# Sheaves on Manifolds

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# Contents

# Literature

These are lecture notes for a course given in the winter semester 2023/24 at Münster university. They are currently work in progress and will be updated continually. Below we list some relevant literature for the course. This will also be expanded over time.

1. [Lan21] for a lot of the $\infty$-categorical basics we will rely on.

2. [Lur18, Appendix D.7 and Section 21.1.2] for details on dualisable and compactly assembled $\infty$-categories.

3. [Cla14] for more details on dualisable and compactly assembled categories, including the intrinsic characterisation and applications.

4. [MFO22, K-theory of inverse limits, by A. Efimov] for $K$-theory of dualizable categories.

5. [Lur17a, Appendix A.1] for the shape of a topos.

6. [Sch23] for a general discussion of six-functor formalisms.

# Chapter 1

# Overview of the course

The main goal of this course is to describe the six-functor formalism and Verdier duality for topological spaces using newly introduced concepts from $\infty$-category theory. We will also try to shed now light on some classical aspects, like shape theory and Kashiwara-Shapira's Ind-sheaves from this perspective. If time permits we will also discuss the theory of microsupport. On a technical level the course will mostly deal with these new $\infty$-categorical concepts. We will also explain how to apply these concepts to algebraic K-theory and explain recent results of Efimov and Bartels–Nikolaus.

The course is aimed at graduate students and postdocs and we will require a solid knowledge of $\infty$-category theory. While we will recall some concepts that we need (such as presentable $\infty$-categories) we assume that the reader is familiar with the basic concepts, such as limits and colimits, adjunctions and the Yoneda lemma. We also assume that the reader is familiar with the $\infty$-category of spectra, that will be crucial later in the course. Let us start by giving an overview some results and topics covered in the course.

## 1.1 The six functors on spaces

Let us first describe the six-functor formalism that we are after. Let $X$ be a locally compact Hausdorff space. Then we can consider the $\infty$-category

$$\mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$$

of sheaves on $X$ with values in the $\infty$-categorical derived category of $\mathbb{Z}$. Concretely such a sheaf is given by a functor

$$\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{D}\mathbb{Z}$$

which satisfies descent, i.e. $\mathcal{F}(\emptyset) = 0$, for two open sets $U, V \subseteq X$ we have that

$$
\begin{array}{ccc}
\mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V)
\end{array}
$$

is a pullback and that for an increasing union of open subsets $\{U_i\}_{i\in I}$ indexed by a filtered poset $I$ the map

$$\mathcal{F}\left(\bigcup_i U_i\right) \to \lim_I \mathcal{F}(U_i)$$

is an equivalence in $\mathcal{DZ}$. Note that everything we say in this chapter will be more generally true for sheaves with values in any presentable, stable $\infty$-category in place of $\mathcal{DZ}$, but for concreteness we stick with $\mathcal{DZ}$ here.

**Remark 1.1.1.** The $\infty$-category $\mathrm{Shv}(X; \mathcal{DZ})$ is closely related to the derived category $\mathcal{D}(\mathrm{Shv}(X, \mathrm{Ab}))$ of sheaves of abelian groups on $X$, but generally not equivalent. The latter is a Bousfield localization of $\mathrm{Shv}(X; \mathcal{DZ})$, more precisely it is equivalent to the $\infty$-category of hypersheaves on $X$. If $X$ is paracompact and has finite covering dimension, then the two $\infty$-categories are equivalent though.

The category $\mathcal{DZ}$ has some extra structure, namely it has a symmetric monoidal structure $\otimes$ given by the tensor product of sheaves. This is defines as the sheafification of the pointwise tensor product of functors. It turns out that this is a closed symmetric monoidal structure, that is for any pair of sheaves $\mathcal{F}, \mathcal{G}$ on $X$ there exists another sheaf $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \in \mathrm{Shv}(X; \mathcal{DZ})$ with the universal property that maps $\mathcal{H} \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ in $\mathrm{Shv}(X; \mathcal{DZ})$ are naturally the same as maps $\mathcal{H} \otimes \mathcal{F} \to \mathcal{G}$. The functors $\otimes$ and $\underline{\mathrm{Hom}}$ are functors number 1 and 2 in our six-functor formalism. Now for any continuous map $f : Y \to X$ we have the pushforward functor

$$f_* : \mathrm{Shv}(Y; \mathcal{DZ}) \to \mathrm{Shv}(X, \mathcal{DZ}) \qquad (f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) \ .$$

For example for $f : Y \to \mathrm{pt}$ we have that $\mathrm{Shv}(\mathrm{pt}, \mathcal{DZ}) = \mathcal{DZ}$ and $f_*\mathcal{F} = \mathcal{F}(Y)$ is given by global section and thus also written as $\Gamma(\mathcal{F})$. This functor has a left adjoint

$$f^* : \mathrm{Shv}(X, \mathcal{DZ}) \to \mathrm{Shv}(Y; \mathcal{DZ})$$

given by pullback of sheaves. Concretely $(f^*\mathcal{F})$ is given by the sheafification of the presheaf $U \mapsto \mathrm{colim}_{V \supseteq f(U) \text{ open}} \mathcal{F}(V)$. For example if $f : U \to X$ is the inclusion of an open set, then $f^*\mathcal{F}$ is simply the restriction of $\mathcal{F}$ to opens in $U$ and thus sometimes written as $\mathcal{F}|_U$. For the inclusion $f : \{x\} \to X$ of a point the pullback $f^*\mathcal{F}$ is the stalk and written as $\mathcal{F}_x$. For the projection $f : X \to \mathrm{pt}$ the pullback $f^*C$ for $C \in \mathcal{DZ}$ is given by the constant sheaf with value $C$, that is the sheafification of the presheaf that is constant with value $C$. We shall also write this as $\underline{C}$.

The functors $f^*$ and $f_*$ are functors 3 and 4 of the six functors. Finally for a map $f : Y \to X$ there is also the functor

$$f_! : \mathrm{Shv}(Y, \mathcal{DZ}) \to \mathrm{Shv}(X, \mathcal{DZ})$$

of proper pushforward defined as

$$(f_!\mathcal{F})(U) = \mathrm{colim}_{K \subseteq f^{-1}U \text{ s.t. } K \to U \text{ proper}} \mathrm{fib}\left(\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}U \setminus K)\right) \ .$$

If $f : Y \to \mathrm{pt}$ then we have that

$$f_! \mathcal{F} = \mathrm{colim}_{K \subseteq Y \text{ compact}} \mathrm{fib}\left(\mathcal{F}(Y) \to \mathcal{F}(Y \setminus K)\right)$$

is given by 'global sections with compact support' and written as $\Gamma_c(\mathcal{F})$. There is a natural map

$$f_! \mathcal{F} \to f_* \mathcal{F}$$

which is immediate from the definitions (as the map from the fibre to the first term) and which is an equivalence if $f$ is proper.

**Proposition 1.1.2.** *If $i : U \to X$ is the inclusion of an open set, then the functor $i_!$ is given by 'extension by zero', that is $i_! \mathcal{F}$ is the sheafification of the presheaf*

$$V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & else \end{cases}$$

**Theorem 1.1.3** (Proper Base change)**.** *If we have a pullback diagram of locally compact Hausdorff spaces*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{f'} & & \downarrow{f} \\ X' & \xrightarrow{g} & X \end{array}$$

*then for $\mathcal{F} \in \mathrm{Shv}(Y, \mathcal{D}\mathbb{Z})$ we have that*

$$g^* f_!(\mathcal{F}) \simeq f'_! g'^*(\mathcal{F}).$$

In particular we have for $f : Y \to X$ and $x \in X$ that

$$(f_! \mathcal{F})_x = \Gamma_c(i^* \mathcal{F})$$

for $i : Y_x \to Y$ the inclusion of the fibre of the point. The functor $f_!$ is functor number 5 and it turns out that it has a mysterious right adjoint

$$f^! : \mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(Y; \mathcal{D}\mathbb{Z}) \ .$$

which is functor number 6 and called the exceptional inverse image functor. In general the functor $f^!$ is tricky to describe, but if $f : U \to X$ is the inclusion of an open subset then $f^! = f^*$ as one easily sees from Proposition 1.1.2 since extension by zero is more or less by definition left adjoint to $f^*$. We can summarize the situation by saying that we have for $f : Y \to X$ that

1. $f_! = f_*$ if $f$ is proper

2. $f^! = f^*$ if $f$ is an open immersion.

We claim that properties (1) and (2) already uniquely determine the adjunction $(f_!, f^!)$ for all maps $f$ provided we also require functoriality, that is $(fg)_! = f_! g_!$. To see this we use the following assertion:

**Lemma 1.1.4.** *Every map $f : Y \to X$ of locally compact Hausdorff spaces can be factored as $Y \xrightarrow{i} \overline{Y} \xrightarrow{p} X$ where $i$ is an open immersion and $p$ is proper.*

*Proof.* We take the one point compactification $Y'$ of $Y$. Then we consider the graph of $f$ inside of $Y \times X$ and take its closure $\overline{Y}$ inside of $Y' \times X$. The projection $\overline{Y} \to X$ is then proper, and the inclusion $Y \to \overline{Y}$ open. $\square$

Now for a given factorization we have that

$$f_! = p_! i_! = p_* i_!$$

where $i_!$ is left adjoint to $i^*$. This uniquely determines the functor $f_!$.

**Remark 1.1.5.** One can wonder whether this is well-defined and how coherently this definition can be made. It is a remarkable observation by Gaitsgory-Rozenblyum and Liu-Zheng as well as Mann that one can in fact use this as a definition of $f_!$ and produce a highly coherent six functor formalism using that.

**Remark 1.1.6.** The $\infty$-category $\mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$ for course makes sense for every topological space $X$, the conditions of being locally compact Hausdorff are not needed for that. The adjunction $f^* \dashv f_*$ also makes sense in this generality for each continuous map. However for the adjunction $f_! \dashv f^!$ to be defined and well behaved one then needs conditions on the map $f$ that are automatically satisfies in the LCH case: it needs to be locally proper, see [SS14].

There is another way to recover the adjunction $f_! \dashv f^!$ from the adjunction $f^* \dashv f_*$ which is more categorical in nature than the geometric construction given above.

Let us describe the idea, which will be the central theme of this lecture course. For every presentable, stable $\infty$-category $\mathcal{C}$ there is a 'dual' category $\mathcal{C}^\vee$. One of the central themes of the first few weeks of the lecture will be to study this duality and particularly which categories are 'dualizable' (meaning that $\mathcal{C} \simeq (\mathcal{C}^\vee)^\vee$). The dual of the category of sheaves on a locally compact Hausdorff space is the $\infty$-category $\mathrm{coShv}(X; \mathcal{D}\mathbb{Z})$ of cosheaves with values in $\mathcal{D}\mathbb{Z}$, that is functors

$$\mathcal{F} : \mathrm{Open}(X) \to \mathcal{D}\mathbb{Z}$$

which satisfy 'codescent', that is $\mathcal{F}(\emptyset) = 0$, for two open sets $U, V \subseteq X$ we have that

$$
\begin{array}{ccc}
\mathcal{F}(U \cap V) & \longrightarrow & \mathcal{F}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cup V)
\end{array}
$$

is a pushout and that for an increasing union of open subsets $\{U_i\}_{i \in I}$ indexed by a filtered poset $I$ the map

$$\text{colim}_I \, \mathcal{F}(U_i) \to \mathcal{F}\left(\bigcup_i U_i\right)$$

is an equivalence in $\mathcal{D}\mathbb{Z}$.

**Theorem 1.1.7** (Lurie, Verdier duality). *There is a canonical equivalence*

$$\mathbb{D} : \text{Shv}(X, \mathcal{D}\mathbb{Z}) \simeq \text{coShv}(X, \mathcal{D}\mathbb{Z})$$

*sending $\mathcal{F} \in \text{Shv}(X, \mathcal{D}\mathbb{Z})$ to the cosheaf*

$$U \mapsto \Gamma_c(F|_U) \ .$$

Here we have used the functor $f_!$ implicitly in this equivalence, namely to define $\Gamma_c$. But we will see in the lecture course that this self-duality of $\text{Shv}(X, \mathcal{D}\mathbb{Z})$ is a completely intrinsic property of the symmetric monoidal $\infty$-category $\text{Shv}(X, \mathcal{D}\mathbb{Z})$: it is a locally rigid category. The rough idea is that for locally rigid categories $\mathcal{C}$ there is an equivalence to $\mathcal{C}^\vee$ informally induced by passing to internally dual objects. We will make this rigoros later in the course.

The point now is that for a continuous map $f : X \to Y$ the adjunction $f^* \dashv f_*$ dualizes to an adjunction on the categories of cosheaves:

$$f_+ : \text{coShv}(X) \rightleftarrows \text{coShv}(Y) : f^+ \ .$$

Concretely the functor $f_+$ is given by $f_+(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. Now using the duality of Theorem 1.1.7 for $X$ and $Y$ we get an induced adjunction between $\text{Shv}(X)$ and $\text{Shv}(Y)$. This is the adjunction $f_! \dashv f^!$. Said more abstractly: the adjunction $f_! \dashv f^!$ is the dual to $f^* \dashv f_*$ using the fact that categories of sheaves are canonically self-dual. The self-duality is induced by the tensor product.

Using the six functor formalism we we can define sheaf cohomology, compactly supported sheaf cohomology, sheaf homology and locally finite sheaf homology (aka Borel Moore homology) of a locally compact Hausdorff space $X$ with coefficients in $\mathbb{Z}$ as

$$H^*(X, \mathbb{Z}) := p_* p^* \mathbb{Z} \qquad H^*_c(X, \mathbb{Z}) := p_! p^* \mathbb{Z}$$
$$H_*(X, \mathbb{Z}) = p_! p^! \mathbb{Z} = p_+ p^+ \mathbb{Z} \qquad H^{\text{lf}}_*(X, \mathbb{Z}) = p_* p^! \mathbb{Z}$$

where $p : X \to \text{pt}$ is the unique map to the point and $\mathbb{Z}$ denotes the constant sheaf/cosheaf with value $\mathbb{Z} \in \mathcal{D}\mathbb{Z}$ on the point. [1] For the definition of homology it is maybe useful to think in terms of cosheaves using the $(-)_+ \dashv (-)^+$ adjunction to be convinced that this is a reasonable definition of homology. We will see that homology and cohomology are dual to each other as a consequence of the general properties of the six functor formalism. More

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[1]We are slighly conflating the object of $\mathcal{D}(\mathbb{Z})$ and its homology here for the purpose of exposition. We should really write the (co)chains instead of (co)homology.

precisely if $X$ is locally nice (e.g. a CW complex) then cohomology is the dual of homology. In general locally finite homology is the dual of compactly supported cohomology, e.g. for the Cantor set where the first statement fails (see next Section).

This generalizes as follows: we define the *dualizing sheaf* of $X$ as $\omega_X := p^!(\mathbb{Z})$ . Then we can define a functor:

$$D : \mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \to \mathrm{Shv}(X; \mathcal{D}\mathbb{Z})^{\mathrm{op}} \qquad \mathcal{F} \mapsto \underline{\mathrm{Hom}}(\mathcal{F}, \omega_X)$$

and we refer to $D\mathcal{F}$ as the *Verdier dual of $\mathcal{F}$*. The functor $D$ is left adjoint to $D^{\mathrm{op}}$, that is

$$\mathrm{Map}_{\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})}(\mathcal{F}, D\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathcal{D}\mathbb{Z})}(\mathcal{G}, D\mathcal{F})$$

but generally the map $\mathcal{F} \to D^2\mathcal{F}$ is not an equivalence (it is for many sheaves though, such as $p^*\mathbb{Z}$ on nice spaces $X$). While mysterious on the sde of sheaves, under the equivalence to cosheaves, $\mathrm{Hom}(-, f^!(\mathbb{Z}))$ corresponds to $\mathrm{Hom}(-, f^+(\mathbb{Z}))$, i.e. the dual of global sections. This means that under the equivalence $\mathrm{Shv}(X; \mathcal{D}\mathbb{Z}) \simeq \mathrm{coShv}(X; \mathcal{D}\mathbb{Z})$ the functor $D$ sends a sheaf to the pointwise dual if the associated cosheaf. Using this we find that:

**Proposition 1.1.8.** *We have for $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X; \mathcal{D}\mathbb{Z})$ and $C \in \mathcal{D}\mathbb{Z}$:*

$$p_* D\mathcal{F} = Dp_! \mathcal{F}$$
$$\underline{\mathrm{Hom}}(\mathcal{F}, D\mathcal{G}) = \underline{\mathrm{Hom}}(\mathcal{G}, D\mathcal{F})$$
$$p^! DC = Dp^* C$$

**Example 1.1.9.** Combining the first and last assertion of Proposition 1.1.8 we obtain that

$$p_* p^! DC = Dp_! p^* C$$

for any $C \in \mathcal{D}\mathbb{Z}$ . Specifically for $C = \mathbb{Z}$ this shows the claim we already made above, namely that locally finite homology is always the dual of compactly supported cohomology.

Applying the first assertion to $\mathcal{F} = Dp^*C$ and also using the third we get

$$p_* D^2 p^* C = Dp_! p^! DC \ .$$

If we assume that $p^*\mathbb{Z}$ agrees with its bidual (which is the case for sufficiently nice spaces such as CW complexes), then this yields the claim that the homology considered above is indeed a predual of cohomology.

Again we shall see that all these things make sense in an arbitrary locally rigid $\infty$-category. Verdier duality becomes particularly useful when we understand the dualizing complex:

**Theorem 1.1.10** (Poincaré duality). *Let $X$ be a (homology) manifold of dimension $n$. Then $p^!(-)$ is equivalent to $p^*(-) \otimes \omega_X$ and $\omega_X$ is locally equivalent to $\mathbb{Z}[n]$.*[2]

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[2]More precisely it is given by the $n$-fold shift of the orientation sheaf.

## 1.2  Efimov K-theory

For any small stable $\infty$-category $\mathcal{C}$ there is an algebraic $K$-theory spectrum[3]. We assume that the small stable $\infty$-categories are idempotent complete and denote the $\infty$-category of small, idempotent complete stable $\infty$-categories by $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Then K-theory is a functor

$$\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$$

where Sp is the $\infty$-category of spectra. We will review the definition of $K$-theory in the lecture. For a ring $R$ we shall write $\mathrm{K}(R) := K(\mathcal{D}^{\mathrm{perf}}R)$.

A dualizable stable $\infty$-category on the other hand is a presentable stable $\infty$-category, i.e. a large category. One key fact is that there is a functor from small stable $\infty$-categories to dualizable ones, which sends $\mathcal{C}$ to its Ind-category $\mathrm{Ind}(\mathcal{C})$. Morally this freely adds filtered colimits (equivalently infinite sums) to $\mathcal{C}$. This defines a full faithful embedding

$$\mathrm{Ind} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{dual}}$$

where the target is the category of dualisable, stable $\infty$-categories whose definition is the first major goal of the course. For example Ind takes the perfect derived $\infty$-category $\mathcal{D}^{\mathrm{perf}}R$ of any ring $R$ (or more generally qcqs scheme) to the derived $\infty$-category $\mathcal{D}(R)$. As we have mentioned before, the object $\mathrm{Shv}(X, \mathcal{D}\mathbb{Z})$ for a locally compact Hausdorff space is an object in $\mathrm{Cat}_\infty^{\mathrm{dual}}$. It does not lie in the image of Ind as we will also see. Other examples of objects of interest in $\mathrm{Cat}_\infty^{\mathrm{dual}}$ are the categories of nuclear modules associated with analytic rings as defined by Clausen-Scholze.

**Theorem 1.2.1** (Efimov). *There is a functor*

$$\mathrm{K}^{\mathrm{cont}} : \mathrm{Cat}_\infty^{\mathrm{dual}} \to \mathrm{Sp}$$

*that extends K-theory, i.e. such that $\mathrm{K}^{\mathrm{cont}} \circ \mathrm{Ind}$ is equivalent to $K$-theory. This functor sends Verdier sequences to fibre sequences and is essentially uniquely determined by these properties.*

This result now allows us to take K-theory for the interesting categories such as sheaves or nuclear modules. It also shows that $\mathrm{K}^{\mathrm{cont}}(\mathcal{D}R) = \mathrm{K}(\mathcal{D}^{\mathrm{perf}}R) = \mathrm{K}(R)$. For the former the foundational result of Efimov is the following:

**Theorem 1.2.2** (Efimov). *For any locally compact Hausdorff space $X$ there is an equivalence*

$$\mathrm{K}^{\mathrm{cont}}(\mathrm{Shv}(X, \mathcal{D}\mathbb{Z})) \simeq \Gamma_c\left(X, \underline{\mathrm{K}\mathbb{Z}}\right) .$$

Here $\underline{\mathrm{K}\mathbb{Z}}$ is the constant sheaf on the $K$-theory spectrum $\mathrm{K}\mathbb{Z}$ of the integers, considered as an object of

$$\mathrm{Shv}(X, \mathrm{Sp})$$

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[3]For the experts: we always mean non-connective $K$-theory here

and we use that for sheaves of spectra there is an analogous six functor formalism as for sheaves with values in $\mathcal{D}\mathbb{Z}$. If we denote the map $X \to \mathrm{pt}$ by $p$ we can also write $\Gamma_c(X, \underline{K\mathbb{Z}})$ as $p_! p^*(K\mathbb{Z})$. We refer to this spectrum as the compactly supported $K\mathbb{Z}$-cohomology of $X$. If $X$ is compact then this is the equivalent to the cohomology

$$\Gamma(X, \underline{K\mathbb{Z}}) = p_* p^* K\mathbb{Z}$$

Here we have to be careful though, since this is sheaf cohomology. If we allow arbitrary compact Hausdorff spaces this might behave quite differently than the $K\mathbb{Z}$ cohomology of the associated anima $\mathrm{Sing}(X) \in \mathrm{An}$. For example the Cantor set $X$ has for any spectrum $E$ that

$$\Gamma(X, \underline{E}) = \bigoplus_\omega E \qquad \text{and} \qquad E^{\mathrm{Sing}(X)} = \prod_\omega E \ .$$

However, if the space $X$ is sufficiently nicely behaved, e.g. a CW-complex, then this distinction goes away. Slightly more generally we will see that for any topological space $X$ that is locally of constant shape (we will explain what that means) there is an associated anima $\mathrm{Shape}(X) \in \mathrm{An}$ such that

$$\Gamma(X, \underline{E}) \simeq E^{\mathrm{Shape}(X)}$$

Being locally of constant shape for $X$ is equivalent to the assertion that the functor

$$p^* : \mathrm{Shv}(\mathrm{pt}; \mathrm{An}) \to \mathrm{Shv}(X, \mathrm{An})$$

admits a left adjoint $p_\natural$ (recall that it always admits a right adjoint $p_*$). This condition and its analogue for sheaves of spectra will play a crucial role for us. We will see that in the stable case the left adjoint $p_\natural$ is automatically given by $p_!(- \otimes \omega_X)$.

**Remark 1.2.3.** If one is willing to work with pro-anima instead of anima then one can in fact define $\mathrm{Shape}(X)$ for any locally compact Hausdorff space $X$ and get $\Gamma(X, \underline{E}) \simeq E^{\mathrm{Shape}(X)}$.

## 1.3 Completed Cosheaves

The $\infty$-category $\mathrm{Cat}_\infty^{\mathrm{dual}}$ of dualisable, stable $\infty$-categories has a lot of interesting structure which we will study in the lecture.

- $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has all colimits and limits.

- There is the notion of Verdier sequences or short exact sequence that behaves a lot like short exact sequences of abelian groups.

- Every object $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{dual}}$ admits a 2-term 'resolution' by compactly generated stable $\infty$-categories, that is for fixed $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{dual}}$ there is a short exact sequence

$$0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0$$

  with $\mathcal{D}$ and $\mathcal{E}$ compactly generated. Concretely we can choose $\mathcal{D}$ to be $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ and $\mathcal{E}$ as $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}})$. We will explain what that means in the course.

In some sense we can think of compactly generated stable $\infty$-categories as 'injective objects' in $\mathrm{Cat}_\infty^{\mathrm{dual}}$.[4] In this sense we have an injective resolution and then for example the continuous $K$-theory functor $\mathrm{K}^{\mathrm{cont}} : \mathrm{Cat}_\infty^{\mathrm{dual}} \to \mathrm{Sp}$ of Efimov is defined using these resolutions, i.e. it is a sort of right derived functor of $\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$.

- $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has a tensor product $\otimes$ that make it symmetric monoidal. For example we have that $\mathrm{Shv}(X; \mathrm{Sp}) \otimes \mathrm{Shv}(Y; \mathrm{Sp}) \simeq \mathrm{Shv}(X \times Y; \mathrm{Sp})$. One can study dualisable objects within $\mathrm{Cat}_\infty^{\mathrm{dual}}$ and these turn out to be exactly the smooth and proper dualisable stable $\infty$-categories. The category $\mathrm{Shv}(X)$ is proper under very mild conditions on $X$ but essentially never smooth.

- Since $\mathrm{Cat}_\infty^{\mathrm{dual}}$ has a tensor product we can speak about commutative algebra objects in $\mathrm{Cat}_\infty^{\mathrm{dual}}$. These are symmetric monoidal, presentable, stable $\infty$-categories with specific properties. Among those we will study a subclass called *(locally) rigid categories*. This notions extends the notion of rigidity for small symmetric monoidal categories. We will see that Verdier duality essentially is the statement that $\mathrm{Shv}(X; \mathrm{Sp})$ is locally rigid for any locally compact Hausdorff space $X$. It is rigid precisely if $X$ is compact.

- The tensor product is closed, that is there is an inner hom $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D})$ for any pair of stable, dualisable $\infty$-categories. In general this inner hom is a bit hard to understand, but again one can use injective resolutions to get a handle on it. Specifically the $\infty$-categories of nuclear modules of Clausen-Scholze can be seen to be (a slight variant of) the inner hom in dualisable category between well-understood categories, e.g.

$$\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}\mathbb{Z}}^{\mathrm{dual}}\left((\mathcal{D}\mathbb{Z})_p^\wedge, \mathcal{D}\mathbb{Z}\right) \simeq \lim_{n \to \infty}^{\mathrm{dual}} \mathcal{D}(\mathbb{Z}/p^n)$$

We will specificially study

$$\widehat{\mathrm{coShv}}(X; \mathcal{D}) := \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Shv}(X; \mathrm{Sp}), \mathcal{D})$$

the 'dual' of the stable $\infty$-category of sheaves. We will give a concrete description of $\widehat{\mathrm{coShv}}(X; \mathcal{D})$ and relate it to Ind-sheaves (which are an $\infty$-categorical version of Kashiwara-Shapiras Ind sheaves). The main result about $\widehat{\mathrm{coShv}}(X; \mathcal{D})$ is the following:

**Theorem 1.3.1** (Bartels–Efimov–Nikolaus)**.** *We have that* $\mathrm{K}^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X; \mathcal{D}\mathbb{Z}))$ *is the locally finite* $\mathrm{K}(\mathbb{Z})$*-homology of* $X$ *(aka. Borel-Moore homology), that is:*

$$\mathrm{K}^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X; \mathcal{D}\mathbb{Z})) \simeq p_* p^! \mathrm{K}\mathbb{Z}$$

*for* $p : X \to \mathrm{pt}$.

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[4]We will see that there is in fact a better notion of injective, but for the purpose of this introduction thinking if injective resolutions gives a good intuition.

# Chapter 2

# Categorical structures

## 2.1 Presentable $\infty$-categories

Presentable $\infty$-categories are big categories (*all* sets, *all* modules, etc.), that are in a sense still generated by small objects.

**Definition 2.1.1.** Let $\kappa$ be a regular cardinal (e.g. $\kappa = \omega$, the first countable ordinal, or $\kappa = \omega_1$, the first uncountable ordinal).

1. An $\infty$-category $I$ is $\kappa$-*filtered* if any map $K \to I$ from a $\kappa$-small simplicial set extends over the right cone $K * \Delta^0 \to I$.

2. For an $\infty$-category $\mathcal{C}$ with small colimits, an object $X \in \mathcal{C}$ is $\kappa$-*compact* if the canonical map
$$\mathrm{colim}_{i \in I} \mathrm{Map}_{\mathcal{C}}(X, Y_i) \to \mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim}_{i \in I} Y_i)$$
is an equivalence for any $\kappa$-filtered small $I$ and any functor $Y : I \to \mathcal{C}$. We write $\mathcal{C}^{\kappa}$ for the full subcategory on $\kappa$-compact objects.

We also call a functor $I \to \mathcal{C}$ with $I$ *kappa*-filtered a $\kappa$-*filtered diagram* in $\mathcal{C}$, speak of $\kappa$-filtered colimits, etc. If $\kappa = \omega$, we simply say filtered and compact.

**Example 2.1.2.**     1. The $\kappa$-compact objects in Set are precisely the $\kappa$-small sets, i.e. those with cardinality smaller than $\kappa$. The collection of $\kappa$-small subsets of a given set $S$ forms a $\kappa$-filtered category, since the union of less than $\kappa$ many $\kappa$-small subsets of $S$ is still $\kappa$-small (this is where regularity of $\kappa$ enters).

2. Similarly, $\kappa$-compact objects in $\mathrm{Mod}(R)$ are modules with a presentation with less than $\kappa$ many generators and relations, and retracts of those.

3. $\kappa$-compact objects in $\mathcal{D}(R)$ are those which are equivalent to complexes of projectives with less than $\kappa$ many generators in total.

4. $\kappa$-compact objects in An, the $\infty$-category of anima (homotopy types) are those anima which can be represented by simplicial sets with less than $\kappa$ many nondegenerate simplices (or CW complexes with less than $\kappa$ many cells), and retracts of those.

**Lemma 2.1.3.** *1. In An, $\kappa$-small limits commute with $\kappa$-filtered colimits.*

*2. $\kappa$-small colimits of $\kappa$-compact objects are again $\kappa$-compact.*

*Proof.* The first statement is [Lur17b, Proposition 5.3.3.3], and is a special property of An or more general $\infty$-topoi. As a quick reality check though, for a $\kappa$-small *product* we may check that
$$\operatorname{colim}_{i \in I} \prod_{j \in J} X_{ij} \to \prod_{j \in J} \operatorname{colim}_{i \in I} X_{ij}$$
is an equivalence: A point in the left term consists of a choice of $i$ and for every $j$ a point in $X_{ij}$. In the right term, we instead have for every $j$, a choice of point in $X_{i(j),j}$ for some $i(j)$ depending on $j$. The former is more restrictive, but if $I$ is $\kappa$-filtered and $J$ $\kappa$-small, then the $i(j)$ have a common upper bound in $I$, so any point in the target really comes from the source. (To turn this into a proof, we of course need to argue also about homotopies between points etc.)

The second follows from the first: Let $X : K \to \mathcal{C}$ be a $\kappa$-small diagramof $\kappa$-compact objects, and $Y : I \to \mathcal{C}$ a $\kappa$-filtered diagram. We may write

$$\operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_K X_k, \operatorname{colim}_I Y_i)$$
$$\simeq \lim_K \operatorname{Map}_{\mathcal{C}}(X_k, \operatorname{colim}_I Y_i)$$
$$\simeq \lim_K \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(X_k, Y_i)$$
$$\simeq \operatorname{colim}_I \lim_K \operatorname{Map}_{\mathcal{C}}(X_k, Y_i)$$
$$\simeq \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_K X_k, Y_i)$$

$\square$

We will now see a way to freely adjoin $\kappa$-filtered colimits to a given category. Recall first how to adjoin *all* small colimits:

**Lemma 2.1.4.** *Let $\mathcal{C}$ be a small $\infty$-category.*

1. *The Yoneda embedding $j : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is fully faithful.*

2. *For any $\mathcal{D}$ with all small colimits, restriction along $j$ induces an equivalence between*

$$\operatorname{Fun}^{\mathrm{colim}}(\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}), \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

*where the left hand denotes small colimit-preserving functors. The inverse is given by left Kan extension along $j$.*

*Proof.* [Lur17b, Proposition 5.1.3.1 and Theorem 5.1.5.6] $\square$

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This means that a colimit-preserving functor is determined by its restriction to the Yoneda image, and any functor on the Yoneda image may be extended to a colimit-preserving one by Kan extension. Note that $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ does not preserve any nontrivial colimits, as colimits in the latter are formed pointwise.

**Definition 2.1.5.** For a small $\infty$-category $\mathcal{C}$, we define $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ as the smallest full subcategory containing $j(\mathcal{C})$ and closed under $\kappa$-filtered colimits.

**Lemma 2.1.6.** *For any $\mathcal{D}$ with $\kappa$-filtered colimits, restriction along $j$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}_{\kappa-\mathrm{filt}}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

*Proof.* [Lur17b, Proposition 5.3.5.10] □

**Remark 2.1.7.** In $\mathrm{Ind}_\kappa(\mathcal{C})$, we have objects $jX$ for every $X \in \mathcal{C}$, but also $\kappa$-filtered colimits, so we may form

$$\mathrm{colim}_{i \in I} jX_i$$

for a $\kappa$-filtered diagram $I \to \mathcal{C}$. Even if $\mathrm{colim}_{i \in I} X_i$ exists, this does not agree with $j \, \mathrm{colim}_{i \in I} X_i$, so we think of this as a new "formal" colimit we adjoin to $\mathrm{Ind}_\kappa(\mathcal{C})$. Mapping spaces may be computed as

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{i \in I} jX_i, \mathrm{colim}_{i' \in I'} jY_{i'})$$
$$= \lim_{i \in I} \mathrm{colim}_{i' \in I'} \mathrm{Map}_\mathcal{C}(X_i, Y_{i'}).$$

It turns out that every object in $\mathrm{Ind}_\kappa(\mathcal{C})$ is in fact of the form $\mathrm{colim}_{i \in I} jX_i$ for some filtered diagram $I \to \mathcal{C}$.

If $\mathcal{C}$ already admits $\kappa$-small colimits, there is another more intrinsic description of $\mathrm{Ind}(\mathcal{C})$.

**Lemma 2.1.8.** *If $\mathcal{C}$ admits $\kappa$-small colimits, $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ consists precisely of those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ which preserve $\kappa$-small limits.*

*Proof.* [Lur17b, Corollary 5.3.5.4] □

**Corollary 2.1.9.** *If $\mathcal{C}$ admits $\kappa$-small colimits, $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ preserves them.*

*Proof.* If $X : K \to \mathcal{C}$ is a $\kappa$-small diagram,

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j \, \mathrm{colim}_{k \in K} X_k, F) = F(\mathrm{colim}_{k \in K} X_k),$$

and

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\mathrm{colim}_{k \in K} jX_k, F) = \lim_{k \in K} F(X_k).$$

Since $F \in \mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ preserves $\kappa$-small limits, these are equivalent. □

This also leads to another universal property of $\mathrm{Ind}_\kappa$: Adjoining *all* colimits, relative to already having $\kappa$-small colimits. This is related to the fact that every colimit can canonically be written as $\kappa$-filtered colimit of $\kappa$-small colimits, hence the heuristic

$$\text{all colimits} \;=\; \kappa\text{-filtered colimits} \;+\; \kappa\text{-small colimits},$$

for example

$$\text{all colimits} \;=\; \text{filtered colimits} \;+\; \text{small colimits},$$

**Lemma 2.1.10.** *If $\mathcal{C}$ admits $\kappa$-small colimits, $\mathrm{Ind}_\kappa(\mathcal{C})$ admits all small colimits, and for any $\mathcal{D}$ which admits small colimits, restriction along $j$ gives an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{colim}_{\kappa-\mathrm{sm}}}(\mathcal{C}, \mathcal{D}).$$

*Proof.* [Lur17b, Example 5.3.6.8] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

A lot of categories in daily life are Ind of something. For example, every set is a filtered colimit of finite sets, every group is a filtered colimit of finitely presented groups, every anima is a filtered colimit of (retracts of) anima represented by finite simplicial sets. In all those cases, we see that they are Ind of their compact objects.

**Definition 2.1.11.** We call a category $\mathcal{C}$ with small colimits *$\kappa$-compactly generated* if $\mathcal{C}^\kappa$ is small, and the canonical $\kappa$-filtered colimit preserving functor

$$k : \mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$$

is an equivalence.

**Lemma 2.1.12.** *For any $\mathcal{C}$ with $\kappa$-filtered colimits,*

$$k : \mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$$

*is fully faithful.*

*Proof.* We need to check that

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(F, G) \to \mathrm{Map}_{\mathcal{C}}(kF, kG)$$

is an equivalence for all $F$ and $G$. For any $G$, the collection of $F$ for which this holds is closed under $\kappa$-filtered colimits since $k$ commutes with colimits. So it suffices to check in the case $F = jX$ with $X \in \mathcal{C}^\kappa$. But then the left hand side commutes with $\kappa$-filtered colimits in $G$ since these are formed pointwise, and the right hand side since $kjX = X$ is $\kappa$-compact. So we may assume $G = jY$, and the result follows from the fact that $j$ is fully faithful. $\quad\square$

**Lemma 2.1.13.** *Let $\mathcal{C}$ be an $\infty$-category which admits small colimits and where $\mathcal{C}^\kappa$ is small. The following are equivalent:*

1. *$\mathcal{C}$ is $\kappa$-compactly generated.*

2. *Every object of $\mathcal{C}$ can be written as small colimit of $\kappa$-compact objects.*

3. *If $X \to Y$ is a morphism in $\mathcal{C}$ such that*

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Z, Y)$$

*is an equivalence for every $\kappa$-compact $Z$, then $X \to Y$ is an equivalence.*

*Proof.* We first show $1 \Leftrightarrow 2$. Since $k$ is fully faithful, $k$ being an equivalence is equivalent to $k$ being essentially surjective. If every object $X$ of $\mathcal{C}$ can be written as colimit $\mathrm{colim}_K Z_k$ of $\kappa$-compact objects $Z_k$, we may assume it to be a $\kappa$-filtered colimit of $\kappa$-compact objects by rewriting it as $\mathrm{colim}_{K' \subseteq K} \mathrm{colim}_{K'} Z_k$, where $K'$ ranges over the $\kappa$-filtered system of $\kappa$-small simplicial subsets of $K$. If $X = \mathrm{colim}_I Z_i$ is a $\kappa$-filtered colimit of $\kappa$-compact objects, $k$ takes $\mathrm{colim}_{i \in I} j Z_i$ to $X$. Conversely, every object in $\mathrm{Ind}_{\kappa}(\mathcal{C})$ is of this form, and so if $k$ is essentially surjective, every object in $\mathcal{C}$ is a $\kappa$-filtered colimit of $\kappa$-compact objects.

For $1 \Leftrightarrow 3$, we have the restricted Yoneda embedding $j' : \mathcal{C} \to \mathrm{Ind}_{\kappa}(\mathcal{C}^{\kappa})$ taking $X \mapsto \mathrm{Map}_{\mathcal{C}}(-, X)$. (This is a $\kappa$-small limit preserving functor from $\mathcal{C}^{\kappa, \mathrm{op}} \to \mathrm{An}$, so lies in $\mathrm{Ind}_{\kappa}(\mathcal{C}^{\kappa})$.) We claim that $k$ is left adjoint to $j'$. Indeed, both $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C}^{\kappa})}(F, j'X)$ and $\mathrm{Map}_{\mathcal{C}}(kF, X)$ are $\kappa$-filtered colimit preserving functors $\mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathrm{An}^{\mathrm{op}}$ in $F$. So to produce an equivalence between them, it suffices to do so on the image of $j : \mathcal{C}^{\kappa} \to \mathrm{Ind}(\mathcal{C}^{\kappa})$, and we have $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C}^{\kappa})}(jZ, j'X) \simeq \mathrm{Map}_{\mathcal{C}}(Z, X)$ by the Yoneda lemma.

Fully faithfullness of $k$ gives that

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C}^{\kappa})}(F, G) \to \mathrm{Map}_{\mathcal{C}}(kF, kG) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(F, j'kG)$$

is an equivalence for any $F$, $G$. By Yoneda, this means that the unit $G \to j'kG$ is an equivalence. If 3 is satisfied, $j'$ detects equivalences. Since the counit $kj'X \to X$ is taken by $j'$ to the inverse equivalence to the unit $j'X \to j'kj'X$, this means that $kj'X \to X$ is an equivalence and so $k$ is essentially surjective. Conversely, if $k$ is an equivalence, of course its adjoint $j'$ is too, and so in particular it detects equivalences. $\square$

The last criterion in particular is extremely useful. For example, we directly see:

**Corollary 2.1.14.** *If $\mathcal{C}$ has small colimits and is $\kappa$-compactly generated for some $\kappa$, it is also $\kappa'$-compactly generated for some $\kappa' > \kappa$.*

**Definition 2.1.15.** We call an $\infty$-category *presentable* if it admits all small colimits, and is $\kappa$-compactly generated for some $\kappa$.

**Example 2.1.16.** If $\mathcal{D}$ is presentable and $\mathcal{C}$ is small, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presentable (one may check that if $\mathcal{C}$ is $\kappa$-small, $\kappa$-filtered colimits and $\kappa$-compact objects in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are taken pointwise).

For example, $\mathrm{An}$ is presentable, and so also $\mathrm{PShv}(X; \mathrm{An})$ is presentable for a topological space $X$, since $\mathrm{Open}(X)$ is small.

So these are big categories which can in some sense be accessed by a small category. (There is the weaker notion of accessible $\infty$-categories, where we only require existence of $\kappa$-filtered colimits instead of all.) One of the main reasons for the importance of presentable $\infty$-categories is the following:

**Theorem 2.1.17** (Adjoint functor theorem). *1. A functor $\mathcal{C} \to \mathcal{D}$ between presentable $\infty$-categories admits a right adjoint if and only if it preserves colimits.*

*2. A functor $\mathcal{D} \to \mathcal{C}$ between presentable $\infty$-categories admits a left adjoint if and only if it preserves limits and $\kappa$-filtered colimits for some $\kappa$. (The latter condition is also called* accessibility *of the functor.)*

*Proof.* [Lur17b, Corollary 5.5.2.9] □

For example, the adjoint functor theorem implies that the "diagonal" functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a right adjoint if $\mathcal{C}$ is small and $\mathcal{D}$ presentable, hence that presentable $\infty$-categories also admit small limits.

A good notion of morphisms between presentable $\infty$-categories is given by pairs of adjoints.

**Definition 2.1.18.** $\mathrm{Pr}^{\mathrm{L}}$ denotes the (big!) category whose objects are presentable $\infty$-categories, and whose morphisms are left adjoint (or colimit-preserving) functors.

Equivalently, one may define $\mathrm{Pr}^{R}$, and passage to the right adjoint gives an equivalence $\mathrm{Pr}^{L,\mathrm{op}} \simeq \mathrm{Pr}^{R}$.

To get more examples for presentable $\infty$-categories, we consider the following notion:

**Definition 2.1.19.** A (left) *Bousfield localisation* of an $\infty$-category $\mathcal{C}$ consists of a pair of adjoint functors

$$\mathcal{C} \underset{L}{\overset{R}{\longleftrightarrow}} \mathcal{D}$$

where $R$ is fully faithful.

Of course, this data is determined already by one of the two functors, by uniqueness of adjoints. It is therefore relatively easy to describe a Bousfield localisation of $\mathcal{C}$, simply by giving the full subcategory $\mathcal{D}$.

$$\mathrm{Map}_{\mathcal{D}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(RX, RY) \simeq \mathrm{Map}_{\mathcal{D}}(LRX, Y)$$

is an equivalence, so $LRX \simeq X$. An object $Y$ lies in the essential image of $R$ if and only if $Y \to RLY$ is an equivalence. If $W$ denotes the class of all morphisms in $\mathcal{C}$ which are sent to equivalences in $\mathcal{D}$ by $L$, then $\mathrm{Map}_{\mathcal{C}}(-, Y)$ takes $W$ to equivalences if and only if $Y$ is in the essential image of $R$: In one direction, this is just the equivalence, in the other, assume that $\mathrm{Map}_{\mathcal{C}}(-, Y)$ takes $W$ to equivalences, then this applies in particular to $Y \to RLY$, so the identity on $Y$ factors through $Y \to RLY$. But then $Y \to RLY \to Y$ and, using the adjunction, $RLY \to Y \to RLY$ are both the identity and $Y \simeq RLY$.

**Lemma 2.1.20.** *If*

$$\mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$$

*is a Bousfield localisation, and $W$ the class of morphisms in $\mathcal{C}$ which are sent to equivalences under $L$, then precomposition with $L$ provides an equivalence*

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc}}(\mathcal{C}, \mathcal{E})$$

*where the right hand side denotes the full subcategory on functors taking $W$ to equivalences. The equivalence also restricts to an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc},\mathrm{colim}}(\mathcal{C}, \mathcal{E}).$$

*Proof.* [Lur17b, Proposition 5.2.7.12] $\qquad\square$

This justifies the name localisation. We may similarly specify a Bousfield localisation by providing a collection of morphisms $W$ in $\mathcal{C}$ and letting $\mathcal{D}$ be the full subcategory on all $W$-local objects, i.e. $Y \in \mathcal{C}$ where $\mathrm{Map}_{\mathcal{D}}(-, Y)$ takes $W$ to equivalences.

**Lemma 2.1.21.** *If $W$ is a (small!) set of morphisms in $\mathcal{C}$, $\mathcal{C}$ is presentable, and $\mathcal{D}$ the full subcategory of $W$-local objects, then $\mathcal{D}$ is also presentable and a Bousfield localisation of $\mathcal{C}$. It is universally characterized by $L$ inducing an equivalence*

$$\mathrm{Fun}^{L}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W-\mathrm{loc},L}(\mathcal{C}, \mathcal{E})$$

*Proof.* [Lur17b, Proposition 5.5.4.2(3) and Remark 5.5.1.6] $\qquad\square$

**Example 2.1.22.** For a space $X$, $\mathrm{PShv}(X; \mathrm{An}) \supseteq \mathrm{Shv}(X; \mathrm{An})$ is a presentable Bousfield localisation: Sheaves are exactly those presheaves which are local with respect to the morphisms

1. $\emptyset \to j(\emptyset)$

2. $j(U) \amalg_{j(U \cap V)} j(V) \to j(U \cup V)$

3. $\mathrm{colim}_{i \in I} j(U_i) \to j(\bigcup_{i \in I} U_i)$

which form a set.

Note that we see from this description also that colimit-preserving functors $\mathrm{Shv}(X; \mathrm{An}) \to \mathcal{E}$ are the same as $W$-local colimit-preserving functors $\mathrm{PShv}(X; \mathrm{An}) \to \mathcal{E}$, and hence the same as functors $\mathrm{Open}(X) \to \mathcal{E}$ with $F(\emptyset)$ initial, $F(U) \amalg_{F(U \cap V)} F(V) \simeq F(U \cup V)$ and $\mathrm{colim}_I F(U_i) \simeq F(\bigcup U_i)$, i.e. cosheaves with values in $\mathcal{E}$!

In fact, every presentable $\infty$-category is more or less of that form. One has:

**Proposition 2.1.23.** *An $\infty$-category $\mathcal{C}$ is presentable if and only if it arises as a presentable Bousfield localisation of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ for some small $\infty$-category $\mathcal{C}_0$.*

*Proof.* If $\mathcal{C}$ is presentable, it is $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ for some $\kappa$. But $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ is itself a presentable Bousfield localisation of $\mathrm{Fun}(\mathcal{C}^{\kappa,\mathrm{op}}, \mathrm{An})$: Since it can be described as full subcategory on $\kappa$-small limit preserving functors, it consists of the objects local with respect to the maps

$$\mathrm{colim}_{k \in K} jX_k \to j(\mathrm{colim}_{k \in K} X_k)$$

for all $\kappa$-small diagrams in $\mathcal{C}^\kappa$. These form a set. (Or more precisely, there is a set of representatives up to equivalence.) $\qquad\square$

**Remark 2.1.24.** Exhibiting $\mathcal{C}$ as Bousfield localisation of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ is a kind of generators-and-relations presentation of $\mathcal{C}$, since it says that colimit-preserving functors out of $\mathcal{C}$ are determined by an arbitrary functor out of the small $\infty$-category $\mathcal{C}_0$ (the generators), such that its colimit extension is taking the small class $W$ (the relations) to equivalences.

**Corollary 2.1.25.** *If $\mathcal{C}$ and $\mathcal{D}$ are presentable, the category*

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

*consisting of left adjoint (i.e. colimit-preserving) functors is itself presentable.*

*Proof.* $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is clearly closed under small colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and therefore admits all small colimits. Writing $\mathcal{C}$ as presentable localisation of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ at a set of morphisms $W$, we see that
$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$$
is the full subcategory on functors whose colimit extension $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An}) \to \mathcal{D}$ takes $W$ to equivalences. If $\kappa$ is bigger than the size of $\mathcal{C}_0$ and $W$, one sees that $\kappa$-filtered colimits and $\kappa$-compact objects in are formed pointwise here. So the $\kappa$-compact objects form a small category and $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is compactly generated. $\qquad\square$

This means that $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ provides an inner Hom to the category $\mathrm{Pr}^L$. We will see later that there is a tensor product left adjoint to this Hom.

We close this discussion of presentability by an application of the adjoint functor theorem regarding generators of a category.

**Lemma 2.1.26.** *Let $\mathcal{C}$ be a presentable $\infty$-category and $S$ a set of objects in $\mathcal{C}$. Then the following are equivalent:*

1. *The smallest full subcategory of $\mathcal{C}$ closed under small colimits and containing $S$ is $\mathcal{C}$ itself.*

2. *If $X \to Y$ is a morphism such that $\mathrm{Map}_\mathcal{C}(Z, X) \to \mathrm{Map}_\mathcal{C}(Z, Y)$ is an equivalence for each $Z \in S$, then $X \to Y$ is an equivalence.*

*Proof.* Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the smallest full subcategory of $\mathcal{C}$ closed under colimits and containing $S$. Then $\mathcal{C}_0$ has arbitrary small colimits, and is $\kappa$-compactly generated where $\kappa$ is such that all objects of $S$ are $\kappa$-compact. So it is presentable, and $i: \mathcal{C}_0 \to \mathcal{C}$ has a right adjoint $R$.

Since $i$ is fully faithful, $X \to RiX$ is an equivalence for each $X \in \mathcal{C}_0$, and $Y \in \mathcal{C}$ lies in $\mathcal{C}_0$ if and only if $iRY \to Y$ is an equivalence. Now assume 2, this implies that $R$ is conservative. But $R(iRY \to Y)$ is the inverse to the unit $RY \to RiRY$, so an equivalence, and so every $Y$ is in the image and $\mathcal{C}_0 = \mathcal{C}$. Conversely, if 1 holds, $i : \mathcal{C}_0 \to \mathcal{C}$ is an equivalence, so $R$ is. If $X \to Y$ induces an equivalence on $\mathrm{Map}_\mathcal{C}(Z, -)$ for all $Z \in S$, it does so for all $Z \in \mathcal{C}_0 = \mathcal{C}$, and Yoneda applies. $\qquad\square$

If the equivalent conditions of the Lemma hold, we say that $S$ *generates* $\mathcal{C}$. For example, $\mathcal{C}$ is $\kappa$-compactly generated if and only if $\mathcal{C}^\kappa$ generates $\mathcal{C}$.

**Lemma 2.1.27.** *If for any $Z \in S$, also $Z \otimes S^n = \mathrm{colim}_{S^n} Z \in S$ (for example if $S$ is closed under finite colimits), then $S$ generates $\mathcal{C}$ if and only if the following holds: For any morphism $X \to Y$ in $\mathcal{C}$ where in each diagram*

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

*with $A, B \in S$, the dashed lift exists, $X \to Y$ is an equivalence.*

*Proof.* If $\mathcal{C}$ is generated by $S$, maps out of $S$ detect equivalences. Given $X \to Y$ with the lifting condition, it therefore suffices that $\mathrm{Map}_\mathcal{C}(A, X) \to \mathrm{Map}_\mathcal{C}(A, Y)$ is an equivalence for any $A \in S$. The lifting problem for the diagram

$$
\begin{array}{ccc}
A \otimes S^{n-1} & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
A & \longrightarrow & Y
\end{array}
$$

translates to a lifting problem for the diagram

$$
\begin{array}{ccc}
S^{n-1} & \longrightarrow & \mathrm{Map}_\mathcal{C}(A, X) \\
\downarrow & \nearrow & \downarrow \\
\mathrm{pt} & \longrightarrow & \mathrm{Map}_\mathcal{C}(A, Y).
\end{array}
$$

If such lifts exist always, this means that all relative homotopy groups of the pair $\mathrm{Map}_\mathcal{C}(A, X) \to \mathrm{Map}_\mathcal{C}(A, Y)$ are trivial, hence that this map is an equivalence.

Conversely, assume the lifting condition detects equivalences, and we need to prove that then $\mathcal{C}$ is generated by $S$. So we need to prove that the $\mathrm{Map}_\mathcal{C}(A, -)$ together detect equivalences. Let $X \to Y$ be a morphism inducing equivalences on all $\mathrm{Map}_\mathcal{C}(A, -)$. Since a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

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is a point in the pullback $\mathrm{Map}(B, Y) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(A, X)$, but

$$
\begin{array}{ccc}
\mathrm{Map}(B, X) & \longrightarrow & \mathrm{Map}(A, X) \\
\downarrow & & \downarrow \\
\mathrm{Map}(B, Y) & \longrightarrow & \mathrm{Map}(A, Y)
\end{array}
$$

is a pullback diagram since the vertical maps are equivalences, any such square admits a lift $B \to X$. But this means that $X \to Y$ satisfies the lifting condition, and so $X \to Y$ is an equivalence. $\qquad \square$

## 2.2 Compactly assembled $\infty$-categories

Recall that an object $X$ in an $\infty$-category $\mathcal{C}$ is called compact, if the functor

$$
\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{An}
$$

commutes with filtered colimits. Here the convention is that if we drop the cardinal $\kappa$ then it is always implicitly assumed to be $\omega$.

Let us instead call an object *weakly compact* if every map $X \to \mathrm{colim}_{i \in I} Z_i$ factors through finite stage $Z_i$, or equivalently:

**Definition 2.2.1.** $X \in \mathcal{C}$ is called *weakly compact* if

$$
\pi_0 \mathrm{colim}_{i \in I} \mathrm{Map}_{\mathcal{C}}(X, Z_i) \to \pi_0 \mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim}_{i \in I} Z_i)
$$

is surjective for any filtered diagram $Z : I \to \mathcal{C}$.

**Lemma 2.2.2.** *If filtered colimits in $\mathcal{C}$ commute with finite limits, weakly compact objects are compact.*

*Proof.* Write $Z = \mathrm{colim}_{i \in I} Z_i$. If $X \to Z_i$ and $X \to Z_j$ are two maps lifting the same $X \to Z$, they provide a map $X \to Z_i \times_Z Z_j$. Writing this as a filtered colimit of $Z_i \times_{Z_k} Z_j$ (over the $k \in I$ with $i, j \to k$, we see that both maps become homotopic in some $Z_k$. So the map is actually bijective on $\pi_0$. Now if we inductively know that

$$
\mathrm{colim}_{i \in I} \mathrm{Map}_{\mathcal{C}}(X, Z_i) \to \mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim}_{i \in I} Z_i)
$$

is an equivalence on $\pi_k$, for $k \leq n$ and *any* $Z_i$, then for any $f : X \to Z_i$, we may form $Z_j' = \mathrm{eq}(\ X \rightrightarrows Z_j\ )$ (indexed over $I_{i/}$), and since

$$
\mathrm{Map}_{\mathcal{C}}(X, Z_j') \simeq \mathrm{Map}_{\mathcal{C}}(X, X) \times_{\Omega_f} \mathrm{Map}_{\mathcal{C}}(X, Z_j),
$$

one deduces that the map for $Z$ is even an equivalence on $\pi_k$ for $k \leq n + 1$. $\qquad \square$

We now would like to formulate a corresponding notion for morphisms.

**Definition 2.2.3.** A morphism $f : X \to Y$ in an $\infty$-category is called *weakly compact*, if for every morphism $Y \to Z = \mathrm{colim}_{i \in I} Z_i$ where $I$ is filtered the composite $X \to Y \to \mathrm{colim}_{i \in I} Z_i$ factors over a finite stage $Z_{i_0} \to Z$.

One could ask for a more structured version of this akin to the definition of compact objects, i.e. that such a factorisation exists in families of maps in some sense.

**Definition 2.2.4.** A morphism $f : X \to Y$ is *strongly compact* if for every filtered colimit $Z = \mathrm{colim}_{i \in I} Z_i$ there exists a lift as indicated:

$$
\begin{array}{ccc}
\mathrm{colim}_i \, \mathrm{Map}_{\mathcal{C}}(Y, Z_i) & \xrightarrow{\ f^*\ } & \mathrm{colim}_i \, \mathrm{Map}_{\mathcal{C}}(X, Z_i) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{C}}(Y, Z) & \xrightarrow{\ \ f^* \ \ } & \mathrm{Map}_{\mathcal{C}}(X, Z)
\end{array}
$$

Note that this lift here is of course up to homotopy, so in the $\infty$-category of anima.

Clearly strongly compact implies weakly compact: In the definition of compact we just ask for such a lift on a single point of $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ and ignore the upper triangle. The converse is (probably) not true in general, even if filtered colimits in $\mathcal{C}$ commute with finite limits. We will however develop below the notion of *compactly assembled* categories, and somewhat surprisingly will see that in those categories the two notions coincide.

**Example 2.2.5.** 1. Assume that a morphism $X \to Y$ factors over a weakly compact object $K \in \mathcal{C}$, i.e. is of the form $X \to K \to Y$. Then it is weakly compact. To see this we simply observe that for a given morphisms $Y \to Z = \mathrm{colim}_{i \in I} Z_i$ the composition $K \to Y \to Z$ already has to factor over a finite stage by weak compactness of $K$. In fact, if $K$ is compact, $X \to Y$ is even strongly compact, since we can get a lift in the diagram by considering

$$
\begin{array}{ccccc}
\mathrm{colim}_i \, \mathrm{Map}_{\mathcal{C}}(Y, Z_i) & \longrightarrow & \mathrm{colim}_i \, \mathrm{Map}_{\mathcal{C}}(K, Z_i) & \longrightarrow & \mathrm{colim}_i \, \mathrm{Map}_{\mathcal{C}}(X, Z_i) \\
\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow \\
\mathrm{Map}_{\mathcal{C}}(Y, Z) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(K, Z) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, Z)
\end{array}
$$

and noting that the morphism in the middle is an equivalence, so we get a lift by following the inverse of this morphism.

2. If $K$ is compactly generated, then the converse is also true, namely that the compact morphisms agree with strongly compact morphisms and are precisely those which factor over a compact object. To see this let $f : X \to Y$ be weakly compact. We write $Y = \mathrm{colim}_{i \in I} Y_i$ as a filtered colimit of compact objects. Then the map $f : X \to Y$ factors by definition of compactness as $X \to Y_i \to Y$ and this gives the desired factorization.

The whole idea of compact morphisms is to generalize the previous example to the non compactly generated case. We will see that there are many $\infty$-categories which don't have many compact objects, but a lot of compact morphisms.

**Example 2.2.6.** Consider the category $\text{Open}(X)$ of open subsets of a locally compact Hausdorff topological space $X$.[1] Clearly the object $X$ is (weakly or strongly) compact in this category precisely if $X$ is a compact topological space (and more generally an open $U$ is compact if and only if it is compact as a topological space). We claim that a morphism

$$U \subseteq V$$

in $\text{Open}(X)$ is weakly compact precisely if there exists a compact subset $K \subseteq X$ with $U \subseteq K \subseteq V$.

Assume such a $K$ exists. Then for every morphism $V \subseteq \bigcup W_i$ with $W_i$ a filtered system of opens in $X$ we find an $i_0$ such that $K$ and thus also $U$ is already contained in $W_{i_0}$.[2] Assume conversely that $U \subseteq V$ is weakly compact and write $V$ as a filtered union of compact subspaces (which is possible by the assertion that $X$ is locally compact). Then $U$ already lies in a finite stage and this gives us our $K$.

We have the following easy assertions:

**Lemma 2.2.7.** *1. An object $X \in \mathcal{C}$ is weakly/strongly compact iff the identity $X \to X$ is weakly/strongly compact.*

*2. If $f : X \to Y$ is weakly/strongly compact then for arbitrary morphisms $W \to X$ and $Y \to Z$ the composition $W \to X \to Y \to Z$ is also weakly/strongly compact.*

*Proof.* 1. If $X$ is weakly/strongly compact, $X \to X$ is weakly/strongly compact. It remains to show that if $X \to X$ is weakly/strongly compact, $X$ is weakly/strongly compact. For the weak statement, observe that we directly see that any $X \to \text{colim}\, Z_i$ factors through a finite stage, and for the strong statement, consider the diagram

$$
\begin{array}{ccc}
\text{colim}\,\text{Map}_{\mathcal{C}}(X, Z_i) & \longrightarrow & \text{colim}\,\text{Map}_{\mathcal{C}}(X, Z_i) \\
\downarrow & \nearrow & \downarrow \\
\text{Map}_{\mathcal{C}}(X, \text{colim}\, Z_i) & \longrightarrow & \text{Map}_{\mathcal{C}}(X, \text{colim}\, Z_i)
\end{array}
$$

which encodes directly that the dashed map is a homotopy inverse to the vertical map.

2. Given $Z \to \text{colim}_I U_i$, by weak compactness of $X \to Y$ we find a lift $X \to U_i$ of the map $X \to Y \to Z \to \text{colim}_I U_i$. Precomposing with $W \to X$, we get the desired lift of $W \to Z \to \text{colim}_I U_i$. The strong statement is similarly obtained by composing diagrams. $\qquad\square$

**Definition 2.2.8.** We say that an object $X$ of an $\infty$-category $\mathcal{C}$ is called weakly/strongly *compactly exhaustible* if it can be written as a sequential colimit

$$X = \text{colim}\,(X_0 \to X_1 \to X_2 \to ...)$$

where all the transition maps $X_i \to X_{i+1}$ are weakly/strongly compact.

---

[1]This is a presentable $\infty$-category!

[2]For this direction we don't need locally compact and Hausdorff

**Example 2.2.9.** Consider the category $\mathrm{Open}(X)$ of a topological space $X$. An object $U$ in this category is weakly or strongly compactly exhaustible iff the space $U$ is compactly exhaustible in the usual sense of topology, that is there exists a sequence of compact topological subspaces $K_0 \subseteq K_1 \subseteq ... \subseteq U$ with union $U$ and such that each $K_i$ is contained in the interior of $K_{i+1}$.

**Example 2.2.10** (Almost mathematics)**.** Assume $A$ is a local ring with maximal ideal $\mathfrak{m} \subseteq A$ with $\mathfrak{m} \otimes_A^L \mathfrak{m} = \mathfrak{m}$ (for example $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathfrak{m}$ flat). The kernel of

$$\mathrm{Mod}(A) \to \mathrm{Mod}(A/\mathfrak{m})$$

forms a full subcategory $\mathrm{aMod}_{\mathfrak{m}}(A)$ closed under colimits, generated by $\mathfrak{m}$. Compact objects in $\mathrm{aMod}_{\mathfrak{m}}(A)$ are exactly the ones which are finitely presented as modules. By Nakayama, these are all trivial, so there are no nonzero compact objects. However, in the example $A = \mathbb{Z}_p[p^{1/p^\infty}]$ any of the inclusions $p^v\mathfrak{m} \to \mathfrak{m}$ with $v > 0$ factors through a finitely generated free module $p^v A$, and so is a (strongly) compact morphism. In particular, we may write $\mathfrak{m}$ as colimit of

$$p\mathfrak{m} \subseteq p^{1/p}\mathfrak{m} \subseteq p^{1/p^2}\mathfrak{m} \subseteq \dots,$$

so $\mathrm{aMod}_{\mathfrak{m}}(A)$ is generated by (strongly) compactly exhaustibles.

Clearly every compact object is (strongly) compactly exhaustible, but the converse does not hold. We now can state the main result about compactly assembled $\infty$-categories:

**Theorem 2.2.11** (Clausen, Lurie)**.** *For a presentable $\infty$-category $\mathcal{C}$ the following are equivalent:*

1. *$\mathcal{C}$ is generated under colimits by strongly compactly exhaustible objects*

2. *Filtered colimits in $\mathcal{C}$ are exact and $\mathcal{C}$ is generated under colimits by weakly compactly exhaustible objects*

3. *The colimit functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint*

4. *$\mathcal{C}$ is $\omega_1$-compactly generated and the colimit functor $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a left adjoint*

5. *$\mathcal{C}$ is a retract in $\mathrm{Pr}^{\mathrm{L}}$ of a compactly generated $\infty$-category.*

6. *Filtered colimits in $\mathcal{C}$ distribute over small limits, i.e. we have*

$$\lim_K \mathrm{colim}_I F \simeq \mathrm{colim}_{I^K} \lim_K F$$

*for $K$ arbitrary and $I$ filtered.* [3]

---

[3]Equivalently, it suffices to ask that filtered colimits *commute* with finite limits and distribute over small products. The former is a version of Grothendieck's AB5 axiom and the latter is a version of Grothendieck's AB6 axiom.

Here $\mathrm{Ind}(\mathcal{C})$ of the (locally small) category $\mathcal{C}$ is as in the small case just the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ generated by representables under small filtered colimits. The universal property holds as in the small case.

We will prove this result in the next section. But for the moment let us draw some corollaries and give some examples.

**Definition 2.2.12** (Lurie, Clausen). An $\infty$-category is called *compactly assembled* if it is presentable and satisfies the equivalent conditions of Theorem 2.2.11.

Note that by Theorem 2.2.11 every compactly assembled $\infty$-category is $\omega_1$-compactly generated. The converse is not true, but we have:

**Example 2.2.13.** Every compactly generated $\infty$-category is compactly assembled. This follows by Theorem 2.2.11(5).

**Example 2.2.14.** The partially ordered set $[0,1]$ has all suprema and is therefore presentable. Its only compact object is 0, but it is compactly assembled: Every "positive length" morphism is compact. Observe that Theorem 2.2.11 says that $[0,1]$ must be a retract of a compactly generated category. Indeed, if $C$ is the Cantor set, there is a surjective continuous increasing map $f : C \to [0,1]$, and an increasing map $g : [0,1] \to C$ with $g(x) = \inf f^{-1}(x)$. Both preserve suprema and are therefore morphisms in $\mathrm{Pr}^{\mathrm{L}}$, and $f \circ g = \mathrm{id}$. Finally, if we think of the Cantor set as decimal numbers in base 3 all of whose digits are 0 or 2, compact objects are exactly the ones that end in infinitely many 0's, and these are dense, so $C$ is compactly generated.

**Remark 2.2.15.** A poset $P$ that is compactly assembled as a category is classically called a continuous poset, see [**?**]. In this case one says if $x < y$ is compact that $x$ is way below $y$.

This inspired Joyal and Johnston's [**?**] 1-categorical treatment of compactly assembled ordinary categories, which they call continuous categories (and drop the presentability condition).

**Example 2.2.16.** The category of sheaves $\mathrm{Shv}(X, \mathrm{An})$ of anima on a locally compact Hausdorff space is compactly assembled. We first claim that if $U \subseteq V$ is an inclusion of open subsets of $X$ with a compact $U \subseteq K \subseteq V$ in between, we have for a filtered colimit $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$ in $\mathrm{Shv}(X)$:

$$
\begin{array}{ccccc}
\mathrm{colim}_i \mathrm{Map}(\underline{V}, \mathcal{F}_i) & \longrightarrow & \mathrm{colim}_i \Gamma(K; \mathcal{F}_i|_K) & \longrightarrow & \mathrm{colim}_i \mathrm{Map}(\underline{U}, \mathcal{F}_i) \\
\downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow \\
\mathrm{Map}(\underline{V}, \mathcal{F}) & \longrightarrow & \Gamma(K; \mathcal{F}|_K) & \longrightarrow & \mathrm{Map}(\underline{U}, \mathcal{F})
\end{array}
$$

This immediately shows that $\underline{U} \to \underline{V}$ is strongly compact. As in example 2.2.9, this shows that $\underline{U}$ for any compactly exhaustible open is weakly compactly exhaustible. In a locally compact Hausdorff space, every point admits a neighbourhood basis of compactly exhaustible opens (this inductively uses that if we have a compact $K$ contained in an open $U$ with compact closure we may squeeze an open $V$ with $K \subseteq V$ and $\overline{V} \subseteq U$ in between). So $\underline{U}$ for any open is a colimit of strongly compactly exhaustible objects, and the category $\mathrm{Shv}(X, \mathrm{An})$ is generated by strongly compactly exhaustible objects.

**Example 2.2.17.** On the other hand, for a Hausdorff space $X$, $\mathrm{Shv}(X)$ is typically *not* compactly generated. The functor $\mathrm{Open}(X) \to \mathrm{Shv}(X)$, $U \mapsto \underline{U}$, is accessible and limit-preserving, so it has a left adjoint $\mathrm{Shv}(X) \to \mathrm{Open}(X)$, which takes a sheaf $\mathcal{F}$ to the union of all opens $U$ where $\mathcal{F}(U)$ is nonempty. As $\mathrm{Open}(X) \to \mathrm{Shv}(X)$ commutes with filtered colimits, $\mathrm{Shv}(X) \to \mathrm{Open}(X)$ preserves compact objects. So if $\mathrm{Shv}(X)$ is compactly generated, so is $\mathrm{Open}(X)$, But $U \in \mathrm{Open}(X)$ is compact only if $U$ is compact. If every open is a union of open and compact subspaces, this means that $X$ is locally profinite. Conversely, if $X$ is locally profinite, every open can be written as union of open and compact subspaces, and so $\mathrm{Shv}(X)$ is generated by $\underline{U}$ for such subspaces, which give compact objects.

## 2.3 Proof of Theorem 2.2.11

In this section, we prove Theorem 2.2.11. We will roughly follow the strategy in the following graph:

$$(1) \Rightarrow (2)$$
$$\nwarrow \quad \not\swarrow$$
$$(3) \Leftrightarrow (6)$$
$$\nearrow \quad \searrow {\scriptstyle \text{using } (2)}$$
$$(5) \Leftarrow (4)$$

### 2.3.1 $1 \Rightarrow 2$

**Lemma 2.3.1.** *Assume that $X \in \mathcal{C}$ is strongly compactly exhaustible as witnessed by a sequential colimit $X = \mathrm{colim}_{\mathbb{N}} X_n$. Let $Y$ be the colimit of an arbitrary filtered diagram $Y = \mathrm{colim}_I Y_i$. Then we have that the colimit functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence*

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{\mathbb{N}} jX_n, \mathrm{colim}\, jY_i) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(X, Y).$$

*Proof.* We have that $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is given as the inverse limit of

$$\ldots \to \mathrm{Map}_{\mathcal{C}}(X_2, Y) \to \mathrm{Map}_{\mathcal{C}}(X_1, Y) \to \mathrm{Map}_{\mathcal{C}}(X_0, Y) \ .$$

We can now use strong compactness of the morphisms to factor each of the maps in this diagram as

$$\mathrm{Map}_{\mathcal{C}}(X_{n+1}, Y) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_n, Y_i) \to \mathrm{Map}_{\mathcal{C}}(X_n, Y).$$

Thus the limit agrees with the limit over

$$\ldots \to \mathrm{Map}(X_2, Y) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_1, Y_i) \to \mathrm{Map}(X_1, Y) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_0, Y_i) \to \mathrm{Map}(X_0, Y)$$

which in turn agrees with the limit over

$$\ldots \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_1, Y_i) \to \mathrm{colim}_i \mathrm{Map}_{\mathcal{C}}(X_0, Y_i)$$

and thus the mapping space in the Ind-category. $\qquad\square$

**Corollary 2.3.2.** *For a strongly compactly exhaustible object $X$ the presentation as as a sequential colimit $X = \operatorname{colim} X_i$ is unique as an ind-object.*

*Proof.* For two presentations we lift the identity to isomorphisms using the previous lemma. □

**Lemma 2.3.3** ($1 \Rightarrow 2$ in Theorem 2.2.11)**.** *If $\mathcal{C}$ is generated by strongly compactly exhaustible objects, it is generated by weakly compactly exhaustible objects and filtered colimits in $\mathcal{C}$ are exact.*

*Proof.* Since strongly compactly exhaustible objects are in particular weakly compactly exhaustible, the only nontrivial implication to show is that filtered colimits in $\mathcal{C}$ are exact. If $K$ is finite and $I$ is filtered, and $F : K \times I \to \mathcal{C}$ some functor, we need to show that

$$\operatorname{colim}_I \lim_K F \to \lim_K \operatorname{colim}_I F$$

is an equivalence. Since strongly compactly exhaustible objects generate $\mathcal{C}$ by assumption, it suffices to show that the above map induces an equivalence on $\operatorname{Map}_{\mathcal{C}}(X, -)$ for $X = \operatorname{colim}_{\mathbb{N}} X_n$ strongly compactly exhausted.

By Lemma 2.3.2, this yields the map

$$\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{n \in \mathbb{N}} j X_n, \operatorname{colim}_{i \in I} j \lim_{k \in K} F(k, i)) \to \lim_{k \in K} \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{n \in \mathbb{N}} j X_n, \operatorname{colim}_{i \in I} j F(k, i))$$

which evaluates to

$$\lim_{n \in \mathbb{N}} \operatorname{colim}_{i \in I} \lim_{k \in K} \operatorname{Map}_{\mathcal{C}}(X_n, F(k, i)) \to \lim_{n \in \mathbb{N}} \lim_{k \in K} \operatorname{colim}_{i \in I} \operatorname{Map}_{\mathcal{C}}(X_n, F(k, i))$$

which is an equivalence since in An, filtered colimits commute with finite limits. □

### 2.3.2 $2 \Rightarrow 3$

In order to prove the existence of a left adjoint of the colimit functor $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$, we will see that it suffices to establish the analogue of for objects which are weakly compactly exhausted, under the additional assumption that in $\mathcal{C}$ filtered colimits are exact.

For that, we will first recast the definition of weakly compact morphisms in terms of Ind, and introduce a variant.

**Lemma 2.3.4.** *A morphism $X \to Y$ in $\mathcal{C}$ is weakly compact if and only if for each $Z \in \operatorname{Ind}(\mathcal{C})$ and any map $Y \to kZ$, we have a lift in the following diagram.*

$$
\begin{array}{ccc}
jX & \dashrightarrow & Z \\
\downarrow & & \downarrow \\
jY & \longrightarrow & jkZ.
\end{array}
$$

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*Proof.* If $Z = \text{colim}_I\, jZ_i$, $kZ = \text{colim}_I Z_i$. Since $jX$ is compact in $\text{Ind}(\mathcal{C})$, such a factorisation is exactly the same as a finite stage $i \in I$ and a factorisation

$$
\begin{array}{ccc}
X & \dashrightarrow & Z_i \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \text{colim}_I Z_i.
\end{array}
$$

in $\mathcal{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Definition 2.3.5.** Let $K$ be some simplicial set, and $X, Y \in \text{Fun}(K, \mathcal{C})$. We call $X \to Y$ *locally weakly compact* if for any $Z \in \text{Fun}(K, \text{Ind}(\mathcal{C}))$ we have a factorisation

$$
\begin{array}{ccc}
jX & \dashrightarrow & Z \\
\downarrow & & \downarrow \\
jY & \longrightarrow & jkZ
\end{array}
$$

in $\text{Fun}(K, \text{Ind}(\mathcal{C}))$.

**Remark 2.3.6.** If $Z$ is of the form $\text{colim}_I\, jZ_i$, i.e. represented by a diagram $\text{Fun}(K \times I, \mathcal{C})$ with $I$ filtered, then

$$
\text{Map}_{\text{Fun}(K, \text{Ind}(\mathcal{C}))}(jX, Z) = \text{colim}_{j \in \text{Fun}(K, I)} \text{Map}_{\text{Fun}(K, \mathcal{C})}(X, Z_{j(-)})
$$

where $Z_{j(-)}$ denotes the composite $K \xrightarrow{j, Z} I \times \text{Fun}(I, \mathcal{C}) \to \mathcal{C}$.

So local compactness here means that $X \to Y \to \text{colim}_I Z_i$ factors "locally" through a finite stage.

**Lemma 2.3.7.** *If $X \to Y$ in $\text{Fun}(K, \mathcal{C})$ is locally weakly compact, then $\text{colim}_K X \to \text{colim}_K Y$ is weakly compact.*

*Proof.* We need to prove that for any $Z \in \text{Ind}(\mathcal{C})$ we have a dashed lift in

$$
\begin{array}{ccc}
j\,\text{colim}_K X & \dashrightarrow & Z \\
\downarrow & & \downarrow \\
j\,\text{colim}_K Y & \longrightarrow & jkZ.
\end{array}
$$

By adjunction, this is the same as a dashed lift in

$$
\begin{array}{ccc}
jX & \dashrightarrow & \text{const}\, Z \\
\downarrow & & \downarrow \\
jY & \longrightarrow & j\,\text{const}\, kZ,
\end{array}
$$

which is a special case of local weak compactness. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Lemma 2.3.8.** *Assume in $\mathcal{C}$, filtered colimits are exact. If $K$ is $n$-dimensional and $X \to Y$ in $\mathrm{Fun}(K, \mathcal{C})$ is the composite of $n+1$ pointwise weakly compact maps, $X \to Y$ is locally weakly compact.*

*Proof.* We prove the following inductive version: Assume we have a map $X \to Y$ in $\mathrm{Fun}(K, \mathcal{C})$ and already a lift

$$
\begin{array}{ccc}
jX & \dashrightarrow & Z \\
\downarrow & & \downarrow \\
jY & \longrightarrow & j\operatorname{colim} Z
\end{array}
$$

in $\mathrm{Fun}(K^{(n-1)}, \mathrm{Ind}(\mathcal{C}))$, i.e. on the $(n-1)$-skeleton. Then we claim that after precomposing with a pointwise weakly compact map $X' \to X$, we obtain a lift on all of $K$. The obstruction to extending the above lift over $K$, i.e. over the remaining $n$-simplices, is as follows. For an $n$-simplex $k_0 \to \ldots \to k_n$ in $K$, we have an $S^{n-2}$ worth of maps $jX_{k_0} \to \operatorname{colim} jZ_{k_n}$, with a provided homotopy identifying them on the colimit, i.e. a map

$$
jX_{k_0} \to (\operatorname{colim} jZ_{k_n})^{S^{n-2}} \times_{(\operatorname{colim} Z_{k_n})^{S^{n-2}}} \operatorname{colim} Z_{k_n},
$$

which we need factor through $\operatorname{colim} jZ_{k_n}$. We may write this as a single map

$$
X_{k_0} \to Z_{k_n,i}^{S^{n-2}} \times_{(\operatorname{colim} Z_{k_n})^{S^{n-2}}} \operatorname{colim} Z_{k_n} = \operatorname{colim}_{i \to j} Z_{k_n,i}^{S^{n-2}} \times_{Z_{k_n,j}^{S^{n-2}}} Z_{k_n,j}.
$$

for some $i$. If we precompose with a levelwise compact map $X' \to X$, we obtain a factorisation

$$
X'_{k_0} \to Z_{k_n,i}^{S^{n-2}} \times_{Z_{k_n,j}^{S^{n-2}}} Z_{k_n,j} \to Z_{k_n,j}
$$

for a single $j$, which factors the obstruction map

$$
jX'_{k_0} \to (\operatorname{colim} jZ_{k_n})^{S^{n-2}} \times_{(\operatorname{colim} Z_{k_n})^{S^{n-2}}} \operatorname{colim} Z_{k_n}
$$

through $\operatorname{colim} jZ_{k_n}$, proving existence of the desired lift.

By induction on the skeleta of $K$, this proves the full statement. $\square$

**Corollary 2.3.9.** *Assume in $\mathcal{C}$, filtered colimits are exact, and $X_\bullet \in \mathrm{Fun}(\mathbb{N}, \mathcal{C})$ is a sequential diagram of weakly compact morphisms. Then*

$$
\pi_0 \operatorname{Map}_{\mathrm{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, Y) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_{\mathbb{N}} X_n, kY)
$$

*is an equivalence for any ind-object $Y$.*

*Proof.* Write $Y = \operatorname{colim}_I jY_i \in \mathrm{Ind}(\mathcal{C})$, and let $\operatorname{const} Y \in \mathrm{Fun}(\mathbb{N}, \mathrm{Ind}(\mathcal{C}))$ be the constant diagram. We also have $jX_\bullet \in \mathrm{Fun}(\mathbb{N}, \mathrm{Ind}(\mathcal{C}))$. The map $jX_{\bullet - 1} \to jX_\bullet$ is pointwise compact, and since $\mathbb{N}$ is equivalent to a 1-dimensional diagram, $jX_{\bullet - 2} \to jX_\bullet$ is locally compact in $\mathrm{Fun}(\mathbb{N}, \mathrm{Ind}(\mathcal{C}))$. So we have a lift in the diagram

$$
\begin{array}{ccc}
jX_{\bullet - 2} & \dashrightarrow & \operatorname{const} Y \\
\downarrow & & \downarrow \\
jX_\bullet & \longrightarrow & \operatorname{const} jkY,
\end{array}
$$

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where the bottom map comes from the map $X_\bullet \to kY = \text{colim}_I Y_i$. The top map corresponds to a map $\text{colim}_{\mathbb{N}} jX_{\bullet-2} \to Y$ in $\text{Ind}(\mathcal{C})$. Since $\text{colim}_{\mathbb{N}} jX_{\bullet-2} \to \text{colim}_{\mathbb{N}} jX_\bullet$ is an equivalence, this proves surjectivity.

For injectivity, we work with $\text{Fun}(\mathbb{N} \times \Delta^1, \text{Ind}(\mathcal{C}))$. Giving two maps $\text{colim}_{\mathbb{N}} jX_\bullet \to Y$ in $\text{Ind}(\mathcal{C})$ lifting $\text{colim}_{\mathbb{N}} X_\bullet \to kY = \text{colim}_I Y_i$ corresponds to a dashed lift in

$$
\begin{array}{ccc}
jX_\bullet & \dashrightarrow & \text{const}\, Y \\
\downarrow & & \downarrow \\
jX_\bullet & \longrightarrow & \text{const}\, jkY,
\end{array}
$$

on restrictions to $\mathbb{N} \times \partial\Delta^1$. As in the proof of Lemma 2.3.8, this lift extends to all of $\mathbb{N} \times \Delta^1$ after precomposing with $jX_{\bullet-2} \to jX_\bullet$. Under the adjunction between $\text{colim}_{\mathbb{N}}$ and $\text{const}$, this yields that any pair of maps $\text{colim}_{\mathbb{N}} jX_\bullet \to Y$ lifting the given $\text{colim}_{\mathbb{N}} X_\bullet \to kY$ is homotopic (since $jX_{\bullet-2} \to jX_\bullet$ is an equivalence under $\text{colim}_{\mathbb{N}}$). $\qquad\square$

**Lemma 2.3.10.** *If $X = \text{colim}\, X_n$ is weakly compactly exhausted and filtered colimits are exact in $\mathcal{C}$,*

$$
\text{Map}_{\text{Ind}(\mathcal{C})}(\text{colim}_{\mathbb{N}} jX_n, Y) \to \text{Map}_{\mathcal{C}}(\text{colim}_{\mathbb{N}} X_n, kY)
$$

*is an equivalence for any Ind-object $Y$.*

*Proof.* We claim the following general fact: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving finite limits, and let $X \in \mathcal{C}$ be an object such that

$$
F : \pi_0 \text{Map}_{\mathcal{C}}(X, Y) \to \pi_0 \text{Map}_{\mathcal{D}}(F(X), F(Y))
$$

is an isomorphism for all $Y \in \mathcal{C}$. Then

$$
F : \text{Map}_{\mathcal{C}}(X, Y) \to \text{Map}_{\mathcal{D}}(F(X), F(Y))
$$

is an equivalence for all $Y \in \mathcal{C}$. To see this, we prove inductively that

$$
F : \pi_i(\text{Map}_{\mathcal{C}}(X, Y); f) \to \pi_i(\text{Map}_{\mathcal{D}}(F(X), F(Y)); f)
$$

is an isomorphism for all $Y$, $f$ and $i$. Assume this is known for all $Y$, $f$ and $i \leq n$. Then for some $Y$ and $f$ consider $Y' = \text{eq}(f, f : X \to Y)$. Since

$$
\Omega_f \text{Map}_{\mathcal{C}}(X, Y) \to \text{Map}_{\mathcal{C}}(X, Y') \to \text{Map}_{\mathcal{C}}(X, X)
$$

is a split fiber sequence (and same for the corresponding sequence in $\mathcal{D}$), we have that $F$ gives a diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{n+1}(\text{Map}_{\mathcal{C}}(X, Y); f) & \longrightarrow & \pi_n(\text{Map}_{\mathcal{C}}(X, Y')) & \longrightarrow & \pi_n(\text{Map}_{\mathcal{C}}(X, X)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_{n+1}(\text{Map}_{\mathcal{D}}(FX, FY); f) & \longrightarrow & \pi_n(\text{Map}_{\mathcal{D}}(FX, FY')) & \longrightarrow & \pi_n(\text{Map}_{\mathcal{D}}(FX, FX)) & \longrightarrow & 0.
\end{array}
$$

By assumption, the right vertical maps are isomorphisms, so also the left vertical map.

Since by Corollary 2.3.9, the $\pi_0$ condition is satisfied in the present situation, the theorem follows. $\square$

**Corollary 2.3.11.** *If in $\mathcal{C}$ filtered colimits are exact, a morphism $X \to Y$ which factors as*

$$X = X_0 \to X_1 \to \ldots \to Y$$

*with all $X_n \to X_{n+1}$ weakly compact (i.e. into "infinitely many weakly compact morphisms"), is strongly compact.*

*Proof.* Let $Z = \operatorname{colim}_{i \in I} j Z_i$ be some Ind-object. For some $Y \to kZ$, we have

$$
\begin{array}{ccc}
\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(jY, Z) \longrightarrow & \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim} jX_n, Z) \longrightarrow & \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(jX, Z) \\
\downarrow & \downarrow{\scriptstyle\simeq} & \downarrow \\
\operatorname{Map}_{\mathcal{C}}(Y, kZ) \longrightarrow & \operatorname{Map}_{\mathcal{C}}(\operatorname{colim} X_n, Z) \longrightarrow & \operatorname{Map}_{\mathcal{C}}(X, kZ)
\end{array}
$$

$\square$

**Corollary 2.3.12.** *If in $\mathcal{C}$ filtered colimits are exact, weakly compactly exhaustible objects are $\omega_1$-compact.*

*Proof.* If $X = \operatorname{colim} X_n$ is weakly compactly exhausted, and $\operatorname{colim}_I Y_i$ is an $\omega_1$-filtered colimit, we have

$$
\begin{aligned}
& \operatorname{Map}_{\mathcal{C}}(X, \operatorname{colim}_I Y_i) \\
= {} & \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, \operatorname{colim}_I jY_i) \\
= {} & \operatorname{colim}_I \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\operatorname{colim}_{\mathbb{N}} jX_n, jY_i) \\
= {} & \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(X, Y_i)
\end{aligned}
$$

since countable colimits of compact objects are $\omega_1$-compact. $\square$

**Lemma 2.3.13** (2 $\Rightarrow$ 3 in Theorem 2.2.11). *If $\mathcal{C}$ is generated by weakly compactly exhaustible objects and filtered colimits in $\mathcal{C}$ are exact, $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint.*

*Proof.* By the pointwise criterion for existence of adjoints, it suffices to show that for each $X \in \mathcal{C}$, $\operatorname{Map}_{\mathcal{C}}(X, k(-))$ is a representable functor $\operatorname{Ind}(\mathcal{C}) \to \operatorname{An}$, i.e. there exists $X' \in \operatorname{Ind}(\mathcal{C})$ with a equivalence
$$\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(X', Y) \cong \operatorname{Map}_{\mathcal{C}}(X, kY)$$
natural in $Y$. The collection of $X$ for which such $X'$ exists is closed under colimits, since limits of representable functors are representable. It also contains compactly exhaustible objects by Lemma 2.3.10. So it contains every $X \in \mathcal{C}$ and the claim follows. $\square$

We also use the notion of locally weakly compact morphisms in $\operatorname{Fun}(K, \mathcal{C})$ to show the following:

**Lemma 2.3.14.** *If filtered colimits are exact in $\mathcal{C}$, weakly compactly exhaustible objects are closed under countable colimits.*

*Proof.* Every countable colimit can be written as a sequential colimit of finite colimits: Enumerating all simplices of a countable simplicial set $K$, the simplicial subset $K_n \subseteq K$ spanned by the first $n$ simplices is finite, and so

$$\operatorname{colim}_K F = \operatorname{colim}_{\mathbb{N}} \operatorname{colim}_{K_n} F|_{K_n}.$$

To prove closure under finite colimits it suffices to show closure under pushouts, since the initial object is clearly compact. Given a diagram $B \leftarrow A \rightarrow C$ of weakly compactly exhaustible objects, we may write $A = \operatorname{colim} A_n$, $B = \operatorname{colim} B_n$, $C = \operatorname{colim} C_n$, and then use Lemma 2.3.10 to lift $A \rightarrow B$ to a natural transformation $A_n \rightarrow B_{i(n)}$. Here we may assume $i : \mathbb{N} \rightarrow \mathbb{N}$ to be cofinal, and hence reindex $B_n$ to have an actual natural transformation $A_n \rightarrow B_n$, same for $A_n \rightarrow C_n$.

We thus have a diagram $B_\bullet \leftarrow A_\bullet \rightarrow C_\bullet$, i.e. a sequential diagram $\operatorname{Fun}(K, \mathcal{C})$ where $K = \bullet \leftarrow \bullet \rightarrow \bullet$, consisting of pointwise weakly compact maps. Since $K$ is 1-dimensional, the composite of any two successive maps is locally weakly compact in $\operatorname{Fun}(K, \mathcal{C})$, and so the composite of any two successive maps in $B_\bullet \amalg_{A_\bullet} C_\bullet$ is weakly compact. So $B \amalg_A C$ is weakly compactly exhausted.

For sequential colimits we proceed analogously, applying Lemma 2.3.10 inductively to write a sequential diagram $A_0 \rightarrow A_1 \rightarrow \ldots$ of compactly exhaustible objects as a sequential colimit of sequential diagrams $A_i = \operatorname{colim}_n A_{i,n}$ where the maps $A_{i,n} \rightarrow A_{i,n+1}$ are compact. This has the diagonal entries as a cofinal subdiagram, and the maps between them are compact since compact morphisms form a 2-sided ideal. $\square$

### 2.3.3  $3 \Rightarrow 1$ and $3 \Leftrightarrow 6$

We now prove that if $k : \operatorname{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$ has a left adjoint $\hat{\jmath}$, $\mathcal{C}$ is generated by strongly compactly exhaustible objects. To do so, we first derive basic properties of such a left adjoint, and then characterize compact morphisms in terms of it.

**Lemma 2.3.15.** *If $k : \operatorname{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$ admits a left adjoint, it is fully faithful, and $\operatorname{id} \rightarrow k \circ \hat{\jmath}$ is an equivalence.*

*Proof.* We have adjunctions $\hat{\jmath} \dashv k \dashv j$, and $j$ is fully faithful. Since the composite

$$\operatorname{Map}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(jX, jY) \simeq \operatorname{Map}_{\mathcal{C}}(kjX, Y)$$

is composition with the counit $kj \rightarrow \operatorname{id}$, but also an equivalence, $kj \simeq \operatorname{id}$.

We now get adjunctions

$$\operatorname{Map}_{\mathcal{C}}(k\hat{\jmath}X, Y) \simeq \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\hat{\jmath}X, jY) \simeq \operatorname{Map}_{\mathcal{C}}(X, kjY) \simeq \operatorname{Map}_{\mathcal{C}}(X, Y),$$

i.e. a natural equivalence $X \rightarrow k\hat{\jmath}X$. Unwinding the equivalences above, we find that it comes from the unit of the adjunction $\hat{\jmath} \dashv k$. $\square$

**Lemma 2.3.16.** *If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint, and $X \to Y$ is a morphism in $\mathcal{C}$, the following are equivalent.*

1. *$X \to Y$ is strongly compact.*

2. *$X \to Y$ is compact.*

3. *$jX \to jY$ factors through $jX \to \hat{\jmath}Y \to jY$ in $\mathrm{Ind}(\mathcal{C})$.*

4. *$\hat{\jmath}X \to \hat{\jmath}Y$ factors through $jX$ in $\mathrm{Ind}(\mathcal{C})$.*

*Proof.* If $X \to Y$ is strongly compact, it is also compact. Recall that a morphism $X \to Y$ is compact if and only if for any $Z \in \mathrm{Ind}(\mathcal{C})$, we have a dashed lift in

$$
\begin{array}{ccc}
jX & \dashrightarrow & Z \\
\downarrow & & \downarrow \\
jY & \longrightarrow & jkZ.
\end{array}
$$

In particular, we may apply this to $Z = \hat{\jmath}Y$ to obtain a factorisation $jX \to \hat{\jmath}Y \to jY$.

Finally, observe that $X \to Y$ is strongly compact if we find a lift in

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jY, Z) & \longrightarrow & \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, Z) \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Map}_{\mathcal{C}}(Y, kZ) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, Z)
\end{array}
$$

We may replace the bottom line by $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\hat{\jmath}Y, Z)$ etc. and use Yoneda to see that this is equivalent to finding a lift in

$$
\begin{array}{ccc}
jY & \longleftarrow & jX \\
\uparrow & \nearrow & \uparrow \\
\hat{\jmath}Y & \longleftarrow & \hat{\jmath}X.
\end{array}
$$

By assumption, we are given a factorisation $jX \to \hat{\jmath}Y \to jY$, making the top triangle commute. The bottom triangle then commutes automatically, since we may apply the adjunction and the canonical equivalences $kj \simeq \mathrm{id} \simeq k\hat{\jmath}$ to translate into commutativity of the diagram

$$
\begin{array}{ccc}
Y & \longleftarrow & X \\
{\scriptstyle\mathrm{id}}\uparrow & \nearrow & \uparrow{\scriptstyle\mathrm{id}} \\
Y & \longleftarrow & X.
\end{array}
$$

Similarly, if we are given a factorisation $\hat{\jmath}X \to jX \to \hat{\jmath}Y$, the other triangle of the diagram commutes automatically using the adjunction $k \dashv j$. $\qquad\square$

**Lemma 2.3.17** (3 $\Leftrightarrow$ 6 of Theorem 2.2.11). *For presentable $\mathcal{C}$, the following are equivalent:*

1. $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ *admits a left adjoint.*

2. *In $\mathcal{C}$, filtered colimits distribute over small limits, i.e.*

$$\mathrm{colim}_{I^K} \lim_K F \simeq \lim_K \mathrm{colim}_I F.$$

3. *In $\mathcal{C}$, filtered colimits are exact and distribute over small products, i.e.*

$$\mathrm{colim}_{\prod_J I} \prod_J F \simeq \prod_J F \, \mathrm{colim}_I F.$$

*Proof.* $1 \Rightarrow 2$: If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, it preserves limits. Let $F : K \times I \to \mathcal{C}$ be a diagram with $I$ filtered. Then since limits and filtered colimits in $\mathrm{Ind}(\mathcal{C})$ are formed pointwise, the Ind object $\lim_K \mathrm{colim}_I jF(k, i)$ agrees with $\mathrm{colim}_{I^K} j \lim_K F(k, i)$, using that filtered colimits distribute over small limits in An. Applying $k$ and using the assumption that it commutes with limits, we learn that

$$\lim_K \mathrm{colim}_I F(k, i) \simeq \mathrm{colim}_{I^K} \lim_K F(k, i)$$

as desired.

$2 \Rightarrow 3$: If $K$ is finite, the diagonal map $I \to I^K$ is cofinal (this is essentially the definition of filtered). So filtered colimits distributing over finite limits is equivalent to filtered colimits commuting with finite limits. Distributing over products is of course also a special case.

$3 \Rightarrow 1$: Similar to $1 \Rightarrow 2$, the condition 3 implies that $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ commutes with finite limits and products, so general limits. If $\mathcal{C}$ is $\kappa$-compactly generated, $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ factors through the restriction map $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}^\kappa)$. We have the diagram of adjoints

$$
\begin{array}{ccc}
\mathrm{Ind}(\mathcal{C}) & \xrightleftharpoons{\quad} & \mathcal{C} \\
\updownarrow & \swarrow\nearrow_{k} & \\
\mathrm{Ind}(\mathcal{C}^\kappa). & &
\end{array}
$$

Limits in $\mathrm{Ind}(\mathcal{C}^\kappa)$ are computed by embedding into $\mathrm{Ind}(\mathcal{C})$, taking the limit there, and reflecting back using the restriction. Now $k$ commuting with limits implies that $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$ commutes with limits. It also commutes with colimits, so it admits a left adjoint by the adjoint functor theorem, and so $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint. $\square$

**Lemma 2.3.18** ($3 \Rightarrow 1$ of Theorem 2.2.11)**.** *If $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{j}$, $\mathcal{C}$ is generated by strongly compactly exhaustible objects.*

*Proof.* Note that Lemma 2.3.17 above implies that filtered colimits in $\mathcal{C}$ are exact. So weakly compactly exhaustible objects in $\mathcal{C}$ are closed under finite colimits by Lemma 2.3.14, and agree with strongly compactly exhaustible objects by Lemma 2.3.16. So by Lemma 2.1.27, it suffices to prove that if $X \to Y$ is a morphism in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

admits a lift whenever $A, B$ are strongly compactly exhaustible, then $X \to Y$ is an equivalence. Since $\hat{\jmath}$ is fully faithful, it suffices to check that $\hat{\jmath}X \to \hat{\jmath}Y$ is an equivalence in $\mathrm{Ind}(\mathcal{C})$. Since $\mathrm{Ind}(\mathcal{C})$ is generated by the Yoneda image which is closed unter finite colimits, we need to prove that any diagram

$$
\begin{array}{ccc}
jA & \longrightarrow & \hat{\jmath}X \\
\downarrow & \nearrow & \downarrow \\
jB & \longrightarrow & \hat{\jmath}Y
\end{array}
$$

admits a dashed lift, for arbitrary $A, B \in \mathcal{C}$.

Now observe that $\mathrm{Ind}(\mathcal{C})^{\Delta^1}$ is compactly generated (since objects of the form $\emptyset \to jB$ and $jA \to jA$ are generators by adjunctions, and are compact; also note that the compact objects are exactly those of the form $jA \to jB$). So any object $F \to G$ of $\mathrm{Ind}(\mathcal{C})^{\Delta^1}$ is a filtered colimit of arrows $jA \to jB$ from $\mathcal{C}^{\Delta^1}$. Applying this to $\hat{\jmath}X \to \hat{\jmath}Y$, we may write this as

$$(\hat{\jmath}X \to \hat{\jmath}Y) = \mathrm{colim}_I(jA_i \to jB_i).$$

Next, observe that applying $k$ yields

$$(X \to Y) = \mathrm{colim}_I(A_i \to B_i),$$

and applying (colimit-preserving) $\hat{\jmath}$ again yields

$$(\hat{\jmath}X \to \hat{\jmath}Y) = \mathrm{colim}_I(\hat{\jmath}A_i \to \hat{\jmath}B_i).$$

So the canonical maps

$$\mathrm{colim}_I(\hat{\jmath}A_i \to \hat{\jmath}B_i) \to \mathrm{colim}_I(jA_i \to jB_i) \to (\hat{\jmath}X \to \hat{\jmath}Y)$$

are equivalences. The map from $jA \to jB$ above therefore factors as

$$
\begin{array}{ccccccc}
jA & \longrightarrow & \hat{\jmath}A_i & \longrightarrow & jA_i & \longrightarrow & \hat{\jmath}X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
jB & \longrightarrow & \hat{\jmath}B_i & \longrightarrow & jB_i & \longrightarrow & \hat{\jmath}Y
\end{array}
$$

for some $i \in I$. We may apply this argument inductively to factor the original diagram as

$$
\begin{array}{ccccccccc}
jA_0 & \longrightarrow & \hat{\jmath}A_1 & \longrightarrow & jA_1 & \longrightarrow & \ldots & & \hat{\jmath}X \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
jB_0 & \longrightarrow & \hat{\jmath}B_1 & \longrightarrow & jB_1 & \longrightarrow & \ldots & & \hat{\jmath}Y.
\end{array}
$$

The maps $A_n \to A_{n+1}$ and $B_n \to B_{n+1}$ here are compact, and the horizontal colimits here agree with $\mathrm{colim}_n \hat{\jmath}A_n = \hat{\jmath}\,\mathrm{colim}\,A_n$ and $\hat{\jmath}\,\mathrm{colim}\,B_n$. Since $\hat{\jmath}$ is fully faithful and we may lift against strongly compactly exhaustible objects by assumption, we therefore find a compatible lift $\mathrm{colim}\,\hat{\jmath}B_n \to \hat{\jmath}X$. Precomposing, we find a lift out of the original $jB_0 = jB$. $\qquad\square$

## 2.3.4 $3 \Leftrightarrow 4 \Leftrightarrow 5$

**Lemma 2.3.19** ($3 \Rightarrow 4$ of Theorem 2.2.11). *If* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ *admits a left adjoint,* $\mathcal{C}$ *is* $\omega_1$-*compactly generated and* $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ *admits a left adjoint.*

*Proof.* By Lemma 2.3.18 and 2.3.3, $\mathcal{C}$ is generated by weakly compactly exhaustible objects and filtered colimits are exact. By Corollary **??**, this implies that $\mathcal{C}$ is generated by $\omega_1$-compact objects. As in the proof of Lemma 2.3.17, this means that the colimit functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Ind}(\mathcal{C}^{\omega_1})$, and its left adjoint factors through $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ as well, giving the desired adjoint. $\qquad\square$

**Lemma 2.3.20** ($4 \Rightarrow 5$ of Theorem 2.2.11). *If* $\mathcal{C}$ *is* $\omega_1$-*compactly generated and* $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ *admits a left adjoint,* $\mathcal{C}$ *is a retract in* $\mathrm{Pr}^{\mathrm{L}}$ *of a compactly generated category.*

*Proof.* $k : \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ has a right adjoint given by the restricted Yoneda embedding $j'$, which is fully faithful since $\mathcal{C}$ is $\omega_1$-compactly generated. As in the proof of Lemma 2.3.15, this implies that the left adjoint $\hat{\jmath}$ is fully faithful, and $\hat{\jmath} \circ k \simeq \mathrm{id}_{\mathcal{C}}$. This is the desired retraction. $\qquad\square$

**Lemma 2.3.21** ($5 \Rightarrow 3$ of Theorem 2.2.11). *If* $\mathcal{C}$ *is a retract in* $\mathrm{Pr}^{\mathrm{L}}$ *of a compactly generated category,* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ *has a left adjoint.*

*Proof.* If $\mathcal{C}$ is compactly generated, it is of the form $\mathrm{Ind}(\mathcal{C}_0)$. In that case, a left adjoint can be described as $\mathrm{Ind}(j)$, the Ind-extension of the functor $\mathcal{C}_0 \to \mathrm{Ind}(\mathrm{Ind}(\mathcal{C}_0))$. Indeed, both

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(j)(-), Y), \quad \mathrm{Map}_{\mathcal{C}}(-, kY)$$

are functors $\mathcal{C} \to \mathrm{An}^{\mathrm{op}}$ which preserve filtered colimits, and they agree on the objects coming from $\mathcal{C}_0$, so we have the desired adjunction.

If more generally $\mathcal{C}$ is a retract of a compactly generated $\mathcal{C}'$, with both functors $\mathcal{C} \xrightarrow{i} \mathcal{C}' \xrightarrow{r} \mathcal{C}$ colimit-preserving, the functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is a retract:

$$
\begin{array}{ccccc}
\mathrm{Ind}(\mathcal{C}) & \xrightarrow{\mathrm{Ind}(i)} & \mathrm{Ind}(\mathcal{C}') & \xrightarrow{\mathrm{Ind}(r)} & \mathrm{Ind}(\mathcal{C}) \\
\downarrow{\scriptstyle k} & & {\scriptstyle \hat{\jmath}}\uparrow\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k} \\
\mathcal{C} & \xrightarrow{i} & \mathcal{C}' & \xrightarrow{r} & \mathcal{C}
\end{array}
$$

Using the middle adjunction, we have the following composite for $X \in \mathcal{C}$ and $Y \in \text{Ind}(\mathcal{C})$:

$$\text{Map}_{\mathcal{C}}(X, kY)$$
$$\downarrow i$$
$$\text{Map}_{\mathcal{C}'}(iX, ikY)$$
$$\downarrow \simeq$$
$$\text{Map}_{\mathcal{C}'}(iX, k\,\text{Ind}(i)Y)$$
$$\downarrow \simeq$$
$$\text{Map}_{\text{Ind}(\mathcal{C}')}(\hat{\jmath}iX, \text{Ind}(i)Y)$$
$$\downarrow \text{Ind}(r)$$
$$\text{Map}_{\text{Ind}(\mathcal{C})}(\text{Ind}(r)\hat{\jmath}iX, Y)$$
$$\downarrow k$$
$$\text{Map}_{\mathcal{C}}(X, kY)$$

using that $k \circ \hat{\jmath} \simeq \text{id}$. If all morphisms here were invertible, we would have exhibited $\text{Ind}(r) \circ \hat{\jmath} \circ i$ as left adjoint of $k$. We don't have that, but the composite is still the identity. So we have exhibited $\text{Map}_{\mathcal{C}}(X, k(-))$ as retract of $\text{Map}_{\text{Ind}(\mathcal{C})}(\text{Ind}(r)\hat{\jmath}iX, -)$. By Yoneda, this shows that $\text{Map}_{\mathcal{C}}(X, k(-))$ is itself corepresentable, by a retract of $\text{Ind}(r)\hat{\jmath}iX$, and so we have that $k$ admits a left adjoint. $\square$

## 2.4 Properties of compactly assembled ∞-categories

Having proved the equivalence of all characterisations of compactly assembled ∞-categories, we record a number of observations:

**Proposition 2.4.1.** *In a compactly assembled ∞-category, the following are equivalent for a morphism $X \to Y$:*

1. *$X \to Y$ is weakly compact*

2. *$X \to Y$ is strongly compact*

3. *$jX \to jY$ factors over $\hat{\jmath}Y$*

4. *$\hat{\jmath}X \to \hat{\jmath}Y$ factors over $jX$*

5. *$\hat{\jmath}X \to \hat{\jmath}Y$ is compact in $\text{Ind}(\mathcal{C}^{\omega_1})$.*

*Proof.* The first four points are Lemma 2.3.16. For the last point, observe that $\text{Ind}(\mathcal{C}^{\omega_1})$ is compactly generated, so $\hat{\jmath}X \to \hat{\jmath}Y$ is compact if and only it factors over some $jZ$. Since the map $\hat{\jmath}X \to jZ$ canonically factors through $jX$, this is equivalent to 4. $\square$

**Definition 2.4.2.** In a compactly assembled $\infty$-category, we simply speak of compact maps if one of the equivalent conditions of proposition 2.4.1 is satisfied. A (compactly) assembled map $X \to Y$ is given by a compact map together with a lift $jX \to \hat{j}Y$.

We write
$$\mathrm{Map}_{\mathcal{C}}^{\mathrm{ca}}(X, Y) := \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, \hat{j}Y)$$
for the *space of compactly assembled maps*.

Note that this is *not* a full subspace of $\mathrm{Map}_{\mathcal{C}}(X, Y)$, so for example a homotopy between compact maps is more data than just a homotopy between the underlying maps.

**Proposition 2.4.3.** *In a compactly assembled $\infty$-category $\mathcal{C}$, every compact morphism $X \to Y$ factors as a composite of two compact morphisms $X \to Z \to Y$, and even extends to a $\mathbb{Q} \cap [0,1]$-indexed diagram $X_\alpha$ (with $X_0 = X$ and $X_1 = Y$).*

*Proof.* A compact map $X \to Y$ gives a map $jX \to \hat{j}Y$. We may write $\hat{j}Y = \mathrm{colim}_{i \in I} jY_i$ as a filtered colimit of representables. Applying $k$, we find that $\mathrm{colim}_{i \in I} Y_i \simeq Y$, and applying $\hat{j}$, we find that $\mathrm{colim}_{i \in I} \hat{j}Y_i \simeq \hat{j}Y$, so

$$\mathrm{colim}_{i \in I} \hat{j}Y_i \to \mathrm{colim}_{i \in I} jY_i \to \hat{j}Y$$

are equivalences. Now the map $jX \to \hat{j}Y$ factors as

$$jX \to \hat{j}Y_i \to jY_i \to \hat{j}Y,$$

which witnesses compactness of $X \to Y_i \to Y$. For the second statement, observe that $\mathbb{Q} \cap [0,1]$ can be realized as ascending union of discrete subposets, where in each step we just add finitely many points between two points. Beginning with $\{0, 1\}$ and $X \to Y$, we may inductively extend over each of these discrete subposets, and then obtain a diagram indexed over their colimit. $\square$

**Example 2.4.4.** In a general $\infty$-category, compact morphisms do not necessarily factor through multiple compact morphisms. For example, consider the poset

$$\mathbb{N} + \mathbb{N} \times \mathbb{N} + \{\infty\},$$

where $\mathbb{N} \times \mathbb{N}$ carries the canonical partial ordering (componentwise instead of lexicographical). Then every subset has a supremum, so this is a presentable category. The morphism

$$(0, 0) \to \infty$$

is compact, since every family $x_i$ of elements with supremum $\infty$ contains elements from $\mathbb{N} \times \mathbb{N} + \{\infty\}$, which receive a morphism from $(0, 0)$. Since both $(0, 0)$ and $\infty$ can be expressed as suprema of elements strictly smaller than themselves, they are not compact. Any factorisation of $(0, 0) \to \infty$ into two compact morphisms must therefore take the form

$$(0, 0) \to (a, b) \to \infty$$

with $a > 0$ or $b > 0$. But the morphisms $(a, b) \to \infty$ with $a > 0$ or $b > 0$ are never compact. For example, if $a > 0$, we have $\infty = \sup(0, n)$ but $(a, b) \not\leq (0, n)$ for any $n$.

So $(0, 0) \to \infty$ is an example of a compact morphism that cannot be factored into compact morphisms.

**Lemma 2.4.5.** *Every compact morphism $X \to Y$ factors as $X \to K \to Y$ where $K$ is $\omega_1$-compact. We can even arrange it such that the morphisms $X \to K$ and $K \to Y$ are compact.*

*Proof.* Since the $\infty$-category is $\omega_1$-compactly assembled we can write $Y$ as a filtered colimit $Y = \operatorname{colim} Y_i$ with $Y_i$ $\omega_1$-compact. Since $X \to Y$ is compact it factors over a finite stage: $X \to Y_i \to Y$ and we set $K = Y_i$. This proves the first part.

For the second part we first factor $X \to Y$ as a composition of three compact morphisms $X \to X_0 \to Y_0 \to Y$ using Proposition 2.4.3. Then we apply the previous consideration to get a factorization $X_0 \to K \to Y_0$ and conclude since $X \to X_0$ is compact that also $X \to K$ is compact and similarly for $K \to Y$. $\qquad\square$

**Proposition 2.4.6.** *In a compactly assembled $\infty$-category, the following are equivalent for an object $X$:*

1. *$X$ is weakly compactly exhaustible*

2. *$X$ is strongly compactly exhaustible*

3. *$X$ is $\omega_1$-compact*

4. *$X$ may be written as colimit of a $\mathbb{Q}_{\geq 0}$-indexed diagram where all "positive length" morphisms are compact*

5. *$X$ may be written as colimit of a $\mathbb{Q}$-indexed diagram where all "positive length" morphisms are compact.*

*Furthermore, in any of the $\mathbb{N}$, $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}$-indexed diagrams above, we may choose all objects to be $\omega_1$-compact.*

*Proof.* Since strongly and weakly compact morphisms agree, 1. and 2. are equivalent. We have also seen in Corollary 2.3.12 that compactly exhaustible objects are always $\omega_1$-compact. Conversely, let $X$ be $\omega_1$-compact. Write $X = \operatorname{colim}_{i \in I} X_i$ where $X_i$ are weakly compactly exhaustible. Since the compactly exhaustible objects are closed under countable (i.e. $\omega_1$-small) colimits by Lemma 2.3.14, we may assume $I$ to be $\omega_1$-filtered here. But then the identity on $X$ factors through one of the $X_i$. So $X$ is a retract of a compactly exhaustible object, but retracts can be written as countable colimits as well.

If we have $X = \operatorname{colim}(X_0 \to X_1 \to \ldots)$ compactly exhausted, we may first factor each morphism into compact morphisms through $\omega_1$-compact objects, and hence assume that the $X_i$ are $\omega_1$-compact. Factoring, we may extend each $X_i \to X_{i+1}$ to a $\mathbb{Q} \cap [i, i+1]$-indexed diagram of $\omega_1$-compact objects and compact morphisms, and therefore have extended the diagram to a $\mathbb{Q}_{\geq 0}$ diagram with the same colimit. Finally, $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}$ contain each other as cofinal subsets ($\mathbb{Q}_{>0} \subseteq \mathbb{Q}_{\geq 0}$ is isomorphic to $\mathbb{Q}$). $\qquad\square$

So we may view compactly assembled $\infty$-categories as a special kind of $\omega_1$-compactly generated ones: The ones where in addition, every $\omega_1$-compactly generated object can be compactly exhausted.

**Lemma 2.4.7.** *Let $\mathcal{D}$ be a category with filtered colimits and $\mathcal{C}$ be compactly assembled. A functor $F : \mathcal{C} \to \mathcal{D}$ commutes with filtered colimits if and only if its* Ind-*extension*

$$\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$$

*is local with respect to $\hat{j}X \to jX$. (I.e. takes those morphisms to equivalences)*

*Proof.* If the Ind-extension $F' : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ of $F : \mathcal{C} \to \mathcal{D}$ is local, we have $F \simeq F' \circ j \simeq F' \circ \hat{j}$, but the latter is a composition of two filtered-colimit preserving functors.

Conversely, assume $F$ preserves filtered colimits. Writing $\hat{j}X \simeq \mathrm{colim}_I jX_i$, we learn that $X \simeq \mathrm{colim}_I X_i$ by applying $k$, and

$$F'(\hat{j}X) \simeq \mathrm{colim}_I F(X_i) \simeq F(X) \simeq F'(jX)$$

by filtered-colimit preservation of $F$, so $F'$ is local with respect to the morphisms $\hat{j}X \to jX$. $\qquad\square$

**Proposition 2.4.8.** *If $F : \mathcal{C} \to \mathcal{D}$ is an arbitrary functor with $\mathcal{C}$ compactly assembled and $\mathcal{D}$ admits all filtered colimits, the category of filtered-colimit preserving functors $G : \mathcal{C} \to \mathcal{D}$ with natural transformation $G \to F$ admits a terminal object given by $k \circ \mathrm{Ind}(F) \circ \hat{j}$. In other words: the natural transformation*

$$k \circ \mathrm{Ind}(F) \circ \hat{j} \to F$$

*is the* filtered colimit assembly map.

*Proof.* Arbitrary functors $\mathcal{C} \to \mathcal{D}$ correspond to filtered-colimit preserving functors $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$. Filtered-colimit preserving functors $\mathcal{C} \to \mathcal{D}$ correspond to filtered-colimit preserving functors $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ which are local with respect to the maps $\hat{j}X \to jX$. These form a (big) left Bousfield localisation, where the left adjoint is given by restricting along $\hat{j}$ and passing to the Ind-extension again. Applying this to the Ind-extension $F' = k \circ \mathrm{Ind}(F)$ of $F$, we obtain the Ind-extension of $F' \circ \hat{j} = k \circ \mathrm{Ind}(F) \circ \hat{j}$ as claimed. $\qquad\square$

This way of passing from a functor to a colimit-preserving one in a universal way is in general known as "assembly" of the functor, because the new functor is typically "assembled" from the restriction of the old functor to some class of objects. For example, if $\mathcal{C}$ is even compactly generated, the universal filtered-colimit preserving functor over $F$ is just the Ind-extension of $F|_{\mathcal{C}^\omega}$. The above formula describes the assembly of $F$ as the unique filtered-colimit preserving functor which is on compactly exhausted objects described by

$$\mathrm{colim}(X_0 \to X_1 \to \ldots) \mapsto \mathrm{colim}(FX_0 \to FX_1 \to \ldots)$$

**Corollary 2.4.9.** *For a compactly assembled category $\mathcal{C}$, $\mathrm{Map}^{\mathrm{ca}}_{\mathcal{C}}(X, Y)$ coincides with the filtered colimit assembly of $\mathrm{Map}_{\mathcal{C}}(X, -)$.*

*Proof.* The assembly of $\mathrm{Map}_{\mathcal{C}}(X, -)$ takes $Y$ to

$$(k \circ \mathrm{Ind}(\mathrm{Map}_{\mathcal{C}}(X, -)))(\hat{j}Y).$$

The functor $k \circ \mathrm{Ind}(\mathrm{Map}_{\mathcal{C}}(X, -)) : \mathcal{C} \to \mathrm{An}$ is the Ind-extension of the functor $\mathrm{Map}_{\mathcal{C}}(X, -)$. This coincides with $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, -)$ since it has the correct value on representables and preserves filtered colimits in $\mathrm{Ind}(\mathcal{C})$. So the assembly takes $Y$ to

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(jX, \hat{j}Y) = \mathrm{Map}_{\mathcal{C}}^c(X, Y)$$

as claimed. $\qquad\square$

## 2.5 The category of presentable $\infty$-categories

**Lemma 2.5.1.** *$\mathrm{Pr}^L$ has small limits, and they are formed "underlying" (i.e. the forgetful functor $\mathrm{Pr}^L \to \mathrm{Cat}_\infty$ preserves limits). Analogously, $\mathrm{Pr}^R$ has small limits and they are formed underlying.*

*Proof.* [Lur17b, todo] $\qquad\square$

**Corollary 2.5.2.** *Colimits in $\mathrm{Pr}^L$ are not formed underlying. Instead, they are formed by passing to the opposite diagram of right adjoints (along the contravariant equivalence $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$) and passing to the limit instead.*

**Example 2.5.3.** Even though the coproduct $\mathcal{C} \amalg \mathcal{D}$ formed in $\mathrm{Cat}_\infty$ is just the disjoint union, $\mathcal{C} \amalg \mathcal{D}$ formed in $\mathrm{Pr}^L$ agrees with the product $\mathcal{C} \times \mathcal{D}$, and more generally $\coprod_I \mathcal{C}_i \simeq \prod_I \mathcal{C}_i$ for *any* set $I$. An explanation for the different behaviour is that a presentable $\infty$-category needs to have small colimits, and in the disjoint union of $\mathcal{C}$ and $\mathcal{D}$, we for example don't have a coproduct of objects coming from different components.

**Definition 2.5.4.** We write $\mathrm{Pr}^L_\kappa$ for the (non-full) subcategory of $\mathrm{Pr}^L$ consisting of all $\kappa$-compactly generated categories with morphisms given by left adjoint functors $F : \mathcal{C} \to \mathcal{D}$ which take $\mathcal{C}^\kappa$ into $\mathcal{D}^\kappa$.

**Lemma 2.5.5.** *$\mathrm{Pr}^L_\kappa$ is equivalent to the full subcategory of the $\infty$-category $\mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa)}$ of small categories with $\kappa$-small colimits, and $\kappa$-small colimit preserving functors spanned by the idempotent complete $\infty$-categories.[4]*

*Proof.* The inverse equivalences are given by $\mathrm{Ind}_\kappa$ and $(-)^\kappa$. [Lur17b, todo] $\qquad\square$

**Lemma 2.5.6.** *For a pair of adjoint functors $L : \mathcal{C} \to \mathcal{D} : R$, we have:*

*1. If the right adjoint $R$ preserves $\kappa$-filtered colimits, $L$ preserves $\kappa$-compact objects.*

---

[4]Note that for $\kappa > \omega$ idempotent completeness is automatic since splitting idempotents can be achieved by a sequential colimits, so in this case $\mathrm{Pr}^L_\kappa \simeq \mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa)}$. But for $\kappa = \omega$ this makes a difference.

2. If $\mathcal{C}$ is $\kappa$-compactly generated and $L$ preserves $\kappa$-compact objects, $R$ preserves $\kappa$-filtered colimits.

*Proof.* If $X$ is compact and $R$ preserves $\kappa$-filtered colimits, then $\mathrm{Map}_{\mathcal{D}}(LX, -) \simeq \mathrm{Map}_{\mathcal{C}}(X, R(-))$ commutes with $\kappa$-filtered colimits, so $LX$ is compact. For the other statement, if $Y_i$ is a $\kappa$-filtered diagram, to check that $\mathrm{colim}\, RY_i \simeq R(\mathrm{colim}\, Y_i)$ if $\mathcal{C}$ is $\kappa$-compactly generated, it suffices to apply $\mathrm{Map}_{\mathcal{C}}(X, -)$ for $\kappa$-compact $X$, which leads to $\mathrm{colim}\,\mathrm{Map}_{\mathcal{D}}(LX, Y_i)$ on both sides since $LX$ is $\kappa$-compact. $\square$

**Lemma 2.5.7.** *The forgetful functor* $\mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Pr}^{\mathrm{L}}$ *(and hence also the functor* $\mathrm{Ind}_\kappa : \mathrm{Cat}^{\mathrm{Rex}(\kappa)}_\infty \to \mathrm{Pr}^{\mathrm{L}}$*) preserves colimits.*

*Proof.* Let $\mathcal{C}_i \to \mathcal{C}$ be a colimit cone in $\mathrm{Pr}^{\mathrm{L}}$ over a diagram in $\mathrm{Pr}^{\mathrm{L}}_\kappa$. We need to prove that it is a colimit cone in $\mathrm{Pr}^{\mathrm{L}}_\kappa$. Passing to right adjoints, it suffices to check that the limit of $\kappa$-compactly generated categories along $\kappa$-filtered colimit preserving right adjoint functors is itself $\kappa$-compactly generated, and universal among $\kappa$-filtered colimit preserving left adjoint functors into the diagram. The first statement follows since the right adjoint functors out of the limit are jointly conservative, and so their left adjoints (which preserve $\kappa$-compact objects) take generators to generators collectively. The other statement follows since $\kappa$-filtered colimits and limits in the limit of categories are formed pointwise. $\square$

**Example 2.5.8.** For a ring $R$, we have the category of perfect complexes $\mathcal{D}(R)^\omega$. We have a functor $BG \to \mathrm{Cat}^{\mathrm{Rex}}_\infty$ encoding the trivial $G$-action on $\mathcal{D}(R)^\omega$. By the above, its colimit is given by the compact objects in the colimit of the trivial $G$-action on its Ind-category $\mathcal{D}(R)$, viewed as diagram $BG \to \mathrm{Pr}^{\mathrm{L}}$. This limit may be computed as the limit of the right adjoint diagram, which is the functor category $\mathrm{Fun}(BG, \mathcal{D}(R)) \simeq \mathcal{D}(R[G])$. So the colimit of the original diagram $BG \to \mathrm{Cat}^{\mathrm{Rex}}_\infty$ is $\mathcal{D}(R[G])^\omega$.

**Example 2.5.9.** The functor $\mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Pr}^{\mathrm{L}}$ does *not* preserve limits. As an example, let $\kappa = \omega$ and consider the pullback diagram

$$
\begin{array}{ccc}
\mathrm{aMod}_{\mathfrak{m}}(A) & \longrightarrow & \mathrm{Mod}(A) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Mod}(A/\mathfrak{m}).
\end{array}
$$

in $\mathrm{Pr}^{\mathrm{L}}$. This is not a pullback diagram in $\mathrm{Pr}^{\mathrm{L}}_\omega$, as the top left corner is not even compactly generated. (In fact, the pullback in $\mathrm{Pr}^{\mathrm{L}}_\omega$ is 0, as any compact-object preserving functor into the kernel of $\mathrm{Mod}(A) \to \mathrm{Mod}(A/\mathfrak{m})$ is zero.) We will however see later that this is a limit in the category of compactly assembled $\infty$-categories, but $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}$ does also not generally preserve limits.

## 2.6 The category of compactly assembled $\infty$-categories

**Proposition 2.6.1.** *1. A filtered-colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$ between compactly assembled $\infty$-categories preserves compact morphisms if and only if it commutes*

*with $\hat{\jmath}$, more precisely that the natural transformation $\hat{\jmath} \circ F \to \mathrm{Ind}(F) \circ \hat{\jmath}$ makes the diagram*

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{Ind}(F)} \mathrm{Ind}(\mathcal{D})$$
$$\hat{\jmath}\Big\uparrow \qquad\qquad \hat{\jmath}\Big\uparrow$$
$$\mathcal{C} \xrightarrow{\quad F \quad} \mathcal{D}$$

*commute.*

2. *A colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ between presentable $\infty$-categories, where $\mathcal{C}$ is compactly assembled, preserves strongly compact morphisms if and only if its right adjoint $R$ commutes with filtered colimits.*

*Proof.* For the first statement, first assume that $F$ preserves compact morphisms. If $X = \mathrm{colim}_{\mathbb{N}} X_n$ is compactly exhausted, $FX = \mathrm{colim}_{\mathbb{N}} FX_n$ is, too, and we have $\hat{\jmath}FX \simeq \mathrm{colim}\, jFX_n \simeq \mathrm{Ind}(F) \mathrm{colim}\, jX_n \simeq \hat{\jmath}X$. So the canonical transformation

$$\hat{\jmath} \circ F \to \mathrm{Ind}(F) \circ \hat{\jmath}$$

is an equivalence on compactly exhaustible objects, and since both functors commute with filtered colimits, also general objects. Conversely, if $F$ commutes with $\hat{\jmath}$, a witness $jX \to \hat{\jmath}Y$ of compactness of $X \to Y$ is taken by $\mathrm{Ind}(F)$ to a morphism $jFX \to \hat{\jmath}FY$ witnessing compactness of $FX \to FY$.

For the second statement, first assume that the right adjoint $R$ preserves filtered colimits. One may directly check from the definition of strongly compact morphisms that $F$ preserves strongly compact morphisms. Conversely, assume that $F$ preserves strongly compact morphisms. To check that

$$\mathrm{colim}_{i \in I} RZ_i \to R(\mathrm{colim}_{i \in I} Z_i)$$

is an equivalence for any filtered diagram, it suffices to check this after $\mathrm{Map}_{\mathcal{C}}(X, -)$ for a strongly compactly exhausted $X = \mathrm{colim}_{n \in \mathbb{N}} X_n$. But we have

$$\mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim}_{i \in I} RZ_i) = \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{\mathbb{N}} jX_n, \mathrm{colim}_I jRZ_i)$$
$$= \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{\mathbb{N}} jFX_n, \mathrm{colim}_I jZ_i)$$
$$= \mathrm{Map}_{\mathcal{C}}(FX, \mathrm{colim}\, Z_i) = \mathrm{Map}_{\mathcal{C}}(X, R\, \mathrm{colim}\, Z_i).$$

$\square$

**Definition 2.6.2.** A left adjoint functor between compactly assembled categories that satisfies the equivalent conditions of Proposition 2.6.1 is called compactly assembled. We denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ the non-full subcategory of $\mathrm{Pr}^{\mathrm{L}}$ spanned by the compactly assembled categories and compactly assembled functors.

We note that since each compactly assembled category is $\omega_1$-compactly generated, and compactly assembled functors preserve $\omega_1$-compact objects, we find that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ is actually a non-full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\omega_1}$. We have an equivalence

$$\mathrm{Pr}^{\mathrm{L}}_{\omega_1} \simeq \mathrm{Cat}_{\infty}^{\mathrm{Rex}(\omega_1)}$$

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where the latter is the $\infty$-category of small $\infty$-categories that admit $\omega_1$-small (i.e countable) colimits and functors that preserve them (see [Lur17b, Proposition 5.5.7.8]). The equivalence is implemented by taking $\omega_1$-compact objects and vice versa by taking $\mathrm{Ind}_{\omega_1}$. It follows that we can think of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ also equivalently as some category of small categories. We would like to make this perspective explicit now.

**Definition 2.6.3.** A small $\infty$-category is called compactly assembled in the small sense if it admits countable colimits and every object is a sequential colimit along compact morphisms[5]. A functor between such is called compactly assembled in the small sense if it preserves $\omega_1$-small colimits and compact morphisms. We denote the $\infty$-category of small compactly assembled $\infty$-categories by $\mathrm{Cat}^{\mathrm{ca}}_{\infty}$.

**Proposition 2.6.4.** *We have an equivalence* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \simeq \mathrm{Cat}^{\mathrm{ca}}_{\infty}$.

*Proof.* If $\mathcal{C}$ is compactly assembled, then $\mathcal{C}^{\omega_1}$ admits countable colimits (this doesn't use anything), and all objects in $\mathcal{C}^{\omega_1}$ can be written as sequential colimit of $\omega_1$-compact objects along compact morphisms by Proposition 2.4.6. So $\mathcal{C}^{\omega_1}$ is compactly assembled in the small sense.

Conversely, assume $\mathcal{C}$ is $\omega_1$-compactly generated and $\mathcal{C}^{\omega_1}$ is compactly assembled in the small sense. If $X = \mathrm{colim}\, X_n$ is compactly exhausted in $\mathcal{C}^{\omega_1}$, then

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(\mathrm{colim}_n jX_n, Y) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_n X_n, kY).$$

Indeed, every $Y$ in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ can be written as $\omega_1$-filtered colimit of $\omega_1$-compact $Y$, and both sides commute with $\omega_1$-filtered colimits in $Y$. So it suffices to check this for $Y$ an $\omega_1$-compact object in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$. These can always be represented as countable filtered diagrams, and the same argument as used to prove Lemma 2.3.10 proves the above statement in that case. We thus get a well-defined functor $\hat{j} : \mathcal{C}^{\omega_1} \to \mathrm{Ind}(\mathcal{C}^{\omega_1})$ taking a compactly exhausted $\mathrm{colim}\, X_n$ to $\mathrm{colim}\, jX_n$, and its $\mathrm{Ind}_{\omega_1}$-extension provides a functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1})$ left adjoint to the colimit functor. $\qquad\square$

We will not really use this perspective here, since we believe that in most examples of compactly assembled categories, such as $\mathrm{Shv}(X)$, the presentable $\infty$-category is the more natural object to define than its $\omega_1$-compact objects.

We have the full faithful inclusion functor

$$i : \mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$$

since every compactly generated $\infty$-category is compactly assembled and the morphisms are the same by Proposition 2.6.1.

**Theorem 2.6.5.** *The category* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ *admits all colimits and the inclusion functor* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}$ *creates colimits.*

---

[5]Compact here is meant to be checked against all countable filtered colimits

*Proof.* Consider a diagram

$$I \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \qquad i \mapsto \mathcal{C}_i$$

and take the colimit of the composition $I \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}$. We denote this colimit by $\mathcal{C}$. Equivalently, $\mathcal{C}$ is the limit of the right adjoint diagram in $\mathrm{Pr}^{R}$. To argue that the original diagram is a colimit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ (i.e. that $\mathcal{C}$ is compactly assembled, it is a diagram of compactly assembled functors, and is an initial such cone), we may equivalently check that the right adjoint diagram is a limit diagram in the category whose objects are compactly assembled $\infty$-categories, and whose morphisms are filtered-colimit preserving right adjoint functors.

Since limits and filtered colimits in a limit of categories along such functors are formed levelwise, the characterisation of compactly assembled categories from Theorem 2.2.11(6) is obviously stable under limits along such functors.

$\square$

**Corollary 2.6.6.** *The functor* $(-)^{\omega_1} : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Cat}_\infty$ *preserves* $\omega_1$*-filtered colimits.*

*Proof.* This is true for the functor $(-)^{\omega_1} : \mathrm{Pr}^{\mathrm{L}}_{\omega_1} \to \mathrm{Cat}_\infty$ by [Lur17b, Proposition 5.5.7.11] and since $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}_{\omega_1}$ preserves small colimits (as they are computed in $\mathrm{Pr}^{\mathrm{L}}$ on both sides). $\square$

**Example 2.6.7.** The functor $(-)^{\omega_1} : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Cat}_\infty$ does not commute with sequential colimits. For example, let $\mathcal{C}_n = \prod_{i=1}^{n} \mathrm{An}$. Then $\mathrm{colim}_n^{\mathrm{ca}} \mathcal{C}_n = \prod_{\mathbb{N}} \mathrm{An}$ has $\omega_1$-compact objects given by $\prod_{\mathbb{N}} \mathrm{An}^{\omega_1}$, which disagrees with $\mathrm{colim}_n \prod_{i=1}^{n} \mathrm{An}^{\omega_1}$.

**Example 2.6.8.** Also, the functor $(-)^\omega : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Cat}_\infty$ does not commute with $\omega$-filtered colimits. To see this, consider the following example of posets (found by Konrad Bals). Let

$$P_n = \{\pm\infty\} \cup \{(x,y) \in \mathbb{R}^2 \mid y \geq n\},$$

With the componentwise order on $\mathbb{R}^2$, and $\pm\infty$ terminal and initial respectively. This is presentable, as we have arbitrary suprema. The only compact object is $-\infty$, as any other object can be written as filtered colimit from strictly below, for example $(x,y) = \mathrm{colim}_{\mathbb{N}}(x - \frac{1}{n}, y)$. So $P_n$ is not compactly generated, but compactly assembled: Any of the morphisms $(x,y) \to (x',y')$ with $x' > x$ and $y' > y$, as well as any of the morphisms $(x,n) \to (x',n)$ with $x' > x$, are compact. The inclusion functors

$$R : P_{n+1} \to P_n$$

preserve suprema and infima, and so admit both adjoints. The left adjoint $L : P_n \to P_{n+1}$ is therefore compactly assembled (it is the map which "rounds up the $y$-coordinate to $n+1$". We have

$$\mathrm{colim}_n(\ldots \to P_n \xrightarrow{L} P_{n+1} \to \ldots) \simeq \lim_n(\ldots \leftarrow P_n \xleftarrow{R} P_{n+1} \leftarrow \ldots) \simeq \{\pm\infty\}.$$

So in the colimit, both $\pm\infty$ are compact, even though only $-\infty$ is compact in any of the $P_n$.

**Theorem 2.6.9** (Ramzi)**.** *The $\infty$-category* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ *is itself presentable.*

The rest of the section will consist of a proof of this theorem. We consider the adjunction

$$\mathrm{Pr}^{\mathrm{L}}_{\omega} \rightleftarrows \mathrm{Pr}^{\mathrm{L}}_{\omega_1} \tag{2.1}$$

where the left adjoint is the inclusion and the right adjoint takes the ind-completion of the $\omega_1$-compact objects. This follows since we have for $\mathcal{C}$ compactly generated and $\mathcal{D}$ $\omega_1$-compactly generated:

$$\mathrm{Fun}^{\omega}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}^{\omega_1})) \simeq \mathrm{Fun}^{\mathrm{Rex}(\omega)}(\mathcal{C}^{\omega}, \mathcal{D}^{\omega_1})$$

and

$$\mathrm{Fun}^{\omega_1}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Rex}(\omega_1)}(\mathcal{C}^{\omega_1}, \mathcal{D}^{\omega_1})$$

This two are now naturally equivalent since $\mathcal{C}^{\omega_1}$ is the free $\infty$-category with $\omega_1$-small colimits on $\mathcal{C}^{\omega}$. ($\mathcal{C}^{\omega_1} \simeq \mathrm{Ind}^{\omega_1}_{\omega}(\mathcal{C}^{\omega})$, which is obtained by closing representables in $\mathrm{Fun}((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{An})$ under $\omega_1$-small $\omega$-filtered colimits.) The adjunction (2.1) induces a comonad $C$ on $\mathrm{Pr}^{\mathrm{L}}_{\omega_1}$.

**Proposition 2.6.10** (Ramzi). *The inclusion functor $i : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}_{\omega_1}$ is comonadic and the induced comonad is equivalent to the comonad $C$ described above.*

*Proof.* We first claim that the right adjoint functor of the adjunction (2.1) if composed with the fully faithful inclusion $\mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ is left adjoint to the functor $i : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}_{\omega_1}$. This follows by verifying the universal property for $\mathcal{C}$ compactly assembled and $\mathcal{D}$ $\omega_1$-compactly generated (TODO, actually not completely easy). Then it follows by abstract non-sense that the two comonads agree and it only remains to check the comonadicity. Do this (follow Maxime's Lemma 3.6) $\qquad \square$

This roughly describes compactly assembled categories as $\omega_1$-generated categories with a map $\mathcal{D} \to \mathrm{Ind}(\mathcal{D}^{\omega_1})$ preserving $\omega_1$-compact objects, and some higher coherences. Of course, this structure map is exactly $\hat{\jmath}$.

Note that this gives a completely abstract characterisation of compactly assembled $\infty$-categories. It also gives some justification to the name.

*Proof of Theorem 2.6.9.* It is coalgebras for some comonad. The comonad is accessible since it is induced by an adjunction between the presentable $\infty$-categories $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ and $\mathrm{Pr}^{\mathrm{L}}_{\omega_1}$. Now it follows from abstract nonsense that comodules for an accessible comonad on a presentable $\infty$-category are presentable. More precisely the $\infty$-category of comodules is a partially lax limit and thus presentable by [Lur17b, 5.4.7.3., 5.4.7.11.]. $\qquad \square$

## 2.7 Limits of compactly assembled categories

As a result of the presentability of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ we can deduce that it also admits all small limits. But limits are in fact somehow hard to understand (presentability only gives an abstract existence proof). In fact, the functor $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}$ definitely does not preserve limits. It can happen, that the presentable limit (e.g. an infinite product) of compactly assembled categories is again compactly assembled. But it will generally still not agree with the limit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$.

If necessary we shall denote the limit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ by $\lim^{\mathrm{ca}}$ and the limit in $\mathrm{Pr}^{\mathrm{L}}$ (which agrees with the limit in $\mathrm{Cat}_\infty$) by $\lim$ to make clear what is meant.

The idea to understand limits in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ is due to Clausen and basically constructs some sort of 'right adjoint' functor from presentable categories to compactly assembled categories. One can think of this as some 'compactly assembled core' sitting inside of each presentable category. Then the limit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ is given by taking the limit in $\mathrm{Pr}^{\mathrm{L}}$ and then taking this 'core'.

**Definition 2.7.1.**   1. We call a class of morphisms $S$ in a presentable $\infty$-category $\mathcal{C}$ an *ideal* if for $f, g, h$ composable in $\mathcal{C}$ and $g \in S$, we have $fgh \in S$.

2. For an ideal $S$, we write $S^{\mathbb{Q}}$ for the sub-ideal of those morphisms $f : X_0 \to X_1$ which extend over a $[0,1] \cap \mathbb{Q}$-indexed diagram all of whose nonidentity morphisms are in $S$, and call $S^{\mathbb{Q}}$ the *factorizable* morphisms in $S$. If $S = S^{\mathbb{Q}}$ we call $S$ factorizable.

3. We call an ideal *accessible* if there exists a cardinal $\kappa$ such that each morphism in $S^{\mathbb{Q}}$ factors over a $\kappa$-compact object of $\mathcal{C}$.

4. We call $S$ a *precompact ideal* if it is accessible, contains the identity on the initial object, and we have the following pushout condition: Given a diagram $F_0 \leftarrow F_1 \to F_2$ of functors $[0,1] \cap \mathbb{Q} \to \mathcal{C}$ with all positive-length morphisms in $S$, the pushout $F_0 \amalg_{F_1} F_2$ takes $0 \to 1$ to a morphism in $S$.[6]

Observe that compact morphisms in a compactly assembled category form a precompact ideal (which is factorizable). In analogy with compactly exhaustible objects, we call an object of $\mathrm{Ind}(\mathcal{C})$ *S-exhaustible* if it can be written as colimit

$$\mathrm{colim}_{\alpha \in \mathbb{Q}} jX_\alpha$$

of a $\mathbb{Q}$-indexed diagram where all "positive length" morphisms are in $S$. (Observe that if $S$ is factorizable, these agree with the objects that can be written as colimit of $\mathbb{N}$-indexed diagrams where the morphisms are in $S$. In particular, in the compactly assembled case they are exactly the objects which are of the form $\hat{j}X$ with $X$ compactly exhaustible.)

Note that the $S$-exhaustible objects only depend on $S^{\mathbb{Q}}$.

**Lemma 2.7.2.** *Assume $X = \mathrm{colim}_{\alpha \in \mathbb{Q}} jX_\alpha$ and $Y$ are two $S$-exhaustible Ind-objects, and we have a map $X \to Y$ of Ind-objects. Then we can represent $Y = \mathrm{colim}_{\alpha \in \mathbb{Q}} jY_\alpha$ and $X \to Y$ by a morphism of $\mathbb{Q}$-indexed diagrams.*

*Proof.* By cofinality, we have $X = \mathrm{colim}_{n \in \mathbb{N}} jX_n$. Write $Y = \mathrm{colim}_{\alpha \in \mathbb{Q}} jY_\alpha$. From the formula for mapping spaces in $\mathrm{Ind}(\mathcal{C})$, we find some $f : \mathbb{N} \to \mathbb{Q}$ and represent $X \to Y$ as natural transformation $jX_n \to jY_{f(n)}$. We may assume $f$ to be strictly increasing. Since all $[a,b] \cap \mathbb{Q}$ for $a, b \in \mathbb{Q}$ are isomorphic, we may reindex the diagram giving $Y$ to assume $f(n) = n$.

---

[6]Of course, this directly implies that all positive-length morphisms lie in $S$, since the restriction to $[a,b] \cap \mathbb{Q}$ is again the same situation.

Now let $g : \mathbb{Q} \to \mathbb{Q}$ be a morphism which takes $(-\infty, 0] \cap \mathbb{Q}$ isomorphically to $(0, 1] \cap \mathbb{Q}$, and $(n, n+1] \cap \mathbb{Q}$ isomorphically to $(n+1, n+2] \cap \mathbb{Q}$. The natural transformation $jX_n \to jY_n$ gives (by pre- and postcomposing) rise to a natural transformation $jX_\alpha \to jY_{g(\alpha)}$. Restricting $Y_\bullet$ along $g$, we obtain the desired representative. $\square$

**Lemma 2.7.3.** *For a precompact ideal $S$, $S$-exhaustible objects are closed under finite colimits.*

*Proof.* The initial object is $S$-exhaustible, so we need to show that they are closed under pushouts. Given a diagram $B \leftarrow A \to C$ of $S$-exhaustibles, Lemma 2.7.2 allows us to represent them as $\mathbb{Q}$-indexed diagrams compatibly. In the $\mathbb{Q}$-indexed diagram representing the pushout, we then have that every positive length morphism is in $S$ by the axioms for precompact ideals. So the pushout is itself $S$-exhaustible. $\square$

**Theorem 2.7.4** (Clausen). *Let $\mathcal{C}$ be a presentable $\infty$-category and $S$ be a set of morphisms in $\mathcal{C}$. Then there exists a terminal compactly assembled $\infty$-category $(\mathcal{C}, S)^{\mathrm{ca}}$ with a left adjoint functor $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ which sends compact morphisms in $(\mathcal{C}, S)^{\mathrm{ca}}$ to morphisms in $S$.*

*(Meaning that functors $\mathcal{D} \to \mathcal{C}$ from compactly assembled $\mathcal{D}$ which take compact morphisms in $\mathcal{D}$ to $S$, factor essentially uniquely through a compactly assembled functor $\mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$.)*

We will refer to $(\mathcal{C}, S)^{\mathrm{ca}}$ as the compactly assembled core of $\mathcal{C}$ with respect to $S$.

*Proof.* We first define $(\mathcal{C}, S)^{\mathrm{ca}}$ as a full subcategory of $\mathrm{Ind}(\mathcal{C})$ as the subcategory generated under colimits by $S$-exhaustible objects, and $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ as the restriction of the "colimit" functor.

Note the collection of $S$-exhaustible objects is small: We have a $\kappa$ such that every morphism in $S^{\mathbb{Q}}$ factors over a $\kappa$-compact object of $\mathcal{C}$. So every $X_n \to X_{n+1}$ factors over a $\kappa$-compact object of $\mathcal{C}$.Thus every $S$-exhaustible object is also a colimit of a sequential diagram of $\kappa$-compact objects of $\mathcal{C}$. Since $\mathcal{C}^\kappa$ is small by presentability, this shows that the collection of $S$-exhaustible objects is small. It also shows that the $S$-exhaustible objects lie in $\mathrm{Ind}(\mathcal{C}^\kappa) \subseteq \mathrm{Ind}(\mathcal{C})$. Since this inclusion is closed under colimits this also shows that $(\mathcal{C}, S)^{\mathrm{ca}} \subseteq \mathrm{Ind}(\mathcal{C}^\kappa)$ and so it is a full subcategory of a presentable $\infty$-category generated by a set of objects under colimits. As a result $(\mathcal{C}, S)^{\mathrm{ca}}$ is itself presentable.

Next we claim that $(\mathcal{C}, S)^{\mathrm{ca}}$ is compactly assembled. To do this, it suffices to prove that the generators $\mathrm{colim}_{\mathbb{Q}} jX_\alpha$ are compactly exhaustible. Indeed, we claim that for each $\beta \in \mathbb{Q}$, the canonical map

$$\mathrm{colim}_{\mathbb{Q}_{<\beta}} jX_\alpha \to \mathrm{colim}_{\mathbb{Q}_{<\beta+1}} jX_\alpha$$

is compact in $(\mathcal{C}, S)^{\mathrm{ca}}$. It lies in $(\mathcal{C}, S)^{\mathrm{ca}}$ since the posets $\mathbb{Q}_{<\beta}$ are isomorphic to $\mathbb{Q}$ (and so these are both themselves $S$-exhaustible), and is compact since it factors through the compact object $jX_\beta$ in the ambient category $\mathrm{Ind}(\mathcal{C})$ which has the same colimits.

Now we need to prove that $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ takes compact morphisms into $S$. So let $X \to Z$ be any compact morphism. We may factor it into two compact morphisms $X \to Y \to Z$, and write $Z = \mathrm{colim}_{i \in I} Z_i$ as filtered colimit of $S$-compactly exhaustible objects, using that

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$S$-exhaustibles are closed under finite colimits (Lemma 2.7.3). Compactness now gives us factorisations as follows:

$$
\begin{array}{ccc}
X & \longrightarrow & \mathrm{colim}_{\mathbb{Q}_{<\beta}} \, jZ_{i,\alpha} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z_i = \mathrm{colim}_{\mathbb{Q}} \, jZ_{i,\alpha} \\
\downarrow & & \downarrow \\
Z & \overset{\simeq}{\longrightarrow} & \mathrm{colim}_{i \in I} \, Z_i
\end{array}
$$

Since the functor to $\mathcal{C}$ takes the top right vertical arrow into $S$ (in $\mathcal{C}$, it factors through $Z_{i,\beta} \to Z_{i,\beta+1}$), and $S$ is an ideal, $X \to Z$ is taken into $S$.

Finally, we need to argue that postcomposition with $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ provides an equivalence between compactly assembled functors $\mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$ and functors $\mathcal{D} \to \mathcal{C}$ which take compact morphisms into $\mathcal{C}$. For that we observe that if we let $C$ be the class of compact morphisms in $\mathcal{D}$, then

$$(\mathcal{D}, C)^{\mathrm{ca}}$$

is the full subcategory of $\mathrm{Ind}(\mathcal{D})$ generated by, and hence coinciding with, the image of $\hat{\jmath}$. So for a functor $\mathcal{D} \to \mathcal{C}$ taking $C$ into $S$,

$$
\begin{array}{ccc}
(\mathcal{D}, C)^{\mathrm{ca}} & \longrightarrow & (\mathcal{C}, S)^{\mathrm{ca}} \\
\downarrow{\simeq} & & \downarrow \\
\mathcal{D} & \longrightarrow & \mathcal{C}
\end{array}
$$

provides the compactly assembled functor $\mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$. Write $\mathrm{Fun}^L((\mathcal{D}, C), (\mathcal{C}, S))$ for left adjoint functors taking $C$ into $S$, then the above diagram witnesses that the composite

$$\mathrm{Fun}^L((\mathcal{D}, C), (\mathcal{C}, S)) \to \mathrm{Fun}^{\mathrm{ca}}(\mathcal{D}, (\mathcal{C}, S)^{\mathrm{ca}}) \to \mathrm{Fun}^L((\mathcal{D}, C), (\mathcal{C}, S))$$

is an equivalence. Conversely, a compactly assembled functor $\mathcal{D} \to (\mathcal{C}, S)^{\mathrm{ca}}$ is uniquely determined by the corresponding diagram above, so the other composite is also an equivalence. $\qquad\square$

**Addendum 2.7.5.** *If all morphism in $S$ are strongly compact in $\mathcal{C}$ then $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is fully faithful.*

*Proof.* In this case we see from Lemma 2.3.2 that for any pair of $S$-exhaustible objects in $\mathrm{Ind}(C)$ the induced map on mapping spaces from $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is fully faithful. It follows that $((\mathcal{C}, S)^{\mathrm{ca}})^{\omega_1} \to \mathcal{C}$ is fully faithful and since it lands in fact in $\mathcal{C}^{\omega_1}$ the claim follows. $\qquad\square$

**Addendum 2.7.6.** *If $\mathcal{C}$ is already compactly assembled and $S$ contains all compact morphisms in $\mathcal{C}$, then $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ also admits a left adjoint and is a (left and right) Bousfield localisation.*

*Proof.* If $S$ contains compact morphisms, $\hat{\jmath}X$ is $S$-exhaustible for any $X \in \mathcal{C}$. Since $\hat{\jmath}$ is even left adjoint to $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$, it is left adjoint to $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$. So $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ admits a fully faithful left adjoint. It also admits a right adjoint, which is then also fully faithful. $\square$

Observe that the situation of Addendum 2.7.6 applies in particular to the class $S$ of all morphisms which factor over an $\omega_1$-compact object. In that case, $(\mathcal{C}, S)^{\mathrm{ca}}$ agrees with $\mathrm{Ind}(\mathcal{C}^{\omega_1})$, the functor to $\mathcal{C}$ is the colimit functor, and the fully faithful adjoints are our $\hat{\jmath}$ and $j$. One may think of Addendum 2.7.6 as describing a smaller version of this situation.

**Lemma 2.7.7.** $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ *is an equivalence if and only if $\mathcal{C}$ is compactly assembled and the following two classes of morphisms in $\mathcal{C}$ agree:*

1. *The compact morphisms.*

2. *Morphisms in $S^{\mathbb{Q}}$.*

*Proof.* If the two classes of morphisms agree and $\mathcal{C}$ is compactly assembled, $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is fully faithful and essentially surjective since $S$-exhaustibles and compactly exhaustibles agree. Conversely, denote the two classes of morphisms by $C$ and $S^{\mathbb{Q}}$, and assume that $(\mathcal{C}, S)^{\mathrm{ca}} \to \mathcal{C}$ is an equivalence. Then $\mathcal{C}$ is compactly assembled, since $(\mathcal{C}, S)^{\mathrm{ca}}$ is. Also, compact morphisms are taken into $S$, and so $C \subseteq S$, and since compact morphisms can be factored, also $C \subseteq S^{\mathbb{Q}}$.

This shows that every compactly exhaustible object is also $S$-exhaustible. So $\hat{\jmath} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ takes values in $(\mathcal{C}, S)^{\mathrm{ca}}$. As it is left adjoint to the colimit functor, which is an equivalence by assumption, we get that $\hat{\jmath}$ is the inverse equivalence. For a diagram $X_\alpha$ with $\alpha \in \mathbb{Q}$ and all nonidentity morphisms in $S$, we have

$$\mathrm{colim}_{\alpha \in \mathbb{Q}} \, jX_\alpha \in (\mathcal{C}, S)^{\mathrm{ca}},$$

and this is taken to $\mathrm{colim}\, X_\alpha$ in $\mathcal{C}$ by the colimit functor. Applying the inverse equivalence, we learn

$$\hat{\jmath} \, \mathrm{colim}\, X_\alpha \simeq \mathrm{colim}\, jX_\alpha$$

for any $\mathbb{Q}$-indexed diagram with nonidentity morphisms in $S$. Now let $X_0 \to X_1$ be an arbitrary morphism from $S^{\mathbb{Q}}$, and $X_\alpha$ with $\alpha \in [0,1] \cap \mathbb{Q}$ an extension to a diagram with nonidentity morphisms in $S$. In $\mathrm{Ind}(\mathcal{C})$, we have that $jX_0 \to jX_1$ factors over

$$\mathrm{colim}_{\alpha < 1} \, jX_\alpha \simeq \hat{\jmath} \, \mathrm{colim}_{\alpha < 1} X_\alpha,$$

which shows that $X_0 \to X_1$ is compact, finishing the proof. $\square$

Now we would like to explain how to compute limits in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$. To this end we assume that we have a diagram

$$I \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \qquad i \mapsto \mathcal{C}_i$$

and form the limit $\mathcal{C} = \lim_i \mathcal{C}_i$ of the underlying diagram in $\mathrm{Pr}^{\mathrm{L}}$. This limit is in fact computed in $\mathrm{Cat}_\infty$, so given as coCartesian sections of the Grothendieck construction of the functor $I \to \mathrm{Cat}_\infty$. Now we have the limit projections

$$p_i : \mathcal{C} \to \mathcal{C}_i$$

and we may form a precompact ideal of "pointwise compact morphisms":

**Lemma 2.7.8.** *Let $\mathcal{C}_i$ be a diagram in $\mathrm{Pr}^L$ with precompact ideals $S_i$ and functors preserving the precompact ideals. Then*

$$S = \{f : c \to c' \mid p_i(f) \in S_i \text{ for all } i\}$$

*forms a precompact ideal in the limit $\mathcal{C}$.*

*Proof.* Both the ideal condition and the pushout condition can be checked pointwise, since the functors $p_j : \lim^L \mathcal{C}_i \to \mathcal{C}_j$ create colimits. For the accessibility condition, we need to show that there exists $\kappa$ such that for any diagram $X : [0,1] \cap \mathbb{Q} \to \lim^L \mathcal{C}_i$ where all $p_i(X_\alpha \to X_{\alpha'})$ for $\alpha < \alpha'$ are compact, $X_0 \to X_1$ factors through $\kappa$-compact $Y$.

We take $Y = \mathrm{colim}_{\alpha < 1} X_\alpha$. Since the $p_i$ create colimits, this colimit is formed pointwise. Since a sequential colimit along compact maps is $\omega^1$-compact, all $p_i(Y)$ are $\omega_1$-compact. The pointwise $\omega_1$-compact objects are contained in the $\kappa$-compact objects of $\lim^L \mathcal{C}_i$ for some $\kappa$ depending only on the diagram, finishing the proof. $\square$

**Proposition 2.7.9.** *The $\infty$-category $(\mathcal{C}, S)^{\mathrm{ca}}$ is the limit of the diagram $I \to \mathrm{Pr}^L_{\mathrm{ca}}$. In formulas*

$$\lim{}^{\mathrm{ca}} \mathcal{C}_i = (\lim \mathcal{C}_i, S)^{\mathrm{ca}}$$

*Proof.* Compactly assembled morphisms $\mathcal{D} \to (\lim \mathcal{C}_i, S)^{\mathrm{ca}}$ are the same as morphisms $\mathcal{D} \to \lim \mathcal{C}_i$ which take compact morphisms into $S$. By definition of $S$, this is the same as a collection of compactly assembled morphisms $\mathcal{D} \to \mathcal{C}_i$. $\square$

Note that this in particular shows that $\lim^{\mathrm{ca}} \mathcal{C}_i$ is a full subcategory of $\mathrm{Ind}(\lim \mathcal{C}_i)$. This will be important later on.

**Example 2.7.10.** We compute the limit of

$$\ldots \to \mathcal{D}(\mathbb{Z}/p^n) \to \mathcal{D}(\mathbb{Z}/p^{n-1}) \to \ldots$$

in $\mathrm{Pr}^L_{\mathrm{ca}}$ which is called the $\infty$-category of nuclear modules over the analytic ring $\mathbb{Z}_p$ and denoted $\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p)$. [7]

In $\mathrm{Pr}^L$, the limit is $\mathcal{D}(\mathbb{Z})^\wedge_p$, the category of $p$-complete derived abelian groups (an object $X \in \mathcal{D}(\mathbb{Z})$ is $p$-complete if $X \simeq \lim_\mathbb{N} X/p^n$. Equivalently, if $\lim(\ldots X \xrightarrow{p} X \ldots) \simeq 0$). Note that this category is compactly generated: Shifts of $\mathbb{Z}/p$ are compact, and generate, since if $\mathrm{Map}(\mathbb{Z}/p[n], X) = 0$ for all $n$, $p : X \to X$ is an equivalence, but under completeness, this means $X = 0$. Also, the right adjoint functors $\mathcal{D}(\mathbb{Z}/p^n) \to \mathcal{D}(\mathbb{Z})^\wedge_p$ are just restriction, and so commute with filtered colimits (since anything hit by restriction is already $p$-complete). So the limit cone is in $\mathrm{Pr}^L_{\mathrm{ca}}$.

---

[7]There is a slight difference to the original nuclear category of Clausen-Scholze, which is why we include the tilde in the notation. We will explain this sublety later.

However, $\mathcal{D}(\mathbb{Z})_p^\wedge$ is not the limit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ (universality fails). Instead, the limit is given by $(\mathcal{D}(\mathbb{Z})_p^\wedge, S)^{\mathrm{ca}}$, where $S$ is the class of morphisms $X \to Y$ such that all $X/p^n \to Y/p^n$ are compact (over $\mathbb{Z}/p^n$).

This $S$ contains identities on compact objects in $\mathcal{D}(\mathbb{Z})_p^\wedge$ (such as $\mathbb{Z}/p^n$), but it also contains the identity on $\mathbb{Z}_p$. So $j\mathbb{Z}_p$ is among the $S$-exhaustibles. We also have more surprising nontrivial objects in $\lim^{\mathrm{ca}} \mathcal{D}(\mathbb{Z}/p^n)$, for example the colimit

$$\mathbb{Q}_p = \mathrm{colim}(j\mathbb{Z}_p \xrightarrow{p} j\mathbb{Z}_p \xrightarrow{p} \ldots)$$

Note that in this case, we do actually have that

$$\lim{}^{\mathrm{ca}} \mathcal{D}(\mathbb{Z}/p^n) \to \lim{}^{L} \mathcal{D}(\mathbb{Z}/p^n)$$

is a Bousfield localisation, related to the fact that the original limit cone was already in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$. This localisation for example kills $\mathbb{Q}_p$.

There are some limits that are preserved by the functor $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}} \to \mathrm{Pr}^{\mathrm{L}}$ and thus computed as limits of underlying $\infty$-categories. An obvious such limit is the product $\mathcal{C} \times \mathcal{D}$, which agrees with the coproduct and which is computed underlying since compact morphisms in the product are simply pairs of compact morphisms. That is, $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$ is semi-additive.

**Example 2.7.11.** For infinite products the map

$$\prod{}^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$$

is not an equivalence in general, even though the target is compactly assembled, since it is also the coproduct in $\mathrm{Pr}^{\mathrm{L}}$ and therefore also in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{ca}}$. We claim that the compact morphisms in $\prod \mathcal{C}_i$ are given by those morphisms which are levelwise compact and finitely supported, that is given by a morphism $X \to \emptyset \to Y$ almost everywhere. To see this we test against levelwise colimits and the colimit

$$(Y_0, \emptyset, \emptyset, \emptyset, \ldots) \to (Y_0, Y_1, \emptyset, \emptyset, \ldots) \to (Y_0, Y_1, Y_2, \emptyset, \ldots) \ .$$

There are in general certainly a lot of levelwise compact maps, which are not finitely supported. Thus we are in the situation of Addendum 2.7.6 and we see that the functor $\prod^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$ is a left and right Bousfield localization. The class of levelwise compact morphisms is also factorizable, thus by Lemma 2.7.7 we see that the map

$$\prod{}^{\mathrm{ca}} \mathcal{C}_i \to \prod \mathcal{C}_i$$

is an equivalence precisely if every levelwise compact morphism is already compact. This is only the case if the product is finite (i.e. almost all of the $\mathcal{C}_i$ are given by the point).

## 2.8  Tensor product on $\mathrm{Pr}^L$

As promised in the introduction, compactly assembled categories admit a further characterization as *dualizable* categories, with respect to a symmetric-monoidal structure on $\mathrm{Pr}^L$. The correct generality for this is in the stable setting, where the examples we care about take place. In this section, we want to discuss the required tensor product on $\mathrm{Pr}^L$ and the notion of stable $\infty$-categories. This tensor product is due to Lurie [Lur17a] and generally all results in this section are due to him.

**Definition 2.8.1.** For presentable $\mathcal{C}, \mathcal{D}, \mathcal{E}$ write $\mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for the full subcategory of the functor category consisting of all functors which are colimit-preserving in both variables separately.

**Definition 2.8.2.** The tensor product of two presentable $\infty$-categories $\mathcal{C}, \mathcal{D}$ is a presentable $\infty$-category $\mathcal{C} \otimes \mathcal{D}$ together with a functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

preserving colimits in each variable separately, such that precomposition induces an equivalence

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

for each presentable $\mathcal{E}$.

As usual, the universal property makes this essentially unique if it exists. We will see shortly that this is the case.

We also see directly that

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})),$$

so that the tensor product exhibits $\mathrm{Fun}^L$ as corresponding internal Hom. Since $\mathrm{Fun}^L(\mathrm{An}, \mathcal{C}) \simeq \mathcal{C}$, we see that An is the neutral element.

**Example 2.8.3.** If $\mathcal{C} = \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ and $\mathcal{D} = \mathrm{Fun}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{An})$, then

$$\mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) = \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})) = \mathrm{Fun}(\mathcal{C}_0, \mathrm{Fun}(\mathcal{D}_0, \mathcal{E})) = \mathrm{Fun}(\mathcal{C}_0 \times \mathcal{D}_0, \mathcal{E}),$$

which agrees with $\mathrm{Fun}^L(\mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An}), \mathcal{E})$. So we see that

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An}).$$

**Example 2.8.4.** Given a left Bousfield localisation $\mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}' \subseteq \mathcal{C}$ consists of the full subcategory of $W_{\mathcal{C}}$-local objects for some set of morphisms $W_{\mathcal{C}}$, and analogously for $\mathcal{D} \to \mathcal{D}'$, then

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'$$

is also a left Bousfield localisation, where $\mathcal{C}' \otimes \mathcal{D}'$ can explicitly be described as the full subcategory local with respect to $W_{\mathcal{C}} \otimes \mathcal{D}^{\kappa} \cup \mathcal{C}^{\kappa} \otimes W_{\mathcal{D}}$, with $\kappa$ large enough that $\mathcal{C}$ and $\mathcal{D}$ are $\kappa$-compactly generated.

Indeed, we have

$$\mathrm{Fun}^L(\mathcal{C}' \otimes \mathcal{D}', \mathcal{E}) = \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C}' \times \mathcal{D}', \mathcal{E}) = \mathrm{Fun}^{\mathrm{biL}, W_{\mathcal{C} \otimes \mathcal{D}} - \mathrm{loc}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) = \mathrm{Fun}^{L, W_{\mathcal{C} \otimes \mathcal{D}} - \mathrm{loc}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$$

Since we have seen that every presentable $\infty$-category can be written as Bousfield localisation of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$, the two examples combine to give existence of tensor products generally. In fact, we can provide a more useful formula:

**Lemma 2.8.5.** $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

*Proof.* Writing $\mathcal{C}$ and $\mathcal{D}$ as left Bousfield localisations of $\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ and $\mathrm{Fun}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{An})$ with respect to some generating equivalences, we have

$$
\begin{aligned}
\mathrm{Fun}^{\mathrm{lim}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) &\subseteq \mathrm{Fun}^{\mathrm{lim}}(\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})^{\mathrm{op}}, \mathcal{D}) \\
&= \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \\
&\subseteq \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Fun}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{An})) \\
&= \mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An}),
\end{aligned}
$$

where both inclusions are characterized by locality conditions. Tracing these through to the rightmost term, one sees that $\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ agrees with the full subcategory of $\mathrm{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathrm{An})$ on local objects, i.e. agrees with the left Bousfield localisation. $\square$

**Proposition 2.8.6.** *For topological spaces $X$ and $Y$ we have $\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y) \simeq \mathrm{Shv}(X \times Y)$.*

*Proof.* $\mathrm{Shv}(X)$ arises as Bousfield localisation of $\mathrm{PShv}(X) = \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{An})$, analogously for $Y$. So $\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y)$ arises as Bousfield localisation of

$$
\mathrm{Fun}((\mathrm{Open}(X) \times \mathrm{Open}(Y))^{\mathrm{op}}, \mathrm{An}).
$$

The locality condition is precisely descent in each variable. Since "boxes" $U \times V$ provide a basis of the topology of $X \times Y$, and the coordinatewise descent condition generates the same Grothendieck topology, this agrees with $\mathrm{Shv}(X \times Y)$. $\square$

**Proposition 2.8.7.** $\mathrm{Shv}(X) \otimes \mathcal{C} \simeq \mathrm{Shv}(X; \mathcal{C})$.

*Proof.* From Lemma 2.8.5, we get

$$
\mathrm{Shv}(X) \otimes \mathcal{C} \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathrm{Shv}(X)^{\mathrm{op}}, \mathcal{C}),
$$

which is a full subcategory of $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C})$ on functors whose extension in $\mathrm{Fun}^{\mathrm{lim}}(\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{An})^{\mathrm{o}}$ satisfies a locality condition. Unwinding definitions, this full subcategory is precisely $\mathrm{Shv}(X; \mathcal{C})$. $\square$

Although $\mathrm{Shv}(X)$ for us means $\mathrm{Shv}(X; \mathrm{An})$, this allows us to approach sheaves with values in various categories.

**Example 2.8.8.** Set is a left Bousfield localisation of An, at the class $W$ of $\pi_0$-isomorphisms. This exhibits $\mathrm{Set} \otimes \mathrm{Set}$ as localisation of $\mathrm{An} \otimes \mathrm{An} \simeq \mathrm{An}$, and inspection shows that it is again at the same class of morphisms, so $\mathrm{Set} \otimes \mathrm{Set} \simeq \mathrm{Set}$. Consequently,

$$
\mathrm{Shv}(X; \mathrm{Set}) \otimes \mathrm{Shv}(Y; \mathrm{Set}) = \mathrm{Shv}(X) \otimes \mathrm{Set} \otimes \mathrm{Shv}(Y) \otimes \mathrm{Set} \simeq \mathrm{Shv}(X \times Y; \mathrm{Set})
$$

**Lemma 2.8.9.** *Writing* $\mathrm{An}_*$ *for the category of pointed objects* $\mathrm{An}_{\mathrm{pt}/}$, *we have*

$$\mathrm{An}_* \otimes \mathcal{C} = \mathcal{C}_*,$$

*the category of pointed objects in* $\mathcal{C}$ *(slice under the terminal object). In particular,* $\mathrm{An}_* \otimes \mathrm{An}_* \simeq \mathrm{An}_*$.

*Proof.* We may write $\mathrm{An}_*$ as Bousfield localisation of $\mathrm{An}^{\Delta^1}$, consisting of the full subcategory of $\mathrm{An}^{\Delta^1}$ on all arrows whose first entry is pt. (The left adjoint $\mathrm{An}^{\Delta^1} \to \mathrm{An}_*$ takes $A \to X$ to $X/A$). This exhibits $\mathrm{Fun}^{\lim}((\mathrm{An}_*)^{\mathrm{op}}, \mathcal{C})$ as full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ with objects characterized by some locality condition. Unwrapping things, we exactly find the full subcategory of arrows where the first entry is pt, i.e. $\mathcal{C}_*$. $\qquad\square$

**Definition 2.8.10.** An $\infty$-category $\mathcal{C}$ is called *stable* if it has finite limits and colimits, a zero object, and a square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pullback diagram if and only if it is a pushout diagram.

For example, this holds in derived categories. Conversely, stable $\infty$-categories behave a lot like derived (or triangulated) categories, for example we can write pullbacks as fibers, etc. It can be seen that the condition on squares is equivalent to the suspension (or shift) functor $\Sigma : \mathcal{C} \to \mathcal{C}$ being an equivalence, where $\Sigma X$ is defined by the pushout

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

A kind of universal example (in presentable stable categories) is therefore given as follows:

**Definition 2.8.11.** The category of spectra $\mathrm{Sp}$ is the colimit

$$\mathrm{colim}(\mathrm{An}_* \xrightarrow{\Sigma} \mathrm{An}_* \xrightarrow{\Sigma} \dots)$$

in $\mathrm{Pr}^L$.

Note that this colimit can be computed as limit in $\mathrm{Pr}^R$ of the right adjoint functors

$$\lim(\dots \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*),$$

i.e. explicitly a spectrum consists of a sequence of pointed anima $X_n$ with equivalences $X_n \simeq \Omega X_{n+1}$.

**Lemma 2.8.12.** *1. The canonical map* $\mathrm{An} \to \mathrm{Sp}$ *induces an equivalence* $\mathrm{Sp} \simeq \mathrm{Sp} \otimes \mathrm{Sp}$. *The inverse equivalence makes* $\mathrm{Sp}$ *into a commutative algebra in* $\mathrm{Pr}^L$.

2. *A presentable category $\mathcal{C}$ is stable if and only if the canonical map*

$$\mathcal{C} \to \mathrm{Sp} \otimes \mathcal{C}$$

*is an equivalence. This makes $\mathcal{C}$ into a module over $\mathrm{Sp}$ in $\mathrm{Pr}^{\mathrm{L}}$ (with symmetric-monoidal structure given by $\otimes$).*

*Proof.* Tensoring

$$\mathrm{Sp} \simeq \mathrm{colim}(\mathrm{An}_* \xrightarrow{\Sigma} \dots)$$

with $\mathcal{C}$, we obtain

$$\mathrm{Sp} \otimes \mathcal{C} \simeq \mathrm{colim}(\mathcal{C}_* \xrightarrow{\Sigma} \dots).$$

If $\mathcal{C}$ is stable, we have $\mathcal{C}_* \simeq \mathcal{C}$, and $\Sigma$ is an equivalence, so in that case the colimit is just $\mathcal{C}$ again. In particular, this applies to $\mathcal{C}$. Conversely, $\mathrm{Sp} \otimes \mathcal{C}$ is stable, since the suspension functor on it can be written as $\Sigma \otimes \mathrm{id}_{\mathcal{C}}$, but $\Sigma : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence. $\qquad\square$

**Definition 2.8.13.** We write $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ for the full subcategory of $\mathrm{Pr}^{\mathrm{L}}$ on stable presentable categories.

By the above, these are exactly the modules over the idempotent $\mathrm{Sp}$. In particular, it is also symmetric-monoidal, with the same tensor product, and unit $\mathrm{Sp}$.

Note that if $\mathcal{C}$ is stable, $\mathcal{C}^\kappa$ is also stable: It suffices to check that $\kappa$-compact objects are closed under fibers, and if $X \to Y \to Z$ is a fiber sequence, $\Sigma^{-1}Y \to \Sigma^{-1}Z \to X$ is a cofiber sequence. Conversely, if $\mathcal{C}$ is small stable, $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable.

Examples of presentable stable $\infty$-categories include derived categories of rings. We may also think of $\mathrm{Sp}$ in such a way: Since $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$, $\mathrm{Sp}$ is an algebra object in $\mathrm{Pr}^{\mathrm{L}}$. This gives $\mathrm{Sp}$ a symmetric-monoidal structure, whose unit is called $\mathbb{S}$, the sphere spectrum. Explicitly, it is the image of $S^0$ under the canonical left adjoint $\mathrm{An}_* \to \mathrm{Sp}$. Since every object in a symmetric-monoidal category is canonically a module over the unit, we have $\mathrm{Sp} \simeq \mathrm{Mod}(\mathbb{S})$.

The right adjoint to the left adjoint functor $\mathrm{Sp} \to \mathcal{D}(R)$ corresponding to the unit $R[0] \in \mathcal{D}(R)$ (which is strong monoidal) takes $R[0]$ to an algebra object in $\mathrm{Sp}$, known as Eilenberg-MacLane spectrum $HR$. This right adjoint in fact induces an equivalence $\mathcal{D}(R) \simeq \mathrm{Mod}(HR)$.

**Example 2.8.14.** For commutative ring spectra $R, S$, one has $\mathrm{Mod}(R) \otimes \mathrm{Mod}(S) \simeq \mathrm{Mod}(R \otimes_{\mathbb{S}} S)$. In particular, for ordinary rings $R, S$, we have $\mathcal{D}(R) \otimes \mathcal{D}(S) \simeq \mathrm{Mod}(HR \otimes_{\mathbb{S}} HS)$.

The tensor product of two Eilenberg-MacLane spectra is rarely itself Eilenberg-MacLane (except in the rational case). So even if we only start with "ordinary derived categories", the tensor product in $\mathrm{Pr}^{\mathrm{L}}$ leads us to more general $\infty$-categories. However, the functor $\mathrm{CRing} \to \mathrm{CAlg}(\mathrm{Sp})$ taking an ordinary ring $R$ to the Eilenberg-MacLane spectrum $HR$ is fully faithful (Eilenberg-MacLane spectra are in a sense discrete), and so we will sometimes drop the $H$ and view ordinary rings as ring spectra, i.e. as algebras over $\mathbb{S}$.

**Definition 2.8.15.** If $\mathcal{C}$ is a commutative algebra in $\mathrm{Pr}^{\mathrm{L}}$, and $\mathcal{D}, \mathcal{E}$ are modules over it, the relative tensor product is defined as

$$\mathcal{D} \otimes_{\mathcal{C}} \mathcal{E} = \mathrm{colim}(\ \mathcal{D} \otimes \mathcal{E} \Leftleftarrows \mathcal{D} \otimes \mathcal{C} \otimes \mathcal{E} \Lleftarrow \mathcal{D} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{E} \dots\ )$$

**Lemma 2.8.16.** *For ordinary rings $R, S$, we have*

$$\mathcal{D}(R) \otimes_{\mathcal{D}(\mathbb{Z})} \mathcal{D}(S) = \mathrm{Mod}_{\mathcal{D}(\mathbb{Z})}(R \otimes_{\mathbb{Z}}^{L} S),$$

*in particular, if $R, S$ are Tor-independent, this simplifies to $\mathcal{D}(R \otimes_{\mathbb{Z}} S)$.*

Note that $\mathcal{D}(\mathbb{Z})$, as opposed to Sp, is not idempotent in $\mathrm{Pr}^{\mathrm{L}}$, as $\mathcal{D}(\mathbb{Z}) \otimes \mathcal{D}(\mathbb{Z}) \simeq \mathrm{Mod}(H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z})$, but $H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$ is very different from $H\mathbb{Z}$! So the $\mathcal{D}(\mathbb{Z})$-linear tensor product almost always differs from the underlying tensor product, and $\mathcal{D}(\mathbb{Z})$-linearity is really additionaly structure on a category, as opposed to stability.

$\mathcal{D}(\mathbb{Z})$-linear categories often arise from so-called *dg-categories*, which are a version of higher categories enriched in chain complexes rather than anima. Let us explain this enriched perspective a bit. For an algebra $\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}$ the unit functor $\mathrm{An} \to \mathcal{C}$ has a right adjoint $U : \mathcal{C} \to \mathrm{An}$ that is lax symmetric monoidal.

**Proposition 2.8.17.** *For a $\mathcal{C}$-module $\mathcal{D}$ in $\mathrm{Pr}^{\mathrm{L}}$ there is an extension*

$$\tag{2.2}$$



*characterised by the universal property that*

$$\mathrm{Map}_{\mathcal{C}}(Z, \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(A, B)) \simeq \mathrm{Map}_{\mathcal{D}}(Z \otimes A, B)).$$

*for $Z \in \mathcal{C}$ and $A, B \in \mathcal{D}$.*

*Proof.* We define a functor

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \qquad (A, B) \mapsto (Z \mapsto \mathrm{Map}_{\mathcal{D}}(Z \otimes A, B))$$

and note that in fact it lands in $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \simeq \mathcal{C}$. This then defines the desired functor with the universal property. The commutativity of the triangle 2.2 follows since the functor $U$ is given under the equivalence $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \simeq \mathcal{C}$ by evaluation at the tensor unit of $\mathcal{C}$. $\square$

One can in fact also lift the composition of maps

$$\mathrm{Map}_{\mathcal{C}}(b, c) \times \mathrm{Map}_{\mathcal{C}}(a, b) \to \mathrm{Map}_{\mathcal{C}}(a, c)$$

to maps

$$\mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(b, c) \otimes \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(a, b) \to \mathrm{Map}_{\mathcal{D}}^{\mathcal{C}}(a, c) .$$

Details left for the reader, also see [**?**] for a highly coherent statement in the sense of enriched categories.

In the case that $\mathcal{C} = \mathrm{Sp}$ we get mapping spectra and for simplicity we write $\mathrm{map}_{\mathcal{C}}(A, B)$ for $\mathrm{Map}_{\mathcal{C}}^{\mathrm{Sp}}(A, B)$. In fact, for any small stable $\infty$-category $\mathcal{C}$ we get a mapping spectrum functor

$$\mathrm{map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$$

for example by restricting the mapping spectrums functor from $\mathrm{Ind}(\mathcal{C})$.

## 2.9 Dualizable stable ∞-categories

In this section we would like to analyse when objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ are dualisable. These will exactly be the stable and compactly assembled ∞-categories. Recall that an object $\mathcal{C}$ in a symmetric monoidal ∞-category category is called dualisable if there exists another object $\mathcal{C}^{\vee}$ and maps

$$\mathrm{ev} : \mathcal{C}^{\vee} \otimes \mathcal{C} \to \mathbb{1} \qquad \mathrm{coev} : \mathbb{1} \to \mathcal{C} \otimes \mathcal{C}^{\vee}$$

such that the compositions

$$\mathcal{C} \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} \mathcal{C} \otimes \mathcal{C}^{\vee} \otimes \mathcal{C} \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathcal{C}$$

$$\mathcal{C}^{\vee} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} \mathcal{C}^{\vee} \otimes \mathcal{C} \otimes \mathcal{C}^{\vee} \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \mathcal{C}^{\vee}$$

are both homotopic to the identity. If the ambient ∞-category is closed symmetric monoidal, then we have that $\mathcal{C}^{\vee}$ has to be equivalent to $\underline{\mathrm{Hom}}(\mathcal{C}, \mathbb{1})$ and the map ev is the canonical evaluation map $\underline{\mathrm{Hom}}(\mathcal{C}, \mathbb{1}) \otimes \mathcal{C} \to \mathbb{1}$. In particular for the category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the dual of an object $\mathcal{C}$ has to be $\mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Sp})$.

**Theorem 2.9.1** (Lurie). *If $\mathcal{C}$ is stable and presentable, then $\mathcal{C}$ is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ if and only if it is compactly assembled.*

Before we give the proof of this theorem we note that by a specialisation of Theorem 2.2.11 being compactly assembled for a stable ∞-category is equivalent to:

1. $\mathcal{C}$ is generated under colimits by weakly (strongly) compactly exhaustible objects

2. The colimit functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint

3. The colimit functor $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a fully faithful left adjoint

4. $\mathcal{C}$ is a retract in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of a compactly generated, stable ∞-category.

5. Filtered colimits in $\mathcal{C}$ *distribute* over small products, i.e. we have

$$\prod_{K} \mathrm{colim}_{I} F \simeq \mathrm{colim}_{\prod_{K} I} \prod_{K} F$$

for $I$ filtered.

The slight changes follow since in stable ∞-categories colimits are automatically exact and since Ind of a stable ∞-category is also stable.

**Lemma 2.9.2.**

1. *If $\mathcal{C}_0$ is small stable, and $\mathcal{D}$ is presentable, $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D})$.*

2. *If $\mathcal{C}_0$ is small stable, and $\mathcal{D}$ is presentable and stable, $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^{L}(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{D})$.*

*Proof.* For the first statement, we have

$$\mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{D} \simeq \mathrm{Fun}^{\lim}(\mathrm{Ind}(\mathcal{C}_0)^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D})$$

by the universal property of Ind.

If additionally $\mathcal{D}$ is stable,

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{L}(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{D}) \qquad \square$$

Note that this implies

$$\mathrm{Fun}^{L}(\mathcal{D}, \mathrm{Ind}(\mathcal{C}_0) \otimes \mathcal{E}) \simeq \mathrm{Fun}^{L}(\mathcal{D}, \mathrm{Fun}^{L}(\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{E})) \simeq \mathrm{Fun}^{L}(\mathcal{D} \otimes \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}), \mathcal{E}).$$

This suggests that $\mathrm{Ind}(\mathcal{C}_0)$ is in fact dual to $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$:

**Lemma 2.9.3.** *If $\mathcal{C}_0$ is small stable, $\mathrm{Ind}(\mathcal{C}_0)$ is a dualizable object in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$, with dual $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$.*

*Proof.* We have an evaluation map, obtained as the functor

$$\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_0) \simeq \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}} \otimes^{\mathrm{Rex}(\omega)} \mathcal{C}_0) \to \mathrm{Sp}$$

which is the Ind-extension of the functor $\mathrm{map}_{\mathcal{C}_0} : \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{Sp}$ (using that it is exact in both arguments).

We also have a coevaluation map $\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \simeq \mathrm{Ind}(\mathcal{C}_0 \otimes^{\mathrm{Rex}(\omega)} \mathcal{C}_0^{\mathrm{op}})$. Since the target is stable, such a morphism is given by a single object. As objects are given by finite-limit preserving functors $(\mathcal{C}_0 \otimes^{\mathrm{Rex}(\omega)} \mathcal{C}_0^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{An}$, i.e. functors $\mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{An}$ which preserve finite limits in both arguments, this can again be given by $\mathrm{Map}_{\mathcal{C}_0}$.

One may then check that the "snake identities" are satisfied, which we won't do here. $\square$

*Proof of Theorem 2.9.1.* By the above, compactly generated categories are dualizable in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$. Since compactly assembled categories are retracts of compactly generated categories, they are also dualizable.[8]

Conversely, assume $\mathcal{C}$ is dualizable with dual $\mathcal{C}^{\vee}$. With $\kappa$ large enough so that $\mathcal{C}$ is $\kappa$-compactly generated, we have that

$$\mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathcal{C}$$

is a left Bousfield localisation. So also

$$\mathrm{Ind}(\mathcal{C}^{\kappa}) \otimes \mathcal{C}^{\vee} \to \mathcal{C} \otimes \mathcal{C}^{\vee}$$

is a left Bousfield localisation (see Example 2.8.4), in particular it is essentially surjective. Since in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$, a functor out of $\mathrm{Sp}$ is given precisely by an object, this means the counit $\mathrm{Sp} \to \mathcal{C} \otimes \mathcal{C}^{\vee}$ lifts to $\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}^{\kappa}) \otimes \mathcal{C}^{\vee}$. Under duality, this means that the identity $\mathcal{C} \to \mathcal{C}$ lifts to a functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\kappa})$, i.e. that $\mathcal{C}$ is a retract of a compactly generated category, finishing the proof. $\square$

---

[8]In general if the ambient category is idempotent closed or admits inner Hom's then retracts of dualizable objects are dualizable. This follows by simply using the 'restricted' evaluation and coevaluation maps.

**Definition 2.9.4.** We denote the category of dualisable, stable presentable $\infty$-categories and compactly assembled functors by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$.

Note that for a left adjoint functor between stable $\infty$-categories being compactly assembled is equivalent to being strongly left adjoint, that is the right adjoint admits a further right adjoint. This follows by Proposition 2.6.1 and the adjoint functor theorem since right adjoint functors between stable $\infty$-categories automatically preserve finite colimits.

We recall that a *Verdier sequence* of stable $\infty$-categories

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

is a sequence that is a fibre and cofibre sequence in the category $\mathrm{Cat}^{\mathrm{ex}}_\infty$ of stable $\infty$-categories and exact functors. Equivalently this means that $\mathcal{C} \to \mathcal{D}$ is fully faithful, the image is closed under retracts in $\mathcal{D}$ and the functor $\mathcal{D} \to \mathcal{E}$ exhibits $\mathcal{E}$ as the Dwyer-Kan localization at the morphisms in $\mathcal{D}$ whose fibre lies in $\mathcal{C}$. (I.e. it is universal among functors taking those morphisms to equivalences.) This Dwyer-Kan localization is also denoted as $\mathcal{E}/\mathcal{D}$ and called the Verdier quotient (it it is the cofibre of stable $\infty$-categories). Conversely

- every exact Dwyer-Kan localization $p : \mathcal{E} \to \mathcal{D}$ between stable $\infty$-categories is a Verdier quotient, i.e. sits in a Verdier sequence with $\mathcal{C} = \ker(p)$.

- Every exact, full faithful functor $\mathcal{C} \to \mathcal{D}$ of stable $\infty$-categories whose image is closed under retracts in $\mathcal{D}$ is a Verdier kernel, i.e. participates in a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ with $\mathcal{E} = \mathcal{D}/\mathcal{C}$.

We denote the $\infty$-category of small idempotent complete stable $\infty$-categories and exact functors by $\mathrm{Cat}^{\mathrm{perf}}_\infty$. Then a Karoubi sequence is a fibre and cofibre sequence in $\mathrm{Cat}^{\mathrm{perf}}_\infty$. Concretely that is a sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

where $\mathcal{C} \to \mathcal{D}$ is fully faithful and the morphism $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is an idempotent completion. Note that generally a Karoubi sequence is not a Verdier sequence since $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ doesn't have to be essentially surjective (equivalently $\mathcal{D} \to \mathcal{E}$ is not essentially surjective.)

However, if $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are all presentable and all functors are left adjoint, then a sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

is Verdier if and only if it is Karoubi (where we apply the notion from $\mathrm{Cat}^{\mathrm{perf}}_\infty$ to the large version $\widehat{\mathrm{Cat}}^{\mathrm{perf}}_\infty$) if and only if $\mathcal{D} \to \mathcal{E}$ is a Bousfield localization with kernel $\mathcal{C}$.

**Example 2.9.5.** A sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of small idempotent complete stable $\infty$-categories is a Karoubi sequence iff

$$\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$$

is a Verdier sequence, i.e. $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is a Bousfield localization with kernel $\mathrm{Ind}(\mathcal{C})$. Note also that in this case all the functors are strongly left adjoint, since they are compactly assembled, so the right adjoint commutes with filtered colimits, but by exactness also with finite colimits.

**Definition 2.9.6.** A functor of small, cocomplete $\infty$-categories $\mathcal{D} \to \mathcal{E}$ is called *homological epimorphism* if the induced functor

$$\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$$

is a left Bousfield localization. A map of ring spectra $R \to S$ is called homological epimorphism if the base-change functor $\mathrm{Perf}(R) \to \mathrm{Perf}(S)$ is a homological epimorphism, equivalently if the base-change functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is a left Bousfield localization.

Note that by the last Example we have that Karoubi quotients are homological epis.

**Lemma 2.9.7.** *For a map $R \to S$ of ring spectra with fibre $I$ the following are equivalent:*

1. $R \to S$ *is a homological epimorphism.*

2. *The multiplication $S \otimes_R S \to S$ is an equivalence.*

3. *We have $I \otimes_R S \simeq 0$.*

4. *The map $I \otimes_R I \to I$ induced by the multiplication is an equivalence.*

*Proof.* The right adjoint to the base-change functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is the restriction functor. This is fully faithful if and only if the counit of the adjunction is an equivalence, i.e. $S \otimes_R M \to M$ is an equivalence for any $S$-module $M$. As we may write the left hand side $(S \otimes_R S) \otimes_S M$, it suffices to check this for $M = S$, i.e. that $S \otimes_R S \to S$ is an equivalence. This proves $1 \Leftrightarrow 2$.

For the implication $2 \Leftrightarrow 3$, we consider the fiber sequence

$$I \otimes_R S \to R \otimes_R S \to S \otimes_R S,$$

noting that $S \simeq R \otimes_R S \to S \otimes_R S$ splits the multiplication $S \otimes_R S \to S$. Finally, for the implication $3 \Leftrightarrow 4$, we consider the fiber sequence

$$I \otimes_R I \to I \otimes_R R \to I \otimes_R S.$$

$\square$

We say that $I$ is an idempotent ideal in this case. Note that (by viewing ordinary rings as Eilenberg MacLane spectra), this contains the "almost mathematics" situation of a surjective map of rings $R \to S$ with kernel a flat ideal with $I^2 = I$, but also localisations of $R$, where the fiber will typically be a derived object with $\pi_{-1}$, see the Examples below.

**Example 2.9.8.** There are homological epimorphisms $p : \mathcal{D} \to \mathcal{E}$ between small stable $\infty$-categories, in fact between ring spectra, such that $\ker(p) \to \mathcal{D} \to \mathcal{E}$ is not a Karoubi sequence. In fact, $\ker(p)$ might be zero. For example, if $R$ is a local ring with flat maximal ideal $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathfrak{m}$, the kernel of $\mathrm{Perf}(R) \to \mathrm{Perf}(R/\mathfrak{m})$ consists of perfect complexes over $R$ which have zero base-change to $R/\mathfrak{m}$, but all of these are zero by Nakayama (cf. Example 2.2.10). This is still a homological epimorphism though.

**Lemma 2.9.9.** *If*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

*is a Verdier sequence, and either $F$ or $G$ admits a right adjoint, the other does, too, and*

$$\mathcal{C} \xrightarrow{R_G} \mathcal{B} \xrightarrow{R_F} \mathcal{A}$$

*is again a Verdier sequence. The analogous statement holds for left adjoints.*

*Proof.* If $G$ admits a right adjoint $R_G$, it is fully faithful, since $G$ is a localisation. We define a functor $R_F : \mathcal{B} \to \mathcal{A}$ by

$$R_F(b) = \mathrm{fib}(b \to R_G(G(b))),$$

noting that this lies in the kernel of $G$. One checks directly that this is right adjoint to $F$, since $\mathrm{Map}(F(a), R_G(G(b))) \simeq \mathrm{Map}(G(F(a)), G(b)) \simeq 0$.

Conversely, assume we have a right adjoint $R_F$ to $F$. We define an endofunctor $\mathcal{B} \to \mathcal{B}$ by taking $b \mapsto \mathrm{cofib}(F(R_F(b)) \to b)$. This functor annihilates $\mathcal{A}$, so factors uniquely through a functor $R_G : \mathcal{C} \to \mathcal{B}$. We now have that $G \circ R_G \simeq \mathrm{id}$, and a cofiber sequence

$$F \circ R_F \to \mathrm{id} \to R_G \circ G,$$

from which we directly get that $G \to G \circ R_G \circ G$ and $R_G \to R_G \circ G \circ R_G$ are equivalences, proving the adjunction.

Finally, $R_F$ is a right Bousfield localisation since $F$ is fully faithful, and $R_G$ is the inclusion of its kernel, as the above cofiber sequence proves that an object is in the image of $R_G$ if and only if it is in the kernel of $R_F$. $\quad\square$

**Proposition 2.9.10.** *For every homological epimorphism $\mathcal{D} \to \mathcal{E}$ between small stable $\infty$-categories the kernel*

$$K = \ker(\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E}))$$

*is dualisable. Conversely every dualisable, stable $\infty$-category arises in this way.*

*Proof.* For a homological epimorphism between stable $\infty$-categories the functor $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is strongly left adjoint. This is for example because the functor is compactly assembled, so its right adjoint preserves filtered colimits, and since everything is exact, all colimits. It follows from the previous Lemma 2.9.9 that the functor

$$K \to \mathrm{Ind}(\mathcal{D})$$

is also strongly left adjoint. In particular, $K$ is a retract of $\mathrm{Ind}(\mathcal{D})$ in $\mathrm{Pr}^L$.

Conversely we can write a given dualisable stable $\infty$-category $K$ as the kernel of the projection

$$\mathrm{Ind}(K^{\omega_1}) \to \mathrm{Ind}(K^{\omega_1})/K$$

where $K \to \mathrm{Ind}(K^{\omega_1})$ is given by $\hat{\jmath}$. This shows the converse. This quotient will be investigated systematically in Section 4.3. $\quad\square$

**Definition 2.9.11.** Let $R \to S$ be a map of ring spectra with fiber $I$, and assume the map $S \otimes_R S \to S$ is an equivalence. Following the logic of Lemma 2.9.7, we may think of $I$ as an idempotent ideal in $R$, in some higher-algebraic sense[9]. Then we denote the kernel of $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ by $\mathrm{Mod}(R, I)$.

We shall see soon that in fact every dualisable, stable $\infty$-category is of this form. Note that every map of pairs $R \to S$ to $R' \to S'$ with ideals $I, J$ induces a map $\mathrm{Mod}(R, I) \to \mathrm{Mod}(S, J)$ which is strongly left adjoint. The latter fact can be seen using that compact morphisms in $\mathrm{Mod}(R)$ are precisely given by those morphisms that factor in $\mathrm{Mod}(R)$ through a compact object and the functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ preserves compact objects.

**Example 2.9.12.** Let $R$ be a ring spectrum with an element $x \in \pi_*(R)$. Then there is a localization $R \to R[x^{-1}]$ which universally inverts $x$. Note that in the absence of commutativity (or at least an Ore condition) this is a bit hard to describe explicitly. Nevertheless we claim that $R \to R[x^{-1}]$ is a homological epimorphism.

To see this we consider the functor

$$\mathrm{Mod}(R) \to \mathrm{Mod}(R[x^{-1}])$$

induced from base change along $R \to R[x^{-1}]$. The target can be identified with those $R$-modules $M$ on which $x$ acts invertibly which follows from the universal property.[10] This functor is the Bousfield localization at the maps $R \xrightarrow{x} R$ given by right multiplication with $x$ (which are left $R$-module maps).

It follows that the kernel is generated by $R/x$, hence compactly generated. In the commutative case, this subcategory can be explicitly descibed as those $R$-modules $M$ with $x$-power torsion homotopy groups. There are a number of classical notations for this category like $\mathrm{Mod}(R)^{x\text{-nil}}$ or $\mathrm{Mod}(R \text{ on } R/x)$.

**Example 2.9.13.** Let $A \to B$ a surjective map of perfect $\mathbb{F}_p$-algebras. Then $A \to B$ is a homological epimorphism. This follows since

$$B \otimes_A B$$

is itself a perfect animated ring (the Frobenius is the tensor product of the Frobenii). But the higher homotopy groups of a perfect animated $\mathbb{F}_p$-algebra are always trivial since the Frobenius acts by zero on higher homotopy groups. So we only need to check that the map out of the *underived* tensor product $B \otimes_A B \to B$ is an isomorphism which is clear from surjectivity of $A \to B$. This also works of $A$ is only perfectoid ($B$ still is a perfect $\mathbb{F}_p$-algebra) since again the tensor product $B \otimes_A B \simeq B \widehat{\otimes}_A B$ is perfect which follows since it is perfectoid and characteristic $p$.

---

[9]One can make this structure precise, but we won't need it here, since we may equivalently characterize it in terms of $S$

[10]Note that in general $R \to R[x^{-1}]$ is not flat, even for ordinary rings [**?**]. Thus we have to take the derived base change.

**Example 2.9.14.** If $R$ is an ordinary commutative ring with flat ideal $I$ with $I^2 = I$, $\mathcal{D}(R) \to \mathcal{D}(R/I)$ is a homological epimorphism. It follows that the kernel

$$\mathcal{D}(R, I) = \ker(\mathcal{D}(R) \to \mathcal{D}(R/I))$$

is compactly assembled. This is the derived version of almost mathematics.

**Example 2.9.15** (Wodzicki)**.** Let $A$ be a $C^*$-algebra with a closed ideal $I \subseteq A$. Then $I$ is idempotent. We will discuss this at length later.

**Proposition 2.9.16.** *Consider a diagram*

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow p \\ \mathcal{D} & \xrightarrow{q} & \mathcal{E} \end{array}$$

*in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ *such that* $p : \mathcal{C} \to \mathcal{E}$ *is a left Bousfield localization (i.e. Verider quotient). Then the pullback in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ *is equivalent to the pullback in* $\mathrm{Pr}^{\mathrm{L}}$ *and the induced map* $\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D}$ *is a left Bousfield localization as well.*

*Proof.* Using the fact that $\mathcal{C} \to \mathcal{E}$ is a left Bousfield localization we can and will consider $\mathcal{E}$ as a full subcategory of $\mathcal{C}$ of local objects. Thus we consider $\mathcal{D} \to \mathcal{E}$ as a functor $\mathcal{D} \to \mathcal{C}$ and will implicitly use this perspective.

We consider the pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ in $\mathrm{Pr}^{\mathrm{L}}$. This can explicitly be described as the $\infty$-category of pairs of an object $d \in \mathcal{D}$ together with a morphism $c \to q(d)$ in $\mathcal{C}$ such that $c \to q(d)$ is a local equivalence. We claim that the right adjoint to the projection $\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D}$ is given by sending $d \in \mathcal{D}$ to the pair $(d, \mathrm{id}_{q(d)})$. This can easily be checked. Also clearly this right adjoint is fully faithful. Thus we see that the functor $p' : \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{D} \to \mathcal{D}$ is also a Bousfield localization. Note that so far we have not used that either the categories or the functors are stable or even compactly assembled.

Now we use that we have a full inclusion

$$\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \subseteq \mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$$

where the right hand side denotes the lax pullback, that is the $\infty$-category of all triples consisting of objects $d \in \mathcal{D}, c \in \mathcal{C}$ and a map $q(d) \to p(c)$. This inclusion is fully faithful (the image consist of those object where the map is an equivalence) and colimit preserving. We claim that the right adjoint is again colimit preserving, which then exhibits $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ as a retract of $\mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$. To see this we claim that the right adjoint is given by

$$(c, d, q(d) \to p(c)) \mapsto (c \times_{p(c)} q(d), d)$$

and this functor clearly commutes with filtered colimits. To see this we simply verify the universal property.

Thus in order to show that $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is dualisable it suffices to show that $\mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$ is dualisable. Using a retract we can reduce to the case that all three are compactly generated. But then it is easy to see that the lax arrow category is again compactly generated.

Finally we claim that a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is levelwise compact iff it is compact which finishes the proof. We see since the inclusion

$$\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \subseteq \mathcal{C} \overrightarrow{\times}_{\mathcal{E}} \mathcal{D}$$

is strongly left adjoint, that the inclusion detects compact morphisms. Finally it is easy to see that the compact morphisms in the lax pullback are precisely the levelwise compact. $\square$

**Corollary 2.9.17.** *Assume that $\mathcal{C} \to \mathcal{E}$ is a Bousfield localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$. Then the kernel, that is the objects in $\mathcal{C}$ mapping to $0$, is also dualisable and it is the fibre in dualisable $\infty$-categories.*

**Warning 2.9.18.** *If we have a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ with strongly left adjoint functors, then if $\mathcal{D}$ is dualisable, then so are $\mathcal{C}$ and $\mathcal{E}$ since both are retracts of $\mathcal{D}$ in $\mathrm{Pr}^{\mathrm{L}}$. But the converse fails: if $\mathcal{C}$ and $\mathcal{E}$ are dualisable, then $\mathcal{D}$ might not be dualisable.*

We have the functor

$$\mathrm{Cat}^{\mathrm{perf}}_{\infty} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$$

sending $\mathcal{C}$ to $\mathrm{Ind}(\mathcal{C})$. This is fully faithful. Every object in the $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ is the kernel of a Bousfield localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$. In other words: everyone in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ admits a 'canonical' resolution by objects in the essential image.

**Construction 2.9.19.** *There is a functor $(-)^{\vee} : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ such that the diagram*

$$
\begin{array}{ccc}
\mathrm{Cat}^{\mathrm{perf}}_{\infty} & \longrightarrow & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}} \\
\downarrow{\scriptstyle \mathrm{op}} & & \downarrow{\scriptstyle (-)^{\vee}} \\
\mathrm{Cat}^{\mathrm{perf}}_{\infty} & \longrightarrow & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}
\end{array}
$$

*commutes. On objects this functor send $\mathcal{C}$ to the dual $\mathcal{C}^{\vee}$ and an internal left adjoint functor $F : C \to D$ is send to the dual of the right adjoint $R_F$. This indeed gives a strongly left adjoint functor: since $F$ is left adjoint to $R_F$ internal to $\mathrm{Pr}^{\mathrm{L}}$ we get that $R_F^{\vee}$ is left adjoint to $F^{\vee}$ inside of $\mathrm{Pr}^{\mathrm{L}}$ and thus $R_F^{\vee}$ lies in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$.*

*The commutativity of the square then follows pointwise since the dual of $\mathrm{Ind}(\mathcal{C})$ is given by $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ and the dual of a morphism $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is given by restriction along $F^{\mathrm{op}}$ as one immediately verifies using the definition of the pairing. Now finally to see that this square commutes as a square of functors of $\infty$-categories we see that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ is an equivalences, since it is idempotent (one can skip the passing to the left adjoint step to see this) and by the previous claim it restricts to an equivalence of the full subcategories $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. But the only non-trivial self equivalence of $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ is given by opping.*

## 2.10  $H$-unital ring spectra

Recall from Lemma 2.9.7 that a map of ring spectra $R \to S$ is a homological epimorphism preicsely if the corresponding fibre $I$ is idempotent. In this section we shall highlight some of the properties that this fibre has. First note that the fibre $I$ is a non-unital ring spectrum. A non-unital ring spectrum is an algebra for an operad, the non-unital version of $\mathbb{E}_1$. It has been shown by Lurie that being unital is merely a property and not extra structure for a ring spectrum, in other words: if a unit exists, then it is unique [Lur17a]. Also for every non-unital ring spectrum $A$ there is a unitalization $A^+$ which is a spectrum is $A \oplus \mathbb{S}$ and the $\infty$-category of non-unital ring spectra is equivalent to the $\infty$-category of ring spectrum augmented over $\mathbb{S}$ through the functor $A \mapsto (A^+ \to \mathbb{S})$.

**Definition 2.10.1.** A non-unital ring spectrum $A$ is called $H$-unital (homologically unital) if the multiplication map

$$A \otimes_{A^+} A \to A$$

is an equivalence or equivalently if $A^+ \to \mathbb{S}$ is a homological epimorphism.

**Example 2.10.2.** Assume that $A$ is unital. Then it is $H$-unital, since in this case $A^+ = A \times \mathbb{S}$ and the augmentation is the projection to the second factor. This is clearly a homological epimorphism.

**Example 2.10.3.** Assume that we have a filtered diagram $i \mapsto A_i$ of ring spectra that are unital, but the transition maps might be non-unital. Then we call the colimit $A = \operatorname{colim} A_i$ in non-unital ring spectra a locally unital ring spectrum. We claim that $A$ is $H$-unital. To see this, observe that $A^+ \simeq \operatorname{colim}_i A_i^+$ as unital rings, and hence

$$A \otimes_{A^+} A \simeq \operatorname{colim} A_i \otimes_{A_i^+} A_i \simeq \operatorname{colim} A_i \simeq A.^{11}$$

For example we can consider the ring

$$M_\infty(R) = \operatorname{colim}_{n \to \infty} M_n(R)$$

of $(\infty \times \infty)$-matrices over a given ring $R$ in which almost all entries are zero.

**Proposition 2.10.4** (Tamme)**.** *Assume that*

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R' & \longrightarrow & S'
\end{array}
$$

*is a pullback diagram, and $R \to S$ is a homological epimorphism. Then*

  *1. $S' \simeq S \otimes_R R'$.*

---

[11] The same argument works for sifted colimits. We thank Claudius Heyer for pointing this out.

2. $R' \to S'$ is a homological epimorphism.

3. The square

$$\begin{array}{ccc} \mathrm{Mod}(R) & \longrightarrow & \mathrm{Mod}(S) \\ \downarrow & & \downarrow \\ \mathrm{Mod}(R') & \longrightarrow & \mathrm{Mod}(S') \end{array}$$

   is a pullback, in particular the induced functor

$$\mathrm{Mod}(R, I) \to \mathrm{Mod}(R', I)$$

   is an equivalence, where $I = \mathrm{fib}(R \to S) = \mathrm{fib}(R' \to S')$.

*Proof.* Let $I$ be the fiber of the horizontal maps. Since $R \to S$ is a homological epimorphism, we have $I \otimes_R S \simeq 0$ and thus

$$I \otimes_R S' \simeq (I \otimes_R S) \otimes_S S' \simeq 0.$$

From the fiber sequence

$$I \otimes_R I \to I \otimes_R R' \to I \otimes_R S'$$

we thus learn $I \otimes_R R' \simeq I$. Now the fiber sequence

$$I \otimes_R R' \to R \otimes_R R' \to S \otimes_R R'$$

tells us that $S \otimes_R R' \simeq S'$.

For the second statement, we observe

$$S' \otimes_{R'} I \simeq (S \otimes_R R') \otimes_{R'} I \simeq S \otimes_R I \simeq 0.$$

For the last statement, the functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R') \times_{\mathrm{Mod}(S')} \mathrm{Mod}(S)$ is fully faithful for any pullback of rings, and admits a right adjoint taking a triple of $M \in \mathrm{Mod}(R')$, $N \in \mathrm{Mod}(S)$ and equivalence $\varphi : S' \otimes_{R'} M \simeq S' \otimes_S N$ to the pullback

$$\begin{array}{ccc} P & \longrightarrow & N \\ \downarrow & & \downarrow \\ M & \longrightarrow & S' \otimes_{R'} M, \end{array}$$

viewed as $R$-module. To prove the claim it suffices to check that this right adjoint is conservative. So assume $P = 0$, and base-change the above pullback diagram along $R \to S$. We have $S \otimes_R N \simeq N$, $S \otimes_R M \simeq S' \otimes_{R'} M$, and $S \otimes_R S' \otimes_{R'} M \simeq S' \otimes_{R'} M \simeq S' \otimes_{R'} M$. So we see that

$$\begin{array}{ccc} S \otimes_R P & \longrightarrow & N \\ \downarrow & & \downarrow \\ S' \otimes_{R'} M & \longrightarrow & S' \otimes_{R'} M \end{array}$$

is a pullback, hence $N = 0$. Thus $S' \otimes_{R'} M = 0$ and the above pullback also proves $M = 0$. $\qquad\square$

**Corollary 2.10.5.** *If $R \to S$ is a morphism of ring spectra whose fiber $I$ is H-unital, then $R \to S$ is a homological epimorphism and $\mathrm{Mod}(R, I) = \mathrm{Mod}(I^+, I)$. In particular the $\infty$-category $\mathrm{Mod}(R, I)$ is independent of $R$.*

*Proof.* Apply to the pullback square

$$
\begin{array}{ccc}
I^+ & \longrightarrow & \mathbb{S} \\
\downarrow & & \downarrow \\
R & \longrightarrow & S.
\end{array}
$$

$\square$

There is a sort of converse to the last proposition.

**Proposition 2.10.6** (Tamme)**.** *Assume that*

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R' & \longrightarrow & S'
\end{array}
$$

*is a pullback diagram where all rings and the horizontal fibers are connective. Then if $R' \to S'$ is a homological epimorphism, $R \to S$ is, too. In particular, if $R \to S$ is a homological epimorphism between connective rings with $\pi_0(R) \to \pi_0(S)$ surjective, the fiber $I$ of $R \to S$ is H-unital.*

*Proof.* By [Lur18, Proposition 16.2.2.1], the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}(R)_{\geq 0} & \xrightarrow{\ F\ } & \mathrm{Mod}(S)_{\geq 0} \\
\downarrow & & \downarrow \\
\mathrm{Mod}(R')_{\geq 0} & \xrightarrow{\ F'\ } & \mathrm{Mod}(S')_{\geq 0}
\end{array}
$$

is a pullback diagram[12].

It follows then by abstract nonsense (as in the proof of Proposition 2.9.16) that the upper map is also a Bousfield localization, thus by unfolding what that means that $S \otimes_R S \simeq S$. This shows the claim. Note that this also shows that in fact, the square of module categories without passing to connective objects is a pullback by Proposition 2.10.4. $\square$

**Example 2.10.7.** The previous statement is wrong without connectivity assumptions. A counterexample is $k[x] \to k[x^{\pm}]$: if the fibre $I$, which has $\pi_{-1}$, was H-unital then we claim that the square

$$
\begin{array}{ccc}
\mathrm{Mod}(k) & \longrightarrow & \mathrm{Mod}(k[x^{-1}]) \\
\downarrow & & \downarrow \\
\mathrm{Mod}(k[x]) & \longrightarrow & \mathrm{Mod}(k[x^{\pm}])
\end{array}
$$

---

[12]This is wrong without the connectivity conditions on the rings and the fiber! Think of $k = k[x] \times_{k[x^{\pm 1}]} k[x^{-1}]$, but the pullback on module categories is given by quasicoherent sheaves on $\mathbb{P}^1$.

would have to be a pullback (which it isn't as argued in Footnote 12). To see this note that if $I$ was $H$-unital then the map $k \to k[x^{-1}]$, which has the same fibre would be a homological epimorphism and thus Proposition 2.10.4 the square would be a pullback.

One can also give a concrete argument for the case $k = \mathbb{Q}$, i.e. consider $R = \mathbb{Q}[x]$ and $S = \mathbb{Q}[x^{\pm 1}]$. The fiber $I$ has homotopy groups given by $\mathbb{Q}[x^{\pm 1}]/\mathbb{Q}[x]$ in degree $-1$. $I$ is H-unital if and only if $I \otimes_{I^+} S = 0$, which we may equivalently compute in the rational world (i.e. $I \otimes_{I \oplus \mathbb{Q}} \mathbb{Q}$). This is computed as colimit of the semi-simplicial diagram

$$I \; \substack{\longleftarrow \\ \longleftarrow} \; I \otimes I \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \; I \otimes I \otimes I \qquad \dots,$$

i.e. it has a filtration with $n$-th associated graded given by $I^{\otimes(n+1)}[n]$, which is concentrated in degree $-1$. So the resulting spectral sequence degenerates and leads to a countably infinitely dimensional $\pi_{-1}$. In particular, $I \otimes_{I^+} S \neq 0$ and $I$ is not H-unital.

**Definition 2.10.8.** For a ring spectrum $A$ we define the category of $H$-unital modules over $A$ as

$$\mathrm{Mod}_H(A) := \mathrm{Mod}(A^+, A) = \ker(\mathrm{Mod}(A^+) \to \mathrm{Mod}(\mathbb{S}))$$

If $A$ is $H$-unital then $A \in \mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(A)$ is a stable, dualisable $\infty$-category. A map $A \to B$ of $H$-unital ring spectra induces a functor $\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$ which is strongly left adjoint. The map $A \to B$ is called Morita equivalence if the induced functor is an equivalence.

A $H$-unital module is the same as a module $M$ over $A$ (which by definition means a module over $A^+$) such that

$$A \otimes_{A^+} M \simeq M$$

One can define a tensor product for non-unital modules using semi-simplicial realisations. Then this even reads as

$$A \otimes_A M \simeq M \ .$$

However we warn the reader that one should be very careful with this semi-simplicial Bar resolution since tensoring over a non-unital ring can behave quite pathological and unexpected. Therefore we prefer to write $\otimes_{A_+}$ instead.

**Example 2.10.9.** Assume that $A$ is a unital ring spectrum. Then

$$\mathrm{Mod}_H(A) = \mathrm{Mod}(A)$$

since $\mathrm{Mod}(A^+) = \mathrm{Mod}(A) \oplus \mathrm{Mod}(\mathbb{S})$ and the map $A^+ \to \mathbb{S}$ induces projection to the second summand.

But note that $\mathrm{Mod}_H(A)$ only depends on the underlying non-unital ring $A$. In particular for a non-unital map $\varphi : A \to B$ between unital rings we get an induced strongly left adjoint functor

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

which is somehow suprising. One can explicitly describe this functor also without reference to $H$-unitality of course, namely this functor is induced by a $B - A$-bimodule by Morita

theory, see Proposition 2.11.1 in the next section for a quick recap. This bimodule is given by the idempotent $e = \varphi(1)$ in $B$. That is

$$Be := \operatorname{colim}(B \xrightarrow{\cdot e} B \xrightarrow{\cdot e} B \xrightarrow{\cdot e} ...)$$

where this is a colimit in $B-A$-bimodules. Note that $Be$ is a retract of $B$ as a $B$-$A$-bimodule.

**Example 2.10.10.** Let $R$ be a ring (spectrum) and consider the map

$$R \to M_n(R)$$

where $M_n(R)$ is the ring spectrum of $n \times n$-matrices, i.e. $\operatorname{end}_R(R^n)$. The map is given by sending $R$ to matrices where all terms are zero except the upper left corner. This is a non-unital map. The corresponding idempotent $e$ is given by the matrix with a 1 on the upper left element and zero's everywhere else. The corresponding submodule is given by $R^n$ as a $M_n(R)$-$R$-bimodule. We therefore see that the induced functor

$$\operatorname{Mod}(R) \to \operatorname{Mod}(M_n(R))$$

is an equivalence, i.e. that the map $\varphi$ is a Morita equivalence.

**Lemma 2.10.11.** *Let $i \mapsto A_i$ be a filtered diagram of unital rings along non-unital ring maps with colimit $A$. Then the canonical functor*

$$\operatorname{colim}^{\mathrm{Pr}^{\mathrm{L}}} \operatorname{Mod}(A_i) \to \operatorname{Mod}_H(A)$$

*is an equivalence.*

*Proof.* This canonical functor sits in a diagram

$$
\begin{array}{ccc}
\operatorname{colim}^{\mathrm{Pr}^{\mathrm{L}}} \operatorname{Mod}(A_i) & \longrightarrow & \operatorname{Mod}_H(A) \\
\downarrow & & \downarrow \\
\operatorname{colim}^{\mathrm{Pr}^{\mathrm{L}}} \operatorname{Mod}(A_i^+) & \longrightarrow & \operatorname{Mod}(A^+) \\
\downarrow & & \downarrow \\
\operatorname{colim}^{\mathrm{Pr}^{\mathrm{L}}} \operatorname{Sp} & \longrightarrow & \operatorname{Sp}
\end{array}
$$

We have that

$$A^+ = \operatorname{colim} A_i^+$$

and thus that $\operatorname{Mod}(A^+) = \varprojlim^{\mathrm{Pr}^{\mathrm{R}}} \operatorname{Mod}(A_i^+) = \operatorname{colim}^{\mathrm{Pr}^{\mathrm{L}}} \operatorname{Mod}(A_i^+)$. Thus the lower two horizontal functors are equivalences. The claim now follows from the assertion that the vertical sequences are fibres sequences. Since this is clear for the right hand sequences we need to argue why the left hand sequence is a fibre sequence, i.e. why taking the kernel commutes with the filtered colimit. This however follows since the kernel of the left vertical map is the cokernel of the right adjoint, since the map is an strongly left adjoint Bousfield localization (seen by the fact that the lower maps are equivalences). Cokernels clearly commute with colimits. $\qquad\square$

**Example 2.10.12.** We find that $\mathrm{Mod}(M_\infty(R)) \simeq \mathrm{Mod}(R)$, that is the map $R \to M_\infty(R)$ is a Morita equivalence.

**Proposition 2.10.13.** *If $A$ is locally unital (filtered colimit of unital rings along non-unital maps) then $\mathrm{Mod}_H(A)$ is compactly generated. Conversely every compactly generated, stable $\infty$-category is equivalent to $\mathrm{Mod}_H(A)$ for some locally unital ring spectrum $A$.*

*Proof.* For $A = \mathrm{colim}\, A_i$ we get by Lemma 2.10.11 that

$$\mathrm{Mod}_H(A) = \underrightarrow{\mathrm{colim}}^{\mathrm{Pr}^L} \mathrm{Mod}(A_i) \ .$$

All the transition maps are strongly left adjoint, thus the colimit is also compactly generated, see Lemma 2.5.7. Now for a general compactly generated, stable $\infty$-category $\mathcal{C}$ we choose a set of compact generators $\{X_i\}_{i \in I}$, that is

$$\mathcal{C} = \langle X_i \mid i \in I \rangle$$

where the brackets denote the stable subcategory generated by the elements under colimits. We then also have that

$$\mathcal{C} = \mathrm{colim}_{F \subseteq I \text{finite}} \langle X_i \mid i \in F \rangle$$

and the subcategories $\mathcal{C}_F = \langle X_i \mid i \in F \rangle$ admit by definition a finite set of generators $\{X_i\}_{i \in F}$. But then they also have a single generator $X_F := \bigoplus_{i \in F} X_i$ and are therefore equivalent to $\mathrm{Mod}(E_F)$ where

$$E_F = \mathrm{end}_{\mathcal{C}}(X_F)$$

is the endomorphism spectrum of $X_F$. The maps $\mathcal{C}_F \to \mathcal{C}_{F'}$ for $F \subseteq F'$ are induced from the non-unital maps

$$E_F \to E_{F'}$$

which send an endomorphism $f$ of $X_F$ to the endomorphism $f \oplus 0$ of $X_{F'} = X_F \oplus X_{F' \backslash F}$. In particular we find that

$$\mathcal{C} = \mathrm{colim}^{\mathrm{Pr}^L} \mathrm{Mod}(E_F)$$

and so that

$$\mathcal{C} = \mathrm{Mod}_H(E)$$

where $E = \mathrm{colim}\, E_F$. $\qquad\square$

**Remark 2.10.14.** The non-unital ring spectrum $E$ from the last proof can also be described somewhat explicitly: the underlying spectrum is given by

$$E = \bigoplus_{i,j} \mathrm{map}_{\mathcal{C}}(X_i, X_j)$$

The multiplication map $E \otimes E \to E$ is induced by the maps

$$\mathrm{map}_{\mathcal{C}}(X_i, X_j) \otimes \mathrm{map}_{\mathcal{C}}(X_k, X_l) \to E$$

which for $j = k$ are given by composition (considered as an element of $\mathrm{map}_{\mathcal{C}}(X_i, X_l)$) and for $j \neq k$ by the zero map.

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**Theorem 2.10.15.** *Every dualisable, stable $\infty$-category $\mathcal{C}$ is equivalent to $\mathrm{Mod}_H(E)$ for some H-unital ring spectrum $E$.*

*Proof.* We choose a set $(X_i)_{i \in I}$ of $\omega_1$-compact objects of $\mathcal{C}$ with the property that they represent all equivalence classes of objects of $\mathcal{C}^{\omega_1}$, and consider

$$E = \bigoplus_{i,j} \mathrm{map}^c(X_i, X_j)$$

where $\mathrm{map}^c_\mathcal{C}(X, Y) = \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(jX, \hat{j}Y)$. We claim this is a non-unital ring. Indeed, consider the localisation sequence

$$\mathcal{C} \xrightarrow{\hat{j}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}.$$

By construction, the objects $jX_i$ are compact generators of $\mathrm{Ind}(\mathcal{C}^{\omega_1})$, and hence also of the quotient $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}$. By Proposition 2.10.13, this yields equivalences $\mathrm{Ind}(\mathcal{C}^{\omega_1}) \simeq \mathrm{Mod}_H(E')$ and $\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} \simeq \mathrm{Mod}_H(E'')$, where

$$E' = \bigoplus_{i,j \in I} \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX_i, jX_j)$$

$$E'' = \bigoplus_{i,j \in I} \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}}(jX_i, jX_j),$$

as well as a non-unital ring homomorphism $E' \to E''$ (since the $E'$ and $E''$ arose as colimit of endomorphisms of $\bigoplus_{i \in I} X_i$).

Using the adjoints in the above Verdier sequence, one sees

$$\mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C}}(jX_i, jX_j) \simeq \mathrm{map}_{\mathrm{Ind}(\mathcal{C}^{\omega_1})}(jX_i, jX_j/\hat{j}X_j),$$

and so the fiber of $E' \to E''$ is the $E$ defined above. In particular, $E$ inherits a non-unital ring structure.

We now claim that $E$ is H-unital. Assuming this, we get that $E' \to E''$ is an H-epi, and that $\mathrm{Mod}_H(E) \simeq \mathrm{fib}(\mathrm{Mod}((E')^+) \to \mathrm{Mod}((E'')^+))$. Since we also have a pullback

$$
\begin{array}{ccc}
\mathrm{Mod}_H(E') & \longrightarrow & \mathrm{Mod}_H(E'') \\
\downarrow & & \downarrow \\
\mathrm{Mod}((E')^+) & \longrightarrow & \mathrm{Mod}((E'')^+)
\end{array}
$$

this identifies $\mathcal{C} \simeq \mathrm{Mod}_H(E)$.

To see that $E$ is H-unital, we need to check that the augmented semi-simplicial object

$$E \longleftarrow E \otimes E \rightleftarrows E \otimes E \otimes E \qquad \cdots$$

exhibits $E$ as colimit (as the left Kan extension of this diagram to $\Delta^{\mathrm{op}}$ is exactly the bar construction computing $E \otimes_{E^+} E$). For any object $Y$, we have a left $E$-module $M(Y) = \bigoplus_i \mathrm{map}^c(X_i, Y)$. We will show indeed that

$$M(Y) \longleftarrow E \otimes M(Y) \rightleftarrows E \otimes E \otimes M(Y) \qquad \cdots$$

is a colimit diagram. Assume first that $Y$ is $\omega_1$-compact, and write $Y = \operatorname{colim}_n Y_n$ along compact maps. We may assume $Y_n$ to be $\omega_1$-compact, and hence $Y_n = X_{i_n}$ for some sequence $i_n$. We choose witnesses $jX_{i_n} \to \hat{j}X_{i_{n+1}}$ representing these compact maps. We have a map

$$M(X_{i_n}) \to E \otimes M(X_{i_{n+1}})$$

induced by $\mathbb{S} \to \operatorname{map}^c(X_{i_n}, X_{i_{n+1}})$. These satisfy the identities of an "extra degeneracy" up to postcomposing with $M(X_{i_n}) \to M(X_{i_{n+1}})$, i.e. we get a dashed lift in

$$
\begin{array}{ccc}
M(X_{i_n}) & \longleftarrow & |E \otimes M(X_{i_n}) \ldots | \\
\downarrow & & \downarrow \\
M(X_{i_{n+1}}) & \longleftarrow & |E \otimes M(X_{i_{n+1}}) \ldots |.
\end{array}
$$

In the colimit, that means we have an equivalence

$$M(Y) \simeq |E \otimes M(Y) \ldots |.$$

Since both sides commute with $\omega_1$-filtered colimits and $Y$ was an arbitrary $\omega_1$-compact object, this more generally follows for arbitrary $Y$, in particular for $Y = \bigoplus X_i$, where we have $M(Y) \simeq E$. $\square$

In particular we see that every dualisable category is an almost category $\operatorname{Mod}(R, I)$ with $I$ $H$-unital (so that the category doesn't even depend on $R$).

**Remark 2.10.16.** We see that

## 2.11   $H$-unital Morita Theory

In this section we will analyse functors between categories of the form $\operatorname{Mod}_H(A)$ in terms of non-unital ring spectra. Let us first recall and extend usual Morita theory for ring spectra. For an ordinary land analogue see [**?**].

**Proposition 2.11.1.**   *1. For unital rings $A$, $B$, we have*

$$\operatorname{Fun}^L(\operatorname{Mod}(A), \operatorname{Mod}(B)) \simeq \operatorname{BiMod}(B, A).$$

2. *For unital rings $A$, $B$, we have that $\operatorname{Map}_{\operatorname{Ring}}(A, B)$ agrees with the space of pairs consisting of a left adjoint functor $\operatorname{Mod}(A) \to \operatorname{Mod}(B)$ together with an equivalence $F(A) \simeq B$. Equivalently a commutative diagram*

$$
\begin{array}{ccc}
& \text{An} & \\
& \downarrow \quad \searrow & \\
\operatorname{Mod}(A) & \longrightarrow & \operatorname{Mod}(B)
\end{array}
$$

*in $\operatorname{Pr}^L$.*

3. *For $H$-unital rings $A$, $B$, we have that*

$$\mathrm{Fun}_H^L(\mathrm{Mod}(A), \mathrm{Mod}(B)) \simeq \mathrm{BiMod}_H(B, A),$$

*where the right hand side is given by the full subcategory of $B^+$-$A^+$-bimodules which lie in $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ when viewed as left or right modules.*

4. *For $H$-unital rings $A$, $B$ where $A$ admits a unit, $\mathrm{Map}_{\mathrm{Ring}^{\mathrm{nu}}}(A, B)$ agrees with the space of left adjoint functors $F : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ together with maps $F(A) \xrightarrow{i} B \xrightarrow{r} F(A)$ exhibiting $F(A)$ as retract of $B$.*

*Proof.* For the first statement, observe that $F(A)$ is a left $B$-module, but also has $\mathrm{end}(A)$ acting by functoriality. As $\mathrm{end}(A) = A$ acting from the right, we have a $B$-$A$-bimodule structure on $F(A)$. As every map $A \to X$ gives a map $F(A) \to F(X)$, we have a natural transformation

$$F(A) \otimes_A X \to F(X),$$

which is an equivalence if $F$ preserves colimits.

For the second statement, observe that for a ring homomorphism $A \to B$, the associated functor $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is given by

$$B \otimes_A -,$$

i.e. corresponds to the $B$-$A$ bimodule $B$. Conversely, given a functor $F$ with $F(A) \simeq B$ as left module, the functor provides a ring homomorphism

$$A \to \mathrm{end}_{\mathrm{Mod}(A)}(A) \to \mathrm{end}_{\mathrm{Mod}(B)}(B) \simeq B$$

which describes the right $A$-module structure on $F(A)$. So ring homomorphisms correspond precisely to functors with an isomorphism $F(A) \simeq B$.

For the third statement, observe that the Verdier sequence

$$\mathrm{Mod}_H(A) \overset{\longleftarrow}{\underset{\longrightarrow}{}} \mathrm{Mod}(A^+) \overset{\longleftarrow}{\underset{\longrightarrow}{}} \mathrm{Mod}(\mathbb{S})$$

exhibits $\mathrm{Mod}_H(A)$ as Bousfield localisation of $\mathrm{Mod}(A^+)$ with kernel the modules restricted along $A^+ \to \mathbb{S}$. This means that exact functors $\mathrm{Mod}_H(A) \to \mathcal{C}$ correspond to exact functors $\mathrm{Mod}(A^+) \to \mathcal{C}$ which annihilate modules restricted from $\mathbb{S}$. So left adjoint functors

$$\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$$

correspond to left adjoint functors

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

which take values in $\mathrm{Mod}(B^+)$ and annihilate modules restricted from $\mathbb{S}$. Translated to bimodules, the first condition just means $\mathbb{S} \otimes_{B^+} M \simeq 0$. For the second it is necessary that $M \otimes_{A^+} \mathbb{S} = 0$, but also sufficient, since if $N$ is restricted from $\mathbb{S}$, $M \otimes_{A^+} N \simeq M \otimes_{A^+} \mathbb{S} \otimes_{\mathbb{S}} N$.

For the final statement observe that for a nonunital ring homomorphism $A \to B$, the restriction of the base-change $B^+ \otimes_{A^+} -$ to $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ is computed as the composite

$$\mathrm{Mod}_H(A) \to \mathrm{Mod}(A^+) \to \mathrm{Mod}(B^+) \to \mathrm{Mod}_H(B),$$

i.e. is given by the $B^+$-$A^+$-bimodule $B \otimes_{A^+} A$. If $A$ is unital, we have an $A^+$-module homomorphism $A^+ \to A$ exhibiting $A$ as retract of $A^+$, so $B \otimes_{A^+} A$ as retract of $B$. Conversely, if $F(A)$ is a retract of $B$, in particular of $B^+$, we get a nonunital ring map

$$A \to \mathrm{end}_{B^+}(F(A)) \to \mathrm{end}_{B^+}(B^+) = B^+$$

which lifts to the fiber of $B^+ \to \mathbb{S}$. (The first map is unital, the second one is nonunital and arises from the retraction). $\square$

Now we would also like to understand functors between dualisable stable $\infty$-categories using zig-zag's of maps of non-unital rings. As a warm up, we first prove a version for unital rings.

**Proposition 2.11.2.** *For $A$ and $B$ unital rings every strongly left adjoint functor $F$ : $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is induced by a zig-zag $A \to C \overset{\simeq}{\leftarrow} B$ of non-unital maps, where $C$ is also unital and $B \to C$ is a Morita equivalence.*

*Proof.* For a functor $F$ we define

$$C := \mathrm{end}_B(F(A) \oplus B)$$

Since $F(A) \oplus B$ is a compact generator of $\mathrm{Mod}(B)$ we have that

$$\mathrm{Mod}(B) \simeq \mathrm{Mod}(C)$$

induced by the map $B \to C$ induced by the split inclusion $B \to F(A) \oplus B$. Similarly we have a map

$$A = \mathrm{end}_A(A) \to \mathrm{end}_B(F(A)) \to \mathrm{end}_B(F(A) \oplus B)$$

of non-unital rings which induces the functor $F$. $\square$

We would like to show a converse to the latter statement. To this end we note that there is a natural notion of 2-morphisms between non-unital morphisms $f, g : A \to B$ where $A$ and $B$ are unital. For simplicity lets first assume that $A$ and $B$ are discrete. Then such a 2-morphism is given by given by an element $b \in f(1)Bg(1)$ such that

$$f \cdot b \simeq b \cdot g .$$

We claim that such a 2-morphism is the same as a natural transformation of the functors

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

induced by $f$ and $g$. This follows since by Proposition 2.11.1 and Example 2.10.9 such a natural transformation is given by a $B$-$A$-bimodule map $Bf(1) \to Bg(1)$. Every left $B$-module map $Bf(1) \to Bg(1)$ is given by right multiplication with an element $b \in f(1)Bg(1)$ and this is a right $A$-module map precisely if for every $a \in A$ we have that $f(a)b = bg(a)$. Now if $A$ and $B$ are not discrete anymore we find similar that the space of natural transformations between the induced functors can be expressed as elements [13] $b \in f(1)Bg(1)$ together with an equivalence $f \cdot b \simeq b \cdot g$.

Using this notion of 2-morphisms we can define an $(\infty, 2)$-category of unital algebras, non-unital maps and 2-morphisms. We denote the $\infty$-categorical core, i.e. the largest $(\infty, 1)$-category contained in this $(\infty, 2)$-category by $\mathrm{Alg}_2^{\mathrm{u}}$. Concretely the 2-morphisms in $\mathrm{Alg}_2^{\mathrm{u}}$ are given by elements $b \in f(1)Bg(1)$ that are units in the sense that there exists a $b' \in g(1)Bf(1)$ such that $bb' \simeq f(1)$ and $bb \simeq g(1)$. Now we have a functor of $\infty$-categories

$$\mathrm{Alg}_2^{\mathrm{u}} \to \mathrm{Pr}_\omega^{\mathrm{L}} \qquad A \mapsto \mathrm{Mod}(A)$$

which sends the class of Morita equivalences $W$ to equivalences of $\infty$-categories. Thus we get an induced functor

$$\mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \to \mathrm{Pr}_{\mathrm{st},\omega}^{\mathrm{L}} \ .$$

This functor lands in the full subcategory of $\mathrm{Pr}_{\mathrm{st},\omega}^{\mathrm{L}}$ consisting of those compactly generated $\infty$-categories that admit a compact generator, aka monogenic ones.

**Proposition 2.11.3.** *The functor* $\mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \to \mathrm{Pr}_{\mathrm{st},\omega}^{\mathrm{L}}$ *is fully faithful with essential image the monogenic stable $\infty$-categories.*

Note that this is a statement about $\infty$-categories. Both categories extend in fact naturally to $(\infty, 2)$-categories and one could also make a $(\infty, 2)$-categorical statement here. But we will not attempt to formulate or proof such a statement here to avoid the use of DK-localizations for $(\infty, 2)$-categories.

Our proof of Proposition 2.11.3 relies on the following statement and will be given below.

**Lemma 2.11.4.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\infty$-categories and $W$ the class of morphisms send to equivalences by $F$. Assume that for every pair of objects $X, Y \in \mathcal{C}$ the induced functor*

$$\mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} \mathrm{Map}_{\mathcal{C}}(X, \hat{Y}) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

*is an equivalence. Then the induced functor $\mathcal{C}[W^{-1}] \to \mathcal{D}$ is fully faithful.*

*Proof.* Consider the left Kan extension functor $L : \mathrm{Fun}(\mathcal{C}, \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An})$ which is left adjoint to the fully faithful restriction functor $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{An})$. By general nonsense we have that for every fixed object $X$ the corepresentable functor $\underline{X} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ is send by $L$ to the corepresentable $\underline{X} \in \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An})$. For an arbitrary object $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ we construct $\widehat{F} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$

$$\widehat{F}(Y) = \mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} F(\hat{Y})$$

---

[13]By an element in a spectrum $A$ we mean a map $\mathbb{S} \to A$ or equivalently a point in $\Omega^\infty A$.

and maps from $\widehat{F}$ into any functor $G$ which lies in the image of the restriction

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{An})$$

(i.e. $G$ sends $W$ to equivalences) are equivalent to maps from $F$ to $G$. Thus if $\widehat{F}$ has the property that it sends $W$ to equivalences, then $\widehat{F} = L(F)$.

We now apply this construction to $\underline{X} \in \mathrm{Fun}(\mathcal{C}, \mathrm{An})$ and the assumption of the statement implies that

$$\widehat{\underline{X}} \simeq \mathrm{Map}_{\mathcal{D}}(F(X), F(-))$$

indeed sends $W$ to equivalences. Therefore we have that $\widehat{\underline{X}}$ is given by the left Kan extension, that the mapping space in $\mathcal{C}[W^{-1}]$ which finishes the proof. $\qquad\square$

**Remark 2.11.5.** A more explicit description of the colimit $\mathrm{colim}_{Y \xrightarrow{\simeq} \hat{Y}} \mathrm{Map}_{\mathcal{C}}(X, \hat{Y})$ from the previous statement is given as the geometric realization of the $\infty$-category of spans

$$X \to \hat{Y} \xLeftarrow{\simeq} Y \ .$$

This follows using that the $\infty$-category of such spans is the unstraightening of the functor

$$\left( Y \xrightarrow{\simeq} \hat{Y} \right) \mapsto \mathrm{Map}_{\mathcal{C}}(X, \hat{Y}) \ .$$

*Proof of Proposition 2.11.3.* We want to apply Lemma 2.11.4. To this end we have to show that the functor

$$\mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \tag{2.3}$$

is an equivalence after realizing the source, which is the zigzag-category whose objects are unital algebras. Morphisms are zigzags of the form $A \to C \xLeftarrow{\simeq} B\}$ with $\simeq$ indicating that the morphism is a Morita equivalence. A 2-morphism in this category is a diagram



where the triangles can are filled by 2-morphisms in $\mathrm{Alg}_2^{\mathrm{u}}$.

To see that (2.5) is an equivalence we use the construction from Proposition 2.11.2 to produce a functor in the opposite direction:

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \to \mathrm{ZigZag}(A, B) \quad F \mapsto \left( A \to C(F) \xLeftarrow{\simeq} B \right)$$

where $C(F) = \mathrm{end}_B(F(A) \oplus B)$, which is clearly functorial in natural equivalences. [14]

---

[14]Note that this functor in fact doesn't use the 2-morphisms in $\mathrm{Alg}^u$, so it lands in in the zigzag category associated with the smaller $\infty$-category of unital algebras and non-unital maps.

Now by construction (see the proof of Proposition 2.11.2) we see that the composition

$$\text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq} \to \text{ZigZag}(A, B) \to \text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq}$$

is equivalent to the identity. It therefore remains to also show that the composition

$$|\text{ZigZag}(A, B)| \to \text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq} \to |\text{ZigZag}(A, B)|$$

is homotopic to the identity. To this end, it suffices to construct a zigzag of natural morphisms in $\text{ZigZag}(A, B)$ from any span $A \xrightarrow{\varphi} C \xleftarrow{\psi} B$ to the induced span $A \to C(F) \xleftarrow{\simeq} B$ with $F$ the induced functor from the span. Let us first work out what $C(F)$ is: by definition it is given by $\text{end}_B(F(A) \oplus B)$ where $F$ is the functor

$$\text{Mod}(A) \xrightarrow{\varphi^*} \text{Mod}(C) \xleftarrow{\psi^*} \text{Mod}(B)$$

induced from the span. Since the right hand functor is an equivalence we have

$$C(F) = \text{end}_B(F(A) \oplus B) \simeq \text{end}_C(\varphi^*(A) \oplus \psi^*(B))$$

with the maps $A \to C(F)$ and $B \to C(F)$ given by the maps $A \to \text{end}_C(\varphi^*(A)) \to C(F)$ and $B \to \text{end}_C(\psi^*(B)) \to C(F)$. Recall that $\varphi^*(A)$ is a retract of $C$ and $\psi^*(B)$ is a retract of $C$ as well (see Example 2.10.9). Thus we have maps of non-unital rings $\text{end}_C(\psi^*(B)) \to C$ and $\text{end}_C(\varphi^*(A)) \to C$ as well as $C(F) = \text{end}_C(\varphi^*(A) \oplus \psi^*(B)) \to \text{end}_C(C \oplus C)$ we now consider the diagram in $\text{Alg}_2^{\text{u}}$ given as



where $i_0$ always denotes an 'inclusion' into the first summand (i.e. the map on endomorphisms obtained by the inclusion) and $i_1$ the inclusion into the second summand. Note that this a diagram of unital rings and non-unital maps except for the lower left triangle. This does not commute, but we claim that there is a 2-morphism in $\text{Alg}_2^{\text{u}}$ filing it. Concretely this 2-morphism is given by the element

$$\begin{pmatrix} 0 & \text{id}_C \\ 0 & 0 \end{pmatrix} \in \text{end}_C(C \oplus C)$$

which conjugates one map into the other (and in fact lies in the correct summand of $\text{end}_C(C \oplus C)$). Alternatively one can also consider the induced diagram on module categories and see that it commutes (almost all the non-unital rings are generators of $\text{Mod}(C)$). The left lower triangle commutes since both maps $C \to \text{end}_C(C \oplus C)$ induced the same functor, as the corresponding bimodules are both given by $C^2$ with left $\text{end}_C(C \oplus C)$-action and right $C$-action. This finishes the proof. $\qquad \square$

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**Remark 2.11.6.** Given the previous statement, one might ask to which extend the category of spans $A \to C \xleftarrow{\simeq} B$ without the conjugation 2-morphisms already models the homotopy type of the space

$$\mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}_H(A), \mathrm{Mod}_H(B))^{\simeq},$$

that is whether the map from the realization of the category of these smaller spans to the space of strongly left adjoint functors is an equivalence. The proof of the previous proposition shows that this map admits a section. We however believe that it is not an equivalence, since we believe that for $A = B$ the two spans

$$A \xrightarrow{\mathrm{id}} A \xleftarrow{\mathrm{id}} A \qquad A \xrightarrow{i_1} M_2(A) \xleftarrow{i_2} A$$

are not equivalent in the realization of the smaller span category. Here $i_1$ and $i_2$ are the inclusions into the upper left and lower right corner of matrices. But these spans both induce the identity functor $\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(A)$.

More generally let us denote category of unital algebras and non-unital maps by $\mathrm{Alg}_1^{\mathrm{u}}$. We believe that the functor $\mathrm{Alg}_1^{\mathrm{u}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st,mono}}$ which takes the $\infty$-category of modules is not a Dwyer-Kan localization since we see no reason that the two maps

$$A \to M_2(A)$$

in $\mathrm{Alg}_1^{\mathrm{u}}$ become equivalent in the DK localization at the Morita equivalences (we also can't prove the opposite though). However, our previous proof shows that the functor

$$\mathrm{Alg}_1^{\mathrm{u}}[W^{-1}] \to \mathrm{Alg}_2^{\mathrm{u}}[W^{-1}] \simeq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st,mono}}$$

admits a section.

Now we would like to turn to the case of locally unital rings.

**Construction 2.11.7.** *For a locally unital ring spectrum $A = \mathrm{colim}\, A_i$ we consider the system of idempotents $e_i = f(1)$ in $A$. These define retract diagrams*

$$Ae_i \hookrightarrow A \twoheadrightarrow Ae_i$$

*as left $A$-modules. Similarly we have retract diagrams*

$$Ae_i \hookrightarrow Ae_j \twoheadrightarrow A_i \tag{2.4}$$

*for $i \to j$ in $I$. The system $i \mapsto Ae_i$ in fact forms a functor from $I$ to the $\infty$-category of retracts (where the morphisms are retract diagrams). Then we have that*

$$e_i Ae_i = \mathrm{end}_{A^+}(Ae_i)$$

*are unital rings and the map $e_i Ae_i \to e_j Ae_j$ for $i \to j$ in $I$ extends to non-unital ring map using the retract diagram (2.4). We then have that $A = \mathrm{colim}\, e_i Ae_i$, that is we may replace the diagram $A_i$ by the new diagram $e_i Ae_i$ to obtain $A$ as a colimit.*

*Now from an external perspective what we have done is the following: in the category* $\mathrm{Mod}_H(A)$ *the object* $A$ *is a generator, but not compact in general. However, we can write the* $A$-module $A$ *as a filtered colimit* $A = \mathrm{colim}\, Ae_i$ *where all the maps are part of retract diagrams and then we get that the non-unital ring* $A$ *is given by*

$$A = \underrightarrow{\mathrm{colim}}_i \mathrm{end}_{A^+}(Ae_i) \ .$$

*But note that* $\mathrm{end}_{A^+}(A) \neq A$. *So this gives a way of recovering* $A$ *from the category* $\mathrm{Mod}_H(A)$ *together with the filtered diagram* $i \mapsto Ae_i$.

*Finally note that if* $I = \mathbb{N}$, *so that the diagram is sequential we can in fact write* $A$ *as an* $A$-module *as the direct sum*

$$A = \bigoplus_{i \in \mathbb{N}} A(e_i - e_{i-1}) \qquad e_{-1} := 0$$

*and so we see that* $A$ *is then the countable sum of generators and we are exactly in the situation of the proof of Proposition 2.10.13.*

**Proposition 2.11.8.** *For* $A$ *and* $B$ *locally unital every strongly left adjoint functor* $F :$ $\mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B)$ *is induced by a zig-zag* $A \to C \xleftarrow{\simeq} B$ *of locally unital rings where* $B \to C$ *is a Morita equivalence.*

*Proof.* We consider the filtered diagrams

$$A = \mathrm{colim}_{i \in I} Ae_i \qquad B = \mathrm{colim}_{j \in J} Be_j$$

in $\mathrm{Mod}_H(A)$ and $\mathrm{Mod}_H(B)$ as in Construction 2.11.7. We can assume without loss of generality that $I = J$, e.g. by passing to the product. Now we define

$$C := \mathrm{colim}_I \mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i)$$

where $i \mapsto F(Ae_i) \oplus Be_i$ also forms a retract style diagram in $\mathrm{Mod}_H(B)$. Again the elements $F(Ae_i) \oplus Be_i$ are compact generators since $F$ is strongly left adjoint and the $Be_i$ already form a generating set. Thus we have that

$$\mathrm{Mod}_H(B) \simeq \mathrm{Mod}_H(C)$$

induced by the map $B \to C$ induced by the split inclusion $Be_i \to F(Ae_i) \oplus Be_i$. Similarly we have a map

$$\mathrm{end}_{A^+}(Ae_i) \to \mathrm{end}_{B^+}(F(Ae_i)) \to \mathrm{end}_{B^+}(F(Ae_i) \oplus Be_i)$$

which induces the functor $F$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We would like to combine Propositions 2.10.13 and 2.11.8 into DK localization statement similar to Proposition 2.11.3. To this end we define a 2-morphism between non-unital maps

$f, g : A \to B$ of non-unital ring spectra generally as a natural transformation of induced functors

$$f^*, g^* : \mathrm{Mod}_H(A) \to \mathrm{Mod}_H(B) \ .$$

We will use this for $H$-unital and for locally unital ring spectra and using the invertible 2-morphisms define $\infty$-categories $\mathrm{Alg}_2^{\mathrm{lu}}$ and $\mathrm{Alg}_2^{\mathrm{H}}$. Again, similar to the case $\mathrm{Alg}_2^u$ these are the $(\infty, 1)$-cores of very natural $(\infty, 2)$-categories which are the more canonical objects. But for simplicity we will restriction to the $\infty = (\infty, 1)$-categorical realm here.

**Theorem 2.11.9.** *The functors*

$$\mathrm{Alg}_2^{\mathrm{lu}} \to \mathrm{Pr}_\omega^{\mathrm{L}} \qquad and \qquad \mathrm{Alg}_2^{\mathrm{H}} \to \mathrm{Pr}_{\mathrm{dual}}^{\mathrm{L}} \ .$$

*given by* $\mathrm{Mod}_H$ *are Dwyer–Kan localizations.*

In order to prove this Theorem we need the following auxiliary construction.

**Construction 2.11.10.** *Let $R$ be a non-unital ring spectra. We want to define another non-unital ring spectrum $M_n(R)$ of $n \times n$-matrices over $R$. As a spectrum this is simply given as $R^{n^2}$ but we would like to give it a non-unital ring structure. To this end we consider the unital ring spectra $M_n(R^+)$ of $n \times n$-matrices over the unitalizations $R^+$ and $M_n(\mathbb{S})$ over the sphere.*

*The morphism $R^+ \to \mathbb{S}$ induced a map of ring spectra*

$$M_n(R^+) \to M_n(\mathbb{S})$$

*and we define $M_n(R)$ to be the fibre.*

**Lemma 2.11.11.**   *1. The map $i_k : R \to M_n(R)$ given by inclusion into the $k$-th diagonal entry is a Morita equivalence and all the functors*

$$i_k^* : \mathrm{Mod}_R \to M_n(R)$$

*are equivalent.*

*2. If $R$ is locally unital, then so is $M_n(R)$.*

*3. If $R$ is $H$-unital, then so is $M_n(R)$.*

*Proof.* TO BE WRITTEN □

*Proof of Theorem 2.11.9.* Let us first prove the case of locally unital rings. We proceed similar to the proof of Proposition 2.11.3, namely we want to apply Lemma 2.11.4. To this end we have to show that the functor

$$\mathrm{ZigZag}(A, B) \to \mathrm{Fun}^{\mathrm{sL}}(\mathrm{Mod}(A), \mathrm{Mod}(B))^{\simeq} \tag{2.5}$$

is an equivalence after realizing the source, which is the zigzag-category whose objects are locally unital algebras. We use the construction from Proposition 2.11.8 to produce a functor in the opposite direction:

$$\text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq} \to \text{ZigZag}(A, B) \quad F \mapsto \left( A \to C(F) \xleftarrow{\simeq} B \right)$$

where $C(F) = \text{colim end}_{B^+}(F(Ae_i) \oplus Be_i)$, which is clearly functorial in natural equivalences. Note that we fix $A = \text{colim}_I Ae_i$ and $B = \text{colim}_I Be_i$ once and for all. The composition

$$\text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq} \to \text{ZigZag}(A, B) \to \text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq}$$

is by construction equivalent to the identity and it remains to also show that the composition

$$|\text{ZigZag}(A, B)| \to \text{Fun}^{\text{sL}}(\text{Mod}(A), \text{Mod}(B))^{\simeq} \to |\text{ZigZag}(A, B)|$$

is homotopic to the identity. We proceed exactly as in the proof of Proposition 2.11.3 and note that for a given ZigZag $A \xrightarrow{\varphi} C \xleftarrow{\psi} B$ we have that

$$C(F) = \text{colim end}_{B^+}(F(Ae_i) \oplus Be_i) \simeq \text{colim end}_{C^+}(\varphi^*(Ae_i) \oplus \psi^*(Be_i))$$

and we consider the natural diagram



## 2.12 The symmetric monoidal structure on $\text{Pr}^{\text{L}}_{\text{ca}}$.

The characterisation of stable and compactly assembled categories as dualisable objects suggests that the tensor product on $\text{Pr}^{\text{L}}$ descends to one on $\text{Pr}^{\text{L}}_{\text{ca}}$, although it doesn't directly imply it (since it only works in the stable case, and doesn't say anything about morphisms). We first prove the following lemma:

**Lemma 2.12.1.** *1. A presentable category is compactly assembled if and only if we find compactly generated $\mathcal{C}'$ and a pair of adjoint functors*

$$\mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{C}'$$

*where both $L$ and $R$ are in $\text{Pr}^{\text{L}}$, and $L$ is fully faithful.*

2. *A left adjoint functor $F : \mathcal{C} \to \mathcal{D}$ between compactly assembled categories is compactly assembled if and only if we find a diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \underset{L}{\overset{R}{\rightleftarrows}} & \mathcal{C}' \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F'} \\
\mathcal{D} & \underset{L}{\overset{R}{\rightleftarrows}} & \mathcal{D}'
\end{array}
$$

*where all morphisms are in* $\mathrm{Pr^L}$, $\mathcal{C}'$ *and* $\mathcal{D}'$ *are compactly generated,* $F'$ *preserves compact objects, and the left adjoint functors $L$ are fully faithful.*

*Proof.* We essentially know the first statement already: Retracts of compactly generated categories are compactly assembled, and in the other direction any compactly assembled category comes with the adjunction $\mathcal{C} \underset{\hat{\jmath}}{\overset{k}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}^{\omega_1})$. Naturality of this also proves one direction of the second statement, since compactly assembled $F : \mathcal{C} \to \mathcal{D}$ commutes with $\hat{\jmath}$ and $k$.

For the final step, assume a diagram

$$
\begin{array}{ccc}
\mathcal{C} & \underset{L}{\overset{R}{\rightleftarrows}} & \mathcal{C}' \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F'} \\
\mathcal{D} & \underset{L}{\overset{R}{\rightleftarrows}} & \mathcal{D}'.
\end{array}
$$

The functors $L$ have a filtered-colimit-preserving right adjoint, so they preserve compact morphisms. They also detect compact morphisms: Since $L$ is fully faithful and preserves colimits, to test whether a morphism is compact in $\mathcal{C}$ (or $\mathcal{D}$), we may test this after applying $L$. Since $F'$ preserves compact morphisms, this shows that $F$ preserves compact morphisms. $\qquad\square$

Essentially, this lemma says that objects and morphisms in $\mathrm{Pr^L_{ca}}$ are characterized as nicely controlled retracts of objects and morphisms in $\mathrm{Pr^L_\omega}$, formed in $\mathrm{Pr^L}$, since fully faithfullness of $L$ implies $RL \simeq \mathrm{id}$. In general, it is probably wrong that arbitrary retracts of morphisms in $\mathrm{Pr^L_\omega}$ lie in $\mathrm{Pr^L_{ca}}$, if we drop the adjointness.

**Proposition 2.12.2.** *The tensor product of* $\mathrm{Pr^L}$ *restricts to a symmetric-monoidal structure on* $\mathrm{Pr^L_{ca}}$, *characterized by corepresenting functors* $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ *which are "bi-compactly assembled": They preserve colimits in each variable, and take $f \times g$ to compact morphisms whenever $f$ and $g$ are compact.*

*Proof.* The tensor product of compactly generated categories is compactly generated. This follows from the fact that $\mathrm{Cat}_\infty^{\mathrm{Rex}(\kappa)}$ also admits a tensor product (where $\kappa$-small colimit preserving functors out of $\mathcal{C}_0 \otimes \mathcal{D}_0$ correspond to functors out of $\mathcal{C}_0 \times \mathcal{D}_0$ which preserve

$\kappa$-small colimits in each argument), see [**?**, Section 4.8.1] for a much more general statement. By checking universal properties, one directly sees

$$\mathrm{Ind}_\kappa(\mathcal{C}_0) \otimes \mathrm{Ind}_\kappa(\mathcal{D}_0) \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0 \otimes^{\mathrm{Rex}(\kappa)} \mathcal{D}_0).$$

It follows that tensor products of compactly assembled $\infty$-categories stay compactly assembled, due to the characterisation as retracts of compactly generated categories.

For morphisms, we argue similarly: A pair of adjoint functors $\mathcal{C} \underset{\hat{\jmath}}{\overset{k}{\longleftrightarrow}} \mathrm{Ind}(\mathcal{C}^{\omega_1})$ where the left adjoint is fully faithful stays such after tensoring with some $\mathcal{E}$, since it is characterized by natural transformations $LR \to \mathrm{id}$ and $\mathrm{id} \to RL$, the latter of which is an equivalence. Now take $F : \mathcal{C} \to \mathcal{D}$ compactly assembled and $\mathcal{E}$ compactly assembled. The previous lemma gives adjunctions $\mathcal{E} \longleftrightarrow \mathcal{E}'$, and

$$
\begin{array}{ccc}
\mathcal{C} & \longleftrightarrow & \mathcal{C}' \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F'} \\
\mathcal{D} & \longleftrightarrow & \mathcal{D}'.
\end{array}
$$

with $\mathcal{C}', \mathcal{D}'$ and $\mathcal{E}'$ compactly generated, and $F'$ compact-object preserving. Tensoring and composing, we obtain a diagram

$$
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{E} & \longleftrightarrow & \mathcal{C}' \otimes \mathcal{E}' \\
\downarrow{\scriptstyle F \otimes \mathrm{id}} & & \downarrow{\scriptstyle F' \otimes \mathrm{id}} \\
\mathcal{D} \otimes \mathcal{E} & \longleftrightarrow & \mathcal{D}' \otimes \mathcal{E}'.
\end{array}
$$

This shows that $F \otimes \mathrm{id}$ is also compactly assembled.

Finally, for the universal property, we consider $\mathcal{C}, \mathcal{D}$ compactly assembled, and let $\mathcal{C} \longleftrightarrow \mathcal{C}'$ be an adjunction as above, analogously for $\mathcal{D}$. In the diagram

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{D} & \longleftrightarrow & \mathcal{C}' \times \mathcal{D}' \\
\downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle F' \otimes \mathrm{id}} \\
\mathcal{C} \otimes \mathcal{D} & \longleftrightarrow & \mathcal{C}' \otimes \mathcal{D}'.
\end{array}
$$

we see that the top horizontal inclusion takes a pair of compactly assembled morphisms to a morphism in $\mathcal{C}' \times \mathcal{D}'$ factoring through a pair of compact objects. Since the bottom horizontal functor detects compact morphisms, this shows that $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ takes pairs of compact morphisms to compact morphisms. This also shows that a compactly assembled functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ gives rise to a "bi-compactly assembled" functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$. Finally, we need to prove that, given a functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ for which $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is "bi-compactly assembled", the functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is compactly assembled. Since such a functor in particular restricts to a functor

$$\mathcal{C}^{\omega_1} \times \mathcal{D}^{\omega_1} \to \mathcal{E}^{\omega_1},$$

we obtain a diagram

$$\mathcal{C} \otimes \mathcal{D} \xleftarrow{\hspace{1cm}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \otimes \mathrm{Ind}(\mathcal{D}^{\omega_1})$$

$$\Big\downarrow{\scriptstyle\otimes} \qquad\qquad\qquad \Big\downarrow$$

$$\mathcal{E} \xleftarrow{\hspace{1cm}} \mathrm{Ind}(\mathcal{E}^{\omega_1}).$$

This exhibits the functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ as compactly assembled. $\qquad\square$

**Corollary 2.12.3.** *For any compactly assembled $\infty$-category $\mathcal{C}$ and every locally compact Hausdorff space $X$ the $\infty$-category $\mathrm{Shv}(X;\mathcal{C})$ is compactly assembled.*

*Proof.* According to Example 2.2.16 we find that sheaves of anima is compactly assembled. Then the claim follows from the assertion that

$$\mathrm{Shv}(X;\mathcal{C}) = \mathrm{Shv}(X;\mathrm{An}) \otimes \mathcal{C}$$

which is Proposition 2.8.7 combined with Proposition 2.12.2. $\qquad\square$

**Lemma 2.12.4.** *For compactly assembled $\mathcal{C},\mathcal{D}$, let $S$ be the class of morphisms in $\mathrm{Fun}^L(\mathcal{C},\mathcal{D})$ consisting of all $\eta : F \to G$ with the property that for any compact morphism $X \to Y$ in $\mathcal{C}$, we have that the composite $F(X) \to G(Y)$ in the square*

$$\begin{array}{ccc} F(X) & \xrightarrow{\;\eta_X\;} & G(X) \\ \downarrow & & \downarrow \\ F(Y) & \xrightarrow{\;\eta_Y\;} & G(Y) \end{array}$$

*is compact. Then $S$ forms a precompact ideal.*

*Proof.* $S$ is clearly an ideal and contains the identity on the initial object. The pushout condition is also easily seen.

For the accessibility, we take a diagram of $F_\alpha$, $\alpha \in [0,1] \cap \mathbb{Q}$, such that for any compact $x \to y$ and any $\alpha < \alpha'$, the composite $F_\alpha(x) \to F_{\alpha'}(y)$ is compact. We need to prove that $F_0 \to F_1$ factors through some $\kappa$-compact $G$ where $\kappa$ is independent of the choice of $F_\alpha$. We take $G = \mathrm{colim}_{\alpha<1} F_\alpha$. For $X$ $\omega_1$-compact, we may write $X = \mathrm{colim}\, X_n$ along compact maps. Using any sequence $\alpha_n$ tending to 1 from below, we see that

$$G(x) = \mathrm{colim}_n F_{\alpha_n}(X_n)$$

is a sequential colimit along compact maps, hence $\omega_1$-compact. This proves that $G$ takes $\omega_1$-compact objects to $\omega_1$-compact objects. Since $\mathcal{C}$ is $\omega_1$-compactly generated, $G$ is $\kappa$-compact in $\mathrm{Fun}^L(\mathcal{C},\mathcal{D})$ for some $\kappa$ only depending on the size of $\mathcal{C}^{\omega_1}$. $\qquad\square$

**Definition 2.12.5.** For compactly assembled $\mathcal{C},\mathcal{D}$, we define an internal Hom by

$$\underline{\mathrm{Hom}}^{\mathrm{ca}}(\mathcal{C},\mathcal{D}) = (\mathrm{Fun}^L(\mathcal{C},\mathcal{D}),S)^{\mathrm{ca}},$$

with $S$ as above.

**Lemma 2.12.6.** *This is actually an internal Hom, i.e.*

$$\operatorname{Fun}^{\mathrm{ca}}(\mathcal{C}, \underline{\operatorname{Hom}}^{\mathrm{ca}}(\mathcal{D}, \mathcal{E})) \simeq \operatorname{Fun}^{\mathrm{ca}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}).$$

*Proof.* By the universal property of compactly assembled cores, the left hand side agrees with the full subcategory of $\operatorname{Fun}^L(\mathcal{C}, \operatorname{Fun}^L(\mathcal{D}, \mathcal{E}))$ on all functors taking compact morphisms into $S$, i.e. under the adjunction to $\operatorname{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ to all functors taking pairs of compact morphisms to compact morphisms. But this is the same as $\operatorname{Fun}^{\mathrm{ca}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ by the previous lemma. $\square$

**Example 2.12.7.** The compact full subcategory of compact objects in $\underline{\operatorname{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$ agrees with $\operatorname{Fun}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$. This is either seen by looking at $\operatorname{Fun}^{\mathrm{ca}}(\operatorname{An}, \underline{\operatorname{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})) = \operatorname{Fun}^{\mathrm{ca}}(\operatorname{An} \otimes \mathcal{C}, \mathcal{D})$, or directly by observing that the compact objects of $(\mathcal{C}, S)^{\mathrm{ca}}$ are exactly given by those objects of $\mathcal{C}$ whose identity lies in $S$, which in the case of $\underline{\operatorname{Hom}}^{\mathrm{ca}}(\mathcal{C}, \mathcal{D})$ gives exactly those functors $F \in \operatorname{Fun}^L(\mathcal{C}, \mathcal{D})$ which preserve compact morphisms.

**Example 2.12.8.** If $\mathcal{C}$ is already compactly generated, $\eta : F \to G$ in $\operatorname{Fun}^L(\mathcal{C}, \mathcal{D})$ is in $S$ if and only if $\eta : F(X) \to G(X)$ is compact for each $\mathcal{C}^\omega$, i.e. $S$ consists exactly of the pointwise compact morphisms in $\operatorname{Fun}^{\mathrm{Rex}}(\mathcal{C}^\omega, \mathcal{D})$. If $\mathcal{D}$ is also compactly generated, those are exactly the morphisms which factor pointwise through a compact object. It does still not follow that $\underline{\operatorname{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D})$ agrees with $\operatorname{Ind}(\operatorname{Fun}^{\mathrm{Rex}}(\mathcal{C}^\omega, \mathcal{D}^\omega))$, since such a natural transformation does not necessarily factor through a pointwise compact functor. For $\mathcal{C} = \operatorname{Mod}(A)$ and $\mathcal{D} = \operatorname{Mod}(B)$, $\operatorname{Fun}^L(\mathcal{C}, \mathcal{D}) = \operatorname{BiMod}(B, A)$, and this unwinds to the following: $S$ consists of those morphisms of bimodules $M \to M'$ for which $M \otimes_A N \to M' \otimes_A N$ factors through a compact $B$-module for each $N$, which does not necessarily agree with those $M \to M'$ which factor through a $B$-$A$ bimodule which is compact as $B$-module.

**Proposition 2.12.9.** *The symmetric monoidal structure on $\operatorname{Pr}^L_{\mathrm{ca}}$ induces a closed symmetric monoidal structure on $\operatorname{Pr}^L_{\mathrm{dual}}$ such that the functor*

$$- \otimes \operatorname{Sp} : \operatorname{Pr}^L_{\mathrm{ca}} \to \operatorname{Pr}^L_{\mathrm{dual}}$$

*is strong symmetric monoidal and such that the fully faithful inclusion $\operatorname{Pr}^L_{\mathrm{dual}} \to \operatorname{Pr}^L_{\mathrm{ca}}$ is closed, that is preserves inner homs.*

*Proof.* One checks immediately that $\otimes$ and $\underline{\operatorname{Hom}}^{\mathrm{ca}}$ restrict to stable compactly assembled categories. $\square$

We will denote the inner hom in $\operatorname{Pr}^L_{\mathrm{dual}}$ also by $\underline{\operatorname{Hom}}^{\mathrm{dual}}$ to make clear that we are in the stable setting (although it agrees with $\underline{\operatorname{Hom}}^{\mathrm{ca}}$).

**Definition 2.12.10.** A dualisable, stable $\infty$-category $\mathcal{C}$ is called smooth if the functor $\operatorname{Sp} \to \mathcal{C}^\vee \otimes \mathcal{C}$ is strongly left adjoint. It is called proper if the functor $\mathcal{C}^\vee \otimes \mathcal{C} \to \operatorname{Sp}$ is strongly left adjoint.

**Proposition 2.12.11.** *The smooth and proper dualisable stable $\infty$-categories are precisely the dualisable objects of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$.*

*For a smooth and proper dualisable $\infty$-category $\mathcal{C}$ we have for any dualisable stable $\infty$-category $\mathcal{D}$*

$$\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \otimes \mathcal{D}$$

*in particular the dual of $\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}}$ agrees with the dual in $\mathrm{Pr}^{\mathrm{L}}$.*

*Proof.* The functor $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{dual}} \to \mathrm{Pr}^{\mathrm{L}}$ is strong symmetric monoidal, thus preserves dualisable objects, duals and inner homs out of dualisable objects. $\square$

**Example 2.12.12.** One interesting example is given as $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^{\wedge}, \mathrm{Sp})$. We have $\mathrm{Fun}^L(\mathrm{Sp}_p^{\wedge}, \mathrm{Sp}) = \mathrm{Sp}_p^{\wedge}$, for example since $\mathrm{Sp}_p^{\wedge}$ is compactly generated and $(\mathrm{Sp}_p^{\wedge})^{\omega}$ is the category of compact $p$-power torsion spectra, which agrees with its opposite (along Spanier-Whitehead duality, i.e. $\mathrm{map}(-, \mathbb{S})$).

Indeed, the equivalence takes $X \in \mathrm{Sp}_p^{\wedge}$ to the functor $\mathrm{Sp}_p^{\wedge} \to \mathrm{Sp}$ taking $Y \mapsto \mathrm{fib}(X \otimes Y \to X \otimes Y[1/p])$.

From the above description of $\underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^{\wedge}, \mathrm{Sp})$, we get that it is $(\mathrm{Sp}_p^{\wedge}, S)^{\mathrm{ca}}$ where $S$ consists of the class of morphisms $X \to X'$ for which $X \otimes Y \to X' \otimes Y$ is compact in $\mathrm{Sp}$ for all compact $p$-power torsion spectra $Y$. Since those are generated as a stable subcategory by $\mathbb{S}/p$, it agrees furthermore with $(\mathrm{Sp}_p^{\wedge}, S')^{\mathrm{ca}}$ where we take $S'$ to be all $X \to X'$ where $X/p \to X'/p$ is compact (yet another choice would be those where $X/p^n \to X'/p^n$ is compact for all $n$).

Recall that $\widetilde{\mathrm{Nuc}}(\mathbb{Z}_p)$ similarly consisted of $(\mathcal{D}(\mathbb{Z})_p^{\wedge}, S)^{\mathrm{ca}}$ for the class of morphisms $X \to X'$ where $X/p^n \to X'/p^n$ is compact for each $n$ (and we could have shown that $n = 1$ suffices, actually). So we define

$$\widetilde{\mathrm{Nuc}}(\mathbb{S}_p) := \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp}_p^{\wedge}, \mathrm{Sp}).$$

Note that we can't directly write this as limit analogous to the $\mathbb{Z}$ case, since $\mathbb{S}/p^n$ is not a ring[15]. Similarly, we can't write the $\mathbb{Z}_p$ case as $\underline{\mathrm{Hom}}^{\mathrm{dual}}$ since we would need a $\mathcal{D}(\mathbb{Z})$-linear version of $\underline{\mathrm{Hom}}^{\mathrm{dual}}$. This will be one of our next goals.

Note that the compact objects in $\widetilde{\mathrm{Nuc}}(\mathbb{S}_p)$ are given by the full subcategory of $\mathrm{Sp}_p^{\wedge}$ on those $X$ where $X/p$ is compact. These agree with the compact $\mathbb{S}_p^{\wedge}$-modules. However, we also have objects such as

$$\mathrm{colim}_{\alpha \in \mathbb{Q}} \, j\widehat{\bigoplus}_{n \geq 1} \mathbb{S}_p^{\wedge} \in \mathrm{Ind}(\mathrm{Sp}_p^{\wedge})$$

where the map from $\alpha \to \alpha'$ is given on the $n$-th summand by multiplication with $p^{\lfloor \alpha' n \rfloor - \lfloor \alpha n \rfloor}$. This is an $S$-exhaustible object, hence an $\omega_1$-compact object in $\widetilde{\mathrm{Nuc}}(\mathrm{Sp}_p^{\wedge})$, but it can't be written as colimit of compact objects since none of the maps factor through a compact $\mathbb{S}_p^{\wedge}$-module.

---

[15]There is of course a free $E_1$-ring $\mathbb{S}//p^n$, but this is yet another description, since $\mathbb{Z}//p^n \neq \mathbb{Z}/p^n$ even over $\mathbb{Z}$

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