SNAITH'S CONSTRUCTION OF COMPLEX K-THEORY

The purpose of this note is to explain a purely "algebraic" construction (starting with $\mathbb{CP}^\infty$), due to Snaith, of complex K-theory as a spectrum.

Consider the space $\mathbb{CP}^\infty$; this is a topological abelian group (as a $K(\mathbb{Z}, 2)$) with a multiplication map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ and a unit map $e : * \to \mathbb{CP}^\infty$. It follows that if we take $\mathbb{CP}^\infty_+$ (that is, if we add a disjoint basepoint), then we can make $\mathbb{CP}^\infty_+$ into an commutative algebra object in the category of *pointed* spaces, via the map

$$\mathbb{CP}^\infty_+ \wedge \mathbb{CP}^\infty_+ \simeq (\mathbb{CP}^\infty \times \mathbb{CP}^\infty)_+ \stackrel{m_+}{\to} \mathbb{CP}^\infty_+,$$

and the unit map

$$e_+ : S^0 \to \mathbb{CP}^\infty_+.$$

Taking suspension spectra, we find that $\Sigma^\infty \mathbb{CP}^\infty_+$ is a commutative ring spectrum: in fact, even an E_∞ ring spectrum because $\mathbb{CP}^\infty$ was a strictly commutative topological monoid. The claim is going to be that we can build K from this spectrum.

The first thing to check is whether $\Sigma^\infty \mathbb{CP}^\infty_+$ is complex-oriented, because K-theory is. To give a complex orientation, we have to give a map

$$\Sigma^{-2}\mathbb{CP}^\infty \to \Sigma^\infty \mathbb{CP}^\infty_+$$

which restricts to the unit map on $\Sigma^{-2}\mathbb{CP}^1 \simeq S^0$. Now, we don't quite have this, but we do definitely have a map

$$\mathbb{CP}^\infty \to \Sigma^\infty \mathbb{CP}^\infty_+$$

which restricts on $\mathbb{CP}^1$ to a certain element $\beta \in \pi_2(\Sigma^\infty \mathbb{CP}^\infty_+)$. (In fact, we have a stable splitting $\Sigma^\infty \mathbb{CP}^\infty_+ \simeq S \oplus \Sigma^\infty \mathbb{CP}^\infty$.)

This is in the wrong degree to be a complex orientation; however, if β were invertible, then we would get a complex orientation. In fact, since $\Sigma^\infty \mathbb{CP}^\infty_+$ is an E_∞-ring spectrum, there is a good theory of localization, and we can formally invert β to give an E_∞-ring spectrum $R[\beta^{-1}]$ together with a map of E_∞-ring spectra $R \to R[\beta^{-1}]$. Then we have:

Theorem 1 (Snaith [3])**.** *K-theory, as a ring spectrum, is the localization $\Sigma^\infty \mathbb{CP}^\infty_+[\beta^{-1}]$.*

The proof here is not the original proof; I learned this argument from Michael Hopkins (though any errors are mine).

1. The formal group law of $\Sigma^\infty \mathbb{CP}^\infty_+[\beta^{-1}]$

To show that we get K-theory from Snaith's construction, let's look at the formal group law of the ring spectrum $\Sigma^\infty \mathbb{CP}^\infty_+[\beta^{-1}]$: this is the key observation. For simplicity, let's write

$$R = \Sigma^\infty \mathbb{CP}^\infty_+, \ R[\beta^{-1}] = \Sigma^\infty \mathbb{CP}^\infty_+[\beta^{-1}].$$

Then $R[\beta^{-1}]$ is complex-oriented, via the map

$$\mathbb{CP}^\infty \to R \to R[\beta^{-1}],$$

which gives an element of $R[\beta^{-1}]^0(\mathbb{CP}^\infty)$ whose restriction to S^2 is an invertible element of $\pi_* R[\beta^{-1}]$.

Let x be the identity map $x : \Sigma^\infty \mathbb{CP}^\infty_+ \to R$. Then the complex orientation corresponds to $x - 1$ under the stable splitting $\Sigma^\infty \mathbb{CP}^\infty_+ = S \oplus \Sigma^\infty \mathbb{CP}^\infty$. Here 1 means the unit map $S \to \Sigma^\infty \mathbb{CP}^\infty_+$.

Proposition 1. The formal group law of $R[\beta^{-1}]$ is the multiplicative one.

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Proof. To see this, we have to compute the pull-back

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{m} \mathbb{CP}^\infty \to R \to R[\beta^{-1}].$$

But we have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{CP}^\infty_+) \wedge (\mathbb{CP}^\infty_+) & \xrightarrow{x \wedge x} & R \wedge R \ , \\ \downarrow m_+ & & \downarrow \\ \mathbb{CP}^\infty_+ & \xrightarrow{x} & R \end{array}$$

from the *definition* of the product structure on the ring spectrum R. In other words, the pull-back of the map $\mathbb{CP}^\infty_+ \to R$ to $\mathbb{CP}^\infty_+ \wedge \mathbb{CP}^\infty_+$ is just the *product*, i.e.

$$m_+^*(x) = p_1^*(x)p_2^*(x) \in R^*(\mathbb{CP}^\infty).$$

This works just as well in $R[\beta^{-1}]$. Since the complex orientation on $R[\beta^{-1}]$ corresponds to $x-1$, we find

$$m_+^*(x-1) = p_1^*(x)p_2^*(x) - 1 = p_1^*(x-1)p_2^*(x-1) + p_1^*(x-1) + p_2^*(x-1);$$

in other words, the formal group law of $R[\beta^{-1}]$ is as claimed.

□

We don't yet know what the coefficient ring of $R[\beta^{-1}]$ is, though. Note that if we use the convention that a complex orientation lives in $R[\beta^{-1}]^2(\mathbb{CP}^\infty)$, then the formal group law becomes $x+y+\beta xy$.

2. $\pi_* R[\beta^{-1}]$ is torsion-free and evenly graded

Classical K-theory is complex oriented and even-periodic, and comes with a map of ring spectra

$$MU \to K$$

classifying the multiplicative formal group law $f(x,y) = x+y+txy$ as well, over $\pi_* K \simeq \mathbb{Z}[t,t^{-1}]$ (or the multiplicative formal group law $f(x,y) = x+y+xy$ over $\pi_0 K \simeq \mathbb{Z}$ if one uses even periodicity). Landweber exactness gives an isomorphism for every spectrum X,

$$K_*(X) \simeq MU_*(X) \otimes_{\pi_* MU} \pi_* K.$$

As we saw in the previous section, $R[\beta^{-1}]$ is also complex oriented (via a map $MU \to R[\beta^{-1}]$) classifying the multiplicative formal group law $x+y+\beta xy$, and we would like to use Landweber exactness again of the multiplicative formal group law to write

$$(R[\beta^{-1}])_*(X) \simeq MU_*(X) \otimes_{\pi_* MU} \pi_* R[\beta^{-1}]$$

for any spectrum X. Unfortunately, we can't do this: we don't know that $\pi_* R[\beta^{-1}]$ is torsion-free, so we can't apply the exact functor theorem to $R[\beta^{-1}]$ yet.

So we need a result:

Proposition 2. $\pi_* R[\beta^{-1}]$ is torsion-free, and $\pi_{odd} R[\beta^{-1}] = 0$.

The latter statement implies that $R[\beta^{-1}]$ is *even periodic.*

Proof. The strategy here is to compute $\pi_* K \wedge R[\beta^{-1}]$ and to use the Landweber-exactness of K. Namely,

$$\pi_* K \wedge \mathbb{CP}^\infty_+ \simeq K_*(\mathbb{CP}^\infty) \simeq \mathbb{Z}[t,t^{-1}]\{\alpha_0, \alpha_1, \alpha_2, \dots\}, \quad \deg \alpha_i = 2i$$

is a free module on $\pi_* K \simeq \mathbb{Z}[t,t^{-1}]$, because K is complex-oriented. We can even get the algebra structure because it is (pre)dual to the coalgebra structure on $K^*(\mathbb{CP}^\infty)$. Localizing at β, we find that $\pi_* K \wedge R[\beta^{-1}]$ is itself torsion-free.

Now we want to go from here to concluding that $\pi_* R[\beta^{-1}]$ is torsion-free. In fact, we have an isomorphism given by Landweber exactness (of K-theory)

$$K_*(R[\beta^{-1}]) \simeq MU_*(R[\beta^{-1}]) \otimes_{\pi_* MU} \pi_* K.$$

The ring $MU_* R[\beta^{-1}]$ is given by $(\pi_* R[\beta^{-1}])[b_1, b_2, \dots]$: that is, it classifies the formal group law which is obtained from the multiplicative group law on $\pi_* R[\beta^{-1}]$ by a universal strict change of coordinates $x + b_1 x^2 + \dots$, i.e. the map $\pi_* MU \to MU_* R[\beta^{-1}]$ classifies this formal group law.

It follows from the tensor product description that $K_*(R[\beta^{-1}])$ is the ring classifying (on $\pi_*R[\beta^{-1}]$ algebras) an isomorphism of the multiplicative formal group law. More precisely, we can say that, given a graded-commutative ring C_*, to give a map $K_*(R[\beta^{-1}]) \to C_*$ is equivalent to giving maps (of graded-commutative rings) $\pi_*R[\beta^{-1}] \to C_*$ and $\mathbb{Z}[t, t^{-1}] \to C$, together with a sequence of elements (of appropriate even degree) $b_1, b_2, \dots \in C_*$ which give a *strict isomorphism* $x + b_1x^2 + \dots$ between the two formal group laws $x + y + \beta xy$ and $x + y + txy$ over C_{even}. In particular, we find that, as graded-commutative rings,

$$K_*(R[\beta^{-1}]) \simeq MU_*(K) \otimes_{\pi_*MU} \pi_*R[\beta^{-1}],$$

since the latter ring also has an equivalent description. Since we know that the left-hand-side is torsion-free and concentrated in even degrees, we want to conclude the same about $\pi_*R[\beta^{-1}]$.

We will prove this prime by prime. Fix a prime p. Then we have an isomorphism $K_*(R[\beta^{-1}])_{(p)} \simeq MU_*(K)_{(p)} \otimes_{\pi_*MU_{(p)}} \pi_*R[\beta^{-1}]_{(p)}$. Note that the two maps $\pi_*MU_{(p)} \to MU_*(K)_{(p)}, \pi_*MU_{(p)} \to \pi_*R[\beta^{-1}]_{(p)}$ have the property that they invert the element v_1 (which is the coefficient of x^p in the p-series of the formal group law), as they classify the multiplicative formal group laws. It follows that

$$K_*(R[\beta^{-1}])_{(p)} \simeq MU_*(K)_{(p)} \otimes_{\pi_*MU_{(p)}[v_1^{-1}]} \pi_*R[\beta^{-1}]_{(p)}. \tag{1}$$

Lemma 1. The map $\pi_*MU_{(p)}[v_1^{-1}] \to MU_*(K)_{(p)}$ is faithfully flat.

Proof. Write $L = \pi_*MU$ for the Lazard ring. Then the map $L \to MU_*(K)$ classifies the formal group law obtained from the multiplicative formal group law $f(x, y) = x + y + txy$ via a universal strict change of coordinates $x + b_1x^2 + \dots$. Alternatively, it classifies the formal group law obtained from the multiplicative formal group law $x + y + xy$ via a universal (not necessarily strict) change of coordinates $t(x + b_1x^2 + \dots)$ where t is invertible. Another way of saying this is that we have a pull-back square

$$\begin{array}{ccc} \mathrm{Spec} MU_*(K) & \longrightarrow & \mathrm{Spec} L \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z} & \longrightarrow & M_{FG} \end{array},$$

where M_{FG} is the moduli stack of formal groups.[1] The horizontal bottom map classifies the multiplicative formal group $x + y + xy$ over $\mathbb{Z}$.

If we localize at p, we get a pull-back square

$$\begin{array}{ccc} \mathrm{Spec} MU_*(K)_{(p)} & \longrightarrow & \mathrm{Spec} L_{(p)} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z}_{(p)} & \longrightarrow & M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)} \end{array},$$

and the map $\mathrm{Spec}\mathbb{Z}_{(p)} \to M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)}$ factors through the open substack of $M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)}$ given by the invertibility of v_1. So we have another pull-back square

$$\begin{array}{ccc} \mathrm{Spec} MU_*(K)_{(p)} & \longrightarrow & \mathrm{Spec} L_{(p)}[v_1^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z}_{(p)} & \longrightarrow & (M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)})[v_1^{-1}] \end{array},$$

and consequently, the assertion of the lemma will follow if we show that

$$\mathrm{Spec}\mathbb{Z}_{(p)} \to (M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)})[v_1^{-1}]$$

is faithfully flat. However, the map is flat (essentially by Landweber exactness), and it is faithfully flat because there is a unique "point" of the stack $(M_{FG} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}_{(p)})[v_1^{-1}]$ (in view of Lazard's classification of formal groups over an algebraically closed field). This proves the lemma. □

For this point of view on M_{FG} and Landweber exactness, see [2] and [1].

Anyway, we now find from the isomorphism (1) and the faithful flatness proved in the lemma that in fact, $\pi_*R[\beta^{-1}]_{(p)}$ must be torsion-free and concentrated in even degrees. This completes the proof. □

[1] *Not* formal group laws!

3. Completion of the proof

In the previous section, we showed that the ring spectrum $R[\beta^{-1}]$ was complex-oriented and even periodic, with no torsion in its homotopy groups. From this, it will be straightforward to show that it must be K.

We can produce a map of ring spectra

$$R \to K$$

by using the virtual bundle on $\mathbb{CP}^\infty_+$ given by $\mathcal{O}(1)$ on $\mathbb{CP}^\infty$ and 0 on $*$, to define the map $\Sigma^\infty \mathbb{CP}^\infty_+ \to K$. In other words, it is the map

$$\Sigma^\infty \mathbb{CP}^\infty_+ \simeq S \oplus \Sigma^\infty \mathbb{CP}^\infty \to K$$

sending S to K via the unit and $\Sigma^\infty \mathbb{CP}^\infty$ to K via $\mathcal{O}(1) - 1$. Alternatively, it is given by the element of $K^*(\mathbb{CP}^\infty)$ classified by $\mathcal{O}(1)$. Note that the pull-back of $\mathcal{O}(1)$ to $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ under the multiplication map $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ is precisely $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. This implies that $R \to K$ is a morphism of ring spectra.

This is in fact a map of *complex-oriented* ring spectra, and, since it sends β to the (invertible) Bott element in $\pi_2 K$, it factors through a map of complex-oriented ring spectra[2]

$$\phi : R[\beta^{-1}] \to K.$$

To prove Snaith's theorem, we need to see that it is an equivalence. That is, we need to show that it is an isomorphism on π_*. We know that $\phi_* : \pi_* R[\beta^{-1}] \to \pi_* K$ is a morphism of torsion-free rings. Now $\pi_0 R[\beta^{-1}] \simeq \mathbb{Z}$ (to see this, one localizes at $\mathbb{Q}$ after which the result is immediate as $\pi_* R \otimes \mathbb{Q} \simeq H_*(\mathbb{CP}^\infty; \mathbb{Q})$, and one notes that there is a map $\pi_0 R[\beta^{-1}] \to \pi_0 K \simeq \mathbb{Z}$). Consequently, the map $\pi_0 R[\beta^{-1}] \to \pi_0 K$ must be an isomorphism, and thus $R[\beta^{-1}] \to K$ is a weak equivalence by periodicity and $\pi_{odd} R[\beta^{-1}] = 0$. This completes the proof of Snaith's theorem.

References

1. Michael Hopkins, *Complex oriented cohomology theories and the language of stacks*, Lecture notes available at `http://www.math.rochester.edu/u/faculty/doug/otherpapers/coctalos.pdf`.
2. Jacob Lurie, *Chromatic homotopy theory*, Lecture notes available at `http://math.harvard.edu/~lurie/252x.html`.
3. Victor Snaith, *Localized stable homotopy of some classifying spaces*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 2, 325–330. MR 600247 (82g:55006)

[2] In fact, $R[\beta^{-1}]$ as a spectrum is the homotopy colimit $R \xrightarrow{\beta} \Sigma^{-2} R \xrightarrow{\beta} \dots$ and the map $R \to K$ uniquely extends to the whole diagram, and thus to the homotopy colimit as $K^{odd}(R) = 0$. One sees that it is a morphism of ring spectra similarly.