Saunders Mac Lane

Lectures on category theory

Bowdoin Summer School 1969

Notes taken by Ellis Cooper

$\boxed{\text{Maclane 1}}$

Category (by axioms)

objects $a \in C$ arrows $f \in C$

every arrow f has domain a (source obj)
codomain b (target obj)

$$f : a \longrightarrow b$$

composition

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, dashed, "\exists\, g \circ f"'] & & c \end{tikzcd}$$

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ex. obj is set
arrow is fn, $g(f(x))$.

For ea obj. b $\exists$ $b \circlearrowleft 1_b$ identity

Axioms. (Assoc.) If

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right] \arrow[rrr, bend right=40] & b \arrow[r, "g"] \arrow[rr, bend left] & c \arrow[r, "h"] & d \end{tikzcd}$$

Then $h \circ (g \circ f) = (h \circ g) \circ f$.

(Ident.) If

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, "g"] \arrow[loop below, "1_b"] & c \end{tikzcd}$$

Then $\left|\begin{array}{l} 1_b \circ f = f \\ g \circ 1_b = g \end{array}\right.$

$\hom(a, b) = \{ f \mid f : a \to b \}$ also a start.

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ex. $\mathbb{E}$ns = cat. of sets

distinguish not only graph but also codomain.

ex. $\mathcal{G}$ = cat. of groups. obj. is a gp.
arrow is homomorphism into.

ex. $\mathbb{A}b$ = cat. of abelian gps. obj. is ab. gp.
arrows same as $\mathcal{G}$ ($\mathbb{A}b$ "full" in $\mathcal{G}$).

ex. $\mathbb{R}ing$ —————— - (with id.).

An arrow is an <u>isomorphism</u> $a \xrightarrow{f} b$ - $\iff$
$\exists\, g : b \longrightarrow a$ s.t. $g \circ f = 1_a$, $f \circ g = 1_b$.

<u>monomorphism</u> $\iff$
(cancel on left) $fh = fk \implies h = k$

<u>epimorphism</u> $\iff$
(cancel on right) $hf = kf \implies h = k$.

remark. surjective $\implies$ epi.
Prop. When "epi $\Rightarrow$ surj" for $\mathbb{R}ing$, $\mathbb{G}rp$, $\mathbb{A}b$, $\mathbb{E}ns$.

defn. $C \xrightarrow{T} X$ <u>functor</u> is a morphism of cat.

obj fn $\quad a \rightsquigarrow Ta \in X$
arrow fn $\quad f \rightsquigarrow Tf \in X$

if
$$\begin{tikzcd} a \arrow[d, "f"] \\ b \end{tikzcd}$$
then
$$\begin{tikzcd} Ta \arrow[d, "Tf"] \\ Tb \end{tikzcd}$$

if
$$\begin{tikzcd} b \arrow[loop right, "1b"] \end{tikzcd}$$
then $T(1_b) = 1_{Tb}$

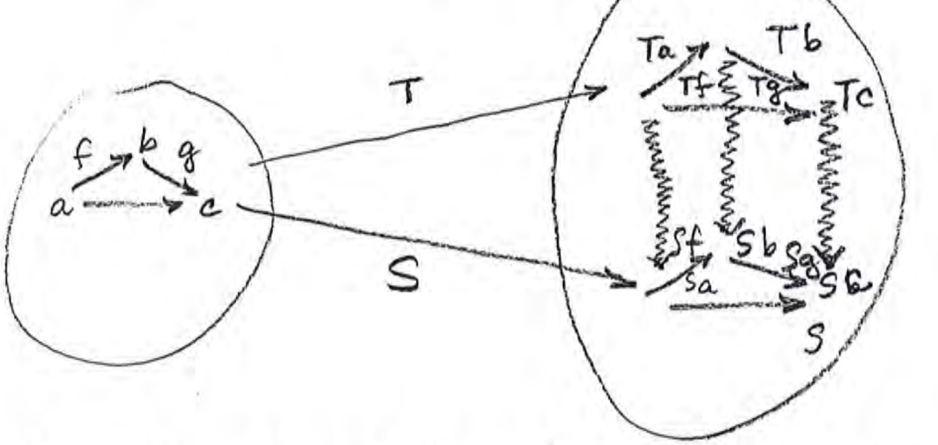
if
$$\begin{tikzcd} \cdot \arrow[d] \\ \cdot \arrow[d] \\ \cdot \end{tikzcd}$$
then
$$\begin{tikzcd} T \arrow[d, "T"] \\ T \arrow[d, "T"] \\ T \end{tikzcd}$$
$T(g \circ f) = Tg \circ Tf$.

ex.

$$\begin{tikzcd}
 & (X,x) \arrow[r, "f"] \arrow[d, "``T\text{''}"] \arrow[ddl, "``S\text{''}"'] & (Y,y) \arrow[dr] & & \text{in } \underbrace{\mathbb{T}\text{op w. b.-p.}}_{\mathbb{T}\text{op}^*} \\
 & & & (Z,z) & \\
 & \pi_1(X,x) \arrow[r, "\pi_1 f"] & \pi_1(Y,y) \arrow[dr] & & \text{in } \mathcal{G}p \\
H_1(X,x) & \text{Fund. Gp.} & & \pi_1(Z,z) &
\end{tikzcd}$$

$$\boxed{\mathbb{T}\text{op}_* \xrightarrow{\pi_1} \text{Gp}}$$

$$\begin{tikzcd}
C \arrow[r, shift left, "T"] \arrow[r, shift right, "S"'] & X
\end{tikzcd}$$



defn. A <u>natural</u> <u>transformation</u> $\alpha : T \longrightarrow S$ is assigns to ea. obj. c an arrow $\alpha_c : Tc \longrightarrow Sc$ so that, if $a \xrightarrow{f} b$ then

$$\begin{tikzcd}
Ta \arrow[r, "Tf"] \arrow[d, "\alpha_a"'] & Tb \arrow[d, "\alpha_b"] \\
Sa \arrow[r, "Sf"] & Sb
\end{tikzcd} \Bigg| \, .$$

ex. $\pi_1(X, x) \longrightarrow H_1(X, x)$.

ex. let $\mathbb{V}ect_F$ = vector spaces over field F.

$$\begin{tikzcd} V \arrow[r, mapsto] & V^* \arrow[d, equal] & : \mathbb{V}ect_F \arrow[r] & \mathbb{V}ect_F \\ & \{T \mid V \xrightarrow{T} \mathbb{F}\}. & & \end{tikzcd}$$

$$V \xrightarrow{f} W \quad \longmapsto \quad W^* \xrightarrow{f^*} V^*$$

dual is "contravariant functor".

If $V \xrightarrow{\alpha_V} V^{**}$ then $**$ <u>is</u> a functor.

$$\boxed{(\alpha_V(a))t = t(a).}$$

<u>"NATURAL"</u>.

"free-est possible"

$$\begin{tikzcd} V \in \mathbb{V}ect_K \arrow[dd, squiggly, "G"] & & FX \arrow[d, "Ff"] \\ & & FY \\ G(V) \in \mathbb{E}ns & \ni & X \arrow[uu, squiggly, "F"] \arrow[d, "f"] \\ & & Y \arrow[ull, bend left] \end{tikzcd}$$

FX = the vector space basis X, ie, elmts are fin. lin. combs. $\sum^n k_i\langle x_i\rangle \,\Big|\, \begin{matrix} k_i \in K \\ x_i \in X \end{matrix}$.

<u>underlying</u> <u>set</u>

$G \circ F = T$

TX = underlying set of FX. ("swells")

$$\begin{tikzcd}
x \in X \arrow[r, "f"] \arrow[d, "\eta_X"] & Y \ni f(x) \arrow[d, "\eta_Y"] \\
\langle x \rangle \in FX \arrow[r, "Tf"'] & FY \ni \langle f(x) \rangle
\end{tikzcd}$$

$\eta_X(x)$ "insertion of generators" .

Prop. η is a nt. xf.

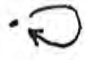
$\eta : I \longrightarrow T$; find $T \circ T \xrightarrow{\bar{\mu}} T$.

$\sum_j l_j \langle \sum_i k_{ij} \langle x_{ij} \rangle \rangle \quad \sum_j \sum_i l_j k_{ij} \langle\langle x_{ij} \rangle\rangle$

$(\mathbb{E}us, T, \eta, \mu)$ Triple . TRIAD here .

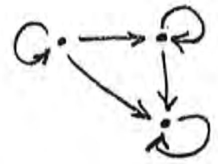
More categories. $\mathbb{0}$

TINY cat.



$\mathbb{1}$

$\mathbb{2}$



$\mathbb{3}$

PRE-ORDER P a $\overset{\text{class}}{\text{set}}$ and relation $a \le b$ s.t. transitive & reflexive.

PARTIAL ORDER antisymmetric pre-order. (POSET).

Prop. Pre-order is a cat. wh. obj's are elmts of P and arrows are opposites of [illegible] $\le$.

Cor. So every Posset. $a \overset{\leftarrow}{\le} b$.

in pre-σ

card. no. hom $(a, b) = \begin{cases} 0 \\ 1 \end{cases}$.

Maclane 2

✓ $$\text{Exercise. }\underbrace{(\text{never } \dots x_3\in x_2\in x_1\in x_0)}_{7.} \overset{\checkmark\ \Rightarrow}{\Longleftrightarrow} \begin{pmatrix}\text{given } x\ \exists\, y\in x\\ \text{s.t. } y\cap x=\emptyset\end{pmatrix}.$$

Z (Zermelo) : $x, y, z, \dots$ sets $\in$ $=$

✓1. (Extensionality) $x=y$ iff $\forall t\,(t\in x \Leftrightarrow t\in y)$

✓2. $\exists\ \emptyset$

✓3. $x, y\ \exists\ \{x,y\}$

✓4. $\forall u\ \exists \bigcup_{t\in u} t = \{s \mid \exists t\ s\in t\in u\}$

✓5. $u\ \exists\ \mathcal{P}u = \{x \mid x\subset u\}$

✓6. $\exists$ inf. set

7. (regularity)

✓8. (comprehension) given a property $P(x)$ and set a
$\exists\ \{x \mid x\in a\ \&\ P(x)\}$

ZC. Ax. of choice

Paul Cohen <u>Set Theory & Continuum Hypothesis</u>

J.L. Krivine <u>Th. Ax. Des Ensembles</u>

$\mathcal{P}u$ = set of all subsets of u

$R_0 = \emptyset$ $\qquad$ $R_{n+1} = \mathcal{P}(R_n)$

$\vdots$ $\qquad$ $R_\omega = R_0 \cup R_1 \cup \cdots$

$\vdots$

(On 6): define ordinal numbers

$0 = \emptyset \quad 1 = \{\emptyset\} \quad 2 = \{0, 1\} \quad 3 = \{0, 1, 2\} \quad \ldots$

$$n+1 = n \cup \{n\}$$

$$\text{succ}\ \alpha = \alpha \cup \{\alpha\} .$$

Ax of Inf. "$\exists\ \omega$" ie., $\exists\ x$ a set s.t. $\emptyset \in x$ & $t \in x \Rightarrow \text{succ}\, t \in x$

Replacement Axiom. "Any image of a set is a set."

Need to explicate "property" in 8.

(Well formed) formula $\Theta, \Psi, \eta, \varsigma \ldots$

$x \in y$, $x = y$, Θ, Ψ fs so are $\Theta \,\&\, \Psi$
Θ or Ψ
not(Θ)

$(\exists t)(\Theta(t))$
$(\forall t)$ $\quad \ldots$

RAx. If $\Theta(x,y)$ is a formula, functional in y, i.e., $\Theta(x, y_1) \,\&\, \Theta(x, y_2) \Rightarrow y_1 = y_2$, if a is a set then $\exists\ \{y \mid \exists\, x \in a \text{ s.t. } \Theta(x,y)\}$.

$$\left.1 - 8 - \underbrace{ZC}_{9} - \underset{8^*}{RAx}\ \right\}\ \text{Zermelo - Fraenkel}$$

Thm $8^* \Rightarrow 8$.

prf. $\theta(x,y) = (x = y \text{ and } P(y))$. | 1–8* } ZFC

Defn. A <u>universe</u> $\mathcal{U}$ is a set with

1) $x \in t \in \mathcal{U} \implies x \in \mathcal{U}$
2) $\omega \in \mathcal{U}$
3) $v \in \mathcal{U} \implies \mathcal{P}v \in \mathcal{U}$
4) $v \in \mathcal{U} \implies \bigcup_{t \in v} t \in \mathcal{U}$
5) $\begin{matrix} x \in \mathcal{U} \\ a \subset \mathcal{U} \end{matrix}$ $f : x \rightarrow a$ surjection $\implies a \in \mathcal{U}$

Properties of $\mathcal{U}$:

Thm $x, y \in \mathcal{U} \implies \{x,y\} \in \mathcal{U}$.

prf. $\{0,1\} \longrightarrow \{x,y\} \subset \mathcal{U}$

So is $\langle x,y \rangle \equiv \{\{x\},\{x,y\}\}$ in $\mathcal{U}$ for all $x,y \in \mathcal{U}$.

Remark. for (5) necessary to define in ZFC notion of ordered pair – product – function:

<u>function</u> $f : u \rightarrow v$ is a tuple $\langle u, G_f, v \rangle$ where $G_f \subset u \times v$ and . . . .

~~⓪~~ $w \subset u \in \mathcal{U} \implies w \in \mathcal{U}$ } by ① $w \subset u \Leftrightarrow \forall x, x \in w \Rightarrow x \in u$.

EXERCISE. ① $u, v \in \mathcal{U} \implies \left\{ \begin{matrix} u \times v \in \mathcal{U} \\ v^u \in \mathcal{U} \end{matrix} \right\}$ $x \in u \in \mathcal{U} \Rightarrow x \in \mathcal{U}$

$u \times v = \{\{\{x\},\{x,y\}\} \mid x \in u, y \in v\}$

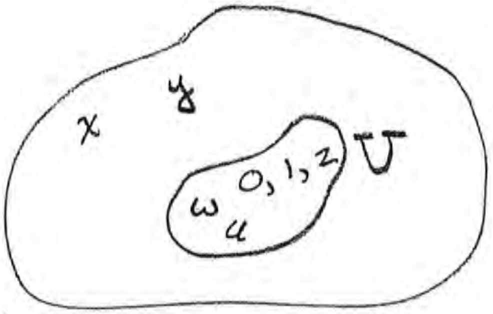
~~②~~ If $I \in \mathcal{U}$, $x_i \in \mathcal{U}$ then $\bigcup_{i \in I} x_i \in \mathcal{U}$

(4) + (5)

▷ Ex. ③ (2) $\Longleftrightarrow$ (4) & (5)

+ $\prod x_i \in U$.

Define. A set x is <u>small</u> $\Longleftrightarrow$ $x \in U$.



✓▷ Theorem. The small sets satisfy ZFC.

ZFCU axiom. $\exists$ a "particular named" universe.

$\mathbb{E}\mathrm{ns}$ = category of small sets.

| ob $\mathbb{E}\mathrm{ns} = U$
| $f : u \longrightarrow v$ is $\langle u, G_f, v \rangle$

ZFC

MacLane 3

(Defn by diagrams): e.g. a monoid is semi-gp w id.

or, defn. Mon is a set M and fn's

$$M\times M \xrightarrow{\mu} M$$

$$1 \xrightarrow{id} M \qquad \text{s.t.}$$

assoc. law :-

$$\begin{tikzcd}
M\times(M\times M) \arrow[r, "1\times\mu"] \arrow[d, phantom, "\cong\ \text{can. isomp.}"] & M\times M \arrow[dd, "\mu"] \\
(M\times M)\times M \arrow[d, "\mu\times 1"'] & \\
M\times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

identity law :-

(left unit)

$$\begin{tikzcd}
M \arrow[r, "\cong"] \arrow[d, equal] & 1\times M \arrow[d, "id\times M"] \\
M & M\times M \arrow[l]
\end{tikzcd}$$

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morphism of monads by diagram

$$\begin{tikzcd}
M \arrow[d, "f"] \\
M'
\end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd}
M\times M \arrow[r, "\mu"] \arrow[d, "f\times f"] & M \arrow[d, "f"] \\
M'\times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

$$\& \quad \begin{tikzcd}
 & M \arrow[d, "f"] \\
1 \arrow[ur] \arrow[r] & M'
\end{tikzcd}$$

✓ exercise. define group and group acts on a set by diagrams.

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remark.. by drawing these in a category $\mathcal{C}$ defines a <u>monoid-object</u> in $\mathcal{C}$ [when $\mathcal{C}$ has products].

About $\mathcal{C}$ have used fact that × is ftr of 2 variables

example. Ab ; say M,N ab gps.

$$\begin{tikzcd} M \arrow[d, "f"] & N \arrow[d, "g"] & M\otimes N \arrow[d, "f\otimes g"] \\ M' & N' & M'\otimes N' \end{tikzcd}$$

[a monoid object in Ab,⊗ is a ring.]

(Defn of cat. by <u>hom-sets</u>) :

A <u>cat.</u> $\mathcal{E}$ is a set $\mathcal{E}$ of objects a, b, c, ... w to ea pair (a,b) a set hom(a,b) [set of all $a \xrightarrow{f} b$] with, for each triple a, b, c a function

$$\hom(b,c) \times \hom(a,b) \longrightarrow \hom(a,c)$$

and to ea obj a an arrow $1 \xrightarrow{id} \hom(a,a)$.

conditions: assoc. law

$$\begin{tikzcd}
\hom(c,d) \times (\hom(b,c) \times \hom(a,b)) \arrow[r] \arrow[d, equal] & \hom(c,d) \times \hom(a,c) \arrow[dd] \\
(\hom(c,d) \times \hom(b,c)) \times \hom(a,b) \arrow[d] & /// \\
\hom(b,d) \times \hom(a,b) \arrow[r] & \hom(a,d)
\end{tikzcd}$$

also 1 is left & right identity.

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remark. One may pedantically assume all hom-sets disjoint.

(hom-set version of functor):

<u>ftr</u> $\mathcal{C} \xrightarrow{T} X$

$$\text{data} \begin{cases} \text{is } \underline{\text{fn}} \quad T : O_{\mathcal{C}} \longrightarrow O_X \\ \text{and } \underline{\text{fn}}\text{'s } \ T : \hom(a,b) \longrightarrow \hom(Ta,Tb) \end{cases}$$

so that

conditions

$$\begin{tikzcd}
\hom(b,c) \times \hom(a,b) \arrow[r] \arrow[d] & \hom(a,c) \arrow[d] \\
{} \arrow[r] & {}
\end{tikzcd}$$

☑ EXERCISE. define nt. xf. by hom-sets .

$\boxed{\text{ZFCU}}$ U is a set closed under everything.

Recall $x \in U \iff x$ is small. $x \in U \Rightarrow x \subset U$;

So $$U \subset \underbrace{\wp U}_{\text{classes}} \subset \underbrace{\wp\wp U}_{\substack{\text{classes of}\\ \text{classes}}} \subset \cdots \quad .$$

$\mathbb{E}$ns is the cat. of all small sets; Gp, ... etc.
Cat is the cat. of small categories [$O_{\mathcal{C}}$, $A_{\mathcal{C}}$ are sets]. Gp is large but its hom sets are small.

Th$^{\underline{m}}$ If $\mathcal{C}$ has small hom sets then $\hom(a,-)$ and $\hom(-,b)$ are functors to $\mathbb{E}$ns.

covariant hom ftr

Fix a.
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\hom(a{,}-)"] & \mathbb{E}\text{ns} \\ b \arrow[r, squiggly] & \hom(a,b) \end{tikzcd}$$
"compose with t"

$$\left[t_* = t\cdot\text{—} = \hom(a,t) \right].$$

Fix b.
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\hom(-{,}b)"] & \mathbb{E}\text{ns} \\ a \arrow[r, mapsto] \arrow[d, "f"] & \hom(a,b) \\ a' \arrow[r, mapsto] & \hom(a',b) \arrow[u, "\text{—}\cdot f"'] \end{tikzcd}$$

$$\left[f^* = \text{—}\cdot f = \hom(f,b) \right].$$

▷ Ques. Can there be a cat. whose hom sets are not small?

example of yes. ftr cat. let $X, \mathcal{C}$ large.

$X^{\mathcal{C}}$ ob = all $T : \mathcal{C} \longrightarrow X$

arrow = nt. xf.
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "T"{name=U}] \arrow[r, bend right=50, "S"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

so α is a "fn" on obj's of $\mathcal{C}$ to arrs of X.

So hom set is a class of classes [$\mathbb{A}b^{\mathbb{T}op}$].

To get "hom-functors" in this situation, define for V any set

$\mathbb{E}ns_V$ = cat of all sets $x \in V$.

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defn. Opposite cat. $\mathcal{C}^{op}$ to $\mathcal{C}$ has same obj's and reversed arrows,

Obsrvt: think of saying "obs & arrs same, but different dom & cod fns"

$f : a \xrightarrow[\mathcal{C}]{} b$ $\qquad$ $\bar{f} : b \xleftarrow[\mathcal{C}^{op}]{} a$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & c \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[dl, "\bar{f}"'] & \\ a & & c \arrow[ul, "\bar{g}"'] \arrow[ll, "\bar{f} \circ \bar{g} \equiv \overline{g \circ f}"] \end{tikzcd}$$

Any contravariant $\mathcal{C} \longrightarrow X$ is a functor $\mathcal{C}^{op} \longrightarrow X$.

defn. $\mathcal{C} \times \mathcal{C}'$ product of cats.

objs : pairs $\langle c, c' \rangle$

arrs : pairs $\langle \downarrow \downarrow \rangle$
$\langle d, d' \rangle$

$$\begin{tikzcd}
 & \mathcal{C} \times \mathcal{C}' \arrow[dl, "P"'] \arrow[dr, "P'"] & \\
\mathcal{C} & & \mathcal{C}' \\
 & X \arrow[ul, "T"] \arrow[ur, "T'"'] \arrow[uu, dashed, "\exists! S"'] &
\end{tikzcd}$$

} projections.

} functors

$S x = \langle T x, T' x \rangle$.

ex.

⊗ on $\mathbb{A}b$ is really ftr of two variables, ie ftr of 1 var on product

$$\mathbb{A}b \times \mathbb{A}b \xrightarrow{\otimes} \mathbb{A}b \ .$$

ex.

also $\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\text{hom}} \mathbb{E}ns$

$$\begin{tikzcd}
a & b \arrow[d, "t"] & \langle a, b \rangle \arrow[d] \arrow[r, mapsto] & \text{hom}(a, b) \arrow[d, "\text{hom}(s,t)"] & f \arrow[d, mapsto] \\
a' \arrow[u, "s"] & b' & \langle a', b' \rangle \arrow[r, mapsto] & \text{hom}(a', b') & t \circ f \circ s
\end{tikzcd}$$

▷

$$\begin{tikzcd}
\hom(a,b) \arrow[r, "\hom(a,t)"] \arrow[d, "\hom(s,b)"'] \arrow[dr, "\hom(s,t)"] & \hom(a,b') \arrow[d, "\hom(s,b')"] \\
\hom(a',b) \arrow[r] & \hom(a',b')
\end{tikzcd}$$

✓▷ exercise. Notes §9, find the nonsense.

(Cat of rectangular matrices):

objs: $\omega = \{0, 1, 2, \dots\}$

$\hom(m,n)$ = all $n \times m$ matrices

□

composition is matrix multiplication.

* example. R some ring.

consider cat Mod_R = set of left R-modules

$$\mathrm{hom}_R(a,b) = \{ f \mid f : a \to b \text{ R-mod hom.} \}$$

Now "$\mathrm{hom}_R(a,b)$" is an object of $\mathbb{A}b$.

▷ composition is a bilinear function,
[distributive]

hence really, comp is

$$\mathrm{Hom}(b,c) \otimes \mathrm{Hom}(a,b) \longrightarrow \mathrm{Hom}(a,c).$$

$$\text{So } \mathrm{Mod}_R \left| \begin{array}{l} \text{set of objects} \\ a, b \in \mathrm{Mod}_R \rightarrow \text{ab gp } \mathrm{Hom}(a,b) \end{array} \right.$$

▷ Modules are a super cat in $\mathbb{A}b$.

$\boxed{\text{Maclane } 4}$

V cat. with ftr of 2 var's $V \times V \xrightarrow{\otimes} V : (a,b) \longmapsto a \otimes b$,
$1 \longrightarrow V$, $(a\otimes b)\otimes c = a \otimes (b \otimes c)$, $1 \otimes a = a \otimes 1 = a$

$\left[\ \mathbb{E}\mathrm{ns} : \otimes = \times \ ;\ \mathbb{A}\mathrm{b} : \otimes = \otimes \ \right].$

Monoid object in V : $\begin{cases} m \in \mathrm{obj}\, V \\ 1)\ m \otimes m \xrightarrow{\mu} m \\ 2)\ 1 \xrightarrow{\eta} m \end{cases}$

$$\begin{tikzcd} a \arrow[r, "f"] & d \\ a\otimes b \arrow[u] \arrow[d] \arrow[r, "f\otimes g"] & c \otimes d \arrow[u] \arrow[d] \\ b \arrow[r, "g"] & c \end{tikzcd}$$

s.t. (1)

(assoc)

$$\begin{tikzcd} m \otimes m \otimes m \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & m \otimes m \arrow[d, "\mu"] \\ m \otimes m \arrow[r, "\mu"'] & m \end{tikzcd}$$

(2)

(lft & rt Id)

$$\begin{tikzcd} 1 \otimes m \arrow[r] \arrow[dr, equal] & m \otimes m \arrow[d, "\mu"] & m \otimes 1 \arrow[l] \arrow[dl, equal] \\ & m & \end{tikzcd}$$

$\Bigg[$ mon obj in $\mathbb{E}$ns, × = ordinary monoid
mon obj in $\mathbb{A}$b, ⊗ = ring
mon obj in $\mathbb{A}$b, × = EXERCISE $\Bigg]$ — pointed ab. gp. ?

$$\begin{tikzcd} m \otimes m \arrow[r, "\mu"] \arrow[d, "f\otimes f"'] & m \arrow[d, "f"] & 1 \arrow[l, "\eta"'] \arrow[d] \\ m' \otimes m' \arrow[r, "\mu'"'] & m' & 1 \arrow[l, "\eta'"] \end{tikzcd}$$

} morphism of mon obj's

EXERCISE. what is a gp in cat. of gps } FIELD (most of the time.)

a "super cat in V" (has a "×" and a "1") is a set O obj's a, b, . . .

SEE back of prec. p. ✳ and to ea pair (a,b) a "set of arrows" Hom(a,b) ∈ V [[illegible]]

data

$$\mathrm{Hom}(a,b) \times \mathrm{Hom}(a,b) \xrightarrow{\mu_{a,b}} \mathrm{Hom}(a,b) \quad V\text{-arrows}$$

$$1 \xrightarrow{\eta_a} \mathrm{Hom}(a,a) \quad V\text{-arrows}$$

s.t.

conditions

①

$$\begin{tikzcd}
& (\mathrm{Hom}(a,b) \times \mathrm{Hom}(a,b)) \times \mathrm{Hom}(a,b) \arrow[dl] \arrow[dr, "\mu_{a,b} \times id"] & \\
\mathrm{Hom}(a,b) \times (\mathrm{Hom}(a,b) \times \mathrm{hom}(a,b)) \arrow[d, "id \times \mu_{a,b}"] & & \mathrm{Hom}(a,b) \times \mathrm{Hom}(a,b) \arrow[d, "\mu_{a,b}"] \\
\mathrm{Hom}(a,b) \times \mathrm{Hom}(a,b) \arrow[rr, "\mu_{a,b}"'] & & \mathrm{Hom}(a,b)
\end{tikzcd}$$

② id law

Thm A cat obj in Ens is a cat with small sets.

What about ~~cat~~ super cat obj in Ab, ⊗ ?

[A mon is cat w 1 obj ;, a mon obj in Ab, ⊗ is a ring.]

[A super cat in Ab, ⊗ is a category, and each hom (a,b) has add. ab. gp. str., and composition is distributive.

Or, super cat in /Ab,⊗ is a cat $\begin{cases} \text{obj's} \\ \text{arr's} \end{cases}$

+ given $a \underset{f'}{\overset{f}{\rightrightarrows}} b$ get $a \xrightarrow{f+f'} b$;

+ hom(a,b) = ab. gp. under this + .

$\left\{ \begin{array}{l} \underline{\text{Pre-additive cat}} \\ \underline{{}^{+}\text{Category}} \end{array} \right\}$

Now to speak of mp's of $^{+}$Cats.

Let C, X $^{+}$Cats.

$$\begin{tikzcd}
a \arrow[d, bend right, mapsto] & \text{obj } C \arrow[d] & \hom(a,b) \arrow[d, "H_{a,b}"] \\
Ha & \text{obj } X & \hom(Ha,Hb)
\end{tikzcd}$$

$\leftarrow$ AN ARROW IN /Ab,⊗

A mp. of $^{+}$Cats is fn H + fn's $H_{a,b}$ so that

$$\begin{tikzcd}
\hom(b,c) \otimes \hom(a,b) \arrow[r] \arrow[d, "H_{b,c} \otimes H_{a,b}"] & \hom(a,c) \arrow[d, "H_{a,c}"] \\
\hom(Hb,Hc) \otimes \hom(Ha,Hb) \arrow[r] & \hom(Ha,Hc)
\end{tikzcd}$$

** Action of a mon on C in $\mathbb{E}$ns is an ordinary action; action of a mon in $\mathbb{A}$b, $\otimes$ is a module.

and

$$\begin{tikzcd}
1 \arrow[r, "\eta_a"] \arrow[dr, "\eta_{Ha,a}"'] & \hom(a,a) \arrow[d, "H_{a,a}"] \\
 & \hom(Ha,Ha)
\end{tikzcd}$$.

So a <u>+functor</u> ${}^+C \longrightarrow {}^+X$ is

data{(1) an ordinary functor H

condition{(2) if $a \overset{f}{\underset{f'}{\rightrightarrows}} b$ then $H(f+f') = Hf + Hf'$,

i.e., an <u>additive functor</u> .

<u>Ex's.</u> Let a a right R module.

$$\begin{tikzcd}
\mathbb{M}od_R \arrow[r] & Ab \\
c \arrow[r, mapsto] & a \otimes_R c
\end{tikzcd}$$

K comm. ring. $Mod_K \longrightarrow \mathbb{M}od_K$.

Let $\mathbb{M}$ a monoid object in V, C obj of V. An <u>action</u> of M on C is $M \otimes C \xrightarrow{\nu} C$ so that

$$\begin{tikzcd}
M \otimes M \otimes C \arrow[r, "1 \otimes \nu"] \arrow[d, "\mu \otimes 1"'] & M \otimes C \arrow[d, "\nu"] \\
M \otimes C \arrow[r, "\nu"'] & C
\end{tikzcd}$$,

and (id works right)

$$\begin{tikzcd}
M \otimes C \arrow[d, "\nu"'] & 1 \otimes C \arrow[l, "\eta \otimes 1"'] \arrow[dl, equal] \\
C &
\end{tikzcd}$$.

EXERCISE. explain hom. of action.

Let S super cat on V. An action of S is fn L, where L assigns to ea obj a of S an obj L(a) of V, and for ea pair (a,b) fn

** back prec. p.

$$\text{hom}(a,b) \otimes L(a) \xrightarrow{\nu_{a,b}} L(b)$$

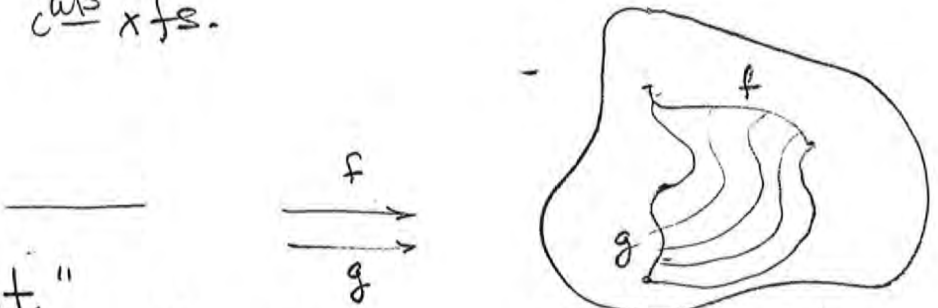
s.t. . . . .

What is this? Take $V = \mathbb{E}ns$; then S is a cat., and L assigns to an obj of super cat a set, and $Lf : La \to Lb$ may be defined —— !!

▷ Thus an action of S is a functor $L : S \longrightarrow \mathbb{E}\mathrm{ns}$. [compare to modules]

Say in $\mathbb{T}\mathrm{op}$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$ cnts xfs.



Defn. $f \overset{\text{"homot."}}{\sim} g$ iff $\exists\ X \times I \xrightarrow{F} Y$ cnts

so that $\left| \begin{array}{l} F(x, 0) = f(x) \\ F(x, 1) = g(x) \end{array} \right.$.

Look seriously at homotopy classes $\mathrm{cls}(f) = \{ g : X \longrightarrow Y \mid g \sim f \}$.

Make new cat. obj's are top sp.'s and arr's are homot. cls's of mf's.

Arr's are <u>not</u> <u>fn's</u> !

$\mathbb{G}\mathrm{p} \overset{?}{\subset} \mathbb{E}\mathrm{ns}$. NO. But $U : \mathbb{G}\mathrm{p} \longrightarrow \mathbb{E}\mathrm{ns}$ is "<u>faithful</u>", injective on hom sets.

<u>full</u> if surjective on hom sets.

Defn. A <u>concrete category</u> is a cat. C together w a faithful fctr to Ens.

PROBLEM. Is every (small) cat. ≅ to a concrete cat. ? ["Does every gp have a rep as a perm. gp.?"]

▷* RESTATED. Given sm cat D; is there a fth fctr D to sets ? (YES) [redacted]

RE STATED. Given <u>any</u> cat (w small hom sets); . . . . . ? (NO) [redacted] Freyd p. 108

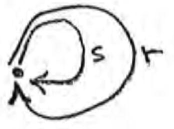
AGAIN a QUESTION: Is every + cat concrete? (replace Ens by Ab). [redacted]

YES

MacLane 5

[the "1" in $1 \otimes a = a$ for $\mathbf{Ab}, \otimes$
is the integers, $\mathbb{Z} \otimes a \cong a$.]

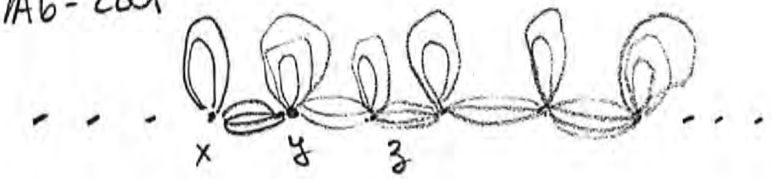
<u>ring</u> is



$s \cdot r$
$s + r$

<u>left</u> <u>R-module</u> is A ab gp
$\forall r \in R \quad L(r) : A \longrightarrow A$
$a \rightsquigarrow L(r)a = ra$

$\mathbf{Ab}$-cat



$L \big\downarrow$ functor to $\mathbf{Ab}$

$\mathbf{Ab}\ L(x) \xrightarrow{Lf} L(x) \qquad L(z)$

(How to live without objects and without elements.)

<u>Defn</u> of cat arr's only: a cat C is set of arr's and composition $k = g \circ f$ sometimes defined

$[\Gamma(g, f, k)]$. <u>Axioms</u> ① $h(gf)$ ② } assoc.
③ } id.

<u>defn</u> e is an <u>id</u> iff
i) ef defined $\Rightarrow ef = f$
ii) ge " $\Rightarrow ge = g$

▷ for any arr f there are ids e_L, e_R s.t. $e_L f$ & $f e_R$ are defined.

"assoc whenever it can"

"id acts that way whenever it can"

①

Th$^{\underline{m}}$ Obj-Arr cat is arr-only cat.

✓ EXERCISE. define nt. xf. by arr only.

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "T"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathbb{B} \end{tikzcd}$$ for any arr. f of $\mathbb{A}$ get e_L, e_R + require $\eta(e_L) \circ Tf = Sf \circ \eta(e_R)$.

(Now, wo elmts): —— Think of $\mathbb{E}$ns (cat sm sets).

$\overset{g}{\underset{h}{\rightrightarrows}} a \xrightarrow{f} b$ <u>monic</u> iff $[f(x) = f(y) \Rightarrow x = y]$ $fg = fh \Rightarrow g = h$.

Let 1 be set w one elmt $\{\bullet\} = 1$.

Defn. <u>t</u> <u>terminal</u> <u>in</u> <u>cat</u> <u>$\mathbf{c}$</u> iff for any ob a

$a \xrightarrow{\exists!} t$.

In $\mathbb{E}$ns, terminal = one pt set.

In $\mathbb{A}$b, " = " " gp.

Let 0 be set w no elmt.

Defn. i <u>initial</u> <u>in</u> <u>cat</u> <u>$\mathbf{c}$</u> iff for any ob a

$i \xrightarrow{\exists!} a$.

In $\mathbb{E}$ns, 0 is initial.

In $\mathbb{A}$b, 1 pt gp.

In $\mathbb{R}$ng, $\mathbb{Z}$.

Th$^{\underline{m}}$ i, t "essentially unique."

DUALITY PRINCIPLE. Let ETAC be elementary theory of abstract categories.

$\left|\begin{array}{l} a, b, \ldots \text{ obj's \& arrs \& comp \& id.} \\ \text{well formed formulas of ETAC} = \Theta \end{array}\right.$

define Θ^{dual} = formulas with $\left|\begin{array}{l} \text{arr's reversed} \\ \text{dom by cod} \\ \vdots \\ \text{term by init} \\ \vdots \\ \text{monic by epic} \end{array}\right.$

"Meta Thm 1" If ETAC yields Θ, then ETAC yields Θ^{dual}.

---

or, "Meta Thm 2" $C \models \Theta$ iff $C^{op} \models \Theta^{dual}$.

or, $C \models \Theta^{dual} \iff C^{op} \models \Theta$.

.... $C \longrightarrow C^{op}$ fts ...

---

defn of COPRODUCT.

a cop of a, b is a diag

$$\begin{tikzcd} & a \amalg b \arrow[d, dashed, "\exists!"] & \\ a \arrow[ur] \arrow[r] & c & b \arrow[ul] \arrow[l] \end{tikzcd}$$

s.t. for any other diag

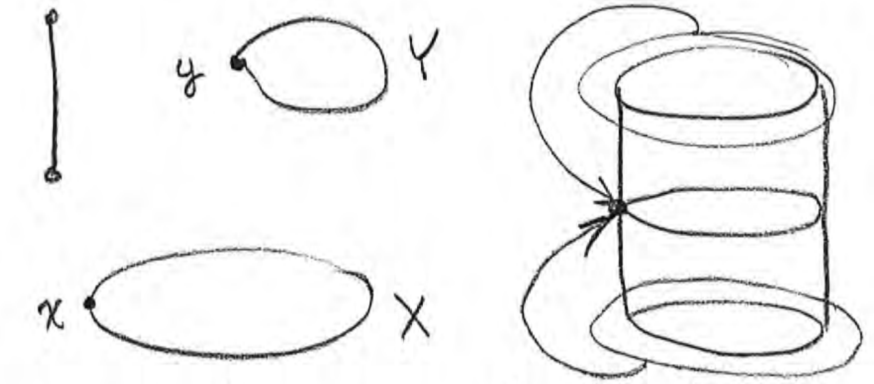
examples. in $\mathbb{E}$ns cop is disj union.
in Cat what is it? Take disj of obj's.

exercise. What is coproduct in Ab, Grp, comm. rings, rings ?

direct sum (Grp)
direct sum (Ab)
tensor product (comm. rings)
? (rings)

$\mathbb{E}\mathrm{ns}_*$ = ptd sets ; $\mathbb{T}\mathrm{op}_*$ = pt top sp's.
"base point"

$X_x \ast Y_y$ smash product $= X\times Y \big/ (x_0\times Y)\cup(X\times y_0)$



POON

what is categorical meaning of smash ?

define $X\times Y \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X\times Y$ $\alpha(x,y) = (x_0, y)$, $\beta(x,y) = (x, y_0)$; then $X\times Y \to X \ast Y$ is coequalizer

---

A cat w fin. prods has prod wi any finite list of obj's ; equiv. to . . . . . .

---

what is categorical description of :

$\mathbb{T}$op
$$\begin{tikzcd} E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{plbk}"] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$
p onto c$^{\text{lts}}$ $E_f = \{\langle x,e\rangle \mid f(x)=p(e)\} \subset X\times E$

W.S. Hatcher Foundations of Mathematics

Maclane 6

On the identity $(g'\circ g)\bullet(f'\circ f) = (g'\bullet f')\circ(g\bullet f)$:

example. Say letters are square matrices;

$$g\cdot f = \begin{pmatrix}\boxed{g} & 0\\ 0 & \boxed{f}\end{pmatrix} \; ; \; \circ = \text{mtx comp.}$$

▷
$$\begin{pmatrix}\boxed{g'} & \\ & \boxed{}\end{pmatrix} = \begin{pmatrix}\boxed{g'} & 0\\ 0 & \boxed{f'}\end{pmatrix}\begin{pmatrix}\boxed{g} & 0\\ 0 & \boxed{f}\end{pmatrix}$$

example. Top gp G. letters are loops $g : I \longrightarrow G$

$$(g\cdot g')(t) = g(t)\cdot g'(t)$$
$$(g\circ g') = \sim\sim\sim$$

▷✓ exercise. show "interchange" holds .

▷ exercise. find coproduct in fields 𝔽ld?

find initial obj in 𝔽ld.

find coproduct in Preorders

& Monoid .

Cat :

(i) describe horiz & vert comp of nt xfs

$$\begin{tikzcd} C \arrow[r, bend left=40, "R"{name=R}] \arrow[r, "S"{name=S, description}] \arrow[r, bend right=40, "T"'{name=T}] \arrow[rr, bend right=70, "T'\circ T"'] & B \arrow[r, bend left=40, "R'"{name=R2}] \arrow[r, "S'"{name=S2, description}] \arrow[r, bend right=40, "T'"'{name=T2}] & A \arrow[Rightarrow, from=R, to=S, "\alpha"] \arrow[Rightarrow, from=S, to=T, "\beta"] \arrow[Rightarrow, from=R2, to=S2, "\alpha'"] \arrow[Rightarrow, from=S2, to=T2, "\beta'"] \arrow[Rightarrow, from=R, to=T, bend right, "\beta\cdot\alpha"'] \end{tikzcd}$$

$c \xrightarrow{f} c'$ ntl of α is

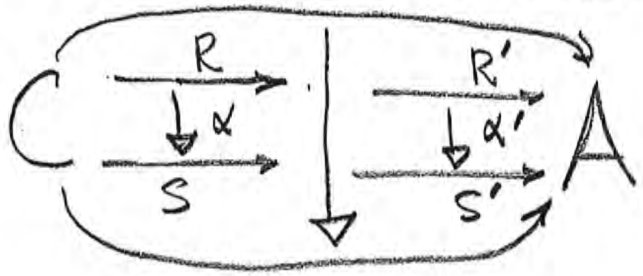
$$\begin{tikzcd} Rc \arrow[r, "Rf"] \arrow[d, "\alpha_c"] \arrow[dd, bend right=60, "(\beta\circ\alpha)_c"'] & Rc' \arrow[d, "\alpha_{c'}"] \arrow[dd, bend left=60, "(\beta\cdot\alpha)_{c'}"] \\ Sc \arrow[r, "Sf"] \arrow[d, "\beta_c"] & Sc' \arrow[d, "\beta_{c'}"] \\ Tc \arrow[r, "Tf"] & Tc' \end{tikzcd}$$

$(\beta\circ\alpha)_c \equiv \underbrace{\beta_c \circ \alpha_c}_{\text{comp in } \mathcal{B}}$.

Write B^{C} the <u>functor</u> <u>category</u> $\begin{cases} \text{objs : ftrs } C^{B} \\ \text{ntxfs} \end{cases}$

$\mathrm{Nat}(R,S) \equiv \{\alpha \mid \alpha : R \Rightarrow S\}$.

exercise. Those ex's on ftr cats in MacLane notes, $((B\times B')^{C}$ "=" $B^{C}\times B'^{C}$ done etc) $(A^{B})^{C} \cong A^{B\times C}$ "done"

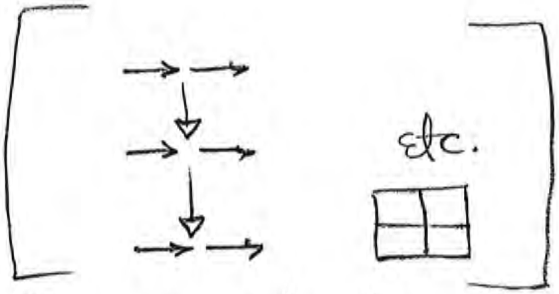


$$\begin{tikzcd} Rc \arrow[d, "\alpha_c"] & R'b \arrow[d, "\alpha'_b"] \\ Sc & S'b \end{tikzcd}$$

$\underline{\alpha' \bullet \alpha}$: is a nt xf.

Set of all nt xfs is (the set of arrows of) two different cats (Double cat).

✓ Th$^{\underline{m}}$ $(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$



$\boxed{\text{Gōdement's fifth rule}}$

rhs & lhs are nt xfs $R'R \to T'T$; for any $c \in C$ we have

$$\begin{tikzcd}
R'Rc \arrow[r, "\alpha'_{Rc}"] \arrow[d, "R'\alpha_c"'] \arrow[dr, "(\alpha' \circ \alpha)_c"] \arrow[dd, bend right=60, "R'(\beta \circ \alpha)_c ="'] & S'Rc \arrow[r] \arrow[d] \arrow[dl, phantom, "///"] \arrow[dr, phantom, "///"] & T'Rc \arrow[d, "T\alpha_c"] \\
R'Sc \arrow[r] \arrow[d, "R'\beta_c"'] \arrow[dr, phantom, "///"] & S'Sc \arrow[r] \arrow[d] \arrow[dr, "(\beta' \circ \beta)_c"] \arrow[dr, phantom, "///"'] & T'Sc \arrow[d, "T'\beta_c"] \\
R'Tc \arrow[r, "\alpha'_{Tc}"'] \arrow[rr, bend right=40, "\overset{\Vert}{(\beta' \cdot \alpha')_{Tc}}"'] & S'Tc \arrow[r, "\beta'_{Tc}"'] & T'Tc
\end{tikzcd}$$

[some question about •'s & o's]

So interchange law holds for category of categories.

Defn. A <u>double</u> <u>category</u> (arrows only) is a set of "double arrows" which is an arr only cat for two different compositions $\underset{\text{"vert"}}{\circ}$ & $\underset{\text{"hor"}}{\bullet}$,

✓▷ such that $(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$ holds whenever $\begin{array}{|c|c|}\hline \alpha & \alpha' \\ \hline \beta & \beta' \\ \hline \end{array}$ are defined. Then both sides are defined and equal.

* Write a ftr for its identity ; $R ; R \xrightarrow[\text{same}]{\Longrightarrow} R$.

Write a cat for id of id. $\sim\!\sim\!\sim\!\sim$.

▷ Thm. Cat is a double cat in wh every horiz id is also vertical identities.

Defn. A <u>2-dimensional</u> <u>category</u> is cat satisfying condition of Thm.

---

* Also, $\left\{\begin{matrix}\text{horizontal}\\ \text{vertical}\end{matrix}\right\}$ ids are closed under

$\left\{\begin{matrix}\text{vertical}\\ \text{horizontal}\end{matrix}\right\}$ composition.

example. Ens : double arrow is a commutative square

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "///"] & {} \arrow[d, "g'"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd}.$$

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[d] \arrow[r, phantom, "\circ"] & {} \arrow[r] \arrow[d] & {} \arrow[d] \arrow[r, phantom, "="] & {} \arrow[rr] \arrow[d] & & {} \arrow[d] \\
{} \arrow[r] & {} & {} \arrow[r] & {} & {} \arrow[rr] & & {}
\end{tikzcd}$$ etc vert.

$$g\,\overset{1}{\underset{1}{\square}}\,g \xrightarrow[f']{g\,\overset{f}{\square}\,g'} g'\,\overset{1}{\underset{1}{\square}}\,g'$$ etc. vert.

~~example of 2 cat. arrows are lozenges.~~ ? ? ?

On Cat :—

$$\begin{tikzcd} R \arrow[d, "\alpha"] \\ S \end{tikzcd} \quad \begin{tikzcd} R' \arrow[d, "\alpha'"] \\ S' \end{tikzcd} \quad \left(\begin{tikzcd} \text{generally} \\ R'R \arrow[d, "\alpha'\cdot\alpha"'] \\ S'S \end{tikzcd}\right).$$ say $\alpha' = R' \begin{tikzcd} R' \arrow[d] \\ R' \end{tikzcd}$ identity ntxf.

this yields composition of ftrs w ntxfs.

$$(\alpha'\cdot\alpha)_c = \alpha'_{Sc}\circ R'(\alpha_c)$$

$$(\alpha'\cdot\alpha)_c = (\alpha'S)_c\circ(R'\alpha)_c$$

$$\alpha'\cdot\alpha = (\alpha'S)\text{"}\circ\text{"}(R'\alpha)$$

also $$(\alpha'\bullet\alpha)_c = S'(\alpha_c)\circ\alpha'_{Rc}$$
$$(\alpha'\bullet\alpha)_c = (S'\alpha)_c\circ(\alpha'R)_c$$
$$\alpha'\bullet\alpha = (S'\alpha)\,\text{"}\circ\text{"}(\alpha'R) \quad .$$

Thus
$$(\alpha'S)\cdot(R'\alpha) = \alpha'\bullet\alpha = (S'\alpha)\cdot(\alpha'R) \; .$$

~~This should come from interchange law:~~

~~$(\beta'\circ\beta)\cdot(\alpha'\circ\alpha) \overset{\text{Int.}}{=} (\beta'\cdot\alpha')\circ(\beta\cdot\alpha)$~~

~~But it doesn't.~~ .

$C^{\mathbf{2}}$ | objs : arr's of C
arr's : comm. squ's in C

Ens for Ehresmann is cat of fu's .

$$\begin{tikzcd} C \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d] & c' \\ Rc \arrow[r, "Rf"] \arrow[d, "\alpha_c"'] & Rc' \arrow[d, "\alpha_{c'}"] \\ Sc \arrow[r, "Sf"'] & Sc' \end{tikzcd} \text{ etc.}$$

α is really a ftr $C \longrightarrow B^{\mathbf{2}}$.

$B^{2\times C}$ $\qquad$ $(B^{2})^{C}$

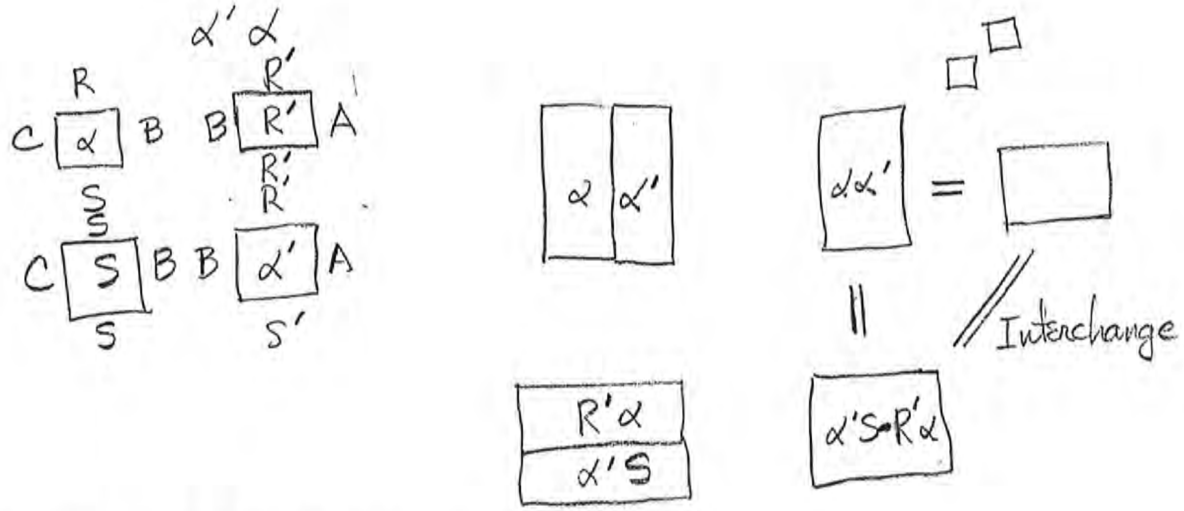
⑥

MacLane 7

$$\begin{tikzcd}
C \arrow[r, bend left=50, "R"{name=U}] \arrow[r, bend right=50, "S"'{name=D}] \arrow[rr, bend left=80, ""{name=T}] \arrow[rr, bend right=80, ""'{name=W}] & B \arrow[r, "R'"] & A
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\arrow[Rightarrow, from=T, to=W, "R'\alpha"]
\end{tikzcd}$$

"mult. nt. xf. w ftr front + back"

$$\begin{tikzcd}
C \arrow[r, bend left=50, "R"{name=U}] \arrow[r, bend right=50, "S"'{name=D}] & B \arrow[r, bend left=50, "R'"{name=U2}] \arrow[r, bend right=50, "S'"'{name=D2}] & A
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\arrow[Rightarrow, from=U2, to=D2, "\alpha'"]
\end{tikzcd}$$



(UNIVERSAL:) GIVEN

$$\begin{tikzcd}
D \arrow[d, "S"] \\
c \in C
\end{tikzcd}$$

a <u>universal</u> <u>arrow</u> <u>from</u> <u>c</u> <u>to</u> <u>S</u> is pair $(r \in D, r \xrightarrow{u} Sd)$ such that for any $c \xrightarrow{f} Sd$, $f = Sf' \circ u$ for unique f'.

$$\begin{tikzcd}
c \arrow[r, "u"] \arrow[dr, "f"'] & Sr \arrow[d, dashed, "Sf'"] & \exists ! & r \arrow[d, "f"] \\
 & Sd & & d
\end{tikzcd}$$

①

example. K_X = vectorspace basis X , $\sum a_i \langle x_i \rangle$

$$\begin{tikzcd}
K_X \in \mathbb{V}ect_K \arrow[d, "U"] \\
X \in \mathbb{E}ns
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & UK_X \arrow[d] & K_X \arrow[d] \\
& UV & V
\end{tikzcd}$$

Given $D \xrightarrow{S} C \ni c$ define cat.

obj's pairs $\langle e, g \rangle$ $c \xrightarrow{g} Se$

cat of <u>objects</u> <u>S</u> <u>under</u> <u>c</u> $c/S\_$

[comma cat]

A universal arrow is an initial obj in S under c.

(coproduct in C):

$$\begin{tikzcd}
C \arrow[d, "\Delta"] & c \arrow[r, "f"] \arrow[d, mapsto] & c' \arrow[d, mapsto] \\
\langle a,b \rangle \in C \times C & \langle c,c \rangle \arrow[r, "\langle f,f \rangle"] & \langle c',c' \rangle
\end{tikzcd}$$

A universal arrow from $\langle a,b \rangle$ to Δ is $\langle a,b \rangle \xrightarrow{u} \langle c,c \rangle$
s.t. $\forall$ $\langle a,b \rangle \xrightarrow{v} \langle x,x \rangle$ $\exists!$ $c \xrightarrow{f} x$ s.t. $v = \Delta f \circ u$

["cop generated by images of factors"]

A *product* in C is

$$\begin{tikzcd}
\langle a,b\rangle & \Delta r \arrow[l, "u"'] & r \\
 & \Delta c \arrow[ul, "t"] \arrow[u, dashed, "\Delta t'"'] & c \arrow[u, "t'"']
\end{tikzcd}$$

"universal arrow from S to c" [co-universal]

---

example. ⇊ denotes cat $0 \rightrightarrows 1$.

$$\begin{tikzcd}
 & C \arrow[d] \arrow[r, phantom, "\Delta"] & c \arrow[d, mapsto] \\
a \overset{f}{\underset{g}{\rightrightarrows}} b \in & C^{\downdownarrows} & c \overset{1}{\underset{1}{\rightrightarrows}} c
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & b \arrow[d] \\
a' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & b'
\end{tikzcd}$$

Universal arrow from Δ to $a \overset{f}{\underset{g}{\rightrightarrows}} b$:

$$\begin{tikzcd}
\left(a \overset{f}{\underset{g}{\rightrightarrows}} b\right) & \Delta r \arrow[l, "u"'] & = r \overset{1}{\underset{1}{\rightrightarrows}} r \\
 & \Delta c \arrow[ul, "w"] \arrow[u, dashed]
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b & \Delta c & c \overset{1}{\underset{1}{\rightrightarrows}} c \\
c \arrow[u, "t"] \arrow[r, equal] & c \arrow[u, "s"']
\end{tikzcd}$$

An arrow w is $t : c \to a$ s.t. $ft = gt$.

[redacted]

$X \in \mathrm{Ens}$; $E \subset X \times X$ an equ rel. $X \times X \overset{p}{\underset{q}{\rightrightarrows}} X$ . . .

discuss co-equalizer in sets w.r.t. equ rel.

---

pullback [cartesian square; produit fibré]
pushout co-cartesian " somme fibré

constructed from products & equalizers.

"Product" in preorder regarded as cat.

$a \leq b \qquad a \longrightarrow b$

glb = product
lub = coproduct. preorder wo glbs has no prods.

Group as an arrow cat, 1 obj only.
"$a \pi a$" nothing exists.

---

plbk as an arr.

$$\begin{tikzcd} C \arrow[d, "\Delta"] & X \arrow[d, phantom, ""] \\ C^{\cdot \to \cdot \leftarrow \cdot} & X = X = X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] & Z & Y \arrow[l] \\ R \arrow[u] \arrow[r, equal] & R \arrow[u] \arrow[r, equal] & R \arrow[u] \end{tikzcd}$$

EXERCISE. in a group plbks exist but not prod.
" " "topsp." " " " " "

X top sp. obj's : op sets of X

arr's : $V \subseteq U$

inclusion

$$\begin{tikzcd} \boxed{?} \arrow[r, hook] \arrow[d, hook] & W \arrow[d, hook] \\ V \arrow[r, hook] & U \end{tikzcd}$$

$V \cap W$.

<u>Kernel pair</u> [<u>congruence</u> <u>relation</u>]

$$\begin{tikzcd} ? \arrow[r, shift left, "\overset{\text{to define}}{u}"] \arrow[r, shift right, "v"'] & a \arrow[r, "\overbrace{f}^{\text{given}}"] & b \\ c \arrow[u, dashed, "\exists!"] \arrow[ur, shift left] \arrow[ur, shift right] & & \end{tikzcd}$$

s.t. $fu = fv$

re-draw as

$$\begin{tikzcd} ? \arrow[r, "u"] \arrow[d, "v"'] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

" pullback of f over itself " .

$$\begin{tikzcd} A \arrow[d, "\overline{U}"] \\ \mathbb{E}\mathrm{ns} \end{tikzcd}$$

define <u>universal element</u> <u>of $\overline{U}$</u> is $\langle r \in A , x \in \overline{U}r \rangle$

s.t. .... $\forall\ s \in A, y \in \overline{U}s$

$\exists!\ r \xrightarrow{u} s$ s.t. $(\overline{U}u)x = y$.

example. V, W fixed $S \in \mathbb{V}\text{ect}$

$$S \overset{U}{\longmapsto} \text{Bilin}(V, W \text{ to } S)$$

univ. elmt. is $\langle r, x \rangle$ $r = $ "$V \otimes W$" and

$x \in \text{Bilin}(V, W \text{ to } S)$

$$\begin{tikzcd}
V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed] \\
 & S
\end{tikzcd}$$

What is difference between univ elmts and univ arrs ?

<u>claim</u> a univ. elmt may be regarded as a univ. arr

$$\begin{tikzcd}
A \arrow[d, "U"] \\
\mathbb{E}\text{ns}
\end{tikzcd}
\qquad \langle r, \underbrace{x \in Ur}_{1 \xrightarrow{x} Ur} \rangle$$

a ue may be regarded as the arrow of a ua.

~~[illegible]~~

conversely,
univ. arrow is univ. elmt for $\hom(c, S-)$

---

exercise. Interpret as universal:

- integral group ring of a group
- tensor alg
- exterior alg
- polynomial ring one variable
- Stone-Cêch compactification
- rep of every Boolalg as alg of sets
- field of quotients of integral domain
- factor gp G/N

MacLane 8

Comma category In Cat have (fin) prod's, have equalizers, therefore pullbacks. Given

$$\begin{tikzcd} E \arrow[dr, "T"'] & & D \arrow[dl, "S"] \\ & C & \end{tikzcd}$$

data

~~$Te \xrightarrow{f} Sd$~~

defn. (T, S) (or T/S) is

$$\begin{tikzcd} \text{obj} & \langle e, \arrow[d] & Te \arrow[r, "f"] \arrow[d, "\text{s.t.} \ /\!/\!/"] & Sd, \arrow[d] & d \rangle \arrow[d] \\ \text{arr} & \langle e', & Te' \arrow[r, "f'"] & Sd', & d' \rangle \end{tikzcd}$$

example. If $E = C = D$, $T = S = 1$ then $1/1 = C^{\mathbf{2}}$.

example. for $c \in C$

$$\begin{tikzcd} \mathbb{1} \arrow[dr, "c"'] & & D \arrow[dl, "S"] \\ & C & \end{tikzcd}$$

c/S | obj $c \xrightarrow{f} Sd$

example.

for $c, c' \in C$

$$\begin{tikzcd} \mathbb{1} \arrow[dr, "c"'] & & \mathbb{1} \arrow[dl, "c'"] \\ & C & \end{tikzcd}$$

$c/c' = C(c, c')$.

obj's of (T, S) are $\langle \underset{e}{\overset{\text{ob}\,E}{\ni}}, \underset{f}{\overset{\text{ar}\,C}{\ni}}, \underset{d}{\overset{\text{ob}\,D}{\ni}} \rangle$ | dom f = e, cod f = d

$$\begin{tikzcd} & & T/S \arrow[dll] \arrow[d] \arrow[drr] & & \\ E \arrow[dr] & /\!/\!/ & C^{\mathbf{2}} \arrow[dl] \arrow[dr] & /\!/\!/ & D \arrow[dl] \\ & C & & C & \end{tikzcd}$$

} universal ; "pullback".

Can "pull it back in pieces."

$$\begin{tikzcd} D \arrow[d, "S"] \\ c \in C \end{tikzcd}$$

ua c to S is $\langle r \in D, c \xrightarrow{u} Sr \rangle$ s.t.

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[dr] & Sr \arrow[d, dashed, "\exists! f'"] \\ & Sd \end{tikzcd}$$

.

ua is ue in $\hom(c, S-)$; also in. obj. in c/S .

A ua gives a bijection

$$\begin{tikzcd} \hom(r,d) \arrow[r] & \hom(c,Sd) \\ f : r \to d \arrow[r, squiggly] & Sf' \circ u \end{tikzcd}$$

natl in d

Thm there is nt. isomp. $\hom_D(r,-) \cong \hom_C(c, S-)$.

Define: $T : D \longrightarrow \mathbb{E}ns$ is representable iff $\exists\, r, \varphi$ s.t. $\hom(r,-) \underset{\varphi}{\cong} T$ nt isomp.

Given a nt isomp $\Phi_x : \hom(r,x) = \hom(c,Sx)$

take $x = r$

$$\begin{tikzcd} \hom(r,x) \arrow[r, "="] & \hom(c,Sx) \\ (1_r \arrow[u, "\exists! f'"] \arrow[r, squiggly] & u : c \to Sr) \arrow[u, "f"'] \\ \Phi_r : \hom(r,r) \arrow[r] & \hom(c,Sr) \end{tikzcd}$$

Thus ua is same as nt. isomp. $\boxed{u = \Phi_r(1_r)}$

Given Φ any nt xf $\hom_D(r,-) \longrightarrow T$ $\quad \begin{array}{c} D \\ \downarrow \\ \mathbb{E}ns \end{array}$

we say everything determined by image of 1_r

i.e.

Yoneda lemma. Given $r \in D$ & $T : D \longrightarrow \mathbb{E}ns$ then

$$Nt(\hom_D(r,-), T) \underset{bij}{\overset{\theta}{\cong}} T(r)$$

[This Thm is for when hom sets of D land in $\mathbb{E}ns$.]

Furthermore, Thm is "natural" in θ.

$$\begin{tikzcd}
\langle T, r \rangle \arrow[r, mapsto] & T(r) \\
\mathbb{E}ns^{D} \times D \arrow[r, "ev"] \arrow[r, bend right=50, "{Nt(Hom(r,-),T)}"'] \arrow[r, phantom, shift right=3, "\Uparrow \theta"] & \mathbb{E}ns
\end{tikzcd}$$

Dual of Yoneda lemma.

✓ Given C, $C^{op} \xrightarrow{T} \mathbb{E}ns$

there is bij $Nt(hom_C(-, s), T) \cong T(s)$.

---

defn <u>Yoneda map</u> $\quad u : C \times C^{op} \to C \times \mathbb{E}ns^{C} \to \mathbb{E}ns$

$$\begin{tikzcd}
C^{op} \arrow[r, "Y"] & \mathbb{E}ns^{C} \\
x \arrow[r, mapsto] & hom_C(x, -)
\end{tikzcd}$$

✓ Thm Y is full and faithful and injective on objects.

[This imbeds any cat into a thing w "all props" of $\mathbb{E}ns$.]

✓ EXERCISE. In $\mathbb{C}at$ there exist coequalizers.

data

$$\begin{tikzcd} A \arrow[d, shift left, "G"] \\ X \arrow[u, shift left, "F"] \end{tikzcd}$$

defn. F is left adjoint of G iff $\exists\ I \xrightarrow{\eta} GF$ ntrf. so that η_X ua from X to G —

<u>Invere</u> <u>limits</u> $I \xrightarrow{F} C$ ftr $F \in C^I$.

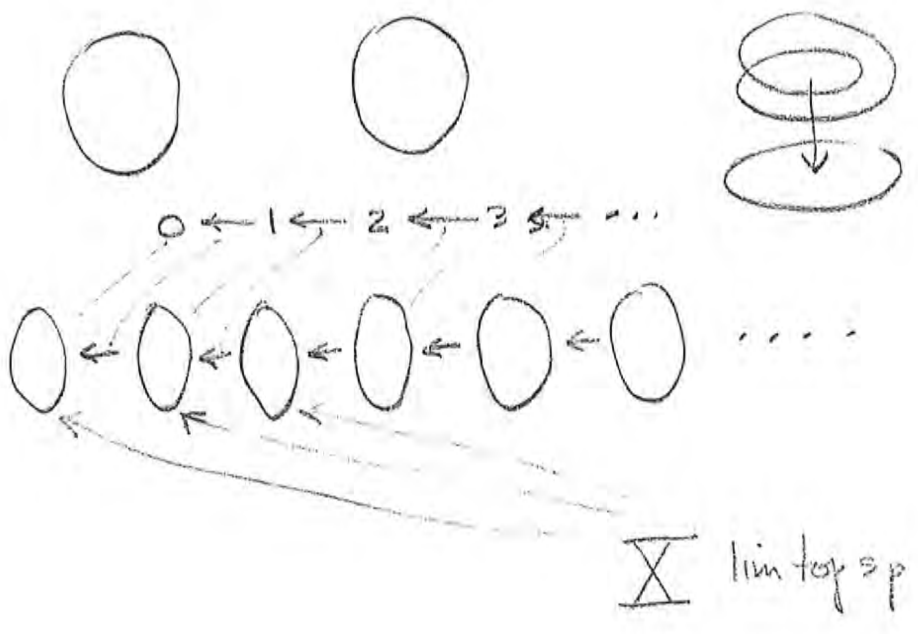
$$\begin{tikzcd} C \arrow[d, "\Delta"] \\ F \in C^I \end{tikzcd}$$

defn. a limit for F is a ua τ from Δ to F,

$c \in C \quad \tau : \Delta c \Longrightarrow F$ & univ.

---

example. <u>2-adic solenoid</u>



algebraic example.

$$\begin{tikzcd}
1 & 2 \arrow[l] & 3 \arrow[l] & & n \arrow[l] \\
\mathbb{Z}_{p} & \mathbb{Z}_{p^2} & \mathbb{Z}_{p^3} & \cdots & \mathbb{Z}_{p^n} & \cdots \\
 & & & & R \arrow[llllu] \arrow[lllu] \arrow[llu]
\end{tikzcd}$$

[res mod p on arrs]

p-adic integers.

Limits & Colimits

MacLane 9

$$I \xrightarrow{F} C$$

<u>Limit</u> is universal cone from Δ to F

$$\begin{tikzcd} C \arrow[d, "\Delta"] & c \arrow[d, squiggly] \\ C^I & \text{const. ftr.} \end{tikzcd}$$

$$\begin{tikzcd} I & & & & & \\ i \arrow[dr, "s"] & & & & c \arrow[llld] \arrow[ddll] \arrow[d, dashed, "\exists !"] & \\ & j & FI \quad F_i \arrow[dr, "Fs"'] & & \lim F \arrow[ll, "\tau_i"'] \arrow[dl, "\tau_j"] & F \xLeftarrow{\tau} \Delta \\ & & & F_j & & \end{tikzcd}$$

product is when I is a set.

<u>Colimit</u> (dualize arrows in C)

example of coequalizer
consider full sbcat of Cat where every endomp is an id. $\mathbb{1}$, $\mathbb{2}$ are like this.

$$\mathbb{1} \rightrightarrows \mathbb{2} \xrightarrow{\text{coequ.}} \mathbb{1}$$

P-adic numbers

$$\mathbb{Z}_p \leftarrow \mathbb{Z}_{p^2} \leftarrow \cdots$$

inverse limit in
cat. of rings (comm, id)

$$\overbrace{a_0 + a_1 + a_2 p^2 + \cdots \quad \text{"convergence"}}^{p\text{-adic integers}}$$

Ex. F field t symbol.

"Formal power series coeffs in F & symbol t"

$$a = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

$$b = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n + \cdots$$

$\left.\begin{matrix} a+b \\ a \cdot b \end{matrix}\right\}$ a ring. an integral domain.

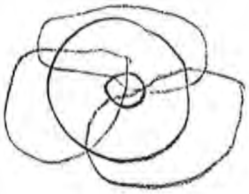
This ring is an inverse limit:

Let $F[t]$ = polyn. ring in t.

$$\begin{tikzcd}
 & & F\{\{t\}\} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots & F[t]/t^{n-1} & F[t]/t^{n} & \cdots & \cdots
\end{tikzcd}$$

"From this get to algebraic numbers, alg geom."

Ex. Cech cohomology. space X



Ex. getting alg objects by colimits finitely generated objs

(Th^m from last time):

$$\begin{tikzcd}
P \arrow[rr] \arrow[dd] & & Y \arrow[dd, "g"] \\
& X \times Y \arrow[ld] \arrow[ru] \arrow[rd, shift left] \arrow[rd, shift right] & \\
X \arrow[rr, "f"'] & & A
\end{tikzcd}$$

pullback is gotten by prod & equ.

$$\begin{tikzcd}
A_1 & A_2 \arrow[l, "f_1"'] & A_3 \arrow[l, "f_2"'] & \cdots \arrow[l, "f_3"'] & A_n \arrow[l] & A_{n+1} \arrow[l] & \arrow[l] \\
(\overset{\psi}{a_1} & \overset{\psi}{a_2} & \overset{\psi}{a_3} & \cdots\cdots\cdots & \overset{\psi}{a_n}) & & \\
& & & & \prod_{n=1}^{\infty} A_n \arrow[llu] \arrow[u] & &
\end{tikzcd}$$

less conceptually

to const. limit of $\rightsquigarrow$ ftr.

projections

(more conceptually)

take equalizer for every pair of projections.

Th^m if C has (small) products and {equs of pairs w common left inverse ~~equalizers of pairs~~} then it has all (small) limits.

pf. Given $I \xrightarrow{F} C$ make

$$\begin{tikzcd}
& & F_k \arrow[rddd, bend left=60, equal] \arrow[r, phantom, "///s"] & \\
& \prod_{i\in I} F_i \arrow[ru] \arrow[rr, dashed, shift left, "\exists! u"] \arrow[rr, dashed, shift right, "\exists! v"'] \arrow[rd, "p_j"'] & & \prod_{s:j\to k} F_k \arrow[dd, "p_s"] \\
E \arrow[rruu, squiggly] \arrow[ru, dashed, "equ(u{,}v)"'] \arrow[rrd, squiggly] & & F_j \arrow[d, "Fs"'] \arrow[r, phantom, "///s"] & \\
D \arrow[u, dashed] \arrow[rr] & & F_k \arrow[r, equal] & F_k = cod\, s
\end{tikzcd}$$

③

Th^m If C has finite products + equ's of pairs then it has finite limits.

Also dual Th^m cops, coequs of prs ⟹ colims.

Defn. C is $\left\{\begin{matrix}\text{small} \\ \text{finitely}\end{matrix}\right\}$ complete iff

C has all $\left\{\begin{matrix}\text{sm} \\ \text{fin}\end{matrix}\right\}$ lims.

example. | Ens, Cat, Gps } all large cats.

Is there sm cat wh is sm complete?

A preorder sm-comp means it has all small glbs.

→ can be shown it is a complete preorder.

See Freyd

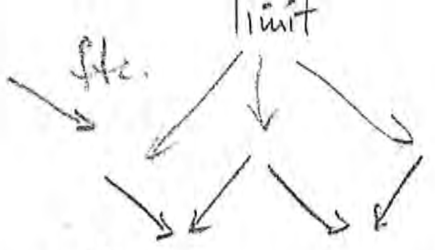
Exercise. If C is sm comp so is C/a ; $C^{I\,\text{sm}}$.

exercise. $(a \times b) \times c = a \times b \times c$

generalize

make precise

"iterated limits can be done in pieces"



$\mathbf{Cat}^{\mathcal{D}}$

$$\begin{tikzcd}
Fx & A \arrow[r, phantom, "\ni"] \arrow[d, shift left, "G"] & a \arrow[d, squiggly] \\
x \arrow[u, squiggly] \arrow[r, phantom, "\in"] & X \arrow[u, shift left, "F"] & Ga
\end{tikzcd}$$

defn. an <u>adjunction from F to G</u> is bijection

$$\hom_A(Fx, a) \xrightarrow{\Phi} \hom_X(x, Ga)$$

natural in x, a

Th<u>m</u>. Let
$$\begin{tikzcd}
D \arrow[d, "S"] \\
c \in C
\end{tikzcd}$$
sit. of ua.

Given for ea $c \in C$ a ua $u : c \xrightarrow{u_c} Sr_c$ from c to S.

$$\begin{tikzcd}
c \arrow[r, "u_c"] \arrow[dr, "f"'] & Sr_c \arrow[d, dashed, "\exists! f'"] \\
& Sd
\end{tikzcd}$$

Then $\exists !$ $T : C \longrightarrow D$ with $Tc = r_c$ so that u is natural.

pf.

$$\begin{tikzcd}
c \arrow[r, "u_c"] \arrow[d, "h"'] \arrow[dr] \arrow[dd, bend right=60] & Tr_c \arrow[d, "STh"] \\
c' \arrow[r, "u_{c'}"'] \arrow[d, "h'"'] & Tr_{c'} \arrow[d] \\
c'' \arrow[r] & Tr_{c''}
\end{tikzcd}$$

$$\begin{tikzcd}
r_c = \text{"}Tc\text{"} \arrow[d, "\text{"}Th\text{"}"] \\
r_{c'} = \text{"}Tc'\text{"}
\end{tikzcd}$$

defn. When this is done T is <u>left</u>-<u>adjoint</u> of S.

STATED SUCCINTLY

Thm Given
$$\begin{tikzcd} D \arrow[d, "S"] \\ C \end{tikzcd}$$
+ for ea $c \in C$ a ua from c to S. $\left(c \xrightarrow{u_c} STc\right)$.

Then $\exists!$ way to make "T" ā ftr so that the bijection $\sim\sim\sim$ is an adjunction.

SWAN Tu W Th $7^{\underline{30}}$

MacLane 10

Adjoint Functors

$$\begin{tikzcd} A \ni a \arrow[d, shift left, "G"] \\ x \in X \arrow[u, shift left, "F"] \end{tikzcd}$$

defn an adjunction F to G is

$$\hom_A(Fx, a) \cong \hom_X(x, Ga)$$

natural in x and a. [both sides bifts]

$$\begin{tikzcd} Fx' \arrow[d, "Fs"'] & & & x' \arrow[d, "s"'] & \\ Fx \arrow[r, "f"] & a \arrow[d, "h"] \arrow[r, squiggly] & {} & x \arrow[r, "f^{\#}"] & Ga \\ & a' & & & \end{tikzcd}$$

F left G right

$$(h \circ f)^{\#} = Gh \circ f^{\#}$$
$$(f \circ Fs)^{\#} = f^{\#} \circ s$$

example. in $\mathbb{E}$ns $\hom(X \times Y, Z) \cong \hom(X, \hom(Y, Z))$

▷ COUNIT UNIT $f : X \times Y \longrightarrow Z$ $(f^{\#}(x))(y) = f(x,y)$.

example. $\mathbb{V}\text{ect}(FX, V) \cong \mathbb{E}\text{ns}(X, UV)$

▷ UNIT = insertion of generators

example. modules ⊗ replaces ×.

▷ example. in any cat w products, diag + prod

$$\begin{tikzcd} C \times C \arrow[d, shift left, "\times"] \\ C \arrow[u, shift left, "\Delta"] \end{tikzcd}$$

[COUNIT] ⇝ gives projections

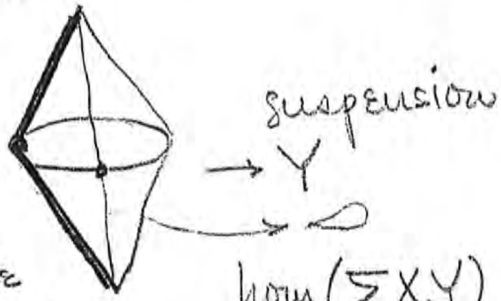
example.

$$\begin{tikzcd} \mathbb{T}\text{op}_* \arrow[d, shift left, "\Omega"] \\ \mathbb{T}\text{op}_* \arrow[u, shift left, "\Sigma"] \end{tikzcd}$$

▷ UNIT + ? COUNIT

suspension

space of loops w cpt-op top.



$\hom(\Sigma X, Y) \cong \hom(X, \Omega Y$

▷ $\left[\Sigma X = S^1 \# X\right]$

Properties of adjunction.

$$\begin{tikzcd}
1_{FX} \in \hom(Fx, Fx) \arrow[d, "- \circ f"] \arrow[r, phantom, "\cong"] & \hom(x, GFx) \arrow[d, "- \circ Gf"] \\
f \in \hom(Fx, a) \arrow[r, phantom, "\cong"] & \hom(x, Ga)
\end{tikzcd}$$

$X \xrightarrow{\eta_x = (1_{Fx})^\#} GFx$

$$\boxed{f^\# = Gf \circ \eta_x}$$

▷ η_X is ua from x to G.

▷✓ claim $\eta : I_X \Longrightarrow GF$ is nt. xf.

▷✓ claim $\varepsilon_a = (1_{Ga})^\flat$ $\quad \varepsilon : FG \longrightarrow I_A$ is nt. xf.

UNIT of ADJ.
(front adjunction)

COUINIT of ADJ.
(back adjunction)

$\eta : I_X \Longrightarrow GF \qquad \varepsilon : FG \Longrightarrow I_A$

$$\underbrace{F \xRightarrow{F\eta} FGF \xRightarrow{\varepsilon F} F} \qquad \boxed{\varepsilon F \cdot F\eta = 1}$$

$$\varepsilon_{Fx} \circ F\eta_x = (\eta_x)^\flat = 1.$$

$$G \xRightarrow{\eta G} GFG \xRightarrow{G\varepsilon} G \qquad \boxed{G\varepsilon \cdot \eta G = 1}$$

So an adjunction gives ε, η + relations $\boxed{}$.

Corollary 1. if F, F' both left adjoint to G, then $F \underset{\exists\theta}{\cong} F'$.

[$G : A \longrightarrow X$ has left adj $\Longleftrightarrow$ for all $x \in X$, $\hom(x, G\_)$ is representable]

Th

$\text{Th}^{\underline{m}}$ An adjunction is determined by [illegible]

(i) F, G + natural bijection

(ii) F, G functors $\eta : I \longrightarrow GF$ ea η_x universal

(iii) [cf. $\text{Th}^{\underline{m}}$ end of last time]

[G ftr F_0 on obj's only ea x. $\eta_x : x \longrightarrow GF_0 x$ univ. then F extends]

▷ (ii)′ $F : X \longrightarrow A$ $\sim\sim\sim$ } dual.
(iii)″

▷ (i) improved to: F_0, G + bij ntl in a then F extends. USE Yoneda

[$\hom(r, a) \to \hom(r, a)$ ntl in a

(iv) ε, η s.t. $\varepsilon F F\eta = 1$ $G\varepsilon \eta G = 1$

pf. (iv) $\Rightarrow$ (i) define $f \rightsquigarrow f^\# = Gf \circ \eta_X$

define $g^b \leftsquigarrow g$

So its just ftrs + nt xfs.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend right] & X \arrow[r] \arrow[rr, bend left] & A \arrow[r] & X
\end{tikzcd}$$

exercise. defn of adjunction in terms of isomp of comma cats.

PRIZE FOR ADJUNCTION WEEK: find a new one.

---

example. preorders

$$\begin{tikzcd}
P \arrow[r, bend left, "L"] & Q \arrow[l, bend left, "R"]
\end{tikzcd}$$

$\xrightarrow{}$ ftr

$p \le p'$ $\Rightarrow$ $f(p) \le f(p')$

$p \to p'$

Galois correspondence

$$\hom_Q(Lp, q) \cong \hom_P(p, Rq)$$

$$Lp \le q \iff p \le Rq$$

naturality collapses

example of example. U set. $\mathcal{P}(U) = \{ S \subset U)$ V also.

$S \le S'$

$$U \xrightarrow{f} V \qquad \begin{tikzcd} P(U) \arrow[r, shift left, "f_*"] & P(V) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

$$\boxed{f_* S \subset T \text{ iff } S \subset f^* T}$$

$f^!$ $f_!$ $\left\{\begin{matrix}\text{lower}\\ +\\ \text{upper}\end{matrix}\right\}$shriek

---

example. U a set. G gp acting on U.

$$\begin{tikzcd}
\mathcal{P}(U) \arrow[r, bend right, "L"'] & \mathcal{P}(G)^{op} \arrow[l, bend right, "R"']
\end{tikzcd}$$

$S \subset U$ $\quad LS = \{\sigma \in G \mid \text{all } s \in S \ \sigma s = s\}$

"fixing gp of S"

$S \subseteq S'$ $\quad LS \leq LS'$

$RH = \{x \in U \mid \text{all } \sigma \in H \ \sigma x = x\}$

$LS \leq H$ iff $S \subseteq RH$

$S \subset RH$

find η, ε.

prove adjointness

Maclane 11

$$\begin{tikzcd} A \arrow[d, shift left, "G"] \arrow[loop right, "I \xLeftarrow{\varepsilon} FG"] \\ X \arrow[u, shift left, "F"] \arrow[loop right, "I \xRightarrow{\eta} GF"] \end{tikzcd}$$

$$\begin{tikzcd} & FGF \arrow[dr, Rightarrow] & \\ F \arrow[ur, Rightarrow] \arrow[rr, equal] & & F \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[rr, equal] \arrow[dr, Rightarrow] & & G \\ & GFG \arrow[ur, Rightarrow] & \end{tikzcd}$$

---

example. <u>preorders</u>

$$\begin{tikzcd} Q \arrow[d, shift left, "R"] \\ P \arrow[u, shift left, "L"] \end{tikzcd}$$

$I \leq RL$

$p \leq RLp$

$LRq \leq q$

adj. sit. in preorder is called <u>Galois connection</u>

(Everything in logic is adj sit.)

U a set. $\mathcal{P}(U)$ Bool alg of subsets of U. $\emptyset$ is initial elmt. U is terminal elmt. has products: $S \cap T \begin{matrix} \leq S \\ \leq T \end{matrix}$.

$-\cap T : \mathcal{P}U \to \mathcal{P}U$ is a functor.

Does it have a lft adjt ?

$\hom(Fx, a) \cong \hom(x\ Ga)$

$Fx \leq a \Longleftrightarrow x \leq Ga$

$\boxed{} \leq A \iff X \leq A \cap T$

maybe ⓡⓣ adjt :

$X \cap T \leq A \iff X \leq T' \cup A$

So $\boxed{-\cap T \text{ is left adjoint of } T' \cup -}$

<u>Alg of Lgc</u> $\mathbb{T}$ elementary theory.

p, q, r sentences. $p \le q$ iff p entails q in theory $\mathbb{T}$.

cat. | S = set of sentences is <u>preorder</u>. false sentence is <u>in. obj.</u>

<u>product</u>: $c \le p, c \le q$ iff $c \le p \,\&\, q$ i.e. CONJUNCTION.

<u>coproduct</u>: disjunction

ftr $-\&q$: $\quad p \,\&\, q \le r$ iff $p \le \underbrace{(\text{not } q \text{ or } r)}_{q \text{ implies } r}$

Must extend to quantifiers :— $\quad \exists x \quad \forall x$

$$\begin{tikzcd}
\langle x,y\rangle \arrow[d, squiggly] & X \times Y \arrow[d, "\pi"] & \mathcal{P}(X\times Y) \arrow[d, bend right, "L"'] \arrow[d, bend left, "R"] \\
x & X & \mathcal{P}X \arrow[u, "\pi^*"']
\end{tikzcd}$$

$\pi^* S = \{\langle x,y\rangle \mid x \in S\}$

look for lft, rt adjts to π^* :

$$\underbrace{L(T) \le S} \quad \text{iff } T \le \pi^* S$$

$$\underbrace{\{x \mid \exists y \ \langle x,y\rangle \in T\}}_{\substack{LT \\ \text{existential} \\ \text{quantifier}}} = \pi(T) \subset S$$

$$\pi^* S \le T \quad \text{iff} \quad \underbrace{S \le RT}$$

$$S \le \underbrace{\{x \mid \forall y \ \langle x\, y\rangle \in T\}}_{\substack{RT \\ \text{universal} \\ \text{quantifier}}}$$

so with inverse images around, ok.

Translate to properties.

defn $\mathcal{C}$ a cat. s.t. $\mathcal{C} \longrightarrow \mathbb{1}$, $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ have both got right adjts, as well as $\mathcal{C} \xrightarrow{- \times b} \mathcal{C}$ a rt adjt, is called cartesian closed category.

t

$$\begin{tikzcd} & a \times b \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd}$$

$\hom(x \times b, y) \cong \hom(x, y^b)$

$- \times b \quad - \quad (\ )^b$

example. $\mathbb{E}$ns

example. Cat.

example. $\mathcal{P}U$ as in Bool alg.

["term obj, fin prods, & hom ftr y^b"]

e.g. in top. need convergence spaces

☑ EXERCISE. U set. $\mathbb{E}\text{ns}/U \overset{w}{\underset{\text{objs}}{}} \left\{ \begin{matrix} X \\ \downarrow \\ U \end{matrix} \right\}$ in cart. cl. cat.

☑ EXERCISE. develop elementary theory of cart. cl. / cat. cartesian closed calculus

1) expon. rules
2) evaluation ftr
3) relation B^A to $\hom(A,B)$?

$A^{\mathbb{1}} = A$, $\mathbb{1}^A = \mathbb{1}$, $(A \times B)^C$, $(A^B)^C$, A^0, $A \times \mathbb{1} = A$

$A \xrightarrow{g} B \to X^A \xleftarrow{X^f} X^B$, $A^X \xrightarrow{f^X} B^X$

▷ EXERCISE. generalize $\mathcal{P}U \xleftarrow{\text{cart cl cat.}} \mathcal{P}V \longleftarrow \mathcal{P}W$

(category of categories $\mathfrak{C}_2$)

category of cart cl cats $= \mathcal{C}_4$.

$\mathbb{E}$ns $\quad U \longrightarrow V \longrightarrow W$

$\mathbb{E}\text{ns}_f$ cat of finite sets.

full $\bigcup^{f}$ $\mathbb{N}$ $\quad \varnothing \quad 1 = \{\varnothing\} \quad 2, \ldots \quad n = \{0, \ldots, n-1\}$.

ea n one set w n elmts.

arr $n \longrightarrow m$ is any fn.

$\mathbb{N}$ a skeleton of $\mathbb{E}\text{ns}_f$.

$$\begin{tikzcd} \mathbb{E}ns \arrow[d, shift left, "S"] \\ \mathbb{N} \arrow[u, shift left, "\underset{\text{inclusion}}{J}"] \end{tikzcd}$$

$S\{a_0, \ldots, a_{n-1}\} = \{0, \ldots, n-1\}$.

ADJNT

---

defn. $A \underset{T}{\overset{S}{\rightleftarrows}} X$ an equivalence is pair of cats + pair of ftrs and isomorphisms $TS \cong I$ $ST \cong I$.

[less than an isomp.]

defn. An adjoint equivalence is $A \underset{T}{\overset{S}{\rightleftarrows}} X$ and $\eta : I \cong TS$ $\varepsilon : ST \cong I$ so that S, T, η, ε is adj.

So have $\eta^{-1} : TS \cong I$

$\varepsilon^{-1} : I \cong ST$.

▷ reverse the identities of η, ε .

Thus S left adj of T

+ T rt adj of S .

Th^m Given $S : A \longrightarrow \underline{X}$, following are equivalent

1° S is part of an adj^t equ $ST\varepsilon\eta$

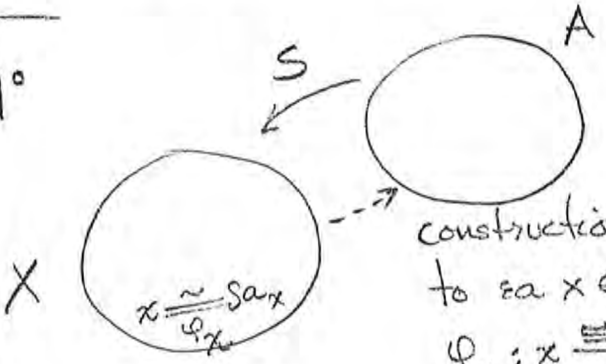
2° S is part of an equivalence $ST \cong I \quad TS \cong I$

3° S is full faithful and to every $x \in \underline{X}$ there is $a \in A$ s.t. $Sa \cong x$.

1° ⇒ 2° triv.

2° ⇒ 3° $ST \cong I$ so ST faithful so T is faithful

3° ⇒ 1°



construction of adj^t:
to ea $x \in \underline{X}$ choose an isomp

$$\begin{tikzcd} \varphi_x : x \arrow[r, "\cong"] \arrow[dr, "f"'] & Sa_x \arrow[d, dashed] \\ & Sb \end{tikzcd}$$

↘ & S is full & faithful

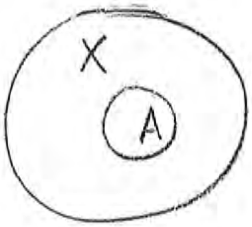
So every ↘ $f \circ \varphi_x^{-1}$ is S of a unique $a_x \xrightarrow{f'} b$

$$f = Sf' \circ \varphi_x$$

$T : x \rightsquigarrow a_x$
w unit φ_x .

need counit isomp.

special case. A full sbcat $\subset X$ &
every $x \cong$ to some $a \in A$



$J: A \subset X$ incl of full

has $\left\{\begin{matrix}\text{left adjt}\\ \text{left inverse}\end{matrix}\right\}$

example of adjnt. $\boxed{\text{July 9}}$ $\boxed{\text{Maclane 12}}$

$$\begin{tikzcd}
R\otimes M & R\text{-}\mathrm{Mod} \arrow[d, shift left, "U"] & A \arrow[d, squiggly] \\
M \arrow[u, squiggly, "\text{left adjnt}"] & \mathrm{Ab} \arrow[u, shift left] & M
\end{tikzcd}$$

$$\hom(R\otimes_{\mathbb{Z}} M, A) \cong \hom_R(R, \hom_{\mathbb{Z}}(M,A))$$

$$\cong \hom_{\mathbb{Z}}(M,A)$$

$$\left[\ \hom_{\mathbb{Z}}(M, UA)\ \right]$$

also rt adjnt

$$\begin{tikzcd}
R\text{-}\mathrm{Mod} \\
\mathrm{Ab} \arrow[u, bend right, "\hom_{\mathbb{Z}}(R,-)"']
\end{tikzcd}$$

~~unit & counit~~

Logic. G. Gentzen — proof theory.

rules of inference : $p \longrightarrow r$ for p entails r.

$$\frac{p \longrightarrow q \quad p \longrightarrow r}{p \longrightarrow q \,\&\, r} \quad \Bigg| \quad \frac{p \longrightarrow q \,\&\, r}{p \longrightarrow q} \quad \frac{p \longrightarrow q \,\&\, r}{r \longrightarrow r}$$

states adjntness

---

exercise (Paré) defn idempotents split in $\mathcal{B}$

$$e : b \longrightarrow b\,,\ e^2 = e \implies \exists x y \quad \begin{tikzcd} b \arrow[rr] \arrow[dr, "y"'] & & b \\ & {} \arrow[ur, "x"'] & \end{tikzcd} \quad \begin{array}{l} xy = e \\ yx = 1 \end{array}$$

① $e: b \to b$ idem.

e splits iff $? \xrightarrow{x} b \overset{e}{\underset{1}{\rightrightarrows}} b$ is equalizer,

$(? = b)$ or $b \overset{e}{\underset{1}{\rightrightarrows}} b \xrightarrow{y} ?$ is coequalizer

② idems split in $\mathcal{B}$ iff they split in $\mathcal{B}^{\mathcal{A}}$

③ say $\mathcal{B} \underset{G}{\overset{U}{\rightleftarrows}} A$ with $e: GU \Rightarrow I$ -

$h: 1 \Rightarrow UG$

and
$$\begin{tikzcd}
U \arrow[r, "hU"] \arrow[rr, bend right] & UGU \arrow[r, "Ue"] & U
\end{tikzcd}$$

where idems split in $\mathcal{B}$ ($\therefore$ $\mathcal{B}^{\mathcal{A}}$).

Conclusion:
$$\begin{tikzcd}
G \arrow[r, Rightarrow, "Gh"] \arrow[dr, Rightarrow, "y"'] & GUG \arrow[r, Rightarrow, "eG"] & G \\
& F \arrow[ur, Rightarrow, "x"'] &
\end{tikzcd}$$
is idem. in $\mathcal{B}^{\mathcal{A}}$

So splits

& F is left adjn to U via

defns
$$\begin{tikzcd}
I \arrow[rr, "\eta"] \arrow[dr, "h"'] & & UF \\
& UG \arrow[ur, "Uy"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
FU \arrow[rr] \arrow[dr, "xU"'] & & I \\
& GU \arrow[ur, "e"'] &
\end{tikzcd}$$

④ formulate with equl's & coequl's.

$$\hom(A\otimes B, C) \cong \hom(A, \hom(B,C))$$

$-\otimes B$ is left adjoint to $\hom(B,-)$

<u>adjunction</u> <u>with</u> <u>a</u> <u>parameter</u> <u>B</u>

$$\begin{tikzcd} X \arrow[r, bend left, "FG\varepsilon\eta"] \arrow[r, bend right, "F'G'\varepsilon'\eta'"'] & B \end{tikzcd}$$

Given $\sigma : F \Longrightarrow F'$ nat. xf.

[illegible]

CONSTRUCTIONS

① $\exists!\,\tau_1$ s.t.

$$\begin{tikzcd} \hom(Fx,a) \arrow[r, "\phi"] & \hom(x,Ga) \\ \hom(F'x,a) \arrow[u, "-\circ\sigma_x"] \arrow[r, "\phi'"'] \arrow[ur, phantom, "\text{s.t. } ///"] & \hom(x,G'a) \arrow[u, dashed, "\hom(x,\tau_a)"'] \end{tikzcd}$$

by Yoneda

② $\exists!\,\tau_2$

$$\begin{tikzcd} G' \arrow[r, Rightarrow, dashed, "\tau_2"] \arrow[d, Rightarrow, "\eta G'"'] \arrow[dr, phantom, "///"] & G \\ GFG' \arrow[r, Rightarrow, "G\sigma G'"'] & GF'G' \arrow[u, Rightarrow, "G\varepsilon'"'] \end{tikzcd}$$

$G\varepsilon' G\sigma G' \eta G'$

③ $\exists\,\tau_3$

$$\begin{tikzcd} FG' \arrow[r, Rightarrow, dashed, "F\tau_3"] \arrow[d, Rightarrow, "\sigma G'"'] \arrow[dr, phantom, "///"] & FG \arrow[d, Rightarrow, "\varepsilon"] \\ F'G' \arrow[r, Rightarrow, "\varepsilon'"'] & I_A \end{tikzcd}$$

by univ. of ε

Thm given σ, $\exists\, \tau_1, \tau_2, \tau_3$ make diags commute.

Furthermore ① ⟹ ② ⟹ ③. Trivial that ② ⟹ ①.

i.e. $\tau_1 = \tau_2 = \tau_3$ in conclusion.

Must dualize.

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\sigma"] \arrow[d, Rightarrow, "F\eta'"'] & F' \\
FG'F' \arrow[r, Rightarrow, "F\tau F'"'] & FGF' \arrow[u, Rightarrow, "\varepsilon F'"']
\end{tikzcd}$$

$\exists^{op}$.

Thus σ, τ determine ea other.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \arrow[r, bend left=40, "FG\varepsilon\eta"{name=U}] \arrow[r, bend right=20, "F'G'\varepsilon'\eta'"{name=M, below}] \arrow[r, bend right=90, looseness=2, "F''G''\varepsilon''\eta''"{name=D, below}] & B
\arrow[Rightarrow, from=U, to=M, "\langle\sigma,\tau\rangle"]
\arrow[Rightarrow, from=M, to=D, "\langle\sigma',\tau'\rangle"]
\end{tikzcd}$$

defn. <u>transformation</u> <u>of</u> <u>adjunctions</u>

τ is the <u>conjugate</u> to σ.

Th<u>m</u> the composite is one.

▷ composite adjunction

find unit & counit

$$\begin{tikzcd}[column sep=huge]
X \arrow[r, bend left=40, "FG\varepsilon\eta"] \arrow[r, bend right=40, "F'G'\varepsilon'\eta'"'] \arrow[r, phantom, "\Downarrow \langle\sigma,\tau\rangle"] \arrow[rr, bend left=70] & B \arrow[r, bend left=40, "\bar{F}\bar{G}\bar{\varepsilon}\bar{\eta}"] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow \langle\bar{\sigma},\bar{\tau}\rangle"] & A
\end{tikzcd}$$

▷ Th$^{\underline{m}}$ composition horiz. of conjugates.

▷ Th$^{\underline{m}}$ Interchange.

$$\left(\langle\bar{\sigma}\,\bar{\tau}\rangle\langle\sigma\,\tau\rangle\right)\cdot\left(\langle\bar{\sigma}'\,\bar{\tau}'\rangle\langle\sigma'\,\tau'\rangle\right)$$

$$\| $$

$$(\qquad\qquad)(\qquad\qquad)$$

~~▷ This requires no proof. project to σ & [illegible] do it [illegible] then [illegible] which is already done~~

requires composition <u>two</u> ways

$\mathbb{A}dj$ = cat of adjnt pairs w conjugate pairs for maps.

$$\begin{tikzcd}
 & \mathbb{A}dj \arrow[dl] \arrow[dr] & \\
\mathbb{C}at^{op}_{op} & & \mathbb{C}at
\end{tikzcd}$$

} forgetful ftrs, one contravariant

op vert
op horiz

⑤

Application.

adjn w parameter :

assume $F : X \times P \longrightarrow A$ biftr. $G : P \times A \longrightarrow X$ ftr for ea $p \in P$ (not nec. biftr)

$$\hom(F(x,p),a) \underset{\text{ntl in } x,a}{\cong} \hom(x, G(p,a)).$$

Th^m. If $F(-,p)$ ea p has rt adjnt $G(p,-)$, then G can be made a biftr uniquely so that the adjn is natural in all three variables. $G : P^{op} \times A \longrightarrow X$.

Example of adjts :

Maclane 13

sep pts from cl's $\qquad T_0$

$$\begin{tikzcd}
\text{Cpt Haus} \arrow[r, hook, "J"] & \text{Compl Reg} \arrow[l, bend left, "\beta"] \arrow[r, hook, "J"] & \text{Top}_0 \arrow[l, bend left, "\gamma"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[dr] & \beta X \arrow[d, dashed, bend left] \\
& I
\end{tikzcd}$$

Stone-Cech

ua from 1 to $J\beta$

$\beta \dashv J$

inclusion has left adjoint
"reflexive subcategory"

$Y \qquad \gamma Y$

collapse what can't be separated

New defn of cat. To define by diagrams.

Monoid obj in mult. cat.

$C, \otimes, K$

$\otimes : C \times C \longrightarrow C$ assoc.

& K obj in C s.t. $K \otimes C \cong C \cong C \otimes K$

M

$$\begin{tikzcd}
M \otimes M \arrow[r, "\mu"] & M & K \arrow[l, "\eta"]
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes M\otimes M \arrow[r] \arrow[d] \arrow[dr, phantom, "''"] & M\otimes M \arrow[d] \\
M\otimes M \arrow[r] & M
\end{tikzcd}$$

& lft id.

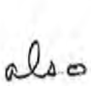
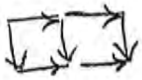
Mon obj in Ens, ×, 1 is ordinary monoid
" " " Ab, ⊗ Z " ring

①

a <u>category object</u> in a category C :

$$\begin{tikzcd} A \arrow[d, shift right, "\partial_1"'] \arrow[d, shift left, "\partial_0"] \\ O \end{tikzcd} \qquad \begin{tikzcd} \Phi \arrow[r] \arrow[d] \arrow[dr, phantom, "plbk"] & A \arrow[d, "\partial_1"] \\ A \arrow[r, "\partial_0"'] & O \end{tikzcd}$$

gives composables.

Pullback lemma.  plbk &  plbk then  is plbk also  is plbk. Also

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "g"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

is plbk.

Given cat C w plbks, an <u>O-graph in C</u> for O∈C is a diag

morphism of O-grphs

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_0"'] \arrow[d, dashed] & O \\ B \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_0"'] & \end{tikzcd}$$

$Grph_O(C)$ cat. w ⊗ :

$$\begin{tikzcd} A \otimes_O B \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "plbk"] & B \arrow[d, "g_0"] \\ A \arrow[r, "f_1"'] & O \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_O B \arrow[r, shift left, "f_1 p \quad f_0 p?"] \arrow[r, shift right, "g_0 q \quad g_1 q?"'] & O \end{tikzcd}$$

<u>Claim</u> $Grph_O(C)$ not only has ⊗ but also is assoc & unit.

[Hence multipl. cat.]

assoc :

$$\begin{tikzcd} & & A \otimes_O B \arrow[dl] \arrow[dr] & & (A \otimes_O B) \otimes_O C \arrow[ll] \arrow[d] & \\ & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & C \arrow[d] & \\ O & & O & & O & \end{tikzcd}$$

etc.

To find identity : "$A \otimes_O \, ? = O$"

$$\begin{tikzcd}
 & & A \otimes_O O \cong A \arrow[dl, "1"'] \arrow[dr, "f_o"] & & \\
 & A \arrow[dl] \arrow[dr] & & O \arrow[dl, equal] \arrow[dr, equal] & \\
O & & O & & O
\end{tikzcd}$$

} trivial O-graph in $\mathcal{C}$.

Defn. A <u>cat</u> <u>obj</u> <u>in</u> $\underline{\mathcal{C}}$ (cat w plbks) is $O \in \mathcal{C}$ and M, μ, η a monoid in $\mathrm{Grph}_O(\mathcal{C})$.

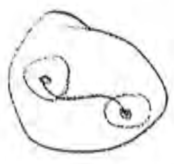
What does this mean?

Prop. A cat. obj in $\mathbb{E}$ns is a sm cat.

What is a cat obj in ______ Top ?

Exercise. Find examples.

Gray :
O a space;
M = Moore paths
$[0,a] \longrightarrow O$ cnts



(Arr only defn of a category):

A

set of "arrows"

$g\circ f$ sometimes defined

$h\circ(g\circ f) = (h\circ g)\circ f$ if either side or $g\circ f$ or $h\circ g$ are defined

identity

$\forall f \ \exists e_0 \ e_1$ s.t. $f e_0 = f \quad e_1 f = f \quad e_0 \xrightarrow{f} e_1$

$$A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} A \qquad \begin{matrix} f \mapsto e_0 \\ f \mapsto e_1 \end{matrix}$$

(DOUBLE CATEGORY):

set D of "double arrows"

s.t. an arr's cat in 2 ways

AX1

$\circ_V$ vertical $\quad \circ_H$ horizontal

$\partial_0^V \partial_1^V \qquad \partial_0^H \partial_1^H$

✓ draw the picture for Cat.

Cat
$$\begin{tikzcd} A \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & B \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & C \end{tikzcd}$$

AX2 | interchange holds

✓ AX3 | ∂_i^H is a "vertical functor"

∂_i^V " " "horizontal functor"

▷ Th$^{\underline{m}}$ $\partial_i^H \partial_j^V = \partial_j^V \partial_i^H$, $\forall i,j$.

Example of double cat in any C, □C D is all commutative squares in C.

Or just take double cat. of all pullbacks in C.

Or double cat of adjnt pairs.

▷ Th$^{\underline{m}}$ A cat obj in Cat <u>is</u> a double cat.

---

get a nicer defn of dub cat so things go smoother (easier)

Maclane 14

Th<u>m</u> Any right adjoint preserves limits.

$$\begin{tikzcd} A \arrow[d, "G"] \\ X \end{tikzcd}$$

$J \xrightarrow{T} A$ have limit, i.e., univ. cone τ from a to J.

Apply G; then $G\tau$ is univ cone from Ga to $G \cdot T$.

<u>G a rtadjt</u>

$$\begin{tikzcd} T_i \arrow[d, "Ts"'] & a = \lim F \arrow[l, "\tau_i"'] \arrow[dl, "\tau_j"] \\ T_j & \end{tikzcd}$$

$GT_i \longleftarrow$

$\cdots$

$\hom(Fx, a) \cong \hom(x, Ga)$

Dually – left adjts preserve colimits.

---

prf in Cat :

$$\begin{tikzcd} & A^J \arrow[dl, shift right, "G^J"'] \arrow[dr, shift left, "\lim"] & \\ X^J \arrow[ur, shift right, "F^J"'] \arrow[dr, shift left, "\lim"] & & A \arrow[ul, shift left, "\Delta"] \arrow[dl, shift right, "G"'] \\ & X \arrow[ul, shift left, "\Delta"] \arrow[ur, shift right, "F"'] & \end{tikzcd}$$

"up to the J" is a double functor and carries adjunctions to adjunctions.

<u>claim</u> two compositions of adjts are $=$.
This shows "$G \lim = \lim G^J$".

also "J to the blank" is a "double ftr" with things twisted

When can you build left adjoints ?

(Freyd Adjoint Functor Theorem)

$G : A \longrightarrow X$ has left adjoint only if G preserves limits in A (carries univ. cone to a univ. cone).

Instead of getting adj lets get ua x to G, i.e., get in. elmt of x/G.

Thm 1° C cat w/ sm lims & sm homsets & (soln set cond) $\exists$ sm set $\{b_i \mid i \in I\}$ of obj of C and $c \in C \Rightarrow \exists i$ s.t. $\exists\, b_i \longrightarrow c$. Then $\exists$ in obj in C.

$$\begin{tikzcd}
v \arrow[r, "e"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & w = \prod_{i \in I} b_i \arrow[d] \arrow[dr] & \\
c & b_i \arrow[r, dashed] & c
\end{tikzcd}$$

w is "weakly initial" ($\exists$ arrow ; need !).

Look at hom(w, w). v, e equalizes all hom(w,w), ie $\forall f, g$ $w \overset{f}{\underset{g}{\rightrightarrows}} w$, $fe = ge$ & univ.

v is not only weakly initial; for suppose $v \overset{s}{\underset{t}{\rightrightarrows}} c$. Can equalize s,t w/ $u \xrightarrow{e_1} v \overset{s}{\underset{t}{\rightrightarrows}} c$. $\exists$ $w \xrightarrow{r} u \xrightarrow{e_1} v \xrightarrow{e} w$

$V \xrightarrow{e} W \underset{1}{\overset{ee_1r}{\rightrightarrows}} W$ is equ; since e equs every pair; $ee_1re = 1e = e$

e mno ∴ $e_1re = 1$. e_1 mno

∴ e_1 isomp, (need only show $ree_1 \overset{?}{=} 1$;

$\cancel{e_1}ree_1 = \cancel{e_1}$ so). Thus $s = t$.

EISENBUD:

$\boxed{\text{From } e_1re = 1 \quad ee_1 = te_1 \quad se_1re = te_1re \quad s = t}$

Thm $G : A \xrightarrow{\text{sm compl., sm homsets}} X$ preserves sm limits;

then G has a left adjoint iff to each x

there is sm set $I = I_x$ in A s.t.

$\forall\ x \xrightarrow{f} Ga\ \exists\ s \in I_x, \exists\ s \xrightarrow{h} a$ s.t. $f = Gh \circ \lambda$ for some $x \xrightarrow{\lambda} Gs$

Reduction of this to prev Thm: take $C = x/G$.

Remains to show C is sm complete.

So show has products & pairwise equs.

EXERCISE: GIVE DIRECT PROOF.

E. Cooper: When I can't do something, I feel that I know nothing.
What do you do when you can't do something?

S. Eilenberg: I hang myself.

1929 self adj lin xfs (Stone)

1938 Stone Čech

? 1940 top w arrs (Hurewicz ?)

1945 cats, ftrs, nt xf

1948 universal constructions (Samuel)

Bourbaki

1950 Abelian cats

1954 homological alg (Grothendieck)

1957 Some remarks (Kan)

1958 adjnt ftr

simplicial sets

under Eilenberg at Columbia

$$X \longrightarrow G(P, F(X, P))$$

$\boxed{\text{Maclane 15}}$

$c, b \in \mathcal{C}$ sm compl + cocompl cat

X a set.

$$\begin{tikzcd} \coprod_X c & c \arrow[l, "i_x"'] \arrow[d, "f"] \\ & b \end{tikzcd}$$

$$\hom_{\mathcal{C}}(\coprod_X c, b) \cong \hom_{\mathbb{E}ns}(X, \hom(c,b)).$$

$$f \rightsquigarrow \{x \rightsquigarrow f \circ i_x\}$$

Also

<u>dual</u> $\hom_{\mathbb{E}ns}(X, \hom(c,b)) \cong \hom_{\mathcal{C}}(c, \prod_X b)$.

BOTH ADJUNCTIONS W PARAM.

$G : P^{op} \times A \to X$

$\eta_x : x \to G(p, F(x,p))$ "ntl in p"

What does this mean?

$$\begin{tikzcd}[row sep=large] & \hom(F(x,p),a) \arrow[r, phantom, "\cong"] & \hom(x,G(p,a)) \\ p \arrow[d, "t"'] & & \\ p' & \hom(F(x,p'),a) \arrow[uu, bend left, "-\circ F(x,t)"] \arrow[r, phantom, "\cong"] & \hom(x,G(p',a)) \arrow[uu, bend right, "G(t,a)\circ -"'] \end{tikzcd}$$

$$\underbrace{\begin{matrix} a \\ \| \\ F(x,p') \end{matrix}}$$

$$1_{F(x,p')} \rightsquigarrow \eta_{x,p'}$$

$$\begin{tikzcd} 1_{F(x,p')} \arrow[d, squiggly, "F(x,t)"] \\ \mathbf{F(x,p)} \end{tikzcd}$$

$$\therefore\ G(p, F(x,t)) \circ \eta_{x,p} \overset{\checkmark}{=} G(t, \overset{\overset{F(x,p')}{\|}}{a}) \circ \eta_{x,p'}$$

$$\begin{tikzcd}
x \arrow[r, "\eta_{x,p}"] \arrow[d, "\eta_{x,p'}"'] & G(p, F(x,p)) \arrow[d, "G(p, F(x,t))"] \\
G(p', F(x,p')) \arrow[r, "G(t, F(x,p'))"'] & G(p, F(x,p'))
\end{tikzcd}$$

General defn. Say $T : P^{op} \times P \longrightarrow C$ (dighe?)

Suppose have $1 \xrightarrow{\eta_p} T(p,p)$ nt xf.

η is natural iff "supernatural"

$$\begin{tikzcd}
p \arrow[d, "t"] & 1 \arrow[r, "\eta_p"] \arrow[d, "\eta_{p'}"'] & T(p,p) \arrow[d, "T(p,t)"] \\
p' & T(p',p') \arrow[r, "T(t,p)"'] & T(p,p')
\end{tikzcd}$$

---

example. $X^Y \times Y \xrightarrow{ev} X$

⊛ $X \overset{ftr}{\rightsquigarrow} X^Y \times Y'$ ⊛⊛ $X \underset{ftr}{\overset{1}{\rightsquigarrow}} X$

ev is a spntl xf. from ⊛ to ⊛⊛

exercise. How do you compose supernaturals?

$\boxed{\text{Dubuc adjoint ftr Th}^{\underline{m}}}$

$$\begin{tikzcd}
C \arrow[d] \\
\mathbb{1} \arrow[u, bend left=60, "\text{initial}"] \arrow[u, bend right=60, "\text{terminal}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d] \\
C \times C \arrow[u, bend left=60, "\text{cop}"] \arrow[u, bend right=60, "\text{prod}"']
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[d] \\
C^{\rightrightarrows} \arrow[u, bend left=60, "\text{coequalizer}"] \arrow[u, bend right=60, "\text{equalizer}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d] \\
C^{J} \arrow[u, bend left=60, "\text{colim} \rightarrow"] \arrow[u, bend right=60, "\text{lim} \leftarrow"']
\end{tikzcd}$$

Adj ftr $\text{Th}^{\underline{m}}$

Left adjoint

initial (univ arr)   got by | product / equalizer

" initials are limits of id "

$\text{Th}^{\underline{m}}$ C has $\underbrace{\text{in obj}}_{\text{left adj}}$ iff $\underbrace{C \xrightarrow{1_C} C \text{ has a limit}}_{\text{rht adj}}$

$\boxed{+\ c_0 = \lim 1_C}$

prf. $(\Rightarrow)$

This means

✓

$$\begin{tikzcd}
 & c_0 \arrow[dl, "\tau_a"'] \arrow[dr, "\tau_b"] & \\
a \arrow[rr, "\forall f"'] \arrow[rr, phantom, shift left=3, "\equiv"] & & b
\end{tikzcd}$$

$\therefore\ \tau$ is ntxf $c_0 \Longrightarrow \text{id}$.

$\tau_{c_0} = 1_{c_0}$

($\Leftarrow$)

$\tau : c_0 \Longrightarrow \mathrm{id}$ univ.

$$\begin{tikzcd} & \lim \mathrm{id} = c_0 \arrow[dl, "\tau_a"'] \arrow[dr, "\tau_b"] & \\ a \arrow[rr, "\mathrm{all}\, f"'] \arrow[rr, phantom, "///", shift left=4] & & b \end{tikzcd}$$

suppose there is another $c_0 \xrightarrow{f} a$:

$$\begin{tikzcd} c_0 \arrow[dr, "g"] & & c_0 \arrow[dr, "\bar{\tau}_a"] & \\ & a & & a \end{tikzcd}$$

need to show $\tau_{c_0} = 1$

$$\begin{tikzcd} & & c_0 \arrow[dd, "\tau_{c_0}"] \\ c_0 \arrow[r, dashed, "\gamma"] \arrow[urr, bend left, "1_{c_0}"] \arrow[drr, bend right, "\tau_{c_0}"'] & c_0 \arrow[ur, "\tau_{c_0}"] \arrow[dr, "\tau_{c_0}"'] \arrow[r, phantom, "\overset{///}{\text{univ.}}"] & {} \\ & & c_0 \end{tikzcd}$$

s.t. $\begin{cases} \tau_{c_0}\gamma = 1_{c_0} \\ \tau_{c_0}\gamma = \tau_{c_0} \end{cases}$ $\therefore 1_{c_0} = \tau_{c_0}$

$$\begin{tikzcd} & c_0 \arrow[dl, "\tau_{c_0} = 1"'] \arrow[dr, "\tau_a"] & \\ c_0 \arrow[rr, "g"'] & & a \end{tikzcd}$$

$\Longrightarrow g = \tau_a$.

certainly $\tau_{c_0} = \tau_{c_0} 1_{c_0}$ so $\exists !\, c_0 \xrightarrow{\gamma} c_0$

c_0 produces a cone ; that cone is terminal

Now for Freyd again :

(in. obj. form)

M any set of sets w $\emptyset, 1, 2 \in M$.

Define C is M-complete $\Longleftrightarrow$ any ftr from a set in M to C has a limit (?)

Th$\underline{\text{m}}$ Say C is M-complete + $\exists$ set $S \subset \mathrm{obj}\, C$ wh is weakly in ($\forall c \in C$ $\exists$ $s \in S$ + arr $s \to c$)

$\Longrightarrow$ $\exists$ in obj in C.

$$\begin{tikzcd} J \arrow[r, "T"] & A \arrow[d, "G"] \\ & X \end{tikzcd}$$

"G preserves limit of T" means if $a \underset{\text{lim cone}}{\xRightarrow{\tau}} T$ then $Ga \underset{\text{is lim cone}}{\xRightarrow{G\tau}} GT$ '

Say "G reflects limit of T" iff

$$a \underset{\text{cone}}{\xRightarrow{\tau}} T \ \& \ Ga \underset{\text{lim cone}}{\xRightarrow{G\tau}} GT \Longrightarrow a \underset{\text{lim cone}}{\xRightarrow{\tau}} T$$

Say "G creates limit of T"

if given $x \underset{\text{lim cone}}{\xRightarrow{\sigma}} GT$ then $\exists!\ a \xRightarrow{\tau} T$ w $G\tau = \sigma$

& τ is lim cone.

Consider $\underbrace{{}^{c_0}/C}_{} \xrightarrow{Q} C$

obj
$$\begin{tikzcd} c_0 \arrow[r, "f"] \arrow[d, equal] & a \arrow[d, dashed, "h"] & \rightsquigarrow & a \arrow[d, "h"] \\ c_0 \arrow[r, "g"'] & b & \rightsquigarrow & b \end{tikzcd}$$

lemma. Q creates ~~limits~~ limits.

Consider ${}^{x}/G \xrightarrow{Q} A$ (take a set of $x \longrightarrow Ga$)

Then Q creates limits if G pres lims

⑤

Creating limits

MacLane 16

$$B \xrightarrow[H]{} C$$

$\mathcal{J}$ collection of categories J .

defn H creates limits over $\mathcal{J}$ if

given $J \xrightarrow{T} B \xrightarrow{H} C \qquad c \overset{\tau}{\Longrightarrow} HT$

and lim cone for composite HT $\wedge$ then $\exists!$

cone $b \overset{\sigma}{\Longrightarrow} T$ over T wh is lim cone s.t. $H\sigma = \tau$

$Hb = c$
$H\sigma_j = \tau_j \ \forall j \in J$

$$\begin{tikzcd} A \arrow[d, "G"] & & (X,G) \arrow[dl] \arrow[d, "P"] \arrow[dr] & & \langle x \xrightarrow{f} Ga, a\rangle \arrow[d, squiggly] & \forall x \in X, \forall f \\ X & A & X \arrow[d] & X^2 & x & \\ & & \text{over } x \text{ is } x/G & & & \end{tikzcd}$$

lemma. "Projections of commas create limits."

G preserves lims in $\mathcal{J}$ then

$Q_x : x/G \longrightarrow A$ creates lims .

$$\begin{tikzcd}
J \arrow[r, "T"] \arrow[rr, bend right, "\hat{T}"'] & x/G \arrow[r, "Q"] & A \arrow[r, "G"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
i \arrow[dr, "s"'] & & x \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & & & \hat{T}_i \arrow[dr, "\hat{T}_s"] & \\
& j & G\hat{T}_i \arrow[rr, "G\hat{T}_s"'] & & G\hat{T}_j & & \hat{T}_j
\end{tikzcd}$$

$T = \langle \tau : x \Rightarrow G\hat{T}, \hat{T} \rangle$

Cone over T ? an obj $x \xrightarrow{h} Gc \in x/G$

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] \arrow[rr, "h"] & & Gc \arrow[dlll] \arrow[dl] \\
G\hat{T}_i \arrow[rr] & & G\hat{T}_j &
\end{tikzcd}$$

⋮

Given T & lim cone β over $\hat{T}$, $\beta : c \Rightarrow \hat{T}$

$$\begin{tikzcd}
& G\hat{T} \\
x \arrow[ur, Rightarrow, "\tau"] \arrow[dr, dashed, "\exists!"'] & \\
& Gc \arrow[uu, Rightarrow, "G\beta"']
\end{tikzcd}$$

since G preserves.

another proof beginning

$$\begin{tikzcd} G \\ \ddot{A} \arrow[d] \\ X \end{tikzcd}$$

$$\begin{tikzcd}
 & J \arrow[r, "T"] & x/G \arrow[r, "Q"] \arrow[d, "\bar{G}"'] \arrow[dr, phantom, "\text{is PLBK!}"] & A \arrow[d, "G\ \text{preserves}"] & \overset{\text{conclude this creates}}{\Rightarrow QT} \\
 & & x/X \arrow[r, "\substack{Q_0 \\ \text{creates}}"'] & X & \Rightarrow GQT \\
 & \exists \Rightarrow \bar{G}T \arrow[uur, squiggly] & & &
\end{tikzcd}$$

Applications

Initial obj = lim of id .

Thm Every left adjt is a limit .

pf. $A \xrightarrow{G} X$ (vertical) has left adjt iff 1° G pres limits

2° $\forall x$

$x/G \xrightarrow{Q} A$

has a limit

(⟹) When this is the case,

$\left(\eta_x : x \longrightarrow GFx \quad \text{in obj in } x/G \right)$

$$\boxed{\begin{array}{c}\text{Left adjt is given by} \\ Fx = \lim\left(x/G \xrightarrow{Q_x} A \right)\end{array}}$$

& since Q creates limits η_x comes from this creation.

Freyd Adjnt Ftr Thm

$$\begin{tikzcd}A \arrow[d, "G"] \\ X\end{tikzcd}$$

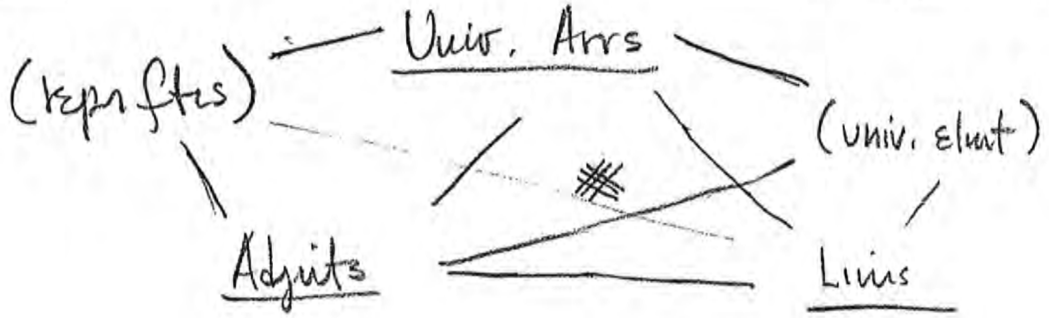
1° A has all $\mathcal{J}$ lims

2° G pres all $\mathcal{J}$ lims

3° $\forall_x \exists$ set in $\mathcal{J}$ of $x \to G$ s.t.

Then G has left adjnt.

pf. suffices to show $x/G \xrightarrow{Q} A$ has a lim.



" The real understanding is some several dimensional inter-rel. of these concepts "

✓✗ exercise. ① problem in §7 (Linton)

~~② define dual cat~~

~~$A \longrightarrow B$ cat ftrs~~

~~C D~~

$$\begin{tikzcd}
J' \arrow[r, "W"] & J \arrow[r, "T"] \arrow[r, bend right=60, "T'"'] \arrow[r, phantom, "\Downarrow \tau"] & A \arrow[r, "H"] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[d, dashed] \arrow[dr, Rightarrow] & & \\
\lim T \arrow[r, Rightarrow, "\gamma_T", "\text{univ. cone}"'] & T \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \operatorname{colim} T \arrow[d, dashed] \\
& & b
\end{tikzcd}$$

$\xleftarrow{\quad}$ $\xrightarrow{\quad}$

} diag in A^J

(lim as a ftr):

$$\begin{tikzcd}
\lim HT \arrow[r, Rightarrow] & HT \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "H(\gamma^T)"] & \operatorname{colim} HT \arrow[d, dashed, "\operatorname{colim} H"] \\
H \lim T \arrow[u, dashed, "\lim H"] \arrow[ur, Rightarrow, "H(\gamma_T)"'] & & \operatorname{colim} T
\end{tikzcd}$$

$$\begin{tikzcd}
\lim TW \arrow[r, Rightarrow] & TW \\
\lim T \arrow[u, dashed] \arrow[ur, Rightarrow, "\gamma_T W"'] &
\end{tikzcd}$$

$\lim : A^J \longrightarrow A : T \rightsquigarrow \lim T$

$$\begin{tikzcd}
\lim T \arrow[r] \arrow[d, dashed, "\lim \tau"'] & T \arrow[d, Rightarrow, "\tau"] \\
\lim T' \arrow[r, Rightarrow] & T'
\end{tikzcd}$$

"partial ftr"
but for A J-complete.

lim, colim : $A^J \longrightarrow A$.

Not the way, yet.

Invent a cat Diag A "union of all A^J, $J \in \mathcal{J}$".

Obj of Diag A is J and a fct $J \xrightarrow{T} A$.

$$\begin{tikzcd} J \arrow[r, "T"] \arrow[d, "S"'] & A \arrow[dl, phantom, "\overset{\tau}{\Leftarrow}"] \\ J' \arrow[ur, "T'"'] & \end{tikzcd}$$

Arr

$$\begin{tikzcd} J \arrow[r, "T"] \arrow[d, "S"'] & A \arrow[d, equal] \arrow[dl, phantom, "\overset{\tau}{\Leftarrow}"] \\ J' \arrow[r, "T'"'] & A \end{tikzcd}$$

$\tau : T'S \Rightarrow T$
or
$\tau : T \Rightarrow T'S$?

say A J-complete.

Now say lim : Diag A $\longrightarrow$ A .

something like "Cat/A"

MacLane 17

$$\begin{tikzcd}
\mathbf{Grp} \arrow[d, shift right] \\
\mathbf{Mon} \arrow[u, bend right]
\end{tikzcd}$$

forgetful w rt adjnt : into units of M

EXERCISE (Palmquist)

▷ Thm $X^{\mathbf{2}} \xrightarrow{\text{take ends}} X \times X$ creates limits. [nothing new]

EXERCISE (Gray)

✓ $\mathbf{Cat}/\!/A \xrightarrow{\text{colim}} A$ Find an adjnt.

$$\begin{tikzcd}
J \arrow[rr, "\Sigma"] \arrow[dr, "T"'] \arrow[rr, phantom, shift right=3, "\Rightarrow"] & & J' \arrow[dl, "T'"] \\
& A &
\end{tikzcd}$$

$I = \{t \mid 0 \le t \le 1\} \in \mathbf{Top}_{CR}$

<u>Urysohn</u> $x, y \in X$ $x \neq y \Rightarrow \exists X \to I$ s.t. $f(x) \neq f(y)$

arrows: <u>defn</u> c is a <u>coseparator</u> in C iff

given $a \underset{t}{\overset{s}{\rightrightarrows}} b \xrightarrow{\exists f} c$ s.t. $fs \neq ft$

$s \neq t$

["<u>cogenerator</u>"]

<u>defn</u> c is a <u>generator</u> [<u>separator</u>]

$$x \xrightarrow{\exists f} a \underset{t}{\overset{s}{\rightrightarrows}} b \quad sf \neq tf.$$

$s \neq t$

Special Ajnt Ftr Th^m^

$$\begin{tikzcd} A \arrow[d, "G"] \\ X \end{tikzcd}$$

, ① $\mathcal{J}$ a class of cats, A $\mathcal{J}$-complete.

② G is $\mathcal{J}$ c^nts^.

③ $\forall a \in A\ \exists$ set indexed by $I \in \mathcal{J}$ $\begin{tikzcd} \cdot \arrow[r, "m_i"] & a \\ \cdot \arrow[u, dashed, "\cong_i"] \arrow[ur, "m"'] & \end{tikzcd}$ monos for every mono $\exists \cong$

A is $\mathcal{J}$ well-powered

④ A has a coseparator.

Then $\exists$ F ⊣ G.

⑤ X, A has hom sets in $\mathcal{J}$

---

defn f generates a

$$\begin{tikzcd} s \arrow[dr, "h"] & \\ & a \end{tikzcd} \quad \& \quad \begin{tikzcd} & Gs \arrow[d, "Gh"] \\ x \arrow[ur, dashed] \arrow[r, "f"'] & Ga \end{tikzcd} \qquad \begin{matrix} h\text{ mno} \\ \Downarrow \\ h \text{ iso} \end{matrix}$$

▷ "if you have cosep then you can get soln set"

SEE FREYD

KAN EXTENSIONS

$$\begin{tikzcd} C \arrow[drr, dashed, "?\ \text{extend}"] & & \\ \text{sbcat}\ \cup & & \\ M \arrow[rr, "T"'] & & A \end{tikzcd} \quad \longrightarrow\!\!\triangleright \quad \begin{tikzcd} C \arrow[rr, dashed, "?\exists\ \text{ext}\ S"] & & A \\ M \arrow[u, "\text{ftr}\ K"] \arrow[rr, "T"'] \arrow[ur, phantom, "\boxed{\Uparrow \alpha}"] & & A \end{tikzcd}$$

given K, T does there exist $\alpha : T \Rightarrow SK$ wh is universal

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\alpha'"'] & SK \arrow[d, dashed] \\ & S'K \end{tikzcd} \qquad \begin{tikzcd} S \arrow[d, dashed, "\exists !"'] \\ S' \end{tikzcd}$$

Then S, α is left Kan extension of T along K

②

Convert to adjnt sit from univ arr sit.

$$\begin{tikzcd}
 & A^{C} \arrow[d, "A^{K}"] & \ni & S \arrow[d, squiggly, "(\text{restrict})"] \\
T \in & A^{M} \arrow[u, dashed, bend left, "\text{leftadnt}"] & \ni & SK
\end{tikzcd}$$

$$Nt[\underbrace{Kan(T), S]}_{\text{To find.}} \cong Nt[T, SK^{-}] .$$

$K : M \longrightarrow C$ $\qquad (K, C)$ comma cat.

obj $\langle m, f : Km \longrightarrow c, c \rangle$

projections:

$$\begin{tikzcd}
 & \langle m, f : Km \to c, c \rangle \arrow[dl, squiggly] \arrow[d, squiggly] \arrow[dr, squiggly, "Q"] & \\
m & (f) & c
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, shift right, "L"'] \arrow[d, "T"'] \arrow[drr, "K"] & (K, C) \arrow[l, shift right, "P"'] \arrow[dr, "Q"] & \\
A & \Rightarrow & C \arrow[ll, "S"]
\end{tikzcd}$$

$PL = id$

$m \longmapsto \langle m, id : Km \longrightarrow Km, m \rangle$

$$Nt[T, SK] \cong Nt[TP, SQ]$$

thus a problem in comma cat.

$$\begin{tikzcd}[column sep=small]
\alpha : T \arrow[d, equal] \Rightarrow SK \arrow[d, equal] & & \beta : TP \Longrightarrow SQ \\
TPL \quad SQL & \arrow[l, squiggly] & \beta L : TPL \Longrightarrow SQL
\end{tikzcd}$$

③

So need only $g \circ \alpha \longrightarrow \beta$.

let $f = \langle m, f : Km \to c, c \rangle$

$\alpha : T \Rightarrow SK : M \longrightarrow A$

$(K, c) \longrightarrow A$

$\alpha_m : Tm \longrightarrow SKm$

$TP \qquad SQ$

diddle

Sichler ex. of adjnt ftr

$\boxed{\text{MacLane 18}}$

cat of <u>Relation</u>

$$\begin{tikzcd} X^{set} \arrow[d] & R \arrow[d] \arrow[r, phantom, "\subset"] & X \times X \arrow[d] \\ X' & R' \arrow[r, hook] & X' \times X' \end{tikzcd}$$

forgetful $\mathbb{R}\mathrm{el} \longrightarrow \mathbb{E}\mathrm{ns}$

$(X,R) \longmapsto X$

has adjnts on left + on right

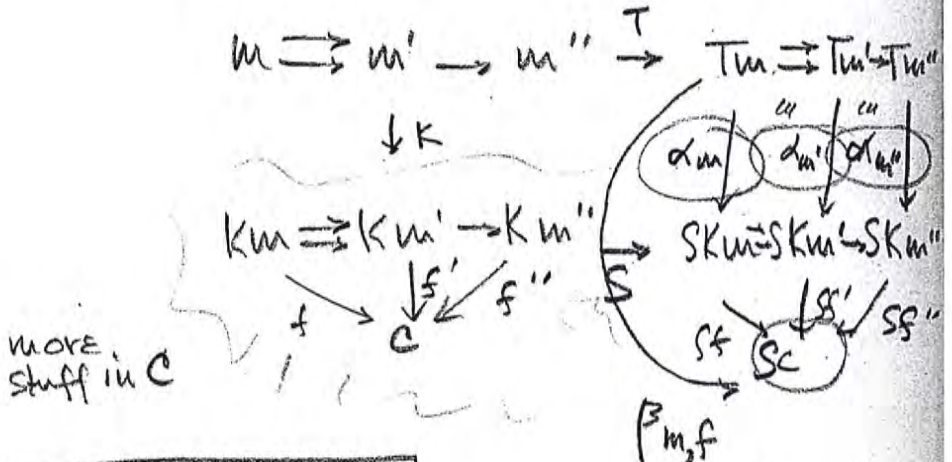
Also

$$\begin{tikzcd} \mathrm{Symm}\ \mathbb{R}\mathrm{el} \arrow[d] \\ \mathbb{R}\mathrm{el} \arrow[u, bend left=60] \arrow[u, bend right=60] \end{tikzcd}$$

from inside + outside

<u>Kan extension</u>

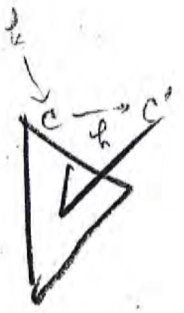
$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "K"'] \arrow[dr, phantom, "\Swarrow \alpha"] & A \arrow[d, phantom, "\|"] \\ C \arrow[r, dashed, "S"'] & A \end{tikzcd}$$



$\boxed{\langle m, f \rangle \in K/c}$

Define $\overline{S} = \sum_K T : C \longrightarrow A$ by

$$\begin{tikzcd}
 & \overline{S}c \arrow[d, dashed, "\exists!"'] & = & \underrightarrow{\text{colim}}\,(K/c \xrightarrow{P_c} M \xrightarrow{T} A) \arrow[d, "\text{if} \nearrow \text{exists}"'] \\
c \arrow[d, "h"'] & \overline{S}c & = & \underrightarrow{\text{colim}}\,(K/c' \xrightarrow{P_{c'}} M \xrightarrow{T} A) \\
c' & & &
\end{tikzcd}$$



$\underline{\text{colim}}$ $\underline{\text{cone}}$ $\overline{\beta}_{m,f} : TP_c\langle m,f\rangle \to \text{Colim}\, TP_c$

"POINTWISE COLIMIT"

$\overline{S}\,,\ \overline{\beta} : TP_c \Longrightarrow \overline{S}$

Have constructed $\overline{S}$ $\overline{\beta}$

$$\begin{tikzcd}
TP_c \arrow[r, Rightarrow, "\overline{\beta}"] \arrow[dr, "\text{univ.}"'] & \overline{S}c \arrow[d, dashed, "\exists! h"] \\
 & Sc
\end{tikzcd}$$

▷ h a nt xf.

Use Thm:

$$\begin{tikzcd}
M & (K, id_C) \arrow[l, "P"'] \arrow[d, "Q"] \\
 & C
\end{tikzcd}$$

$Nt[T, SK] \cong Nt[TP, SQ]$

$\alpha : T \Longrightarrow SK$ β

②

So: Th^m Given

$$\begin{tikzcd} & C \\ M \arrow[r, "T"'] \arrow[u, "K"] & A \end{tikzcd}$$

and ea c

$\exists$ colim $(K/c \longrightarrow M \longrightarrow A)$,

then T has lft Kan ext $\bar{S}$ over K

and $\bar{S}c = \operatorname{colim}(TP_c)$.

✓ α_m : colim cone @ $\langle m,$ ⌝

? Cor. M small & A sm cocplt $\Longrightarrow$?
C has sm homsets

---

lemma. $J \xrightarrow{H} A$, J has term obj T.

Then colim $H = HT$.

✓ pf.

$$\begin{tikzcd} i \arrow[dr] & & j \arrow[dl] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} Hi \arrow[dr] & & Hj \arrow[dl] \\ & HT & \end{tikzcd} \Bigg\} \text{univ.}$$

✓ Cor. If K full & fth then the universal $\alpha : T \Longrightarrow \bar{S}K$ $(\bar{S} = \Sigma_K T)$ is nt. isomp. !

calculate $\bar{S}Km = \operatorname{colim}(\underbrace{K/Km}_{Km \xrightarrow{f} Km} \longrightarrow M \xrightarrow{T} A)$

Take $f = id_{Km}$. use K full . fthfl .

How did Kan do this ?

EXERCISE . Dualize everything .

<u>lemma</u>. $Nt[\underset{\alpha}{T, SK}] \cong Nt[\underset{\lambda}{\underbrace{hom(Km, c) \times Tm}_{\coprod_{f \in hom(Km,c)} Tm}}, Sc]$

$$\begin{tikzcd} G : M^{op} \times M \times C \arrow[r] \arrow[d, Rightarrow, "\lambda"'] & A \\ C \arrow[ru] & \end{tikzcd}$$

sup ut×f

---

prf. given α make λ .

$$\begin{tikzcd} \coprod_{f \in hom(Km,c)} Tm \arrow[rr, dashed, "\lambda_{m,c}"] & & Sc \\ Tm \arrow[u, "i_f"] \arrow[rr, dashed] \arrow[rru, "///"] \arrow[rd, "\alpha_m"'] & & Sc \\ & SKm \arrow[ru, "Sf"'] & \end{tikzcd}$$

show λ sp. nt xf :

$$Km \xrightarrow{Kh} Km' \xrightarrow{f'} c$$

$$\begin{tikzcd} m \arrow[d, "h"] \\ m' \end{tikzcd}$$

$$\begin{tikzcd}
\coprod_{f'\in(Km',c)} Tm \arrow[rrrr, "\exists!"] \arrow[dddd, "\cong"] & & & & \coprod_{f\in(Km,c)} Tm \arrow[dddd] & \\
& Tm \arrow[ul, "\iota_{f'}"'] \arrow[rr, equal, "\text{NTL of }\alpha"'] \arrow[d, "Th"'] \arrow[dr, phantom, "\cong"] & \arrow[u, mapsto] & Tm \arrow[ur, "\iota_{f'}\circ Kh"] \arrow[d, "\alpha_m"] \arrow[dr, phantom, "\text{defn }\lambda_{m,c}"] & & \\
& Tm' \arrow[r, "\alpha_{m'}"'] \arrow[ddl, "\iota_f"] & SKm' \arrow[ddrr, "Sf'"] & SKm \arrow[l, "SKh"'] \arrow[ddr, "Sf'"] \arrow[drr, "\lambda_{m,c}", "\boxed{SKh}"'] & & \\
& & & & & SKm' \arrow[dl, "Sf"] \\
\coprod_{f'\in(Km',c)} Tm' \arrow[rrrr, "\text{defn }\lambda_{m',c}", "\lambda_{m',c}"'] & & & & Sc &
\end{tikzcd}$$

▷

⟹ Make α from λ.

Thus to get Kan ext need univ gadget.

$$M^{op} \times M \xrightarrow{H} \underset{\underset{a}{\rotatebox{90}{$\in$}}}{A}$$

$\lambda : H \Longrightarrow a$ sp. nt. xf.

$\lambda_m : H(m,m) \longrightarrow a$

defn. e is <u>coend</u> of H .

$$\begin{tikzcd} e, \bar{\lambda} & H \arrow[r, Rightarrow, "\bar{\lambda}"] & e \arrow[d, dashed, "\exists! \text{ in } A"] \\ & \forall \lambda \arrow[r] & a \end{tikzcd}$$

sp nt.

$$\begin{tikzcd} & H(m',m) \arrow[dl, "h^*"'] \arrow[dr, "h_*"] & \\ H(m,m) \arrow[d, "\lambda_m"'] \arrow[drr] & & H(m',m') \arrow[dll, "\lambda_{m'}"'] \arrow[d] \\ a & & e = \int^{m} H(m,m) \arrow[ll, dashed, bend left, "\exists"] \end{tikzcd}$$

<u>Day - Kelley</u>

<u>Yoneda</u>

$\bar{S}c = \int^{m} \coprod Tm$ coend of a coproduct.

---

objs M $\mathring{\cup}$ orrs M

M twiddles & H twiddles

$$\int^{m} H(m,m) = \text{colim-}(\tilde{H} : \tilde{M} \longrightarrow A)$$

$$\begin{tikzcd} C \arrow[dr, dashed, "S"] & \\ M \arrow[u, "K"] \arrow[r, "T"'] \arrow[ur, phantom, "\Uparrow\alpha"] & A \end{tikzcd}$$

$$T \overset{\alpha}{\Longrightarrow} SK$$

Thm
$$Nt(T,SK) \overset{\exists \text{bij}}{\cong} \hom_A\Big(\overbrace{\hom(Km,c)\times Tm}^{\coprod_{(Km,c)} Tm}, Sc\Big)$$

pf.
$$\prod_{(Km,c)} \hom_A(Tm,Sc) = \hom\big(\hom(Km,c), \hom(Tm,Sc)\big) \quad \cong \text{adj}$$

$$\big\Vert \ \text{Yoneda}$$

fix m
$\hom(Km,-) \to \hom(Tm,S-)$

$$\hom(Tm, SKm)$$

TRIADS

$$\varepsilon : FG \Longrightarrow id_A$$

$$\begin{tikzcd} A \arrow[d, shift left, "G"] \\ X \arrow[u, shift left, "F"] \arrow[loop below] \end{tikzcd}$$

$$\eta : id_X \Longrightarrow \underset{\substack{\Vert \\ T}}{GF}$$

$$T = GF$$

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[rr, Rightarrow, bend right] & FGF \arrow[r, Rightarrow, "\varepsilon F"] & F \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[rr, Rightarrow, bend right] & GFG \arrow[r, Rightarrow, "G\varepsilon"] & G \end{tikzcd}$$

$G\varepsilon F$

$$\underset{\substack{\Vert \\ T^2}}{TT = GFGF} \longrightarrow \underset{\substack{\Vert \\ T}}{GF}$$

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d] \\ T^2 \arrow[r] \arrow[ur, phantom, "\text{Assoc}"] & T \end{tikzcd}$$

$\boxed{\text{ident}}$

defn a <u>triad</u> T, η, μ, X is an endofr + two nt xfs s.t. . . . ie <u>a monoid object in X^X</u> wh multiplication is composition of functors.

Godement
<u>Huber</u>
Kleisli
Eilenberg-Moore } <u>were asked</u> <u>the</u> <u>right question</u> (by Hilton)

Given T, η, μ in X define a <u>T-algebra</u> is

$$\langle x, \underset{\text{"action"}}{h : Tx \longrightarrow x} \rangle \text{ s.t.}$$

(i)

$$\begin{tikzcd}
TTx \arrow[r, "Th"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "h"] \\
Tx \arrow[r, "h"'] & x
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, equal] & Tx \arrow[d, "h"] \\
& x
\end{tikzcd}$$

~~Comma cat.~~

Comparison lemma.

$$\begin{tikzcd}
 & A \arrow[r, "K"] \arrow[d, shift left, "G"] & A' \arrow[d, shift left, "G'"] & \\
\varepsilon, \eta & X \arrow[u, shift left, "F"] \arrow[r, "L"'] & X' \arrow[u, shift left, "F'"] & \varepsilon', \eta'
\end{tikzcd}$$

$KF = F'L$
&
$LG = G'K$

Then the foll are equ :

1°
$$\begin{tikzcd}
\hom(Fx, a) \arrow[r, phantom, "\cong"] \arrow[d, "K"'] & \hom(x, Ga) \arrow[d, "L"] \\
\hom(KFx, Ka) \arrow[d, phantom, "\shortparallel"] & \hom(Lx, LGa) \arrow[d, phantom, "\shortparallel"] \\
\hom(F'Lx, Ka) \arrow[r, phantom, "\cong"] & \hom(Lx, GKa)
\end{tikzcd}$$

2° $L\eta = \eta' L$

3° $K\varepsilon = \varepsilon' K$

IS THIS AN ADJ SQU?

$\boxed{\text{Yes, a special case}}$

Freyd Seminar T W Th 7^{30}
here

Maclane 20

given $F : P^{op} \times P \longrightarrow A$; $c \in A$.

defn *tent vertex c over F* is $c \xrightarrow{t_p} F(p,p)$ for ea p

and for ea $p \xrightarrow{h} q$

then have $c \xrightarrow{t_{p,q}} F(p,q)$
$\forall p,q$ + $p \xrightarrow{h} q$

$$\begin{tikzcd}
c \arrow[r, "t_p"] \arrow[d, "t_q"'] & F(p,p) \arrow[d, "\hom(p,h)"] \\
F(q,q) \arrow[r, "\hom(h,q)"'] & F(p,q)
\end{tikzcd}$$

The *end* of F is the *universal tent* .

$$\text{End of } F = \int_p F(p,p) \xrightarrow{u_p} F(p,p)$$

---

exercise. prove, if $F, G : A \longrightarrow D$ then

① ✓ $Nt(F,G) = \int_a \hom_D(Fa, Ga)$

② (Yoneda) if $G : A^{op} \longrightarrow c$ then

$$\int^b \hom(a,b) \times G(b) = G(a)$$

③ $F : A \longrightarrow C \Rightarrow \int^a \hom(a,b) \times F(a) = F(b)$

④

$$\begin{tikzcd} & C \arrow[dr, dashed, "\bar{S}"] & \\ M \arrow[ur, "K"] \arrow[rr, "T"'] & & A \end{tikzcd}$$

Right Kan extension

$$T \Longleftarrow SK$$

✓ not.: $S = \lceil_K T$.

$$(\lceil_K T)c = \int_m Tm^{\hom(c,Km)} \quad \left(a^X = \prod_X a\right)$$

---

exercise. Show that (under suitable conditions) a left adjunt can be repr. as a Right Kan ext.

ie given

$$\begin{tikzcd} A \arrow[d, "G"] \\ X \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dr, dashed] & \\ A \arrow[ur, "G"] \arrow[rr, "id_A"'] & & A \end{tikzcd}$$

$$F = \lceil_G (id_A)$$

$$\begin{tikzcd} A \arrow[d, shift left, "G"] \\ X \arrow[u, shift left, "F"] \end{tikzcd}$$

$\varepsilon : FG \Longrightarrow id$

adjn.

$\eta : id \Longrightarrow GF$

$GF = T : X \longrightarrow X$

$\eta : I \longrightarrow T$

$\mu : T^2 \longrightarrow T$

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

"T-algs" X^T | ob $\langle x, h : Tx \xrightarrow{\text{structure map}} x \rangle$ s.t. □ ▼

arr

$$\begin{tikzcd} \langle x, \arrow[d] & h : Tx \arrow[d] & \cdots & x \rangle \arrow[d] \\ \langle x', & h' : Tx' & \longrightarrow & x' \rangle \end{tikzcd}$$

$$\begin{tikzcd}
\langle Tx, T^2x \xrightarrow{\mu} Tx\rangle & X^T \arrow[d, bend left, "G^T"] \\
x \arrow[u, squiggly] & X \arrow[u, bend left, "F^T"]
\end{tikzcd}$$

unit $\eta^T = \eta$

counit $\varepsilon^T_{\langle X, h\rangle} = h$ the str. map.

$$\begin{tikzcd}
A \arrow[r, "?K"] \arrow[d, dash] & X^T \arrow[d, dash] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}$$

$\boxed{\varepsilon^T K = K\varepsilon}$ expresses that K is mp of adjus.

$$\begin{tikzcd}
a \arrow[r, squiggly, "K"] \arrow[d, "f"'] & \langle Ga, \quad ? \quad \rangle \arrow[d, "f"'] \\
b & \langle Gb, \quad \quad \rangle
\end{tikzcd}$$

$$\begin{tikzcd}
Ga \arrow[d, "Gf"'] \\
Gb
\end{tikzcd}$$

we have $Kf = Gf$, but $? = \varepsilon^T_{\langle Ga, ?\rangle} = \varepsilon^T_{Ka} = K\varepsilon_a = G(\varepsilon_a)$.

▷ Check that its the right thing.

Examples. Let G a fixed group. A <u>G-action</u> is set X & $G \times X \longrightarrow X$ ... So G-$\mathbb{A}$ct is cat of G-actions.

$$\begin{tikzcd}
G\text{-}\mathbb{A}\mathrm{ct} \arrow[d, shift left, "U"] \\
\mathbb{E}\mathrm{ns} \arrow[u, shift left, "F"]
\end{tikzcd}$$

a T-alg is ..... a set w an action.

Same for
$$\begin{tikzcd}
R\text{-}\mathbb{M}\mathrm{od} \arrow[d, dash] \\
\mathbb{A}\mathrm{b}
\end{tikzcd}$$

$$\begin{tikzcd}
FX & \mathbb{S}\mathrm{mgp} \arrow[d, shift left, "U"] \\
X & \mathbb{E}\mathrm{ns} \arrow[u, shift left]
\end{tikzcd}$$

FX = free smgp generators $x \in X$; elmts are "words" $\langle x_1, \ldots, x_n \rangle \in X^{n \geq 1}$

$\langle x_1, \ldots, x_n \rangle \langle y_1, \ldots, y_m \rangle =$
$\langle x_1 \ldots x_n y_1 \ldots y_m \rangle$.

Product by juxtaposition. $\eta : \underset{(X \longrightarrow UFX)}{x \longmapsto \langle x \rangle}$.

$\varepsilon : FGS \longrightarrow S \quad \langle s_1 \rangle \langle s_2 \rangle \cdots \langle s_n \rangle \rightsquigarrow s_1 \cdots s_n$

Triad: ......

What is a T-alg for Smgp? It will be

set X & str map

$$\begin{tikzcd} & TX \arrow[r, "h"] \arrow[d, equal] & X \\ & \dot{\bigcup}_{n\geq 1} X^n \arrow[r, "h"] & X \end{tikzcd}$$

thus str map "is" a fam of mps $h_n : X^n \longrightarrow X$.

⋮

" Total description of a semigrp "

Look at <u>all</u> operations generated by given operations.

For a G-act can give action by action of generators

$$\begin{tikzcd} R\text{-Mod} \arrow[d, shift left, "G"] \\ \text{Ens} \arrow[u, shift left] \arrow[loop right, "T"] \end{tikzcd}$$

TX = set of mps X→R finitely non-zero

$= \text{hom}_{fin}(X, R)$

$= \{ t = \sum t_x \langle x \rangle \quad \}$

$X \xrightarrow{\text{unit}} TX : x \mapsto \langle x \rangle$

add
mult.

$$T^2X \xrightarrow{\mu} TX \; : \; \sum_t u_t \left\langle \sum [illegible] \langle x \rangle \right\rangle \rightsquigarrow$$

WHAT IS A T-ALG?

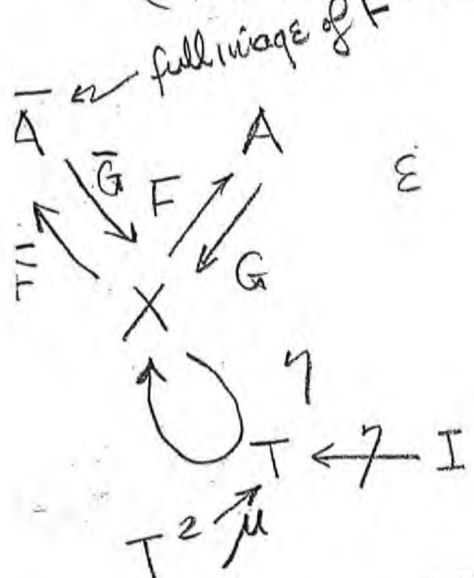
It's a set X & str. map $h : \underset{\substack{\| \\ \text{fin formal} \\ \text{lin combs.}}}{TX} \longrightarrow X$

h adds em up.

Its not much to think of.

$\boxed{\text{DUBUC } 4^{\underline{IS}}}$ $\boxed{\text{MacLane 21}}$

(Kleisli construction)

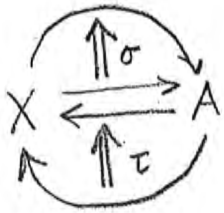


$\bar{G} = G|\bar{A}$.

$Th^{\underline{m}}$ Given triad T, η, μ in X, it can be realized by adjt pair $X \underset{G}{\overset{\bar{F}}{\rightleftarrows}} \bar{X}$ where F is 1-1 on objects

(Comparison) If $A \rightleftarrows X$ is any other realization $\exists!\ \bar{X} \longrightarrow A$ wh is mp of adjs.

$$\left[\begin{tikzcd} \bar{X} \arrow[rr, dashed, "\exists!"] \arrow[dr, shift left] & & A \arrow[dl, shift left] \\ & X \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}\right]$$



Every realization of triple is between Kleisli & Eil-Moore

~~question~~ ~~qu~~ does Every cat of adjts (over a cat) ~~wt~~ w in & term obj a cat of realiz of a triple cat of real of a triple have in & term obj's

free fctr of in obj is isomp

$$\begin{tikzcd} \bar{X} \arrow[r, "\exists!"] \arrow[dr, equal] & A \arrow[r, "\exists!"] \arrow[d, equal] & X^T \arrow[dl, equal] \\ & X & \end{tikzcd}$$

3rd Kleisli

defn X_T | objs $\bar{x}$ for $x \in X$

arr: $\hom(\bar{x}, \bar{y}) = \left\{ k \mid k : Tx \to Ty \text{ s.t. } \right.$

$$\begin{tikzcd}
T^2x \arrow[r, "Tk"] \arrow[d, "\mu_x"'] \arrow[dr, phantom, "///"] & T^2y \arrow[d, "\mu_y"] \\
Tx \arrow[r, "k"'] & Ty
\end{tikzcd} \left. \right\}$$

$$\begin{tikzcd}
X_T \arrow[dr, shift left, "G_T"] & \bar{x} \arrow[r, squiggly, mapsto] & x \\
& X \arrow[ul, shift left, "F_T"] & k \arrow[r, squiggly, mapsto] & k
\end{tikzcd}$$

NOTE: THIS IS NOT NT.

exercise, Find total description of comm. rings and of rings.

Affine modules a set M & $\forall\, k_1, \dots, k_n$ $\sum k_i = 1$

$\exists$ operation $M^n \longrightarrow M$

$$x_1, \dots, x_n \rightsquigarrow \sum k_i x_i$$

$$1x = x$$

(Beck) PET Th$^{\underline{m}}$ (Precise Equivalent Triad)

" When is comparison K an equivalence ? "

$A \in \mathbb{A}b$. have

$$\begin{tikzcd} FGFGA \arrow[r, shift left] \arrow[r, shift right] & FGA \arrow[r, "epi"] & A \arrow[r] & 0 \end{tikzcd} \Big\}\ \text{sets}$$

" To get a canonical resolution of A "

In cat A say $a \overset{f}{\underset{g}{\rightrightarrows}} a'$ is a parallel pair.

say $a \overset{f}{\underset{g}{\rightrightarrows}} a' \xrightarrow{k}$ s.t. $kf = kg$.

is a fork.

An exact fork is a fork wh is coequ.

A split fork is
$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & a' \arrow[r, "k"] \arrow[l, bend left, "t"] & b \arrow[l, bend left, "s"] \end{tikzcd}$$
s.t. $\begin{cases} ks = 1 \\ ft = 1 \\ gt = sk \end{cases}$

$\boxed{\text{a certain animal.}}$

▷ lemma. Every split fk is ex. fk.

[split fks are "absolute coequalizers"]

---

example. X^T T-alg $\langle x, h : Tx \longrightarrow x \rangle$

$$\begin{tikzcd} Tx \arrow[r, "h"] & x \\ T^2x \arrow[u, "Th"] \arrow[r, "\mu"'] \arrow[ur, phantom, "\cdots"] & Tx \arrow[u, "h"'] \end{tikzcd}$$

$\langle Tx, \mu \rangle \xrightarrow{h} \langle x, h \rangle$ is a T-alg map

Have

(*) $$\begin{tikzcd} \langle T^2 x, \mu_{Tx} \rangle \arrow[r, shift left, "\mu_x"] \arrow[r, shift right, "Th"'] & \langle Tx, \mu \rangle \arrow[r, "h"] & \langle x, h \rangle \end{tikzcd}$$

why is μ_x a T-alg map?

$$\begin{tikzcd} T^3 x \arrow[r, "\mu_{Tx}"] \arrow[d, "T\mu_x"'] \arrow[dr, phantom, "''"] & T^2 x \arrow[d, "\mu_x"] \\ T^2 x \arrow[r, "\mu"'] & Tx \end{tikzcd}$$ by defn of Tria

This (*) is a fork in X^T.

$$\begin{tikzcd} X^T \arrow[d, "G^T \text{ forgetful}"] \\ X \end{tikzcd}$$

G^T applied to fk * gives fk in X :

▷ $$\begin{tikzcd} T^2 X \arrow[r, shift left, "\mu_x"] \arrow[r, shift right, "Th"'] & Tx \arrow[r, "h"] \arrow[l, bend left=50, "\eta_{Tx}"] & x \arrow[l, bend left=50, "\eta_x"] \end{tikzcd}$$ $\left\{ \begin{array}{l} h\eta_x = 1 \\ \underline{\qquad} \\ \underline{\qquad} \end{array} \right.$

<u>claim</u> This is split fk.

(*) <u>Canonical</u> <u>presentation</u> <u>of</u> <u>T-alg</u> .

$$\begin{tikzcd}
a \in A \arrow[rr, shift left, "K"] \arrow[dr, shift left, "G"] & & X^T \ni \langle x, h \rangle \arrow[ll, dashed, shift left, "\check{K}"] \arrow[dl, shift left, "G^T"] \\
& X \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] \arrow[loop right, "T, \eta, \mu"] &
\end{tikzcd}$$

$F \dashv G$

$$Ka = \langle Ga, GFGa \xrightarrow{\varepsilon_{Ga}} Ga \rangle .$$

When is K an $\left\{\begin{matrix}\text{equivalence} \\ \text{isomorphism}\end{matrix}\right\}$ of cats?

So ask rather, first, has K left adjt?

$$\hom_{X^T}(\langle x, h \rangle, Ka)$$

$$g \rightsquigarrow g^\flat = \{ f \mid f : Fx \to a \ \&$$

f coequalizes

$$\begin{tikzcd}
FGFx \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{Fx}"'] & Fx \arrow[r] \arrow[dr, "f"'] & \check{a} \arrow[d, dashed] \\
& & a
\end{tikzcd} \}$$

$$\rightsquigarrow \{ k \mid k : \check{a} \to a \text{ s.t. } \check{a} = \text{coequ} \}$$

So assume A has coequalizers.

$$\hom_A(\ ? \ , a)$$

$$\begin{tikzcd}
\langle x, & h: Tx \arrow[r] \arrow[d, "Tg"'] \arrow[dr, phantom, "="] & x \rangle \arrow[d, "g"] \\
\langle Ga, \arrow[from=u, "g"'] & \varepsilon_{Ga}: GFGa \arrow[r] & Ga \rangle
\end{tikzcd}$$

$$g \circ h = G\varepsilon_a \cdot Tg$$
$$= \varepsilon_{Ga} \cdot GFg$$
$$\in \hom_X(x, Ga)$$
$$\cong$$
$$\hom_A(Fx, a)$$

$$(gh)^b = [G(\varepsilon_a \cdot Fg)]^b$$
$$\|$$
$$g^b \circ Fh = (\varepsilon_a \circ Fg)\varepsilon_{GFx}$$
$$\|$$
$$\vdots$$

so define $\check{K}\langle x, h\rangle = \check{a}$.

$$\boxed{\text{Thus if A has coeqns K has a left adjnt.}}$$

$$\begin{tikzcd}
Fx, a \arrow[loop above, "1"] & x, Ga \arrow[loop below, "1"]
\end{tikzcd}$$

$1^{\#} = \eta$

$1^b = \varepsilon$

Now want to know when this adjnt pair is an adjnt equiv.

want $\left\{ \begin{matrix} I \xrightarrow{\bar{\eta}} K\check{K} \\ \check{K}K \xrightarrow{\bar{\varepsilon}} I \end{matrix} \right\}$ both isomps.

Assume G reflects coequs

"G tests for coequs"

So what have we got ?

Th$^{\underline{m}}$ If $A \underset{G}{\overset{F}{\leftrightarrows}} G$ adjt pair, G is CTT{

every pap in A w coequ in X has coequ in A

and G pres & refl coequs.

Then the comparison K is an equivalence of cats.

But more precisely, the only coequs necessary were of these:

$$\boxed{\begin{tikzcd} FGFx \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{Fx}"'] & Fx \end{tikzcd}}$$

ie forks in A wh G image is split fork.

Th$^{\underline{m}}$ Given $F \dashv G$ + comparison K
consider in A pops $\underset{g}{\overset{f}{\rightrightarrows}}$ wh are "split by G"

[ie $\underset{Gg}{\overset{Gf}{\rightrightarrows}}$ $\longleftarrow$ embedding in spl't-fk]

① If A has coequalizers of all G-split pops, then K has a left adjnt

② If G preserves coequs of same stuff then unit : $I \longrightarrow K\check{K}$ is isomp.

③ If G reflects coequs of same stuff then counit $\check{K}K \longrightarrow I$ isomp.

" If A has + G pres + refl coequs of G-split pops then K is an equivalence of cats "

Precise Isomoph Th^m for Triads

Maclane 23

Given
$$\begin{tikzcd} A^{\varepsilon} \arrow[d, shift left, "G"] \\ X^{\eta} \arrow[u, shift left, "F"] \arrow[loop right, "{T, \eta, \mu = G\varepsilon F}"] \end{tikzcd}$$
$F \dashv G$

Construt
$$\begin{tikzcd} X^{T} \arrow[d, shift left, "G^{T}"] \\ X \arrow[u, shift left, "F^{T}"] \end{tikzcd}$$
T-algs.

Want K is an isomorphism iff G creates coequalizers of all parallel pairs split by G.

split fork
$$\begin{tikzcd} {} \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & {} \arrow[r, "e"] \arrow[l, bend left, "t"] & {} \arrow[l, bend left, "s"] \end{tikzcd}$$
s.t.
$$\begin{cases} ef' = eg' \\ es = 1 \\ f't = 1 \\ g't = se \end{cases}$$

$$\begin{tikzcd} {} \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & {} \arrow[r, dashed, "e"] \arrow[l, dashed, bend left] & {} \arrow[l, dashed, bend left] \end{tikzcd}$$

G creates coequ of $\overset{f}{\underset{g}{\rightrightarrows}}$ means given $\overset{Gf}{\underset{Gg}{\rightrightarrows}} \xrightarrow{e}$ etc.

$$\check{K}\langle x, h\rangle = \check{a} \xleftarrow{\check{K}} X^{T} \quad \langle x, h : Tx \longrightarrow x\rangle$$

$$\begin{tikzcd} FGFx \arrow[r, shift left, "\varepsilon Fx"] \arrow[r, shift right, "Fh"'] & Fx \arrow[r, dashed, "\overset{\exists!}{e}"] & \overset{\exists!}{\check{a}} \end{tikzcd}$$

$$\left.\begin{tikzcd} \langle T^2x, \mu_{Tx}\rangle \arrow[r, shift left, "\mu_x"] \arrow[r, shift right, "Th"'] & \langle Tx, \mu_x\rangle \arrow[r, "h"] & \langle x, h\rangle \end{tikzcd}\right\} \text{in } X^{T}$$

F ↖ ⟝ $\qquad$ ↙ G^{T}

$$\begin{tikzcd} T^2x \arrow[r, shift left, "\mu_x"] \arrow[r, shift right, "Th"'] & Tx \arrow[r, "h"] \arrow[l, bend left, "\eta_{Tx}"] & x \arrow[l, bend left, "\eta_x"] \end{tikzcd}$$

split fk

$Ge = h$

$G\check{a} = x$

(To chk isomph): ① $K\check{K}a = a$ $\quad [x = Ga]$

$$\begin{tikzcd}
 & & & \langle Ga, G\varepsilon_a : GFGa \longrightarrow Ga\rangle \arrow[dl, mapsto] \\
FGFGa \arrow[r, shift left] \arrow[r, shift right] \arrow[d, leftrightarrow] & FGa \arrow[r, dashed, "e_a"] & a & \\
GFGFGa \arrow[r, shift left] \arrow[r, shift right] & GFGa \arrow[r, "G\varepsilon_a"] & Ga &
\end{tikzcd}$$

② $\check{K}K\langle x, h\rangle =$

$\mathbb{R}$ $a<b$
iff
$\exists p \neq 0, a+p=b$
$\forall a,b$ $a<b$ or $a=b$ or

$$K\check{a} = \langle G\check{a}, G\varepsilon_{\check{a}} : GFG\check{a} \longrightarrow G\check{a}\rangle$$

$$\overset{\textstyle \langle\, x}{\shortparallel}$$

need $G\check{a} = h$. happy to show $e = \varepsilon_{\check{a}}$.

$$e : Fx \longrightarrow \check{a}$$

use formula for # in adjunction $F \dashv G$

$$e^{\#} = Ge \circ \eta_x = h \circ \eta_x = 1 \text{ by split fork}$$

so $e^{\#} = 1 \Rightarrow e = \varepsilon_{\check{a}}$.

Now must show K isomp $\Rightarrow$ ..... .

Tierney $3^{\underline{00}}$ M

MacLane 241

$J \xrightarrow{D} K$ <u>initial ftr</u> iff for any $K \xrightarrow{T}$,
$\lim T \longrightarrow \lim TD$ <u>isomp</u> .

EXERCISE. Show D is initial iff for ea $k \in K$
D/k is non-empty and "connected"

(PITT significance): When is $A \xrightarrow{G} X$ und ftr for cat of alg's of some triple? Precisely when G has left adjt, and G creates coeqs of paps split by G.

Application: Compact Haus sp.

$$\begin{tikzcd} \mathbb{CH} \arrow[d, shift left, "G"] \\ \mathrm{Ens} \arrow[u, shift left, "\rho"] \end{tikzcd}$$

<u>Claim</u> This is PITT. Left adjt is just Stone-Čech compactification. So left to show creation.

CH $$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$ need ! top W s.t. e $c^{\underline{nts}}$

Ens $$\begin{tikzcd} {|X|} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & {|Y|} \arrow[r, "e"] \arrow[l, bend left, "t"] & W \arrow[l, bend left, "s"] \end{tikzcd}$$ $\left\{\begin{array}{l} es=1 \\ ft=1 \\ gt=se \\ ef=eg \end{array}\right.$

Paré : lemma. $X^{cpt} \xrightarrow{f^{c\underline{nts}}} Y^{Haus} \Rightarrow f^{cl}$

" $c^{\underline{nts}}$ & cl " means $f(\bar{S}) = \overline{f(S)}$
"commutes w closure".

A top on X is a closure operation on $\mathcal{P}(X)$

$$\left|\begin{array}{l} \bar{\emptyset} = \emptyset \\ \bar{S} \supset S \\ \bar{\bar{S}} = \bar{S} \\ \overline{S \cup T} = \bar{S} \cup \bar{T} \end{array}\right.$$

Any ftr applied to e gives e' coequ.

$\mathcal{P}$ is a ftr.

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left, "\mathcal{P}f"] \arrow[r, shift right, "\mathcal{P}g"'] \arrow[d, "cl"'] & \mathcal{P}Y \arrow[r, "\mathcal{P}e^{\ coequ}"] \arrow[d, "cl"'] & \mathcal{P}W \arrow[d, dashed, "\exists!\,(\bar{\ })_W"] \\ \mathcal{P}X \arrow[r, shift left, "\mathcal{P}f"] \arrow[r, shift right, "\mathcal{P}g"'] \arrow[ur, phantom, "'''"] & \mathcal{P}(Y) \arrow[r, "\mathcal{P}e", "coequ"'] & \mathcal{P}W \end{tikzcd}$$

▷ <u>claim</u> $(\overline{\ \ })_W$ is a closure operation.

$\therefore$ W is top. sp. & $Y \xrightarrow{e} W$ is $c^{\underline{nts}}$ cl.

Need to show W $\underbrace{\text{cpt}}$ & Hsdf.

$c^{\underline{nts}}$ image of cpt is cpt.
e is a surjection.

let $w_1, w_2 \in W$ $\quad e^{-1}(w_1)^{cl} \cap e^{-1}(w_2)^{cl} = \emptyset$

~~[illegible]~~

$U_i \supset e^{-1}(w_i)$ $\quad U_i$ op & disj

take complements U_1' U_2' ———

PITT - Birkhoff: varietal cat. Then S full sbcat is varietal iff $\to$ {closed under sm prod, sub-obj, quot}

$X = \mathcal{E}ns$ $\quad T$ = triad in $\mathcal{E}ns$ $\quad A$ full sbcat $\subset X^T$

~~[illegible]~~ Then $A \xrightarrow{G|A} X$ makes $A \cong$ some $X^{T'}$.

iff
1° A cl under prod
2° " " " subobj
3° " " " quots

TW Alg Th
Th Dub cat
F Found

Functorial Pasting?

MacLane 25

Defn An algebraic theory is a category A | objs $1, a, a^2, a^3, \ldots$

& ea a^n "given as a product of n a's"

ie $\exists\ a^n \xrightarrow{p_i} a \quad i=1,\ldots,n$ s.t. a^n is a product

$$\left[\therefore\ a^{n+m} = a^n \times a^m\right]$$

defn An arr $\omega : a^n \longrightarrow a$ is n-ary operation.

Any other arr

$$\begin{tikzcd} a^n \arrow[r, "\alpha"] \arrow[dr, "p_i\alpha"'] & a^m \arrow[d, "p_i"] \\ & a \end{tikzcd} \quad i=1,\ldots,m$$

Given m n-ary ops $a^n \xrightarrow{\omega_i} a$ $\exists!\ a^n \longrightarrow a^m$ denote $\langle \omega_1, \ldots, \omega_n \rangle^{\#}$.

Thus any other arr is an m-tuple $\langle \omega_1, \ldots, \omega_n \rangle^{\#}$ of n-ary ops.

For $f : m \longrightarrow n$ in fin sets there is unique induced

$$\begin{tikzcd} a^n \arrow[r, "f^*"] \arrow[d, "p_{f(i)}"'] & a^m \arrow[d, "p_i"] \\ a \arrow[r, equal] & a \end{tikzcd} \quad i=1,\ldots,m$$

.

example. $\mathbb{F}\mathrm{in}^{op}$ is an alg th.

Prop. For any alg th A, $\exists \mathbb{F}\mathrm{in}^{op} \longrightarrow A$ embeds.

Prop. An alg th A is a cat cont $\mathbb{F}\mathrm{in}^{op}$

same objs
same products

.

---

example. th of a monoid.

$(xy)z = x(yz)$
$\exists \{(xy)z\} \xleftrightarrow[\text{bij}]{} \{x(yz)\}$.

Say T triad in Ens.

T "has rank $\aleph_0$" means: consider all fin subsets of arb set X. Claim $TX = \operatorname{colim}_{y \overset{fin}{\subset} X} Ty$.

Formal identities in th A

$a^n \xrightarrow{\omega} a \quad f: k \to n \quad f^*: a^n \to a^k$

takes as objs the elmts
takes as maps the equalities $a=b$

M a set $\quad M\times M \xrightarrow{\mu} M$ a bin mult.

$\underline{M}$ cat | objs all bin parenthesized words of elmts of M

~~[illegible]~~

$\hom(X,Y) = \begin{cases} \text{the equality } X=Y \text{ iff } "\mu"X = "\mu"Y \\ \emptyset \text{ otherwise} \end{cases}$

assertion: μ is associative iff $-xy: \underline{M} \to \underline{M}$ ~~and~~ is right adjt to $yx-: \underline{M} \to \underline{M}$

① $f^*\langle \omega_1, \dots, \omega_k\rangle^{\#} = \langle \omega_{f_1}, \dots, \omega_{f_k}\rangle^{\#}$

pf. $\quad a^n \longrightarrow$

② $\langle \omega_1 \circ f_1^*, \dots, \omega_n \circ f_m^* \rangle^{\#} =$ something.

Th of a monoid

$$a^n \xrightarrow{\omega_n} a \qquad x_1 \ldots x_n \mapsto \text{their prod}$$

A, B th's. An arr $A \xrightarrow{H} B$ is a ftr s.t.

$Ha = b$
$Ha^n = b^n$
$Hp^i = \text{"}p^i\text{"}$

id on obj's
and for any $f : m \to n$
w $f^* : a^n \to a^m$
then $Hf^* = \text{"}f^*\text{"}$

Given a th A; an A-alg X is a prod pres ftr $X : A \to \mathrm{Ens}$

$$\begin{tikzcd}
a^0 = 1 \arrow[r, mapsto] & \{\bullet\} \\
a^1 \arrow[r, mapsto] & X = |X| \text{ und. set of A-alg} \\
a^2 \arrow[u, shift left] \arrow[u, shift right] & X^2 \arrow[u, shift left] \arrow[u, shift right] \\
a^n \arrow[r, mapsto] &
\end{tikzcd}$$

A th. Cat of all A-algs are the

$$\begin{tikzcd}
\underbrace{\mathrm{FUN}_\pi(A, \mathrm{Ens})}_{\text{prod pres ftrs}} \subset \mathrm{Ens}^A \arrow[r, "\text{fgt}"] & \mathrm{Ens} \\
& |X|
\end{tikzcd}$$

Semantics takes Th to its cat of models

$\boxed{\text{MacLane 26}}$

Alg Th's Structure is Left Adjt To Semantics

A th is cat | objs $\overset{\text{term.}}{1}, a, a^2, \dots, a^n, \dots$

$\exists\ a^n \longrightarrow a$ proj's

& they make a^n a prod

$\mathbb{F}\text{in}^{op} \hookrightarrow A$ always

mp of th's is prod pres ftr s.t. H(gen obj) = gen obj.

Or,

$$\begin{tikzcd}
 & \mathbb{F}\text{in}^{op} \arrow[dl] \arrow[dr] & \\
A \arrow[rr] & & B
\end{tikzcd}$$

Example.

$$\begin{tikzcd}
C \arrow[d, shift left, "G"] \\
\mathcal{E}\text{ns} \arrow[u, shift left, "F"]
\end{tikzcd}$$

"fin gen fr alqs"

$\eta_x : x \xrightarrow{\text{(keeps th from collapsing}} GFx$ mno. $\boxed{\Leftrightarrow \text{there is } c \in C \text{ s.t. } g(c) > \{\bullet\}}$

Then let A full sbcat w all obj's F(n), $n = \{0, \dots, n\} \in \mathcal{E}\text{ns}$. Then A^{op} is a th.

$\mathbb{T}\text{hy}$ = cat of all alg ths ; $\mathbb{F}\text{in}^{op}$ is in. obj.

[Phillip Hall - Th of clones 1947]

A. Nerode 1956 - <u>COMPOSITON</u> thesis

An <u>A-alg</u> X is a prod pres ftr $X : A \to Ens$.

mp :

$$\begin{tikzcd} X : A \arrow[dr] \arrow[dd, Rightarrow, "\text{nt. xf. } h"'] & \\ & Ens \\ Y : A \arrow[ur] & \end{tikzcd}$$

$X \xRightarrow{h} Y$;

$$\begin{tikzcd} a^n \arrow[d, "p_i"] & {|X|^n} \arrow[r, "h_n"] \arrow[d, "p_i"'] \arrow[dr, phantom, "///"] & {|Y|^n} \arrow[d, "p_i"] \\ a & {|X|} \arrow[r] & {|Y|} \end{tikzcd}$$

$$\begin{tikzcd} a^n \arrow[d, "\omega"] & {|X|^n} \arrow[r] \arrow[d, "\omega_X"'] \arrow[dr, phantom, "///"] & {|Y|^n} \arrow[d, "\omega_Y"] \\ a & {|X|} \arrow[r] & {|Y|} \end{tikzcd}$$

$\Longleftrightarrow$ h nt. xf

$$A\text{-}Alg = Fun_{\pi}(A, Ens) \xrightarrow{U} Ens$$

For ea th have <u>its</u> cat of algs ;

So have ftr.

$$(\mathbb{T}hy)^{op} \xrightarrow[\text{SEMANTICS}]{\mathcal{S}} \overbrace{\mathbb{L}\,cat/\mathcal{E}ns}^{?} \quad \left|\ \begin{array}{l} obj \begin{tikzcd} C \arrow[d, "U"] \\ \mathcal{E}ns \end{tikzcd} \\ + \\ \text{condition} \end{array}\right.$$

$$\begin{tikzcd}
\mathcal{S}A \arrow[d, bend right, "\mathcal{S}H"'] \arrow[r, phantom, "= \ A\text{-}\mathbb{A}lg ="] & \mathbb{F}un_{\pi}(A, \mathcal{E}ns) \\
\mathcal{S}B \arrow[r, equal] & \mathbb{F}un_{\pi}(B, \mathcal{E}ns) \arrow[u]
\end{tikzcd}$$

Need adjt to $\mathcal{S}$. $h : X \Longrightarrow Y$ nt xf.

$$\begin{tikzcd}
a^n \arrow[d, "\omega"] & {|X|^n} \arrow[r, "h"] \arrow[d, "\omega_X"'] \arrow[dr, phantom, "''"] & {|Y|^n} \arrow[d, "\omega_Y"] \\
a & {|X|} \arrow[r] & {|Y|}
\end{tikzcd} \quad \Bigg\} \text{read this two ways}$$

$$U : C \longrightarrow \mathcal{E}ns$$
$$\overset{X}{a} \longrightarrow X(a) = |X|$$

$$U^n : X \longmapsto |X|^n$$

$$\omega : U^n \Longrightarrow U \quad \text{nt. xf.}$$

An n-ary op is just nt xf.

Will show, conversely,

<u>lemma</u>. If A alg th, $C \xrightarrow{U} \text{Ens}$; then any $\theta : U^n \longrightarrow U$ nt xf is ω for some n-ary op $a^n \longrightarrow a$ of the th.

$$\left[\begin{array}{l} \qquad \theta_X : U^n X \longrightarrow UX \\ \text{ie} \quad \theta_X = \omega_X \quad \forall X \end{array}\right]$$

pf. Look at fr algs; Let $X = \hom_A(a^n, -)$ the fr alg. θ ntl means

$$\begin{tikzcd}
(UX)^n \arrow[d, equal] \arrow[r, phantom, "="] & U^n X = \overset{1_{a^n}}{\hom_A(a^n, a^n)} \arrow[d, "\theta_X"] \arrow[rd, squiggly, "\text{Yoneda}"] & \\
{[\hom_A(a^n, a)]^n} & UX = \hom_A(a^n, a) & \omega : a^n \longrightarrow a
\end{tikzcd}$$

for some ω.

Take any $Y : A \xrightarrow{\times} \text{Ens}$ an alg.

$U^n(Y) = Y(a^n)$

$$\begin{tikzcd}
\hom(a^u, a^u) \arrow[d, phantom, "\shortparallel"] & & h : X \longrightarrow Y \\
U^u(X) \arrow[rr, "U^u(h)"] \arrow[d, "\theta_X"'] & & U^u(Y) = Y(a^u) \ni^{\forall} t \, \exists h \arrow[d, "\theta_Y"] \\
U(X) \arrow[rr, "Uh"'] & & U(Y) = Y(a) \\
\omega : a^u \to a \arrow[u, phantom, sloped, "\in"] & &
\end{tikzcd}$$

commutes for every $h : \overbrace{\hom_A(a^u, -)}^{X} \longrightarrow Y$

$\Updownarrow$ Yoneda again

▷

calculate $\theta_Y(t)$. $t \in Y(a^u)$

Take corr. to t the $h_t : \hom(a^u, -) \to Y$.

By defn $h_t(1_{a^u}) = t$.

Thus $\omega \rightsquigarrow (h_t)\omega = Y(\omega)t = \theta_Y t$

so $\theta_Y = Y_\omega$ <u>done</u> .

<u>Claim</u> We have now the desired adjnt.

$$\mathrm{Thy} \underset{\hat{\mathcal{S}}}{\overset{\mathcal{S}}{\rightleftarrows}} \underbrace{\mathbb{L}\,\mathrm{cat}/\mathcal{E}ns}_{\substack{\text{not cat of all fgt ftrs} \\ \text{but ..... sm wt x fs}}}$$

▷ where $\hat{\mathcal{S}}U$ is a th, namely

objs $1, b, b^2, \ldots, b^n, \ldots$

$$\hom_{\hat{\mathcal{S}}U}(b^n, b) = \mathrm{Nt}(U^n, U)$$

indeed

$$\hom_{\hat{\mathcal{S}}U}(b^n, b^m) = \mathrm{Nt}(U^n, U^m)$$

$$\begin{tikzcd} \mathcal{C} \arrow[d, bend right=50, "U^n"', ""{name=L}] \arrow[d, bend left=50, "U^m", ""'{name=R}] \\ \mathcal{E}ns \arrow[from=L, to=R, Rightarrow, "\theta"] \end{tikzcd}$$

FOUNDATIONS

<u>sm</u> <u>hom</u> <u>sets</u>

the lemma gives

$\varepsilon_A : \hat{\mathcal{S}}\mathcal{S}A \cong A$ counit. Need unit:

$\eta_U : U \longrightarrow \mathcal{S}\hat{\mathcal{S}}U$ is constructed by :

Given
$$\begin{tikzcd} c \in C \arrow[d, "U"'] \arrow[drr, dashed, "\eta_U"] & & \\ \mathcal{E}ns & & B\text{-alg} \ni X \arrow[ll, "U_B"] \end{tikzcd}$$

$\hat{\mathcal{S}} U = B$ | $\hom_B(b^n, b) = \mathrm{Nt}(U^n, U)$

define $X = \eta_U(c)$ to be what ?

it is prod pres ftr $B \longrightarrow \mathcal{E}ns$

So $X(b^n)$ is a set ?

$\forall$for
$$\begin{tikzcd} b^n \arrow[d, "\theta"] \\ b \end{tikzcd} \qquad \begin{tikzcd} X(b^n) \arrow[d, "X_\theta"] \\ X(b) \end{tikzcd}$$

JUST TAKE
$$\begin{tikzcd} X(b^n) = (Uc)^n \arrow[d] \\ Uc \end{tikzcd}$$

& $X\theta = \theta_c$

must wave at basic identities.

Th$^{\underline{m}}$ to be proved :

$$\begin{tikzcd}
A \arrow[d, "H"', "x"] \arrow[r, dash, dotted] & \mathcal{S}A \arrow[r, phantom, "\subset"] \arrow[d, dashed, bend right] & \mathrm{Ens}^A \\
B & \mathcal{S}B \arrow[u, "\mathcal{S}H"'] \arrow[r, phantom, "\subset"] & \mathrm{Ens}^B \arrow[u, bend right, "\mathrm{Ens}^H"']
\end{tikzcd}$$

the map

- A-algs over Ens

B-algs over Ens

prove: $\underbrace{\text{Kan ext}}_{\text{of } x \text{ pres ftr}}$ landes in $\mathcal{S}B$.

"Every alg ftr has an adjnt"

$\boxed{\text{MacLane 27}}$

Th$^{\underline{m}}$ (Lawvere) Every alg ftr has a left adjt obtained by restricting Kan extension.

all models of th A

$$\begin{tikzcd}
A \arrow[d, "H"', "\times"] & \mathcal{S}(A) \arrow[r, phantom, "\subset"] \arrow[d, dashed, bend right] & Ens^{A} \arrow[d, dashed, bend right] \\
B & \mathcal{S}(B) \arrow[r, phantom, "\subset"] \arrow[u, "\mathcal{S}H"'] & Ens^{B} \arrow[u, "Ens^{H}"']
\end{tikzcd}$$

all models of th B

✓ In Ens, fin prod commute w colimits, i.e.,

~~$\underset{\text{fin prod}}{X \times} \Big(\underset{\text{coprod}}{\coprod_{\alpha} y_{\alpha}} \Big) \overset{\text{disj}\,\cup}{\cong} \coprod_{\alpha} \times \coprod y_{\alpha}$~~

$$\boxed{X \times \coprod_{\alpha} y_{\alpha} \cong \coprod_{\alpha} (X \times y_{\alpha})}$$

$$\begin{tikzcd}
A \arrow[r, "X"] \arrow[d, "H"', "\times"] & Ens \\
B \arrow[ur, dashed, "Y"]
\end{tikzcd}$$

left Kanext of X along H

$Yb = \operatorname{colim} \left(H/b \longrightarrow A \xrightarrow{X} Ens \right)$

lemma.
$$\begin{tikzcd}
A \arrow[r, "X"] \arrow[d, "H"', "\times"] & Ens \\
B
\end{tikzcd}$$
If X pres fin prod so does Y.

prf. Let $b' \xleftarrow{p'} b = b' \times b'' \xrightarrow{p''} b''$ prod in B.

$Yb \quad H/b \longrightarrow A \longrightarrow Ens$

$Yb' \quad H/b' \longrightarrow A \longrightarrow Ens$

$Yb'' \quad H/b'' \longrightarrow A \longrightarrow Ens$

$$\begin{tikzcd}
H/b' \times H/b'' \arrow[rr, dashed, shift left, "\boldsymbol{\Pi}"] \arrow[d, "Q' \times Q''"] & & H/b \arrow[ll, dashed, shift left, "\boldsymbol{\Delta}"] \arrow[d, "Q"] \\
A \times A \arrow[rr, shift left, "\Pi"] & & A \arrow[ll, shift left, "\Delta"]
\end{tikzcd}$$

jack it up

$$\Delta \dashv \Pi$$

$$\begin{tikzcd}
\Big\langle Ha' \arrow[d, "f'"] & , & Ha'' \arrow[d, "f''"] \Big\rangle \\
b' & & b''
\end{tikzcd}$$

$$\begin{tikzcd}
& H(a' \times a'') \arrow[dl, "\text{canonical}"] \\
Ha' \times Ha'' \arrow[d, "f' \times f''"] & \\
b = b' \times b'' &
\end{tikzcd}$$

$\boldsymbol{\Pi}$ (maps to)

~~Prod~~

$$\begin{tikzcd}
Ha \arrow[d] & Ha \arrow[d] \\
b \arrow[d] & b \arrow[d] \\
b' & b''
\end{tikzcd}
\qquad
\begin{tikzcd}
Ha \arrow[d, "f"] \\
b
\end{tikzcd}$$

$\boldsymbol{\Delta}$

$$\langle p'f, p''f \rangle \longleftarrow\!\!\shortmid f$$

$$\boxed{\boldsymbol{\Delta} \dashv \boldsymbol{\Pi}}$$

$$\hom(\Delta f, (f', f'')) \overset{?}{\cong} \hom(f, \Pi(f', f''))$$

~~directed~~

▷

cofinal !!

$$\begin{tikzcd}
H/_{b'} \times H/_{b''} \arrow[r, "\Pi"] \arrow[d, "Q' \times Q''"'] \arrow[dd, bend right, ""] \arrow[ddr, bend right=60, ""] \arrow[ddr, bend left=80, ""] \arrow[dr, phantom, "/\!/\!/ \text{ for a reason}"] & H/_b \arrow[d, "Q"] \arrow[dd, bend left=60, "\text{colim ?} = Yb"] \\
A \times A \arrow[r, "\pi"] \arrow[d, "X \times X"'] \arrow[dr, phantom, "/\!/\!/ \text{ by hyp.}"] & A \arrow[d, "X"] \\
\mathcal{E}ns \times \mathcal{E}ns \arrow[r, "\pi", "\text{prod in sets}"'] & \mathcal{E}ns
\end{tikzcd}$$

$j/_D$?

cofinal: $I \xrightarrow{D} J$ <u>cofinal</u> iff $D/_j$ conn $\neq \emptyset$ $\forall j$.

look at adjn of Π ; show $f/_\Pi$ conn. $\neq \emptyset$

use unit of adjn : univ map

$$\begin{tikzcd}
f \arrow[r, "\eta_f"] \arrow[dr] & \Pi \Delta f \arrow[d] \\
 & {}
\end{tikzcd}$$

DIFFICULTY : how do you calculate colimit of $(\vert\!\!\leftarrow)$?

$L' = XQ' \quad L'' = XQ''$

▷

$\text{colim}(L' \times L'') \cong \text{colim}\, L' \times \text{colim}\, L''$

Let H_* be left adjnt of $\mathcal{S}H$ made by Th$^{\underline{m}}$.

$$\begin{tikzcd}
\mathcal{S}(A) \arrow[d, shift left, "\mathcal{S}H"] \\
\mathcal{S}(B) \arrow[u, shift left, "H_*"]
\end{tikzcd}$$

Poincaré-Birkhoff-Witt Th$^{\underline{m}}$ "Says Lie alg embeds in enveloping alg" ie

$$L \xrightarrow{\eta_L} \mathcal{S}(H)(H_*L) \quad \text{mno.}$$

["In cat th look for adjnts, not embeds"]

Problems: ① (Lawvere) For wh alg ftrs H is the unit η of the adjn $H_* \dashv \mathcal{S}H$ always mno?

② (Burnside) G gp n fixed integer.

Say $g^n = 1 \ \forall g \in G$.

If G fin gen, is then G fin?

For wh th maps is semantics "fin gen alg preserving"

$\mathcal{S}H$ is the algebraic functor

Problem answered by Lawvere. | Given $\mathcal{C}^{cat}$ when is it $\cong \mathcal{S}A$ some th A?

Tensor product of th's A, B.

Cop is easy to define, $A \amalg B$

"objs are ints; ops are all of A, all of B, & everything out of them"

$$A \otimes B = A \amalg B / \equiv$$

ea op of A interchanges w ea op of B

Abelian theories
linear theories
affine theories
many theories

---

hypertheories cat T s.t.

1° fin prods (term obj)

2° two special objs — | a the generator &
| 2

2 has $1 \overset{\text{true}}{\underset{\text{false}}{\rightrightarrows}} 2$

An arrow $x \longrightarrow 2$ is a property

$x^2 \longrightarrow 2$ is a binary relation

condition

$$\begin{tikzcd}
1 \times x \arrow[dr, "\text{true} \times x"'] & & 1 \times x \arrow[dl, "\text{false} \times x"] \\
& 2 \times x &
\end{tikzcd}$$

a cop. $\forall x$

or $\boxed{2 \times x = x + x}$

$\hom_T(x, 2)$ is a Boolean alg.

$\therefore$ a category.

$a + (b - c)$

$+ \quad -$

$a + b - c$

$(a+b) - c$

lemma for Lawvere. colim $T'\bar{\times}T''$ = colim $T'\times T''$ $\boxed{\text{MacLane 28}}$

$$\begin{tikzcd} \mathrm{Ens} & J'\times J'' \arrow[l, "T'\bar{\times}T''"'] \arrow[d, "T'\times T''"] & (j';j'') \\ T'j'\times T''j'' & \mathrm{Ens}\times\mathrm{Ens} & \\ & (T'j', T''j'') & \end{tikzcd}$$

A lecture on Foundations

ZFCU — universe

$x\in U \iff x^{\mathrm{sm}}$

Ens = cat of sm sets

Had claimed, & still claim, that ZFCU is adequate.

$\mathbb{C}\mathrm{at}$ = cat of sm cats.

a large cat is one whose set of obj's $\subset U$ not $\in U$.

$$\begin{tikzcd} A & & \\ (\mathbb{F}\mathrm{in})^{\mathrm{op}} \arrow[u] & \mathbb{T}\mathrm{hy} \arrow[r, bend left, "\mathcal{S}"] & C \arrow[d, "U"] \arrow[l, bend left, "\hat{\mathcal{S}}"] \\ & & \mathrm{Ens} \end{tikzcd}$$

$\mathcal{S}(A) = \mathrm{Fun}_{\pi}(A, \mathrm{Ens})$ large cat

$\mathbb{L}\mathrm{cat}/\mathrm{Ens}$ = "large cats over Ens" ie large cat w fgt ftr

Correction on Burnside: Novikov & Adian Izvestia 300–400 pp.

What in thunder do you mean by that Maclane?

Pocat | cat of posets
has fgt ftr ;
has both adjts
[left adjt = chaos]

TOPOLOGY
not Algebra

Freudenthal- Cêch th diagrams in top before war '37 —
Dutch journal Compositio...

Diagram is a sentence :

"In C a product of a, b"

"C has binary products"

$$\begin{tikzcd}
\forall a & \exists c \arrow[l, dashed] \arrow[r, dashed] & \forall b \\
& \forall' d \arrow[ul, "\forall'"] \arrow[u, dashed, "\exists!"'] \arrow[ur, "\forall'"'] &
\end{tikzcd}$$

cf. cat of discourse

---

① Van der Waerden
② Hausdorff —
③ Hilbert
④ Russell-Whitehead

TWIST THE LIONS TAIL

School

items $x, y, z \to \langle x, y \rangle$ item ordpr

classes A, B, C $x \in A$

Axioms

① $\langle x\ y \rangle = x'\ y'$ iff $x = x'\ y = y'$

② $A = B$ $\forall$ items x, $x \in A \Leftrightarrow x \in B$

$(A \subset B \ \& \ B \subset A)$

③ $\exists\ \emptyset$

④ $\exists\ \{x\}$ $\{x, y\}$

⑤ $\exists\ A \times B$

comprehension ⑥ $\exists\ \{x \mid x \in A \ \& \ \psi(x)\}$

for ea formula ψ

w limited quantifiers $(\exists t \in A)$

$(\forall s \in B)$

No quantifiers on classes

Model

Example ZF

item = set

class = set

Example. ZF take some U

form U $U \times U$ $U \times (U \times U)$ etc
$(U \times U) \times U$

item: any elmt of one of these

class: any subset of one of these

<u>homogeneous</u> <u>set theory</u>

Adjoint Fth Th$^{\underline{m}}$ can be done in School.

$2^{30} - 3^{00}$ NOTES
3^{00} - ISBELL

MacLane 29

Pultr

classically, <u>rel</u> <u>on</u> <u>X</u> is $R \subset X \times X$

Gabriel Theory $\mathbb{T}$ sm cat, dist. obj. and $\mathcal{D}$ a set of limits in A
$\qquad\qquad\qquad\qquad$ A $\qquad\qquad$ a

$\mathcal{D} : D \longrightarrow A$

a $\mathbb{T}$-alg is ftr $A \longrightarrow$ Ens wh pres lims from $\mathcal{D}$

$$\left.\begin{array}{l} a \\ a^2 \\ r \end{array}\right\} \mathcal{D} \qquad r \xrightarrow{i} a^2 \qquad \text{pllbk} \quad \begin{tikzcd} r \arrow[r, "1"] \arrow[d, "1"'] & r \arrow[d, "i"] \\ r \arrow[r, "i"'] & a^2 \end{tikzcd}$$

take lim pres ftr of ↗ to Ens.

✓ show you get mono as image of i.

CAT W STR.

- Double categories
- 2-dimensional categories
- multiplicative categories $\otimes$
  - closed cat V $\quad - \otimes a$ has rt adjt $(-)^a$
- V-based cat

①

Double category

In alg th we know what $\omega : a^n \longrightarrow a$ interchanges w $\theta : a^m \longrightarrow a$. We have tensor $A \otimes A$, double theory of A.

Th$^{\underline{m}}$ (Eckmann-Hilton) a double monoid M is a commutative monoid.

$$\begin{tikzcd} M \times M \arrow[r, shift left, "\bullet"] \arrow[r, shift right, "\circ"'] & M & \{*\} \arrow[l, shift right, "1_\bullet"'] \arrow[l, shift left, "1_\circ"] \end{tikzcd}$$

and four interchanges:

① 1 w $1'$:

$$\begin{tikzcd} \{* = (M^0)^0 \arrow[r] \arrow[d] & M^0 = \{*\} \arrow[d, "1_\bullet"] \\ M \arrow[r, "1_\circ"'] & M \end{tikzcd}$$

ie, same units.

② $\bullet$ int w $\circ$:

$$\begin{tikzcd} (M^2)^2 \arrow[r] \arrow[d] & M^2 \arrow[d, "+"] \\ M^2 \arrow[r, "\bullet"'] & M \end{tikzcd}$$

$$(x \bullet y) \circ (a \bullet b) = (x \circ a) \bullet (y \circ b)$$

set $x = b = 1$

so $y \circ a = a \bullet y$

set $y = a = 1$

$x \circ b = x \bullet b$

Actually, can get associativity by set $a=1$.
So start w double multiplicative system w unit.

Cor. Grp in cat of gps is an ab grp.

"A <u>double</u> <u>category</u> is two cat str wh interchange".

Example. Take C any cat). Consider comm squares in C :

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "\alpha"'] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

wh f, g mono ; α, β epi .

Recall
obj + arr defn of cat:

$$\begin{tikzcd} A \underset{O}{\times} A \arrow[r, "\circ"] & A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & O \arrow[l, bend left=60, "id"] \end{tikzcd}$$

defn A <u>double</u> <u>graph</u> is four sets H, V, S, O

$$\begin{tikzcd} & & S \arrow[dl, shift right, "e_1"'] \arrow[dl, shift left, "e_0"] \arrow[dr, shift left, "d_0"] \arrow[dr, shift right, "d_1"'] & & \\ \text{squares} & H \arrow[dr, shift left, "d_0"] \arrow[dr, shift right, "d_1"'] & & V \arrow[dl, shift right, "e_1"'] \arrow[dl, shift left, "e_0"] & \text{vert arrs} \\ \text{horiz arrs} & & O & & \end{tikzcd}$$

and $e_i d_j = d_j e_i \quad \forall\, i, j$.

1° e_i is a map of d-graphs
ie graph in cat of graphs

set of meshes

<u>defn</u> double category: four sets S, H, V, O :

vertically composable squares

$$\begin{tikzcd}
 & S \underset{V}{\times} S \arrow[dl, shift left, "e_1"'] \arrow[dl, shift right, "e_0"] \arrow[dr, "\circ"] & & S \underset{H}{\times} S \arrow[dl, "\bullet"'] \arrow[dr, shift left, "d_1"] \arrow[dr, shift right, "d_0"'] & \\
H \underset{\circ}{\times} H \arrow[dr, "\circ"] & & S \arrow[dl, shift left, "e_1"'] \arrow[dl, shift right, "e_0"] \arrow[dr, shift left, "d_1"] \arrow[dr, shift right, "d_0"'] & & V \underset{\bullet}{\times} V \arrow[dl, "\bullet"'] \\
 & H \arrow[ur, bend left, "id_V"] \arrow[dr, shift left, "d_0"] \arrow[dr, shift right, "d_1"'] & & V \arrow[ul, bend right, "id_H"'] \arrow[dl, shift left, "e_1"'] \arrow[dl, shift right, "e_0"] & \\
 & & O \arrow[ul, bend left, "id_H"] \arrow[ur, bend right, "id_V"'] & &
\end{tikzcd}$$

$d_0 \, id_H = d_0$

<u>axioms</u> (1) cat for $\bullet$ & id_V e_i
(2) cat for $\circ$ & id_H d_i

	d_i	id_H	$\circ$
e_i	$e_i d_j = d_j e_i$	$e_i \, id_H = id_H \, e_i$	$d_i(a \circ b) = d_i a \circ d_i b$ "if $a \circ b$ defined"
id_V	$d_i \, id_V = id_V \, d_i$	$id_H \, id_V = id_V \, id_H$	
$\bullet$	$d_i(x \bullet y) = d_i x \bullet d_i y$ "when $x \bullet y$ defined"	$id_H(x \bullet y) \overset{\shortparallel}{} id_H x \bullet id_H y$	"$\circ$ int. w $\bullet$"
$\bigtriangledown$			

defn $\boxed{\begin{matrix} x & a \\ y & b \end{matrix}}$ meshes iff all vert + horiz compositions $\begin{matrix} x \\ y \end{matrix}$ $\begin{matrix} a \\ b \end{matrix}$ xa yb are defined.

axiom of interchange: when mesh then

$$(x\cdot y)\circ(a\cdot b) = (x\circ a)\cdot(y\circ b).$$

Theorem. A double cat is same thing as cat obj in cat [in two ways].

$$\begin{tikzcd}
A\times_C A \arrow[d] \\
A \arrow[d, shift left] \arrow[d, shift right] \\
C \arrow[u, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
H\times H \arrow[r] & H \arrow[r, shift left] \arrow[r, shift right] & O \arrow[l, bend left]
\end{tikzcd}$$

write it out boy.

---

defn 2-dimensional cat (2-cat) is a double cat w $V = O$ (+ e_0, e_1 are identities)

$$\begin{tikzcd}
\text{ntxf} & S \arrow[d, shift left] \arrow[d, shift right] \\
\text{fn} & H \arrow[d, shift left] \arrow[d, shift right] \\
\text{cat} & O
\end{tikzcd}$$

$\boxed{A \xleftarrow{FG\varepsilon\eta} X}$

example. Adj

example. Cat

$$\begin{tikzcd}
A & B \arrow[l, bend right, "T"'] \arrow[l, bend left, "S"] \arrow[l, phantom, "\Uparrow\eta"]
\end{tikzcd}$$

"vert arrs are just ids"

So we can think of dub cat as partially defined quaternary operation $\boxed{\begin{smallmatrix} x & a \\ y & b \end{smallmatrix}}$.

Conjecture I: squares $\boxed{a}$ every square has a border

NW	N	NE
W	a	E
SW	S	SE

axiom. border of a border

composition: $\boxed{\begin{matrix} x & a \\ y & b \end{matrix}}$ mesh iff borders can be read from

x	$Ex = Wa$	a
$Sx = Ny$		
y		b

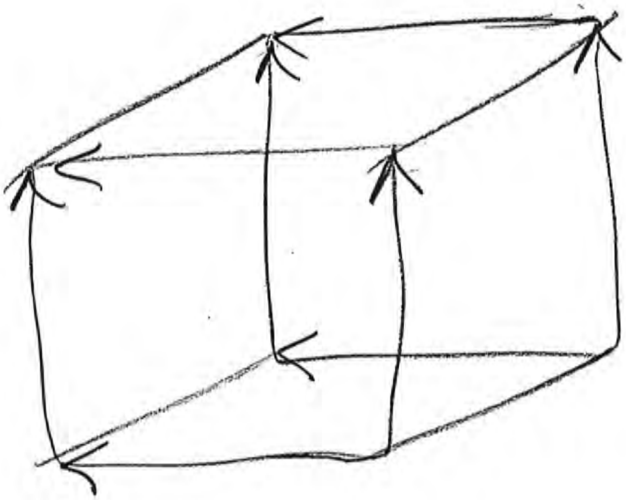
& identities.
& associativity:

Given $\begin{matrix} x & a & u \\ y & b & v \\ z & c & w \end{matrix}$ mesh

$$\begin{pmatrix} \begin{pmatrix} x & a \\ y & b \end{pmatrix} & \begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} z & c \end{pmatrix} & \begin{pmatrix} w \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (x) & (a \ u) \\ \begin{pmatrix} y \\ z \end{pmatrix} & \begin{pmatrix} b & v \\ c & w \end{pmatrix} \end{pmatrix}.$$

---

in dub cat, basic "comm squ" is cube:



$\boxed{\text{MacLane 30}}$

Example of Gabriel theory [smcat w specified limits].

G-Theory of a cat:

equalizer of . . . .

$$\begin{tikzcd} A \times_0 A \times_0 A \arrow[r, shift left] \arrow[r, shift right] & A \times_0 A \arrow[r, "\circ"] & A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & O \arrow[l, bend left=60] \end{tikzcd}$$

$$\begin{tikzcd} A \times_0 A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{plbk}"] & A \arrow[d, "d_1"] \\ A \arrow[r, "d_0"'] & O \end{tikzcd}$$

Multiplicative [Monoidal] Categories

Cat M and biftr $M \times M \xrightarrow{\otimes} M$

$$\begin{tikzcd} \langle x, y \rangle \arrow[r, squiggly] \arrow[d, "\langle , \rangle"] & x \otimes y \arrow[d, "- \otimes -"] \\ \langle x', y' \rangle \arrow[r, squiggly] & x' \otimes y' \end{tikzcd}$$

$\otimes$ associative
+ has obj. k a unit } up to isomp. i.e. $\exists\ \alpha_{XYZ} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$

+ $\exists$ $\rho_L : k \otimes x \cong x$, $\rho_R : x \cong x \otimes k$

monoid object in M: $m \otimes m \xrightarrow{\mu} m \xleftarrow{\eta} k$ s.t. . . . . .

$$\begin{tikzcd} m \otimes (m \otimes m) \arrow[r] \arrow[d, "\wr"] & m \otimes m \arrow[dd] \\ (m \otimes m) \otimes m \arrow[d] & \\ m \otimes m \arrow[r] & m \end{tikzcd}$$

Add conditions on α to assure total "coherence".

$$\begin{tikzcd}
 & a\otimes(b\otimes(c\otimes d)) \arrow[r, "\alpha_{a,b,c\otimes d}"] \arrow[dl, "\alpha_{a,b\otimes c,d}"'] & (a\otimes b)\otimes(c\otimes d) \arrow[dd] \\
a\otimes((b\otimes c)\otimes d) \arrow[d] & /// & \\
(a\otimes(b\otimes c))\otimes d \arrow[rr] & & ((a\otimes b)\otimes c)\otimes d
\end{tikzcd}$$

and similarly on λ, ρ :

$$\begin{tikzcd}
k\otimes(a\otimes b) \arrow[r] \arrow[d] & (k\otimes a)\otimes b \arrow[dl] \\
a\otimes b &
\end{tikzcd}$$

+ other side

▷ Exercise. Draw condition for a sphere.
[see repeats of above + ntl of α]

Theorem. "All diagrams in α, λ, ρ commute."

Symmetric Multiplicative categories

Above, plus nt isomp $\gamma_{xy} : x\otimes y \cong y\otimes x$,

w cond. | $\gamma^2 = \mathrm{id}$;
mix γ and α : $\rightarrow$

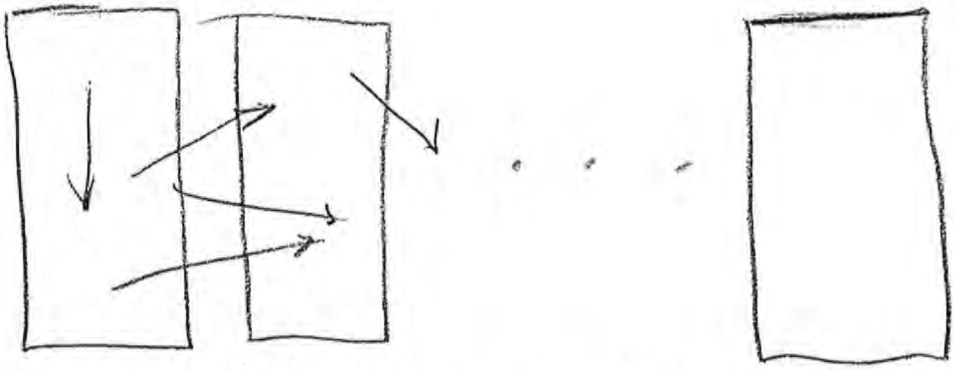
$$\begin{tikzcd}
a\otimes(b\otimes c) \arrow[rr, "\alpha"] \arrow[d, "\gamma"'] & & (a\otimes b)\otimes c \arrow[rr, "\gamma"] & & c\otimes(a\otimes b) \arrow[dd, "\alpha"] \\
a\otimes(c\otimes b) \arrow[dr, "\alpha"'] & & /// & & \\
& (a\otimes c)\otimes b \arrow[rrr, "\gamma"'] & & & (c\otimes a)\otimes b
\end{tikzcd}$$

and wrt id:

$$\begin{tikzcd}
k\otimes x \arrow[rr, "\gamma"] \arrow[dr, "\lambda"'] & & x\otimes k \arrow[dl, "\rho"] \\
& x &
\end{tikzcd}$$

Theorem. "All diags in $\alpha, \lambda, \rho, \gamma$ commute."

Arrange n-⊗-words by symmetric grp S_n



S_n generated by successor-transpositions.

defn A morphism $\underset{\alpha,\lambda,\rho,\gamma}{\underset{\otimes}{M}} \longrightarrow \underset{\dots}{\underset{\otimes'}{M'}}$ of multcat is

$F\_ \otimes F\_ \Longrightarrow F(\_\otimes\_$

is (1) ftr $M \xrightarrow{F} M'$

(2) nt xf $\varphi : Fa \otimes Fb \longrightarrow F(a \otimes b)$

(3) nt xf $\psi : k' \longrightarrow Fk$

$\Big\}$ WOBBLE of F

plus conditions :

(1)

$$\begin{tikzcd}
Fa \otimes Fb \arrow[r, "\varphi"] \arrow[d, "\gamma"] & F(a \otimes b) \arrow[d, "F\gamma"] \\
Fb \otimes Fa \arrow[r, "\varphi"'] & F(b \otimes a)
\end{tikzcd}$$

(2)

④

LAWVERE 4^{30} | Maclane 31

Alg th of monoids A . A monoid is prod presftr $A \xrightarrow[\times]{X} \mathcal{E}ns$

A mp of monoids is a ftr $F : A \xrightarrow[\times]{} \mathcal{E}ns^{\mathbf{2}}$.

$\underbrace{\text{Multiplicative cat}}$ coherence $\mathrm{Th}^{\underline{m}}$:

data | cat M & bifti $\otimes : M \times M \longrightarrow M$
& obj $k \in M$
& bunch of canonical nt. xfs $(\alpha, \lambda, \rho, \ldots)$

Given any $\otimes$ word W in n-letters there is ftr $W_M : M^n \longrightarrow M$.

For any two such words $\exists$ can map $W_M \Longrightarrow W'_M$.

conditions | if $W = W' \Longrightarrow can = id$
& composite of cans is can

[add symmetry for symm mult cat.]

What is a ftr of mult cats?

$$\begin{tikzcd}(M, \otimes, k, can) \arrow[r, bend right] & (\overline{M}, \overline{\otimes}, \overline{k}, \overline{can})\end{tikzcd}$$

fu ropping

Examples. ① $\underset{\otimes_K}{K\text{-modules}} \longrightarrow \underset{\otimes_{\mathbb{Z}}}{\mathcal{A}b}$

② $\mathcal{E}ns_{\text{mlhl.}}/A \xrightarrow{U} \underset{\times}{\mathcal{E}ns}$

$$\begin{tikzcd} U(X \times_A Y) \arrow[r, "\psi"] & UX \times UY \\ U\begin{pmatrix} A \\ \| \\ A \end{pmatrix} \arrow[r] & 1 \end{tikzcd}$$

morphism of mult cat

ie, "ftr of mult cat" doesn't pres prod, just induces map "φ".

defn

data

a ftr of und cat $F : M \longrightarrow \overline{M}$

$\forall\ a, b \quad \exists \quad \varphi_{ab} : Fa \mathbin{\overline{\otimes}} Fb \longrightarrow F(a \otimes b)$ wl

$\exists \quad \psi : \overline{k} \longrightarrow Fk$

conditions

$$\begin{tikzcd} Fa \mathbin{\overline{\otimes}} (Fb \mathbin{\overline{\otimes}} Fc) \arrow[r, "\overline{\alpha}"] \arrow[d] & (Fa \mathbin{\overline{\otimes}} Fb) \mathbin{\overline{\otimes}} Fc \arrow[d] \\ Fa \mathbin{\overline{\otimes}} (F(b \otimes c)) \arrow[d] & F(a \otimes b) \mathbin{\overline{\otimes}} Fc \arrow[d] \\ F(a \otimes (b \otimes c)) \arrow[r] & F((a \otimes b) \otimes c) \end{tikzcd}$$

[also define comorphism]

Better defn : take any words W, W' in $X_1, \ldots, X_n$.

$$W_M : M^n \longrightarrow M$$

$$W_{\overline{M}} : \overline{M}^n \longrightarrow \overline{M}$$

$$\vdots$$

Let C be a 2-dim cat. [0 1 2 cells]

$$\begin{tikzcd} S \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[d, shift left] \arrow[d, shift right] \\ O \end{tikzcd}$$

to see that a mult cat is just a <u>wobbly</u> <u>monoid</u> in $\mathrm{Cat}, \times$

[ordinary monoid
$$\begin{tikzcd} 3 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 2 \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] & 1 \arrow[l] \arrow[dl] \\ & 1 & \end{tikzcd}$$
]

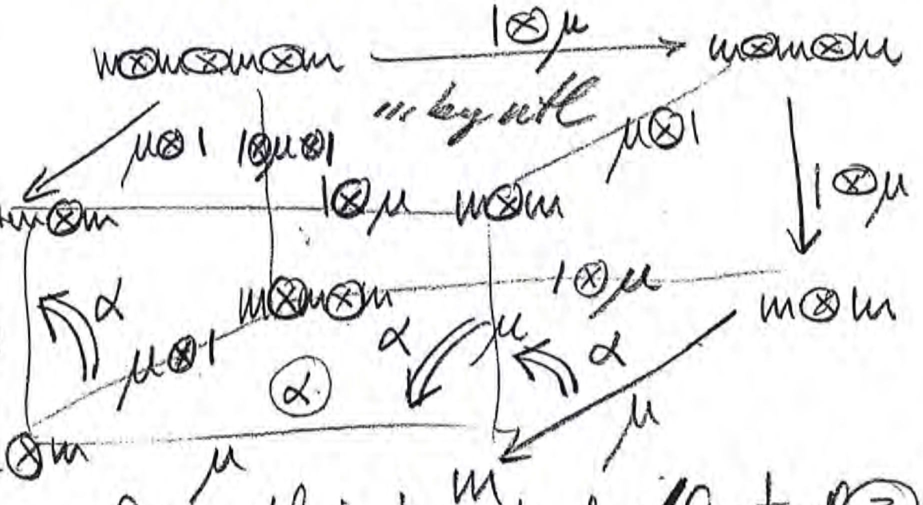
commutes on the nose

so <u>wobble</u> is α, λ, ρ

$$\begin{tikzcd} m^3 \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & m^2 \arrow[d] \arrow[dl, phantom, "\overset{\alpha}{\Swarrow}"] \\ m^2 \arrow[r] & m \end{tikzcd} \quad 2\text{cell}$$

$$\begin{tikzcd} {} \arrow[r, "1\times m"] \arrow[d, equal] & m^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\overset{\lambda}{\Swarrow}"] \arrow[dr, phantom, "\overset{\rho}{\Searrow}"] & {} \arrow[l, "m\times 1"'] \arrow[d, equal] \\ m \arrow[r, equal] & m & m \arrow[l, equal] \end{tikzcd}$$

+ COHerence :



get pentagon from this by back ~~d~~ frontpaths ③

M some mult cat.

An <u>M-based category</u> [cat w homsets in M]

Take homset defn of cat but say homsets are not in Ens but in M.

defn M-based cat is

<u>data</u>

objs a, b, c, ...

to ea pair a, b ∃ $a \backslash b \in M$

and to ea $\langle a, b, c \rangle$ a "composition"

$$(b\backslash c) \otimes (a \backslash b) \longrightarrow (a \backslash c),$$

and identity

$$k \xrightarrow{\eta_a} (a \backslash a)$$

for ea ∈ [illegible].

<u>conditions</u>

$$\begin{tikzcd}
\left[(c\backslash d)\otimes(b\backslash c)\right]\otimes(a\backslash b) \arrow[r] \arrow[d, "\cong \text{ can}"'] & (c\backslash d)\otimes(a\backslash c) \arrow[dd, "\text{assoc.}"'] \\
(c\backslash d)\otimes\left[(b\backslash c)\otimes(a\backslash b)\right] \arrow[d] & \\
(c\backslash d)\otimes(a\backslash c) \arrow[r] & a\backslash d
\end{tikzcd}$$

+ η_a left + rt unit.

$$\underbrace{\underbrace{M \xrightarrow{U} \mathcal{E}ns : m \longmapsto \hom_M(k, m)}_{\text{fgt ful ftr of mult cat}}}_{\text{gives a cat to ea } M \text{ bsd cat } A}$$

M

$$(A, a\backslash b, \circ, \eta) \xrightarrow{F} (X, x\backslash y, \bar{\circ}, \bar{\eta})$$

F set of obj's $\longrightarrow$ set of obj's

$\forall ab$ $\quad a\backslash b \xrightarrow{F_{ab}} Fa\backslash Fb$

&

$$\begin{tikzcd}
b\backslash c \otimes a\backslash b \arrow[r, "\circ"] \arrow[d, "F_{bc} \otimes F_{ab}"'] & a\backslash c \arrow[d, "F_{ac}"] \\
Fb\backslash Fc \otimes Fa\backslash Fb \arrow[r, "\bar{\circ}"'] & Fa\backslash Fc
\end{tikzcd}$$

& do right by η's.

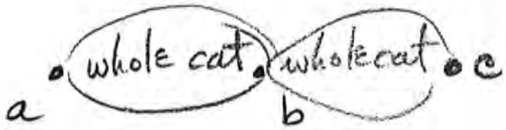
Example. Ab - based cat = additive cat

example. (Cat, ×)- based cat = 2-dim cat

hom(a, b) is a cat

obj s are called arrs

arrs of hom(a, b) called cells.



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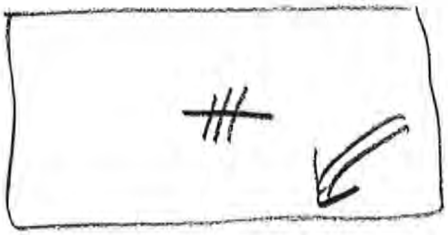
defn BICATEGORY = wobbly cat based cat.

So put in a wobble in assoc. diag.



+ COHERENCE.

Example. Bicat w one obj = mult cat.

"I FONDLY IMAGINED ...."

[illegible]

Δ mult. cat.

\+ assoc. on the nose.

NOT SYMMETRIC.

extend to transfinite ordinals.

OR : mult cat is Cat of pre-orders | refl trans

has sum

$(X, \leq) + (Y, \leqq) =$

$(X \amalg Y, \leq \leqq + \text{"} X \leq Y \text{"})$

objs $0, 1, 2, \ldots$

finite ordinal numbers

$n = \{0, 1, \ldots, n-1\}$

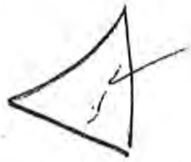
arrs order pres fns $n \xrightarrow{f} m$

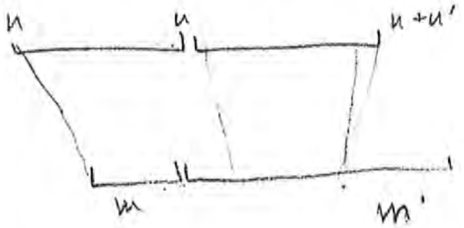
$i \leq j \Rightarrow fi \leq fj$

$\otimes$ "sum"

$$\begin{tikzcd} & n+n' \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "f'"] & \\ m & & m' \end{tikzcd}$$

$(f+f')i = \begin{cases} fi & 0 \leq i \leq n \\ f'i + m & n \leq i \leq n+n' \end{cases}$





$\Delta \subset$ Top

0 1 2 3 4 . . . n

arrs are affine maps of simplices preserving order of vertices

simplicial complexes ⟷ posets

$\Delta \subset$ Cat

0 1 2 3 . . . n

$\mathbb{1}$ $\mathbb{2}$ $\mathbb{3}$

---

1 in Δ is terminal object.

in $\Delta \subset$ Cat, set $$\begin{tikzcd} 0 \arrow[r, "\exists!"', "\eta"] & \mathbb{1} & 2 \arrow[l, "\exists!"', "\mu"] \end{tikzcd}$$

$$\begin{tikzcd} 1+1+1+1 \arrow[r, "1+1+\mu"] \arrow[d, "\mu+1+1"'] \arrow[dr, "\mu^{(3)}"] \arrow[drr, bend right=60, "\mu^{(4)}"'] & 1+1+1 \arrow[r] \arrow[d, "\mu"] \arrow[dr, "\mu^{(3)}"] & 1+1 \arrow[d] \\ 1+1+1 \arrow[r, "\mu"'] & 1+1 \arrow[r] & 1 \end{tikzcd}$$

necessarily commutes so μ is associative & η is left/rht unit.

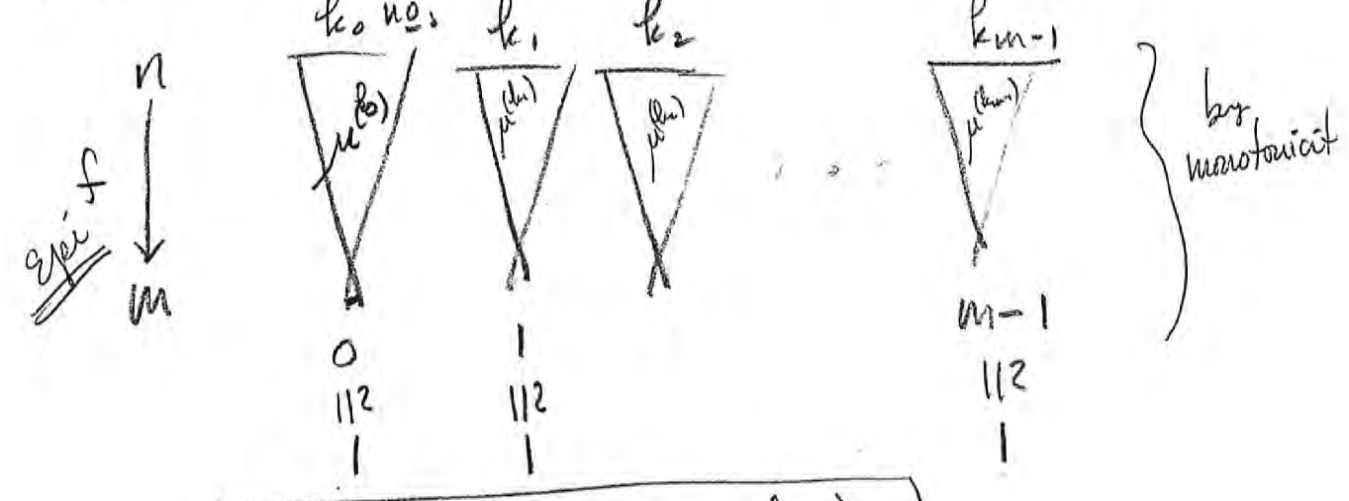
This is a monoid in Δ Cat.

Claim This is universal monoid?

$0 \xrightarrow{\mu^{(0)}} 1 = \eta$

$1 \xrightarrow{\mu(1)} 1 = id$.

Indeed, claim Δ is generated by η & μ. by sums & composite



$$\boxed{\begin{array}{c} f = \mu^{(k_0)} + \cdots + \mu^{(k_{m-1})} \\ k_0 + \cdots + k_{m-1} = n \end{array}}$$

Calculate all compositions from the laws of monoid η, μ.

epi $\overset{\cdot\cdot}{\uparrow}_{n+1} s^i$ ~ collapses vertices SKIP i OR DOUBLE-UP i
"degeneracies" 1 2 ... n

mno $\uparrow_n d^i$ ~ "face maps"
0 1 2 ... n-1

in $\mathbb{\Delta}^{op}$

$$d_i d_j = d_{j-1} d_i \quad i<j$$
$$s_i s_j = s_{j+1} s_i \quad i \leq j$$
$$d_i s_j = s_{j-1} d_i \quad i<j$$
$$d_i s_i = 1$$
$$d_i s_{i+1} = 1$$

simplicial & singular homology

$\mathbb{\Delta}$, $0 \xrightarrow{\eta} 1 \xleftarrow{\mu} 2$ in "the" univ monoid:

of M mult cat : ie $\exists!$ functor taking

(case 1) $\underbrace{\text{strict mult cat}}_{\text{strict ftr preserves unit \& tensor}}$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] & 2 \arrow[l] \arrow[d] \\ k \arrow[r] & a & a\otimes a \arrow[l] \end{tikzcd}$$

$\mathbb{\Delta} \subset \mathbb{C}at$

a ftr $\mathbb{\Delta}^{op} \xrightarrow{X} C$ is a simplicial object in C

$\mathbb{Z}$

"doing"

picks out bunch of figs: & set of occurrences of them in each other.

$(\forall P)(\exists x)(x \in A \Rightarrow x \in P_y)$

$x \in A \cup B$

$(\forall B)(\exists! A)(x \in A \cup B)$

simplicial object in $\mathbb{A}b$:

$$\begin{tikzcd}
\Delta^{op} \arrow[r, "L"] & \mathbb{A}b
\end{tikzcd}$$

$$\begin{tikzcd}
L_0 \\
L_1 \\
\vdots \\
L_n \\
L_{n+1} \arrow[u, shift left=3] \arrow[u, shift right=3] \arrow[u, phantom, shift left=6, "d^n"']
\end{tikzcd}$$

$i<j$

$d_i d_j = d_j d_i \Rightarrow$ if $\partial = \Sigma(-1) d^i$ then $\partial\partial = 0$.

ie a <u>chain complex</u>

Wobbly monoid in a 2-cat $\mathcal{C}$

$$\begin{tikzcd}
k \arrow[r, "\eta"] & a & a \otimes a \arrow[l, "\mu"']
\end{tikzcd}$$

$$\begin{tikzcd}
a^3 \arrow[r] \arrow[d] & a^2 \arrow[d] \arrow[dl, Rightarrow, shorten=6mm, "\beta", "2cell"'] \\
a^2 \arrow[r] & a
\end{tikzcd}$$

Given $\mathcal{C}$ 2-cat

make $\left\{ \begin{matrix} \mathrm{Fun}(\mathcal{C}) \\ \mathrm{Cyl}(\mathcal{C}) \end{matrix} \right\}$

$\boxed{\text{think } \mathcal{C} = \mathbf{Cat}}$

objs $\bullet \leftarrow \bullet$ arrs of $\mathcal{C}$

arrs
$$\begin{tikzcd}
\bullet \arrow[d, "F"'] & \bullet \arrow[l] \arrow[d, "F'"] \\
\bullet & \bullet \arrow[l] \arrow[ul, phantom, "\Uparrow \xi"]
\end{tikzcd}$$
2cells

from F' to F

2-cell (pillow)

cylinder

commutativity condition

For any 2-cat $\mathcal{C}$

EXERCISE

A wobbly monoid in cyl($\mathcal{C}$) is 2 wobbly monoids in $\mathcal{C}$ and a mp between them.

$\mathcal{C} = \mathbf{Cat}$

find the universal wobbly monoid bigger than △

$\boxed{\text{Lawvere } 3^{\underline{00}}}$ $\boxed{\text{MacLane 33}}$

on the nose

Coherence By Skeleton

Take $\otimes$ in skel, say in Ens or Ab ; then iso o.t.n. .

Does it work?

Δ ord no's + order pres mps. $0 \xrightarrow{\eta} 1 \xleftarrow{\mu} 2$ is monoid

For Any monoid in mult cat M, $k \xrightarrow{\eta} a \xleftarrow{\mu} a \otimes a$

$\exists!$ $\Delta \longrightarrow M$ wh is strict ftr [prod to prod, unit to unit]

and takes $0 \longrightarrow 1 \longleftarrow 2$ right to $k \longrightarrow a \longleftarrow a \otimes a$.

This is universal strict monoid.

What is universal wobbly monoid?

Application of Δ to cohomology :

$$\begin{tikzcd} & X^T \arrow[loop right] \arrow[dl, shift left, "U"] \arrow[dr, equal] & \\ X \arrow[ur, shift left, "F"] \arrow[loop right, "{(T,\eta,\mu)}"] & & A \end{tikzcd}$$

$G = FU$, $\varepsilon : G \longrightarrow I$, $\delta : G \longrightarrow G^2$ } COMONOID IN A^A.

COTRIAD $[I \longleftarrow G \longrightarrow G^2]$

TRIAD

Thus $\exists$ $\Delta^{op} \longrightarrow A^A$ } cotriad in A gives simplicial object in A^A.

$$\begin{tikzcd} 0 \arrow[d, "\eta"'] & I \\ 1 & G \arrow[u, "\varepsilon"] \arrow[d, "\delta"'] \\ 2 \arrow[u, "\mu"] & G^2 \end{tikzcd}$$

$$\begin{tikzcd} a\in A : & a & Ga \arrow[l, "\varepsilon_a"'] & G^2a \arrow[l, shift right, "G\varepsilon_a"'] \arrow[l, shift left, "\varepsilon Ga"] & G^3a \arrow[l, shift right=2, "G^2\varepsilon a"'] \arrow[l, "G\varepsilon G a"] \arrow[l, shift left=2, "\varepsilon G^2 a"] & \cdots \end{tikzcd}$$

[It's really $I \longleftarrow G \leftleftarrows G^2 \Lleftarrow G^3 \ldots$]

There are face op's ; there are also the degeneracy ops going other way.

Say <u>coefficient functor</u> $T : A \longrightarrow \mathbb{A}b$:

$$\begin{tikzcd}
Ta & TGa \arrow[l] \arrow[l, bend left, shift left=2] & TG^2a \arrow[l, shift left] \arrow[l, shift right] \arrow[l, bend left, shift left=2] & TG^3a \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, bend left, shift left=2]
\end{tikzcd}$$

Then boundary ops : $\partial = d_0 - d_1 \quad \partial = d_0 - d_1 + d_2$

$$\partial\partial = 0$$

This takes all of cohomology theories starting w standard complexes.

---

example. <u>Group</u> Π. Cohomology of Π :

$$\begin{tikzcd}
\Pi\text{-modules} \arrow[loop left] \arrow[d, shift left, "U"] & \mathbb{Z}(\Pi)\otimes - : a \arrow[r, squiggly] & \mathbb{Z}(\Pi)\otimes a \arrow[l, bend left] \\
\mathbb{A}b \arrow[u, shift left, "\mathbb{Z}(\Pi)\otimes -"] & ? & x \otimes t \arrow[l, "\varepsilon"']
\end{tikzcd}$$

$\otimes \qquad\qquad |$

$$a \leftarrow \mathbb{Z}(\Pi)\otimes a \leftarrow \mathbb{Z}(\Pi)\otimes\mathbb{Z}(\Pi)\otimes a \longleftarrow$$

Barr & Beck ACYCLIC MODELS & TRIPLES — LA JOLLA

$\downarrow$

(Wobbly Ftrs) defn Bicategory B

DATA

1) set O of objects

2) to ea pr objs $a, b \in O$ a cat $a \backslash b$

[an object of it called "arrow" of B
" arr " " " "cell" of B]

$$f : a \longrightarrow b$$

$$\begin{tikzcd} a \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow A"] & b \end{tikzcd}$$

composition: $\bullet$ called vertical.

$$\left[\begin{array}{ll} e_0 A = f & e_1 A = g \\ d_0 f = a & d_1 f = b \end{array}\right]$$

3) for ea triple $a, b, c \in O$ a ftr, composition

$$b \backslash c \times a \backslash b \xrightarrow{\circ} a \backslash c$$

called horizontal

$$\boxed{\circ \text{ interchanges with } \bullet}$$

4) for ea obj a, an identity functor $\mathbb{1} \xrightarrow{\eta_a} a \backslash a$

$$\begin{tikzcd} a \arrow[r, mapsto] & a \arrow[r, bend left=50, "1_a"] \arrow[r, bend right=50, "1_a"'] \arrow[r, phantom, "\Downarrow I_a"] & a \end{tikzcd}$$

Axioms (cat ax w wobble)

1) There is given α s.t.

$$\begin{tikzcd}
c\backslash d \times b\backslash c \times a\backslash b \arrow[r] \arrow[d] & c\backslash d \times a\backslash c \arrow[d] \arrow[dl, Rightarrow, "\exists\, \alpha"', "nt \times f"] \\
b\backslash d \times a\backslash b \arrow[r] & a\backslash d
\end{tikzcd}$$

2) λ, ρ for units.

3) (coherence) α, λ, ρ are.

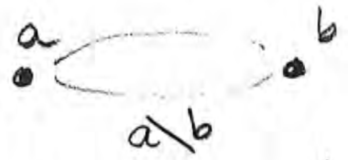
<u>pentagon</u>

$\boxed{\text{BENABOU - Midwest}}$

---

examples. Bicat w one obj is mult cat.
[monoidal cat]

Bicat w 2 objects :



$a\backslash a$ mult cat M   C $b\backslash b$ mult cat N

have actions $\left|\begin{array}{l} C \times M \longrightarrow M \\ N \times C \longrightarrow N \end{array}\right.$

<u>Strict</u> Bicat means $\alpha = \lambda = \rho = 1$.

$$\begin{tikzcd}
 & S \times_O S \arrow[dr] & \\
A \times_O A \arrow[dr] & & S^{\text{cells}} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \\
 & A^{\text{arrs}} \arrow[dr, shift left] \arrow[dr, shift right] & \\
 & & O^{\text{obj}}
\end{tikzcd}$$

This is double cat w $V = O$.

---

example. (A bicat out of modules):

$$\begin{array}{l|l} & \text{objs all rings } R, S, T, \ldots \\ \text{arr} & R \backslash S = {}_S\mathrm{Mod}_R \end{array}$$

composition = tensor over S

---

example. <u>Span</u> in C (fin lims)

O = obj of C

$$a \backslash b = \text{all} \quad \begin{tikzcd}
 & x \arrow[dl] \arrow[dr] \arrow[d] & \\
b & x' \arrow[l] \arrow[r] & a
\end{tikzcd} \quad \text{variable } x$$

$$\text{composition} = \begin{tikzcd}
 & & \cdot \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{pblk}"] & & \\
 & \cdot \arrow[dl] \arrow[dr] & {} & \cdot \arrow[dl] \arrow[dr] & \\
c & & b & & a
\end{tikzcd}$$

[illegible]

defn **Bifunctor** $B \xrightarrow{(F,\varphi)} \overline{B}$ [wobbly ftr]

1) an obj fn $a \in O \longmapsto Fa \in \overline{O}$

2) a cat ftr $a\backslash b \xrightarrow{F_{ab}} Fa\backslash Fb$

3)

∃ ntl xf φ_{abc}

$$\begin{tikzcd}
b\backslash c \times a\backslash b \arrow[r, "\circ"] \arrow[d, "F_{bc} \times F_{ab}"'] \arrow[dr, phantom, "\overset{\varphi_{abc}}{\nearrow}"] & a\backslash c \arrow[d, "F_{ac}"] \\
Fb\backslash Fc \times Fa\backslash Fb \arrow[r, "\bar{\circ}"] & Fa\backslash Fc
\end{tikzcd}$$

& unit OK.

4)

∃ nt xf φ_a

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta_a"] \arrow[dr, "\overline{\eta}_{Fa}"'] & a\backslash a \arrow[d, "F_{aa}"] \arrow[d, phantom, "\overset{\varphi_a}{\Uparrow}"', shift right=4] \\
 & Fa\backslash Fa
\end{tikzcd}$$
.

5) coherence of (F, φ).

---

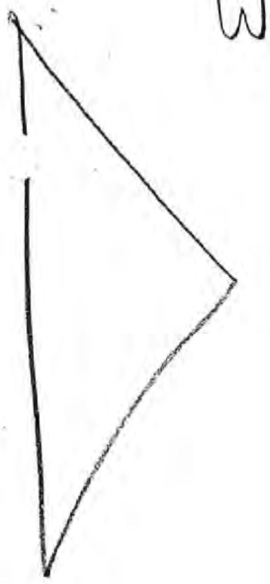
partially ordered sets give wobbly ftr.

" $F(f) \cdot F(f') \underset{\varphi}{\leq} F(f \circ f')$ ".

---

an ordinary cat is a bicat (all cells = 1) so can consider biftrs$^{\text{wobbly}}$ $C^{\text{cat}} \longrightarrow B^{\text{bicat}}$.

What is a bifte



$$S \xrightarrow{(F,\varphi)} M$$

mult cat

set made cat w/ $x \leq y$ $\forall x, y \in S$.

Coherence Conundrum

MacLane 34

Proposal: Ens   Ab

Sk Ens   Sk Ab

$$\begin{tikzcd} & X \bar{\times} Y = \mathrm{copy}(X\times Y) \arrow[dl] \arrow[dr] & \\ X & & Y \arrow[ll, bend right] \end{tikzcd}$$

$$A \,\bar{\otimes}\, (B \,\bar{\otimes}\, C) \underset{\text{on nose}}{=\!=\!=} (A \,\bar{\otimes}\, B) \,\bar{\otimes}\, C$$

Sk Ens DOES NOT have coherence

card $$\begin{tikzcd} P \arrow[loop, "f"] \arrow[loop, "g"] \arrow[loop, "h"] \end{tikzcd} = \aleph_0$$

$$\begin{tikzcd} & P \bar{\times} P = P \arrow[dl, "p_1"'] \arrow[dr] \arrow[d, "f\times g"] & \\ {} & {} & {} \end{tikzcd}$$

$\therefore f = h$

CLOSED CATEGORY

Ens ; Ab ; K-modules ; complexes

$\times$ | hom $\otimes$ | hom $\underset{K}{\otimes}$ | hom_K [shifts have $\partial\partial =$

$(L\otimes M)_k =$

$\sum_p L_{k-p}\otimes M$

$\boxed{\text{So we have cl cats w } \otimes / \text{hom}}$

internal hom

DEFN. a cl cat V is

a) mult cat: $\otimes$ bifctr ; α assoc ; λ, ρ unit mps ;

b) for ea $a \in V$, $\_\otimes a$ has rt adjnt $a \backslash \_$

ie specified adjn $\mathrm{hom}(x\otimes a, y) \cong \mathrm{hom}(x, a\backslash y)$

ie, unit + counit

$\varepsilon : (a\backslash y)\otimes a \longrightarrow y$ [evaluation]

$\delta : x \to a\backslash x\otimes a$

$X \longrightarrow \mathrm{Hom}(A, X\times A)$

$x \qquad a \mapsto (x,a)$

Using a-b derive all usual identities involving $\otimes$ & $\backslash$.

Autonomous cats : Linton J. Alg. 1965

Closed cats : Eil.-Kelly La Jolla

$$\frac{x\otimes a \xrightarrow{f} y}{x \xrightarrow[f^{\#}]{} a\backslash y}$$

"Lawvere Rule of inference"

convert need to define composition

$$b\overset{g}{\backslash} c \otimes a\overset{f}{\backslash} b \longrightarrow a\overset{g\circ f}{\backslash} c \qquad (g\circ f)x = gfx$$

into need for

$$b\backslash c \longrightarrow (a\backslash b)\backslash(a\backslash c) .$$

Everything relative to closed category:

replace "arrows $a \longrightarrow y$" by "object $a\backslash y$".

$$\begin{tikzcd}
(b\backslash c)\otimes((a\backslash b)\otimes a) \arrow[r, "1\otimes\varepsilon"] & (b\backslash c)\otimes b \arrow[r, "\varepsilon"] & c \\
((b\backslash c)\otimes(a\backslash b))\otimes a \arrow[u, "\mathrm{can}"] \arrow[urr, dashed, "t"'] & &
\end{tikzcd}$$

Take $t^{\#} : b\backslash c \otimes a\backslash b \longrightarrow a\backslash c$.

This defines composition.

SUPERIOR DEFN:

$$\begin{tikzcd}
b\backslash c \otimes a\backslash b \otimes a \arrow[r] \arrow[d, "1\otimes\varepsilon"'] \arrow[dr, phantom, "\underline{\text{defn}}"] & a\backslash c\otimes a \arrow[d, "\varepsilon"] \\
b\backslash c \otimes b \arrow[r, "\varepsilon"] & c
\end{tikzcd}$$

<u>Claim</u> It is assoc.

pf.

$$\begin{tikzcd}
((c\backslash d)\otimes(b\backslash c))\otimes(a\backslash b) \arrow[r, shift left, "\overset{?}{=}"] \arrow[r, shift right] \arrow[dr] \arrow[dd, "(\ \ )^{\#}\otimes \mathrm{id}_{(a\backslash b)}"'] & a\backslash d \\
& (c\backslash d)\otimes \\
(b\backslash d)\otimes(a\backslash b) &
\end{tikzcd}$$

multiply this by $-\otimes$

to show $m \overset{f}{\underset{g}{\rightrightarrows}} a\backslash d$ are equal,

show adjoints $m\otimes a \overset{f^b}{\underset{g^b}{\rightrightarrows}} d$ are equal.

$$\begin{tikzcd}
c\backslash d \otimes b\backslash c \otimes a\backslash b \otimes a \arrow[r] \arrow[d] \arrow[ddr] & c\backslash d \otimes a\backslash c \otimes a \arrow[dd] \\
b\backslash d \otimes a\backslash b \otimes a \arrow[d] & \\
a\backslash d \otimes a \arrow[r] & d
\end{tikzcd}$$

Th$^{\underline{m}}$ (Comp assoc &) has left/rt id.

ie for ea a have $k \xrightarrow{\eta_a} a\backslash a$:

since $k \otimes a \xrightarrow{\lambda_a} a$ given by mult cat

Must show truly a unit.

Th$^{\underline{m}}$ says that $\backslash$ introd as adjt to $\otimes$ does have assoc comp & left/rt unit.

ie any cl cat V is a V-based cat w $a\backslash b$.

recall: if $M^{\text{mult cat}}$ then M-based cat $\underline{C}$ is

$$\left|\begin{array}{l} c\backslash c' \in M \\ c'\backslash c'' \times c\backslash c' \xrightarrow{\cdot} c\backslash c'' \\ + \text{ units etc.} \end{array}\right.$$

Really, once have V based cat can get real cat out of it:

To construct

$$\begin{tikzcd}
\mathcal{V}, \otimes \arrow[d, "U"] \\
\mathcal{E}ns, \times
\end{tikzcd}$$

In $\mathbb{A}b$
$\mathbb{Z}$ is unit of $\otimes$ &
$A \cong \hom(\mathbb{Z}, A)$

<u>define</u> $Va = \hom(k, a)$ (wh- $k \otimes a \overset{\lambda}{\cong} a \overset{\rho}{\cong} a \otimes k$

[not nec. faithful]

?∃ $Ua \times Ub \xrightarrow{\varphi} U(a \otimes b)$?

$$\begin{tikzcd}
\hom(k,a) \times \hom(k,b) \arrow[rr, dashed] \arrow[dr] & & \hom(k, a \otimes b) \\
& \hom(k \otimes k, a \otimes b) \arrow[ur, "{\hom(\lambda_k, 1)}"'] &
\end{tikzcd}$$

$f, g \mapsto f \otimes g$

$$k \otimes k \xrightarrow{\lambda_k} k$$

Thm U, φ are a mp of mult cats.

pf. Need to sh coherence.

$$\begin{tikzcd}
\overset{f}{Ua} \times (\overset{g}{Ub} \times \overset{h}{Uc}) \arrow[r, "\alpha", "sets"'] \arrow[d] & (Ua \times Ub) \times Uc \arrow[d, "\varphi"] \\
Ua \times U(b \otimes c) \arrow[d] & U(a \otimes b) \times Uc \arrow[d, "\varphi"] \\
U(a \otimes (b \otimes c)) \arrow[r, "U\alpha"] & U((a \otimes b) \otimes c)
\end{tikzcd}$$

Chase elmnts.

What does U do to a\b ?
What relation to hom(a, b) ?

$$\begin{tikzcd}
U(a \backslash b) \arrow[d, equal] & & hom(a, b) \\
hom(k, a \backslash b) \arrow[d, equal] & & \\
hom(k \otimes a, b) \arrow[uurr, equal, "\lambda"'] \arrow[uurr, dash, bend left] & &
\end{tikzcd}$$

$\text{Th}^{\underline{m}}$ If V cl cat then $a \backslash b$ is lifting (up to 'isomp') of hom ... $V^{op} \times V \longrightarrow \text{Ens}$ ↙

$\mathcal{C}$ cat

$$\begin{tikzcd} C^{op} \times C \arrow[r, dashed, "\exists \text{ lift}"] \arrow[dr, "\text{hom}(-,-)"'] & C \arrow[d, "U"] \\ & \text{Ens} \end{tikzcd}$$

We do this not to get *rid* of arrs, but to *enrich* hom sets.

$$\begin{tikzcd} V \arrow[d, "U"] \\ \text{Ens} \end{tikzcd}$$

$\mathcal{C}$ any V-based cat.

$c, c' \in C$

$c \backslash c' \in V$

$$c' \backslash c'' \otimes c \backslash c' \xrightarrow{\circ} c \backslash c''$$

Claim such $\mathcal{C}$ gives honest cat:

Define $\text{hom}(c, c') = U(c \backslash c')$

An arrow $f : c \longrightarrow c'$ is

$$f \in U(c \backslash c') = \hom(k, c \backslash c')$$

[redacted]

Th$^{\underline{m}}$ Mp of mult cats $M \xrightarrow{(F,\varphi)} \bar{M}$.

$\boxed{\text{Maclane 35}}$

Given C an M-based cat

set C of obj's $a, b, c, \dots$

for ea pr a, b obj $a\backslash b$ of M

& "composition" $b\backslash c \otimes a\backslash b \to a\backslash c$

& units $k \xrightarrow{\eta_a} a\backslash a$

$F : M \to \bar{M}$ ftr

$Fa \,\bar{\otimes}\, Fb \xrightarrow[\varphi_{ab}]{} F(a\otimes b)$

$\bar{k} \xrightarrow{\varphi_k} F(k)$

Then C is an $\bar{M}$-based cat with evident structures:

$\bar{\circ}$

① $a \,\overline{\backslash}\, b = F(a\backslash b)$

②
$$\begin{tikzcd}
F(b\backslash c)\,\bar{\otimes}\,F(a\backslash b) \arrow[d, "\varphi"] \arrow[drr, dashed, "\bar{\circ}\ \text{new composition}"] & & \\
F(b\backslash c \otimes a\backslash b) \arrow[rr, "F\circ"'] & & F(a\backslash c)
\end{tikzcd}$$

③
$$\begin{tikzcd}
\bar{k} \arrow[r, "\varphi"] \arrow[rr, bend right, "\text{new unit}"'] & F(k) \arrow[r, "F\eta_a"] & F(a\backslash a)
\end{tikzcd}$$

$\boxed{\text{pf: cf V§7 notes}}$

pf. must show assoc. & unit diagrams hold for "$\bar{C}$".
Can it be proven w smaller diagrams?

efn. B total theory of single binary mult op :

B is a <u>mult cat</u> | obj $0, 1, 2, 3, \ldots, n, \ldots$

$+$ assoc.

arrs : (i) identities

(ii) $$\begin{tikzcd} 0 \arrow[r, "\eta"] & 1 & 2 \arrow[l, "\mu"'] \end{tikzcd}$$

(iii) all from these

(iv) $n \xrightarrow{\omega} 1$

all parenthesized words in $+$

ew defn a <u>mult cat</u> M is a cat M and for ea op $\omega : n \to 1$ of B & have ftr $\otimes_\omega : M^n \to M$ s.t. $\otimes_1 : M \to M = 1_M$. And, for ea pr ω, ω' of B $\omega, \omega' : n \to 1$ have nt xf

$$\begin{tikzcd} M^n \arrow[r, bend left=50, "\otimes_\omega"{name=U}] \arrow[r, bend right=50, "\otimes_{\omega'}"'{name=D}] & M \arrow[Rightarrow, from=U, to=D, "\alpha^{\omega}_{\omega'}"] \end{tikzcd}$$

s.t. $\alpha^{\omega'}_{\omega''} \circ \alpha^{\omega}_{\omega'} = \alpha^{\omega}_{\omega''}$.

$$\begin{tikzcd} k_1 + \cdots + k_n \arrow[r, "\lambda_1 + \cdots + \lambda_n"] \arrow[rr, bend right, "\omega(\lambda_1 + \cdots + \lambda_n)"'] & n \arrow[r, "\omega"] & 1 \end{tikzcd}$$

$$\otimes_\omega(\otimes_{\lambda_1}+\cdots+\otimes_{\lambda_n}) = \otimes_\omega(\lambda_1+\cdots+\lambda_n)$$

$$\alpha^{\omega}_{\omega'}(\alpha^{\lambda_1}_{\lambda_1'}+\cdots+\alpha^{\lambda_n}_{\lambda_n'}) = \alpha^{\omega(\lambda_1+\cdots+\lambda_n)}_{\omega'(\lambda_1+\cdots+\lambda_n)}$$

[this gives longer defn's + smaller <u>diagrams</u>]

new defn. <u>map</u> <u>of</u> <u>mult</u> <u>cat</u> $M \xrightarrow{(F,\varphi)} \overline{M}$ is

ftr $M \xrightarrow{F} \overline{M}$ s.t. $\exists\ \varphi_\omega$ s.t.

$$\begin{tikzcd}
M^n \arrow[r, "\otimes_\omega"] \arrow[d, "\overline{F}"'] & M \arrow[d, "F"] \\
\overline{M}^n \arrow[r, "\overline{\otimes}_\omega"'] \arrow[ur, phantom, "\overset{\varphi_\omega}{\nearrow}"] & \overline{M}
\end{tikzcd}$$

$\varphi_\omega : \overline{\otimes}_\omega \overline{F} \Rightarrow F\otimes$

<u>conditions</u>

① $\varphi_{\omega(\lambda_1+\cdots+\lambda_n)}$ = what it should be

② if $\omega, \omega' : n \longrightarrow 1$ in B then

$$\begin{tikzcd}
\overline{\otimes}_\omega F \arrow[r] \arrow[d, "\overline{\alpha}^{\omega}_{\omega'} F^n"'] & F\otimes_\omega \arrow[d, "\text{can } F\alpha^{\omega}_{\omega'}"] \\
\overline{\otimes}_{\omega'} F \arrow[r] & F\otimes_{\omega'}
\end{tikzcd}$$

defn. An <u>M-based</u> <u>category</u> C

objs — a set of objs & a map

$$: C \times C \xrightarrow{\nu} M \quad (ft_2)$$

$$\boxed{\langle a, b \rangle \mapsto a \backslash b}$$

$$\begin{tikzcd} C \times_C C \arrow[d, shift left] \arrow[d, shift right] \\ C \end{tikzcd}$$

<u>plbk</u>

composition : for ea word $\omega \in B$

<u>data</u>

$$\begin{tikzcd} (a_1 b_1)(a_2 b_2) \cdots (a_n b_n) \arrow[dd, bend right, mapsto] & & \\ & (C \times C)^n_C \arrow[r, "\nu^n"] \arrow[d] \arrow[dr, phantom, "\overset{\kappa_\omega}{\swarrow}"] & M^n \arrow[d, "\otimes_\omega"] \arrow[d, bend left=90, "\otimes_{\omega'}"] \arrow[d, phantom, shift left=6, "\overset{\alpha^{\omega}_{\omega'}}{\Leftarrow}"] \\ (a_1 \quad b_n) & C \times C \arrow[r, "\nu"] & M \end{tikzcd}$$

<u>condition</u>

① $\kappa_1 = 1$

② [illegible] $\kappa_\omega = \kappa_{\omega'} \circ (\alpha^{\omega'}_{\omega} \cdot \nu^n)$

Now show that $\underbrace{C}_{\text{M based cat}}$ goes to an $\overline{M}$ based cat

by a mp $M \to \overline{M}$ of mult cats.

DUBUC: thus a mult cat is a + preserving ftr $B \xrightarrow{M} \mathbf{Cat}$

[analogy: model of alg th]

JOEL COHEN: make B a 2-cat

makes α's into
$$\begin{tikzcd} n \arrow[r, bend left=50, "\omega"{name=U}] \arrow[r, bend right=50, "\omega'"'{name=D}] & 1 \arrow[from=U, to=D, phantom, "\Downarrow \alpha^{\omega}_{\omega'}"] \end{tikzcd}$$
2-cells.

Cl Cat V is mult cat w $\otimes_\omega$ s.t. ea $\_\otimes a$ has rt adjt $a\backslash\_$, so given $a\backslash b \otimes a \xrightarrow{\varepsilon} b$ evaluation & also given $a \xrightarrow{\delta} y\backslash(a\otimes y)$ + usual identities.

defn. Symmetric Cl Cat: all $\gamma : a\otimes b \cong b \otimes a$ given & coherent.

defn Closed Cartesian Cat: $\otimes$ is a product.

Ex. Ens, X ; $\mathbf{Cat}$ Top is not.

$X \dashv$ ftr cat

⑤

Given C V-based &

$$\begin{tikzcd} & C \arrow[dl, "T"'] \\ C' & \end{tikzcd}
\qquad
\begin{tikzcd} V \arrow[d, "{(F,\varphi)}"] \\ \bar{V} \end{tikzcd}
\qquad
\begin{tikzcd} a\backslash b \arrow[r, mapsto] \arrow[d, "F"'] & Ta\backslash Tb \arrow[d, "F"] \\ \hom(a,b) \arrow[r] & \hom(Ta,Tb) \end{tikzcd}$$

Kelley - Tensor Products in Algebras
[J. of Alg.]

Coherence In Closed Categories | Maclane 36 |

[Kelly & MacLane ; Lambek: using logical methods, i.e., adjunction in a rule of inference

| ogic by G. Gentzen |

Eilenberg-Moore: on "linkages" in J. of Alg

V symm cl cat $\otimes$ k α assoc $a\otimes k \underset{\beta}{\cong} a$

$\gamma: a\otimes b \cong b\otimes a$ coherent

$$\hom(a\otimes b, c) \cong \hom(a, [b,c])$$

$$[b,c]\otimes b \xrightarrow{\varepsilon} c \qquad \delta : x \longrightarrow [x\otimes a, a]$$

ε, δ an adjn.

fn (Shapes of ftrs): (i) k is a shape [the const.]

(ii) 1 is a shape

(ii) if S, T are shapes so is $S\otimes T$ and $[S,T]$

efn Proper shape never [const, non-const]

①

Every shape has list of variables $v(S)$

VI3, $v(k) = \emptyset$ $v(1) = \{*\}$ $v(S\otimes T) = v(S) + v(T)$ [ordinal sum]

$\triangleright v[S,T] = v(S) \overline{+} v(T)$

S a shape gives an actual ftr $|S| : \underbrace{V\times\cdots\times V}_{n = \text{length of } S} \longrightarrow V$

defn **Link** $\xi : S \to T$ is an involution on $V(S) \mp V(T)$ ~~no fixed pts~~

isomp of period 2

defn **compatible links**

cat, $\mathcal{G}$ | objs **shapes**
arrs **links**

This is a cl cat. Have $\otimes$ + $[\,,\,]$ on shapes. Need to produce ε, δ

Goal Thm If T and S are proper shapes and

$$\begin{tikzcd} {|T|} \arrow[r, bend left, "\theta"] \arrow[r, bend right, "\theta'"'] & {|S|} \end{tikzcd}$$

wt xfs, $\Gamma\theta = \Gamma\theta'$,

"graph"

then $\theta = \theta'$, provided θ, θ' are → defn θ **allowable**

$\mathcal{H} \xrightarrow{\Gamma} \mathcal{G}$

degenerates into total Enigma