Physics beyond the standard model with S_2 extra-space

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Abstract

We propose new approaches to construct models of the physics beyond the Standard Model using two-sphere(S^2) as the extra dimensional space. Especially, we are interested in Gauge-Higgs Unification models and Universal Extra Dimensional models and construct models based on these ideas with S^2 extra space. We analyze a gauge-Higgs unification model which is based on a gauge theory defined on a six-dimensional spacetime with an S^2 extra-space. We impose a symmetry condition for a gauge field and non-trivial boundary conditions of the S^2. We provide the scheme for constructing a four-dimensional theory from the six-dimensional gauge theory under these conditions. We then construct a concrete model based on an SO(12) gauge theory with fermions which lie in a 32 representation of SO(12), under the scheme. This model leads to a Standard-Model(-like) gauge theory which has gauge symmetry SU(3) $\times$ SU(2)$_L$ $\times$ U(1)$_Y$($\times$ U(1)^2) and one generation of SM fermions, in four-dimensions. The Higgs sector of the model is also analyzed, and it is shown that the electroweak symmetry breaking and the prediction of W-boson and Higgs-boson masses are obtained. The former attempts of constructing Gauge-Higgs Unification models are also introduced, which are based on Coset Space Dimensional Reduction scheme. The new Universal Extra Dimensional model is also constructed, which is defined on a six-dimensional spacetime with two-sphere orbifold S^2/Z_2 as an extra-space. We specify our model by choosing the gauge symmetry as SU(3)$\times$SU(2)$\times$U(1)$_Y$$\times$U(1)$_X$, introducing field contents in six-dimensions as their zero modes correspond to the Standard model particles, and determining a boundary condition of these fields on orbifold S^2/Z_2. A background gauge field that belongs to U(1)$_X$ is introduced there, which is necessary to obtain massless chiral fermions in four-dimensional spacetime. We then analyze Kaluza-Klein(KK) mode expansion of the fields in our model and derive the mass spectrum of the KK particles. We find that the lightest KK particles are the 1st KK particle of massless gauge bosons at tree level. We also discuss the KK parity of the KK modes in our model and confirm the stability of the lightest KK particle which is important for dark matter physics.

Contents

1 Introduction

The Standard Model(SM) has been successful in describing phenomenology of the elementary particle physics up to the energy of order Tev. Not only did it explain experimental results but it also gave us deeper insights that gauge symmetry governs the interactions among the particles and its spontaneous breaking rises particle masses. The SM, however, is not a satisfactory model since the choice of the gauge group and the contents of the particles are input of the model, and there are at least 18 parameters in the model even without neutrino mass and lepton mixing. Furthermore, there seem to be several problems in the SM, e.g. the hierarchy problem, lack of the candidate of dark mater, and so on. Thus it is suggested that the physics beyond the SM would appear around Tev scale and the SM is an effective theory originated from this high energy physics. We, therefore, are motivated to construct a model describing the physics beyond the SM which solve above problems. Among these problems, the hierarchy problem strongly drives physicists to construct a model beyond the SM. We need to explain the stability of the weak scale to solve the problem. For such an explanation, supersymmetry has been mostly employed and the consequence of these models are extensively explored. There are also other mechanisms, say, little higgs, extra dimensions, and so on. These have not been intensively studied compared to supersymmetric models. Furthermore there is still a room for new type of models. Since the Large Hadron Collider experiment is about to operate, which will explore the physics at TeV scale, it is urgent to investigate all the possible models at that scale.

Among these approaches, a physics beyond the SM with extra-dimension is very interesting. We, especially, are interested in Universal Extra Dimensional(UED) models and Gauge-Higgs Unification(GHU) models and provide new approaches to construct these models.

The gauge-Higgs unification is one of the attractive approaches to the physics beyond the SM since it provides origin of the Higgs sector in the SM [1, 2, 3] (for recent approaches, see Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]). In this approach, the Higgs particles originate from the extra-dimensional components of the gauge field of a gauge theory defined on spacetime with dimensions larger than four. Thus the Higgs sector is embraced into the gauge interactions in the higher-dimensional spacetime and part of the fundamental properties of Higgs scalar is determined from the gauge interactions. We then would obtain the Higgs sector described by fewer free parameter than the SM one and this Higgs sector is expected to be predictive.

Coset Space Dimensional Reduction (CSDR) scheme is one of the attractive approaches to construct a GHU model in this regard [1, 21, 22, 23, 24, 25]. This scheme introduces a compact extra dimensional space which has the structure of a coset of Lie groups, S/R. The Higgs field and the gauge field of the SM are merged into a gauge field of a gauge group G in the higher-dimensional spacetime. The SM fermions are unified into a representation of this gauge group. The particle contents surviving in four dimensional theory are determined by the identification of the gauge transformation as a rotation within the extra-dimensional space. The four dimensional gauge symmetries are determined by embedding of R into G. Since the Higgs originates from extra dimensional components of the gauge field, the Higgs and Yukawa sectors in four dimensional Lagrangian are uniquely determined. Furthermore, as shown in Ref. [27, 28, 29, 30], it is possible to obtain chiral fermions when total dimension, D, of the spacetime is even. The chiral fermions can be obtained even from (pseudo)real representations in $D = 8n + 2$ ($D = 8n + 6$) [27, 30].

Gauge theories in six- and ten-dimensional spacetime with simple gauge group are well investigated within the CSDR scheme. No known model, however, reproduced the particle content of the SM or Grand Unified Theory (GUT) [1, 22, 31, 32, 33, 34, 35, 36, 37, 38]. We then introduce new attempts to construct GHU model within the CSDR scheme. These approaches are based on extending the dimensionality of the space time and candidate of the gauge group; we applied a fourteen-dimensional gauge theory and an eight-dimensional gauge theory as former approach and direct product gauge group in ten-dimensions as latter approach. We also show difficulties in constructing a realistic model in these attempts.

We then provide a new approach of GHU model based on a gauge theory defined on the six-dimensional spacetime with the extra-space which has the structure of two-sphere S^2 to overcome some of the difficulties. We can impose on the fields of this gauge theory the symmetry condition which identifies the gauge transformation as the isometry transformation of S^2 as in the CSDR scheme [1, 21, 22, 23, 24] , since the S^2 has the coset space structure such as S^2=SU(2)/U(1). We then impose on the gauge field the symmetry in order to carry out the dimensional reduction of the gauge sector. The dimensional reduction is explicitly carried out by applying the solution of the symmetry condition, and a background gauge field is introduced as a part of the solution of the symmetry condition [1]. We obtain, by the dimensional reduction, the scalar sector with a potential term which leads to spontaneous symmetry breaking. The symmetry also restricts the gauge symmetry and the scalar contents originated from extra guage field components in four-dimensions. We, however, do not impose the symmetry on the fermion of the gauge theory, in contrast to other CSDR models. We then have massive Kaluza-Klein(KK) modes of fermion in four-dimensions while gauge and scalar fields have no massive KK mode, and would obtain a dark-matter candidate. Generally, the KK modes do not have massless mode because of positive curvature of S^2 [39]. We, however, obtain a massless KK mode because of existence of background gauge field; the fermion components which have the massless mode are determined by the background gauge field.

We also impose on fields of a six-dimensional theory the non-trivial boundary conditions of S^2 together with the symmetry condition in order to overcome the difficulty of breaking a GUT gauge symmetry. A GUT gauge symmetry can be broken to SM gauge symmetry by the non-trivial boundary conditions (for cases with orbifold extra-space, see for example [4, 5, 6, 7, 8, 11, 12, 16, 17, 18, 40, 41]).

We then analyze the gauge theory defined on the six-dimensional spacetime which has S^2 as extra-space, with the symmetry condition and non-trivial boundary conditions. The gauge symmetry, scalar contents and massless fermion contents are determined by the symmetry condition and the boundary conditions. First, we provide the scheme for constructing a four-dimensional theory from the six-dimensional gauge theory. We then construct the model based on SO(12) gauge symmetry and show that SM-Higgs doublet and one generation of massless fermions are obtained in four-dimensions. We also find that the electroweak symmetry breaking is realized and Higgs mass value is predicted by analyzing Higgs sector of the model.

The UED model is the extension of the SM to higher-dimensional spacetime and all the SM particles propagate extra dimensions [42, 43]. Indeed the minimal version of UED has recently been studied very much. It is a model with one extra dimension defined on an orbifold S^1/Z_2. This orbifold is given by identifying the extra spatial coordinate y with $-y$ and hence there are fixed points $y = 0, \pi$. By this identification chiral fermions are obtained. It is shown that this model is free from the current experimental constraints if the scale of extra dimension $1/R$, which is the inverse of the compactification radius R, is larger than 400 GeV [42, 44]. The dark matter can be explained by the first or second Kaluza-Klein (KK) mode [45], which is often the first KK photon, and this model can be discriminated from other models [46]. This model also can give plausible explanations for SM neutrino masses which are embedded in extended models [47].

The UED models with more than five dimensions have not been studied extensively despite the fact that it can explain some problems in the SM. The six dimensional models are particularly interesting. It is known that the number of the generations of quarks and leptons is derived by anomaly cancellations

[48] and the proton stability is guaranteed by a discrete symmetry of a subgroup of 6D Lorentz symmetry [49]. Since the above UED model was proposed as a six dimensional model with extra dimensions of T^2/Z_2, it is very interesting to pursue six dimensional models with an alternative compactification.

As a physically intriguing example, there is a model with two dimensional compact space S^2, which has so far received a little attention (see for some works in this direction on the Einstein-Maxwell theory [50, 51] and the gauge-Higgs unification [1, 52, 53]). In models with two spheres, it is well known that fermions cannot be massless because of the positive curvature and hence they have a mass of O($1/R$) [54, 39]. We cannot overcome the theorem simply by the orbifolding of the extra spaces. In another words, we have no massless fermion on the curved space with positive curvature, but we know a mechanism to obtain a massless fermion on that space by introducing a nontrivial background gauge field [55, 50]. The nontrivial background gauge field can cancel the spin connection term in the covariant derivative. As a result, a massless fermion naturally appears. Furthermore, we note that the background gauge field configuration is energetically favorable since the background gauge kinetic energy lowers a total energy. In order to realize chiral fermions, the orbifolding is required, for instance.

We provide here a new type of UED with S^2 extra dimensions. We will show that we can construct a model in six dimensions with S^2/Z_2 extra space. We extend the SM to the space, employ the method of background, and acquire chiral fermions. Due to this orbifolding, all the bosons of the SM can be massless in the SU(2) limit. This means the lowest states are completely consistent with the SM as they should be. Furthermore, there are KK modes for each particle, and the lightest mode among them is stable due to the KK parity originated from the orbifolding. Besides the complexity stemming from the structure of S^2 instead of S^1, the feature is quite similar up to the first KK mode. The difference appears from the second KK modes.

2 Gauge-Higgs Unification Model using Coset Space Dimensional Reduction

2.1 The brief review of Coset Space Dimensionla Reduction

In this subsection, we recapitulate the scheme of the coset space dimensional reduction (CSDR) and the construction of the four-dimensional theory by CSDR [22].

2.1.1 General case

We begin with a gauge theory with a gauge group G defined on a D-dimensional spacetime M^D. The spacetime M^D is assumed to be a direct product of the four-dimensional spacetime M^4 and a compact coset space S/R such that $M^D = M^4 \times S/R$, where S is a compact Lie group and R is a Lie subgroup of S. The dimension of the coset space S/R is thus $d \equiv D-4$, implying $\dim S - \dim R = d$. This assumption on the structure of extra-dimensional space requires the group R to be embedded into the group SO(d), which is a subgroup of the Lorentz group SO($1, D-1$). Let us denote the coordinates of M^D by $X^M = (x^\mu, y^\alpha)$, where x^μ and y^α are coordinates of M^4 and S/R, respectively. The spacetime index M runs over $\mu \in \{0,1,2,3\}$ and $\alpha \in \{4,5,\cdots,D-1\}$. We define the vielbein $e_M{}^A$ which relates the metric of the manifold M^D (the bulk spacetime), denoted by $g_{MN}(X)$, and that of the tangent space $T_X M^D$ (the local Lorentz frame), denoted by $h_{AB}(X)$, as $g_{MN} = e_M{}^A e_N{}^B h_{AB}$. Here $A = (\mu, a)$, where $a \in \{4,\cdots,D\}$, is the index for the coordinates of $T_X M^D$. We conventionally use $\mu,\nu,\lambda,\cdots$ to denote the indices for M^4; $\alpha,\beta,\gamma,\cdots$ for the coset space S/R; $a,b,c,\cdots$ for the algebra of the group S/R;

$M, N, \cdots$ for (μ, α); and A, B for (μ, a). We introduce a gauge field $A_M(x,y) = (A_\mu(x,y), A_\alpha(x,y))$, which belongs to the adjoint representation of the gauge group G, and fermions $\psi(x,y)$, which lies in a representation F of G. The action S of this theory is given by

$$\begin{aligned} S = & \int d^D X \sqrt{-g} \\ & \times \Big(-\frac{1}{8} g^{MN} g^{KL} \operatorname{Tr} F_{MK}(X) F_{NL}(X) \\ & \quad + \frac{1}{2} i \bar{\psi}(X) \Gamma^A e_A{}^M D_M \psi(X) \Big), \end{aligned} \tag{2.1}$$

where $g = \det g_{MN}$, $F_{MN}(X) = \partial_M A_N(X) - \partial_N A_M(X) - [A_M(X), A_N(X)]$ is the field strength, D_M is the covariant derivative on M^D, and Γ^A is the generators of the D-dimensional Clifford algebra.

The extra-dimensional space S/R admits S as an isometric transformation group, and we impose on $A_M(X)$ and $\psi(X)$ the following symmetry under this transformation in order to carry out the dimensional reduction [21, 57, 58, 59, 60, 61]. Consider a coordinate transformation which acts trivially on x and gives rise to a S-transformation on y as

$$(x, y) \to (x, sy), \tag{2.2}$$

where $s \in S$. We require that this coordinate transformation Eq. (2.2) should be compensated by a gauge transformation. This symmetry, connecting nontrivially the coordinate and gauge transformation, requires R to be embedded into G. The symmetry further leads to the following set of the symmetric condition on the fields:

$$A_\mu(x,y) = g(y;s) A_\mu(x, s^{-1}y) g^{-1}(y;s), \tag{2.3a}$$

$$\begin{aligned} A_\alpha(x,y) = & g(y;s) J_\alpha{}^\beta A_\beta(x, s^{-1}y) g^{-1}(y;s) \\ & + g(y;s) \partial_\alpha g^{-1}(y;s), \end{aligned} \tag{2.3b}$$

$$\psi(x,y) = f(y;s) \Omega \psi(x, s^{-1}y), \tag{2.3c}$$

where $g(y;s)$ and $f(y;s)$ are gauge transformations in the adjoint representation and in the representation F, respectively, and $J_\alpha{}^\beta$ and Ω are the rotation in the tangent space for the vectors and spinors, respectively. These conditions of Eq. (2.3) make the D-dimensional Lagrangian invariant under the S-transformation of Eq. (2.2) and therefore independent of the coordinate y of S/R. The dimensional reduction is then carried out by integrating over the coordinate y to obtain the four-dimensional Lagrangian. The four-dimensional theory consists of the gauge fields A_μ, fermions ψ, and in addition the scalars $\phi_a \equiv e_a{}^\alpha A_\alpha$.The gauge group reduces to a subgroup H of the original gauge group G. The dimensional reduction under the symmetric condition Eq. (2.3) and the assumption $h^{AB} = \operatorname{diag}(\eta^{\mu\nu}, -g^{ab})$, where $\eta_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$ and $g^{ab} = \operatorname{diag}(a_1, a_2, \cdots, a_d)$ with a_i's being positive, leads to the four-dimensional effective Lagrangian L^{eff} given by

$$\begin{aligned} L^{\text{eff}} = & -\frac{1}{4} F^t_{\mu\nu} F^{t\mu\nu} + \frac{1}{2} (D_\mu \phi_a)^t (D^\mu \phi^a)^t + V(\phi) \\ & + \frac{1}{2} i \bar{\psi} \Gamma^\mu D_\mu \psi + \frac{1}{2} i \bar{\psi} \Gamma^a e_a{}^\alpha D_\alpha \psi, \end{aligned} \tag{2.4}$$

where t is the index for the generators of the gauge group G. It is notable that the Lagrangian Eq. (2.4) includes the scalar potential $V(\phi)$, which is completely determined by the group structure as

$$V(\phi) = -\frac{1}{4} g^{ac} g^{bd} \times \mathrm{Tr}\left[\left(f_{ab}{}^{C} \phi_C - [\phi_a, \phi_b] \right) \left(f_{cd}{}^{D} \phi_D - [\phi_c, \phi_d] \right) \right], \tag{2.5}$$

where C and D runs over the indices of the algebra of S, and $f_{ab}{}^{C}$ is the structure constants of the algebra of S. This potential may cause the spontaneous symmetry breaking, rendering the final gauge group K a subgroup of the group H.

The scheme of CSDR substantially constrains the four-dimensional gauge group H and its representations for the particle contents as shown below. First, the gauge group of the four-dimensional theory H is easily identified as

$$H = C_G(R), \tag{2.6}$$

where $C_G(R)$ denotes the centralizer of R in G [21]. Note that this implies $G \supset H \times R$ up to the U(1) factors. Secondly, the representations of H for the Higgs fields are specified by the following prescription. Suppose that the adjoint representations of R and G are decomposed according to the embeddings $S \supset R$ and $G \supset H \times R$ as

$$\mathrm{adj}\, S = \mathrm{adj}\, R + \sum_s r_s, \tag{2.7}$$

$$\mathrm{adj}\, G = (\mathrm{adj}\, H, 1) + (1, \mathrm{adj}\, R) + \sum_g (h_g, r_g), \tag{2.8}$$

where r_ss and r_gs denote representations of R, and h_gs denote representations of H. The representation of the scalar fields are h_gs whose corresponding r_gs in the decomposition Eq. (2.8) are contained also in the set $\{r_s\}$. Thirdly, the representation of H for the fermion fields are determined as follows [64]. Let the group R be embedded into the Lorentz group SO(d) in such a way that the vector representation d of SO(d) is decomposed as

$$d = \sum_s r_s, \tag{2.9}$$

where r_s are the representations obtained in the decomposition Eq. (2.9). This embedding specifies a decomposition of the spinor representation σ_d of SO(d) into irreducible representations σ_is of R as

$$\sigma_d = \sum_i \sigma_i. \tag{2.10}$$

Now the representations of H for the four-dimensional fermions are found by decomposing F according to $G \supset H \times R$ as

$$F = \sum_f (h_f, r_f). \tag{2.11}$$

The representations of our interest are h_fs whose corresponding r_fs are found in $\{\sigma_i\}$ obtained in Eq. (2.10). Note that a phenomenologically acceptable model needs chiral fermions in the four dimensions as the SM does. This is possible only when the coset space S/R satisfies $\mathrm{rank}\, S = \mathrm{rank}\, R$, according to the non-trivial result due to Bott [63]. The chiral fermions are then obtained most straightforwardly when we introduce a Weyl fermion in $D = 2n$ ($n = 1, 2, \cdots$) dimensions and F is a complex representation [27, 28, 29, 30]. Interestingly, they can be obtained even if F is real or pseudoreal representation, provided $D = 4n + 2$ [27, 30]. The four-dimensional fermions are doubled in these cases, and these extra fermions are eliminated by imposing the Majorana condition on the Weyl fermions in $D = 4n+2$ dimensions[27, 30]. From this condition we get chiral fermions for $D = 8n + 2$ $(8n + 6)$ when F is real (pseudoreal). It is therefore interesting to consider $D = 6, 10, 14, 18, \cdots$.

Here we mention the effect of gravity. When we include the effect of gravity and consider dynamics of an extra-space we would find the difficulty to obtain stable extra space. This is the common difficulty of extra-dimensional models and some works have been done on this point. For example it is discussed in terms of radion fields which are the scalar fields originated from higher-dimensional components of metric after compactification [31],[32]. The effect of gravity to CSDR scheme is also discussed in [4], [5]. Although we agree that the effect of gravity is important, we do not discuss about the effect of gravity in this letter since it is beyond the scope of this letter.

2.1.2 The case of ten-dimensional spacetime with direct product gauge group

In this section, we briefly recapitulate the scheme of the coset space dimensional reduction in ten dimensions with a direct product gauge group [22, 56].

We begin with a gauge theory defined on a ten-dimensional spacetime M^{10} with a gauge group $G = G_1 \times G_2$ where G_1 and G_2 are simple Lie groups. Here M^{10} is a direct product of a four-dimensional spacetime M^4 and a compact coset space S/R, where S is a compact Lie group and R is a Lie subgroup of S. The dimension of the coset space S/R is thus $6 \equiv 10-4$, implying $\dim S - \dim R = 6$. This structure of extra-dimensional space requires the group R to be embedded into the group SO(6), which is a subgroup of the Lorentz group SO(1, 9). Let us denote the coordinates of M^{10} by $X^M = (x^\mu, y^\alpha)$, where x^μ and y^α are coordinates of M^4 and S/R, respectively. The spacetime index M runs over $\mu \in \{0, 1, 2, 3\}$ and $\alpha \in \{4, 5, \cdots, 9\}$. We introduce, in this theory, a gauge field $A_M(x, y) = (A_\mu(x, y), A_\alpha(x, y))$, which belongs to the adjoint representation of the gauge group G, and fermions $\psi(x, y)$, which lies in a representation F of G.

The extra-dimensional space S/R admits S as an isometric transformation group as discussed in sec. 2.1.1. The symmetry condition of CSDR is imposed on $A_M(X)$ and $\psi(X)$ in order to carry out the dimensional reduction as discussed in sec. 2.1.1 [21, 57, 58, 59, 60, 61].

The gauge symmetry and particle contents of the four-dimensional theory are substantially constrained by the CSDR scheme. We provide below the prescriptions to identify the four-dimensional gauge group H and its representations for the particle contents in direct product gauge group case.

First, the gauge group of the four-dimensional theory H is easily identified as Eq. (2.6) where $C_G(R)$ denotes the centralizer of R in $G = G_1 \times G_2$ [21]. Thus the four dimensional gauge group H is determined by the embedding of R into G. We then assume that R has also direct product structure $R = R_1 \times R_2$ so that we can embed them into G_1 and G_2. Here, R_1 and R_2 are not necessarily simple. We also assume that four dimensional gauge groups H is obtained from only G_1 up to U(1) factors. This assumption ensures the coupling unification if H is the gauge group of the SM. These conditions imply

$$G = G_1 \times G_2, \tag{2.12}$$
$$R = R_1 \times R_2, \tag{2.13}$$
$$G_1 \supset H \times R_1, \tag{2.14}$$
$$G_2 \supset R_2, \tag{2.15}$$

up to U(1) factors.

Secondly, the representations of H for the scalar fields are specified by the following prescription. Let us decompose the adjoint representation of S according to the embedding $S \supset R_1 \times R_2$ as,

$$\text{adj } S = (\text{adj } R_1, \mathbf{1}) + (\mathbf{1}, \text{adj } R_2) + \sum_s (r_{1s}, r_{2s}), \tag{2.16}$$

where r_{1s} and r_{2s} are representations of R_1 and R_2, respectively. We then decompose the adjoint representation of G_1 and G_2 according to the embeddings $G_1 \supset H \times R_1$ and $G_2 \supset R_2$, respectively;

$$\text{adj } G_1 = (\text{adj } H, \mathbf{1}) + (\mathbf{1}, \text{adj } R_1) + \sum_g (h_g, r_{1g}), \tag{2.17}$$

$$\text{adj } G_2 = \text{adj } R_2 + \sum_g r_{2g}, \tag{2.18}$$

where r_{1g}s and r_{2g}s denote representations of R_1 and R_2, and h_gs denote representations of H. The decomposition of $\text{adj}\, G$ thus becomes

$$\begin{aligned} \text{adj } G =& (\text{adj } G_1, \mathbf{1}) + (\mathbf{1}, \text{adj } G_2) \\ =& (\text{adj } H, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \text{adj } R_1, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \text{adj } R_2) \\ &+ \sum_g (h_g, r_{1g}, \mathbf{1}) + \sum_g (\mathbf{1}, \mathbf{1}, r_{2g}). \end{aligned} \tag{2.19}$$

The representation of the scalar fields are h_gs whose corresponding $(r_{1g}, \mathbf{1})$s in the decomposition Eq. (2.19) are contained also in the set $\{(r_{1s}, r_{2s})\}$ in Eq. (2.16). Note that the trivial representation $\mathbf{1}$s also remain in four-dimensions if corresponding $(\mathbf{1}, r_{2g})$s of Eq. (2.19) are also contained in the set $\{(r_{1s}, r_{2s})\}$ in Eq. (2.16).

Thirdly, the representation of H for the fermion fields is determined as follows [64]. Let the group R be embedded into the Lorentz group SO(6) in such a way that the vector representation $\mathbf{6}$ of SO(6) is decomposed as $\mathbf{6} = \sum_s (r_{1s}, r_{2s})$, where r_{1s} and r_{2s} are the representations obtained in the decomposition Eq. (2.16). This embedding specifies a decomposition of the Weyl spinor representations $\mathbf{4}(\bar{\mathbf{4}})$ of SO(6) under $\text{SO}(6) \supset R_1 \times R_2$ as

$$\mathbf{4} = \sum_i (\sigma_{1i}, \sigma_{2i}) \left(\bar{\mathbf{4}} = \sum_i (\overline{\sigma}_{1i}, \overline{\sigma}_{2i}) \right), \tag{2.20}$$

where $\sigma_{1i}(\overline{\sigma}_{1i})$s and $\sigma_{2i}(\overline{\sigma}_{2i})$s are irreducible representations of R_1 and R_2. We then decompose the $\text{SO}(1,9)$ Weyl spinor $\mathbf{16}$ according to $(\text{SU}(2) \times \text{SU}(2))(\approx \text{SO}(1,3)) \times \text{SO}(6)$ as

$$\mathbf{16} = (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}), \tag{2.21}$$

where $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations of $\text{SU}(2) \times \text{SU}(2)$ correspond to left- and right-handed spinors, respectively. We now decompose a representation F of the gauge group G. We take F_1 and F_2 to be a representation of G_1 and G_2 for the fermions in ten-dimensional spacetime. Decompositions of F_1 and F_2 are

$$F_1 = \sum_f (h_f, r_{1f}), \tag{2.22}$$

$$F_2 = \sum_f r_{2f}, \tag{2.23}$$

Table 1: A complete list of six-dimensional coset spaces S/R with $\operatorname{rank} S = \operatorname{rank} R$ [22]. The brackets in R clarifies the correspondence between the subgroup of R and the subgroup of S. The factor of R with subscript "max" indicates that this factor is a maximal regular subgroup of S.

No.	S/R
(i)	$\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$
(ii)	$\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$
(iii)	$\mathrm{SU}(4)/\mathrm{SU}(3) \times \mathrm{U}(1)$
(iv)	$\mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$
(v)	$\mathrm{G}(2)/\mathrm{SU}(3)$
(vi)	$\mathrm{SO}(7)/\mathrm{SO}(6)$
(vii)	$\mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$
(viii)	$\mathrm{SU}(3) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{U}(1)] \times \mathrm{U}(1)$
(ix)	$(\mathrm{SU}(2)/\mathrm{U}(1))^3$

under $G_1 \supset H \times R_1$ and $G_2 \supset R_2$. Therefore the decomposition of F becomes

$$F = \sum_f (h_f, r_{1f}, r_{2f}). \tag{2.24}$$

The representations for the left-handed(right-handed) fermions are h_fs whose corresponding (r_{1f}, r_{2f})s are found in $\{(\sigma_{1i}, \sigma_{2i})\}(\{(\overline{\sigma}_{1i}, \overline{\sigma}_{2i})\})$ obtained in Eq. (2.20). Note that a phenomenologically acceptable model needs chiral fermions in the four dimensions as the SM does. The chiral fermions are obtained most straightforwardly when we introduce a complex representation of G as F [27, 28, 29, 30]. More interesting is the possibility to obtain them if F is real representation, provided $\operatorname{rank} S = \operatorname{rank} R$ [63]. A pair of Weyl fermions appears in a same representation in this case, and one of the pair is eliminated by imposing the Majorana condition on the Weyl fermions [27, 30]. We thus apply the CSDR scheme to complex or real representations of gauge group G for fermions.

Coset space S/R of our interest should satisfy $\operatorname{rank} S = \operatorname{rank} R$ to generate chiral fermions in four dimensions [63]. This condition limits the possible S/R to the coset spaces collected in Table 1 [22]. The R of coset (i) in Table 1 with subscript "max" indicates that this is the maximal regular subgroup of the S. There, the correspondence between the subgroup of R and the subgroup of S is clarified by the brackets in R. For example, the coset space (iv) suggests direct product of $\mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2)/\mathrm{U}(1)$.

2.1.3 The case of eight-dimensional spacetime

In this section, we briefly recapitulate the scheme of the coset space dimensional reduction in eight dimensions [22, 62].

We begin with a gauge theory defined on an eight-dimensional spacetime M^8 with a simple gauge group G. Here M^8 is a direct product of a four-dimensional spacetime M^4 and a compact coset space S/R, where S is a compact Lie group and R is a Lie subgroup of S. The dimension of the coset space S/R is thus $4 \equiv 8 - 4$, implying $\dim S - \dim R = 4$. This structure of extra-dimensional space requires the group R be embedded into the group SO(4), which is a subgroup of the Lorentz group SO(1, 7). Let us denote the coordinates of M^8 by $X^M = (x^\mu, y^\alpha)$, where x^μ and y^α are coordinates of M^4 and S/R, respectively. The spacetime index M runs over $\mu \in \{0, 1, 2, 3\}$ and $\alpha \in \{4, 5, 6, 7\}$. In this theory, we introduce a gauge field $A_M(x, y) = (A_\mu(x, y), A_\alpha(x, y))$, which belongs to the adjoint representation of the gauge group G, and fermions $\psi(x, y)$, which lies in a representation F of G.

The extra-dimensional space S/R admits S as an isometric transformation group as discussed in sec. 2.1.1. The symmetry condition of CSDR is imposed on $A_M(X)$ and $\psi(X)$ in order to carry out the dimensional reduction as discussed in sec. 2.1.1 [21, 57, 58, 59, 60, 61].

The gauge symmetry and particle contents of the four-dimensional theory are substantially constrained by the CSDR scheme. We provide below the prescriptions to identify the four-dimensional gauge group H and its representations for the particle contents in eight-dimensional case.

First, the gauge group of the four-dimensional theory H is easily identified as Eq. (2.6) [21]. Thus the four dimensional gauge group H is determined by the embedding of R into G. These conditions imply

$$G \supset H \times R, \tag{2.25}$$

up to U(1) factors.

Second, the representations of H for the scalar fields are specified by the following prescription. Let us decompose the adjoint representation of S according to the embedding $S \supset R$ as,

$$\text{adj } S = \text{adj } R + \sum_s r_s. \tag{2.26}$$

We then decompose the adjoint representation of G according to the embeddings $G \supset H \times R$;

$$\text{adj } G = (\text{adj } H, \mathbf{1}) + (\mathbf{1}, \text{adj } R) + \sum_g (h_g, r_g), \tag{2.27}$$

where r_gs and h_gs denote representations of R and H, respectively. The representation of the scalar fields are h_gs whose corresponding r_gs in the decomposition Eq. (2.27) are also contained in the set $\{r_s\}$ in Eq. (2.26).

Third, the representation of H for the fermion fields is determined as follows [64]. The $\mathrm{SO}(1,7)$ Weyl spinor $\mathbf{8}$ is decomposed under its subgroup $(\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R})(\simeq \mathrm{SO}(1,3)) \times (\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2)(\simeq \mathrm{SO}(4))$ as

$$\mathbf{8} = (\mathbf{2}_L, \mathbf{1}, \mathbf{2}_1, \mathbf{1}) + (\mathbf{1}, \mathbf{2}_R, \mathbf{1}, \mathbf{2}_2), \tag{2.28}$$

where $(\mathbf{2}_L, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2}_R)$ representations of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ correspond to left- and right-handed spinors, respectively. The group R is embedded into the Lorentz ($\mathrm{SO}(1,7)$) subgroup $\mathrm{SO}(4)$ in such a way that the vector representation $\mathbf{4}$ of $\mathrm{SO}(4)$ is decomposed as $\mathbf{4} = \sum_s r_s$, where r_ss are the representations obtained in the decomposition Eq. (2.26). This embedding specifies a decomposition of the spinor representations $(\mathbf{2}_1, \mathbf{1})((\mathbf{1}, \mathbf{2}_2))$ of $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \supset R$ as

$$(\mathbf{2}_1, \mathbf{1}) = \sum_i (\sigma_{1i}) \quad \left((\mathbf{1}, \mathbf{2}_2) = \sum_i (\sigma_{2i}) \right). \tag{2.29}$$

We now decompose representation F of the gauge group G for the fermions in eight-dimensional spacetime. Decomposition of F is

$$F = \sum_f (h_f, r_f), \tag{2.30}$$

under $G \supset H \times R$. The representations for the left-handed (right-handed) fermions are h_fs whose corresponding r_fs are found in $\sigma_{1i}(\sigma_{2i})$ obtained in Eq. (2.29).

Table 2: A complete list of four-dimensional coset spaces S/R with rank$S$ = rank$R$. We also list the decompositions of the vector representation **4** and the spinor representation $(\mathbf{2}_1, \mathbf{1}) + (\mathbf{1}, \mathbf{2}_2)$ of $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ under the Rs. The representations of r_s in Eq. (2.26) and σ_{1i} and σ_{2i} in Eq. (2.29) are listed in the columns of "Branches of **4**" and "Branches of **2**", respectively.

S/R		Branches of **4**	Branches of **2**
(i)	Sp(4)/[SU(2) × SU(2)]	$(\mathbf{2}, \mathbf{2})$	$(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$
(ii)	SU(3)/[SU(2) × U(1)]	$\mathbf{2}(\pm 1)$	$\mathbf{2}(0)$ and $\mathbf{1}(\pm 1)$
(iii)	$(\mathrm{SU}(2)/\mathrm{U}(1))^2$	$(\pm\mathbf{1}, \pm\mathbf{1})$	$(\pm\mathbf{1}, 0)$ and $(0, \pm\mathbf{1})$

A phenomenologically acceptable model needs chiral fermions in
four dimensions as the SM does. The SO(1, 7) spinor is not self-dual and its charge conjugate state is in a different representation from itself. Thus the Majorana condition cannot be used to obtain a chiral structure from a vectorlike representation of G. Therefore, we need to introduce complex representation for eight-dimensional fermions. Thus eight-dimensional model possesses a completely different feature from $4n + 2$-dimensional models. We must work on complex representation for eight-dimensional fermions.

Finally coset space S/R of our interest should satisfy $\mathrm{rank}\, S = \mathrm{rank}\, R$ to generate chiral fermions in four dimensions [63]. We list all of four-dimensional coset spaces S/R satisfying the condition and decompositions of SO(4) spinor and vector representation in Table 2.

2.2 The search for models with CSDR

2.2.1 Models on fourteen-dimensional spacetime

In this section, we search for candidates of the coset space S/R, the gauge group G, and its representation F for fermions in the spacetime of the dimensionality $D = 14$ for phenomenologically acceptable models based on CSDR scheme [25]. Such models should induce a four-dimensional theory that has a gauge group $H \supset \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, and accomodates chiral fermions contained in the SM. This requirement constrains the D, S/R, G, F, and the embedding of R in G.

Number of dimensions D should be $2n$ in order to give chiral fermions in four dimensions. We are particularly interested in the case of $D = 4n+2$, where chiral fermions can be obtained in four dimensions even if F is real or pseudoreal. The simplest cases of $D = 6$ and 10 are well investigated. No known model, however, reproduced the particle contents of the SM or GUT. [1, 22, 31, 32, 33, 34, 35, 36, 37, 38]. This is due to the small dimensionality of the vector and spinor representations of SO(d). It is difficult when $d = 2$ and 6 to match r_ss from SO(d) vector and σ_is from SO(d) spinor with r_gs from $\mathrm{adj}\, G$ and r_fs from F, respectively (see Eqs. (2.9)-(2.11)). We consider a higher-dimensional spacetime to enlarge the dimensionality of SO(d) vector and spinor representations. More r_gs and r_fs will satisfy the matching prescription, and hence richer particle contents are obtained. Another merit of higher-dimensional spacetime is the increase of candidates of the coset space and thus of the gauge group. We thus investigate next smallest dimensionality of $D = 4n + 2$, which is $D = 14$.

Coset space S/R of our interest should have dimension $d = D-4 = 10$, implying $\dim S - \dim R = 10$, and should satisfy $\mathrm{rank}\, S = \mathrm{rank}\, R$ to generate chiral fermions in four dimensions [63]. These conditions limit the possible S/R to the coset spaces collected in Table 3. There the correspondence between the subgroup of R and the subgroup of S is clarified by the brackets in R. For example, the coset space (2) suggests direct sum of SO(7)/SO(6) and Sp(4)/[SU(2) × SU(2)]. The factor of R with subscript "max" indicates that this factor is a maximal regular subalgebra of S. For example, the coset (20) in Table 3 indicates that $[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$ is the maximal regular subgroup of Sp(4). We show the embedding of R in SO(10) in Table 4. The representations of r_s in Eq. (2.9) and σ_i in Eq. (2.10) are listed in the columns of "Branches of **10**" and "Branches of **16**", respectively.

The representation F of G for the fermions should be either complex or pseudoreal but not real, since the fermions of real representation do not allow the Majorana condition when $D = 14$ and induces doubled fermion contents after the dimensional reduction [27, 30]. Table 5 lists the candidate groups G and their complex and pseudoreal representations. Here we consider the dimensions of fermion representations less than 1025 since even larger representations yield numerous higher dimensional representations of fermion, under the gauge group of the SM or GUTs, in the four-dimensions. The representations in this table are the candidates of F.

We constrain the gauge group G by the following two criteria once we choose S/R out of the coset spaces listed in Table 3. First, G should have an embedding of R whose centralizer $C_G(R)$ is appropriate as a candidate of the four-dimensional gauge group H (recall Eq. (2.6)). In this paper, we consider the following groups as candidates of H: the GUT gauge groups such as E_6, SO(10), and SU(5); the SM gauge group SU(3) × SU(2) × U(1); and those with an extra U(1). Secondly, we consider only the regular subgroup of G when we decompose it to embed R. We then find that no candidate of G and S/R that satisfy this requirement gives E_6, $\mathrm{E}_6 \times \mathrm{U}(1)$, and SU(5) as H. We notice that the number of U(1)'s in R must be no more than that in H, since the U(1)'s in R is also a part of its centralizer, *i.e.* a part of H. We can thus exclude (26) − (35) in Table 3. The candidates of G for each S/R satisfying the above conditions are summarized in Table 6.

Careful consideration is necessary when there are more than one branch in decomposing G to its regular subgroup $H \times R$, since the different decomposition branches lead to different representations of H and R. Two cases deserve close attention. The first is the decomposition of $\mathrm{SO}(2n + 1)$. It has essentially two distinct branches of decomposition, one being

$$\mathrm{SO}(2n+1) \supset \mathrm{SO}(2k_0+1) \times \prod_i \mathrm{SO}(2k_i). \tag{2.31}$$

and the other being

$$\mathrm{SO}(2n+1) \supset \mathrm{SO}(2n) \supset \prod_i \mathrm{SO}(2k_i), \tag{2.32}$$

An example is the decomposition of Sp(4) ≃ SO(5) into SU(2) × U(1). One of the two branches of decomposition is Sp(4) ⊃ SU(2) × U(1), which is equivalent to SO(5) ⊃ SO(3) × SO(2), corresponding to Eq. (2.31). The other branch is Sp(4) ≃ SO(5) ⊃ SO(4) ≃ SU(2) × SU(2) ⊃ SU(2) × U(1), corresponding to Eq. (2.32). The two branches of decomposition lead to different branching of the representations. The second is the normalization of U(1) charge. The different normalizations provide different representations of H for four-dimensional fields.

$H = \mathrm{SO}(10)(\times \mathrm{U}(1))$

First we search for viable SO(10) models in four dimensions. We list below the combinations of S/R, G and F that provide $H = \mathrm{SO}(10)(\times\mathrm{U}(1))$ and the representations which contain field contents of the SM for the scalars and the fermions. We indicate the coset S/R with its number assigned in Table 3 The embedding of R into G is shown for each candidates since this embedding uniquely determines all

Table 3: A complete list of ten-dimensional coset spaces S/R with $\mathrm{rank}\, S = \mathrm{rank}\, R$. The brackets in R clarifies the correspondence between the subgroup of R and the subgroup of S. The factor of R with subscript "max" indicates that this factor is a maximal regular subalgebra of S.

No.	S/R
(1)	SO(11)/SO(10)
(2)	SO(7) × Sp(4)/SO(6) × [SU(2) × SU(2)]
(3)	G_2 × Sp(4)/SU(3) × [SU(2) × SU(2)]
(4)	SU(6)/SU(5) × U(1)
(5)	SO(9) × SU(2)/SO(8) × U(1)
(6)	SO(7) × SU(3)/SO(6) × [SU(2) × U(1)]
(7)	SU(4) × Sp(4)/[SU(3) × U(1)] × [SU(2) × SU(2)]
(8)	$(\mathrm{Sp}(4))^2$ × SU(2)/$[\mathrm{SU}(2) \times \mathrm{SU}(2)]^2$ × U(1)
(9)	G_2 × SU(3)/SU(3) × [SU(2) × U(1)]
(10)	Sp(4) × Sp(4)/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$ × [SU(2) × SU(2)]
(11)	Sp(4) × Sp(4)/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$ × [SU(2) × SU(2)]
(12)	Sp(6) × SU(2)/[Sp(4) × SU(2)] × U(1)
(13)	G_2 × SU(2)/SU(2) × SU(2) × U(1)
(14)	Sp(6)/Sp(4) × U(1)
(15)	G_2/SU(2) × U(1)
(16)	Sp(4) × SU(3) × SU(2)/[SU(2) × SU(2)] × [SU(2) × U(1)] × U(1)
(17)	SU(4) × SU(3)/[SU(3) × U(1)] × [SU(2) × U(1)]
(18)	SO(7) × $(\mathrm{SU}(2))^2$/SO(6) × $(\mathrm{U}(1))^2$
(19)	SU(5) × SU(2)/[SU(4) × U(1)] × U(1)
(20)	Sp(4) × SU(3)/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$ × [SU(2) × U(1)]
(21)	Sp(4) × SU(3)/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$ × [SU(2) × U(1)]
(22)	SU(3) × Sp(4)/[U(1) × U(1)] × [SU(2) × SU(2)]
(23)	SU(4) × SU(2)/SU(2) × SU(2) × U(1) × U(1)
(24)	G_2 × $(\mathrm{SU}(2))^2$/SU(3) × $(\mathrm{U}(1))^2$
(25)	SU(4)/SU(2) × U(1) × U(1)
(26)	Sp(4) × $(\mathrm{SU}(2))^3$/[SU(2) × SU(2)] × $(\mathrm{U}(1))^3$
(27)	$(\mathrm{SU}(3))^2$ × SU(2)/$[\mathrm{SU}(2) \times \mathrm{U}(1)]^2$ × U(1)
(28)	SU(4) × $(\mathrm{SU}(2))^2$/[SU(3) × U(1)] × $(\mathrm{U}(1))^2$
(29)	Sp(4) × $(\mathrm{SU}(2))^2$/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$ × $(\mathrm{U}(1))^2$
(30)	Sp(4) × $(\mathrm{SU}(2))^2$/$[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$ × $(\mathrm{U}(1))^2$
(31)	SU(3) × SU(3)/[U(1) × U(1)] × [SU(2) × U(1)]
(32)	Sp(4) × SU(2)/[U(1) × U(1)] × U(1)
(33)	SU(3) × $(\mathrm{SU}(2))^3$/[SU(2) × U(1)] × $(\mathrm{U}(1))^3$
(34)	$(\mathrm{SU}(2)/\mathrm{U}(1))^5$
(35)	SU(3) × $(\mathrm{SU}(2))^2$/[U(1) × U(1)] × $(\mathrm{U}(1))^2$

Table 4: The decompositions of the vector representation **10** and the spinor representation **16** of SO(10) under R's which are listed in Table 3 and have two or less U(1) factors. The representations of r_s in Eq. (2.9) and σ_i in Eq. (2.10) are listed in the columns of "Branches of **10**" and "Branches of **16**", respectively. The U(1) charges for the cosets (16) – (35) have a freedom of retaking the linear combination.

S/R		Branches of **10**	Branches of **16**
(1)	SO(10)	$\mathbf{10}$	$\mathbf{16}$
(2)	(SO(6), SU(2), SU(2))	$(\mathbf{6},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{4},\mathbf{2},\mathbf{1}), (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$
(3)	(SU(3), SU(2), SU(2))	$(\mathbf{3},\mathbf{1},\mathbf{1}), (\bar{\mathbf{3}},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{1}), (\bar{\mathbf{3}},\mathbf{1},\mathbf{2}), (\mathbf{1},\mathbf{2},\mathbf{1}), (\mathbf{1},\mathbf{1},\mathbf{2})$
(4)	SU(5)(U(1))	$\mathbf{5}(6)$, $\bar{\mathbf{5}}(-6)$	$\mathbf{1}(-15)$, $\bar{\mathbf{5}}(9)$, $\mathbf{10}(-3)$
(5)	SO(8)(U(1))	$\mathbf{8}_{\mathrm{v}}(0)$, $\mathbf{1}(2)$, $\mathbf{1}(-2)$	$\mathbf{8}_{\mathrm{s}}(-1)$, $\mathbf{8}_{\mathrm{c}}(1)$,
(6)	(SO(6), SU(2))(U(1))	$(\mathbf{6},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{2})(3)$, $(\mathbf{1},\mathbf{2})(-3)$	$(\mathbf{4},\mathbf{2})(0)$, $(\bar{\mathbf{4}},\mathbf{1})(3)$, $(\bar{\mathbf{4}},\mathbf{1})(-3)$
(7)	(SU(3), SU(2), SU(2))(U(1))	$(\mathbf{3},\mathbf{1},\mathbf{1})(-4)$, $(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(4)$, $(\mathbf{1},\mathbf{2},\mathbf{2})(0)$	$(\mathbf{3},\mathbf{1},\mathbf{2})(2)$, $(\bar{\mathbf{3}},\mathbf{2},\mathbf{1})(-2)$, $(\mathbf{1},\mathbf{1},\mathbf{2})(-6)$, $(\mathbf{1},\mathbf{2},\mathbf{1})(6)$
(8)	(SU(2), SU(2), SU(2), SU(2))(U(1))	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(2)$, $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(-2)$	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})(1)$, $(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(-1)$, $(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})(-1)$, $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(1)$
(9)	(SU(3), SU(2))(U(1))	$(\mathbf{3},\mathbf{1})(0)$, $(\bar{\mathbf{3}},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{2})(3)$, $(\mathbf{1},\mathbf{2})(-3)$	$(\mathbf{3},\mathbf{2})(0)$, $(\bar{\mathbf{3}},\mathbf{1})(3)$, $(\bar{\mathbf{3}},\mathbf{1})(-3)$, $(\mathbf{1},\mathbf{2})(0)$, $(\mathbf{1},\mathbf{1})(3)$, $(\mathbf{1},\mathbf{1})(-3)$
(10)	(SU(2), SU(2), SU(2))(U(1))	$(\mathbf{2},\mathbf{2},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{3})(2)$, $(\mathbf{1},\mathbf{1},\mathbf{3})(-2)$	$(\mathbf{2},\mathbf{1},\mathbf{3})(-1)$, $(\mathbf{1},\mathbf{2},\mathbf{3})(1)$, $(\mathbf{1},\mathbf{2},\mathbf{1})(3)$, $(\mathbf{2},\mathbf{1},\mathbf{1})(-3)$
(11)	(SU(2), SU(2), SU(2))(U(1))	$(\mathbf{2},\mathbf{2},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{1},\mathbf{2})(1)$, $(\mathbf{1},\mathbf{1},\mathbf{2})(-1)$, $(\mathbf{1},\mathbf{1},\mathbf{1})(2)$ $(\mathbf{1},\mathbf{1},\mathbf{1})(-2)$	$(\mathbf{1},\mathbf{2},\mathbf{2})(-1)$, $(\mathbf{1},\mathbf{2},\mathbf{1})(0)$, $(\mathbf{1},\mathbf{2},\mathbf{1})(2)$, $(\mathbf{2},\mathbf{1},\mathbf{2})(1)$ $(\mathbf{2},\mathbf{1},\mathbf{1})(0)$, $(\mathbf{2},\mathbf{1},\mathbf{1})(-2)$
(12)	(Sp(4), SU(2))(U(1))	$(\mathbf{4},\mathbf{2})(0)$, $(\mathbf{1},\mathbf{1})(2)$, $(\mathbf{1},\mathbf{1})(-2)$	$(\mathbf{5},\mathbf{1})(-1)$, $(\mathbf{1},\mathbf{3})(-1)$, $(\mathbf{4},\mathbf{2})(1)$
(13)	(SU(2), SU(2))(U(1))	$(\mathbf{4},\mathbf{2})(0)$, $(\mathbf{1},\mathbf{1})(2)$, $(\mathbf{1},\mathbf{1})(-2)$	$(\mathbf{4},\mathbf{2})(1)$, $(\mathbf{5},\mathbf{1})(-1)$, $(\mathbf{1},\mathbf{3})(-1)$
(14)	Sp(4)(U(1))	$\mathbf{4}(1)$, $\mathbf{4}(-1)$, $\mathbf{1}(2)$, $\mathbf{1}(-2)$	$\mathbf{5}(1)$, $\mathbf{4}(-2)$, $\mathbf{4}(0)$, $\mathbf{1}(3)$, $\mathbf{1}(1)$, $\mathbf{1}(-1)$,
(15a)	SU(2)(U(1))	$\mathbf{2}(3)$, $\mathbf{2}(-3)$, $\mathbf{2}(1)$, $\mathbf{2}(-1)$, $\mathbf{1}(-2)$, $\mathbf{1}(2)$	$\mathbf{3}(1)$, $\mathbf{2}(-4)$, $\mathbf{2}(2)$, $\mathbf{2}(-2)$, $\mathbf{2}(0)$, $\mathbf{1}(5)$, $\mathbf{1}(3)$, $\mathbf{1}(-3)$, $\mathbf{1}(1)$, $\mathbf{1}(-1)$
(15b)	SU(2)(U(1))	$\mathbf{4}(1)$, $\mathbf{4}(-1)$, $\mathbf{1}(2)$, $\mathbf{1}(-2)$	$\mathbf{5}(1)$, $\mathbf{4}(-2)$, $\mathbf{4}(0)$, $\mathbf{1}(3)$, $\mathbf{1}(1)$, $\mathbf{1}(-1)$,
(16)	(SU(2), SU(2), SU(2))(U(1), U(1))	$(\mathbf{2},\mathbf{2},\mathbf{1})(0,0)$, $(\mathbf{1},\mathbf{1},\mathbf{2})(3,0)$, $(\mathbf{1},\mathbf{1},\mathbf{2})(-3,0)$, $(\mathbf{1},\mathbf{1},\mathbf{1})(0,2)$ $(\mathbf{1},\mathbf{1},\mathbf{1})(0,-2)$	$(\mathbf{2},\mathbf{1},\mathbf{2})(0,1)$, $(\mathbf{1},\mathbf{2},\mathbf{2})(0,-1)$, $(\mathbf{2},\mathbf{1},\mathbf{1})(3,-1)$, $(\mathbf{2},\mathbf{1},\mathbf{1})(-3,-1)$ $(\mathbf{1},\mathbf{2},\mathbf{1})(3,1)$, $(\mathbf{1},\mathbf{2},\mathbf{1})(-3,1)$
(17)	(SU(3), SU(2))(U(1), U(1))	$(\mathbf{3},\mathbf{1})(0,-4)$, $(\bar{\mathbf{3}},\mathbf{1})(0,4)$, $(\mathbf{1},\mathbf{2})(3,0)$, $(\mathbf{1},\mathbf{2})(-3,0)$	$(\mathbf{3},\mathbf{2})(0,2)$, $(\bar{\mathbf{3}},\mathbf{1})(3,-2)$, $(\bar{\mathbf{3}},\mathbf{1})(-3,-2)$, $(\mathbf{1},\mathbf{2})(0,-6)$, $(\mathbf{1},\mathbf{1})(3,6)$, $(\mathbf{1},\mathbf{1})(-3,6)$
(18)	SO(6)(U(1), U(1))	$\mathbf{6}(0,0)$, $\mathbf{1}(2,0)$, $\mathbf{1}(-2,0)$, $\mathbf{1}(0,2)$, $\mathbf{1}(0,-2)$	$\mathbf{4}(1,-1)$, $\mathbf{4}(-1,1)$, $\bar{\mathbf{4}}(1,1)$, $\bar{\mathbf{4}}(-1,-1)$,
(19)	SU(4)(U(1), U(1))	$\mathbf{4}(0,-5)$, $\bar{\mathbf{4}}(0,5)$, $\mathbf{1}(2,0)$, $\mathbf{1}(-2,0)$	$\mathbf{6}(-1,0)$, $\mathbf{4}(1,5)$, $\bar{\mathbf{4}}(1,-5)$, $\mathbf{1}(-1,10)$, $\mathbf{1}(-1,-10)$
(20)	(SU(2), SU(2))(U(1), U(1))	$(\mathbf{3},\mathbf{1})(0,2)$, $(\mathbf{3},\mathbf{1})(0,-2)$, $(\mathbf{1},\mathbf{2})(3,0)$, $(\mathbf{1},\mathbf{2})(-3,0)$	$(\mathbf{3},\mathbf{2})(0,-1)$, $(\mathbf{3},\mathbf{1})(3,1)$, $(\mathbf{3},\mathbf{1})(-3,1)$, $(\mathbf{1},\mathbf{2})(0,3)$, $(\mathbf{1},\mathbf{1})(3,-3)$, $(\mathbf{1},\mathbf{1})(-3,-3)$
(21)	(SU(2), SU(2))(U(1), U(1))	$(\mathbf{2},\mathbf{1})(1,0)$, $(\mathbf{2},\mathbf{1})(-1,0)$, $(\mathbf{1},\mathbf{2})(0,3)$, $(\mathbf{1},\mathbf{2})(0,-3)$ $(\mathbf{1},\mathbf{1})(2,0)$, $(\mathbf{1},\mathbf{1})(-2,0)$	$(\mathbf{2},\mathbf{2})(-1,0)$, $(\mathbf{1},\mathbf{2})(2,0)$, $(\mathbf{1},\mathbf{2})(0,0)$, $(\mathbf{2},\mathbf{1})(1,3)$, $(\mathbf{2},\mathbf{1})(1,-3)$, $(\mathbf{1},\mathbf{1})(0,3)$, $(\mathbf{1},\mathbf{1})(0,-3)$, $(\mathbf{1},\mathbf{1})(-2,3)$, $(\mathbf{1},\mathbf{1})(-2,-3)$,
(22)	(SU(2), SU(2))(U(1), U(1))	$(\mathbf{2},\mathbf{2})(0,0)$, $(\mathbf{1},\mathbf{1})(a,c)$, $(\mathbf{1},\mathbf{1})(b,d)$, $(\mathbf{1},\mathbf{1})(-a,-c)$ $(\mathbf{1},\mathbf{1})(-b,-d)$, $(\mathbf{1},\mathbf{1})(a+b,c+d)$, $(\mathbf{1},\mathbf{1})(-a-b,-c-d)$	$(\mathbf{2},\mathbf{1})(0,0)$, $(\mathbf{1},\mathbf{2})(0,0)$, $(\mathbf{2},\mathbf{1})(b,d)$, $(\mathbf{2},\mathbf{1})(a,c)$ $(\mathbf{2},\mathbf{1})(-a-b,-c-d)$, $(\mathbf{1},\mathbf{2})(a+b,c+d)$, $(\mathbf{1},\mathbf{2})(-a,-c)$, $(\mathbf{1},\mathbf{2})(-b,-d)$

Table 4: (Continued.)

S/R		Branches of **10**	Branches of **16**
(23)	(SU(2), SU(2))(U(1), U(1))	$(\mathbf{2},\mathbf{2})(0,2)$, $(\mathbf{2},\mathbf{2})(0,-2)$, $(\mathbf{1},\mathbf{1})(2,0)$, $(\mathbf{1},\mathbf{1})(-2,0)$	$(\mathbf{3},\mathbf{1})(-1,0)$, $(\mathbf{1},\mathbf{3})(-1,0)$, $(\mathbf{2},\mathbf{2})(1,-2)$, $(\mathbf{2},\mathbf{2})(1,2)$, $(\mathbf{1},\mathbf{1})(-1,4)$, $(\mathbf{1},\mathbf{1})(-1,-4)$,
(24)	SU(3)(U(1), U(1))	$\mathbf{3}(0,0)$, $\bar{\mathbf{3}}(0,0)$, $\mathbf{1}(2,0)$, $\mathbf{1}(-2,0)$, $\mathbf{1}(0,2)$, $\mathbf{1}(0,-2)$	$\mathbf{3}(1,-1)$, $\mathbf{3}(-1,1)$, $\bar{\mathbf{3}}(1,1)$, $\bar{\mathbf{3}}(-1,-1)$, $\mathbf{1}(1,-1)$, $\mathbf{1}(-1,1)$, $\mathbf{1}(1,1)$, $\mathbf{1}(-1,-1)$
(25)	SU(2)(U(1), U(1))	$\mathbf{2}(-1,2)$, $\mathbf{2}(1,2)$, $\mathbf{2}(-1,-2)$, $\mathbf{2}(1,-2)$, $\mathbf{1}(2,0)$, $\mathbf{1}(-2,0)$	$\mathbf{3}(-1,0)$, $\mathbf{2}(2,2)$, $\mathbf{2}(0,2)$, $\mathbf{2}(0,-2)$, $\mathbf{2}(2,-2)$, $\mathbf{1}(-1,4)$, $\mathbf{1}(-1,-4)$, $\mathbf{1}(-3,0)$, $\mathbf{1}(1,0)$, $\mathbf{1}(-1,0)$
(26)	(SU(2), SU(2))(U(1), U(1), U(1))	$(\mathbf{2},\mathbf{2})(0,0,0)$, $(\mathbf{1},\mathbf{1})(2,0,0)$, $(\mathbf{1},\mathbf{1})(-2,0,0)$, $(\mathbf{1},\mathbf{1})(0,2,0)$, $(\mathbf{1},\mathbf{1})(0,-2,0)$, $(\mathbf{1},\mathbf{1})(0,0,2)$, $(\mathbf{1},\mathbf{1})(0,0,-2)$	$(\mathbf{2},\mathbf{1})(1,1,1)$, $(\mathbf{2},\mathbf{1})(-1,-1,1)$, $(\mathbf{2},\mathbf{1})(1,-1,-1)$, $(\mathbf{2},\mathbf{1})(-1,1,-1)$, $(\mathbf{1},\mathbf{2})(1,-1,1)$, $(\mathbf{1},\mathbf{2})(-1,1,1)$, $(\mathbf{1},\mathbf{2})(1,1,-1)$, $(\mathbf{1},\mathbf{2})(-1,-1,-1)$
(27)	(SU(2), SU(2))(U(1), U(1), U(1))	$(\mathbf{2},\mathbf{1})(3,0,0)$, $(\mathbf{2},\mathbf{1})(-3,0,0)$, $(\mathbf{1},\mathbf{2})(0,3,0)$, $(\mathbf{1},\mathbf{2})(0,-3,0)$, $(\mathbf{1},\mathbf{1})(0,0,2)$, $(\mathbf{1},\mathbf{1})(0,0,-2)$	$(\mathbf{2},\mathbf{2})(0,0,-1)$, $(\mathbf{2},\mathbf{1})(0,3,1)$, $(\mathbf{2},\mathbf{1})(0,-3,1)$, $(\mathbf{1},\mathbf{2})(3,0,1)$, $(\mathbf{1},\mathbf{2})(-3,0,1)$, $(\mathbf{1},\mathbf{1})(3,3,-1)$, $(\mathbf{1},\mathbf{1})(-3,3,-1)$, $(\mathbf{1},\mathbf{1})(3,-3,-1)$, $(\mathbf{1},\mathbf{1})(-3,-3,-1)$
(28)	SU(3)(U(1), U(1), U(1))	$\mathbf{3}(-4,0,0)$, $\bar{\mathbf{3}}(4,0,0)$, $\mathbf{1}(0,2,0)$, $\mathbf{1}(0,-2,0)$, $\mathbf{1}(0,0,2)$, $\mathbf{1}(0,0,-2)$	$\mathbf{3}(2,-1,1)$, $\mathbf{3}(2,1,-1)$, $\bar{\mathbf{3}}(-2,1,1)$, $\bar{\mathbf{3}}(-2,-1,-1)$, $\mathbf{1}(6,1,1)$, $\mathbf{1}(-6,-1,1)$, $\mathbf{1}(-6,1,-1)$, $\mathbf{1}(6,-1,-1)$
(29)	SU(2)(U(1), U(1), U(1))	$\mathbf{3}(2,0,0)$, $\mathbf{3}(-2,0,0)$, $\mathbf{1}(0,2,0)$, $\mathbf{1}(0,-2,0)$, $\mathbf{1}(0,0,2)$, $\mathbf{1}(0,0,-2)$	$\mathbf{3}(-1,1,1)$, $\mathbf{3}(-1,-1,-1)$, $\mathbf{3}(1,1,-1)$, $\mathbf{3}(1,-1,1)$, $\mathbf{1}(3,1,1)$, $\mathbf{1}(3,-1,-1)$, $\mathbf{1}(-3,1,-1)$, $\mathbf{1}(-3,-1,1)$
(30)	SU(2)(U(1), U(1), U(1))	$\mathbf{2}(1,0,0)$, $\mathbf{2}(-1,0,0)$, $\mathbf{1}(2,0,0)$, $\mathbf{1}(-2,0,0)$, $\mathbf{1}(0,2,0)$, $\mathbf{1}(0,-2,0)$, $\mathbf{1}(0,0,2)$, $\mathbf{1}(0,0,-2)$	$\mathbf{2}(1,1,-1)$, $\mathbf{2}(1,-1,1)$, $\mathbf{2}(-1,1,1)$, $\mathbf{2}(-1,-1,-1)$, $\mathbf{1}(2,1,1)$, $\mathbf{1}(2,-1,-1)$, $\mathbf{1}(-2,1,-1)$, $\mathbf{1}(-2,-1,1)$, $\mathbf{1}(0,1,1)$, $\mathbf{1}(0,-1,-1)$, $\mathbf{1}(0,1,-1)$, $\mathbf{1}(0,-1,1)$
(31)	SU(2)(U(1), U(1), U(1))	$\mathbf{2}(3,0,0)$, $\mathbf{2}(-3,0,0)$, $\mathbf{1}(0,2,0)$, $\mathbf{1}(0,-2,0)$, $\mathbf{1}(0,1,3)$, $\mathbf{1}(0,-1,-3)$, $\mathbf{1}(0,1,-3)$, $\mathbf{1}(0,-1,3)$	$\mathbf{2}(0,1,3)$, $\mathbf{2}(0,1,-3)$, $\mathbf{2}(0,0,0)$, $\mathbf{2}(0,-2,0)$, $\mathbf{1}(3,2,0)$, $\mathbf{1}(-3,2,0)$, $\mathbf{1}(3,0,0)$, $\mathbf{1}(-3,0,0)$, $\mathbf{1}(3,-1,3)$, $\mathbf{1}(-3,-1,3)$, $\mathbf{1}(3,-1,-3)$, $\mathbf{1}(-3,-1,-3)$
(32)	(U(1), U(1), U(1))	$(2,0,0)$, $(-2,0,0)$, $(0,-2,0)$, $(0,2,0)$, $(0,0,2)$, $(0,0,-2)$, $(2,2,0)$, $(-2,-2,0)$, $(2,-2,0)$, $(-2,2,0)$	$(3,1,-1)$, $(3,-1,1)$, $(-3,-1,-1)$, $(-3,1,1)$, $(1,3,1)$, $(-1,-3,1)$, $(-1,3,-1)$, $(1,-3,1)$, $(-1,1,1)$, $(1,-1,1)$, $(1,-1,-1)$, $(-1,1,1)$, $(1,1,1)$, $(-1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$

Table 4: (Continued.)

S/R		Branches of **10**	Branches of **16**
(33)	SU(2)(U(1), U(1), U(1), U(1))	**2**(3, 0, 0, 0), **2**(−3, 0, 0, 0), **1**(0, 2, 0, 0), **1**(0, −2, 0, 0), **1**(0, 0, 2, 0), **1**(0, 0, −2, 0), **1**(0, 0, 0, 2), **1**(0, 0, 0, −2)	**2**(0, 1, −1, 1), **2**(0, −1, 1, 1), **2**(0, 1, 1, −1), **2**(0, −1, −1, −1), **1**(3, 1, 1, 1), **1**(−3, 1, 1, 1), **1**(3, −1 − 1, 1), **1**(−3, −1, −1, 1), **1**(3, 1, −1, −1), **1**(−3, 1, −1, −1), **1**(3, −1, 1, −1), **1**(−3, −1, 1, −1)
(34)	(U(1), U(1), U(1), U(1), U(1))	(2, 0, 0, 0, 0), (−2, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, −2, 0, 0, 0), (0, 0, 2, 0, 0), (0, 0, −2, 0, 0), (0, 0, 0, 2, 0), (0, 0, 0, −2, 0), (0, 0, 0, 0, 2), (0, 0, 0, 0, −2)	(1, 1, 1, −1, 1), (−1, −1, 1, −1, 1), (1, 1, −1, 1, 1), (−1, −1, −1, 1, 1), (1, 1, 1, 1, −1), (−1, −1, 1, 1, −1), (1, 1, −1, −1, −1), (−1, −1, −1, −1, −1), (1, −1, 1, 1, 1,), (−1, 1, 1, 1, 1), (1, −1, −1, −1, 1), (−1, 1, −1, −1, 1), (1, −1, 1, −1, −1), (−1, 1, 1, −1, −1), (1, −1, −1, 1, −1), (−1, 1, −1, 1, −1)
(35)	(U(1), U(1), U(1), U(1))	(1, 3, 0, 0), (−1, −3, 0, 0), (−1, 3, 0, 0), (1, −3, 0, 0), (2, 0, 0, 0), (−2, 0, 0, 0), (0, 0, 2, 0), (0, 0, −2, 0), (0, 0, 0, 2), (0, 0, 0, −2)	(2, 0, −1, 1), (−2, 0, 1, 1), (2, 0, 1, −1), (−2, 0, −1, −1), (0, 0, −1, 1), (0, 0, 1, 1), (0, 0, 1, −1), (0, 0, −1, −1), (1, 3, 1, 1), (1, −3, 1, 1), (−1, 3, −1, 1), (−1, −3, −1, 1), (1, 3, −1, −1), (1, −3, −1, −1), (−1, 3, 1, −1), (−1, −3, 1, −1)

Table 5: The gauge groups that have either complex or pseudoreal representations and their complex and pseudoreal representations whose dimension is no larger than 1024 [65]. The groups SU(8) and SU(9) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of S/R in Table 3.

Group	Complex representations	Pseudoreal representations
SU(7)	**21**, **28**, **35**, **84**, **112**, **140**, $\cdots$	
SO(12)		**32**, **32′**, **352**, **352′**
SO(13)		**64**, **768**
Sp(12)		**208**, **364**
E_6	**27**, **351**, **351′**	
SO(14)	**64**, **832**	
Sp(14)		**350**, **560**, **896**
Sp(16)		**544**, **816**
SU(10)	**45**, **55**, **120**, **210**, **220**, **330**, $\cdots$	
SO(18)	**256**	
SO(19)		**512**
Sp(18)		**798**
SO(20)		**512**
SO(21)		**1024**

Table 6: The allowed candidates of the gauge group G for each choice of H and S/R. The top row indicates H and the left column indicates S/R by the number assigned in Table 3.

	SO(10)	SO(10) × U(1)	SU(5) × U(1)	SU(3) × SU(2) × U(1)	SU(3) × SU(2) × U(1) × U(1)
(1)	SO(20)				
(2)	SO(20)				
(4)		SO(20), SO(21)	SU(10)		SU(10), SO(18), SO(19)
(5)		SO(20), SO(21)	SO(18), SO(19)		SO(18), SO(19)
(6)		SO(20), SO(21)	SO(19)		SO(18), SO(19), Sp(18)
(7)					SO(19), Sp(18)
(8)		SO(20), SO(21)	SO(18), SO(19), Sp(18)	Sp(16)	SO(18), SO(19), Sp(18)
(9)					Sp(16)
(10)		SO(18), SO(19)	Sp(16)	SO(14), Sp(14)	Sp(16)
(11)		SO(18), SO(19)	Sp(16)	SO(14), Sp(14)	Sp(16)
(12)		SO(19)	Sp(16)	Sp(14)	Sp(16)
(13)			SO(14), Sp(14)	SO(13), Sp(12)	SO(14), Sp(14)
(14)			Sp(14)	Sp(12)	Sp(16)
(15)		SO(14)	SU(7), SO(13), Sp(12)	SO(10), SO(11), Sp(10)	SU(7), SO(12), SO(13), Sp(12), E_6
(16)					Sp(16)
(17)					Sp(16)
(18)					SU(9), Sp(16)
(19)					SU(9), Sp(16)
(20)					SO(14), Sp(14)
(21)					SO(14), Sp(14)
(22)					SO(14), Sp(14)
(23)					SO(14), Sp(14)
(24)					SU(8), Sp(14)
(25)					SU(7), SO(12), SO(13), Sp(12), E_6

the representations of the scalars and fermions in the four-dimensional theory. In Table 7, we show all the field contents in four dimensions for each combination of $(S/R, G, F)$.

(a) $S/R\,(11) = \mathrm{Sp}(4) \times \mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}} \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$, $G = \mathrm{SO}(19)$, and $F = \mathbf{512}$.
We embed R in the subgroup SU(2)×SU(2)×SU(2)×U(1) of SO(19) according to the decomposition

$$\begin{aligned} \mathrm{SO}(19) &\supset \mathrm{SO}(10) \times \mathrm{SO}(9) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{2.33}$$

Notice that there is another branch of the decomposition such as

$$\begin{aligned} \mathrm{SO}(19) &\supset \mathrm{SO}(18) \supset \mathrm{SO}(10) \times \mathrm{SO}(8) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{2.34}$$

As mentioned at the beginning of this section, it gives different representations of the subgroup SO(10)× SU(2) × SU(2) × SU(2) × U(1) for a representation of SO(19). For example, the adjoint representation **171** of SO(19) is decomposed according to decomposition branch Eq. (2.33) and Eq. (2.34) as follows [65, 66]:

Table 7: The field contents in four dimensions with H = SO(10)(×U(1)) for each combination of $(S/R, G, F)$. Coset spaces are indicated by the number assigned in Table 3. Numbers in a superscript of the representations denote its multiplicity.

14D model			4D model		
S/R	G	F	H	Scalars	Fermions
(1)	SO(20)	**512**	SO(10)	**10**	**16**
(2)	SO(20)	**512**	SO(10)	$\{\mathbf{10}\}^2$	$\{\mathbf{16}\}^2$
(4)	SO(20)	**512**	SO(10) × U(1)	**10**(2), **10**(−2)	**16**(−1), **16**(3), **16**(−5)
(5)	SO(20)	**512**	SO(10) × U(1)	**10**(0), **10**(2), **10**(−2)	**16**(1), **16**(−1)
(6)	SO(20)	**512**	SO(10) × U(1)	**10**(0), **10**(1), **10**(−1)	**16**(0), **16**(1), **16**(−1)
(8)	SO(20)	**512**	SO(10) × U(1)	**10**(0), **10**(0), **10**(2), **10**(−2)	**16**(1), **16**(1), **16**(−1), **16**(−1)
(10)	SO(18)	**256**	SO(10) × U(1)	**10**(0)	**16**(3), **16**(−3), $\overline{\mathbf{16}}$(−3), $\overline{\mathbf{16}}$(3)
(11)	SO(18)	**256**	SO(10) × U(1)	**10**(0)	**16**(2), **16**(−2), $\overline{\mathbf{16}}$(−2), $\overline{\mathbf{16}}$(2)
(11)	SO(19)	**512**	SO(10) × U(1)	**10**(0), **10**(2), **10**(−2)	**16**(1), **16**(−1), $\overline{\mathbf{16}}$(1), $\overline{\mathbf{16}}$(−1)
(15)	SO(14)	**64**	SO(10) × U(1)	(a): **10**(1), **10**(−1), **1**(2), **1**(−2) (b): **10**(3), **10**(−3)	(a): **16**(0), **16**(1), **16**(−1), $\overline{\mathbf{16}}$(0), $\overline{\mathbf{16}}$(−1), $\overline{\mathbf{16}}$(1) (b): **16**(0), **16**(3), **16**(−3), $\overline{\mathbf{16}}$(0), $\overline{\mathbf{16}}$(−3), $\overline{\mathbf{16}}$(3)

$$
\begin{aligned}
\mathbf{171} \;\; = \;\; & (\mathbf{45},\mathbf{1},\mathbf{1},\mathbf{1})(0)+(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})(0)\\
& +(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})(0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})(0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(0)\\
& +(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(2)+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(-2)+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3})(0)\\
& +(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})(2)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})(-2)\\
& +(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})(0)+(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})(2)\\
& +(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})(-2)+(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{3})(0),
\end{aligned} \tag{2.35}
$$

$$
\begin{aligned}
\mathbf{171} \;\; = \;\; & (\mathbf{45},\mathbf{1},\mathbf{1},\mathbf{1})(0)+(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})(0)\\
& +(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})(0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})(0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(0)\\
& +(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})(0)+(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})(0)\\
& +(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(0)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(2)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(-2)\\
& +(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})(1)+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})(-1)\\
& +(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{2})(1)+(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{2})(-1)\\
& +(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1)+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(-1).
\end{aligned} \tag{2.36}
$$

The singlets of $SU(2)\times SU(2)\times SU(2)\times U(1)$, which are $(\mathbf{45},\mathbf{1},\mathbf{1},\mathbf{1})(0)$ and $(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})(0)$, form an adjoint representation of SO(11) which is $(\mathbf{55},\mathbf{1},\mathbf{1},\mathbf{1})(0)$. This indicates that the centralizer of $SU(2)\times SU(2)\times SU(2)\times U(1)$ is not $H = SO(10)\times U(1)$ but $SO(11)\times U(1)$, which is irrelevant to our purpose.

(b) S/R (15a) $= G_2/SU(2)\times U(1)$, $G = SO(14)$, and $F = \mathbf{64}$.

We embed R in the subgroup $SU(2)\times U(1)$ of $G = SO(14)$ according to the decomposition

$$
\begin{aligned}
SO(14) &\supset SO(10)\times SU(2)\times SU(2)\\
&\supset SO(10)\times SU(2)\times U(1).
\end{aligned} \tag{2.37}
$$

There are two branches of embedding which leads to the field contents of the SM in this case, owing to the freedom of the normalization of U(1) charges as mentioned in the beginning part of this section. For example, the adjoint representation of SO(14) can be decomposed according to Eq. (2.37) as [65, 66]

$$
\begin{aligned}
\mathbf{91} = & (\mathbf{45},\mathbf{1})(0)+(\mathbf{1},\mathbf{3})(0)+(\mathbf{1},\mathbf{1})(0)\\
& +(\mathbf{1},\mathbf{1})(2x)+(\mathbf{1},\mathbf{1})(-2x)\\
& +(\mathbf{10},\mathbf{2})(x)+(\mathbf{10},\mathbf{2})(-x),
\end{aligned} \tag{2.38}
$$

where x is an arbitrary number reflecting the freedom of the normalization. The choice of $x = 1$ and $x = 3$ leads to the scalar contents (a) and (b) of Table 7 respectively, as can be seen by comparing the U(1) charges of Eq. (2.38) with those in the row (15a) of Table 4.

(c) S/R (1) $= SO(11)/SO(10)$, $G = SO(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup SO(10) of $G = SO(20)$ according to the decomposition

$$
SO(20) \supset SO(10)\times SO(10). \tag{2.39}
$$

(d) $S/R\,(2) = \mathrm{SO}(7) \times \mathrm{Sp}(4)/\mathrm{SO}(6) \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$, $G = \mathrm{SO}(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ of $G = \mathrm{SO}(20)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(20) &\supset \mathrm{SO}(10) \times \mathrm{SO}(10) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \end{aligned} \tag{2.40}$$

(e) $S/R\,(4) = \mathrm{SU}(6)/\mathrm{SU}(5) \times \mathrm{U}(1)$, $G = \mathrm{SO}(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup $\mathrm{SU}(5) \times \mathrm{U}(1)$ of $G = \mathrm{SO}(20)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(20) &\supset \mathrm{SO}(10) \times \mathrm{SO}(10) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(5) \times \mathrm{U}(1). \end{aligned} \tag{2.41}$$

(f) $S/R\,(5) = \mathrm{SO}(9) \times \mathrm{SU}(2)/\mathrm{SO}(8) \times \mathrm{U}(1)$, $G = \mathrm{SO}(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup $\mathrm{SO}(8) \times \mathrm{U}(1)$ of $G = \mathrm{SO}(20)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(20) &\supset \mathrm{SO}(10) \times \mathrm{SO}(10) \\ &\supset \mathrm{SO}(10) \times \mathrm{SO}(8) \times \mathrm{U}(1). \end{aligned} \tag{2.42}$$

(g) $S/R\,(6) = \mathrm{SO}(7) \times \mathrm{SU}(3)/\mathrm{SO}(6) \times [\mathrm{SU}(2) \times \mathrm{U}(1)]$, $G = \mathrm{SO}(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of $G = \mathrm{SO}(20)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(20) &\supset \mathrm{SO}(10) \times \mathrm{SO}(10) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1) \end{aligned} \tag{2.43}$$

(h) $S/R\,(8) = \{\mathrm{Sp}(4)\}^2 \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)]^2 \times \mathrm{U}(1)$, $G = \mathrm{SO}(20)$, and $F = \mathbf{512}$.

We embed R in the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of $G = \mathrm{SO}(20)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(20) &\supset \mathrm{SO}(10) \times \mathrm{SO}(10) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2)' \times \mathrm{SU}(2)' \\ &\quad\ \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{2.44}$$

(i) $S/R\,(10) = \mathrm{Sp}(4) \times \mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}} \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$, $G = \mathrm{SO}(18)$, and $F = \mathbf{256}$.

We embed R in the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of $G = \mathrm{SO}(18)$ according to the decomposition

$$\begin{aligned} \mathrm{SO}(18) &\supset \mathrm{SO}(10) \times \mathrm{SO}(8) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ &\supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{2.45}$$

(j) $S/R\,(11) = \mathrm{Sp}(4) \times \mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}} \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$, $G = \mathrm{SO}(18)$ and $F = \mathbf{256}$.

We embed R in the subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$of $G=\mathrm{SO}(18)$ according to the decomposition

$$\begin{aligned}\mathrm{SO}(18) &\supset \mathrm{SO}(10)\times\mathrm{SO}(8)\\ &\supset \mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\\ &\supset \mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1).\end{aligned} \tag{2.46}$$

We find ten candidates of $(S/R,G,F)$ which give at least one fermion with representation **16** and scalar with **10** in four dimensions. Other combinations of $(S/R,G,F)$ are excluded since they do not provide both a representation **16** for fermions and a representation **10** for scalars.

In many cases we obtain several **16**s for fermions. Particularly interesting candidates among them are ($G=\mathrm{SO}(20)$, $S/R\,(4)$, $F=\mathbf{512}$) and ($G=\mathrm{SO}(20)$, $S/R\,(6)$, $F=\mathbf{512}$). They give three **16**s corresponding to three generations of fermions. In such cases the extra U(1) symmetry can be interpreted as a family symmetry.

We obtain the scalar field in the **10** representation of SO(10) in all cases. This scalar field contains the SM Higgs. Notice, however, that no scalar content belongs to $\mathbf{16},\mathbf{45},\mathbf{126},\cdots$, which are necessary to break SO(10) to the SM gauge group. This is inevitable for $H=\mathrm{SO}(10)(\times\mathrm{U}(1))$. The gauge group G for $H=\mathrm{SO}(10)(\times\mathrm{U}(1))$ is $\mathrm{SO}(N)$, and SO(10) appears in the decomposition

$$\mathrm{SO}(N)\supset\mathrm{SO}(10)\times\mathrm{SO}(N-10)\supset\cdots. \tag{2.47}$$

Only **1** or **10** representations of SO(10) are obtained from the adjoint representation of $\mathrm{SO}(N)$ under the above decomposition. Thus no scalar can break SO(10) to the SM gauge group. Fortunately, we can construct a phenomenologically acceptable model without these scalar contents by employing the topological symmetry breaking mechanism, known as Hosotani mechanism or Wilson flux breaking mechanism [36, 37, 67, 68, 69, 70, 71, 72, 73]. This mechanism requires extra-dimensional spaces to be non-simply connected. Hence we have to consider the non-simply connected coset spaces such as $(S/R)/T$ instead of the simply connected ones, where T is a suitable discrete symmetry group.

$H=\mathrm{SU}(5)\times\mathrm{U}(1)$

Secondly, we search for viable $\mathrm{SU}(5)\times\mathrm{U}(1)$ models in four dimensions. We list below the combinations of S/R, G and F which provides $H=\mathrm{SU}(5)\times\mathrm{U}(1)$ and representations which contain field contents of the SM for the scalars and the fermions. The embedding of R into G is shown for each candidates since this embedding uniquely determines all the representations of the scalars and fermions in the four-dimensional theory. In Table 8, we show all the field contents in four dimensions for each combination of $(S/R,G,F)$.

(a) $S/R(15)=\mathrm{G}_2/\mathrm{SU}(2)\times\mathrm{U}(1)$, $G=\mathrm{Sp}(12)$ and $F=\mathbf{208}$.
We embed R in the subgroup $\mathrm{SU}(2)\times\mathrm{U}(1)$ of $G=\mathrm{Sp}(12)$ according to the decomposition

$$\begin{aligned}\mathrm{Sp}(12)&\supset\mathrm{Sp}(10)\times\mathrm{Sp}(2)\\ &\supset\mathrm{SU}(5)\times\mathrm{SU}(2)\times\mathrm{U}(1).\end{aligned} \tag{2.48}$$

(b) $S/R(14)=\mathrm{Sp}(6)/\mathrm{Sp}(4)\times\mathrm{U}(1)$, $G=\mathrm{Sp}(14)$, and $F=\mathbf{350}$.

Table 8: The field contents in four dimensions with $H = \mathrm{SU}(5) \times \mathrm{U}(1)$ for each combination of $(S/R, G, F)$. Coset spaces are indicated by the number assigned in Table 3.

14D model			4D model	
S/R	G	F	Scalars	Fermions
(11)	Sp(16)	**544**	$\mathbf{15}(2)$, $\overline{\mathbf{15}}(-2)$, $\mathbf{5}(1)$, $\overline{\mathbf{5}}(-1)$, $\mathbf{1}(0)$	$\{\mathbf{24}(0)\}^2$, $\mathbf{10}(2)$, $\overline{\mathbf{10}}(-2)$, $\mathbf{5}(1)$, $\overline{\mathbf{5}}(-1)$, $\{\mathbf{1}(0)\}^4$
(14)	Sp(14)	**350**	$\mathbf{15}(-2)$, $\overline{\mathbf{15}}(2)$, $\mathbf{5}(-1)$, $\overline{\mathbf{5}}(1)$	$\mathbf{45}(1)$, $\overline{\mathbf{45}}(-1)$, $\mathbf{24}(0)$, $\mathbf{10}(3)$, $\mathbf{10}(-2)$, $\overline{\mathbf{5}}(1)$, $\mathbf{5}(1)$, $\overline{\mathbf{5}}(-1)$
(15)	Sp(12)	**208**	$\mathbf{15}(2)$, $\overline{\mathbf{15}}(-2)$, $\mathbf{5}(1)$, $\overline{\mathbf{5}}(-1)$	$\mathbf{45}(1)$, $\overline{\mathbf{45}}(-1)$, $\mathbf{24}(0)$, $\mathbf{10}(-3)$, $\overline{\mathbf{10}}(3)$, $\mathbf{10}(2)$, $\overline{\mathbf{10}}(-2)$, $\mathbf{5}(1)$, $\overline{\mathbf{5}}(-1)$

We embed R in the subgroup $\mathrm{Sp}(4) \times \mathrm{U}(1)$ of $G = \mathrm{Sp}(14)$ according to the decomposition

$$\begin{aligned} \mathrm{Sp}(14) &\supset \mathrm{Sp}(10) \times \mathrm{Sp}(4) \\ &\supset \mathrm{SU}(5) \times \mathrm{Sp}(4) \times \mathrm{U}(1). \end{aligned} \tag{2.49}$$

(c) $S/R(11) = \mathrm{Sp}(4) \times \mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}} \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$, $G = \mathrm{Sp}(16)$, and $F = \mathbf{544}$.

We embed R in the subgroup $\mathrm{SU}(2)' \times \mathrm{SU}(2)' \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of $G = \mathrm{Sp}(16)$ according to the decomposition

$$\begin{aligned} \mathrm{Sp}(16) &\supset \mathrm{Sp}(10) \times \mathrm{Sp}(6) \\ &\supset \mathrm{Sp}(10) \times \mathrm{Sp}(4) \times \mathrm{SU}(2) \\ &\supset \mathrm{Sp}(10) \times \mathrm{SU}(2)' \times \mathrm{SU}(2)' \times \mathrm{SU}(2) \\ &\supset \mathrm{SU}(5) \times \mathrm{SU}(2)' \times \mathrm{SU}(2)' \times \mathrm{SU}(2) \times \mathrm{U}(1). \end{aligned} \tag{2.50}$$

We find three candidates of $(S/R, G, F)$ that give at least one pair of fermions with representation $\mathbf{10}$ and $\bar{\mathbf{5}}$, and a scalar with $\mathbf{5}$ representation in four dimensions. Other combinations of $(S/R, G, F)$ are excluded since they do not provide these representations for fermions and scalars.

We obtain the scalar field in $\mathbf{5}$ representation of SU(5) for all cases. This scalar field contains the SM Higgs. Notice, however, that no scalar contents belongs to $\mathbf{24}, \cdots$, which are necessary to break SU(5) to the SM gauge group. The lack of such scalars is a general feature for $H = \mathrm{SU}(5) \times \mathrm{U}(1)$. The gauge groups G for $H = \mathrm{SU}(5) \times \mathrm{U}(1)$ are $\mathrm{SU}(N)$, $\mathrm{SO}(N)$, and $\mathrm{Sp}(N)$. These groups are decomposed into subgroups including $\mathrm{SU}(5) \times \mathrm{U}(1)$, and their adjoint representations are decomposed accordingly as well:

$$\begin{aligned} &\mathrm{SU}(N) \supset \mathrm{SU}(5) \times \mathrm{SU}(N-5) \times \mathrm{U}(1) \supset \cdots \\ &\mathrm{adj}\,\mathrm{SU}(5) = (\mathbf{24}, \mathbf{1})(0) + (\mathbf{1}, \mathrm{adj}\,\mathrm{SU}(N-1))(0) + (1,1)(0) \\ &\qquad + (\mathbf{5}, \overline{N-5})(a) + (\bar{\mathbf{5}}, N-5)(-a) \\ &\qquad = \cdots \end{aligned} \tag{2.51}$$

$$
\begin{aligned}
\mathrm{SO}(N) &\supset \mathrm{SO}(10)\times\mathrm{SO}(N-10)\\
&\supset \mathrm{SU}(5)\times\mathrm{SO}(N-10)\times\mathrm{U}(1)\supset\cdots\\
\mathrm{adj}\,\mathrm{SO}(N) &= (\mathbf{45},\mathbf{1})+(\mathbf{1},\mathrm{adj}\,\mathrm{SO}(N-10))\\
&\quad+(\mathbf{10},\mathbf{1})+(\mathbf{1},N-10)\\
&=(\mathbf{24},\mathbf{1})(0)+(\mathbf{1},\mathrm{adj}\,\mathrm{SO}(N-10))(0)+(\mathbf{1},\mathbf{1})(0)\\
&\quad+(\mathbf{10},\mathbf{1})(4)+(\overline{\mathbf{10}},\mathbf{1})(-4)+(\mathbf{5},\mathbf{1})(2)+(\bar{\mathbf{5}},\mathbf{1})(-2)+(\mathbf{1},N-10)(0)\\
&=\cdots
\end{aligned}
\tag{2.52}
$$

$$
\begin{aligned}
\mathrm{Sp}(2N) &\supset \mathrm{Sp}(10)\times\mathrm{Sp}(2N-10)\\
&\supset \mathrm{SU}(5)\times\mathrm{Sp}(2N-1)\times\mathrm{U}(1)\\
&\supset\cdots\\
\mathrm{adj}\,\mathrm{Sp}(2N) &= (\mathbf{55},\mathbf{1})+(\mathbf{1},\mathrm{adj}\,\mathrm{Sp}(2N-10))\\
&\quad+(\mathbf{10},\mathbf{1})+(\mathbf{1},2N-10)\\
&=(\mathbf{24},\mathbf{1})(0)+(\mathbf{1},\mathrm{adj}\,\mathrm{Sp}(2N-10))(0)+(\mathbf{1},\mathbf{1})(0)\\
&\quad+(\mathbf{15},\mathbf{1})(2)+(\overline{\mathbf{15}},\mathbf{1})(-2)+(\mathbf{5},\mathbf{1})(1)+(\bar{\mathbf{5}},\mathbf{1})(-1)+(\mathbf{1},N)(0)\\
&=\cdots\,.
\end{aligned}
\tag{2.53}
$$

Only **1**, **5**, **10**, or **15** representation of SU(5) is obtained from the adjoint representations of SU(N), SO(N), and Sp(N) under the above decompositions. Then, no scalar can break SU(5) to the SM gauge group. Therefore we should employ the flux breaking mechanism to break SU(5) to the SM gauge group.

$H = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$

We find no viable candidate for $H = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. We exclude the coset spaces (16) – (35) in Table 3. They have two or more factors of U(1) in R, and these U(1)'s become the part of $H = C_G(R) = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, which has only one U(1). The single U(1) factor in R becomes $\mathrm{U}(1)_Y$ of the SM gauge group, hence the decomposition of the spinor representation **16** of SO(10) to R need to have U(1) charges whose ratio is $1:2:(-3):(-4):6$. Referring to Table 4, we find that the coset spaces (4) – (15) do not have such U(1) charge and thus are excluded. The explicit analysis of the remaining coset spaces (1), (2) and (3) shows that they do not induce the SM either.

$H = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$

Finally, we search for viable $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ models in four dimensions. We list below the combinations of S/R, G, and F which provide $H = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ and representations of the SM scalars and fermions. Embedding of R in G is also shown for each candidates. Note that we can take a linear combination of the two U(1)'s. The U(1) charges in the decomposition are first chosen to facilitate the decomposition of the group G, then combined to embed R into G, and subsequently organized again to reproduce the hypercharge of the SM. We explicitly show these linear recombinations of U(1) for each candidates. In Table 10, we show all the field contents in four dimensions for each combination of $(S/R, G, F)$.

(a) S/R (15a) $= \mathrm{G}_2/\mathrm{SU}(2)\times\mathrm{U}(1)$, $G = \mathrm{Sp}(12)$, and $F = \mathbf{364}$.

Table 9: The field contents in four dimensions with $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R \times \mathrm{U}(1)_A$. Coset spaces are indicated by the number assigned in Table 3. Numbers in a superscript of the representations denote its multiplicity.

14D model			4D model	
S/R	G	F	Scalars	Fermions
(15a)	Sp(12)	**364**	$(\mathbf{1},\mathbf{2})(-2,3)$, $(\mathbf{1},\mathbf{2})(2,-3)$, $(\mathbf{3},\mathbf{1})(-1,-4)$, $(\bar{\mathbf{3}},\mathbf{1})(1,4)$, $(\mathbf{6},\mathbf{1})(-2,-8)$, $(\bar{\mathbf{6}},\mathbf{1})(2,8)$	$(\mathbf{15},\mathbf{1})(-1,4)$, $(\overline{\mathbf{15}},\mathbf{1})(1,-4)$, $(\mathbf{10},\mathbf{1})(-3,-12)$, $(\overline{\mathbf{10}},\mathbf{1})(3,12)$, $(\mathbf{3},\mathbf{1})(-1,-4)$, $\{(\bar{\mathbf{3}},\mathbf{1})(1,4)\}^3$, $(\mathbf{1},\mathbf{3})(0,0)$, $(\mathbf{1},\mathbf{1})(-4,6)$, $\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\mathbf{1},\mathbf{2})(-2,3)$, $(\mathbf{1},\mathbf{2})(2,-3)$, $(\mathbf{3},\mathbf{3})(-1,-4)$, $(\bar{\mathbf{3}},\mathbf{3})(1,4)$, $(\bar{\mathbf{3}},\mathbf{1})(5,-2)$, $(\mathbf{3},\mathbf{1})(-1,-4)$, $(\mathbf{3},\mathbf{1})(3,-10)$, $(\bar{\mathbf{3}},\mathbf{1})(-3,10)$, $(\mathbf{3},\mathbf{2})(-3,-1)$, $(\bar{\mathbf{3}},\mathbf{2})(3,1)$, $(\mathbf{3},\mathbf{2})(1,-7)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,7)$, $(\mathbf{8},\mathbf{1})(0,0)$, $(\mathbf{6},\mathbf{1})(2,-8)$, $(\bar{\mathbf{6}},\mathbf{1})(-2,8)$
(9)	Sp(16)	**544**	$(\mathbf{1},\mathbf{2})(1,0)$, $(\mathbf{1},\mathbf{2})(-1,0)$	$(\mathbf{1},\mathbf{1})(-2,0)$, $(\mathbf{1},\mathbf{2})(1,0)$, $\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\bar{\mathbf{3}},\mathbf{1})(2,-1)$, $(\mathbf{3},\mathbf{1})(2,1)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-1)$, $(\mathbf{3},\mathbf{2})(-1,1)$, $\{(\mathbf{3},\mathbf{1})(0,1)\}^3$, $\{(\bar{\mathbf{3}},\mathbf{1})(0,-1)\}^3$, $(\mathbf{8},\mathbf{1})(0,0)$, $(\mathbf{6},\mathbf{1})(0,-1)$, $(\bar{\mathbf{6}},\mathbf{1})(0,1)$
(15a)	SO(13)	**768**	$(\mathbf{1},\mathbf{2})(3,3)$, $(\mathbf{1},\mathbf{2})(-3,-3)$, $(\mathbf{3},\mathbf{1})(-2,-6)$, $(\bar{\mathbf{3}},\mathbf{1})(2,6)$	$(\mathbf{3},\mathbf{3})(-2,-4)$, $(\bar{\mathbf{3}},\mathbf{3})(2,4)$, $(\mathbf{1},\mathbf{3})(0,-6)$, $(\mathbf{1},\mathbf{3})(0,6)$, $(\mathbf{3},\mathbf{2})(1,3)$, $(\bar{\mathbf{3}},\mathbf{1})(-4,-6)$, $(\mathbf{3},\mathbf{1})(-2,0)$, $(\bar{\mathbf{3}},\mathbf{1})(2,0)$, $(\mathbf{3},\mathbf{2})(1,3)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-3)$, $(\bar{\mathbf{3}},\mathbf{2})(5,3)$, $(\mathbf{1},\mathbf{1})(0,-6)$, $(\mathbf{1},\mathbf{1})(0,6)$, $(\mathbf{1},\mathbf{2})(3,-3)$, $(\mathbf{1},\mathbf{2})(-3,3)$, $(\mathbf{1},\mathbf{2})(-3,-9)$, $(\mathbf{1},\mathbf{2})(3,9)$, $(\mathbf{3},\mathbf{2})(1,3)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-3)$, $(\mathbf{3},\mathbf{1})(-2,0)$, $(\bar{\mathbf{3}},\mathbf{1})(2,0)$, $(\mathbf{1},\mathbf{1})(0,6)$, $(\mathbf{1},\mathbf{1})(0,-6)$, $(\mathbf{1},\mathbf{2})(3,-3)$, $(\mathbf{1},\mathbf{2})(-3,3)$, $(\mathbf{3},\mathbf{1})(-2,0)$, $(\bar{\mathbf{3}},\mathbf{1})(2,0)$, $(\mathbf{3},\mathbf{2})(1,3)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-3)$, $(\bar{\mathbf{3}},\mathbf{1})(-4,6)$, $(\mathbf{3},\mathbf{2})(1,-9)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,9)$, $(\mathbf{6},\mathbf{1})(2,0)$, $(\bar{\mathbf{6}},1)(-2,0)$, $(\mathbf{6},\mathbf{2})(-1,-3)$, $(\bar{6},2)(1,3)$, $(\mathbf{8},\mathbf{1})(2,0)$, $(\mathbf{8},\mathbf{1})(-2,0)$, $(\mathbf{8},\mathbf{2})(-1,-3)$, $(\mathbf{8},\mathbf{2})(1,3)$, $(\mathbf{3},\mathbf{1})(-2,0)$, $(\bar{\mathbf{3}},\mathbf{1})(2,0)$, $(\mathbf{3},\mathbf{2})(1,3)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-3)$, $(\mathbf{1},\mathbf{1})(0,-6)$, $(\mathbf{1},\mathbf{1})(0,6)$, $(\mathbf{1},\mathbf{2})(3,-3)$, $(\mathbf{1},\mathbf{2})(-3,3)$
(14)	Sp(14)	**350**	$(\mathbf{1},\mathbf{2})(-1,-9/2)$, $(\mathbf{1},\mathbf{2})(1,9/2)$, $(\mathbf{3},\mathbf{2})(-2,11/2)$, $(\bar{\mathbf{3}},\mathbf{2})(2,-11/2)$, $(\mathbf{1},\mathbf{3})(-2,-9)$, $(\mathbf{1},\mathbf{3})(2,9)$	$(\mathbf{6},\mathbf{1})(3,-1)$, $(\mathbf{8},\mathbf{1})(0,0)$, $(\mathbf{1},\mathbf{1})(-2,-9)$, $\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\mathbf{3},\mathbf{1})(-1,10)$, $(\bar{\mathbf{3}},\mathbf{1})(1,-10)$, $\{(\bar{\mathbf{3}},\mathbf{1})(3,-1)\}^2$, $\{(\mathbf{1},\mathbf{2})(-1,-9/2)\}^2$, $\{(\mathbf{1},\mathbf{2})(1,9/2)\}^3$, $(\mathbf{3},\mathbf{2})(-2,11/2)$, $(\mathbf{1},\mathbf{3})(0,0)$, $(\bar{\mathbf{3}},\mathbf{3})(3,-1)$

Table 10: The field contents in four dimensions with $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_\alpha$. Coset spaces are indicated by the number assigned in Table 3. Numbers in superscript of the representations denote its multiplicity. The U(1) charges are rearranged from those of Table 9 so that the charge of $\mathrm{U}(1)_Y$ is proportional to the hypercharge of the Standard Model.

14D model			4D model			
			Scalars		Fermions	
S/R	G	F	SM fields	Extra fields	SM fields	Extra fields
(15a)	Sp(12)	**364**	$(\mathbf{1},\mathbf{2})(3,-32)$, $(\mathbf{1},\mathbf{2})(-3,32)$	$(\mathbf{3},\mathbf{1})(-2,-27)$, $(\bar{\mathbf{3}},\mathbf{1})(2,27)$, $(\mathbf{6},\mathbf{1})(-4,-54)$, $(\bar{\mathbf{6}},\mathbf{1})(4,54)$	$(\mathbf{3},\mathbf{2})(1,-59)$, $(\bar{\mathbf{3}},\mathbf{1})(2,27)$ $(\bar{\mathbf{3}},\mathbf{1})(-4,91)$ $(\mathbf{1},\mathbf{2})(-3,32)$ $(\mathbf{1},\mathbf{1})(6,-64)$	$(\mathbf{15},1)(34/11,-11)$, $(\overline{\mathbf{15}},\mathbf{1})(-34/11,11)$, $(\mathbf{10},\mathbf{1})(-6,-81)$, $(\overline{\mathbf{10}},\mathbf{1})(6,81)$, $\{(\mathbf{3},\mathbf{1})(-2,-27)\}^2$, $(\mathbf{1},\mathbf{3})(0,0)$, $\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\mathbf{1},\mathbf{2})(3,-32)$, $(\mathbf{3},\mathbf{3})(-2,-27)$, $(\bar{\mathbf{3}},\mathbf{3})(2,27)$, $(\mathbf{3},\mathbf{1})(-8,37)$, $(\bar{\mathbf{3}},\mathbf{1})(8,-37)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,59)$, $(\mathbf{3},\mathbf{2})(-5,5)$, $(\bar{\mathbf{3}},\mathbf{2})(5,-5)$, $(\mathbf{8},\mathbf{1})(0,0)$, $\{(\bar{\mathbf{3}},\mathbf{1})(2,27)\}^2$, $(\mathbf{6},\mathbf{1})(-68/11,22)$, $(\bar{\mathbf{6}},\mathbf{1})(68/11,-22)$
(9)	Sp(16)	**544**	$(\mathbf{1},\mathbf{2})(3,-2)$, $(\mathbf{1},\mathbf{2})(-3,2)$		$(\mathbf{1},\mathbf{1})(6,-4)$, $(\mathbf{1},\mathbf{2})(-3,2)$, $(\bar{\mathbf{3}},\mathbf{1})(-4,7)$, $(\bar{\mathbf{3}},\mathbf{1})(2,3)$, $(\mathbf{3},\mathbf{2})(1,-5)$	$\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\mathbf{3},\mathbf{1})(-8,1)$, $\{(\mathbf{3},\mathbf{1})(-2,-3)\}^3$, $\{(\bar{\mathbf{3}},\mathbf{1})(2,3)\}^2$, $(\bar{\mathbf{3}},\mathbf{2})(5,1)$, $(\mathbf{8},\mathbf{1})(0,0)$, $(\mathbf{6},\mathbf{1})(2,3)$, $(\bar{\mathbf{6}},\mathbf{1})(-2,-3)$
(15a)	SO(13)	**768**	$(\mathbf{1},\mathbf{2})(-3,66)$, $(\mathbf{1},\mathbf{2})(3,-66)$	$(\mathbf{3},\mathbf{2})(1,34)$, $(\bar{\mathbf{3}},\mathbf{1})(2,100)$, $(\bar{\mathbf{3}},\mathbf{1})(-4,32)$, $(\mathbf{1},\mathbf{2})(-3,-102)$, $(\mathbf{1},\mathbf{1})(6,36)$, $(\mathbf{3},\mathbf{3})(0,8)$, $(\bar{\mathbf{3}},\mathbf{3})(0,-8)$, $(\mathbf{1},\mathbf{3})(-6,-36)$, $(\mathbf{1},\mathbf{3})(6,36)$, $(\mathbf{3},\mathbf{1})(4,-32)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-34)$, $(\bar{\mathbf{3}},\mathbf{2})(-7,98)$	$(\mathbf{1},\mathbf{1})(-6,-36)$, $(\mathbf{1},\mathbf{2})(-9,30)$, $(\mathbf{1},\mathbf{2})(9,-30)$, $(\mathbf{1},\mathbf{2})(3,102)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-34)$, $(\mathbf{3},\mathbf{1})(4,-32)$, $(\mathbf{1},\mathbf{1})(-6,-36)$	$(\mathbf{1},\mathbf{2})(-9,30)$, $(\mathbf{1},\mathbf{2})(9,-30)$, $(\mathbf{3},\mathbf{1})(4,-32)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-34)$, $(\mathbf{3},\mathbf{2})(-11,38)$, $(\bar{\mathbf{3}},\mathbf{2})(11,-38)$, $(\mathbf{6},\mathbf{1})(-4,32)$, $(\bar{\mathbf{6}},\mathbf{1})(4,-32)$, $(\mathbf{6},\mathbf{2})(-1,-34)$, $(\bar{\mathbf{6}},\mathbf{2})(1,34)$, $(\mathbf{8},\mathbf{1})(-4,32)$, $(\mathbf{8},\mathbf{1})(4,-32)$, $(\mathbf{8},\mathbf{2})(-1,-34)$, $(\mathbf{8},\mathbf{2})(1,34)$, $(\mathbf{3},\mathbf{1})(4,-32)$, $(\bar{\mathbf{3}},\mathbf{2})(-1,-34)$, $(\mathbf{1},\mathbf{1})(-6,-36)$, $(\mathbf{1},\mathbf{2})(-9,30)$, $(\mathbf{1},\mathbf{2})(9,-30)$, $(\bar{\mathbf{3}},\mathbf{1})(2,100)$, $\{(\mathbf{3},\mathbf{2})(1,34)\}^5$, $\{(\bar{\mathbf{3}},\mathbf{1})(-4,32)\}^2$, $\{(\mathbf{1},\mathbf{1})(6,36)\}^2$
(14)	Sp(14)	**350**	$(\mathbf{1},\mathbf{2})(3,-2)$, $(\mathbf{1},\mathbf{2})(-3,2)$		$(\mathbf{1},\mathbf{1})(6,-4)$, $(\mathbf{1},\mathbf{2})(-3,2)$, $(\bar{\mathbf{3}},\mathbf{1})(-4,7)$, $(\bar{\mathbf{3}},\mathbf{1})(2,3)$, $(\mathbf{3},\mathbf{2})(1,-5)$	$\{(\mathbf{1},\mathbf{1})(0,0)\}^2$, $(\mathbf{3},\mathbf{1})(-8,1)$, $\{(\mathbf{3},\mathbf{1})(-2,-3)\}^3$, $\{(\bar{\mathbf{3}},\mathbf{1})(2,3)\}^2$, $(\bar{\mathbf{3}},\mathbf{2})(5,1)$, $(\mathbf{8},\mathbf{1})(0,0)$, $(\mathbf{6},\mathbf{1})(2,3)$, $(\bar{\mathbf{6}},\mathbf{1})(-2,-3)$

We decompose Sp(12) as

$$
\begin{aligned}
\mathrm{Sp}(12) &\supset \mathrm{Sp}(6)\times\mathrm{Sp}(6)\\
&\supset \mathrm{Sp}(6)\times\mathrm{Sp}(4)\times\mathrm{SU}(2)'\\
&\supset \mathrm{SU}(3)\times\mathrm{Sp}(4)\times\mathrm{SU}(2)'\times\mathrm{U}(1)_{\mathrm{a}}\\
&\supset \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)'\times\mathrm{U}(1)_{\mathrm{a}}\\
&\supset \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)'\times\mathrm{U}(1)_{\mathrm{a}}\times\mathrm{U}(1)_{\mathrm{b}}.
\end{aligned}
\tag{2.54}
$$

Accordingly the adjoint representation of Sp(12) is decomposed as [65, 66]

$$
\begin{aligned}
\mathbf{78} \;=\;& (\mathbf{8},\mathbf{1},\mathbf{1})(0,0)+(\mathbf{1},\mathbf{3},\mathbf{1})(0,0)+(\mathbf{1},\mathbf{1},\mathbf{3})(0,0)+(\mathbf{1},\mathbf{1},\mathbf{1})(0,0)\\
&+(\mathbf{1},\mathbf{1},\mathbf{1})(0,0)+(\mathbf{6},\mathbf{1},\mathbf{1})(2,0)+(\bar{\mathbf{6}},\mathbf{1},\mathbf{1})(-2,0)+(\mathbf{3},\mathbf{1},\mathbf{2})(1,0)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})(-1,0)+(\mathbf{3},\mathbf{2},\mathbf{1})(1,0)+(\bar{\mathbf{3}},\mathbf{2},1)(-1,0)+(\mathbf{3},\mathbf{1},\mathbf{1})(1,1)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(-1,-1)+(\mathbf{3},\mathbf{1},\mathbf{1})(1,-1)+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(-1,1)+(\mathbf{1},\mathbf{2},\mathbf{1})(0,1)\\
&+(\mathbf{1},\mathbf{2},\mathbf{1})(0,-1)+(\mathbf{1},\mathbf{1},\mathbf{2})(0,1)+(\mathbf{1},\mathbf{1},\mathbf{2})(0,-1)+(\mathbf{1},\mathbf{1},\mathbf{1})(0,2)\\
&+(\mathbf{1},\mathbf{1},\mathbf{1})(0,-2)+(\mathbf{1},\mathbf{2},\mathbf{2})(0,0)\\
&(\mathrm{SU}(3),\mathrm{SU}(2),\mathrm{SU}(2)')(\mathrm{U}(1)_{\mathrm{a}},\mathrm{U}(1)_{\mathrm{b}}).
\end{aligned}
\tag{2.55}
$$

We take a linear combination of $\mathrm{U}(1)_{\mathrm{a}}$ and $\mathrm{U}(1)_{\mathrm{b}}$, respecting the orthogonality of the two, to obtain U(1) charges listed in Table 4, at the row (15a) and the columns "Branch of **10**" and "Branch of **16**". We define

$$
Q_R \equiv -xQ_{\mathrm{a}} - yQ_{\mathrm{b}}, \tag{2.56a}
$$

$$
Q_A \equiv -2yQ_{\mathrm{a}} + 3xQ_{\mathrm{b}}, \tag{2.56b}
$$

where Q_is ($i \in \{\mathrm{a},\, \mathrm{b},\, R,\, A\}$) denote the charges of $\mathrm{U}(1)_i$. Embedding R in $\mathrm{SU}(2)\times\mathrm{U}(1)_R$, we obtain the decomposition of the adjoint representation,

$$
\begin{aligned}
\mathbf{78} \;=\;& (\bar{\mathbf{8}},\mathbf{1},\mathbf{1})(0,0)+(\bar{\mathbf{1}},\mathbf{3},\mathbf{1})(0,0)+(\bar{\mathbf{1}},\mathbf{1},\mathbf{3})(0,0)\\
&+(\bar{\mathbf{1}},\mathbf{1},\mathbf{1})(0,0)+(\bar{\mathbf{1}},\mathbf{1},\mathbf{1})(0,0)\\
&+(\bar{\mathbf{6}},\mathbf{1},\mathbf{1})(-2x,-4y)+(\bar{\mathbf{6}},\mathbf{1},\mathbf{1})(2x,4y)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})(-x,-2y)+(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})(x,2y)\\
&+(\bar{\mathbf{3}},\mathbf{2},\mathbf{1})(-x,-2y)+(\bar{\mathbf{3}},\mathbf{2},\mathbf{1})(x,2y)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(-x-y,-2y+3x)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(x+y,2y-3x)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(-x+y,-2y-3x)\\
&+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(x-y,2y+3x)\\
&+(\bar{\mathbf{1}},\mathbf{2},\mathbf{1})(-y,3x)+(\bar{\mathbf{1}},\mathbf{2},\mathbf{1})(y,-3x)\\
&+(\bar{\mathbf{1}},\mathbf{1},\mathbf{2})(-y,3x)+(\bar{\mathbf{1}},\mathbf{1},\mathbf{2})(y,-3x)\\
&+(\bar{\mathbf{1}},\mathbf{1},\mathbf{1})(-2y,6x)+(\bar{\mathbf{1}},\mathbf{1},\mathbf{1})(2y,6x)\\
&+(\bar{\mathbf{1}},\mathbf{2},\mathbf{2})(0,0).
\end{aligned}
\tag{2.57}
$$

We find that $y = \pm 2$ provides the SM Higgs doublet by comparing the $\mathrm{U}(1)_R$ charges in the decomposition Eq. (2.57) with those in Table 4. Further investigation shows that we can obtain the SM fermions as well by taking $x = 1$ and $y = 2$. The resulting field contents are summarized in Table 9. We can explicitly obtain appropriate $\mathrm{U}(1)_Y$ hypercharges of the SM particles by taking another linear combination of $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_A$ as

$$Q_Y \equiv -\frac{6}{11}Q_R + \frac{7}{11}Q_A, \tag{2.58a}$$

$$Q_\alpha \equiv 19Q_R + 2Q_A, \tag{2.58b}$$

where Q_Y and Q_α are the charges of $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_\alpha$, respectively. We thereby obtain SM Higgs, SM fermions and other fermions listed as in Table 10.

(b) $S/R\,(9) = \mathrm{G}_2 \times \mathrm{SU}(3)/\mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{U}(1)]$, $G = \mathrm{Sp}(16)$, and $F = \mathbf{544}$.

We embed R in subgroup $\mathrm{SU}(3)_\mathrm{b} \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ of Sp(16) according to the decomposition

$$\begin{aligned}
\mathrm{Sp}(16) \;\supset\;& \mathrm{Sp}(6)_\mathrm{a} \times \mathrm{Sp}(6)_\mathrm{b} \times \mathrm{Sp}(4)\\
\supset\;& \mathrm{SU}(3)_\mathrm{a} \times \mathrm{Sp}(6)_\mathrm{b} \times \mathrm{Sp}(4) \times \mathrm{U}(1)_R\\
\supset\;& \mathrm{SU}(3)_\mathrm{a} \times \mathrm{SU}(3)_\mathrm{b} \times \mathrm{Sp}(4)\\
& \times \mathrm{U}(1)_R \times \mathrm{U}(1)_A\\
\supset\;& \mathrm{SU}(3)_\mathrm{a} \times \mathrm{SU}(3)_\mathrm{b} \times \mathrm{SU}(2) \times \mathrm{SU}(2)\\
& \times \mathrm{U}(1)_R \times \mathrm{U}(1)_A.
\end{aligned} \tag{2.59}$$

The resulting field contents are summarized in Table 9. We explicitly obtain appropriate $\mathrm{U}(1)_Y$ hypercharges of the SM particles by taking combination of $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_A$ as

$$Q_Y \equiv 3Q_A - 2Q_R, \tag{2.60a}$$

$$Q_\alpha \equiv -2Q_A - 3Q_R, \tag{2.60b}$$

where Q_is ($i \in \{R,\, A,\, Y,\, \alpha\}$) denote the charges of $\mathrm{U}(1)_i$. We thereby obtain SM Higgs, SM fermions and other fermions listed in Table 10.

(c) $S/R\,(15\mathrm{a}) = \mathrm{G}_2/\mathrm{SU}(2) \times \mathrm{U}(1)$, $G = \mathrm{SO}(13)$, and $F = \mathbf{768}$.

We decompose SO(13) as

$$\begin{aligned}
\mathrm{SO}(13) \;\supset\;& \mathrm{SU}(4) \times \mathrm{SO}(7)\\
\supset\;& \mathrm{SU}(4) \times \mathrm{SU}(2)'' \times \mathrm{SU}(2)' \times \mathrm{SU}(2)\\
\supset\;& \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_\mathrm{a}\\
\supset\;& \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_\mathrm{a} \times \mathrm{U}(1)_\mathrm{b},
\end{aligned} \tag{2.61}$$

where $\mathrm{SU}(2)'' \sim \mathrm{SO}(3)$ and $\mathrm{SU}(2)' \times \mathrm{SU}(2) \sim \mathrm{SO}(4)$. We obtain U(1) charges listed in Table 4 at the row of (15a) and the column of "Branch of **10**" and "Branch of **16**" by taking a linear combination of $\mathrm{U}(1)_\mathrm{a}$ and $\mathrm{U}(1)_\mathrm{b}$ as

$$Q_R \equiv \frac{3}{2}Q_\mathrm{b} + \frac{1}{2}Q_\mathrm{a} \tag{2.62}$$

$$Q_A \equiv \frac{3}{2}Q_\mathrm{b} - \frac{3}{2}Q_\mathrm{a}, \tag{2.63}$$

where Q_i ($i \in \{\text{a, b}, R, A\}$) denote the charges of $\mathrm{U}(1)_i$. Embedding R in $\mathrm{SU}(2) \times \mathrm{U}(1)_R$, we obtain the field contents summarized in Table 9. We explicitly obtain appropriate $\mathrm{U}(1)_Y$ hypercharges of the SM particles by taking another linear combination $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_A$,

$$Q_Y \equiv -2Q_R + Q_A, \tag{2.64a}$$
$$Q_\alpha \equiv 16Q_R + 6Q_A, \tag{2.64b}$$

where Q_Y and Q_α are the charges of $U(1)_Y$ and $U(1)_\alpha$, respectively. We thereby obtain SM Higgs, SM fermions and other fermions listed in Table 10.

(d) $S/R\,(14) = \mathrm{Sp}(6)/\mathrm{Sp}(4) \times \mathrm{U}(1)$, $G = \mathrm{Sp}(14)$, and $F = \mathbf{350}$.
We decompose Sp(14) as

$$\begin{aligned}
\mathrm{Sp}(14) \;&\supset\; \mathrm{Sp}(10) \times \mathrm{Sp}(4)\\
&\supset\; \mathrm{Sp}(6) \times \mathrm{Sp}(4)' \times \mathrm{Sp}(4)\\
&\supset\; \mathrm{SU}(3) \times \mathrm{Sp}(4)' \times \mathrm{Sp}(4) \times \mathrm{U}(1)_\mathrm{a}\\
&\supset\; \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{Sp}(4) \times \mathrm{U}(1)_\mathrm{a} \times \mathrm{U}(1)_\mathrm{b}.
\end{aligned} \tag{2.65}$$

We obtain U(1) charges listed in Table 4 at the row of (14) and the columns of "Branch of **10**" and "Branch of **16**" by taking a linear combinations of $\mathrm{U}(1)_\mathrm{a}$ and $\mathrm{U}(1)_\mathrm{b}$ as

$$Q_R \equiv \frac{1}{2}(-9Q_\mathrm{b} + 2Q_\mathrm{a}) \tag{2.66a}$$
$$Q_A \equiv -Q_\mathrm{b} - 3Q_\mathrm{a}, \tag{2.66b}$$

where Q_i ($i \in \{\text{a, b}, R, A\}$) denote the charges of $\mathrm{U}(1)_i$. Embedding R in $\mathrm{Sp}(4) \times \mathrm{U}(1)_R$, we obtain the resulting field contents summarized in Table 9. We explicitly obtain appropriate $\mathrm{U}(1)_Y$ hypercharges of the SM particles by taking another linear combination of $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_A$ as

$$Q_Y \equiv -\frac{2}{29}(5Q_R + 21Q_A), \tag{2.67a}$$
$$Q_\alpha \equiv -\frac{2}{29}(14Q_R - 5Q_A), \tag{2.67b}$$

where Q_Y and Q_α are the charges of $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_\alpha$. We thereby obtain SM Higgs, SM fermions and other fermions listed in Table 10.

We find four candidates of $(S/R, G, F)$ which give the SM Higgs doublet and at least one generation of the SM fermions in four dimensions. These models, however, generate numerous undesired fields that does not appear in the particle spectrum of the SM as tabulated in Table 10. These extra fields need to be eliminated to construct a realistic model based on the candidates we found.

2.2.2 Models on ten-dimensional spacetime with direct product gauge group

In this section we obtain the combinations of the coset space S/R and the gauge group G of the ten-dimensional theory [56]. We first obtain the coset space S/R and then we restrict the possible gauge group G for each S/R.

We select the coset space S/R from the ones listed in Table 1 by the following two criteria. First, R should be a direct product of subgroups R_1 and R_2 to have new freedom to embedding of R into G. This criterion exculdes the candidates of S/R (v) and (vi) in Table 1.

Table 11: The decompositions of the vector representation **6** and the spinor representation **4** of SO(6) under Rs which are listed as (i) –(iv) in Table 1. The representations of r_s in Eq. (2.16) and σ_i in Eq. (2.20) are listed in the columns of "Branches of **6**" and "Branches of **4**", respectively.

S/R		Branches of **6**	Branches of **4**
(i)	SU(2)(U(1))	$\mathbf{3}(2), \mathbf{3}(-2)$	$\mathbf{1}(3), \mathbf{3}(-1)$
(ii)	SU(2)(U(1))	$\mathbf{1}(2), \mathbf{1}(-2), \mathbf{2}(1), \mathbf{2}(-1)$	$\mathbf{2}(1), \mathbf{1}(0), \mathbf{1}(-2)$
(iii)	SU(3)(U(1))	$\mathbf{3}(-4), \bar{\mathbf{3}}(4)$	$\mathbf{1}(-6), \mathbf{3}(2)$
(iv)	(SU(2), SU(2))(U(1))	$(\mathbf{2}, \mathbf{2})(0), (\mathbf{1}, \mathbf{1})(2), (\mathbf{1}, \mathbf{1})(-2)$	$(\mathbf{2}, \mathbf{1})(1), (\mathbf{1}, \mathbf{2})(-1)$

Table 12: The embedding of R into $G = G_1 \times G_2$ for the coset spaces (i) and (ii).

(i) $\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$ and (ii) $\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$		
(a)	$G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(2)$,	$G_2 \supset \mathrm{U}(1)$
(b)	$G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{U}(1)$,	$G_2 \supset \mathrm{SU}(2)$

Secondly, the four-dimensional gauge group obtained by Eq. (2.6) should be that of the SM or a GUT with at most one extra U(1) gauge group, *i.e.* the SM-like gauge group $G_{\mathrm{SM}}(\times \mathrm{U}(1))$, where $G_{\mathrm{SM}} \equiv \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, or a GUT-like gauge group $G_{\mathrm{GUT}}(\times \mathrm{U}(1))$, where G_{GUT} is either SU(5), SO(10) or E_6. This criterion exculdes the candidates (vii) – (ix) in Table 1 by the following reasons.

1. We note that the U(1)s in R are also parts of its centralizer, *i.e.* a part of H. We thus exclude the candidate (ix) since we consider the Hs which have at most two U(1) factors.
2. Similarly, as long as we consider the GUT-like and G_{SM} gauge groups, we do not need to consider the candidates (vii) and (viii).
3. The candidates (vii) and (viii) do not allow $H = G_{\mathrm{SM}} \times \mathrm{U}(1)$ either for the following reason. The hypercharge of the SM should be reproduced by a certain linear combination of two U(1)s in R, which should be matched to the spinor representation of SO(6). The dimension of the SO(6) spinor representation is four, and thus no more than four different values of U(1) charges are available. On the other hand the fermion content of the SM has five different values of U(1) charges. Hence, this case never reproduces the hypercharges of the SM fermions.
4. Due to the above three reasons the candidates (i) – (iv) allow neither G_{SM} nor G_{SM} as H.

To summarize, the possible model requires coset space S/R listed in (i) – (iv) of Table 1, with either $H = G_{\mathrm{SM}} \times \mathrm{U}(1)$ or $H = G_{\mathrm{GUT}} \times \mathrm{U}(1)$. In Table 11 we show the embedding of R in SO(6) for these coset spaces. The representations of r_s in Eq. (2.16) and σ_i in Eq. (2.20) are listed in the columns of "Branches of **6**" and "Branches of **4**", respectively. The embedding of R into higher dimensional gauge group $G = G_1 \times G_2$ is listed in Table 12–14. These embeddings are straightforwardly obtained by decomposing gauge group G to its regular subgroup which contains an R-subgroup of G. A detailed discussion about the embeddings is summarized in [25]. For each embedding of R, the candidates of G are summarized in Table 15–18. Note that all the candidates of G in Table 15–18 are subgroup of SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ which are required by superstring theory.

Table 13: The embedding of R into $G = G_1 \times G_2$ for the coset space (iii).

(iii) SU(4)/SU(3) × U(1)		
(a)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ SU(3),	$G_2 \supset$ U(1)
(b)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ U(1),	$G_2 \supset$ SU(3)

Table 14: The embedding of R into $G = G_1 \times G_2$ for the coset space (iv).

(iv) Sp(4) × SU(2)/[SU(2) × SU(2)] × U(1)		
(a)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ SU(2),	$G_2 \supset$ SU(2) × U(1)
(b)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ SU(2) × SU(2),	$G_2 \supset$ U(1)
(c)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ U(1),	$G_2 \supset$ SU(2) × SU(2)
(d)	$G_1 \supset (G_{\rm SM}$ or $G_{\rm GUT}) \times$ SU(2) × U(1),	$G_2 \supset$ SU(2)

Table 15: The candidates of the gauge groups G_1 and G_2 for each of the coset space (i) and (ii) in Table 1. The top row indicates the assigned number of S/R in Table 1 and embedding of R assigned in Table 12. The leftmost column indicates H.

	(i)-(a) and (ii)-(a)	(i)-(b) and (ii)-(b)
SU(3) × SU(2) × U(1) × U(1)	G_1 = SO(10), SO(11), Sp(10) G_2 = SU(2), U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = SU(2)
SU(5) × U(1)	G_1 = No candidate G_2 = SU(2), U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = SU(2)
SO(10) × U(1)	G_1 = SO(13) G_2 = SU(2), U(1)	G_1 = SO(12), SO(13), E_6 G_2 = SU(2)
E_6 × U(1)	G_1 = No candidate G_2 = SU(2), U(1)	G_1 = E_7 G_2 = SU(2)

Table 16: The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iii) in Table 1. The top row indicates the assigned number of S/R in Table 1 and embedding of R assigned in Table 13. The leftmost column indicates H.

	(iii)-(a)	(iii)-(b)
SU(3) × SU(2) × U(1) × U(1)	G_1 = E_6 G_2 = SU(2), U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = G_2, SU(3)
SU(5) × U(1)	G_1 = No candidate G_2 = SU(2), U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = G_2, SU(3)
SO(10) × U(1)	G_1 = No candidate G_2 = SU(2), U(1)	G_1 = SO(12), SO(13) E_6 G_2 = G_2, SU(3)
E_6 × U(1)	G_1 = E_8 G_2 = SU(2), U(1)	G_1 = E_7 G_2 = G_2, SU(3)

Table 17: The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iv) in Table 1. The top row indicates the assigned number of S/R in Table 1 and embedding of R assigned in Table 14. The leftmost column indicates H.

	(iv)-(a)	(iv)-(b)
SU(3) × SU(2) × U(1) × U(1)	G_1 = SO(10), SO(11), Sp(10) G_2 = SU(3), Sp(4), G_2	G_1 = SO(13), Sp(12) G_2 = SU(2), U(1)
SU(5) × U(1)	G_1 = No candidate G_2 = SU(3), Sp(4), G_2	G_1 = No candidate G_2 = SU(2), U(1)
SO(10) × U(1)	G_1 = SO(13) G_2 = SU(3), Sp(4), G_2	G_1 = SO(14), SO(15) G_2 = SU(2), U(1)
E_6 × U(1)	G_1 = No candidate G_2 = SU(3), Sp(4), G_2	G_1 = No candidate G_2 = SU(2), U(1)

Table 18: The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iv) in Table 1. The top row indicates the assigned number of S/R in Table 1 and embedding of R assigned in Table 14. The leftmost column indicates H.

	(iv)-(c)	(iv)-(d)
SU(3) × SU(2) × U(1) × U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = G_2, Sp(4)	G_1 = SU(7), SO(12), SO(13), Sp(12), E_6 G_2 = SU(2)
SU(5) × U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10) G_2 = G_2, Sp(4)	G_1 = SU(7), SO(13) Sp(12), E_6 G_2 = SU(2)
SO(10) × U(1)	G_1 = SO(12), SO(13), E_6 G_2 = G_2, Sp(4)	G_1 = SO(14), SO(15), E_7 G_2 = SU(2)
E_6 × U(1)	G_1 = E_7 G_2 = G_2, Sp(4)	G_1 = E_8 G_2 = SU(2)

Table 19: The complex or real representations of the possible gauge groups [65]. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of S/R and embedding of R in Table 12–14.

Group	Complex representations	Real representations
SU(6)	**6**, **15**, **21**, **56**, **70**, **84**, **105**, **105′**, **120**, **126**, **210**, **210′**, **252**, **280**, **315**, **336**, **384**, **420**, **462**, **490**, **504**, **560**, **700**, **720**, **792**, **840**, **840′**, **840″**, **896**, $\cdots$	**35**, **175**, **189**, **405**, $\cdots$
SO(11)		**11**, **55**, **65**, **165**, **275**, **320**, **330**, **429**, **462**, **935**, $\cdots$
SO(12)		**12**, **66**, **77**, **220**, **352**, **462**, **495**, **560**, **792**, $\cdots$
SO(14)	**64**, **832**, $\cdots$	**14**, **91**, **104**, **364**, **546**, **896**, $\cdots$
SO(15)		**15**, **105**, **119**, **128**, **455**, **665**, $\cdots$
F_4		**26**, **52**, **273**, **324**, $\cdots$
E_6	**27**, **351**, **351′** $\cdots$	**78**, **650**, $\cdots$
E_7		**133** $\cdots$
E_8		**248**, $\cdots$

The representation F_1 of G_1 for the fermions should be either complex or real but not pseudoreal, since the fermions of pseudoreal representation do not allow the Majorana condition when $D = 10$ and induce doubled fermion contents after the dimensional reduction [27]. Table 19 lists the candidate groups G_1 and their complex and real representations whose dimension is no more than 1024. The representations in this table are the candidates of F_1. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of S/R and embedding of R in Table 12–14.

The representation F_2 of G_2 has to be real as well as F_1 to impose the Majorana condition. Without this condition, F_2 can be any representation. We limited ourselves to the case $\dim F = \dim F_1 \times \dim F_2 < 1025$ since larger representations yield numerous higher dimensional representations of fermion under the $G_{\rm SM} \times U(1)$ and $G_{\rm GUT} \times U(1)$.

Now we are ready to investigate the representations for fermions and scalars in four dimensions. We first note that we need a R_2 singlet in SO(6) vector to obtain the Higgs candidate h_g (*cf.* Eq.(2.19) and the discussion below). We can thus exclude the candidates (i) and (iii) of S/R in Table 1 (*cf.* Table 11). In Tables 20–22, we list the possible candidates of G_1, G_2, (F_1, F_2), and the corresponding representations of four-dimensional scalars and fermions for each H, which is either $G_{\rm SM} \times {\rm U}(1)$, ${\rm SU}(5) \times {\rm U}(1)$, ${\rm SO}(10) \times {\rm U}(1)$, or $E_6 \times {\rm U}(1)$. The representations of four dimensional fermions are classified into A, B, and C. The representations of class A are the *standard representations*; $\bar{\mathbf{5}}$ and **10** for SU(5), **16** for SO(10), and **27** for E_6, which lead to the SM fermions after GUT breaking. The representations of class B lead to both of the SM fermions and non-SM fermions after GUT breaking. The representations of class C lead only to non-SM fermions after GUT breaking.

$H = G_{\rm SM} \times {\rm U}(1)$

We investigate all combinations of S/R, G_1 and G_2 in Table 15–18 which provide $H = G_{\rm SM} \times {\rm U}(1)$ in four dimensions. We obtain a representation which is identified as the SM Higgs-doublet in four dimensions from the following cases.

1. R embedded as (ii)-(b), $G_1 = {\rm SU}(6)$ and $G_2 = {\rm SU}(2)$.

2. R embedded as (ii)-(b), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{SU}(2)$.
3. R embedded as (iv)-(c), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{G}_2$.
4. R embedded as (iv)-(c), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{Sp}(4)$.
5. R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{G}_2$.
6. R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{Sp}(4)$.
7. R embedded as (iv)-(d), $G_1 = \mathrm{Sp}(12)$ and $G_2 = \mathrm{SU}(2)$.
8. R embedded as (iv)-(d), $G_1 = \mathrm{E}_6$ and $G_2 = \mathrm{SU}(2)$.

Any of these cases does not reproduce a whole generation of the SM fermions. Therefore we cannot obtain the SM in four dimensions. The difficulty in obtaining the SM is ultimately due to the smallness of the dimension of SO(6) spinor representation.

$H = \mathrm{SU}(5) \times \mathrm{U}(1)$

We investigate the case of $H = \mathrm{SU}(5) \times \mathrm{U}(1)$ and summarize the result in Table 20. We obtain the representation **5** which corresponds to the Higgs scalar in the following cases.

1. R embedded as (ii)-(b), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{SU}(2)$.
2. R embedded as (ii)-(b), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{SU}(2)$.
3. R embedded as (iv)-(c), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{Sp}(4)$.
4. R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{Sp}(4)$.
5. R embedded as (iv)-(d), $G_1 = \mathrm{E}_6$ and $G_2 = \mathrm{SU}(2)$.

As for the fermions, we see that the *standard representations* of SU(5) GUT are not obtained at all for the cases 3, 4, and 5, while they are obtained by combining two representations of F in the cases 1 and 2. For the example of case 1, we can choose $(\mathbf{70}, \mathbf{2})$ and $(\mathbf{280}, \mathbf{1})$ to obtain all the *standard representations*, $\bar{\mathbf{5}}$ and **10**, in four dimensions, along with the extra fermions of class B and C.

$H = \mathrm{SO}(10) \times \mathrm{U}(1)$

We investigate all the combinations of S/R, G_1, and G_2 for $H = \mathrm{SO}(10) \times \mathrm{U}(1)$. We obtain the representation **10** which corresponds to the Higgs scalar in the following cases.

1. R embedded as (ii)-(b), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{SU}(2)$.
2. R embedded as (ii)-(b), $G_1 = \mathrm{E}_6$ and $G_2 = \mathrm{SU}(2)$.
3. R embedded as (iv)-(b), $G_1 = \mathrm{SO}(14)$ and $G_2 = \mathrm{SU}(2)$.
4. R embedded as (iv)-(b), $G_1 = \mathrm{SO}(14)$ and $G_2 = \mathrm{U}(1)$.
5. R embedded as (iv)-(b), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{SU}(2)$.

6. R embedded as (iv)-(b), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{U}(1)$.
7. R embedded as (iv)-(c), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{G}_2$.
8. R embedded as (iv)-(c), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{Sp}(4)$.
9. R embedded as (iv)-(c), $G_1 = \mathrm{E}_6$ and $G_2 = \mathrm{SU}(2)$
10. R embedded as (iv)-(c), $G_1 = \mathrm{E}_6$ and $G_2 = \mathrm{G}_2$.
11. R embedded as (iv)-(d), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{SU}(2)$.
12. R embedded as (iv)-(d), $G_1 = \mathrm{E}_7$ and $G_2 = \mathrm{SU}(2)$.

We further obtain the *standard representations* of the fermions which lead to all the SM fermions of one generation in the cases $1-6$, 8, 11 and 12 (see Table 21).

The case 4 with $F = \mathbf{832}(1)$ is intriguing since we obtain two **16**s and two **144**s, each of which leads to a complete set of the SM fermions of one generation. We thus obtain four generations of fermions which can accommodate the known three generations. Furthermore these representations can form three distinct types of Yukawa coupling: $\mathbf{16} \times \mathbf{16} \times \mathbf{10}$, $\mathbf{144} \times \mathbf{16} \times \mathbf{10}$, and $\mathbf{144} \times \mathbf{144} \times \mathbf{10}$. These couplings may explain the origin of the Yukawa couplings distinguishing the the generations and the mixing among them.

$H = \mathrm{E}_6 \times \mathrm{U}(1)$

The results for $H = \mathrm{E}_6 \times \mathrm{U}(1)$ are listed in Table 22. We obtain representation **27** which corresponds to the Higgs scalar in the following cases.

1. R embedded as (ii)-(b), $G_1 = \mathrm{E}_7$ and $G_2 = \mathrm{SU}(2)$.
2. R embedded as (iv)-(c), $G_1 = \mathrm{E}_7$ and $G_2 = \mathrm{G}_2$.
3. R embedded as (iv)-(d), $G_1 = \mathrm{E}_8$ and $G_2 = \mathrm{SU}(2)$.

The *standard representations* of fermion **27**, which provide all the SM fermions of one generation, are obtained in cases 1 and 3.

Case 1 with $F = (\mathbf{133}, \mathbf{1})$ is interesting since the structure of the SM with three generations may be explained. The Yukawa coupling of this model needs to be in the form $\overline{\mathbf{27}}(-2) \times \mathbf{27}(2) \times \mathbf{78}(0)$. The fermion representation $\mathbf{27} + \mathbf{78}$ of E_6 contains three generations of $\bar{\mathbf{5}} + \mathbf{10}$ in terms of its SU(5) subgroup, giving the origin of the known three generations. Indeed, this fermion content is analyzed in, for example, nonlinear sigma models giving a family unification [74] based on a broken E_7 symmetry [75], under which a reproduction of the observed mixing structure among the three generations of fermions has been attempted [76].

Table 20: The models for H=SU(5) × U(1) which include the SM Higgs-doublet and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified into A, B, and C. The fermion-As contain only the SM fermions; fermion-Bs contain both the SM fermions and extra fermions; fermion-Cs contain only extra fermions.

S/R = Sp(4)/[SU(2) × U(1)], $G_1 \supset$ SU(5) × U(1), $G_2 \supset$ SU(2)

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
SU(6)	SU(2)	(56, 2)	5(6), $\bar{5}$(-6)		15(-3)	35(-3)
		(70, 2)	5(6), $\bar{5}$(-6)	10(-3)	15(-3), 40(-3)	
		(280, 1)	5(6), $\bar{5}$(-6)	$\bar{5}$(-6)	$\overline{70}$(-6)	24(0), 45(-6), 126(0)
						24(0), 126(0)
		(405, 1)	5(6), $\bar{5}$(-6)	$\bar{5}$(-6)	$\overline{70}$(-6)	1(0), 24(0), 200(0)
						5(6), 70(6), 1(0), 24(0), 200(0)
		(840, 1)	5(6), $\bar{5}$(-6)		$\overline{45}$(6)	280'(6), 126(0), 224(0)
						$\overline{105}$(6), 126(0), 224(0)
SO(11)	SU(2)	(11, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)		1(0)
		(55, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)		1(0), 24(0)
		(65, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)		1(0), 24(0)
		(165, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)	$\bar{4}5$(-2)	1(0), 24(0)
		(275, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)	$\bar{7}0$(-2)	1(0), 24(0)
		(320, 2)	5(2), $\bar{5}$(-2)	10(-1), 10(-1)	15(-1), 40(-1)	
		(330, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)	$\bar{4}5$(-2)	1(0), 24(0), 75(0)
		(429, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2), $\bar{5}$(-2)	$\bar{4}5$(-2), $\bar{7}0$(-2)	1(0), 24(0), 24(0)
		(462, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)	$\bar{4}5$(-2), $\bar{5}0$(-2)	1(0), 24(0), 75(0)
		(935, 1)	5(2), $\bar{5}$(-2)	$\bar{5}$(-2)	$\bar{7}0$(-2)	1(0), 24(0), 200(0)

Table 21: The models for H = SO(10) × U(1) which include the SM Higgs and one generation of the SM fermions in four-dimensions. The fermions in four-dimensions are classified into A, B, and C where fermion-As are **16** representation of SO(10); fermion-Bs contain both the SM fermions and extra-fermions; fermion-Cs contain only extra-fermions. We can obtain two types of results for fermions from one combination of (G_1, G_2, F) since we have a freedom to change the overall sign of U(1) charges which appear in the R-decomposition of SO(6) vector and spinor.

S/R = Sp(4)/[SU(2) × U(1)]. $G_1 \supset$ SO(10) × U(1). $G_2 \supset$ SU(2)

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
SO(12)	SU(2)	(12, 1)	10(2), 10(-2)		10(0),	1(2)
					10(0),	1(-2)
		(66, 1)	10(2), 10(-2)		10(2), 45(0)	1(0)
					10(-2), 45(0)	1(0)
		(77, 1)	10(2), 10(-2)		10(2), 54(0)	1(0)
					10(-2), 54(0)	1(0)
		(220, 1)	10(2), 10(-2)		45(2), 10(0), 120(0)	
					45(-2), 10(0), 120(0)	
		(352, 1)	10(2), 10(-2)		54(2), 10(0), 210'(0)	
					54(-2), 1(-2), 10(0), 210'(0)	
		(462, 1)	10(2), 10(-2)		126(2), 210(0)	
					$\overline{126}$(-2), 210(0)	
		(495, 1)	10(2), 10(-2)		120(2), 45(0), 210(0)	
					120(-2), 45(0), 210(0)	
		(560, 1)	10(2), 10(-2)		54(2), 45(2), 10(0), 10(0), 320(0)	1(2)
					54(-2), 45(-2), 10(0), 10(0), 320(0)	1(-2)
		(792, 1)	10(2), 10(-2)		210(2), 120(0), 126(0), $\overline{126}$(0)	
					210(-2), 120(0), 126(0), $\overline{126}$(0)	
E_6	SU(2)	(78, 1)	16(-3), $\overline{16}$(3)	16(-3)	45(0)	1(0)
					45(0)	1(0) $\overline{16}$(3)
		(650, 1)	16(-3), $\overline{16}$(3)	16(3)	$\overline{144}$(3), 45(0), 54(0), 210(0)	1(0)
					144(-3),45(0), 54(0), 210(0)	1(0) $\overline{16}$(-3)

S/R = Sp(4) × SU(2)/[SU(2) × SU(2)] × U(1), $G_1 \supset$ SO(10) × SU(2) × SU(2), $G_2 \supset$ U(1)

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
SO(14)	SU(2)	(64, 2)	10(0)	16(1),16(1)		$\overline{16}$(-1),$\overline{16}$(-1)
	U(1)	64(1)	10(0)	16(1),16(-1)		
		832(1)	10(0)	16(1),16(-1)	144(1) 144(-1)	
SO(15)	SU(2)	(128, 2)	10(0), 1(0)	16(1), 16(-1)		$\overline{16}$(1), $\overline{16}$(-1)
	U(1)	128(1)	10(0), 1(0)	16(1)		$\overline{16}$(1)

S/R = Sp(4) × SU(2)/[SU(2) × SU(2)] × U(1). $G_1 \supset$ SO(10) × SU(2) × U(1). $G_2 \supset$ SU(2)

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
SO(15)	SU(2)	(128,1)	10(2),10(-2)	16(1)	$\overline{16}$(1)	
E_7	SU(2)	(133,1)	10(2),10(-2)	16(1)		

Table 22: The models for $H = \mathrm{E}_6 \times \mathrm{U}(1)$ which include the SM Higgs and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified into A, B and C where fermion-As are **27** representation of E_6; fermion-Bs contain both the SM fermions and extra fermions; fermion-Cs contain only extra-fermions. We can obtain two types of results for fermions from one combination of (G_1, G_2, F) since we have a freedom to change the overall sign of U(1) charges which appear in R-decomposition of SO(6) vector and spinor.

$S/R = \mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{U}(1)$, $G_1 \supset \mathrm{E}_6 \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
E_7	SU(2)	(133,1)	27(2),$\bar{27}(-2)$	27(2)	78(0)	1(0)
					$\overline{27}$(-2), 78(0)	1(0)

$S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G_1 \supset \mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$

G_1	G_2	(F_1, F_2)	Scalars	Fermions-A	B	C
E_8	SU(2)	F(248, 1)	27(-2), $\bar{27}$(2)	27(1)		
					$\bar{27}$(-1)	

2.2.3 Models on eight-dimensional spacetime

In this section we search for realistic models in the CSDR scheme in eight-dimensions [62].

First we investigate four-dimensional gauge group H and higher-dimensional gauge group G for each coset space of (i), (ii) and (iii) listed in Table. 2. The four-dimensional gauge groups acceptable for H are listed in Table. 23. This Table is obtained from the following considerations.

1. The number of U(1)s in H must be more than that in R, since the U(1)s in R are also part of its centralizer, i.e. part of H. Therefore, the number of U(1)s in H must be R or more. We thus exclude SU(5), SO(10), and E_6 for coset space of (ii) and G_{SM}, SU(5), SO(10), E_6, $\mathrm{SU}(5) \times \mathrm{U}(1)$, $\mathrm{SO}(10) \times \mathrm{U}(1)$, and $\mathrm{E}_6 \times \mathrm{U}(1)$ for coset space of (iii).

2. We also exclude G_{SM} for the coset space of (ii) and $G_{\mathrm{SM}} \times \mathrm{U}(1)$ for the coset space of (iii). The hypercharges of the SM should be reproduced by the U(1) charges in R, which means that all the hypercharges must appear in the decomposition of SO(4) spinor. The dimension of the SO(4) spinor representation is however two, and hence more than two different values of U(1) charges are not available. Consequently, these cases never reproduce the five hypercharges of the SM fermions.

3. We allow at most one extra U(1) in four-dimensional gauge group. This excludes $G_{\mathrm{SM}} \times \mathrm{U}(1)$, $\mathrm{SU}(5) \times \mathrm{U}(1)$, $\mathrm{SO}(10) \times \mathrm{U}(1)$, $\mathrm{E}_6 \times \mathrm{U}(1)$, $G_{\mathrm{SM}} \times \mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{SO}(10) \times \mathrm{U}(1) \times \mathrm{U}(1)$, and $\mathrm{E}_6 \times \mathrm{U}(1) \times \mathrm{U}(1)$ for coset space of (i), and $G_{\mathrm{SM}} \times \mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{SO}(10) \times \mathrm{U}(1) \times \mathrm{U}(1)$, and $\mathrm{E}_6 \times \mathrm{U}(1) \times \mathrm{U}(1)$ for coset space of (ii).

The higher-dimensional gauge group G should have the same rank as that of $H \times R$ up to U(1)s and possess complex representations to obtain chiral fermions. We also list the candidates of G in Table 23.

We investigate representations of four-dimensional gauge group in the CSDR scheme. The representations for scalars in four-dimensional spacetime are obtained by comparing Eq. (2.26) and Eq. (2.27),

Table 23: The candidates of H and G.

S/R	H	G
(i)	SU(3) × SU(2) × U(1)	SU(7), E_6
	SU(5)	SU(7), E_6
	SO(10)	SU(8), SO(14)
	E_6	SU(9)
(ii)	SU(3) × SU(2) × U(1) × U(1)	SU(7), E_6
	SU(5) × U(1)	SU(7), E_6
	SO(10) × U(1)	SU(8), SO(14)
	E_6 × U(1)	SU(9)
(iii)	SU(3) × SU(2) × U(1) × U(1) × U(1)	SU(7), E_6
	SU(5) × U(1) × U	SU(7), E_6
	SO(10) × U(1) × U	SU(8), SO(14)
	E_6 × U(1) × U	SU(9)

while those for fermions are similarly obtained by comparing Eq. (2.29) and Eq. (2.30). Note again that F must be complex representation in order to obtain chiral fermions in four dimensions. We limit the dimension of F to 1000 to avoid numerous representations of the fermions under four-dimensional gauge group.

Exhaustive investigation of all combinations of S/R, G, H and F leaves six candidates of models which include at least one generation of known fermions; they are listed in Table 24. We find that all of these viable models give $H = \mathrm{SO}(10)$ (with one or two U(1)s) in four dimensions and that no model lead to a promising four-dimensional theory that gives $H = G_{\mathrm{SM}}$, $H = \mathrm{SU}(5)$ or $H = \mathrm{E}_6$ (with one or two U(1)s). This result is understood as follows. For $H = G_{\mathrm{SM}}$ and SU(5) (with one or two U(1)s), no model reproduces a whole generation of the SM fermions due to the smallness of the dimension of SO(4) spinor representation. For $H = \mathrm{E}_6$ (with one or two U(1)s), no gauge group has E_6 as a regular subgroup and has a complex representation at the same time. On the other hand, the models with $H = \mathrm{SO}(10)$ (with one or two U(1)s) are favored since one generation of the SM fermions can be embedded into only one representation, such as **16** and **144**, and these representations are easily obtained from complex representations of SO(14).

We summarize the details of these viable $H = \mathrm{SO}(10)$ (with one or two U(1)s) models below.

For coset space of (i), we embed $R = \mathrm{SU}(2)\times\mathrm{SU}(2)$ into $G = \mathrm{SO}(14)$ according to the decomposition

$$\mathrm{SO}(14) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(10). \tag{2.68}$$

The decomposition of the adjoint representation of SO(14) according to the decomposition of Eq. (2.68) is

$$\mathbf{91} = (\mathbf{1}, \mathbf{1}, \mathbf{45}) + (\mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\underline{\mathbf{2}, \mathbf{2}}, \mathbf{10}), \tag{2.69}$$

and thus we obtain **10** as the scalar reperesentation in four dimensions. Similarly, we decompose the complex representations **64** and **832** of SO(14) according to the decomposition of Eq. (2.68) as

$$\mathbf{64} =(\underline{\mathbf{2}, \mathbf{1}}, \mathbf{16}) + (\underline{\mathbf{1}, \mathbf{2}}, \overline{\mathbf{16}}), \tag{2.70}$$

$$\begin{aligned}\mathbf{832} =&(\underline{\mathbf{1}, \mathbf{2}}, \mathbf{144}) + (\underline{\mathbf{2}, \mathbf{1}}, \overline{\mathbf{144}}) + (\mathbf{2}, \mathbf{3}, \mathbf{16}) + (\mathbf{3}, \mathbf{2}, \overline{\mathbf{16}}) \\ &+ (\underline{\mathbf{1}, \mathbf{2}}, \overline{\mathbf{16}}) + (\underline{\mathbf{2}, \mathbf{1}}, \mathbf{16}),\end{aligned} \tag{2.71}$$

Table 24: Four-dimensional scalar and fermion representations for each combination of S/R, G, H and F.

S/R	H	G	scalar	F	fermions
(i)	SO(10)	SO(14)	$\mathbf{10}$	$\mathbf{64}$	$\{\mathbf{16}\}^2$
				$\mathbf{832}$	$\{\mathbf{16}\}^2$, $\{\overline{\mathbf{144}}\}^2$
(ii)	SO(10) × U(1)	SO(14)	$\mathbf{10}(1)$, $\mathbf{10}(-1)$	$\mathbf{64}$	$\mathbf{16}(0)$, $\mathbf{16}(1)$, $\mathbf{16}(-1)$
				$\mathbf{832}$	$\{\mathbf{16}(0)\}^2$, $\mathbf{16}(1)$, $\mathbf{16}(-1)$, $\overline{\mathbf{144}}(0)$, $\overline{\mathbf{144}}(1)$, $\overline{\mathbf{144}}(-1)$
(iii)	SO(10) × U(1) × U(1)	SO(14)	$\mathbf{10}(1,1)$, $\mathbf{10}(1,-1)$ $\mathbf{10}(-1,1)$, $\mathbf{10}(-1,-1)$	$\mathbf{64}$	$\mathbf{16}(1,0)$, $\mathbf{16}(-1,0)$, $\mathbf{16}(0,1)$, $\mathbf{16}(0,-1)$
				$\mathbf{832}$	$\{\mathbf{16}(1,0)\}^2$, $\{\mathbf{16}(-1,0)\}^2$, $\{\mathbf{16}(0,1)\}^2$, $\{\mathbf{16}(0,-1)\}^2$, $\overline{\mathbf{144}}(1,0)$, $\overline{\mathbf{144}}(-1,0)$, $\overline{\mathbf{144}}(0,1)$, $\overline{\mathbf{144}}(0,-1)$

and obtain $\{\mathbf{16}\}^2$ from $F = \mathbf{64}$ and $\{\mathbf{16}\}^2 + \{\overline{\mathbf{144}}\}^2$ from $F = \mathbf{832}$ as representations for the left-handed fermion in four dimensions.

For coset space of (ii), we embed SU(2) × U(1) into SO(14) according to the decomposition

$$\begin{aligned} SO(14) &\supset SU(2) \times SU(2) \times SO(10) \\ &\supset SU(2) \times U(1) \times SO(10). \end{aligned} \tag{2.72}$$

The decomposition of the adjoint representation of SO(14) according to the decomposition of Eq. (2.72) is

$$\begin{aligned} \mathbf{91} =& (\mathbf{1},\mathbf{1},\mathbf{45}) + (\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{10}) \\ =& (\mathbf{1},\mathbf{45})(0) + (\mathbf{3},\mathbf{1})(0) + (\mathbf{1},\mathbf{1})(2) + (\mathbf{1},\mathbf{1})(0) + (\mathbf{1},\mathbf{1})(-2) \\ & + (\underline{\mathbf{2}},\mathbf{10})(\underline{1}) + (\underline{\mathbf{2}},\mathbf{10})(\underline{-1}), \end{aligned} \tag{2.73}$$

and thus we obtain $(\mathbf{10}(1))$ and $(\mathbf{10}(-1))$ as the scalar representations in four dimensions. Similarly, we decompose the complex representations $\mathbf{64}$ and $\mathbf{832}$ of SO(14) according to the decomposition of Eq. 2.72) as

$$\begin{aligned} \mathbf{64} =& (\mathbf{2},\mathbf{1},\mathbf{16}) + (\mathbf{1},\mathbf{2},\overline{\mathbf{16}}) \\ =& (\mathbf{2},\mathbf{16})(0) + (\mathbf{1},\overline{\mathbf{16}})(1) + (\mathbf{1},\overline{\mathbf{16}})(-1) \end{aligned} \tag{2.74}$$

$$\begin{aligned} \mathbf{832} =& (\mathbf{1},\mathbf{2},\mathbf{144}) + (\mathbf{2},\mathbf{1},\overline{\mathbf{144}}) + (\mathbf{2},\mathbf{3},\mathbf{16}) + (\mathbf{3},\mathbf{2},\overline{\mathbf{16}}) \\ & + (\mathbf{1},\mathbf{2},\overline{\mathbf{16}}) + (\mathbf{2},\mathbf{1},\mathbf{16}) \\ =& (\underline{\mathbf{1}},\mathbf{144})(\underline{1}) + (\underline{\mathbf{1}},\mathbf{144})(\underline{-1}) + (\underline{\mathbf{2}},\overline{\mathbf{144}})(\underline{0}) + (\mathbf{2},\mathbf{16})(2) \\ & + (\underline{\mathbf{2}},\mathbf{16})(\underline{0}) + (\mathbf{2},\mathbf{16})(-2) + (\mathbf{3},\overline{\mathbf{16}})(1) + (\mathbf{3},\overline{\mathbf{16}})(-1) \\ & + (\underline{\mathbf{1}},\overline{\mathbf{16}})(\underline{1}) + (\underline{\mathbf{1}},\overline{\mathbf{16}})(\underline{-1}) + (\underline{\mathbf{2}},\mathbf{16})(\underline{0}), \end{aligned} \tag{2.75}$$

and obtain $\mathbf{16}(0)$, $\mathbf{16}(1)$ and $\mathbf{16}(-1)$ from $F = \mathbf{64}$ and $\{\mathbf{16}(0)\}^2$, $\mathbf{16}(1)$, $\mathbf{16}(-1)$, $\overline{\mathbf{144}}(0)$, $\overline{\mathbf{144}}(1)$ and $\overline{\mathbf{144}}(-1)$ from $F = \mathbf{832}$ as representations for the left-handed fermion in four dimensions.

For coset space of (iii), we embed U(1) × U(1) into SO(14) according to the decomposition

$$\begin{aligned} \mathrm{SO}(14) &\supset \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(10) \\ &\supset \mathrm{SO}(10)\times\mathrm{U}(1)\times\mathrm{U}(1). \end{aligned} \tag{2.76}$$

The decomposition of the adjoint representation of SO(14) according to the decomposition of Eq. (2.76) is

$$\begin{aligned} \mathbf{91} =&(\mathbf{1},\mathbf{1},\mathbf{45})+(\mathbf{3},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3},\mathbf{1})+(\mathbf{2},\mathbf{2},\mathbf{10}) \\ =&\mathbf{45}(0,0)+\mathbf{1}(2,0)+\mathbf{1}(0,0)+\mathbf{1}(-2,0)+\mathbf{1}(0,2)+\mathbf{1}(0,0) \\ &+\mathbf{1}(0,-2)+\mathbf{10}(1,1)+\mathbf{10}(1,-1)+\mathbf{10}(-1,1)+\mathbf{10}(-1,-1), \end{aligned} \tag{2.77}$$

and thus we obtain $(\mathbf{10}(1,1))$, $(\mathbf{10}(1,-1))$, $(\mathbf{10}(-1,1))$ and $(\mathbf{10}(-1,-1))$ as the scalar representations in four dimensions. Similarly, we decompose the complex representations **64** and **832** of SO(14) according to the decomposition of Eq. (2.76) as

$$\begin{aligned} \mathbf{64} =&(\mathbf{2},\mathbf{1},\mathbf{16})+(\mathbf{1},\mathbf{2},\overline{\mathbf{16}}) \\ =&\mathbf{16}(1,0)+\mathbf{16}(-1,0)+\overline{\mathbf{16}}(0,1)+\overline{\mathbf{16}}(0,-1) \end{aligned} \tag{2.78}$$

$$\begin{aligned} \mathbf{832} =&(\mathbf{1},\mathbf{2},\mathbf{144})+(\mathbf{2},\mathbf{1},\overline{\mathbf{144}})+(\mathbf{2},\mathbf{3},\mathbf{16})+(\mathbf{3},\mathbf{2},\overline{\mathbf{16}})+(\mathbf{1},\mathbf{2},\overline{\mathbf{16}}) \\ &+(\mathbf{2},\mathbf{1},\mathbf{16}) \\ =&\mathbf{144}(0,1)+\mathbf{144}(0,-1)+\overline{\mathbf{144}}(1,0)+\overline{\mathbf{144}}(-1,0) \\ &+\mathbf{16}(1,2)+\mathbf{16}(1,0)+\mathbf{16}(1,-2)+\mathbf{16}(-1,2) \\ &+\mathbf{16}(-1,0)+\mathbf{16}(-1,-2)+\overline{\mathbf{16}}(2,1)+\overline{\mathbf{16}}(2,-1) \\ &+\overline{\mathbf{16}}(0,1)+\overline{\mathbf{16}}(0,-1)+\overline{\mathbf{16}}(-2,1)+\overline{\mathbf{16}}(-2,-1) \\ &+\overline{\mathbf{16}}(0,1)+\overline{\mathbf{16}}(0,-1)+\mathbf{16}(1,0)+\mathbf{16}(-1,0), \end{aligned} \tag{2.79}$$

and obtain $\mathbf{16}(1,0)$, $\mathbf{16}(-1,0)$, $\mathbf{16}(0,1)$ and $\mathbf{16}(0,-1)$ from $F=\mathbf{64}$ and $\{\mathbf{16}(1,0)\}^2$, $\{\mathbf{16}(-1,0)\}^2$, $\{\mathbf{16}(0,1)\}^2$, $\{\mathbf{16}(0,-1)\}^2$, $\overline{\mathbf{144}}(1,0)$, $\overline{\mathbf{144}}(-1,0)$, $\overline{\mathbf{144}}(0,1)$ and $\overline{\mathbf{144}}(0,-1)$ from $F=\mathbf{832}$ as representations for the left-handed fermion in four dimensions.

We can obtain one generation of the SM fermion from all of the candidates listed in Table 24 since the representations **16** and **144** of SO(10) include one generation of the SM fermion. The models with SU(3)/SU(2)×U(1) are particularly intersting in our results. We obtain the three generations of the SM fermions for this coset space with $F=\mathbf{64}$. We also obtain odd number generation in the combination of coset space of (ii) with $F=\mathbf{832}$. This is due to the fact that the SO(4) spinor is not self conjugate which forbid Majorana-Weyl condition and that R, which is SU(2)×U(1), are embedded into SO(4) lopsidedly.

We notice that these models provide anomaly-free theories in four-dimensional spacetime although the four-dimensional theories are chiral; the SO(10) gauge symmetry does not have anomaly, and the anomaly from the extra U(1) symmetry cancels since the traces of U(1) charges and their cubes are zero for all cases.

3 The Standard(-like) Model from an SO(12) Grand Unified Theory in six-dimensions with S_2 extra-space [77]

3.1 Six-dimensional gauge theory with extra-space S^2 under the symmetry condition and non-trivial boundary conditions

In this section, we develop the scheme for constructing a four-dimensional theory from a gauge theory on six-dimensional spacetime which has extra-space as two-sphere S^2 with the symmetry condition and non-trivial boundary conditions.

3.1.1 A Gauge theory on six-dimensional spacetime with S_2 extra-space

We begin with a gauge theory with a gauge group G defined on a six-dimensional spacetime M^6. The spacetime M^6 is assumed to be a direct product of the four-dimensional Minkowski spacetime M^4 and two-sphere S^2 such that $M^6 = M^4 \times S^2$. The two-sphere S^2 is a unique two-dimensional coset space, and can be written as $S^2 = \mathrm{SU}(2)_I/\mathrm{U}(1)_I$, where $\mathrm{U}(1)_I$ is the subgroup of $\mathrm{SU}(2)_I$. This coset space structure of S^2 requires that S^2 has the isometry group $\mathrm{SU}(2)_I$, and that the group $\mathrm{U}(1)_I$ is embedded into the group SO(2) which is a subgroup of the Lorentz group SO(1,5). We denote the coordinate of M^6 by $X^M = (x^\mu, y^\theta = \theta, y^\phi = \phi)$, where x^μ and $\{\theta, \phi\}$ are M^4 coordinates and S^2 spherical coordinates, respectively. The spacetime index M runs over $\mu \in \{0,1,2,3\}$ and $\alpha \in \{\theta, \phi\}$. The metric of M^6, denoted by g_{MN}, can be written as

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -g_{\alpha\beta} \end{pmatrix}, \tag{3.1}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ and $g_{\alpha\beta} = diag(1, \sin^{-2}\theta)$ are metric of M^4 and S^2 respectively. Notice that we omit the radius R of S^2 in this discussion. We define the vielbein e^M_A that connects the metric of M^6 and that of the tangent space of M^6, denoted by h_{AB}, as $g_{MN} = e^A_M e^B_N h_{AB}$. Here $A = (\mu, a)$, where $a \in \{4,5\}$, is the index for the coordinates of tangent space of M^6. The explicit form of the vielbeins are summarized in the Appendix. We introduce a gauge field $A_M(x,y) = (A_\mu(x,y), A_\alpha(x,y))$, which belongs to the adjoint representation of the gauge group G, and fermions $\psi(x,y)$, which lies in a representation F of G. The action of this theory is given by

$$S = \int dx^4 \sin\theta d\theta d\phi \big(\bar{\psi} i \Gamma^\mu D_\mu \psi + \bar{\psi} i \Gamma^a e^\alpha_a D_\alpha \psi - \frac{1}{4g^2} g^{MN} g^{KL} Tr[F_{MK} F_{NL}]\big), \tag{3.2}$$

where $F_{MN} = \partial_M A_N(X) - \partial_N A_M(X) - [A_M(X), A_N(X)]$ is the field strength, D_M is the covariant derivative including spin connection, and Γ_A represents the 6-dimensional Clifford algebra. Here D_M and Γ_A can be written explicitly as,

$$D_\mu = \partial_\mu - A_\mu, \tag{3.3}$$

$$D_\theta = \partial_\theta - A_\theta, \tag{3.4}$$

$$D_\phi = \partial_\phi - i\frac{\Sigma_3}{2}\cos\theta - A_\phi, \tag{3.5}$$

$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{I}_2, \tag{3.6}$$

$$\Gamma_4 = \gamma_5 \otimes \sigma_1, \tag{3.7}$$

$$\Gamma_5 = \gamma_5 \otimes \sigma_2, \tag{3.8}$$

where $\{\gamma_\mu, \gamma_5\}$ are the 4-dimensional Dirac matrices, $\sigma_i (i = 1, 2, 3)$ are Pauli matrices, $\mathbf{I}_d$ is $d \times d$ identity, and Σ_3 is defined as $\Sigma_3 = \mathbf{I}_4 \otimes \sigma_3$.

3.1.2 The symmetry condition and the boundary conditions

We impose on the gauge field $A_M(X)$ the symmetry which connects $\mathrm{SU}(2)_I$ isometry transformation on S^2 and the gauge transformation on the fields in order to carry out dimensional reduction, and the non-trivial boundary conditions of S^2 to restrict four-dimensional theory. The symmetry requires that the $\mathrm{SU}(2)_I$ coordinate transformation should be compensated by a gauge transformation [1, 21]. The symmetry further leads to the following set of the symmetry condition on the fields:

$$\xi_i^\beta \partial_\beta A_\mu = \partial_\alpha W_i + [W_i, A_\mu], \tag{3.9}$$

$$\xi_i^\beta \partial_\beta A_\alpha + \partial_\alpha \xi_i^\beta A_\beta = \partial_\alpha W_i + [W_i, A_\alpha], \tag{3.10}$$

where ξ_i^α is the Killing vectors generating $\mathrm{SU}(2)_I$ symmetry and W_i are some fields which generate an infitesimal gauge transformation of G. Here index $i = 1, 2, 3$ corresponds to that of SU(2) generators. The explicit forms of ξ_i^αs for S^2 are:

$$\begin{aligned} &\xi_1^\theta = \sin\phi, \qquad \xi_1^\phi = \cot\theta\cos\phi, \\ &\xi_2^\theta = -\cos\phi, \qquad \xi_2^\phi = \cot\theta\sin\phi, \\ &\xi_3^\theta = 0, \qquad \xi_3^\phi = -1. \end{aligned} \tag{3.11}$$

The LHSs of Eq (3.9,3.10) are infintesimal isometry $\mathrm{SU}(2)_I$ transformation and the RHSs of those are infintesimal gauge transformation.

The non-trivial boundary conditions are defined so as to remain the action Eq (3.2) invariant, and are written as

$$\psi(x, \pi-\theta, -\phi) = \gamma_5 P \psi(x, \theta, \phi), \tag{3.12}$$

$$A_\mu(x, \pi-\theta, -\phi) = P A_\mu(x, \theta, \phi) P, \tag{3.13}$$

$$A_\theta(x, \pi-\theta, -\phi) = -P A_\theta(x, \theta, \phi) P, \tag{3.14}$$

$$A_\phi(x, \pi-\theta, -\phi) = -P A_\phi(x, \theta, \phi) P, \tag{3.15}$$

$$\psi(x, \theta, \phi+2\pi) = P' \psi(x, \theta, \phi), \tag{3.16}$$

$$A_\mu(x, \theta, \phi+2\pi) = P' A_\mu(x, \theta, \phi) P', \tag{3.17}$$

$$A_\theta(x, \theta, \phi+2\pi) = P' A_\theta(x, \theta, \phi) P', \tag{3.18}$$

$$A_\phi(x, \theta, \phi+2\pi) = P' A_\phi(x, \theta, \phi) P', \tag{3.19}$$

where $P(P')$s act on the representation space of gauge group G and satisfy $P^2 = 1((P')^2 = 1)$; we can take element of $P(P')$ as ± 1.

3.1.3 The dimensional reduction and a Lagrangian in four-dimensions

The dimensional reduction of gauge sector is explicitly carried out by applying the solutions of the symmetry condition Eq (3.9,3.10). These solutions are given by Manton [1] as

$$\begin{aligned}
A_\mu &= A_\mu(x), && (3.20)\\
A_\theta &= -\Phi_1(x), && (3.21)\\
A_\phi &= \Phi_2(x)\sin\theta - \Phi_3\cos\theta, && (3.22)\\
W_1 &= -\Phi_3\frac{\cos\phi}{\sin\theta}, && (3.23)\\
W_2 &= -\Phi_3\frac{\sin\phi}{\sin\theta}, && (3.24)\\
W_3 &= 0, && (3.25)
\end{aligned}$$

and satisfy the following constraints:

$$\begin{aligned}
[\Phi_3, A_\mu] &= 0, && (3.26)\\
[-i\Phi_3, \Phi_i(x)] &= i\epsilon_{3ij}\Phi_j(x), && (3.27)
\end{aligned}$$

where $\Phi_1(x)$ and $\Phi_2(x)$ are scalar fields, and $-i\Phi_3$ are chosen as generator of $\mathrm{U}(1)_I$. Note that the Φ_3 term in Eq. (3.22) corresponds to the background gauge field [78]. Substituting the solutions Eq (3.20)-(3.22) into $A_M(X)$ in action Eq (3.2), we can easily integrate coordinates θ and ϕ in the gauge sector. We then obtain a four dimensional action as

$$\begin{aligned}
S_{4D}^{(gauge)} = \int d^4x \Big(&- \frac{1}{4g^2} Tr[F_{\mu\nu}F^{\mu\nu}(x)] \\
&- \frac{1}{2g^2} Tr[D'_\mu \Phi_1(x) D'^\mu \Phi_1(x) + D'_\mu \Phi_2(x) D'^\mu \Phi_2(x)] \\
&- \frac{1}{2g^2} Tr[(\Phi_3 + [\Phi_1(x), \Phi_2(x)])(\Phi_3 + [\Phi_1(x), \Phi_2(x)])] \Big), \qquad (3.28)
\end{aligned}$$

where $D'_\mu \Phi = \partial_\mu - [A_\mu, \Phi]$. The fermion sector of four-dimensional action is obtained by expanding fermions in normal modes of S^2 and then integrating S^2 coordinate in six-dimensional action. Thus, the fermions have massive KK modes which would be a candidate of dark matter. Generally, the KK modes do not have massless mode because of the positive curvature of S^2 [39]. We, however, can show that the fermion components satisfying the following condition have massless mode:

$$-i\Phi_3 \psi = \frac{\Sigma_3}{2}\psi. \qquad (3.29)$$

Square mass of the KK modes are eigenvalues of square of extra-dimensional Dirac-operator $-i\hat{D}$. In the S^2 case, $-i\hat{D}$ is written as

$$\begin{aligned}
-i\hat{D} &= -ie^{\alpha a}\Gamma_a D_\alpha \\
&= -i\big[\Sigma_1(\partial_\theta + \frac{\cot\theta}{2}) + \Sigma_2(\frac{1}{\sin\theta}\partial_\phi + \Phi_3\cot\theta)\big], \qquad (3.30)
\end{aligned}$$

where $\Sigma_i = \mathbf{I}_4 \times \sigma_i$. Square of $-i\hat{D}$ can be explicitly calculated:

$$(-i\hat{D})^2 = -\big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2 + i(2(-i\Phi_3) - \Sigma_3)\frac{\cos\theta}{\sin^2\theta}\partial_\phi \\ -\frac{1}{4} - \frac{1}{4\sin^2\theta} + \Sigma_3(-i\Phi_3)\frac{1}{\sin^2\theta} - (-i\Phi_3)^2\cot^2\theta\big]. \tag{3.31}$$

We then act this operator on a fermion $\psi(X)$ which satisfy Eq. (3.29), and obtain the reration

$$(-i\hat{D})^2\psi = -\big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\big]\psi. \tag{3.32}$$

The eigenvalues of the RHS operator are less than or equal to zero. Thus the fermion components satisfying Eq. (3.29) have massless mode, while other components only have massive KK mode. Note that the massless mode ψ_0 should be independent of S^2 coordinates θ and ϕ:

$$\psi_0 = \psi(x). \tag{3.33}$$

The existence of massless fermion may indicate the meaning of the symmetry condition; though the energy density of the gauge sector in the appearance of the background fields is higher than that of no background fields, since we have massless fermions, it may consist a ground state as a total in the presence of fermions. We also note that we could impose symmetry condition on fermions [22, 64]. In that case, we obtain the massless condition Eq. (3.29) from symmetry condition of fermion, and the solution of symmetry condition is independent from S^2 coordinate: $\psi = \psi(x)$ with no massive KK mode. Therefore, we can apply the same discussion for this case as our case if we only focus on the massless mode in our scheme.

3.1.4 A gauge symmetry and particle contents in four-dimensions

The symmetry conditions and the non-trivial boundary conditions substantially constrain the four dimensional gauge group and its representations for the particle contents. The gauge symmetry and particle contents in four-dimensions must satisfy the constraints Eq (3.26),(3.27),(3.29) and be consistent with the boundary conditions Eq (3.12)-(3.19). We show the prescriptions to identify four-dimensional gauge symmetry and particle contents below.

First, we show the prescriptions to identify gauge symmetry and field components which satisfy the constrants Eq (3.26),(3.27),(3.29). The gauge group H that satisfy the constraint Eq (3.26) is identified as

$$H = C_G(U(1)_I) \tag{3.34}$$

where $C_G(U(1)_I)$ denotes the centralizer of $\mathrm{U}(1)_I$ in G [21]. Note that this implies $G \supset H = H' \times \mathrm{U}(1)_I$, where H' is some subgroup of G.

Second, the scalar field components which satisfy the constraints Eq. (3.27) are specified by the following prescription. Suppose that the adjoint representations of $\mathrm{SU}(2)_I$ and G are decomposed according to the embeddings $\mathrm{SU}(2)_I \supset \mathrm{U}(1)_I$ and $G \supset H' \times \mathrm{U}(1)_I$ as

$$3(\mathrm{adj}\,\mathrm{SU}(2)) = (0(\mathrm{adj}\,\mathrm{U}(1)_R)) + (2) + (-2), \tag{3.35}$$

$$\mathrm{adj}\,G = (\mathrm{adj}\,H)(0) + 1(0(\mathrm{adj}\,\mathrm{U}(1))_R) + \sum_g h_g(r_g), \tag{3.36}$$

where h_gs denote representation of H', and r_gs denote $\mathrm{U}(1)_I$ charges. The scalar components satisfying the constraints belong to h_gs whose corresponding r_gs in the decomposition Eq. (3.36) are ± 2.

Third, the fermion components which satisfy the constraints Eq. (3.29) are determined as follows [64]. Let the group $\mathrm{U}(1)_I$ be embedded into the Lorentz group SO(2) in such a way that the vector representation 2 of SO(2) is decomposed according to $\mathrm{SO}(2) \supset \mathrm{U}(1)_I$ as

$$2 = (2) + (-2). \tag{3.37}$$

This embedding specifies a decomposition of the weyl spinor representation σ_6=4 of SO(1,5) according to SO(1,5) $\supset$ SU(2) $\times$ SU(2) $\times$ $\mathrm{U}(1)_I$ as

$$\sigma_6 = (2,1)(1) + (1,2)(-1), \tag{3.38}$$

where SU(2) $\times$ SU(2) representations (2,1) and (1,2) correspond to left-handed and right-handed spinors, respectively. We then decompose F according to $G \supset H' \times \mathrm{U}(1)_I$ as

$$F = \sum_f h_f(r_f). \tag{3.39}$$

Now the fermion components satisfying the constraints are identified as h_fs whose corresponding r_fs in the decomposition Eq. (3.39) are (1) for left-handed fermions and (-1) for right-handed fermions.

Finally, we show which gauge symmetry and field components remain in four-dimensions by surveying the consistency between the boundary conditions Eq. (3.12)-(3.19), the solutions Eq. (3.20)-(3.22), and fermion massless mode Eq. (3.33). We then apply Eq (3.20)-(3.22) and Eq. (3.33) to Eq. (3.12)-(3.19), and obtain the parity conditions

$$A_\mu(x) = P^{(\prime)} A_\mu(x) P^{(\prime)}, \tag{3.40}$$
$$-\Phi_1(x) = -P(-\Phi_1(x))P, \tag{3.41}$$
$$-\Phi_1(x) = P'(-\Phi_1(x))P', \tag{3.42}$$
$$\Phi_2(x) + \Phi_3 \cos\theta = -P\Phi_2(x)P + P\Phi_3 P \cos\theta, \tag{3.43}$$
$$\Phi_2(x) - \Phi_3 \cos\theta = P'\Phi_2(x)P' - P'\Phi_3 P' \cos\theta, \tag{3.44}$$
$$\psi(x) = \gamma^5 P\psi(x), \tag{3.45}$$
$$\psi(x) = P'\psi(x). \tag{3.46}$$

We find that gauge fields, scalar fields and massless fermions in four-dimensions should be even for $PA_\mu P$ and $P'A_\mu P'$; $-P\Phi_{1,2}P$ and $P'\Phi_{1,2}P'$; $\gamma_5 P\psi$ and $P'\psi$, respectively. Φ_3 always remains since it is proportional to an $\mathrm{U}(1)_I$ generator and commutes with $P(P')$. Therefore the particle contents are identified as the components which satisfy both the constraints Eq (3.26),(3.27),(3.29) and the parity conditions Eq Eq (3.40)-(3.46). The gauge symmetry remained in four-dimensions can also be identified by observing which components of the gauge fields remain.

3.2 The SO(12) model

In this section, we discuss a model based on a gauge group G=SO(12) and a representation F=32 of SO(12) for fermions. The choice of G=SO(12) and F=32 is motivated by the study based on CSDR which leads to an SO(10) $\times$ U(1) gauge theory with one generation of fermion in four-dimensions [27] (for SO(12) GUT see also [79]).

3.2.1 A gauge symmetry and particle contents

First, we show the particle contents in four-dimensions without parities Eq. (3.12)-(3.19). We assume that U$(1)_I$ is embedded into SO(12) such as

$$SO(12) \supset SO(10) \times U(1)_I. \tag{3.47}$$

Thus we identify SO(10) $\times$ U$(1)_I$ as the gauge group which satisfy the constraints Eq (3.26), using Eq. (3.34). We identify the scalar components which satisfy Eq. (3.27) by decomposing adjoint representation of SO(12):

$$SO(12) \supset SO(10) \times U(1)_I : 66 = 45(0) + 1(0) + 10(2) + 10(-2). \tag{3.48}$$

According to the prescription below Eq. (3.34) in sec. 3.1, the scalar components 10(2)+10(-2) remains in four-dimensions. We also identify the fermion components which satisfy Eq. (3.29) by decomposing 32 representations of SO(12) as

$$SO(12) \supset SO(10) \times U(1)_I : 32 = 16(1) + \overline{16}(-1). \tag{3.49}$$

According to the prescription below Eq. (3.36) in sec. 3.1, we have the fermion components as 16(1) for a left-handed fermion and $\overline{16}$(-1) for a right-handed fermion, respectively, in four-dimensions.

Next, we specify the parity assignment of $P(P')$ in order to identify the gauge symmetry and particle contents that actually remain in four-dimensions. We choose a parity assignment so as to break gauge symmetry as SO(12) $\supset$ SO(10) $\times$ U$(1)_I$ $\supset$ SU(5)$\times$ U$(1)_X$ $\times$ U$(1)_I$ $\supset$ SU(3) $\times$ SU$(2)_L$ $\times$ U$(1)_Y$ $\times$ U$(1)_X$ $\times$ U$(1)_I$, and to maintain Higgs-doublet in four-dimensions. The parity assignment is written in 32 dimensional spinor basis of SO(12) such as

$$\begin{aligned} &SO(12) \supset SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I \\ &32 = (3,2)^{(+-)}(1,-1,1) + (\bar{3},2)^{(+-)}(-1,1,-1) \\ &\quad + (3,1)^{(--)}(4,1,-1) + (\bar{3},1)^{(--)}(-4,-1,1) \\ &\quad + (3,1)^{(-+)}(-2,-3,-1) + (\bar{3},1)^{(-+)}(2,3,1) \\ &\quad + (1,2)^{(++)}(3,-3,-1) + (1,2)^{(++)}(-3,3,1) \\ &\quad + (1,1)^{(--)}(6,-1,1) + (1,1)^{(--)}(-6,1,-1) \\ &\quad + (1,1)^{(-+)}(0,-5,1) + (1,1)^{(-+)}(0,5,-1), \end{aligned} \tag{3.50}$$

where e.g. $(+,-)$ means that the parities (P, P') of the associated components are (even, odd). We find the gauge symmetry in four-dimensions by surveying parity assignment for the gauge field. The parity assignments of the gauge field under $A_\mu \to PA_\mu P(P'A_\mu P')$ are:

$$\begin{aligned}
66 =&(8,1)^{(++)}(0,0,0) + (1,3)^{(++)}(0,0,0) + (1,1)^{(++)}(0,0,0)\\
&+ (1,1)^{(++)}(0,0,0) + (1,1)^{(++)}(0,0,0)\\
&+ \big[(3,2)^{(-+)}(-5,0,0) + (\bar{3},2)^{(-+)}(5,0,0)\\
&+ (3,2)^{(--)}(1,4,0) + (\bar{3},2)^{(--)}(-1,-4,0)\\
&+ (3,1)^{(+-)}(4,-4,0) + (\bar{3},1)^{(+-)}(-4,4,0)\\
&+ \underline{(3,1)^{(+-)}(-2,2,2) + (\bar{3},1)^{(+-)}(2,-2,-2)}\\
&+ \underline{(3,1)^{(++)}(-2,2,-2) + (\bar{3},1)^{(++)}(2,-2,2)}\\
&+ \underline{(1,2)^{(--)}(3,2,2) + (1,2)^{(--)}(-3,-2,-2)}\\
&+ \underline{(1,2)^{(-+)}(3,2,-2) + (1,2)^{(-+)}(-3,-2,2)}\\
&+ (1,1)^{(+-)}(6,4,0) + (1,1)^{(+-)}(-6,-4,0)\big].
\end{aligned} \tag{3.51}$$

The components with an underline are originated from 10(2) and 10(-2) of SO(10) $\times$ U$(1)_I$, which do not satisfy constraints Eq. (3.26), and hence these components do not remain in four-dimensions. Thus we have the gauge field with $(+,+)$ parity components without an underline in four-dimensions, and the gauge symmetry is SU(3) $\times$ SU$(2)_L$ $\times$ U$(1)_Y$ $\times$ U$(1)_X$ $\times$ U$(1)_I$.

The scalar particle contents in four-dimensions are determined by the parity assignment, under $\Phi_{1,2} \to -P\Phi_{1,2}P$ and $P'\Phi_{1,2}P'$:

$$\begin{aligned}
66 =&(8,1)^{(-+)}(0,0,0) + (1,3)^{(-+)}(0,0,0) + (1,1)^{(-+)}(0,0,0)\\
&+ (1,1)^{(-+)}(0,0,0) + (1,1)^{(-+)}(0,0,0)\\
&+ \big[(3,2)^{(++)}(-5,0,0) + (\bar{3},2)^{(++)}(5,0,0)\\
&+ (3,2)^{(+-)}(1,4,0) + (\bar{3},2)^{(+-)}(-1,-4,0)\\
&+ (3,1)^{(--)}(4,-4,0) + (\bar{3},1)^{(--)}(-4,4,0)\\
&+ \underline{(3,1)^{(--)}(-2,2,2) + (\bar{3},1)^{(--)}(2,-2,-2)}\\
&+ \underline{(3,1)^{(-+)}(-2,2,-2) + (\bar{3},1)^{(-+)}(2,-2,2)}\\
&+ \underline{(1,2)^{(+-)}(3,2,2) + (1,2)^{(+-)}(-3,-2,-2)}\\
&+ \underline{(1,2)^{(++)}(3,2,-2) + (1,2)^{(++)}(-3,-2,2)}\\
&+ (1,1)^{(--)}(6,4,0) + (1,1)^{(--)}(-6,-4,0)\big].
\end{aligned} \tag{3.52}$$

Note that the relative sign for the parity assignment of P is different from Eq. (3.51), and that the only underlined parts satisfy the constraints Eq. (3.27). Thus the scalar components in four-dimensions are (1,2)(3,2,-2) and (1,2)(-3,-2,2).

We find massless fermion contents in four-dimensions, by surveying the parity assignment for each components of fermion fields. We introduce two types of left-handed Weyl fermions that belong to 32 representation of SO(12), which have parity assignment $\psi^{(P')} \to \gamma_5 P\psi^{(P')}(P'\psi^{(P')})$ and $\psi^{(-P')} \to$

$\gamma_5 P\psi^{(-P')}(-P'\psi^{(-P')})$ respectively. They have the parity assignment as

$$\begin{aligned}
32_L^{(P')} =& \underline{(3,2)^{(--)}(1,-1,1)_L} + (\bar{3},2)^{(--)}(-1,1,-1)_L \\
&+ \underline{(\bar{3},1)^{(+-)}(-4,-1,1)_L} + (3,1)^{(+-)}(4,1,-1)_L \\
&+ \underline{(\bar{3},1)^{(++)}(2,3,1)_L} + (3,1)^{(++)}(-2,-3,-1)_L \\
&+ \underline{(1,2)^{(-+)}(-3,3,1)_L} + (1,2)^{(-+)}(3,-3,-1)_L \\
&+ \underline{(1,1)^{(+-)}(6,-1,1)_L} + (1,1)^{(+-)}(-6,1,-1)_L \\
&+ \underline{(1,1)^{(++)}(0,-5,1)_L} + (1,1)^{(++)}(0,5,-1)_L, \qquad (3.53) \\
32_R^{(P')} =& (3,2)^{(+-)}(1,-1,1)_R + \underline{(\bar{3},2)^{(+-)}(-1,1,-1)_R} \\
&+ (\bar{3},1)^{(--)}(-4,-1,1)_R + \underline{(3,1)^{(--)}(4,1,-1)_R} \\
&+ (\bar{3},1)^{(-+)}(2,3,1)_R + \underline{(3,1)^{(-+)}(-2,-3,-1)_R} \\
&+ (1,2)^{(++)}(-3,3,1)_R + \underline{(1,2)^{(++)}(3,-3,-1)_R} \\
&+ (1,1)^{(--)}(6,-1,1)_R + \underline{(1,1)^{(--)}(-6,1,-1)_R} \\
&+ (1,1)^{(-+)}(0,-5,1)_R + \underline{(1,1)^{(-+)}(0,5,-1)_R}, \qquad (3.54)
\end{aligned}$$

and

$$\begin{aligned}
32_L^{(-P')} =& \underline{(3,2)^{(-+)}(1,-1,1)_L} + (\bar{3},2)^{(-+)}(-1,1,-1)_L \\
&+ \underline{(\bar{3},1)^{(++)}(-4,-1,1)_L} + (3,1)^{(++)}(4,1,-1)_L \\
&+ \underline{(\bar{3},1)^{(+-)}(2,3,1)_L} + (3,1)^{(+-)}(-2,-3,-1)_L \\
&+ \underline{(1,2)^{(--)}(-3,3,1)_L} + (1,2)^{(--)}(3,-3,-1)_L \\
&+ \underline{(1,1)^{(++)}(6,-1,1)_L} + (1,1)^{(++)}(-6,1,-1)_L \\
&+ \underline{(1,1)^{(+-)}(0,-5,1)_L} + (1,1)^{(+-)}(0,5,-1)_L, \qquad (3.55) \\
32_R^{(-P')} =& (3,2)^{(++)}(1,-1,1)_R + \underline{(\bar{3},2)^{(++)}(-1,1,-1)_R} \\
&+ (\bar{3},1)^{(-+)}(-4,-1,1)_R + \underline{(3,1)^{(-+)}(4,1,-1)_R} \\
&+ (\bar{3},1)^{(-+)}(2,3,1)_R + \underline{(3,1)^{(-+)}(-2,-3,-1)_R} \\
&+ (1,2)^{(+-)}(-3,3,1)_R + \underline{(1,2)^{(+-)}(3,-3,-1)_R} \\
&+ (1,1)^{(-+)}(6,-1,1)_R + \underline{(1,1)^{(-+)}(-6,1,-1)_R} \\
&+ (1,1)^{(--)}(0,-5,1)_R + \underline{(1,1)^{(--)}(0,5,-1)_R}, \qquad (3.56)
\end{aligned}$$

where L(R) means left-handedness(right-handedness) of fermions in four-dimensions, and the underlined parts correspond to the components which satisfy constraints Eq. (3.29). Note the relative sign for parity assignment of P between left-handed fermion and right-handed fermion, and that of P' between $32^{(P')}$ and $32^{(-P')}$. The difference between $32^{(P')}$ and $32^{(-P')}$ is allowed because of the bilinear form of the fermion sector. We thus find that the massless fermion components in four-dimensions are one generation of SM-fermions with right-handed neutrino: $\{(3,2)(1,-1,1)_L, (3,1)(4,1,-1)_R, (3,1)(-2,-3,-1)_R, (1,2)(-3,3,1)_L, (1,1)(-6,1,-1)_R, (1,1)(0,5,-1)_R\}$.

3.2.2 The Higgs sector of the model

We analyze the Higgs-sector of our model. The Higgs-sector $L_{\rm Higgs}$ is the last two terms of Eq. (3.28):

$$\begin{aligned} L_{\rm Higgs} =& -\frac{1}{2g^2}Tr[D'_\mu\Phi_1(x)D'^\mu\Phi_1(x)+D'_\mu\Phi_2(x)D'^\mu\Phi_2(x)] \\ & -\frac{1}{2g^2}Tr[(\Phi_3+[\Phi_1(x),\Phi_2(x)])(\Phi_3+[\Phi_1(x),\Phi_2(x)])], \end{aligned} \tag{3.57}$$

where the first term of LHS is the kinetic term of Higgs and the second term gives the Higgs potential. We then rewrite the Higgs-sector in terms of genuine Higgs field in order to analyze it.

We first note that the Φ_is are written as

$$\Phi_i = i\phi_i = i\phi_i^a Q_a, \tag{3.58}$$

where Q_as are generators of gauge group SO(12), since Φ_is are originated from gauge fields $A_\alpha = iA_\alpha^a Q_a$; for the gauge group generator we assume the normalization Tr(Q_aQ_b)=-2δ_{ab}. Note that we assumed the $-i\Phi_3$ as the generator of U(1)$_I$ embedded in SO(12),

$$-i\Phi_3 = Q_I. \tag{3.59}$$

We change the notation of the scalar fields according to Eq. (3.35) such that,

$$\phi_+ = \frac{1}{2}(\phi_1+i\phi_2), \quad \phi_- = \frac{1}{2}(\phi_1-i\phi_2), \tag{3.60}$$

in order to express solutions of the constraints Eq. (3.27) clearly. The constraints Eq. (3.27) is then rewritten as

$$[Q_I,\phi_+] = \phi_+, \qquad [Q_I,\phi_-] = -\phi_-. \tag{3.61}$$

The kinetic term L_{KE} and potential $V(\phi)$ term are rewritten in terms of ϕ_+ and ϕ_-:

$$L_{KE} = -\frac{1}{g^2}Tr[D'_\mu\phi_+(x)D'^\mu\phi_-(x)], \tag{3.62}$$

$$V = -\frac{1}{2g^2}Tr[Q_I^2 - 4Q_I[\phi_+,\phi_-]+4[\phi_+,\phi_-][\phi_+,\phi_-]], \tag{3.63}$$

where covariant derivative D'_μ is $D'_\mu\phi_\pm = \partial_\mu\phi_\pm - [A_\mu,\phi_\pm]$.

Next, we change the notation of SO(12) generators Q_a according to decomposition Eq (3.51) such that

$$\begin{aligned} Q_G = \{ & Q_i, Q_\alpha, Q_Y, Q, Q_I, Q_{ax(-500)}, Q^{ax(500)} \\ & Q_{ax(140)}, Q^{ax(-1-40)}, Q_{a(4-40)}, Q^{a(-440)} \\ & Q_{a(-22-2)}, Q^{a(2-22)}, Q_{a(-222)}, Q^{a(2-2-2)} \\ & Q_{x(322)}, Q^{x(-3-2-2)}, Q_{x(32-2)}, Q^{x(-3-22)} \\ & Q(640), Q(-6-40)\}, \end{aligned} \tag{3.64}$$

where the order of generators corresponds to Eq (3.51), index $i = 1-8$ corresponds to SU(3) adjoint rep, index $\alpha = 1-3$ corresponds to SU(2) adjoint rep, index $a = 1-3$ corresponds to SU(3)-triplet,

$[\ Q^{x(-3-22)},Q_{y(32-2)}] = -\sqrt{\frac{3}{10}}\ \delta^x_y\ Q_Y\ + -\sqrt{\frac{1}{5}}\ \delta^x_y\ Q\ +\delta^x_y\ Q_I\ +\frac{1}{\sqrt{2}}\ (\sigma^*_\alpha)^x_y\ Q_\alpha$	
$[Q_\alpha,Q_x] = -\frac{1}{\sqrt{2}}\ (\sigma_\alpha)^y_x\ Q_y$	$[Q_\alpha,Q^x] = \frac{1}{\sqrt{2}}\ (\sigma^*_\alpha)^x_y\ Q^y$
$[Q_x,Q_y]=0$	$[Q_Y,Q^x]= -\sqrt{\frac{3}{10}}\ Q^x$
$[Q,Q^x]= -\sqrt{\frac{1}{5}}\ Q^x$	$[Q_I,Q^x] = Q^x$

Table 25: commutation relations of $Q^{x(-3-22)}$, $Q_{x(32-2)}$, Q_α, Q_Y, Q and Q_I

and index $x = 1, 2$ corresponds to SU(2)-doublet. We write $\phi_\pm$ in terms of the genuine Higgs field ϕ_x which belongs to (1,2)(3,2,-2), such that

$$\phi_+ = \phi_x Q^{x(-3-22)} \tag{3.65}$$

$$\phi_- = \phi^x Q_{x(32-2)}, \tag{3.66}$$

where $\phi^x = (\phi_x)^\dagger$. We also write gauge field $A_\mu(x)$ in terms of Qs in Eq. (3.64) as

$$A_\mu(x) = i(A^i_\mu Q_i + A^\alpha_\mu Q_\alpha + B_\mu Q_Y + C_\mu Q + E_\mu Q_I). \tag{3.67}$$

We then need commutation relations of $Q^{x(-3-22)}$, $Q_{x(32-2)}$, Q_α, Q_Y, Q and Q_I in order to analyze the Higgs sector; we summarized them in Table 25.

Finally, we obtain the Higgs sector with genuine Higgs field by substituting Eq. (3.65)-(3.67) into Eq. (3.62, 3.63) and rescaling the fields $\phi \to g/\sqrt{2}\phi$ and $A_\mu \to gA_\mu$, and the couplings $\sqrt{2}g = g_2$ and $\sqrt{6/5}g = g_Y$,

$$L_{Higgs} = |D_\mu \phi_x|^2 - V(\phi), \tag{3.68}$$

where the covariant derivative $D_\mu\phi_x$ and potential $V(\phi)$ are

$$D_\mu\phi_x = \partial_\mu\phi_x + ig_2\frac{1}{2}(\sigma_\alpha)^y_x A_{\alpha\mu}\phi_y + ig_Y\frac{1}{2}B_\mu\phi_x + i\sqrt{\frac{1}{5}}gC_\mu\phi_x - igE_\mu\phi_x, \tag{3.69}$$

$$V = -\frac{2}{R^2}\phi^x\phi_x + \frac{3g^2}{2}(\phi^x\phi_x)^2, \tag{3.70}$$

respectively. Notice that we explicitly write radius R of S^2 in the Higgs potential, and that we omitted the constant term in the Higgs potential. We note that the SU(2)$_L$ × U(1)$_Y$ parts of the Higgs sector has the same form as the SM Higgs sector. Therefore we obtain the electroweak symmetry breaking SU(2)$_L$ × U(1)Y → U(1)$_{EM}$. The Higgs field ϕ^x acquires vaccume expectation value(VEV) as

$$< \phi > = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{3.71}$$

$$v = \sqrt{\frac{4}{3}}\frac{1}{gR}, \tag{3.72}$$

and W boson mass m_W and Higgs mass m_H are given in terms of radius R

$$m_W = g_2 \frac{v}{2} = \sqrt{\frac{2}{3}} \frac{1}{R}, \tag{3.73}$$

$$m_H = \sqrt{3} g v = \sqrt{4} \frac{1}{R}. \tag{3.74}$$

The ratio between m_W and m_H is predicted

$$\frac{m_H}{m_W} = \sqrt{6}. \tag{3.75}$$

4 Universal Extra dimension model on $M^4 \times S^2/Z_2$ spacetime [80]

In this section, we first recapitulate a gauge theory defined on the six-dimensional spacetime which has extra-space as two-sphere orbifold S^2/Z_2. We then construct a six-dimensional Lagrangian for Universal Extra Dimension Model on the spacetime.

4.1 The model

We consider a gauge theory defined on the six-dimensional spacetime M^6 which is assumed to be a direct product of the four-dimensional Minkowski spacetime M^4 and a compact two-sphere orbifold S^2/Z_2, such that $M^6 = M^4 \times S^2/Z_2$. We denote the coordinate of M^6 by $X^M = (x^\mu, y^\theta = \theta, y^\phi = \phi)$, where x^μ and $\{\theta, \phi\}$ are the M^4 coordinates and are the S^2/Z_2 spherical coordinates, respectively. On the orbifold, the point (θ, ϕ) is identified with $(\pi - \theta, -\phi)$. The spacetime index M runs over $\mu \in \{0, 1, 2, 3\}$ and $\alpha \in \{\theta, \phi\}$. The metric of M^6, denoted by g_{MN}, can be written as

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -g_{\alpha\beta} \end{pmatrix}, \tag{4.1}$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ and $g_{\alpha\beta} = diag(R^2, R^2 \sin^2\theta)$ are metric of M^4 and S^2/Z_2 respectively, and R denotes the radius of S^2/Z_2. We define the vielbein e_A^M that connects the metric of M^6 and that of the tangent space of M^6, denoted by h_{AB}, as $g_{MN} = e_M^A e_N^B h_{AB}$. Here $A = (\mu, a)$, where $a \in \{4, 5\}$, is the index for the coordinates of tangent space of M^6. The explicit form of the vielbeins are summarized in the Appendix.

We introduce, in this theory, a gauge field $A_M(x, y) = (A_\mu(x, y), A_\alpha(x, y))$, SO(1,5) chiral fermions $\Psi_\pm(x, y)$, and complex scalar fields $\Phi(x, y)$. The SO(1,5) chiral fermion $\Psi_\pm(x, y)$ is defined by the action of SO(1,5) chiral operator Γ_7, which is defined as

$$\Gamma_7 = \gamma_5 \otimes \sigma_3, \tag{4.2}$$

where γ_5 is SO(1,3) chiral operator and $\sigma_i (i = 1, 2, 3)$ are Pauli matrices. The chiral fermion $\Psi_\pm(x, y)$ satisfies

$$\Gamma_7 \Psi_\pm(x, y) = \pm \Psi_\pm(x, y) \tag{4.3}$$

and is obtained by acting the chiral projection operator of SO(1,5), $\Gamma_\pm$, on Dirac fermion $\Psi(x,y)$, where $\Gamma_\pm$ is defined as

$$\Gamma_\pm = \frac{1 \pm \Gamma_7}{2}. \tag{4.4}$$

We can also write $\Psi_\pm(x,y)$ in terms of SO(1,3) chiral fermion ψ as

$$\Psi_+ = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{4.5}$$

$$\Psi_- = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{4.6}$$

where $\psi_{R(L)}$ is a right(left)-handed SO(1,3) chiral fermion. We should determine the boundary condition of these fields on S^2/Z_2 to specify a model. The boundary conditions for each field can be defined as

$$\begin{aligned}
\Phi(x, \pi-\theta, -\phi) &= \pm\Phi(x,\theta,\phi) && (4.7)\\
A_\mu(x, \pi-\theta, -\phi) &= A_\mu(x,\theta,\phi) && (4.8)\\
A_{\theta,\phi}(x, \pi-\theta, -\phi) &= -A_{\theta,\phi}(x,\theta,\phi) && (4.9)\\
\Psi(x, \pi-\theta, -\phi) &= \pm\gamma_5\Psi(x,\theta,\phi) && (4.10)
\end{aligned}$$

by requiring the invariance of a six-dimensional action under the Z_2 transformation.

The action of the gauge theory is written, in general, as

$$S = \int dx^4 R^2 \sin\theta d\theta d\phi \Big(\bar{\Psi}_\pm i \Gamma^\mu D_\mu \Psi_\pm + \bar{\Psi}_\pm i \Gamma^a e^\alpha_a D_\alpha \Psi_\pm - \frac{1}{4g^2} g^{MN} g^{KL} Tr[F_{MK} F_{NL}] + (D^M\Phi)^* D_M \Phi - V(\Phi) - \lambda \bar{\Psi}_\pm \Phi \Psi_\mp \Big), \tag{4.11}$$

where $F_{MN} = \partial_M A_N(X) - \partial_N A_M(X) - [A_M(X), A_N(X)]$ is the field strength, D_M is the covariant derivative including a spin connection, $V(\Phi)$ is the scalar potential term, and Γ_A represents the 6-dimensional Clifford algebra. Here D_M and Γ_A can be written explicitly as

$$D_\mu = \partial_\mu - iA_\mu, \tag{4.12}$$
$$D_\theta = \partial_\theta - iA_\theta, \tag{4.13}$$
$$D_\phi = \partial_\phi - iA_\phi \; (-i\frac{\Sigma_3}{2}\cos\theta), \tag{4.14}$$
$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{I}_2, \tag{4.15}$$
$$\Gamma_4 = \gamma_5 \otimes i\sigma_1, \tag{4.16}$$
$$\Gamma_5 = \gamma_5 \otimes i\sigma_2, \tag{4.17}$$

where $\{\gamma_\mu, \gamma_5\}$ are the 4-dimensional Dirac matrices, $\mathbf{I}_d$ is $d\times d$ identity, and Σ_3 is defined as $\Sigma_3 = \mathbf{I}_4 \otimes \sigma_3$. We note that the spin connection term in D_ϕ is applied only for fermions.

We discuss the condition to obtain massless chiral fermions in four-dimensional spacetime. The positive curvature of an extra-space gives mass to fermions in four-dimensional spacetime even if we introduce chiral fermions in a higher-dimensional spacetime. The spin connection term for fermions in Eq. (4.14) expresses the existence of positive curvature of S^2 and leads mass term of fermions in four-dimensional spacetime. We thus need some prescription to obtain a massless fermion in four-dimensional spacetime within our model since S^2 has the positive curvature. We then introduce a background gauge field A^B_ϕ which has the form [50, 1, 78]

$$A^B_\phi = \hat{Q}\cos\theta \tag{4.18}$$

where $\hat{Q}$ is a charge of some U(1) gauge symmetry, in order to cancel the mass from the curvature and to obtain massless fermions in four-dimensional spacetime. Indeed, A^B_ϕ cancel the spin connection term for the upper(lower) component SO(1,3) fermion in Eq. (4.5) if the fermion has the charge $Q = +(-)\frac{1}{2}$ and the upper(lower) component gets a massless Kaluza-Klein mode.

We then specify our model. Chose the gauge group G as the standard model gauge group with an extra $U(1)_X$ gauge symmetry , i.e. G=SU(3)×SU(2)×$U(1)_Y$×$U(1)_X$, and introduce background gauge field which belongs to the gauge field of the extra U(1). We must introduce the extra U(1) to obtain all the massless chiral SM fermions in M^4, otherwise some SM fermions have masses, inevitably, from the positive curvature of S^2 when we introduce background gauge field which belongs to $U(1)_Y$.

We introduce fermions $Q(x,y), U(x,y), D(x,y), L(x,y)$ and $E(x,y)$ that belong to representations of SU(3)×SU(2)×$U(1)_Y$, which are the same as the left-handed quark doublet, right-handed up-type quark, right-handed down-type quark, left-handed lepton doublet and right-handed charged lepton. We then assign the extra U(1) charge $Q = \frac{1}{2}$ to these fermions as the simplest case in which all massless SM fermions appear in four-dimensional spacetime. The chirality of SO(1,5) and boundary condition for these fermions are determined to give massless SM fermions in four-dimensional spacetime, as summarized in Table 26.

Table 26: SO(1,5) chirality and boundary conditions for each fermions in six dimensions. The signs for boundary condition express the sign in front of γ_5 in RHS of Eq. (4.10).

Fermions	SO(1,5) chirality	boundary conditions
$Q(x,y)$	$-$	$-$
$U(x,y)$	$+$	$+$
$D(x,y)$	$+$	$+$
$L(x,y)$	$-$	$-$
$E(x,y)$	$+$	$+$

The Higgs field $H(x,y)$ is introduced to not have $U(1)_X$ charge and to be even under the Z_2 action so that Yukawa coupling terms can be constructed.

The action of our model in six-dimensional spacetime is written as

$$S_{6D} = \int dx^4 R^2 \sin\theta d\theta d\phi \Big[(\bar{Q}, \bar{U}, \bar{D}, \bar{L}, \bar{E}) i\Gamma^M D_M (Q, U, D, L, E)^\top$$
$$-g^{MN} g^{KL} \sum_i \frac{1}{4g_i^2} Tr[F_{i\ MK} F_{i\ NL}] + L_{Higgs}(H)$$
$$+[\lambda_u Q\bar{U}H^* + \lambda_d Q\bar{D}H + \lambda_e L\bar{E}H + \text{h.c}] \Big], \quad (4.19)$$

where $i = SU(3), SU(2), U(1)_Y$ and $U(1)_X$, and $L_{Higgs}(H)$ denotes a Lagrangian for Higgs field. The action in four-dimensional spacetime is obtained by integrating the Lagrangian over S^2/Z_2 coordinate.

4.2 KK mode expansion and particle mass spectrum in four-dimensions

In this section we analyze Kaluza-Klein expansion of each field in our model, and derive mass spectrum of the Kaluza-Klein modes.

4.2.1 KK mode expansion of the fermions

The fermions $\Psi(x, y)$ can be expanded in terms of the eigenfunctions of square of Dirac operator $i\hat{D}$ on S^2/Z_2 where the Dirac operator is written as

$$\begin{aligned} i\hat{D} &= ie^{\alpha a} i\sigma_a D_\alpha \otimes \gamma_5 \\ &= -\frac{1}{R}\big[\sigma_1(\partial_\theta + \frac{\cot\theta}{2}) + \sigma_2(\frac{1}{\sin\theta}\partial_\phi + i\hat{Q}\cot\theta)\big] \otimes \gamma_5, \end{aligned} \quad (4.20)$$

where $\hat{Q}$ is the $\text{U}(1)_X$ charge operator in our model. We thus need to derive eigenfunctions of $(i\hat{D})^2$ first. The square of the Dirac operator $(i\hat{D})^2$ is written as

$$(-i\hat{D})^2 = \frac{1}{R^2}\big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2 + 2i(\hat{Q} - \frac{\sigma_3}{2})\frac{\cos\theta}{\sin^2\theta}\partial_\phi$$
$$-\frac{1}{4} - \frac{1}{4\sin^2\theta} + \hat{Q}\sigma_3\frac{1}{\sin^2\theta} - \hat{Q}^2\cot^2\theta\big]. \quad (4.21)$$

We then obtain the eigenvalue equation of $(i\hat{D})^2$ as

$$\frac{1}{R^2}\big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2 + 2i(Q - \frac{\sigma_3}{2})\frac{\cos\theta}{\sin^2\theta}\partial_\phi$$
$$-\frac{1}{4} - \frac{1}{4\sin^2\theta} + Q\sigma_3\frac{1}{\sin^2\theta} - Q^2\cot^2\theta\big]\Psi(\theta,\phi) = -\lambda^2\Psi(\theta,\phi), \quad (4.22)$$

where $-\lambda^2$ express the eigenvalue of $(i\hat{D})^2$ and Q is the $\text{U}(1)_X$ charge of the $\Psi(\theta, \phi)$. We expand $\Psi(\theta, \phi)$ to solve the equation such that

$$\Psi(\theta,\phi) = \sum_m \frac{e^{im\phi}}{\sqrt{2\pi}} \begin{pmatrix} \alpha_{\lambda m}(\theta) \\ \beta_{\lambda m}(\theta) \end{pmatrix} \quad (4.23)$$

where m is an integer. The eigenvalue equation becomes

$$\Big[\frac{d}{dz}(1-z^2)\frac{d}{dz}-\frac{m^2+2m(Q-\frac{\sigma_3}{2})z+(Q-\frac{\sigma_3}{2})^2}{1-z^2}\Big]\begin{pmatrix}\alpha_{\lambda m}(z)\\ \beta_{\lambda m}(z)\end{pmatrix}=-(R^2\lambda^2+Q^2-\frac{1}{4})\begin{pmatrix}\alpha_{\lambda m}(z)\\ \beta_{\lambda m}(z)\end{pmatrix}, \quad (4.24)$$

where we changed the variable as $\theta\to z=\cos\theta$. Here we note that the replacement of m with $-m$ and Q with $-Q$ corresponds to the exchange of $\alpha_{\lambda m}$ and $\beta_{\lambda m}$. We next put $\alpha_{\lambda m}$ and $\beta_{\lambda m}$ in the following form [39],

$$\begin{pmatrix}\alpha_{\lambda m}(z)\\ \beta_{\lambda m}(z)\end{pmatrix}=\begin{pmatrix}(1-z)^{\frac{1}{2}|m+Q-\frac{1}{2}|}(1+z)^{\frac{1}{2}|m-Q+\frac{1}{2}|}\xi_{\lambda m}(z)\\ (1-z)^{\frac{1}{2}|m+Q+\frac{1}{2}|}(1+z)^{\frac{1}{2}|m-Q-\frac{1}{2}|}\eta_{\lambda m}(z)\end{pmatrix}. \quad (4.25)$$

We finally find the equation for $\eta_{\lambda m}$ and $\xi_{\lambda m}$ by applying Eq. (4.25) to Eq. (4.24), as

$$\Big[(1-z^2)\frac{d^2}{dz^2}-2(|m|+1)z\frac{d}{dz}-m^2-|m|+R^2\lambda^2\Big]\xi_{\lambda m}(z)=0, \quad (4.26)$$

and

$$\begin{aligned}&\Big[(1-z^2)\frac{d^2}{dz^2}+\{|m-1|-|m+1|-(|m-1|+|m+1|+2)z\}\frac{d}{dz}\\ &-\frac{1}{2}m^2-\frac{1}{2}|m-1||m+1|-\frac{1}{2}(|m-1|+|m+1|)-\frac{1}{2}+R^2\lambda^2\Big]\eta_{\lambda m}(z)=0\end{aligned} \quad (4.27)$$

where we substitute $\frac{1}{2}$ for Q since all the fermions have this charge in our model. These equations can be attributed to the differential equation for the Jacobi polynomial $P_n^{(\alpha,\beta)}$; the properties of the Jacobi polynomial and their differential equation are summarized in Appendix B. We thus obtain the eigenfunctions $\xi_{\lambda m}$, $\eta_{\lambda m}$ of the form

$$\xi_{\lambda m}(z) = C_\xi^{lm}P_{l-|m|}^{(|m|,|m|)}(z), \quad (4.28)$$

$$\eta_{\lambda m}(z) = C_\eta^{lm}P_{l-|m|}^{(|m+1|,|m-1|)}(z), \quad (4.29)$$

for $m\neq 0$,

$$\xi_{\lambda 0}(z) = C_\xi^{l0}P_l^{(0,0)}(z), \quad (4.30)$$

$$\eta_{\lambda 0}(z) = C_\eta^{l-1}P_{l-1}^{(1,1)}(z), \quad (4.31)$$

(4.32)

for $m=0$, and the eigenvalue λ_l as

$$\lambda=\frac{\sqrt{l(l+1)}}{R} \quad (4.33)$$

.

For any m, here an l is integer which satisfy $l\geq m$ and $C_{\xi(\eta)}^{lm}$s are normalization constants. The normalization of the eigenfunctions are chosen as

$$C_\xi^{lm} = \sqrt{\frac{n!(2l+|m|+1)\Gamma(l+|m|+1)}{2^{2|m|+1}\Gamma(l+1)\Gamma(l+1)}}, \quad (4.34)$$

$$C_\eta^{lm} = i\sqrt{\frac{n!(2l-2|m|+|m+1|+|m-1|+1)\Gamma(l-|m|+|m+1|+|m-1|+1)}{2^{|m+1|+|m-1|+1}\Gamma(l-|m|+|m+1|+1)\Gamma(l-|m|+|m-1|+1)}}, \quad (4.35)$$

so that

$$\int |\alpha_{lm}(z)|^2 dz = \int |\beta_{lm}(z)|^2 dz = 1, \tag{4.36}$$

where we choose relative phase of the normalization constants as defined above for later convenience. We therefore obtain the eigenfunctions of $(i\hat{D})^2$ as

$$\Psi_{lm}(\theta,\phi) = \begin{pmatrix} \tilde{\alpha}_{lm}(z,\phi) \\ \tilde{\beta}_{lm}(z,\phi) \end{pmatrix} = \frac{e^{im\phi}}{\sqrt{2\pi}} \begin{pmatrix} C_\xi^{lm}(1-z)^{\frac{1}{2}|m|}(1+z)^{\frac{1}{2}|m|}P_{l-|m|}^{(|m|,|m|)}(z) \\ C_\eta^{lm}(1-z)^{\frac{1}{2}|m+1|}(1+z)^{\frac{1}{2}|m-1|}P_{l-|m|}^{(|m+1|,|m-1|)}(z) \end{pmatrix}, \tag{4.37}$$

for $m \neq 0$ and

$$\Psi_{lm}(\theta,\phi) = \begin{pmatrix} \tilde{\alpha}_{l0}(z) \\ \tilde{\beta}_{l0}(z) \end{pmatrix} \frac{1}{\sqrt{2\pi}} \begin{pmatrix} C_\xi^{l0} P_l^{(0,0)}(z) \\ C_\eta^{l0}\sqrt{1-z^2}P_{l-1}^{(1,1)}(z) \end{pmatrix}, \tag{4.38}$$

for $m = 0$. These eigenfunctions satisfy the orthogonality relations

$$\int d\Omega(\tilde{\alpha}_{lm})^* \tilde{\alpha}_{l'm'} = \int d\Omega(\tilde{\beta}_{lm})^* \tilde{\beta}_{l'm'} = \delta_{ll'}\delta_{mm'}. \tag{4.39}$$

We note that the eigenfunction for $(l = 0, m = 0)$ has only upper component since $P_{0-1}\ (1,1)(z) = 0$.

We can obtain the KK mode functions for chiral fermions $\Psi_\pm(x,\theta,\phi)$ which satisfy the boundary conditions Eq. (4.10) in terms of $\tilde{\alpha}_{lm}(z,\phi)$, $\tilde{\beta}_{lm}(z,\phi)$, $\tilde{\alpha}_{l0}(z)$ and $\tilde{\beta}_{l0}(z)$ in Eq. (4.37) and (4.38). These KK mode functions are summarized below.

1. The KK mode function for $\Psi_+(x,\theta,\phi)$ which satisfy the boundary condition $\Psi_+(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_+(x,\theta,\phi)$ are

   (a) $m \neq 0$

$$\Psi_{+l|m|}^{(\pm\gamma_5)}(x,\theta,\phi) = \begin{pmatrix} \frac{1}{\sqrt{2}}[\tilde{\alpha}_{lm}(z,\phi) \pm (-1)^{l-|m|}\tilde{\alpha}_{l-m}(z,\phi)]\psi_R^{l|m|}(x) \\ \frac{i}{\sqrt{2}}[\tilde{\beta}_{lm}(z,\phi) \mp (-1)^{l-|m|}\tilde{\beta}_{l-m}(z,\phi)]\psi_L^{l|m|}(x) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\alpha}_{l|m|}^{\pm}(z,\phi)\psi_R^{l|m|}(x) \\ \tilde{\beta}_{l|m|}^{\mp}(z,\phi)\psi_L^{l|m|}(x) \end{pmatrix} \tag{4.40}$$

   (b) m=0

$$\Psi_{+l0}^{(\pm\gamma_5)}(x,\theta,\phi) = \begin{pmatrix} \frac{1}{2}[\tilde{\alpha}_{l0}(z) \pm (-1)^l\tilde{\alpha}_{l0}(z)]\psi_R^{l0}(x) \\ \frac{i}{2}[\tilde{\beta}_{l-10}(z) \pm (-1)^l\tilde{\beta}_{l-10}(z)]\psi_L^{l0}(x) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\alpha}_{l0}^{\pm}(y,\phi)\psi_{R(L)}^{l0}(x) \\ \tilde{\beta}_{l0}^{\mp}(z)\psi_{L(R)}^{l0}(x) \end{pmatrix}. \tag{4.41}$$

2. The KK mode function for $\Psi_-(x,\theta,\phi)$ which satisfy the boundary condition $\Psi_-(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_-(x,\theta,\phi)$ are

   (a) $m \neq 0$

$$\Psi_{-l|m|}^{(\pm\gamma_5)}(x,\theta,\phi) = \begin{pmatrix} \frac{1}{\sqrt{2}}[\tilde{\alpha}_{lm}(z,\phi) \mp (-1)^{l-|m|}\tilde{\alpha}_{l-m}(z,\phi)]\psi_L^{l|m|}(x) \\ \frac{i}{\sqrt{2}}[\tilde{\beta}_{lm}(z,\phi) \pm (-1)^{l-|m|}\tilde{\beta}_{l-m}(z,\phi)]\psi_R^{l|m|}(x) \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}_{l|m|}^{\mp}(z,\phi)\psi_L^{l|m|}(x) \\ \tilde{\beta}_{l|m|}^{\pm}(z,\phi)\psi_R^{l|m|}(x) \end{pmatrix} \tag{4.42}$$

(b) m=0

$$\Psi^{(\pm\gamma_5)}_{-l0}(x,\theta,\phi) = \begin{pmatrix} \frac{1}{2}[\tilde{\alpha}_{l0}(z) \mp (-1)^l \tilde{\alpha}_{l0}(z)]\psi^{l0}_L(x) \\ \frac{i}{2}[\tilde{\beta}_{l-10}(z) \mp (-1)^l \tilde{\beta}_{l-10}(z)]\psi^{l0}_R(x) \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}^{\mp}_{l0}(z)\psi^{l0}_L(x) \\ \tilde{\beta}^{\pm}_{l0}(z)\psi^{l0}_R(x) \end{pmatrix} \tag{4.43}$$

where $\psi(x)$s are SO(1,3) spinors. We can explicitly confirm that these KK mode functions satisfy the boundary conditions by straightforward calculation using

$$\tilde{\alpha}_{lm}(-z,-\phi) = (-1)^{l-|m|}\tilde{\alpha}_{l-m}(z,\phi), \tag{4.44}$$

$$\tilde{\beta}_{lm}(-z,-\phi) = (-1)^{l-|m|}\tilde{\beta}_{l-m}(z,\phi), \tag{4.45}$$

which are obtained by the definition of $\tilde{\alpha}_{lm}(z,\phi)$ and $\tilde{\beta}_{lm}(z,\phi)$. We therefore expand the fermions in six-dimensional space time such that

$$\Psi^{(\pm\gamma_5)}_{+}(x,\theta,\phi) = \sum_{l=0}\sum_{m=0}^{l} \Psi^{(\pm\gamma_5)}_{+l|m|}(x,\theta,\phi) \tag{4.46}$$

for fermions $\Psi_+(x,\theta,\phi)$ which satisfy the boundary condition: $\Psi_+(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_+(x,\theta,\phi)$, and

$$\Psi^{(\pm\gamma_5)}_{-}(x,\theta,\phi) = \sum_{l=0}\sum_{m=0}^{l} \Psi^{(\pm\gamma_5)}_{-l|m|}(x,\theta,\phi) \tag{4.47}$$

for fermions $\Psi_-(x,\theta,\phi)$ which satisfy the boundary condition: $\Psi_-(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_-(x,\theta,\phi)$. We also summarize below the action of the Dirac operator $i\hat{D}$ on the KK modes, since it is useful to analyze the KK mass terms.

1. The KK mode function for $\Psi_+(x,\theta,\phi)$ which satisfy the boundary condition $\Psi_+(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_+(x,\theta,\phi)$ are

   (a) $m \neq 0$

$$i\hat{D}\Psi^{(\pm\gamma_5)}_{+l|m|}(x,\theta,\phi) = iM_l \begin{pmatrix} -\frac{i}{\sqrt{2}}[\tilde{\alpha}_{lm}(z,\phi) \pm (-1)^{l-|m|}\tilde{\alpha}_{l-m}(z,\phi)]\psi^{l|m|}_L(x) \\ \frac{1}{\sqrt{2}}[\tilde{\beta}_{lm}(z,\phi) \mp (-1)^{l-|m|}\tilde{\beta}_{l-m}(z,\phi)]\psi^{l|m|}_R(x) \end{pmatrix} \tag{4.48}$$

   (b) m=0

$$i\hat{D}\Psi^{(\pm\gamma_5)}_{+l0}(x,\theta,\phi) = iM_l \begin{pmatrix} -\frac{i}{2}[\tilde{\alpha}_{l0}(z) \pm (-1)^l \tilde{\alpha}_{l0}(z)]\psi^{l0}_L(x) \\ \frac{1}{2}[\tilde{\beta}_{l-10}(z) \pm (-1)^l \tilde{\beta}_{l-10}(z)]\psi^{l0}_R(x) \end{pmatrix} \tag{4.49}$$

2. The KK mode function for $\Psi_-(x,\theta,\phi)$ which satisfy the boundary condition $\Psi_-(x,\pi-\theta,-\phi) = \pm\gamma_5\Psi_-(x,\theta,\phi)$ are

   (a) $m \neq 0$

$$i\hat{D}\Psi^{(\pm\gamma_5)}_{-l|m|}(x,\theta,\phi) = iM_l \begin{pmatrix} \frac{i}{\sqrt{2}}[\tilde{\alpha}_{lm}(z,\phi) \pm (-1)^{l-|m|}\tilde{\alpha}_{-lm}(z,\phi)]\psi^{l|m|}_R(x) \\ -\frac{1}{\sqrt{2}}[\tilde{\beta}_{lm}(z,\phi) \mp (-1)^{l-|m|}\tilde{\beta}_{l-m}(z,\phi)]\psi^{l|m|}_L(x) \end{pmatrix} \tag{4.50}$$

(b) m=0

$$i\hat{D}\Psi^{(\pm\gamma_5)}_{-l0}(x,\theta,\phi) = iM_l \begin{pmatrix} \frac{i}{2}[\tilde{\alpha}_{l0}(z) \mp (-1)^l \tilde{\alpha}_{l0}(z)]\psi^{l0}_R(x) \\ \frac{-1}{2}[\tilde{\beta}_{l-10}(z) \mp (-1)^l \tilde{\beta}_{l-10}(z)]\psi^{l0}_L(x) \end{pmatrix} \tag{4.51}$$

where $M_l = \frac{\sqrt{l(l+1)}}{R}$. These results respect the choice of the phase between normalization constants of upper and lower components in Eq. (4.34) and (4.35) since the Dirac operator exchange upper and lower components.

We then derive the kinetic terms and the KK mass terms for each KK modes of the fermion in four-dimensional spacetime. The kinetic terms for the fermion KK modes are obtained by expanding the higher-dimensional chiral fermion $\Psi_\pm$ in terms of mode functions Eq. (4.40),(4.41),(4.42), and (4.43) and integrating over θ and ϕ. We thus obtain the kinetic terms such that

1. For $\Psi^{(+\gamma_5)}_{+}(x,\theta,\phi)$

   (a) $m \neq 0$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+l|m|} i\Gamma^\mu \partial_\mu \Psi^{(+\gamma_5)}_{+l|m|} = \bar{\psi}^{l|m|}_R(x) i\gamma^\mu \partial_\mu \psi^{l|m|}_R + \bar{\psi}^{l|m|}_L(x) i\gamma^\mu \partial_\mu \psi^{l|m|}_L \tag{4.52}$$

   (b) $m = 0$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+00} i\Gamma^\mu \partial_\mu \Psi^{(+\gamma_5)}_{+00} = \bar{\psi}^{00}_R(x) i\gamma^\mu \partial_\mu \psi^{00}_R \quad \text{for} \quad l = 0 \tag{4.53}$$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+l0} i\Gamma^\mu \partial_\mu \Psi^{(+\gamma_5)}_{+l0} = \frac{(1+(-1)^l)^2}{4}[\bar{\psi}^{l0}_R(x) i\gamma^\mu \partial_\mu \psi^{l0}_R + \bar{\psi}^{l0}_L(x) i\gamma^\mu \partial_\mu \psi^{l0}_L] \quad \text{for} \quad l \neq 0 \tag{4.54}$$

2. For $\Psi^{(-\gamma_5)}_{-}(x,\theta,\phi)$

   (a) $m \neq 0$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-l|m|} i\Gamma^\mu \partial_\mu \Psi^{(-\gamma_5)}_{-l|m|} = \bar{\psi}^{l|m|}_R(x) i\gamma^\mu \partial_\mu \psi^{l|m|}_R + \bar{\psi}^{l|m|}_L(x) i\gamma^\mu \partial_\mu \psi^{l|m|}_L \tag{4.55}$$

   (b) $m = 0$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-00} i\Gamma^\mu \partial_\mu \Psi^{(-\gamma_5)}_{-00} = \bar{\psi}^{00}_L(x) i\gamma^\mu \partial_\mu \psi^{00}_L \quad \text{for} \quad l = 0 \tag{4.56}$$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-l0} i\Gamma^\mu \partial_\mu \Psi^{(-\gamma_5)}_{-l0} = \frac{(1+(-1)^l)^2}{4}[\bar{\psi}^{l0}_R(x) i\gamma^\mu \partial_\mu \psi^{l0}_R + \bar{\psi}^{l0}_L(x) i\gamma^\mu \partial_\mu \psi^{l0}_L] \quad \text{for} \quad l \neq 0 \tag{4.57}$$

We obtain the mass terms of the fermion KK modes by using Eq. (4.40)-(4.43) and Eq. (4.48)-(4.51) and integrating over θ and ϕ, such that

1. For $\Psi^{(+\gamma_5)}_{+}(x,\theta,\phi)$

   (a) $m \neq 0$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+l|m|} i\hat{D} \Psi^{(+\gamma_5)}_{+l|m|} = M_l[\bar{\psi}^{l|m|}_R(x)\psi^{l|m|}_L + \bar{\psi}^{l|m|}_L(x)\psi^{l|m|}_R] \tag{4.58}$$

(b) $m=0$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+00} i\hat{D}\Psi^{(+\gamma_5)}_{+00} = 0 \quad \text{for} \quad l=0 \tag{4.59}$$

$$\int d\Omega \bar{\Psi}^{(+\gamma_5)}_{+l0} i\hat{D}\Psi^{(+\gamma_5)}_{+l0} = \frac{(1+(-1)^l)^2}{4} M_l[\bar{\psi}^{l0}_R(x)\psi^{l0}_L + \bar{\psi}^{l0}_L(x)\psi^{l0}_R] \quad \text{for} \quad l \neq 0 \tag{4.60}$$

2. For $\Psi^{(-\gamma_5)}_{-}(x,\theta,\phi)$

(a) $m \neq 0$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-l|m|} i\hat{D}\Psi^{(-\gamma_5)}_{-l|m|} = -M_l[\bar{\psi}^{l|m|}_R(x)\psi^{l|m|}_L + \bar{\psi}^{l|m|}_L(x)\psi^{l|m|}_R] \tag{4.61}$$

(b) $m=0$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-00} i\hat{D}\Psi^{(-\gamma_5)}_{-00} = 0 \quad \text{for} \quad l=0 \tag{4.62}$$

$$\int d\Omega \bar{\Psi}^{(-\gamma_5)}_{-l0} i\hat{D}\Psi^{(-\gamma_5)}_{-l0} = -\frac{(1+(-1)^l)^2}{4} M_l[\bar{\psi}^{l0}_R(x)\psi^{l0}_L + \bar{\psi}^{l0}_L(x)\psi^{l0}_R] \quad \text{for} \quad l \neq 0. \tag{4.63}$$

We have thus confirmed that the fermion $\Psi^{(-\gamma_5)}_{-}(x,\theta,\phi)(\Psi^{(+\gamma_5)}_{+}(x,\theta,\phi))$ has the chiral left(right)-handed massless mode.

We must consider Yukawa coupling of Higgs zero mode and fermion KK modes to obtain mass spectrum of the KK particles after the electroweak symmetry breaking. The Yukawa coupling term in six-dimensional spacetime has the form

$$L_{Yukawa} = \int d\Omega \big[\lambda \bar{\Psi}^{(+\gamma_5)}_{+}(x,\theta,\phi) H(x,\theta,\phi) \Psi^{(-\gamma_5)}_{-}(x,\theta,\phi) + \text{h.c}\big], \tag{4.64}$$

and we obtain the coupling of Higgs zero mode and fermion KK mode in four-dimensional spacetime as

$$L_{Yukawa0} = \sum_{lm} \lambda \Big[\bar{\psi}^{l|m|}_R(x) H^{00}(x) \tilde{\psi}^{l|m|}_L(x) + \bar{\psi}^{l|m|}_L(x) H^{00}(x) \tilde{\psi}^{l|m|}_R(x)\Big] + \text{h.c}, \tag{4.65}$$

where we put tilde on fermions which are obtained from $\Psi^{(-\gamma_5)}_{-}$. After the electroweak symmetry breaking, the Higgs zero mode have a vacuum expectation value(v.e.v) and we have the mass term of the kk mode of the form

$$\begin{pmatrix} \bar{\psi}_{lm} & \bar{\tilde{\psi}}_{lm} \end{pmatrix} \begin{pmatrix} M_l & m_f \\ m_f & -M_l \end{pmatrix} \begin{pmatrix} \psi_{lm} \\ \tilde{\psi}_{lm} \end{pmatrix} \tag{4.66}$$

where m_fs express the masses of the SM fermions originated from the Yukawa coupling term. Since this mass term mix ψ and $\tilde{\psi}$ we must diagonalize the mass term. We change the basis of ψ and $\tilde{\psi}$ as

$$\begin{pmatrix} \psi_{lm} \\ \tilde{\psi}_{lm} \end{pmatrix} = \begin{pmatrix} \gamma_5 \cos\alpha_l & \sin\alpha_l \\ -\gamma_5 \sin\alpha_l & \cos\alpha_l \end{pmatrix} \begin{pmatrix} \psi'_{lm} \\ \tilde{\psi}'_{lm} \end{pmatrix} \tag{4.67}$$

to diagonalize the mass term, where

$$\tan 2\alpha_l = \frac{m_f}{M_l}. \tag{4.68}$$

After diagonalizing mass term, we obtain the mass spectrum

$$M_f^l = \pm\sqrt{M_l^2 + m_f^2}. \tag{4.69}$$

We note that the KK mass M_l do not depend on m and ms are not mixed in mass terms, so that degeneracy of KK masses is

$$l+1 \qquad \text{for} \qquad l = \text{even}, \tag{4.70}$$

$$l \qquad \text{for} \qquad l = \text{odd}, \tag{4.71}$$

since m runs 0 to l.

4.2.2 KK mode expansion of gauge field

Let us focus on the quadratic terms of gauge field Lagrangian in (3.2) since we would like to know the mass spectrum of gauge fields. Decomposing the Lagrangian into 4D components, we obtain

$$\begin{aligned}
&L_{\text{gauge}}^{\text{quadratic}} \\
&= -\frac{1}{4g^2}\sin\theta\,\big[R^2(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&-2\left\{(\partial_\mu A_\theta)(\partial^\mu A_\theta) - 2(\partial_\mu A_\theta)(\partial_\theta A^\mu) + (\partial_\theta A_\mu)(\partial_\theta A^\mu)\right\} \\
&-2\left\{(\partial_\mu \tilde{A}_\phi)(\partial^\mu \tilde{A}_\phi) - \frac{2}{\sin\theta}(\partial_\mu \tilde{A}_\phi)(\partial_\phi A^\mu) + \frac{1}{\sin^2\theta}(\partial_\phi A_\mu)(\partial_\phi A^\mu)\right\} \\
&+\frac{2}{R^2\sin^2\theta}\left\{(\partial_\theta \sin\theta\tilde{A}_\phi)(\partial_\theta \sin\theta\tilde{A}_\phi) - 2(\partial_\theta \sin\theta\tilde{A}_\phi)(\partial_\phi A_\theta) + (\partial_\phi A_\theta)(\partial_\phi A_\theta)\right\}\Big]
\end{aligned} \tag{4.72}$$

where we defined $\tilde{A}_\phi$ as $\tilde{A}_\phi \equiv A_\phi/\sin\theta$ for the kinetic term to be canonical. We note that the background field $\langle A_\phi\rangle$ belongs to the $\mathrm{U}(1)_X$ gauge field and hence $[A_{\mu,\theta}, \langle A_\phi\rangle] = 0$. Namely, we have no background gauge field which induces masses of A_μ and A_θ.

In order to fix the gauge, the follwing gauge-fixing Lagrangian that cancels the mixing terms A_μ and $A_\theta, \tilde{A}_\phi$is added.

$$\begin{aligned}
L_{\text{gf}} &= -\sqrt{-g}\frac{1}{2\xi g^2}\left[\partial_\mu A^\mu + \frac{\xi}{R^2\sin\theta}\left(\partial_\theta(\sin\theta A^\theta) + \frac{1}{\sin\theta}\partial_\phi A^\phi\right)\right]^2 \\
&= -\frac{R^2\sin\theta}{2\xi g^2}\left[(\partial_\mu A^\mu)^2 + \frac{\xi^2}{R^4\sin^2\theta}\left(\partial_\theta(\sin\theta A_\theta) + \frac{1}{\sin\theta}\partial_\phi A_\phi\right)^2\right. \\
&\qquad \left. -\frac{2\xi}{R^2\sin\theta}(\partial_\mu A^\mu)\left(\partial_\theta(\sin\theta A_\theta) + \frac{1}{\sin\theta}\partial_\phi A_\phi\right)\right]
\end{aligned} \tag{4.73}$$

where ξ is a gauge-fixing parameter.

Combining (4.72) and (4.73) and after partial integration, we obtain

$$
\begin{aligned}
&L^{\text{quadratic}}_{\text{gauge}} + L_{\text{gf}} \\
&= -\frac{1}{4g^2}\sin\theta\,\Big[R^2(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&-2\left\{(\partial_\mu A_\theta)(\partial^\mu A_\theta) + (\partial_\theta A_\mu)(\partial_\theta A^\mu) + (\partial_\mu \tilde{A}_\phi)(\partial^\mu \tilde{A}_\phi) + \frac{1}{\sin^2\theta}(\partial_\phi A_\mu)(\partial_\phi A^\mu)\right\} \\
&+\frac{2}{R^2\sin^2\theta}\left\{(\partial_\theta \sin\theta \tilde{A}_\phi)(\partial_\theta \sin\theta \tilde{A}_\phi) - 2(\partial_\theta \sin\theta \tilde{A}_\phi)(\partial_\phi A_\theta) + (\partial_\phi A_\theta)(\partial_\phi A_\theta)\right\}\Big] \\
&- \frac{R^2\sin\theta}{2\xi g^2}\left[(\partial_\mu A^\mu)^2 + \frac{\xi^2}{R^4\sin^2\theta}\left(\partial_\theta(\sin\theta A_\theta) + \partial_\phi \tilde{A}_\phi\right)^2\right] \\
&= -\frac{1}{4g^2}\sin\theta\,\Big[R^2(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&-2\left\{(\partial_\theta A_\mu)(\partial_\theta A^\mu) + \frac{1}{\sin^2\theta}(\partial_\phi A_\mu)(\partial_\phi A^\mu)\right\} \\
&-2\left\{(\partial_\mu A_\theta)(\partial^\mu A_\theta) + (\partial_\mu \tilde{A}_\phi)(\partial^\mu \tilde{A}_\phi)\right\} + \frac{2}{R^2\sin^2\theta}\left((\partial_\theta \sin\theta \tilde{A}_\phi) - (\partial_\phi A_\theta)\right)^2\Big] \\
&- \frac{R^2\sin\theta}{2\xi g^2}\left[(\partial_\mu A^\mu)^2 + \frac{\xi^2}{R^4\sin^2\theta}\left(\partial_\theta(\sin\theta A_\theta) + \partial_\phi \tilde{A}_\phi\right)^2\right]. \qquad (4.74)
\end{aligned}
$$

We find that KK mass term for the four-dimensional components of gauge field can be diagonalized by expanding them by spherical harmonics since the extra-kinetic terms can be expressed by the square of angular momentum operator. Extra-components of guage field, however, do not have clear form of extra-kinetic terms to be diagonalized. We then perform following substitutions,

$$
\begin{aligned}
\tilde{A}_\phi(x,\theta,\phi) &= \partial_\theta \phi_1(x,\theta,\phi) + \frac{1}{\sin\theta}\partial_\phi \phi_2(x,\theta,\phi) \qquad (4.75) \\
A_\theta(x,\theta,\phi) &= \partial_\theta \phi_2(x,\theta,\phi) - \frac{1}{\sin\theta}\partial_\phi \phi_1(x,\theta,\phi). \qquad (4.76)
\end{aligned}
$$

Here we note that there is no component which is independent of S^2/Z_2 coordinates since they are forbidden by the boundary conditions. This substitution leads

$$
\begin{aligned}
\frac{1}{\sin\theta}(\partial_\theta \sin\theta \tilde{A}_\phi) - \frac{1}{\sin\theta}\partial_\phi A_\theta &= \frac{1}{\sin\theta}\partial_\theta(\sin\theta \partial_\theta \phi_1) + \frac{1}{\sin^2\theta}\partial_\phi^2 \phi_1, \qquad (4.77) \\
\frac{1}{\sin\theta}(\partial_\theta \sin\theta A_\theta) + \frac{1}{\sin\theta}\partial_\phi \tilde{A}_\phi &= \frac{1}{\sin\theta}\partial_\theta(\sin\theta \partial_\theta \phi_2) + \frac{1}{\sin^2\theta}\partial_\phi^2 \phi_2, \qquad (4.78)
\end{aligned}
$$

where RHS's are expressed by square of angular momentum opperator acting on $\phi_{1(2)}$. The four-dimensional kinetic term of A_ϕ and A_θ is also rewritten as

$$
\begin{aligned}
&\sin\theta[\partial_\mu A_\theta \partial^\mu A_\theta + \partial_\mu \tilde{A}_\phi \partial^\mu \tilde{A}_\phi] \\
&= \sin\theta\Big[-\partial_\mu\phi_1\partial^\mu\Big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi_1) + \frac{1}{\sin^2\theta}\partial_\phi^2\phi_1\Big] \\
&\qquad -\partial_\mu\phi_2\partial^\mu\Big[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi_2) + \frac{1}{\sin^2\theta}\partial_\phi^2\phi_2\Big]\Big] \\
&\qquad -2\partial_\mu(\partial_\theta\phi_2)\partial^\mu(\partial_\phi\phi_1) + 2\partial_\mu(\partial_\theta\phi_1)\partial^\mu(\partial_\phi\phi_2). \qquad (4.79)
\end{aligned}
$$

Note that the last two terms are canceled between them after expanding $\phi_{1(2)}$ by spherical harmonics and performing partial integration. After substitution Eq. (4.75) and (4.76), we obtain

$$
\begin{aligned}
&L_{\text{gauge}}^{\text{quadratic}} + L_{\text{gf}} \\
&= -\frac{1}{4g^2}\sin\theta\Big[R^2(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - 2A^\mu\hat{L}^2 A_\mu \\
&+2\Big\{\partial_\mu\phi_1\partial^\mu(\hat{L}^2\phi_1) + \partial_\mu\phi_2\partial^\mu(\hat{L}^2\phi_2) + \Big\} - \frac{2}{R^2}(\hat{L}^2\phi_1)^2\Big] \\
&- \frac{R^2\sin\theta}{2\xi g^2}\Big[(\partial_\mu A^\mu)^2 + \frac{\xi^2}{R^4}(\hat{L}^2\phi_2)^2\Big], \qquad (4.80)
\end{aligned}
$$

where $\hat{L}^2 = -(1/\sin\theta)\partial_\theta(\sin\theta\partial\theta) - (1/\sin^2)\theta\partial_\phi^2$ is the square of angular momentum operator. It is now clear that diagonal KK mass terms can be obtained by expanding gauge fields using spherical harmonics. The mode expansions are then carried out in the following way.

$$A_\mu(x,\theta,\phi) = \sum_{l,m} A_\mu^{lm}(x) Y_{lm}^+(\theta,\phi), \qquad (4.81)$$

$$\phi_{1(2)}(x,\theta,\phi) = \sum_{l(\neq 0),m} \phi_{1(2)}^{lm}(x)\frac{Y_{lm}^+(\theta,\phi)}{\sqrt{l(l+1)}} \qquad (4.82)$$

where the mode finction $Y_{lm}^{\pm}(\theta,\phi)$ is defined as

$$
\begin{aligned}
Y_{lm}^+(\theta,\phi) &\equiv \frac{(i)^{l+m}}{\sqrt{2}}[Y_{lm}(\theta,\phi) \pm (-1)^l Y_{l-m}(\theta,\phi)] \quad \text{for} \quad m \neq 0 \qquad (4.83)\\
Y_{lm}^-(\theta,\phi) &\equiv \frac{(i)^{l+m+1}}{\sqrt{2}}[Y_{lm}(\theta,\phi) \pm (-1)^l Y_{l-m}(\theta,\phi)] \quad \text{for} \quad m \neq 0 \qquad (4.84)\\
Y_{l0}^{+(-)}(\theta) &\equiv Y_{l0}(\theta) \quad \text{for} \quad m = 0 \text{ and } l = \text{even(odd)} \\
&\equiv 0 \qquad\quad \text{for} \quad m = 0 \text{ and } l = \text{odd(even)} \qquad (4.85)
\end{aligned}
$$

We note that the mode functions $Y_{lm}^{\pm}$ are eigenfunctions with Z_2 parity $\pm$ under Z_2 action $(\theta,\phi) \to (\pi-\theta,-\phi)$ because of the property $Y_{lm}(\pi-\theta,-\phi) = (-1)^l Y_{l-m}(\theta,\phi)$. We further notice that the phase factors $(i)^{l+m(+1)}$ in the mode functions Eq. (4.83) and (4.84) are required since the fields A_μ and $\phi_{1(2)}$ must be real and the spherical harmonics satisfy $(Y_{lm})^* = (-1)^m Y_{l-m}$.

Substituting the mode expansions into the Lagrangian (4.80) and integrating out θ, ϕ coordinates leads to

$$
\begin{aligned}
&L_{\rm gauge}^{\rm quadratic} + L_{\rm gf} \\
&= -\frac{1}{4}\sum_{l,m(\neq 0)} (\partial_\mu A_\nu^{lm} - \partial_\nu A_\mu^{lm})(\partial^\mu A^{\nu lm} - \partial^\nu A^{\mu lm}) \\
&- \frac{1}{4}\sum_{l:{\rm even}} (\partial_\mu A_\nu^{l0} - \partial_\nu A_\mu^{l0})(\partial^\mu A^{\nu l0} - \partial^\nu A^{\mu l0}) \\
&+ \frac{1}{2}\sum_{l,m(\neq 0)} (\partial_\mu \phi_1^{lm})(\partial^\mu \phi_1^{lm}) + \sum_{l(\neq 0):{\rm even}} (\partial_\mu \phi_1^{l0})(\partial^\mu \phi_1^{l0}) \\
&+ \frac{1}{2}\sum_{l,m(\neq 0)} (\partial_\mu \phi_2^{lm})(\partial^\mu \phi_2^{lm}) + \sum_{l(\neq 0):{\rm even}} (\partial_\mu \phi_2^{l0})(\partial^\mu \phi_2^{l0}) \\
&+ \sum_{l,m} \frac{l(l+1)}{2R^2} A_\mu^{lm} A^{\mu lm} - \sum_{l,m(\neq 0)} \frac{l(l+1)}{2R^2}(\phi_1^{lm})^2 - \xi \sum_{l,m(\neq 0)} \frac{l(l+1)}{2R^2}(\phi_2^{lm})^2 \\
&- \sum_{l(\neq 0):{\rm even}} \frac{l(l+1)}{2R^2}\left[(\phi_1^{l0})^2 + \xi(\phi_2^{l0})^2\right] - \frac{1}{2\xi}\sum_{lm(\neq 0)} (\partial_\mu A^{\mu lm})^2 - \frac{1}{2\xi}\sum_{l:{\rm even}} (\partial_\mu A^{\mu l0})^2
\end{aligned}
\tag{4.86}
$$

A rescaling $A_\mu \to R^{-1} A_\mu$ was done so that the gauge kinetic term is canonical. The KK modes of the ϕ_2 are interpreted as Nambu-Goldstone(NG) bosons since their KK masses are proportional to the gauge fixing parameter ξ. These NG bosons will be eaten by KK modes of four dimensional components of gauge field giving their longitudinal component.

Next, let us turn to the Higgs part to incorporate the electroweak symmetry breaking effects. Higgs part of the Lagrangian is given by

$$
\begin{aligned}
L_{\rm Higgs} &= \sqrt{-g}\left[g^{MN}(D_M H)^\dagger D_N H - V(H)\right] \\
&= R^2 \sin\theta \left[\eta^{\mu\nu}(D_\mu H)^\dagger D_\nu H \right. \\
&\left. -\frac{1}{R^2}|D_\theta H|^2 - \frac{1}{R^2 \sin^2\theta}|D_\phi H|^2 - V(H)\right],
\end{aligned}
\tag{4.87}
$$

$$
D_M = \partial_M - i g_2 A_M - \frac{i}{2} g_1 B_M \tag{4.88}
$$

where $g_{1,2}$ and A_M, B_M are the gauge coupling constants and gauge fields of $SU(2)_L, U(1)_Y$ gauge groups, respectively. $V(H)$ denotes a Higgs potential. The gauge boson masses are obtained from the covariant derivatives as in the standard model by putting the Higgs VEV $H^T = (0, \frac{v}{\sqrt{2}})$,

$$
\begin{aligned}
L_{\text{Higgs}} \supset R^2 \sin\theta \frac{1}{4}\frac{1}{2} & \left| \begin{pmatrix} g_2 A_M^3 + g_1 B_M & g_2(A_M^1 - iA_M^2) \\ g_2(A_M^1 + iA_M^2) & -g_2 A_M^3 + g_1 B_M \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\
&= R^2 \sin\theta \left[m_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \right. \\
&\qquad - \frac{1}{R^2}\left(m_W^2 W_\theta^+ W_\theta^- + \frac{1}{2} m_Z^2 Z_\theta Z_\theta \right) \\
&\qquad \left. - \frac{1}{R^2}\left(m_W^2 \tilde{W}_\phi^+ \tilde{W}_\phi^- + \frac{1}{2} m_Z^2 \tilde{Z}_\phi \tilde{Z}_\phi \right) \right] \\
&= \sum_{l,m} \left[m_W^2 W_\mu^{+lm} W^{\mu-lm} + \frac{1}{2} m_Z^2 Z_\mu^{lm} Z^{\mu lm} \right. \\
&\qquad - \left(m_W^2 W_1^{+lm} W_1^{-lm} + \frac{1}{2} m_Z^2 Z_1^{lm} Z_1^{lm} \right) \\
&\qquad \left. - \left(m_W^2 W_2^{+lm} W_2^{-lm} + \frac{1}{2} m_Z^2 Z_2^{lm} Z_2^{lm} \right) \right]
\end{aligned} \tag{4.89}
$$

where $W_{1(2)}$ and $Z_{1(2)}$ are defined by the substitution Eq. (4.75) and (4.76).

Combining the results (4.86) and (4.89), the mass spectrum of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge sector is summarized as follows.

$$
\begin{aligned}
& W_\mu^{lm} : m_W^2 + \frac{l(l+1)}{R^2}, \quad Z_\mu^{lm} : m_Z^2 + \frac{l(l+1)}{R^2}, \quad \gamma_\mu^{lm} : \frac{l(l+1)}{R^2}, \\
& W_1^{lm} : m_W^2 + \frac{l(l+1)}{R^2}, \quad Z_1^{lm} : m_Z^2 + \frac{l(l+1)}{R^2}, \quad \gamma_1^{lm} : \frac{l(l+1)}{R^2}, \\
& W_2^{lm} : m_W^2 + \xi\frac{l(l+1)}{R^2}, \quad Z_2^{lm} : m_Z^2 + \xi\frac{l(l+1)}{R^2}, \quad \gamma_2^{lm} : \xi\frac{l(l+1)}{R^2}.
\end{aligned} \tag{4.90}
$$

Here we mention extra $\mathrm{U}(1)_X$ sector in our model. We notice that the $\mathrm{U}(1)_X$ symmetry is anomalous and is broken at the quantum level, so that its gauge boson should be heavy [81]. We thus expect the $\mathrm{U}(1)_X$ gauge boson and its KK modes are decoupled from the low energy sector of our model.

We, theore, conclude that the lightest KK particles are 1st KK mode of four-dimensional components of massless gauge bosons and that of physical scalar boson originated from extra components of gauge field. These kk particles are the 1st KK mode of photon γ_μ^{11}, scalar photon γ_1^{11}, gluon g_μ^{11} and scalar gluon g_1^{11} at tree level. We can also guess that the 1st KK photon is the promising candidate of the lightest KK particle after including a quantum correction and it would be a good candidate for the dark matter [82].

4.3 KK mode expansion of the Higgs field

Here we discuss the KK mode expansion and mass spectrum of the Higgs field. We thus focus on the kinetic-mass terms of the Higgs field in six-dimensional space time. The kinetic-mass term has the form

$$\begin{aligned} L^{(6D)}_{Higgs-kinetic} &= R^2 \sin\theta \big[\partial^\mu H^\dagger(X)\partial_\mu H(X) \\ &- \frac{1}{R^2}\partial_\theta H^\dagger(X)\partial_\theta H(X) - \frac{1}{R^2\sin^2\theta}\partial_\phi H^\dagger(X)\partial_\phi H(X)\big]. \end{aligned} \tag{4.91}$$

After partial integration, we obtain

$$\begin{aligned} L^{(6D)}_{Higgs-kinetic} &= R^2 \sin\theta \Bigg[\partial^\mu H^\dagger(X)\partial_\mu H(X) \\ &+ \frac{1}{R^2} H^\dagger(X)\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right)H(X)\Bigg], \end{aligned} \tag{4.92}$$

where the derivative operator in the second term has the form of square of angular momentum operator.

The mode expansion of the Higgs field is carried out as follows,

$$H(x,\theta,\phi) = \sum_{l,m} H^{lm}(x) Y^+_{lm}(\theta,\phi) \tag{4.93}$$

since the Higgs field has even parity under the Z_2 action.

Substituting the mode expansion into the Lagrangian Eq. (4.92) and integrating out $\theta and \phi$ coordinates, we find the kinetic and mass terms of the Higgs field in four-dimensional spacetime

$$L^{(4D)}_{Higgs-kinetic-mass} = \partial^\mu (H^{lm}(x))^\dagger \partial_\mu H^{lm}(x) - \frac{l(l+1)}{R^2}(H^{lm}(x))^\dagger H^{lm}(x). \tag{4.94}$$

We, therefore, find the mass spectrum of the Higgs field such that

$$M_l = \sqrt{\frac{l(l+1)}{R^2} + m_H^2}, \tag{4.95}$$

where m_H is the Higgs zero mode mass obtained from the Higgs potential after electro weak symmetry breaking. There are $l+1(l)$ mass degeneracies of the KK modes for even(odd) l since m runs from 0 to l for each l and $Y^+_{l0} = 0$ for $l =$odd.

4.3.1 The KK-parity for each KK modes

We discuss the KK-parity for each KK modes to investigate the stability of the lightest KK particle. In our model, the KK momentum is not conserved due to the orbifolding, but the discrete part is still conserved as a remnant of KK momentum conservation. We can see that there is an additional discrete Z'_2 symmetry of $(\theta,\phi) \to (\theta, \phi+\pi)$, which is different from the previous Z_2 symmetry. This Z'_2 symmetry is understood as the symmetry of interactions under the exchange of two fixed points on S^2/Z_2 orbifold which are points $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$. Note that the fixed points have different ϕ coordinates 0 and π but the same θ coordinate $\pi/2$ so that the Z'_2 action shift only in the ϕ coordinate. The mode functions for each fields in Eqs. (4.46),(4.47),(4.81),(4.82), and (4.93) are transformed under the Z_2' action $(\theta,\phi) \to (\theta,\phi+\pi)$ such that

$$\Psi^{\pm\gamma_5}_{+l|m|}(x,\theta,\phi+\pi) = (-1)^m \Psi^{\pm\gamma_5}_{+l|m|}(x,\theta,\phi), \tag{4.96}$$

$$\Psi^{\pm\gamma_5}_{-l|m|}(x,\theta,\phi+\pi) = (-1)^m \Psi^{\pm\gamma_5}_{+l|m|}(x,\theta,\phi), \tag{4.97}$$

$$Y^{\pm}_{lm}(\theta,\phi+\pi) = (-1)^m Y^{\pm}_{lm}(\theta,\phi+\pi). \tag{4.98}$$

Thus the KK-parity is defined as $(-1)^m$ and we find the KK-parity is conserved as a consequence of the Z_2' symmetry of the Lagrangian in six-dimensional spacetime. We, therefore, can confirm the stability of the lightest KK particle which would be the 1st KK mode of photon γ^{11}_μ since this mode has $m=1$ and cannot decay into SM particles. We summarize the KK particle masses, mass degeneracy and the KK parity in Table. 27

Table 27: Summary of the KK particle masses, mass degeneracy and the KK parity for KK modes of fermion ψ^{lm}, four dimensional gauge bosons A^{lm}_μ, scalar gauge bosons $\phi^{lm}_{1(2)}$ and Higgs boson H^{lm}, where the m_f, m_g and m_H are the zero mode masses for the fermions, the gauge bosons and the Higgs boson respectively, which correspond to the masses of SM particles.

particle	KK mass^2	mass degeneracy	KK parity
$\psi^{lm}(x)$	$\frac{l(l+1)}{R^2}+m_f^2$	$l+1$ for l =even $(0 \le m \le l)$ l for l =odd $(0 < m \le l)$	$(-1)^m$
$A^{lm}_\mu(x)$	$\frac{l(l+1)}{R^2}+m_g^2$	$l+1$ for l =even $(0 \le m \le l)$ l for l =odd $(0 < m \le l)$	$(-1)^m$
$\phi^{lm}_1(x)$	$\frac{l(l+1)}{R^2}+m_g^2$	$l+1$ for l =even$(\neq 0)$ $(0 \le m \le l)$ l for l =odd $(0 < m \le l)$	$(-1)^m$
$\phi^{lm}_2(x)$	$\xi\frac{l(l+1)}{R^2}+m_g^2$	$l+1$ for l =even$(\neq 0)$ $(0 \le m \le l)$ l for l =odd $(0 < m \le l)$	$(-1)^m$
$H^{lm}(x)$	$\frac{l(l+1)}{R^2}+m_H^2$	$l+1$ for l =even $(0 \le m \le l)$ l for l =odd $(0 < m \le l)$	$(-1)^m$

4.3.2 KK particles contribution to Higgs production via gluon fusion

We here examine Higgs boson production process via gluon fusion in our model. Higgs production by gluon fusion is very important because it is the dominant production mode at LHC and it has been studied in various models beyond the SM [88, 89, 90, 91, 92] as well as the SM [93].

Before calculating contributions of KK fermions to one-loop effective couplings between Higgs boson and gluons, it is instructive to recall the SM result. We parameterize the effective coupling between Higgs boson and gluons as

$$\mathcal{L}_{\text{eff}} = C_g^{\text{SM}}\; h\; G^{a\mu\nu} G^a_{\mu\nu}, \tag{4.99}$$

where h is a SM higgs boson and $G^a_{\mu\nu}$ is a gluon field strength tensor. This operator is a dimension six (five) one before (after) electroweak symmetry breaking. The coupling is generated by one-loop corrections (triangle diagram) where quarks are running. The top quark loop diagram gives the dominant contribution and the coupling C_g^{SM} is described in the following instructive form [93]:

$$C_g^{\text{SM}} = -\frac{m_t}{v} \times \frac{\alpha_s F_{1/2}(4m_t^2/m_h^2)}{8\pi m_t} \times \frac{1}{2}, \tag{4.100}$$

where, in the right hand side, the first term $-\frac{m_t}{v}$ is top Yukawa coupling, the second term is from the loop integral with the QCD coupling α_s at QCD vertices, the loop function $F_{1/2}(\tau)$ given by (for $\tau \geq 1$)

$$\begin{aligned} F_{1/2}(\tau) &= -2\tau\left(1 + (1-\tau)\arcsin^2(1/\sqrt{\tau})\right) \\ &\to -\frac{4}{3} \quad \text{for } \tau \gg 1, \end{aligned} \tag{4.101}$$

and 1/2 is a QCD group factor (Dynkin index). Mass of the fermion (top quark) running in the loop appears in the denominator in the second term of (4.100), which is canceled with top quark mass from Yukawa coupling. It is well-known that in the top quark decoupling limit, namely top quark mass m_t is much heavier than Higgs boson mass m_h, $F_{1/2}$ becomes a constant and the resultant effective coupling becomes independent of m_t and m_h.

A calculation of KK mode contributions in a 6D UED model on S^2/Z_2 is completely analogous to the top loop correction in the SM case. To carry out the calculation, we need to know top Yukawa coupling constant and KK mass spectrum of top quark in our model. The relevant top Yukawa coupling and mass terms after electroweak symmetry breaking are given by

$$\begin{aligned} \mathcal{L} \supset \sum_{lm} \Big[& M_l \bar{\psi}_R^{lm} \psi_L^{lm} - M_l \bar{\tilde{\psi}}_R^{lm} \tilde{\psi}_L^{lm} \\ & + \frac{m_t}{v}\left(\bar{\psi}_R^{lm}(x) H^{00} \tilde{\psi}_L^{lm} + \bar{\psi}_L^{lm} H^{00} \tilde{\psi}_R^{lm}\right) \\ & + m_t\left(\bar{\psi}_R^{lm} \tilde{\psi}_L^{lm} + \bar{\psi}_L^{lm} \tilde{\psi}_R^{lm}\right) + \text{h.c.}\Big]. \end{aligned} \tag{4.102}$$

where L, R denote chirality in four dimensional sense. $\psi^{lm}, \tilde{\psi}^{lm}$ are chiral fermions in a six dimensional sense, namely they are classified by the eigenvalues of Γ_7 as $\Gamma_7 \psi^{lm} = \psi^{lm}, \Gamma_7 \tilde{\psi}^{lm} = -\tilde{\psi}^{lm}$. m_t is top quark mass, M_l is top quark KK masses with $M_l^2 = l(l+1)/R^2$, and v is a vacuum expectation value of Higgs field.

The mass terms for KK modes can be written down in a matrix form by using Dirac fermion

$$\begin{pmatrix} \bar{\psi}^{lm} & \bar{\tilde{\psi}}^{lm} \end{pmatrix} \begin{pmatrix} M_l & m_t \\ m_t & -M_l \end{pmatrix} \begin{pmatrix} \psi^{lm} \\ \tilde{\psi}^{lm} \end{pmatrix}, \tag{4.103}$$

which is diagonalized by the change of basis

$$\begin{pmatrix} \psi^{lm} \\ \tilde{\psi}^{lm} \end{pmatrix} = \begin{pmatrix} \gamma_5 \cos\alpha_l & \sin\alpha_l \\ -\gamma_5 \sin\alpha_l & \cos\alpha_l \end{pmatrix} \begin{pmatrix} \psi'^{lm} \\ \tilde{\psi}'^{lm} \end{pmatrix} \tag{4.104}$$

where $\tan 2\alpha_l = m_t/M_l$. Rewriting (4.102) in terms of mass eigenstates ψ'^{lm} and $\tilde{\psi}'^{lm}$, we find that top KK mass eigenvalue is $m_t^{(l)} = \sqrt{\frac{l(l+1)}{R^2} + m_t^2}$ and top Yukawa coupling is $-(m_t \sin 2\alpha_l)/v = -m_t^2/(v m_t^{(l)})$, respectively [87]. Making use of this information, the KK mode contributions in our model are found to be

$$\mathcal{L}_{\text{eff}} = C_g^{\text{KK(UED}_2)}\, h\, G^{a\mu\nu} G^a_{\mu\nu} \tag{4.105}$$

where

$$\begin{aligned}
& C_g^{\text{KK(UED}_2)} \\
= & -\sum_{l=1}^{\infty} n(l) \left[\frac{m_t}{v} \frac{m_t}{m_t^{(l)}} \times \frac{\alpha_s F_{1/2}(4(m_t^{(l)})^2/m_h^2)}{8\pi m_t^{(l)}} \frac{1}{2} \right] \times 2 \\
= & \frac{\alpha_s}{\pi v} \frac{m_t^2}{m_h^2} \sum_{l=1}^{\infty} [(2l+1)\times \\
& \left\{ 1 + \left(1 - \frac{4(m_t^{(2l)})^2}{m_h^2} \right) \arcsin^2 \left(\frac{m_h}{2m_t^{(2l)}} \right) \right\} \\
& + (2l-1)\times \\
& \left\{ 1 + \left(1 - \frac{4(m_t^{(2l-1)})^2}{m_h^2} \right) \arcsin^2 \left(\frac{m_h}{2m_t^{(2l-1)}} \right) \right\}] \\
\simeq & \frac{\alpha_s}{6\pi v} \sum_{l=1}^{\infty} \left[\frac{(2l+1)m_t^2}{\frac{2l(2l+1)}{R^2} + m_t^2} + \frac{(2l-1)m_t^2}{\frac{2l(2l-1)}{R^2} + m_t^2} \right]
\end{aligned} \tag{4.106}$$

where "UED$_{2(1)}$" denotes our 6D UED model on S^2/Z_2 [87] (5D UED model on S^1/Z_2 [83], which will be discussed later), respectively. A factor "2" is multiplied in the second line, since the degrees of freedom of 6D fermion are doubled compared with the SM case. In the second and the third equalities, the mode sum is decomposed into even or odd number term of l since the degeneracy $n(l)$ with respect to m is different, *e.g.* $n(l) = l + 1(l)$ for l : even (odd) [87]. The limit $m_h^2, m_t^2 \ll (1/R)^2$ have been taken in the last line to simplify the results. As expected from the dimensional analysis, the mode sum is logarithmically divergent. Also, note that the KK mode contribution is constructive against the top quark contribution in the SM.

It is interesting to compare our result with that in the minimal UED model on S^1/Z_2 [89], where the KK mode mass spectrum and Yukawa couplings are given by $M_n = \sqrt{(n/R)^2 + m_t^2}$ and $-(m_t^2/M_n v)$, respectively. In this case, we find the effective coupling as

$$\mathcal{L}_{\text{eff}} = C_g^{\text{KK(UED}_1)}\, h\, G^{a\mu\nu} G^a_{\mu\nu} \tag{4.107}$$

where

$$\begin{aligned} & C_g^{\mathrm{KK(UED_1)}} \\ = & -\sum_{n=1}^{\infty}\left[\frac{m_t}{v}\frac{m_t}{M_n}\times\frac{\alpha_s F_{1/2}(4M_n^2/m_h^2)}{8\pi M_n}\times\frac{1}{2}\right]\times 2 \\ \simeq & \frac{\alpha_s}{6\pi v}\sum_{n=1}^{\infty}\frac{m_t^2}{(n/R)^2} \end{aligned} \tag{4.108}$$

where we have taken the limit $m_h^2,\ m_t^2 \ll (1/R)^2$ again to simplify the result. This KK mode contribution is finite and constructive to the top quark one in the SM.

As we have shown, the KK mode loop contribution to the effective coupling between Higgs boson and gluons is constructive similar to the top quark loop contribution in the SM. This fact leads to remarkable effects on Higgs boson search at the LHC. Since the main production process of Higgs boson at the LHC is through gluon fusion, so that the deviation of the effective coupling between Higgs boson and gluons from the SM and other model's predictions directly affects the Higgs boson production cross section.

Let us consider the ratio of the Higgs boson production cross section in the UED model on S^2/Z_2 and S^1/Z_2 to the SM one, which is described as

$$\Delta \equiv \frac{\sigma(gg\to h;\ \mathrm{SM}+\mathrm{KK})}{\sigma(gg\to h;\ \mathrm{SM})} = \left(1+\frac{C_g^{\mathrm{KK(UED_{2(1)})}}}{C_g^{\mathrm{SM}}}\right)^2. \tag{4.109}$$

The numerical results of this ratio as a function of the compactification scale $1/R$ are shown in Fig. 1 where these plots are calculated by using exact expressions of $C_g^{\mathrm{SM}}, C_g^{\mathrm{KK(UED_{2(1)})}}$ not approximated ones. The bold (dashed) line corresponds to the UED model on $S^2/Z_2(S^1/Z_2)$. The horizontal line $\Delta = 1$ is the SM prediction. In this analysis, we have taken Higgs mass to be $m_h = 120, 150, 180$ GeV (from the left to the right in Fig. 1). The results are not sensitive to the Higgs boson mass. The KK fermion contribution is constructive and the Higgs production cross section is increased in the UED scenario in contrast to the case of little Higgs [90] or gauge-Higgs unification [91]. The present UED model on S^2/Z_2 gives rise to more enhanced Higgs production cross section than that of the minimal UED model on S^1/Z_2. This is very natural because the number of KK particles is larger. For $1/R = 1$ TeV, the KK fermion contribution of our model is sizable and the production cross section is more enhanced by around 30(10)% than the SM (minimal UED on S^1/Z_2) prediction. Thus, we have found that our model predicts a remarkable collider signature of Higgs production at LHC. We expect that our prediction will be soon verified by the forthcoming experiment.

In our analysis, we have summed only the first five KK mode contributions. The reason is the following. Our model is a six dimensional model, so the gluon fusion amplitude is logarithmically divergent as mentioned earlier. More specifically, the mode sum behaves as $\log(\Lambda R)$ with the cutoff scale Λ. Therefore, one might worry about the cutoff dependence of the result. We can find an upper bound for the cutoff scale by using naive dimensional analysis. A loop expansion parameter ε of six dimensional theories is given by

$$\varepsilon = \frac{\pi^3}{2(2\pi)^6}g_6^2\Lambda^2 = \frac{\alpha}{8\pi}(R\Lambda)^2 \tag{4.110}$$

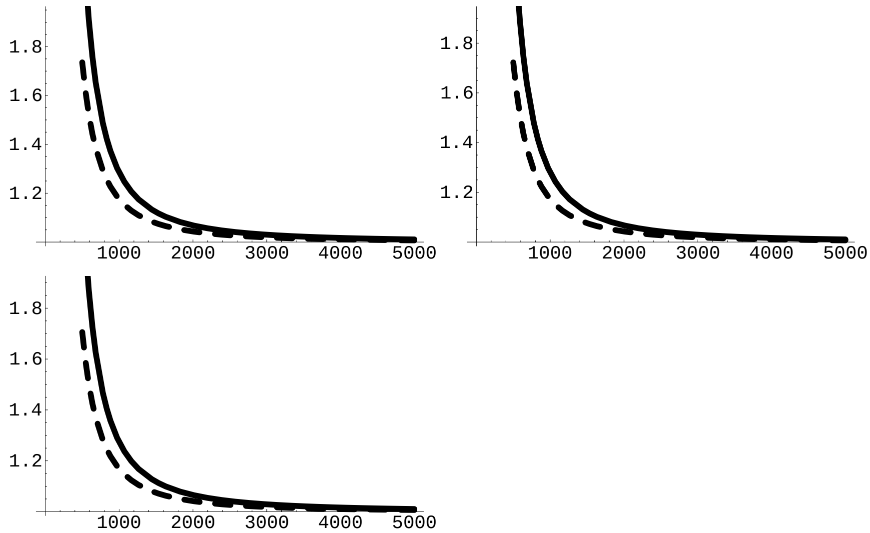

Figure 1: The ratio of the Higgs boson production cross section via gluon fusion in 6D UED on S^2/Z_2, 5D UED on S^1/Z_2 and in the SM, as a function of the compactification scale $1/R$ in a unit of GeV. The vertical axis denotes Δ defined above and $\Delta = 1.0$ corresponds to the SM prediction. The bold (dashed) line corresponds to the 6D UED on S^2/Z_2 (5D UED on S^1/Z_2) prediction, respectively. Higgs mass is taken to be 120, 150, 180 GeV from the left to the right.

where g_6 is a gauge coupling constant in six dimensional gauge theories. The cutoff scale is introduced to make ε dimensionless. $\alpha \equiv g_4^2/(4\pi)(g_4$: 4D gauge coupling constant). Requiring that our theory is perturbative at the cutoff scale, $\varepsilon \lesssim 1$, we obtain

$$R\Lambda \lesssim \sqrt{\frac{8\pi}{\alpha}}. \tag{4.111}$$

The most stringent bound is found for the case that our 4D effective theory becomes strong coupling at the cutoff scale $\alpha \simeq 1$. Thus, we finally obtain the upper bound of the cutoff scale

$$\Lambda \lesssim \sqrt{8\pi}/R \simeq 5/R. \tag{4.112}$$

Here is a comment on collider physics. If we take into account the Higgs boson decay process, our result holds true for the case where Higgs mass is heavier than around 150 GeV. In that situation, Higgs boson will mainly decay into W^+W^- pair via the SM vertex at tree level. Therefore, Δ is unchanged. However, if Higgs boson mass is lighter than 150 GeV, the most promising discovery mode is two photon decay process. This process is also given by one-loop triangle diagram and KK modes of top quark and W-boson contribute. The analysis of Higgs boson decay into two photons is beyond the scope of this paper and is left for future work [94].

5 Summary

We proposed new approaches to construct a Gauge-Higgs Unification(GHU) model and Universal Extra Dimensional(UED) model.

For GHU model, we applied Coset Space Dimensional Reduction(CSDR) scheme since this scheme provides well determined four-dimensional theory starting from higher-dimensional theory. We constructed GHU model with CSDR scheme based on gauge theory in fourteen and eight dimensional spacetime with simple gauge group and gauge theory in ten dimensional spacetime with direct product gauge group. We then exhaustively searched for the phenomenologically acceptable models. Several models are found which lead phenomenologically acceptable theory in four dimensions; four fourteen dimensional models provide SM like gauge symmetry SU(3)×SU(2)×U(1)×U(1) and SM particle contents in four dimensions and many fourteen, ten and eight dimensional models provide GUT like model with particle contents which contain SM Higgs and generations of SM fermions. Apparent difficulties are, however, also found in these models e.g. appearance of extra particles in SM like case and lack of Higgs particle which break GUT gauge symmetry in GUT like case.

We then provide new approach to construct a GHU model, which are based on gauge theory defined on the six-dimensional spacetime which has an S^2/Z_2 extra-space, with the symmetry condition and non-trivial boundary conditions.

We first provided the scheme for constructing a four-dimensional theory from a gauge theory on six-dimensional spacetime which has extra space S^2 with the symmetry condition of gauge field and the non-trivial boundary conditions. We showed the prescriptions to identify the gauge field and the scalar field, which satisfy the symmetry condition and the boundary conditions. A fermion sector of four-dimensional theory is also obtained by expanding fermions in normal mode and integrating the S^2 coordinates, although explicit form was not shown. Massive KK modes of fermions then appear in contrast to scalar and gauge field, which would provide a candidate of dark-matter. They may give a rich phenomena in near future collider experiment. To discuss these matters, we have to find the eigenvalues of Eq. (3.30). We leave this in future work. We also showed that fermions can have massless mode because of the existence of a background gauge field. The fermion components which have massless modes are then determined by the background gauge field and the boundary conditions.

Note that by imposing the symmetry condition, we can get massless fermions. It may indicate the meaning of the symmetry condition; though the energy density of the gauge sector in the appearance of the background fields is higher than that of no background fields, since we have massless fermions, it may consist a ground state as a total in the presence of fermions.

We then constructed the model based on the SO(12) gauge theory with fermions which lies in a 32 representation of SO(12). We showed that $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I$ gauge symmetry is remained in four-dimensions, and that the SM Higgs-doublet is obtained without appearance of extra scalar contents. One generation of SM fermions are successfully obtained by introducing two types of fermions which have different parity assignment under $\theta \to \pi - \theta$. We also analyzed the Higgs sector that are obtained from gauge sector of the six-dimensional gauge theory. The electroweak symmetry breaking is then realized and the Higgs mass value is predicted.

To make our model more realistic, there are several challenges such as eliminating the extra U(1) symmetries and constructing the realistic Yukawa couplings, which are the same as other gauge-Higgs unification models. We, however, can get not only appropriate one-generation fermion fields but also Kaluza-Klein modes. This suggests that we obtain the dark matter candidate in our model.

For UED model, we proposed a approach which is defined on the six-dimensional spacetime whose extra space is the two-sphere orbifold S^2/Z_2 and analyzed the mass spectrum of Kaluza-Klein particles in the model.

We first specified our model in six-dimensional spacetime $M^4 \times S^2/Z_2$. The orbifold S^2/Z_2 is clarified by operating the Z_2 action on S^2 and the feature of the gauge theory on $M^4 \times S^2/Z_2$ is summarized in section 2. There, we mentioned that a massless mode of fermion is obtained if we introduce a background gauge field to cancel the mass of fermions which arise from the spin connection for the positive curvature of S^2. The Lagrangian of our model is then constructed by specifying the gauge symmetry, the field contents and the boundary conditions for each fields. The gauge symmetry is chosen as the SM gauge symmetry with the extra $U(1)_X$ symmetry, which is $SU(3)\times SU(2)\times U(1)_Y \times U(1)_X$, where the extra $U(1)_X$ symmetry is introduced so that all the fermions in our model have the massless modes corresponding to the SM fermions. We then introduced the field contents where the zero modes of the fields correspond to the SM field contents under their boundary conditions. Thus the combinations of chirality and boundary conditions for each fermions are determined to give the zero mode which correspond to the SM fermions as summarized in Table. 26.

We then analyzed the KK mode expansion for fermions, gauge fields, and Higgs field. The fermions are expanded in terms of the linear combinations of the eigenfunctions of the extra-space Dirac operator which contains background gauge field. Those linear combinations are defined to satisfy the boundary conditions of the fermions. After the mode expansion and integrating S^2 coordinates, we obtained the kinetic term and the KK mass term of fermions in four-dimensional spacetime, and confirmed that each fermions have the chiral massless mode. The mass spectrum of the fermion KK mode is then obtained as in Eqs. (4.69),(4.70) and (4.71). The gauge fields are expanded in terms of the linear combinations of the spherical harmonics which satisfy the boundary condition. We obtained the quadratic terms of the gauge fields in four-dimensional spacetime as in Eq. (4.86), after gauge fixing and integration of the S^2 coordinates. We then analyzed the mass spectrum of the gauge fields and summarized the feature of the mass spectrum in Eq. (4.90) and in the sentences below Eq. (4.90). There we noted that the $U(1)_X$ symmetry is anomalous and is broken at the quantum level, so that its gauge boson should be heavy. We thus expect the $U(1)_X$ gauge boson and its KK modes are decoupled from the low energy sector of our model. The Higgs field is also expanded in terms of the linear combinations of the spherical harmonics which satisfy the boundary condition. The mass spectrum of the Higgs KK modes is specified in Eq. (4.95) and sentences below Eq. (4.95). These mass spectrum are summarized in Table 27

We also investigated the KK-parity in our model and found that the KK-parity is defined as $(-1)^m$. This KK-parity is conserved as a result of Z_2' symmetry of the Lagrangian and when the stability of the lightest KK particle with m=odd is confirmed. We, therefore, found that the lightest KK photon γ_μ^{11}, which is the promising candidate of the lightest KK particle, is stable and can be a good candidate of the dark matter. We must take into account the quantum correction to the masses of the KK particles in order to clarify the lightest KK particle and the dark matter candidate. Furthermore we need to derive all interaction terms. However this is beyond the scope of this paper and we leave this for future work. It would be very interesting to study experimental signatures of our model and compare them with other extra dimensional model predictions.

We, furthermore, investigated the main Higgs production process via gluon fusion at LHC in our 6D UED model compactified on S^2/Z_2. Higgs production cross section in our model has 30 (10)% enhancement comparing with the prediction of the SM (minimal UED model on S^1/Z_2) for the compactification scale of order 1 TeV. We expect that our remarkable prediction will be soon verified by a forthcoming LHC experiment.

We, therefore, found that the GHU model and the UED model with S^2 extra space are very interesting in phenomenological point of view and it is very important to these model further.

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A Geometrical quantity on S^2

We summarize the geometrical quantity on S^2 such as vielveins e^a_α, killing vectors ξ^α_a and spin connection R^{ab}_α. The vielveins are expressed as

$$\begin{aligned} e^1_\theta &= R, \\ e^2_\phi &= R\sin\theta, \\ e^1_\phi &= e^2_\theta = 0. \end{aligned} \tag{A.1}$$

The non-zero components of spin connection are

$$R^{12}_\phi = -R^{21}_\phi = -\cos\theta. \tag{A.2}$$

B Summary of the Jacobi polynomial

We summarize the feature of the Jacobi polynomial $P^{\alpha,\beta}_n(z)$ $(\alpha, \beta > -1)$ [39]. The Jacobi polynomial $P^{\alpha,\beta}_n(z)$ obey the differential equation of the form

$$\sigma(y)P'' + \tau(y)P' + \lambda_n P = \frac{1}{\rho^{(\alpha,\beta)}(y)}\frac{d}{dy}[\sigma(y)\rho^{(\alpha,\beta)}(y)P'] + \lambda_n P = 0, \tag{B.1}$$

where $\rho^{\alpha,\beta}$, $\sigma(z)$, $\tau(z)$ and λ_n are given by

$$\rho^{(\alpha,\beta)}(z) = (1-z)^\alpha(1+z)^\beta, \tag{B.2}$$

$$\sigma(y) = 1 - z^2, \tag{B.3}$$

$$\tau(z) = \beta - \alpha - (\alpha+\beta+2)z, \tag{B.4}$$

$$\lambda_n = n(n+\alpha+\beta+1), \tag{B.5}$$

where n is a non-negative integer.

The explicit form of the Jacobi polynomials are given by both the differential and integral of Rodrigues' formulas

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}\frac{1}{\rho^{(\alpha,\beta)}(z)}\frac{d^n}{dz^n}[\sigma(z)^n\rho^{(\alpha,\beta)}(z)], \tag{B.6}$$

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}\frac{1}{\rho^{(\alpha,\beta)}(z)}\frac{n!}{2\pi i}\oint\frac{\sigma^n(z)\rho^{(\alpha,\beta)}(w)}{(w-z)^{n+1}}dw, \tag{B.7}$$

where the contour of complex integration in the second equation must encircle the point z. The orthogonal relation of the Jacobi polynomials is given as

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(z)\rho^{(\alpha,\beta)}(z)dz = \delta_{mn}\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}. \tag{B.8}$$

The recurrence equations for the Jacobi polynomials are summarized such that

$$\frac{d}{dy}P_n^{(\alpha,\beta)}(y) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(y) \qquad (n>0), \tag{B.9}$$

$$(\frac{\alpha}{1-y}-\frac{d}{dy})P_n^{(\alpha,\beta)}(y) = \frac{n+\alpha}{1-y}P_n^{(\alpha-1,\beta+1)}(y), \tag{B.10}$$

$$(\frac{\beta}{1+y}+\frac{d}{dy})P_n^{(\alpha,\beta)}(y) = \frac{n+\beta}{1+y}P_n^{(\alpha+1,\beta-1)}(y). \tag{B.11}$$

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