How do you identify one thing with another?

an intro to Homotopy Type Theory

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The Plan

-2) How do you identify one thing with another?

-1) An intro to type theory

0) Types of identifications

1) Propositions as types

2) The structure identity principle

3) Group Cohomology

n) Topological quantum gates...

jww Hisham Sati & Urs Schreiber

How do you identify one thing with another?

- It depends on what type of thing they are.
  - to identify a real affine space A with $\mathbb{R}^n$, it suffices to choose a point of origin in A.
  - to identify a n-dimensional vector space V with $\mathbb{R}^n$, it suffices to choose coordinates — a basis of V.
  - to identify the abelian group $H^n(S^n;\mathbb{Z})$ with $\mathbb{Z}$, it suffices to give an orientation of S^n.
  - to identify the integer n for which $\pi_4(S^3)$ is isomorphic to $\mathbb{Z}/n$ as $\mathbb{Z}$, it suffices to prove that n equals 2.

equality is a special case of identification

What are identifications?

- Identifications are mathematical objects in their own right!
  - points of origin, coordinate systems, orientations, ...
- To identify two mathematical structures, give an isomorphism between them.
  (a structure-preserving, invertible map.)

  ↳ "Structure identity principle"
- To identify two elements of a set, prove them equal.
  (e.g. numbers, vectors, points in space...

  ↳ "Propositions as types"
- To know how to identify X with Y, we only need to know what type of thing X and Y are.

Type Theory

- A formal system for tracking what type of thing everything is.
  - Types, like $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathrm{Vect}_{\mathbb{R}}$, Mfd, Type...
  - Elements, like $3 : \mathbb{N}$, $\pi : \mathbb{R}$, $\mathbb{R}^n : \mathrm{Vect}_{\mathbb{R}}$, $\mathbb{R}^n : \mathrm{Mfd}$, ...
- Variable elements, like $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"context"}} \vdash x^2+1 \overset{\text{"is a"}}{:} \mathbb{R}$$

- Variable types, $M : \mathrm{Mfd},\ p : M \vdash T_pM : \mathrm{Vect}_{\mathbb{R}}$
- Variable elements of variable type,

$$M : \mathrm{Mfd},\ p : M \vdash v(p) : T_pM$$

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

$$TM := (p:M) \times T_pM$$

Pair types:

- If $B(x)$ is a type when $x:A$, then

$$(x:A) \times B(x) \qquad A \times B$$

is the type of pairs (a,b) with $a:A$ and $b:B(a)$.

$$\mathrm{Vec}(M) := (p:M) \to T_pM$$

Function types:

- If $B(x)$ is a type when $x:A$, then

$$(x:A) \to B(x) \qquad A \to B$$

is the type of functions $x \mapsto f(x)$ where $x:A \vdash f(x):B(x)$.

Inductive Types

- $\mathbb{B}$, the type of booleans
  - ↳ We have $0 : \mathbb{B}$ and $1 : \mathbb{B}$, and
    - To define an element $b : \mathbb{B} \vdash e(b) : X(b)$, it suffices to define $e(0) : X(0)$ and $e(1) : X(1)$.

Assumptions / Conclusion

$$\frac{\Gamma \vdash e_0 : X(0) \qquad \Gamma \vdash e_1 : X(1)}{\Gamma,\ b : \mathbb{B} \vdash \mathrm{ind}(e_0, e_1)(b) : X(b)}$$

$$\mathrm{ind}(e_0, e_1)(0) :\equiv e_0, \quad \mathrm{ind}(e_0, e_1)(1) :\equiv e_1$$

Inductive Types

- $\mathbb{N}$, the type of natural numbers
  - We have $0 : \mathbb{N}$ and $n : \mathbb{N} \vdash n+1 : \mathbb{N}$, and to define $n : \mathbb{N} \vdash f(n) : X(n)$, it suffices to define $f(0) : X(0)$ and $n : \mathbb{N}, x : X(n) \vdash r(n,x) : X(n+1)$

"recursion"

$$\frac{\Gamma \vdash f_0 : X(0) \qquad \Gamma, n : \mathbb{N}, x : X(n) \vdash r(n,x) : X(n+1)}{\Gamma, n : \mathbb{N} \vdash \mathrm{ind}(f_0, r)(n) : X(n)}$$

$$\mathrm{ind}(f_0, r)(0) := f_0, \quad \mathrm{ind}(f_0, r)(n+1) := r(n, \mathrm{ind}(f_0, r)(n))$$

Inductive types: Identifications!

- For $x, y : A$, a type $(x =_A y)$ of ways to identify x with y as elements of A.

∘ We have a reflexive self-identification

$$x : A \vdash \mathrm{refl}_x : (x =_A x)$$

∘ To define an element

$$x : A,\ y : A,\ p : (x =_A y) \vdash f(x, y, p) : X(x, y, p)$$

it suffices to define

$$x : A \vdash r_f(x) : X(x, x, \mathrm{refl}_x)$$

$$f(x, x, \mathrm{refl}_x) := r_f(x).$$

To define an element

$$x:A,\ y:A,\ i:(x =_A y) \vdash f(x,y,i) : X(x,y,i)$$

it suffices to define

$$x:A \vdash f(x,x,\mathrm{refl}_x) : X(x,x,\mathrm{refl}_x)$$

So:

- We can define $\mathrm{inv} : (x =_A y) \to (y =_A x)$ by

  $$\mathrm{inv}(\mathrm{refl}_x) := \mathrm{refl}_x$$

- We can define $\bullet : (x =_A y) \times (y =_A z) \to (x =_A z)$ by

  $$\mathrm{refl}_x \bullet j := j$$

We can define $\mathrm{inv} : (x =_A y) \to (y =_A x)$ by

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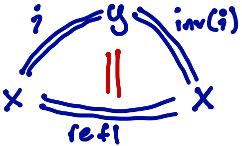
We can define $\bullet : (x =_A y) \times (y =_A z) \to (x =_A z)$ by

$$\mathrm{refl}_x \bullet j := j$$

- We can also define

$$\mathrm{invleft} : (j : (x = y)) \to (j \cdot \mathrm{inv}(j) =_{(x=x)} \mathrm{refl}_x)$$

by $\mathrm{invleft}(\mathrm{refl}_x) := \mathrm{refl}_{\mathrm{refl}_x}$



In total, any type becomes an ∞-groupoid

The structure identity principle says that identifications between mathematical structures are isomorphisms.

But our definition of identification was abstract

Can we prove that, e.g.

$$(\mathbb{R}^n \overset{}{\underset{\mathrm{Vect}_{\mathbb{R}}}{=}} V) \text{ is the same as } \{\text{bases of } V\}?$$

Yes!

Every function is a functor, every construction covariant.

- For $f : A \to B$, define $f_* : (x =_A y) \to (fx =_B fy)$ by
$$f_*(\mathrm{refl}_x) :\equiv \mathrm{refl}_{fx}$$
- If $C : A \to \mathrm{Type}$, define $\mathrm{tr}_C : (x = y) \to (C(x) \to C(y))$
by $$\mathrm{tr}_C(\mathrm{refl}_x) :\equiv \mathrm{id}_{C(x)}$$
- Eg: $$\Lambda^n : \left\{\begin{matrix}n\text{-dim}\\ \mathbb{R}\text{-vector}\\ \text{spaces}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}1\text{-dim}\\ \mathbb{R}\text{-vector}\\ \text{spaces}\end{matrix}\right\}$$
$$V \longmapsto \{\text{alternating } n\text{-forms on } V\}$$

By the SIP, $M : GL_n(\mathbb{R})$ determines $\overline{M} : \mathbb{R}^n = \mathbb{R}^n$.

Then $\Lambda^n_* \overline{M} : \Lambda^n \mathbb{R}^n = \Lambda^n \mathbb{R}^n$ is given by change of basis and is equal to scaling by $1/\det M$.

Every function is a functor, every construction covariant.

For emphasis:

if $\Gamma, V : \left\{\begin{matrix}n\text{-dim}\\ \mathbb{R}\text{-vector}\\ \text{Spaces}\end{matrix}\right\} \vdash C(V) : \mathrm{Type}$ is any construction which may be performed on an n-dimensional vector space, then

$$C(\mathbb{R}^n) \text{ has an action of } GL_n(\mathbb{R})$$

determined entirely by the definition of C.

In fact, we can show that

$$\left(\left\{\begin{matrix}n\text{-dim}\\ \mathbb{R}\text{-vector}\\ \text{Spaces}\end{matrix}\right\} \to \mathrm{Set}\right) \simeq \left\{\begin{matrix}\text{Actions of}\\ GL_n(\mathbb{R})\\ \text{on Sets}\end{matrix}\right\}$$

Propositions as Types

- Equality is a special case of identification
  if A is a set, then $(x=y)$ is a proposition.
- We can define other propositions as types

$$A : \mathrm{Type} \vdash \exists! A := (a:A) \times ((b:A) \to (a=b))$$

$$f : X \to Y \vdash \ulcorner f \text{ is a bijection} \urcorner := (y:Y) \to (\exists! (x:X) \times (y = fx))$$

- Remarkably $\exists! A$ means unique up to unique ident.
  and $\ulcorner f \text{ is a bijection} \urcorner$ means f is an equivalence
  of ∞-groupoids!

Propositions as Types

- A type A is a **proposition** if $\forall x, y : A.\ \exists!(x = y)$

  $\ulcorner A \text{ is a prop.} \urcorner :\equiv (x, y : A) \to \exists!(x = y)$

- A type A is a **set** if $\forall x, y : A$, $(x = y)$ is a prop.

  $\ulcorner A \text{ is a set} \urcorner :\equiv (x, y : A) \to \ulcorner (x = y) \text{ is a prop.} \urcorner$

- A type A is a **groupoid** if $\forall x, y : A.\ (x = y)$ is a set.

  $\ulcorner A \text{ is a groupoid} \urcorner :\equiv (x, y : A) \to \ulcorner (x = y) \text{ is a set} \urcorner$

- A type A is an **$(n+1)$-type** if $\qquad$ (-2)-type means $\exists!$

  $\forall x, y : A.\ ((x = y) \text{ is an } n\text{-type}).$

Propositional Truncation: $\exists$

Thm (UFP, Rijke): For any type A, there is a proposition $\exists A$ and a map $\eta : A \to \exists A$ so that

$$\begin{tikzcd} A \arrow[r, "\forall f"] \arrow[d, "\eta"'] \arrow[dr, phantom, "="] & P \\ \exists A \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

$\leftarrow$ a proposition

Thm (Kraus): To map from $\exists A$ into a ____ X

- Set, we need $f : A \to X$, $c_f : (x, y : A) \to (fx = fy)$
- groupoid, we need that and

$$coh_f : (x, y, z : A) \to \big(c_f(x,y) \cdot c_f(y,z) = c_f(x,z)\big)$$

$$\begin{tikzcd} & fy \arrow[dr, equal, "c_f(y,z)"] & \\ fx \arrow[ur, equal, "c_f(x,y)"] \arrow[rr, equal, "c_f(x,z)"'] \arrow[rr, phantom, bend left=20, "="] & & fz \end{tikzcd}$$

cocycle conditions from pure logic!

Structure Identity Principle:

$$\mathrm{Group} := \left\{\begin{aligned}
&(G : \mathrm{Type}) && \text{Data}\\
&\times (\cdot : G \times G \to G) && \\
&\times (-^{-1} : G \to G) && \text{Structure}\\
&\times (1 : G) && \\
&\times (G \text{ is a set}) && \\
&\times (g,h,k : G) \to ((g \cdot h)\cdot k = g \cdot (h \cdot k)) && \\
&\times (g : G) \longrightarrow (g \cdot 1 = g) \times (1 \cdot g = g) && \text{Properties}\\
&\times (g : G) \longrightarrow (g \cdot g^{-1} = 1) \times (g^{-1} \cdot g = 1) &&
\end{aligned}\right.$$

Univalence Axiom: $(A =_{\mathrm{Type}} B) \xrightarrow{\sim} (f : A \to B) \times \ulcorner f \text{ is a bijection} \urcorner$

$tr_C : (x = y) \to (C(x) \to C(y))$

Lemmas:

- $(a_1, b_1) =_{(a:A)\times B(a)} (a_2, b_2) \simeq (i : a_1 = a_2) \times (tr_B^{\,i}(b_1) = b_2)$
- $(f =_{A \to B} g) \simeq (x : A) \to (fx = gx)$
- E.g. $tr^{\,i}_{B \mapsto (B \times B \to B)}(f) = i \circ f \circ (i^{-1} \times i^{-1}) \ldots$

Group Cohomology

Def: A *delooping* of G is a pointed type $pt : BG$
st: ① $G \cong (pt =_{BG} pt)$ ② $\forall e : BG.\ \exists (pt = e)$

E.g. $\mathbb{R}^n : \left\{\begin{matrix}n\text{-dim} \\ \mathbb{R}\text{-vector} \\ \text{Spaces}\end{matrix}\right\}$ is a $BGL_n(\mathbb{R})$.

$G : \mathrm{Tors}_G$ is a BG

Thm (B-vD-R, B-C-TF-R): If A is *abelian*, it is infinitely deloopable:

set-truncation

$$B^{n+1}A := \underbrace{(X : \mathrm{Type})}_{\text{"Gerbe"}} \times \underbrace{\|X = B^n A\|_0}_{\text{"Band"}}$$

$$pt := (B^n A, |\mathrm{refl}|_0)$$

Group Cohomology

Def: $\tilde{H}^n(G;A) := \|(c : BG \to B^n A) \times (pt = c\,pt)\|_0$

E.g.

$\{$1-dim Hermitian vector spaces$\}$ $\quad (X : Type) \times \|X = Tors_{\mathbb{Z}}\|_0$

Lemma (M.): the map $c : BU(1) \to B^2\mathbb{Z}$ classifying the central extension $0 \to 2\pi\mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ can be defined by

$$V \mapsto \Big((A : Aff_{\mathbb{R}}) \times (e : A \to V) \times \begin{cases} \forall t : T.\ \|e(t)\| = 1 \\ \forall t : T, r : \mathbb{R},\ e(t+r) = e^{2\pi i r} e(t) \end{cases}$$

$$, \text{"if } \phi : C = V,\ (A, e) \mapsto \{a : A \mid e(a) = \phi(1)\}\text{"} \Big)$$

Topological Quantum Gates:

Thm (Sati-Schreiber-M.):

determined by a level κ and weights w_i, $i = 1 \ldots d$

Given a twist $c : BBr_{n+d} \to BU(1)$,

The set of twisted cohomology classes

$$c : BBr_d \vdash H^n_c(Br_n^{2c} ; \mathbb{C}) :=$$

$BBr_n^{2c} := \mathrm{fib}(c)$, $BBr_{n+d} \to BBr_d$

$$\| (V : BU(1)) \to (x : BBr_n^{2c}) \times (c(c,x) = V) \to B^n V \|_0$$

is bijective with the set of $\widehat{su}_2^{\kappa-2}$-conformal blocks for $d+1$ point correlators on the Riemann sphere with weights w_i (and $w_{d+1} = n + \sum w_i$).

And, transport in c is parallel transport along the Knizhnik-Zamolodchikov connection.

Thanks شكرا

Topological Quantum Gates in Homotopy Type Theory

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Anyonic Topological Order in
Twisted Equivariant Differential (TED) K-Theory

(arXiv:2206.13563)

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CENTRAL H-SPACES AND BANDED TYPES

ULRIK BUCHHOLTZ, J. DANIEL CHRISTENSEN, JARL G. TAXERÅS FLATEN, AND EGBERT RIJKE

Textbooks:

- The HoTT Book - UFP
- Introduction to HoTT - Rijke