{\varphi,\tau\times I}$.

Proposition 7.5.3. *For any such $F : (K \times I)^{(n)} \to S^n$ and any n-cell $\sigma \subseteq K$, we have*

$$(\delta d_F)(\sigma) = \deg(F \circ f_{\partial(\sigma\times I)}) + (-1)^n(c_{\varphi_1} - c_{\varphi_0})(\sigma).$$

Proof. By Proposition 7.5.2, $\deg(F\circ f_{\partial(\sigma\times I)}) = c_F(\partial(\sigma\times I))$. Moreover, by definition $c_{\varphi_1}(\sigma) = c_F(\sigma \times \{1\})$ and $c_{\varphi_0}(\sigma) = c_F(\sigma \times \{0\})$. Finally, the boundary formula (Theorem 2.6.4(c)) gives

$$\partial(\sigma \times I) = \partial\sigma \times I + (-1)^n\sigma \times \{1\} - (-1)^n\sigma \times \{0\}.$$

Combining these gives the desired formula. □

Note that if $F : K^{(n)} \times I \to S^n$, then $\deg(F \times f_{\partial(\sigma\times I)}) = 0$. This implies:

Corollary 7.5.4. *If $\varphi_0, \varphi_1 : K \to S^n$ are homotopic maps then $[c_{\varphi_0}] = [c_{\varphi_1}]$ in $H^n(K; \mathbb{Z})$.*

Thus we may replace any $\varphi : K \to S^n$ with a cellular approximation $\varphi_0 : K \to S^n$ and define an obstruction class $\xi_\varphi := [c_{\varphi_0}] \in H^n(K; \mathbb{Z})$.

Theorem 7.5.5 (Hopf)**.** *Let K be a CW-complex of dimension d. Then for $n = 1, d$, there is a bijective correspondence*

$$\begin{aligned} [K : S^n] &\longleftrightarrow H^n(K; \mathbb{Z}) \\ [\varphi] &\longmapsto \xi_\varphi. \end{aligned}$$

Proof. There are two steps to prove:

(1) φ_0 and φ_1 are homotopic if and only if $\xi_{\varphi_0} = \xi_{\varphi_1}$.

(2) For any $\xi \in H^n(K;\mathbb{Z})$, there exists a map $\varphi : K \to S^n$ such that $\xi_\varphi = \xi$.

(1, $\Longrightarrow$) is given by Corollary 7.5.4. To prove ($\Longleftarrow$), assume $\xi_{\varphi_0} = \xi_{\varphi_1}$. By Corollary 7.5.4, we may also assume both φ_0, φ_1 are cellular. Then there is some $\alpha \in C^{n-1}(K)$ such that $\delta\alpha = (-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})$. Define $F : K^{(n)} \times \partial I \cup K^{(n-1)} \times I \to S^n$ as follows: F takes $K^{(n-1)} \to *$ (so that it is a cellular map); $F|_{K\times\{0\}} = \varphi_0$ and $F|_{K\times\{1\}} = \varphi_1$; and for any $(n-1)$-cell τ of K, $\deg F_{e,\tau\times 1} = \alpha(\tau)$. Then by definition,

$$d_F(\tau) = c_F(\tau \times I) = \deg F_{e,\tau\times I} = \alpha(\tau),$$

so we see that $d_F = \alpha$. Hence Proposition 7.5.3 implies that for any n-cell σ,

$$\begin{aligned}(-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})(\sigma) &= (\delta\alpha)(\sigma)\\ &= (\delta d_F)(\sigma)\\ &= \deg(F \times f_{\partial(\sigma\times I)}) + (-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})(\sigma).\end{aligned}$$

Therefore $\deg(F \times f_{\partial(\sigma\times I)}) = 0$, so by properties of degree, when $n = d$, F extends over $K \times I$. When $n = 1$, the statement follows from the proof of (2) below.

(2) First suppose $n = d$. Let c be a cocycle such that $[c] = \xi$. Then we have a sequence of maps

$$K^{(n)} \to K^{(n)}/K^{(n-1)} \to \bigwedge S^n \xrightarrow{\varphi} S^n,$$

where the wedge product is taken over all n-cells τ and the last map φ is given by the wedge of each $\deg c_\varphi(\tau)$. Then $c_\varphi(\tau) = \deg \varphi_{e,\tau} = c(\tau)$, so $\xi_\varphi = \xi$. Now suppose $n = 1$. We can get the same map $\varphi : K^{(1)} \to S^1$ as above but now we want to extend it over a 2-cell σ. By Proposition 7.5.2, the composition $\varphi \circ F_{\partial\sigma} : S^1 \to S^1$ has degree $(\delta c)(\sigma) = 0$ since c is a cocycle. Therefore $\varphi \circ F_{\partial\sigma}$ extends over D^2, so φ extends over σ. This gives us $\varphi : K^{(2)} \to S^1$. To extend this to the entirety of K, use the fact (Corollary 1.2.18) that any map $S^k \to S^1$ extends over D^k for any $k \geq 1$. This defines $\varphi : K \to S^1$ as required, and clearly $c_\varphi = c$. $\square$

This allows us to reprove the result in Example 7.3.3 concerning $\pi_n(S^n)$.

Corollary 7.5.6. *The degree function* $\deg : \pi_n(S^n) \to \mathbb{Z}$ *is a bijection for all* $n \geq 1$.

Corollary 7.5.7. *For any CW-complex* K*, there is an isomorphism of groups*

$$[K, S^1] \cong H^1(K;\mathbb{Z}).$$

7.6 Eilenberg-Maclane Spaces

Theorem 7.6.1. *If* G *is an abelian group and* $n \in \mathbb{N}$*, then there exists a CW-complex* $K(G, n)$ *such that*

$$\pi_k(K(G,n)) = \begin{cases} G, & k = n\\ 0, & k \neq n.\end{cases}$$

Definition. *A space $K(G,n)$ with $\pi_n(K(G,n)) = G$ and $\pi_k(K(G,n)) = 0$ otherwise is called an* **Eilenberg-Maclane space** *of type (G,n).*

Note that by Hurewicz's theorem (7.3.2), for such a space $K(G,n)$ we have

$$\widetilde{H}_k(K(G,n);\mathbb{Z}) = \begin{cases} G & \text{if } k = n \\ 0 & \text{if } k < n. \end{cases}$$

Then by the universal coefficient theorem (3.4.2), $H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n);\mathbb{Z}),G) = \mathrm{Hom}(G,G)$, and the latter group has a distinguished element $1 : G \to G$.

Definition. *The* **fundamental class** *of an Eilenberg-Maclane space $K(G,n)$ is the class $\iota \in H^n(K(G,n);G)$ corresponding to $1 \in \mathrm{Hom}(G,G)$.*

Theorem 7.6.2. *For a space X, define $\Phi, \Psi : [X, K(G,n)] \to H^n(X;G)$ by $\Phi : [f] \mapsto f^*\iota$ and $\Psi : [f] \mapsto$ the obstruction class to a homotopy between f and a constant map. Then $\Phi = \Psi$ and both maps are bijections and natural.*

In particular, the natural isomorphism $H^n(X;G) \cong [X, K(G,n)]$ shows that cohomology is a *representable functor*.

Example 7.6.3. $K(\mathbb{Z},1) = S^1$ by Corollary 1.2.18. Moreover, Corollary 7.5.7 says that $H^1(X;\mathbb{Z}) = [X, S^1]$ for any CW-complex X, confirming the $n = 1$ case of Theorem 7.6.2.

Corollary 7.6.4. *For any pair of abelian groups G, G', there are one-to-one correspondences*

$$[K(G,n), K(G',n)]_0 \longleftrightarrow [K(G,n), K(G',n)] \longleftrightarrow \mathrm{Hom}(G,G').$$

Proof. Send a map $f : K(G,n) \to K(G',n)$ to $f_* : \pi_n(K(G,n)) \to \pi_n(K(G',n))$ which by definition is a map $G \to G'$. Then by Theorem 7.6.2 and the universal coefficient theorem,

$$[K(G,n), K(G',n)] \cong H^n(K(G,n);G') \cong \mathrm{Hom}(H_n(K(G,n);\mathbb{Z}),G') = \mathrm{Hom}(G,G').$$

□

Corollary 7.6.5. *If K and K' are two Eilenberg-Maclane spaces for a pair (G,n), there exists a homotopy equivalence $K \to K'$ which is unique up to homotopy and induces the identity $\pi_n(K) \to \pi_n(K')$.*

Definition. *For integers n, m and abelian groups G, G', a* **cohomology operation** *of type (n, G, m, G') is a natural transformation*

$$\theta : H^n(-;G) \longrightarrow H^m(-;G').$$

Suppose θ is a cohomology operation. Then applying it to the fundamental class of $K(G,n)$ determines a class $\theta(\iota) \in H^m(K(G,n);G') = [K(G,n), K(G',m)]$. Further, if $O(n,G,m,G')$ represents the set of all cohomology operations of this type, then evaluation on ι induces a bijection

$$O(n,G,m,G') \longleftrightarrow H^m(K(G,n);G') = [K(G,n), K(G',m)].$$

Example 7.6.6. It's easy to compute $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty$ by considering the CW-structure of $\mathbb{R}P^\infty$. One can in fact prove that $H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]$, the polynomial ring in one variable with $\mathbb{Z}/2\mathbb{Z}$-coefficients generated by the fundamental class $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$. It turns out that $x^2 \in H^2(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ represents the "Steenrod square" cohomology operation,

$$\begin{aligned} H^1(X; \mathbb{Z}/2\mathbb{Z}) &\longrightarrow H^2(X; \mathbb{Z}/2\mathbb{Z}) \\ \alpha &\longmapsto \alpha^2 = \alpha \cup \alpha. \end{aligned}$$