Algebras, Operads, Combinads

Jean-Louis Loday

IRMA (Cnrs) et Institut de Mathématique Zinbiel, Strasbourg

HOGT Lille, 23 mars 2012

Plan of the talk

I. Types of algebras, types of operads

Plan of the talk

I. Types of algebras, types of operads

II. An example: planar trees and ns operads

Plan of the talk

I. Types of algebras, types of operads

II. An example: planar trees and ns operads

III. Combinatorial patterns and combinads

Plan of the talk

I. Types of algebras, types of operads

II. An example: planar trees and ns operads

III. Combinatorial patterns and combinads

IV. Another example: surjective maps and permutads

Plan of the talk

I. Types of algebras, types of operads

II. An example: planar trees and ns operads

III. Combinatorial patterns and combinads

IV. Another example: surjective maps and permutads

V. Further research:
  a) Feynman diagrams
  b) higher operads: opetopes
    (relationship with quantum groups)

Plan of the talk

I. Types of algebras, types of operads

II. An example: planar trees and ns operads

III. Combinatorial patterns and combinads

IV. Another example: surjective maps and permutads

V. Further research:
   a) Feynman diagrams
   b) higher operads: opetopes
      (relationship with quantum groups)

(thanks to María Ronco and Bruno Vallette)

[JLL-MR] JLL, M.Ronco, Permutads, soumis à J.Algebraic Combinatorics A.

[JLL-BV] JLL and B.Vallette, Algebraic operads, Grundlehren Math.Wiss. 346, Springer, Heidelberg, 2012.

I. Types of algebras, types of operads

Types of algebras

Algebra usually means: unital **associative** algebra

I. Types of algebras, types of operads

Types of algebras

Algebra usually means: unital **associative** algebra

There are other types of algebras:
Lie, Poisson, Jordan, dendriform, Zinbiel, Leibniz, Batalin-Vilkovisky, magmatic, A_∞, etc.

I. Types of algebras, types of operads

Types of algebras

Algebra usually means: unital **associative** algebra

There are other types of algebras:
Lie, Poisson, Jordan, dendriform, Zinbiel, Leibniz, Batalin-Vilkovisky, magmatic, A_∞, etc.

Each one of these types is encoded by an **operad** $\mathcal{P}$, either nonsymmetric (As, Dend, Mag, A_∞) or symmetric.

$$\mathcal{P} : \mathrm{Mod} \to \mathrm{Mod}$$

I. Types of algebras, types of operads

Types of algebras

Algebra usually means: unital **associative** algebra

There are other types of algebras:
Lie, Poisson, Jordan, dendriform, Zinbiel, Leibniz, Batalin-Vilkovisky, magmatic, A_∞, etc.

Each one of these types is encoded by an **operad** $\mathcal{P}$, either nonsymmetric (As, Dend, Mag, A_∞) or symmetric.

$$\mathcal{P} : \mathrm{Mod} \to \mathrm{Mod}$$

1. $\mathcal{P}(V) = \bigoplus_n \mathcal{P}_n \otimes V^{\otimes n}$
2. $\mathcal{P}(V) = \bigoplus_n \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$
3. $\Gamma\mathcal{P}(V) = \bigoplus_n (\mathcal{P}(n) \otimes V^{\otimes n})^{\mathbb{S}_n}$ (B.Fresse)

Types of operads

There are several types of operads:

- ns operads,
- symmetric operads,
- divided power operads
- cyclic operads,
- shuffle operads,
- wheeled operads,
- etc.

Types of operads

There are several types of operads:

- ns operads,
- symmetric operads,
- divided power operads
- cyclic operads,
- shuffle operads,
- wheeled operads,
- etc.

Question: what does encode a type of operads? In other words: want to describe the monad on which an operad is an algebra on.

Types of operads

There are several types of operads:

- ns operads,
- symmetric operads,
- divided power operads
- cyclic operads,
- shuffle operads,
- wheeled operads,
- etc.

Question: what does encode a type of operads? In other words: want to describe the monad on which an operad is an algebra on.

Motivation: Koszul duality.

$A =$ associative alg $\Rightarrow A^! =$ associative alg
$A =$ Lie alg $\Rightarrow A^! =$ commutative alg

Types of operads

There are several types of operads:

- ns operads,
- symmetric operads,
- divided power operads
- cyclic operads,
- shuffle operads,
- wheeled operads,
- etc.

Question: what does encode a type of operads? In other words: want to describe the monad on which an operad is an algebra on.

Motivation: Koszul duality.

$A =$ associative alg $\Rightarrow A^! =$ associative alg
$A =$ Lie alg $\Rightarrow A^! =$ commutative alg

At the operad level: $As^! = As \quad , \quad Lie^! = Com$

Types of operads

There are several types of operads:

- ns operads,
- symmetric operads,
- divided power operads
- cyclic operads,
- shuffle operads,
- wheeled operads,
- etc.

Question: what does encode a type of operads? In other words: want to describe the monad on which an operad is an algebra on.

Motivation: Koszul duality.

$A =$ associative alg $\Rightarrow A^! =$ associative alg
$A =$ Lie alg $\Rightarrow A^! =$ commutative alg

At the operad level: $As^! = As \quad , \quad Lie^! = Com$

Given an operad of some type, which type is the Koszul dual?

Koszul duality for types of operads

In order to perform Koszul duality for types of operads, we need to know how to encode a type of operads, and, then, to construct a Koszul duality theory for these objects.

Koszul duality for types of operads

In order to perform Koszul duality for types of operads, we need to know how to encode a type of operads, and, then, to construct a Koszul duality theory for these objects.

Obviously the relevant object encoding (ns) operads is self-dual, since the Koszul dual of an (ns) operad is an (ns) operad.

Koszul duality for types of operads

In order to perform Koszul duality for types of operads, we need to know how to encode a type of operads, and, then, to construct a Koszul duality theory for these objects.

Obviously the relevant object encoding (ns) operads is self-dual, since the Koszul dual of an (ns) operad is an (ns) operad.

The strategy: look for the free object, discard the variable
ex: $T(V)$ gives As

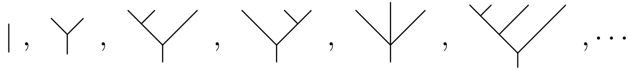
II. An example: planar trees and ns operads

Planar trees

II. An example: planar trees and ns operads

Planar trees

The set of planar rooted trees PT:

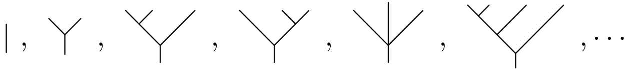


They have a **root**, **leaves**, **vertices**, **inputs of vertices**
The set PT comes with a notion of **substitution** of a tree at a given vertex of another tree

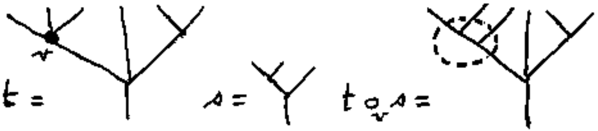
II. An example: planar trees and ns operads

Planar trees

The set of planar rooted trees PT:



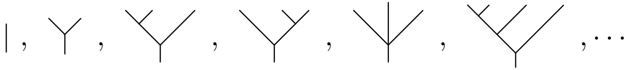
They have a **root**, **leaves**, **vertices**, **inputs of vertices**
The set PT comes with a notion of **substitution** of a tree at a given vertex of another tree



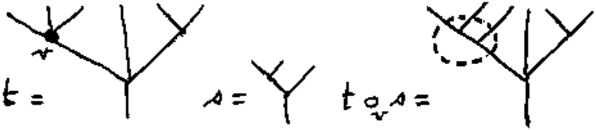
II. An example: planar trees and ns operads

Planar trees

The set of planar rooted trees PT:



They have a **root**, **leaves**, **vertices**, **inputs of vertices**
The set PT comes with a notion of **substitution** of a tree at a given vertex of another tree



Key point: substitution is associative.

$\mathbb{N}$-modules

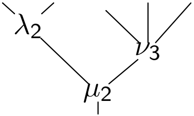
$M = \{M_n\}_{n\geq 1}$ where M_n is a $\mathbb{K}$-module (Hyp $M_1 = 0$ for simplicity)

$\mathbb{N}$-modules

$M = \{M_n\}_{n\geq 1}$ where M_n is a $\mathbb{K}$-module (Hyp $M_1 = 0$ for simplicity)

The functor $\mathbb{PT} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod$ defined by

$\mathbb{PT}(M)_n =$ span of the pbr trees with n leaves decorated by M

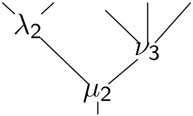


$\mathbb{N}$-modules

$M = \{M_n\}_{n \geq 1}$ where M_n is a $\mathbb{K}$-module (Hyp $M_1 = 0$ for simplicity)

The functor $\mathbb{PT} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod$ defined by

$\mathbb{PT}(M)_n =$ span of the pbr trees with n leaves decorated by M



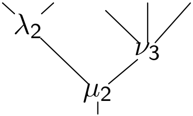
THM The substitution process defines a monoid structure $\Gamma : \mathbb{PT} \circ \mathbb{PT} \to \mathbb{PT}$ on the endofunctor $\mathbb{PT}$, hence $(\mathbb{PT}, \Gamma)$ is a monad on the category of $\mathbb{N}$-modules.

$\mathbb{N}$-modules

$M = \{M_n\}_{n\geq 1}$ where M_n is a $\mathbb{K}$-module (Hyp $M_1 = 0$ for simplicity)

The functor $\mathbb{PT} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod$ defined by

$\mathbb{PT}(M)_n =$ span of the pbr trees with n leaves decorated by M



THM The substitution process defines a monoid structure $\Gamma : \mathbb{PT} \circ \mathbb{PT} \to \mathbb{PT}$ on the endofunctor $\mathbb{PT}$, hence $(\mathbb{PT}, \Gamma)$ is a monad on the category of $\mathbb{N}$-modules.

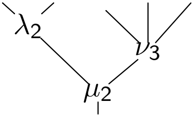
An algebra over the monad $(\mathbb{PT}, \Gamma)$ is a **ns operad**
(this is the *combinatorial definition*, see for instance `[JLL-BV]`).

$\mathbb{N}$-modules

$M = \{M_n\}_{n\geq 1}$ where M_n is a $\mathbb{K}$-module (Hyp $M_1 = 0$ for simplicity)

The functor $\mathbb{PT} : \mathbb{N}\text{-}\mathit{mod} \to \mathbb{N}\text{-}\mathit{mod}$ defined by

$\mathbb{PT}(M)_n =$ span of the pbr trees with n leaves decorated by M



THM The substitution process defines a monoid structure $\Gamma : \mathbb{PT} \circ \mathbb{PT} \to \mathbb{PT}$ on the endofunctor $\mathbb{PT}$, hence $(\mathbb{PT}, \Gamma)$ is a monad on the category of $\mathbb{N}$-modules.

An algebra over the monad $(\mathbb{PT}, \Gamma)$ is a **ns operad**
(this is the *combinatorial definition*, see for instance `[JLL-BV]`).
PROP $\mathbb{PT}(M)$ is the free ns operad on M

Other examples of types of operads

Construction of the free object:

- ns operads use planar rooted trees
- symmetric operads use nonplanar rooted trees
- cyclic operads use nonplanar nonrooted trees
- strictly cyclic operads use planar nonrooted trees
- shuffle operads use ... shuffle trees (Dotsenko-Khoroshkin)

Other examples of types of operads

Construction of the free object:

- ns operads use planar rooted trees
- symmetric operads use nonplanar rooted trees
- cyclic operads use nonplanar nonrooted trees
- strictly cyclic operads use planar nonrooted trees
- shuffle operads use ... shuffle trees (Dotsenko-Khoroshkin)

Question: what is the general framework for all these examples?

III. Combinatorial patterns and combinads (work in progress)

Definition of a **combinatorial pattern** $\mathbb{X}$ over $\mathbb{N}$:

X is a set, whose elements are called *trees* (abuse of terminology)

Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$
and its set of *leaves* $\mathrm{leav}(t)$

Any vertex v comes with its set of *inputs* $\mathrm{in}(v)$

III. Combinatorial patterns and combinads (work in progress)

Definition of a **combinatorial pattern** $\mathbb{X}$ over $\mathbb{N}$:
X is a set, whose elements are called *trees* (abuse of terminology)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$
and its set of *leaves* $\mathrm{leav}(t)$
Any vertex v comes with its set of *inputs* $\mathrm{in}(v)$
For any trees t, s and $v \in \mathrm{vert}(t)$ and bijection $\mathrm{in}(v) \cong \mathrm{leav}(s)$
there is given a new tree denoted $t \circ_v s$ such that
$\mathrm{leaves}(t \circ_v s) = \mathrm{leaves}(t)$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \setminus \{v\}) \cup \mathrm{vert}(s)$

III. Combinatorial patterns and combinads (work in progress)

Definition of a **combinatorial pattern** $\mathbb{X}$ over $\mathbb{N}$:
X is a set, whose elements are called *trees* (abuse of terminology)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$
and its set of *leaves* $\mathrm{leav}(t)$
Any vertex v comes with its set of *inputs* $\mathrm{in}(v)$
For any trees t, s and $v \in \mathrm{vert}(t)$ and bijection $\mathrm{in}(v) \cong \mathrm{leav}(s)$
there is given a new tree denoted $t \circ_v s$ such that
$\mathrm{leaves}(t \circ_v s) = \mathrm{leaves}(t)$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \backslash \{v\}) \cup \mathrm{vert}(s)$
We assume **associativity** of this composition, that is
I. sequential axiom, for $w \in \mathrm{vert}(s)$: $(t \circ_v s) \circ_w r = t \circ_v (s \circ_w r)$
II. parallell axiom, for $w \in \mathrm{vert}(t)$: $(t \circ_v s) \circ_w r = (t \circ_w r) \circ_v s$

III. Combinatorial patterns and combinads (work in progress)

Definition of a **combinatorial pattern** $\mathbb{X}$ over $\mathbb{N}$:
X is a set, whose elements are called *trees* (abuse of terminology)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$
and its set of *leaves* $\mathrm{leav}(t)$
Any vertex v comes with its set of *inputs* $\mathrm{in}(v)$
For any trees t, s and $v \in \mathrm{vert}(t)$ and bijection $\mathrm{in}(v) \cong \mathrm{leav}(s)$
there is given a new tree denoted $t \circ_v s$ such that
$\mathrm{leaves}(t \circ_v s) = \mathrm{leaves}(t)$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \backslash \{v\}) \cup \mathrm{vert}(s)$
We assume **associativity** of this composition, that is
I. sequential axiom, for $w \in \mathrm{vert}(s)$: $(t \circ_v s) \circ_w r = t \circ_v (s \circ_w r)$
II. parallell axiom, for $w \in \mathrm{vert}(t)$: $(t \circ_v s) \circ_w r = (t \circ_w r) \circ_v s$

In the planar case all the sets $\mathrm{leaves}(t), \mathrm{in}(v)$ are completely determined by their cardinality $n \in \mathbb{N}$.

Definition of a combinad

A *combinad* is a monad on $\mathbb{N}$-modules $(\mathcal{X}, \Gamma)$

$$\mathcal{X} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod \quad , \quad \Gamma : \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

where $\mathcal{X}$ is induced by a given combinatorial pattern $\mathbb{X}$

$\mathcal{X}(M)_n = \text{span of the } \mathbb{X}\text{-trees with } n \text{ leaves, vertices decorated by } M$

and the composition Γ is induced by the substitution of the combinatorial pattern.

Definition of a combinad

A *combinad* is a monad on $\mathbb{N}$-modules $(\mathcal{X}, \Gamma)$

$$\mathcal{X} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod \quad , \quad \Gamma : \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

where $\mathcal{X}$ is induced by a given combinatorial pattern $\mathbb{X}$

$\mathcal{X}(M)_n = \text{span of the } \mathbb{X}\text{-trees with } n \text{ leaves, vertices decorated by } M$

and the composition Γ is induced by the substitution of the combinatorial pattern.

An *operad of type* $\mathbb{X}$ is an algebra over the monad $\mathcal{X}$.

Definition of a combinad

A *combinad* is a monad on $\mathbb{N}$-modules $(\mathcal{X}, \Gamma)$

$$\mathcal{X} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod \quad , \quad \Gamma : \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

where $\mathcal{X}$ is induced by a given combinatorial pattern $\mathbb{X}$

$\mathcal{X}(M)_n = \text{span of the } \mathbb{X}\text{-trees with } n \text{ leaves, vertices decorated by } M$

and the composition Γ is induced by the substitution of the combinatorial pattern.

An *operad of type* $\mathbb{X}$ is an algebra over the monad $\mathcal{X}$.

Koszul duality of combinads: mimick `[JLL-BV]` and use rewriting as in Hoffbeck-Dotsenko-Khoroshkin.

Definition of a combinad

A *combinad* is a monad on $\mathbb{N}$-modules $(\mathcal{X}, \Gamma)$

$$\mathcal{X} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod \quad , \quad \Gamma : \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

where $\mathcal{X}$ is induced by a given combinatorial pattern $\mathbb{X}$

$\mathcal{X}(M)_n = \text{span of the } \mathbb{X}\text{-trees with } n \text{ leaves, vertices decorated by } M$

and the composition Γ is induced by the substitution of the combinatorial pattern.

An *operad of type* $\mathbb{X}$ is an algebra over the monad $\mathcal{X}$.

Koszul duality of combinads: mimick [JLL-BV] and use rewriting as in Hoffbeck-Dotsenko-Khoroshkin.
Related to rewriting of polygraphs (work in progress with Ph.Malbos and Y.Guiraud)

Algebras	Operads	Combinads
(associative alg) $T(V)$, $U(\mathfrak{g})$, $S(V)$ $\cdots$	(ns operads) As	
(dendriform alg) $PBT(V)$, $T^{sh}(V)$, $\cdots$	$Dend$	$\mathbb{PT}$
(whatever alg) $\cdots$	$\cdots$	
(Lie alg) $Lie(V)$, $\mathfrak{g}$, sl_n $\cdots$	(symm operads) Lie	
(commutative alg) $S(V)$, $K[x,y]/\sim$, $\cdots$	Com	$\mathbb{T}$
(associative alg) $T(V)$, $\cdots$	Ass	
$\cdots$	$\cdots$	
$\cdots$	(permutads) $PermAs$	
$\cdots$	$q\text{-}permAs$	$\mathbb{SM}$
$\cdots$	$\cdots$	
$\cdots$	(shuffle operads) Ass	
$\cdots$	Com	
$\cdots$	$q\text{-}permAs$	$Sh\mathbb{T}$
$\cdots$	$ShAs$	

IV. Another example: surjective maps and permutads

IV. Another example: surjective maps and permutads

[JLL-MR]

IV. Another example: surjective maps and permutads

[JLL-MR]

The combinatorial pattern of *surjective maps* $\mathbb{X}$:
$\mathbb{X}_n =$ surjective maps $t : \underline{n} \to \underline{k}$
vertices of t: the elements of $\underline{k}$ (shown as $\times$)
inputs of a vertex v of t: the sectors around v (the number is $\#t^{-1}(v) + 1$)

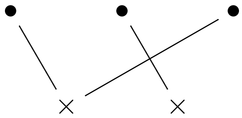
IV. Another example: surjective maps and permutads

[JLL-MR]

The combinatorial pattern of *surjective maps* $\mathbb{X}$:
$\mathbb{X}_n =$ surjective maps $t : \underline{n} \to \underline{k}$
vertices of t: the elements of $\underline{k}$ (shown as $\times$)
inputs of a vertex v of t: the sectors around v (the number is $\#t^{-1}(v) + 1$)

$n = 3$

vertices $k = 2$

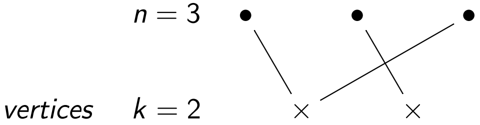


Substitution: given by composition of surjective maps

IV. Another example: surjective maps and permutads

[JLL-MR]

The combinatorial pattern of *surjective maps* $\mathbb{X}$:
$\mathbb{X}_n =$ surjective maps $t : \underline{n} \to \underline{k}$
vertices of t: the elements of $\underline{k}$ (shown as $\times$)
inputs of a vertex v of t: the sectors around v (the number is $\# t^{-1}(v) + 1$)



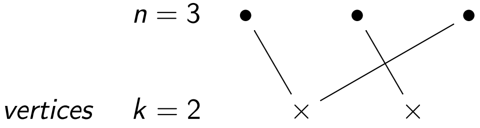
Substitution: given by composition of surjective maps

An algebra over this combinad is called a *permutad*.

IV. Another example: surjective maps and permutads

[JLL-MR]

The combinatorial pattern of *surjective maps* $\mathbb{X}$:
$\mathbb{X}_n =$ surjective maps $t : \underline{n} \to \underline{k}$
vertices of t: the elements of $\underline{k}$ (shown as $\times$)
inputs of a vertex v of t: the sectors around v (the number is $\# t^{-1}(v) + 1$)



Substitution: given by composition of surjective maps

An algebra over this combinad is called a *permutad*.
This is the *combinatorial definition* of a permutad.

Permutad vs shuffle algebra

Prop A permutad is an $\mathbb{N}$-module $\mathcal{P}$ equipped with linear maps

$$\bullet_\gamma : \mathcal{P}_{n+1} \otimes \mathcal{P}_{m+1} \to \mathcal{P}_{n+m+1}, \text{ for } \gamma \in Sh(n, m),$$

verifying:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$ in $Sh(n, m, r)$.

For some shuffles: $\bullet_\gamma = \circ_i$ (consecutive elements in the second set)

Permutad vs shuffle algebra

Prop A permutad is an $\mathbb{N}$-module $\mathcal{P}$ equipped with linear maps

$$\bullet_\gamma : \mathcal{P}_{n+1} \otimes \mathcal{P}_{m+1} \to \mathcal{P}_{n+m+1}, \text{ for } \gamma \in Sh(n, m),$$

verifying:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$ in $Sh(n, m, r)$.

For some shuffles: $\bullet_\gamma = \circ_i$ (consecutive elements in the second set)

This is the *partial definition* of a permutad. This is essentially the definition given by M. Ronco of a *shuffle algebra*, which are in fact *colored algebras*.

`M. Ronco, Shuffle bialgebras, Ann.Inst.Fourier 61 (2011), 799-850.`

PermAs and the permutohedron

The analog of *As* is the permutad generated by a binary operation and the associativity relation, denoted *PermAs*

PermAs and the permutohedron

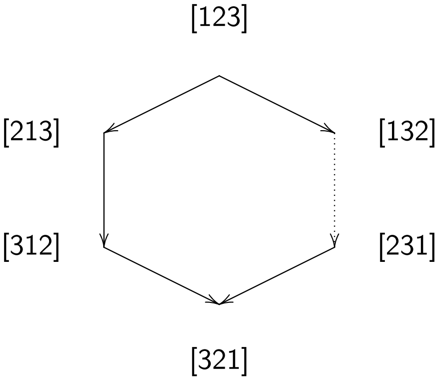
The analog of *As* is the permutad generated by a binary operation and the associativity relation, denoted *PermAs*

THM $PermAs_n$ is one-dimensional (like As_n).

PermAs and the permutohedron

The analog of *As* is the permutad generated by a binary operation and the associativity relation, denoted *PermAs*

THM $PermAs_n$ is one-dimensional (like As_n).
Key lemma: connectedness of a subgraph of the weak Bruhat order graph:



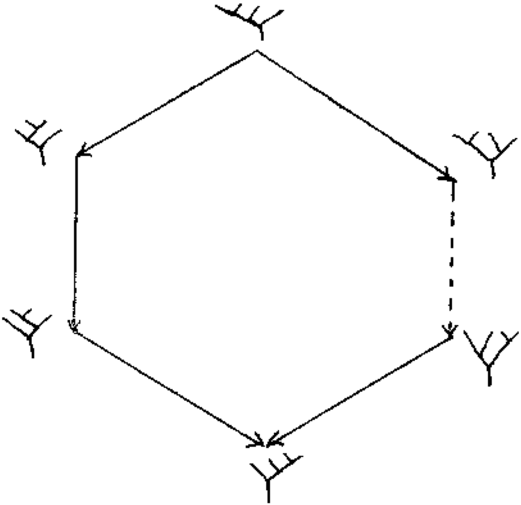
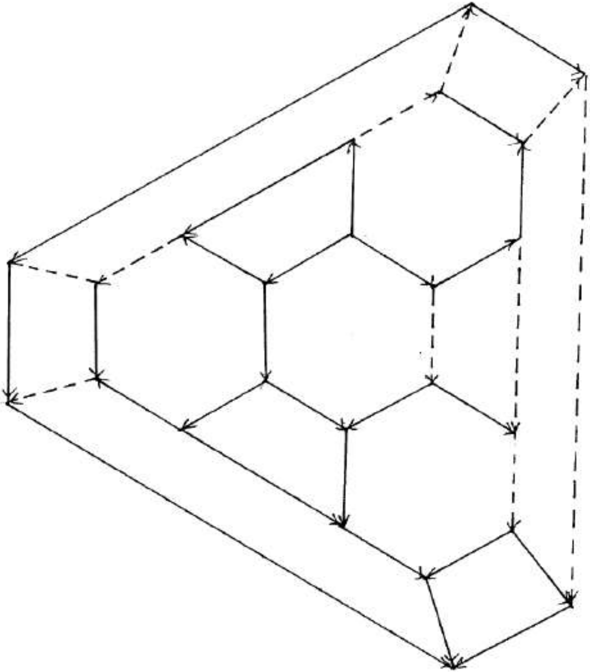


Figure: P^2 and trees



Minimal model of *PermAs*

- Minimal model of the ns operad *As* is A_∞ where

$$(A_\infty)_n = C_*(\text{associahedron})$$

- Minimal model of the permutad *PermAs* is $PermAs_\infty$ where

$$(PermAs_\infty)_n = C_*(\text{permutohedron})$$

Minimal model of *PermAs*

- Minimal model of the ns operad *As* is A_∞ where

$$(A_\infty)_n = C_*(\text{associahedron})$$

- Minimal model of the permutad *PermAs* is $PermAs_\infty$ where

$$(PermAs_\infty)_n = C_*(\text{permutohedron})$$

PROP The permutad with one binary operation and relation

$$(xy)z = q\ x(yz)$$

is Koszul for any q.
(In the operad case, only for $q = 0, 1, \infty$)

V. Further research (work in progress):

a) Feynman diagrams

One can construct a combinad from finite graphs with various decorations **stable by substitution**,
for instance Feynman graphs (QED, φ^4)
(work in progress JLL-N.Nikolov related to vertex algebras)

V. Further research (work in progress):

a) Feynman diagrams

One can construct a combinad from finite graphs with various decorations **stable by substitution**,
for instance Feynman graphs (QED, φ^4)
(work in progress JLL-N.Nikolov related to vertex algebras)

Symmetry choice in a combinatorial pattern at vertices with $n+1$ flags:

- planar rooted $\Rightarrow$ symmetry group $= \{1\}$,
- planar nonrooted $\Rightarrow$ symmetry group $= C_{n+1}$ (cyclic group),
- nonplanar rooted $\Rightarrow$ symmetry group $= \mathbb{S}_n$,
- nonplanar nonrooted $\Rightarrow$ symmetry group $= \mathbb{S}_{n+1}$,
- p inputs, q outputs $\Rightarrow$ symmetry group $= \mathbb{S}_p \times \mathbb{S}_q$.

More general combinatorial patterns

Let $\mathbb{Y}$ be a combinatorial pattern, for instance $\bullet$, $\mathbb{N}$ (ladders). A combinatorial pattern $\mathbb{X}$ over $\mathbb{Y}$ is
a set X of elements such that each $t \in X$ has an underlying set $|t| \in Y$ (replaces n)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$ and its set of *leaves* $\mathrm{leav}(t) \in Y$
Any vertex v comes with its set of *inputs* $\mathrm{in}(v) \in Y$

More general combinatorial patterns

Let $\mathbb{Y}$ be a combinatorial pattern, for instance $\bullet$, $\mathbb{N}$ (ladders). A combinatorial pattern $\mathbb{X}$ over $\mathbb{Y}$ is
a set X of elements such that each $t \in X$ has an underlying set $|t| \in Y$ (replaces n)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$ and its set of *leaves* $\mathrm{leav}(t) \in Y$
Any vertex v comes with its set of *inputs* $\mathrm{in}(v) \in Y$
For any trees t, s and $v \in \mathrm{vert}(t)$ and an isomorphism $\mathrm{in}(v) \cong \mathrm{leav}(s)$ there is given a new tree denoted $t \circ_v s$ such that $|t \circ_v s| = |t|$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \backslash \{v\}) \cup \mathrm{vert}(s)$

More general combinatorial patterns

Let $\mathbb{Y}$ be a combinatorial pattern, for instance $\bullet$, $\mathbb{N}$ (ladders). A combinatorial pattern $\mathbb{X}$ over $\mathbb{Y}$ is

a set X of elements such that each $t \in X$ has an underlying set $|t| \in Y$ (replaces n)

Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$ and its set of *leaves* $\mathrm{leav}(t) \in Y$

Any vertex v comes with its set of *inputs* $\mathrm{in}(v) \in Y$

For any trees t, s and $v \in \mathrm{vert}(t)$ and an isomorphism $\mathrm{in}(v) \cong \mathrm{leav}(s)$ there is given a new tree denoted $t \circ_v s$ such that $|t \circ_v s| = |t|$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \backslash \{v\}) \cup \mathrm{vert}(s)$

We assume **associativity** of this composition, that is

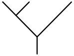
I. sequential axiom, for $w \in \mathrm{vert}(s)$: $(t \circ_v s) \circ_w r = t \circ_v (s \circ_w r)$

II. parallell axiom, for $w \in \mathrm{vert}(t)$: $(t \circ_v s) \circ_w r = (t \circ_w r) \circ_v s$

Combinatorics

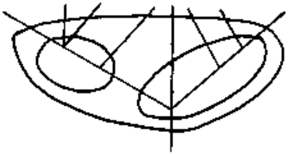
ladder = Dynkin diagram A_n

How to continue the sequence:

$k = 0$	$k = 1$	$k = 2$	$k = 3$
Mod	$\mathbb{N}$-mod	$\mathbb{PT}$-mod	$\mathbb{CPT}$-mod
$\bullet$	$-\bullet-\bullet-\bullet-$		(see picture later)
	ladder	circled ladder = tree	circled tree

Free combinad

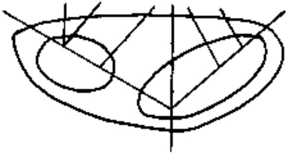
The combinatorial pattern is made of **circled trees**



leaves:	elements	in	Algebras
vertices:	operations	in	Operads
circles:	compositions	in	Combinads

Free combinad

The combinatorial pattern is made of **circled trees**



leaves:	elements	in	Algebras
vertices:	operations	in	Operads
circles:	compositions	in	Combinads

Substitution: as usual

(related to the dendroidal sets of Ieke Moerdijk and Ittay Weiss)

b) higher operads: opetopes

Alg=0-op	Op=1-op	Comb=2-op	$\cdots$
			$\cdots$
		$\mathbb{CPT}(W)$	$\mathbb{CPT}$
	$Mag(M) = \mathbb{PT}(M)$	$\mathbb{PT}$	
$T(V) = As(V)$	$\mathrm{As}=\{\mathrm{As}_n\}_{n\geq 1}$		

$V = \bullet$-module	$\bullet$		
$M = \mathbb{N}$-module	$\mathbb{N}$	—•—•—•—	comb. pattern over $\bullet$
$W = \mathbb{PT}$-module	$\mathbb{PT}$		comb. pattern over $\mathbb{N}$
	$\mathbb{CPT}$		comb. pattern over $\mathbb{PT}$
	etc.		comb. pattern over $\mathbb{CPT}$

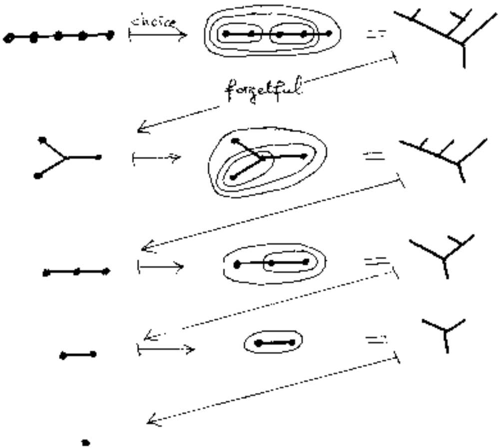
b) higher operads: opetopes

Alg=0-op	Op=1-op	Comb=2-op	$\cdots$
			$\cdots$
		$\mathbb{CPT}(W)$	$\mathbb{CPT}$
	$Mag(M) = \mathbb{PT}(M)$	$\mathbb{PT}$	
$T(V) = As(V)$	$\mathrm{As}=\{\mathrm{As}_n\}_{n\geq 1}$		

$V = \bullet$-module	$\bullet$		
$M = \mathbb{N}$-module	$\mathbb{N}$	$-\bullet-\bullet-\bullet-$	comb. pattern over $\bullet$
$W = \mathbb{PT}$-module	$\mathbb{PT}$		comb. pattern over $\mathbb{N}$
	$\mathbb{CPT}$		comb. pattern over $\mathbb{PT}$
	etc.		comb. pattern over $\mathbb{CPT}$

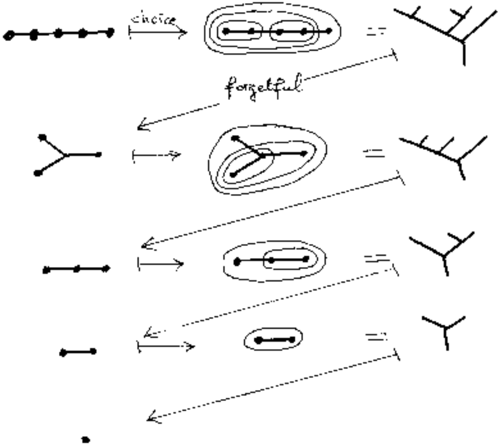
$T(V), Mag(M), \mathbb{CPT}(W)$ are free objects,
As, $\mathbb{PT}$ are associative objects

A tower of binary trees



graphs circled graphs rooted trees

A tower of binary trees $=$ binary opetope



graphs circled graphs rooted trees

Opetopes

Similar objects already appeared in the literature in the work:

`Baez, J.C.; Dolan, J., Higher-dimensional algebra. III. n-categories and the algebra of opetopes. Adv. Math. 135 (1998)`

under the name **opetopes**.

Opetopes

Similar objects already appeared in the literature in the work:

`Baez, J.C.; Dolan, J., Higher-dimensional algebra. III. n-categories and the algebra of opetopes. Adv. Math. 135 (1998)`

under the name **opetopes**.
Not surprizing since the philosophy is the same, except that Baez, Dolan (and then Joyal, Kock, Batanin, etc.) work in the set environment, while we are working in the linear environment.
Main differences: a set is a co-object, because of the diagonal (duplication is possible),
in the linear case there is more freedom (*Lie* is not set-theoretic).

Quantum groups

Number of binary opetopes $a(n)$, for A_n, :

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144		

$10 = 5 \times 2, \quad 144 = (12 \times 5 + 2 \times 6) \times 2$

Quantum groups

Number of binary opetopes $a(n)$, for A_n, :

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144		

This is, up to $n = 5$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

$$U_q^-(sl_n)$$

Canonical bases for quantized enveloping algebras were introduced by George Lusztig and Masaki Kashiwara in the nineties.

`Lusztig, G. Canonical bases arising from quantized enveloping algebras. J.Amer.Math.Soc.3 (1990)`

Quantum groups

Number of binary opetopes $a(n)$, for A_n, :

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144	6 608	

This is, up to $n = 6$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

$$U_q^-(sl_n)$$

Canonical bases for quantized enveloping algebras were introduced by George Lusztig and Masaki Kashiwara in the nineties.

`Lusztig, G. Canonical bases arising from quantized enveloping algebras. J.Amer.Math.Soc.3 (1990)`

Quantum groups

Number of binary opetopes $a(n)$, for A_n, :

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144	6 608	1 044 736

This is, up to $n = 6$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

$$U_q^-(sl_n)$$

Canonical bases for quantized enveloping algebras were introduced by George Lusztig and Masaki Kashiwara in the nineties.
`Lusztig, G. Canonical bases arising from quantized enveloping algebras. J.Amer.Math.Soc.3 (1990)`

CONJECTURE The number of regions of linearity for A_6 is

$$1\,044\,736$$

Quantum groups

Number of binary opetopes $a(n)$, for A_n, $d(n)$, for D_n, etc.:

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144	6 608	1 044 736
d(n)	1	1	2	12	184	8 704	1 395 456

This is, up to $n = 6$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

$$U_q^-(sl_n)$$

Canonical bases for quantized enveloping algebras were introduced by George Lusztig and Masaki Kashiwara in the nineties.

`Lusztig, G. Canonical bases arising from quantized enveloping algebras. J.Amer.Math.Soc.3 (1990)`

CONJECTURE The number of regions of linearity for A_6 is

$$1\,044\,736$$

More: a helpful lemma in order to compute $a(n)$ and $d(n)$ is the following. A *partition* of the graph G is: I and J nonempty connected subgraphs of G such that each vertex is either in I or in J. Define $\varphi(G) =$ sum of "derived graphs" (forgetful $\circ$ choice)

Lemma For any graph G we have

$$\varphi(G) = \sum_{(I,J)} \varphi(I) \vee \varphi(J) \ ,$$

where the sum is over all partitions of G.

This decomposition is close to Carter's method to construct regions of linearity in
`Carter, R.W. Canonical bases, reduced words, and Lusztig's piecewise-linear function. ...(1997).`

Thanks to Richard Green and Robert Marsh for their help.

MERCI !