NOTES ON SUPER MATH

MOSTLY FOLLOWING

BERNSTEIN–DELIGNE–MORGAN

Contents

Date: Long Long Time Ago in a Land Far Away .

The main source. This is the text [BDM] :

- Deligne, Pierre; Morgan, John W. *Notes on supersymmetry* (following Joseph Bernstein).

References

[BDM] Deligne, Pierre; Morgan, John W. *Notes on supersymmetry* (following Joseph Bernstein). Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 41–97, Amer. Math. Soc., Providence, RI, 1999.

1. **Linear algebra**

The super math is the mathematics that obeys certain sign rule. One can introduce super versions of standard objects by writing formulas enriched with some signs and then claiming that these formulas work well. A more systematic approach is the description of the sign rule as an additional structure – a *braiding* – on the tensor category of $\mathbb{Z}_2$-graded vector spaces. A braiding on a tensor category $\mathcal{T}$ provides a notion of *commutative algebras* in the setting of $\mathcal{T}$, as a consequence one obtains notions of $\mathcal{T}$-versions of geometric objects, Lie groups etc., i.e., the standard bag of mathematical ideas. The above braiding (*the super braiding*) gives the *super math.* We will survey the effect of the *super braiding* on linear algebra, geometry (super manifolds) and analysis (integration on super manifolds).

Some unusual aspects. Some concepts develop unexpected subtleties. For instance on super manifolds there are three objects that generalize various aspects of differential forms: (super) differential forms, densities, integral forms.

The odd part contributes in the direction opposite from what one expects. This is familiar in the case of super dimension which is just the Euler characteristic: $even - odd$. However as this principle propagates through more complicated objects it gets more surprising. We will see this when we study integration on super manifolds.

Applications.

(1) Some non-commutative situations are commutative from the super point of view.
(2) Some standard constructions have a more "set-theoretic" interpretation in the super setting :
   (a) The differential forms on a manifold M can be viewed as functions on a super manifold which is the moduli of maps from the super point $\mathbb{A}^{0|1}$ to M.
   (b) The differential forms on the loop space $\Lambda(M)$ are functions on the super manifold which is the moduli of maps from the super circle $S^{1|1}$ to M. This explains the non-trivial structure of a vertex algebra on these differential forms, for instance the vector fields on $S^{1|1}$ give the (a priori sophisticated) structure of $N = 2$ topological vertex algebra.
   (c) Complexes in homological algebra are representations of a certain super group with the underlying manifold $S^{1|1}$.
(3) Supersymmetry: this is a symmetry of a mathematical object which mixes even and odd components. These are more difficult to spot without the super point

of view. For instance integrals with supersymmetry of will be easier to calculate. (Our example will be the baby case of Witten's approach to Morse theory.)

(4) Fermions: elementary particles break into bosons and fermions depending on whether they obey usual mathematics or require super-mathematics.

Development. The underlying structure of this theory is the category $s\mathcal{V}ect_{\Bbbk}$ of super vector spaces over a basic field[(1)] (or ring) $\Bbbk$, with a structure of a tensor category with a super braiding. The next level is the linear algebra in $s\mathcal{V}ect_{\Bbbk}$, it has two notions of liner operators: (i) the *inner* Hom, i.e., $\underline{\text{Hom}}(U,V)$ is a super vector space, it consists of all $\Bbbk$-liner operators, (i) the *categorical* Hom, i.e., $\text{Hom}(U,V) = \text{Hom}_{s\mathcal{V}ect_{\Bbbk}}(U,V)$ is an ordinary vector space, it consists of all $\Bbbk$-liner operators that preserve parity,

1.1. **Super-math as the math in the braided tensor category of super vector spaces.** A *super vector space* is simply a vector space graded by $\mathbb{Z}_2 = \{0,1\}$: $V = V_0 \oplus V_1$, i.e., a representation of the group[(2)] $\{\pm 1\}$. Therefore, μ_2 acts on any category of super objects.[3]

1.1.1. *Parity.* We will say that vectors $v \in V_p$ are *homogeneous of parity p* and we will denote the parity of v by p_v or $\overline{v}$. Another way to keep track of parity is the "fermionic sign" $(-1)^F$. On each super vector space this is the linear operator which is $+1$ on V_0 and -1 on V_1. Here, "F" for fermionic, will sometimes be used to indicate the super versions of standard constructions.

1.1.2. *Sign Rule and super braiding.* The meaning of "super" is that all calculations with super vector spaces have to obey the

"*Sign Rule*: when a passes b, the sign $(-1)^{p_a p_b}$ appears.

More precisely (and more formally) the calculations are done in the tensor category $\mathcal{V}ect^s_{\Bbbk}$ of super vector spaces over $\Bbbk$, enriched by a certain structure called "braiding". The braiding on a tensor category is a (consistent) prescription of what we mean by a natural identification of $V\otimes W$ and $W\otimes V$, i.e., a commutativity isomorphism ("commutativity constraint) $c_{VW}:\ V\otimes W \to W\otimes V$, functorial in V and W.

The braiding in the "ordinary" math is $c_{V,W}(v\otimes w) = w\otimes v$ on $VVect_{\Bbbk}$. The super math is based on the braiding in $(\mathcal{V}ect^s_{\Bbbk}, \underset{\Bbbk}{\otimes})$ given by the sign rule

$$c_{VW}:\ V\otimes W \to W\otimes V, \quad v\otimes w \mapsto (-1)^{p_v p_w} w\otimes v,$$

which is the formalization of the above *Sign Rule.*

[1]Here $\Bbbk$ is even, i.e., there is no parity grading in "numbers".

[2]To cover the case of the arbitrary ground ring $\Bbbk$, the correct group is the group scheme μ_2 of second roots of unity, defined over integers. The difference matters only when 2 is not invertible in $\Bbbk$.

[3]Moreover, μ_2 acts identically on objects, so it lies in the center of that category.

1.1.3. *Braiding gives geometry.* While one can define associative algebras in any tensor category, in order to have a notion of commutative algebras the tensor category needs a braiding. In this way, each braiding gives one version of the notion of commutative algebras, hence one version of standard mathematics.

1.1.4. *General arguments and calculations in coordinates.* While the calculations in a braided tensor category $\mathcal{V}ect^s_{\Bbbk}$ are "natural", and of a general nature (arguments valid for any braiding), working in coordinates will involve applications of this specific commutativity constraint and also careful sign conventions.

1.1.5. *Unordered tensor products in braided tensor categories.* (i) In any tensor category a tensor product of a finite ordered family $\otimes_1^n V_{i_k} \stackrel{\text{def}}{=} V_{i_1}\otimes\cdots\otimes V_{i_n}$ is defined canonically, and the associativity constraint can be viewed as identity.

(ii) In a braided tensor category, the tensor product is also defined for unordered families, here $\otimes_{i\in I} V_i$ is defined as the projective limit of all tensor products given by a choice of order (the consistency property of commutativity constraints ensures that this is a projective system).

1.1.6. *Special property of the super braiding.* The super braiding is very special – it is self-inverse, i.e., $c_{VW} = {c_{WV}}^{-1}$. In particular, $c^2_{VV} = 1$.

1.2. **The effect of the sign rule on linear algebra over the base ring $\Bbbk$.** Some mathematical constructions extend to any tensor category, for instance the notion of an algebra. In our case it gives the following notion: a super $\Bbbk$-algebra is a $\Bbbk$-algebra A with a compatible super structure, i.e., $A_p{\cdot}A_q\subseteq A_{p+q}$.

1.2.1. *Commutativity.* Mathematical constructions related to commutativity require the tensor category to have a braiding.

The *commutator* in an algebra A in a braided category is obtained by applying the multiplication to $a\otimes b - c_{A,A}(b\otimes a)$. So, the (super)commutator in a super-algebra A is

$$[a,b]_F \stackrel{\text{def}}{=} ab - (-1)^{p_a p_b}\, ba.$$

So we say that elements a and b of a super-algebra A super-commute if

$$ab = (-1)^{p_a p_b}\, ba.$$

An abstract reason for usefulness of the notion of super-commutativity is that it allows one to think of some non-commutative situations as if they were commutative, and this in particular gives notions of a super-commutative algebra A, i.e., of a super-space X with the super-commutative algebra of functions $\mathcal{O}(X) = A$.

1.2.2. *Functors between vector spaces, super vector spaces and graded vector spaces.*

- *Inclusion* $\mathcal{V}ect_{\Bbbk}\subseteq\mathcal{V}ect_{\Bbbk}^{s}$. This is an inclusion of a full braided tensor subcategory.
- *Forgetful functor* $\mathcal{V}ect_{\Bbbk}^{s}\xrightarrow{\mathcal{F}}\mathcal{V}ect_{\Bbbk}$. It forgets the super structure, i.e., the $\mathbb{Z}_2$-grading. It is a functor between tensor categories but not the braided tensor categories.
- *Projection to the even part* $\mathcal{V}ect_{\Bbbk}^{s}\xrightarrow{-_0}\mathcal{V}ect_{\Bbbk}$. This is an exact functor but it does not preserve the tensor category structure.
- *Forgetful functors* $\mathcal{V}ect_{\Bbbk}^{\bullet}\xrightarrow{\boldsymbol{s}}\mathcal{V}ect_{\Bbbk}^{s}\xrightarrow{\mathcal{F}}\mathcal{V}ect_{\Bbbk}$ Let $\mathcal{V}ect_{\Bbbk}^{\bullet}$ be the graded vector spaces, i.e., vector spaces V with a $\mathbb{Z}$-decomposition $V=\oplus_{n\in\mathbb{Z}}V_n$. Any graded vector space $V=\oplus_{n\in\mathbb{Z}}V^n$ defines a super vector space
$$\boldsymbol{s}(V)\overset{\text{def}}{=}V \text{ with the decomposition } V=V_0\oplus V_1 \text{ for } V_0\overset{\text{def}}{=}\oplus_{n\ even}V^n \text{ and } V_1\overset{\text{def}}{=}\oplus_{n\ odd}V^n.$$
The standard braiding on $\mathcal{V}ect_{\Bbbk}^{\bullet}$ is the super grading!

We denote by $\underline{1}$ the unit object $\Bbbk$ in $\mathcal{V}ect_{\Bbbk}^{s}$.

1.2.3. *Some notions in a braided tensor category.* In a braided tensor category we automatically have the notions of

- standard classes of algebras: (associative, commutative, unital, Lie),
- standard operations on algebras (tensor product of algebras, opposite algebra),
- modules over algebras,
- linear algebra of such modules,
- symmetric and exterior algebras of modules over commutative algebras
- etc.

1.2.4. *Parity change of super vector spaces.* Operation $\Pi:\ \mathcal{V}ect_{\Bbbk}^{s}\to\mathcal{V}ect_{\Bbbk}^{s}$ is defined by $(\Pi V)_p\overset{\text{def}}{=}V_{1-p}$. So, $\Pi(V)$ can be canonically identified with V as a vector space but the parities have changed.

We will also denote by Π the one dimensional *odd* vector space $\Bbbk\pi$ with a chosen basis π, then the functor Π is canonically identified with the left tensoring functor
$$\Pi V\ \cong\ \Pi\otimes V.$$
Observe that we have made a *choice* of tensoring with Π on the *left*.

1.2.5. *The inner* Hom *and duality in super vector spaces.* There are two related and easily confused concepts.

(1) Hom *for the category of super vector spaces.* For two super vector spaces V and W,
$$\mathrm{Hom}_{\Bbbk}(V,W)\overset{\text{def}}{=}\mathrm{Hom}_{super\ \Bbbk-modules}(V,V)$$

denotes all maps of super vector spaces, i.e., $\Bbbk$-linear maps which preserve the super structure (i.e., the parity). So this is an *ordinary*vector space, i.e., an *even* vector space.

(2) *Inner* Hom *in the category of super vector spaces.* The vector space of all $\Bbbk$-linear maps $\mathrm{Hom}_{\Bbbk}[\mathcal{F}(V),\mathcal{F}(W)]$ has a canonical structure of a super vector space which we denote $\underline{\mathrm{Hom}}_{\Bbbk}(V,W)$.

The relation is given by composing with the projection to the even part: $\mathrm{Hom} = -_0 \circ \underline{\mathrm{Hom}}$, i.e., the even part of the inner Hom consists of maps that preserve parity

$$\underline{\mathrm{Hom}}_{\Bbbk}(V,W)_0 = \mathrm{Hom}_{\Bbbk}[V,W],$$

and the odd part is the maps that reverse the parity.

The *dual super vector space* is defined in terms of inner Hom

$$\check{V} \stackrel{\text{def}}{=} \underline{\mathrm{Hom}}_{\Bbbk}(V,\Bbbk).$$

1.2.6. *Some canonical maps.* The *convention* we use is that *linear operators act on the left* , i.e.,

there is a canonical evaluation map of super vector spaces
$$\underline{\mathrm{Hom}}(U,V)\otimes U \to V, \quad A\otimes u \mapsto Au.$$

(1) <u>The pairing with linear functionals.</u> Applying this to linear functionals yields for each super vector space V, its *evaluation map* (or its *canonical pairing*), which is a map of super vector spaces

$$ev_V : \check{V}\otimes V \to \underline{1}, \quad \langle \omega, v\rangle \stackrel{\text{def}}{=} ev_V(\omega\otimes v) \stackrel{\text{def}}{=} \omega(v).$$

(2) The map $(V\otimes\check{V})\otimes V = V\otimes(\check{V}\otimes V) \to V\otimes\underline{1} \cong V$ gives a map

$$\check{V}\otimes V \to \underline{\mathrm{End}}(V).$$

(3) The map $V\otimes\check{V} \xrightarrow{c_{V,\check{V}}} \check{V}\otimes V \xrightarrow{ev_V} V\otimes\underline{1} \cong V$ gives the *biduality map* for V

$$\iota : V \to \check{\check{V}}, \quad \langle \iota_v, \omega\rangle \stackrel{\text{def}}{=} \langle v, \omega\rangle = (-1)^{p_\omega p_v} \langle \omega, v\rangle, \quad v\in V,\ \omega\in\check{V}.$$

Remarks. (1) If $V\otimes\check{V} \to \underline{\mathrm{Hom}}_{\Bbbk}(V,V)$ is an isomorphism the coevaluation map can be interpreted as a diagonal $\delta_V : \underline{1} \to V\otimes\check{V}$.

(2) We can define the canonical *wrong way maps* such as $\check{V}\otimes_{\Bbbk} V \to \underline{\mathrm{Hom}}_{\Bbbk}(V,V)$, by inserting braiding in appropriate places.

1.2.7. *The trace and the dimension.* We will see in 1.6.1 that the super-trace of a linear operator $T: V\to V$ can be calculated defined using its block decomposition

$$\mathrm{str}_F(T) = \mathrm{tr}(T_{00}) - \mathrm{tr}(T_{11}).$$

In particular, the super-dimension (fermionic dimension) is $s\dim_F(V) = \dim(V_0) - \dim(V_1)$.

1.2.8. *Tensor category of graded super vector spaces.* A graded super vector space is a graded vector space $V = \oplus_{\mathbb{Z}} V_p$ with a super structure on each V_p. There seem to be two (equivalent) ways to choose the braiding on the tensor category of super graded vector spaces.

- Bernstein's convention uses the sign given by the *total parity* $\deg(v) + p_v$

  $$c_{VW}(v\otimes w) = (-1)^{(deg(v)+p_v)(deg(w)+p_w)} w\otimes v.$$

  Then, taking the total parity $(deg(v) + p_v$ is a functor $\boldsymbol{s}$ into the tensor category of super-vector spaces.
- Deligne's convention is that commutativity constraint given by the sign which is the *product of the signs for the degree and for the parity*

  $$c_{VW}(v\otimes w) \overset{\text{def}}{=} (-1)^{deg(v)deg(w)}\cdot(-1)^{p_v p_w}\ w\otimes v = (-1)^{deg(v)deg(w)+p_v p_w}\ w\otimes v,$$

  this means that we combine (multiply) the commutativity constraints due to the $\mathbb{Z}$-grading and the $\mathbb{Z}_2$-grading.

Remark. The above two choices of braidings on the same tensor category are equivalent by an involution ι on the tensor category of graded super vector spaces. ι is given by changing the $\mathbb{Z}_2$-degree by adding the $\mathbb{Z}$-degree. (The tensoring constraint for ι is $\iota_{V,W} = (-1)^{\deg(v)\cdot p_q} :\ \iota(V\otimes W)\xrightarrow{\cong} \iota(V)\otimes\iota(W)$.)

1.2.9. *Symmetric algebras $S(V)$.* For a super vector space V, $S(V)$ is defined as the super commutative algebra freely generated by V. If V is even we are imposing the ordinary commutativity $uv = vu$ and $S(V) = \Bbbk[v_1, ..., v_n]$ for any basis of V. If V is odd, we are imposing the anti-commutativity $uv = -vu$ and therefore $\mathcal{F}[S(V)] = \overset{\bullet}{\wedge}\mathcal{F}(V)$, i.e., if we forget parity this is an ordinary exterior algebra. A more precise formulation is in the odd case is

$$S(V) = \boldsymbol{s}[\overset{\bullet}{\wedge}\mathcal{F}(V)].$$

1.3. **Super algebras.** Let A be a super algebra. The constructions bellow are not ad hoc, there is no smart choice. On one hand these are special case of definitions in general braided categories, and on the other hand they are also forced on us by desire of compatibility of super vector spaces with ordinary vector spaces (see 1.7).

1.3.1. *The opposite algebra A^o.* A^o is given by the following multiplication structure on the super vector spaces A

$$a \underset{A^o}{\cdot} b \ \stackrel{\text{def}}{=}\ (-1)^{p_a p_b}\, b{\cdot}a, \quad a,b\in A^o = A.$$

There is an equivalence of categories $M \mapsto M^o$ of left A-modules and right A^o-modules by $M^0 = m$ as a super vector space and

$$m{\cdot}a \stackrel{\text{def}}{=} (-1)^{p_a p_m}\ a{\cdot}m.$$

A is super commutative iff $A^o = A$. In particular, for super-commutative A left and right modules are the same in the sense that one has an equivalence as above $\mathfrak{m}^l(A) \ni M \mapsto M^o \in \mathfrak{m}^r(A^o) = \mathfrak{m}^r(A)$.

1.3.2. *Tensor product of algebras.* The algebra structure on $A\otimes B$ is

$$(a'\otimes b)(a''\otimes b'') \stackrel{\text{def}}{=} (-1)^{p_{b'} p_{a''}}\ a'a''\otimes b'b''.$$

Algebra structure on a tensor product of algebra requires braiding so that multiplication can be defined by

$$(A\otimes B)\otimes(A\otimes B) =\ A\otimes B\otimes A\otimes B \xrightarrow[\cong]{1\otimes c_{B,A}\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow[\cong]{m_A\otimes m_B} A\otimes B.$$

1.3.3. *Derivatives.* A linear map $\partial:\ A\to A$ is said to be a *(left) derivative of A of parity p* if

$$\partial(ab) =\ (\partial a)b +\ (-1)^{p\cdot p_a}\ a(\partial b).$$

There is also an (equivalent) notion of right derivatives, but we follow the convention that operators act on the left of vectors.

A consequence of this convention, we will (later) write the pairing of a vector field ξ and a 1-form ω (a differential), in the form $\langle \xi,\omega\rangle$, so that it agrees with the left action of vector fields on functions: $\langle \xi, df\rangle =\ \xi(f)$.

1.3.4. *Parity change on modules for a super algebra A.* For a left A-module M, super vector space ΠM has a canonical A-action

$$a \underset{\Pi M}{\cdot} m =\ (-1)^{p_a}\ am.$$

For a right A-module M, the actions on M and ΠM are the same. The reason seems to be that the parity change is viewed as a left tensoring $\Pi(M) =\ \Pi \underset{\Bbbk}{\otimes} M$.

1.4. **Lie algebras and their enveloping algebras.**

1.4.1. *Lie algebras.*

Remark. Non-triviality of $[x,x]=0$ In a super Lie algebra $\mathfrak{g}$ let us specialize the relation $[x,y]+(-1)^{p_x p_y}[y,x]=0$ to $x=y$. If x is even, it says that $[x,x]=0$, however if x is odd it does not say anything. So, condition $[x,x]=0$ (i.e., $\Bbbk{\cdot}x$ is an abelian subalgebra), is non-trivial for odd x.

1.4.2. *Enveloping algebras of Lie algebras.*

Theorem. [Poincare-Birkhoff-Witt]

Proof. The Poincare-Birkhoff-Witt theorem is proved in any tensor category with a $\mathbb{Q}$-structure, by constructing explicitly the enveloping algebra multiplication $*$ on the symmetric algebra $S^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Its relation to the standard product in $S^\bullet(\mathfrak{g})$ is

$$x_1\cdots x_n = \int_{S_n} d\sigma\ x_{\sigma 1} * \cdots * x_{\sigma n} = \frac{1}{|S_n|}\sum_{\sigma\in S_n} x_{\sigma 1} * \cdots * x_{\sigma n}.$$

One defines multiplication $*$ inductively, for $x_i, y_j \in \mathfrak{g}$

$$(x_1\cdots x_p) * (y_1\cdots y_q) \stackrel{\text{def}}{=} x_1 * (x_2 * (\cdots (x_p * (y_1\cdots y_q))\cdots)$$

and

$$x * (y_1\cdots y_q) \stackrel{\text{def}}{=} x y_1\cdots y_q + \int_{S_{q+1}} \sum_{i=1}^{q}\ (q-i+1)\ y_{\sigma 1} * \cdots * [x, y_{\sigma i}] * \cdot * y_{\sigma q}.$$

1.5. **Linear algebra on free modules over super-algebras (inner Hom, free modules and matrices).** Here A is a super-algebra. The subtle parts are only done when A is super-commutative.

1.5.1. *The inner* Hom *for* A*-modules.* Let $\mathcal{F}$ denote forgetting the super structure. For A-modules M,N, the vector space of $\mathcal{F}(A)$-linear maps $\mathrm{Hom}_{\mathcal{F}(A)}[\mathcal{F}M,\mathcal{F}N]$ has a canonical super-structure $\underline{\mathrm{Hom}}_A(M,N)$, such that the even part $\underline{\mathrm{Hom}}_A(M,N)_0$ is the space of maps of A-modules $\mathrm{Hom}_A(M,N)$ (i.e., the even maps in $\underline{\mathrm{Hom}}_A(M,N)$ are those that preserve the super structure). In particular, one has the duality operation on A-modules

$$\check{M} \stackrel{\text{def}}{=} \underline{\mathrm{Hom}}_A(M,A) \quad \text{hence} \quad \mathcal{F}(\check{M}) \ = \ \mathrm{Hom}_{\mathcal{F}(A)}[\mathcal{F}M,\mathcal{F}A].$$

1.5.2. *Free modules.* By a basis of a module over a super-algebra A one means a *homogeneous* basis. Then a free module M over a super-algebra A means a module that has a basis. The *standard free left* A-modules are

$$A^{p|q} \ \stackrel{\text{def}}{=}\ \oplus_1^{p+q}\ Ae_i,$$

with e_i even precisely for $i\le p$.

One has $A^{p|q}\cong\ A^p\oplus\Pi(A^q)$, etc.

1.5.3. *Linear operators.* We will be only interested in supercommutative A, but we take a moment for the general case. In the general case we view A as a right A-module so that $A^{p|q} \stackrel{\text{def}}{=} \oplus_1^{p+q} Ae_i$ is a right A-module (and therefore a left A^o-module). This is convenient because the *inner* Hom *super algebra* $\underline{\text{End}}_A(A^{p|q})$, acts on $A^{p|q}$ on the left.

If A happens to be super-commutative, then the above left action of A^o on $A^{p|q}$ can be viewed as a n action of $A = A^o$, and this is the original action of A on $A^{p|q}$ viewed as a left A-module. So, the above convention using right action gives in this case the standard construction of $\underline{\text{End}}_A(A^{p|q})$ for commutative algebras A.

1.5.4. *Coordinatization and matrices.* In general the Coordinatization of vectors in $A^{p|q}$ is by

$$x = \ x^i e_i \quad \text{with } x^i \in A^o.$$

If A is super commutative this is the ordinary A-Coordinatization.

The Coordinatization of operators uses the right A-action

$$Te_j \ = \ e_i T^i_j, \quad T^i_j \in A.$$

Bloc form of matrices. Since each row and column in a matrix has a parity, the positions in a matrix come with a pair of signs[4]

$$(T^i_j) = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix}.$$

1.6. **Berezinian (super determinant) of free modules and automorphisms of free modules.** Let A be a super-commutative algebra. We consider the notions of trace and its nonlinear analogue, the determinant.

Super trace The notion of trace in the super setting is given by general principles (the braiding of the tensor structure). It applies to the *inner endomorphisms* of a free A-module M of finite rank, and yields an even map of A-modules

$$Tr: \ \underline{\text{Hom}}_A(M,M) \longrightarrow A.$$

In particular it applies to endomorphisms $\text{Hom}_A(M,M) = \underline{\text{Hom}}_A(M,M)_0$ and gives a map of A_0-modules

$$\text{Hom}_A(M,M) = \ \underline{\text{Hom}}_A(M,M)_0 \longrightarrow A_0.$$

Super determinant General principles also determine what the notion of super determinant (called Berezinian) should be. One requires that

- $(\star 1)$ $\det(e^T) = \ e^{Tr(T)}$ when e^T makes sense,
- $(\star 2)$ det is in some sense algebraic (map of algebraic groups).

[4]This should not be confused with parity of matrix coefficients – all of them can be arbitrary elements of A.

We will construct super determinant using its linear algebra characterization as the action of the operator on the top exterior power. The requisite notion of the "super top exterior power" will be constructed in a somewhat ad hoc way. Among all formulas that yield the top exterior power in classical mathematics we observe that one of these produces a rank one module even in the super case.[(5)]

One should notice that super-determinant is defined in a very <u>restrictive situation</u> – on automorphisms of free modules. So, there are two step restrictions (i) to $\mathrm{Hom}_A(M,M)$ ($=$ $\underline{\mathrm{Hom}}_A(M,M)$ rather then all of $\underline{\mathrm{Hom}}_A(M,M)$, and (ii) to only the invertible part $\mathrm{Aut}_A(M,M)$ of $\mathrm{Hom}_A(M,M)$. For instance if A is even then a matrix of $T\in\mathrm{Aut}_A(M,M)$ has $T_{+-}=0=T_{-+}$ while T_{++},T_{--} are some ordinary matrices with coefficients in A. Then the Berezian is given by

$$Ber(T) \;=\; \det(T_{++})\cdot\det(T_{--})^{-1}.$$

Here, M will be a free A-module, hence isomorphic to one of $A^{p|q}$.

1.6.1. *Super trace.* If $\check{M}\underset{A}{\otimes}M\rightarrow \underline{\mathrm{Hom}}_A(M,M)$ is an isomorphism, one has a categorical notion of the trace

$$Tr \overset{\text{def}}{=} [\underline{\mathrm{Hom}}_A(M,M) \overset{\cong}{\longleftarrow} \check{M}\underset{A}{\otimes}M \xrightarrow{ev_M} \underline{1} = A].$$

Lemma. (a) In terms of dual bases e_i,e^i of M and $\check{M}$, this reduces to $Tr(T)=\ (-1)^{p_i}\ \langle e^i,Te_i\rangle$.

(b) In terms of the matrix $(T^i_j)\overset{\overset{?}{\text{def}}}{=}\langle e^i,Te_i\rangle$ that we defined above using the *right* action of A, this is

$$Tr(T)=\ (-1)^{p_i}T^i_i=\ Tr(T_{++})-Tr(T_{--}).$$

(c) The trace of a commutator is still zero: $Tr[A,B]=0$, i.e., $Tr(AB)=(-1)^{p_Ap_B}\ Tr(BA)$.

Proof. (a) and (b) follow from

$$Tr[ev_{M,M}(m\check{\omega})] \;=\; (-1)^{\overline{\omega}\cdot\overline{m}}\cdot\langle\omega,m\rangle.$$

For this one recalls that the map $\check{M}\otimes M \rightarrow \underline{\mathrm{End}}_A(M)$ is a composition $\check{M}\otimes M \xrightarrow{c_{\check{M},M}} M\otimes\check{M} \xrightarrow{ev_{M,M}} \underline{\mathrm{End}}_A(M)$ (remark 1.2.6). So, the operator corresponding to $\omega\otimes m\in\check{M}\otimes M$ is $ev_{M,M}[c_{\check{M},M}\ \omega\otimes m]=\ (-1)^{\overline{\omega}\cdot\overline{m}}\cdot ev_{M,M}(\omega\otimes m)$, and according to the above definition, its trace is $\langle\omega,m\rangle$.

[5]In the end we get a computable formula and we can check the characterizing property ($\star$).

1.6.2. *Berezinian of a free module (= "top exterior power").* Classically, the determinant of a linear operator is its action on the top exterior power. However, for an odd line $L = \Bbbk\theta$, exterior algebra $\overset{\bullet}{\wedge} L = \oplus_0^\infty \Bbbk\theta^n$ has no top power. Instead we will use another classical formula for the top exterior power of a free A-module L. Let V be a vector space over $\Bbbk$, then[6]

$$(0\hookrightarrow V)^! \mathcal{O}_V \overset{\text{def}}{=} \text{Ext}^\bullet_{\mathcal{O}(V)}[\mathcal{O}_0,\ \mathcal{O}(V)] \ =\ \overset{top}{\wedge} V\ [-\dim(V)].$$

LHS is In the super setting this will be the definition of the RHS. So the LHS is the correct super analogue of the top exterior power, that gives the super version of the determinant.

Lemma. Let A be a commutative super-algebra and let $L = A^{p|q}$ be a free A-module.

(a) The graded object

$$Ber(L) \overset{\text{def}}{=} \text{Ext}^\bullet_{S^\bullet_A(\check{L})}[A,\ S^\bullet_A(\check{L})] = \text{Ext}^\bullet_{\mathcal{O}(L)}[\mathcal{O}(0),\ \mathcal{O}(L)]$$

is concentrated in the degree p where it is a free A-module of rank 1, and of parity the same as q.

(b) Berezian is canonically isomorphic to a line bundle

$$Ber(L) \cong\ \Lambda^{\text{top}} L_0 \otimes S^{\text{top}}(L_1)^*.$$

(c) To any ordered basis $e_1, ..., e_n$ one canonically associates a basis $[e_1, ..., e_n]$ of $Ber(L)$.[7]

Proof. (a) **(0)** Any ordered decomposition $L = L_1 \oplus L_2$ induces $S(L_1^*)\otimes_A S(L_2^*) \xrightarrow{\cong} S(L^*)$, and then $Ber(L_1) \overset{L}{\otimes}_A Ber(L_2) \xrightarrow{\cong} Ber(L)$.

(1) *The case* $p|q = 1|0$. If $L = Ae$ then the $S^\bullet_A(\check{L})$-module A (via the augmentation), has a free Koszul resolution[8] $S_A(\check{L}) \underset{A}{\otimes} L^* \xrightarrow{s\otimes e^* \mapsto se^*} S_A(\check{L}) \to A \to 0$, hence

$$\text{Ext}^\bullet_{S^\bullet_A(\check{L})}[A,\ S^\bullet_A(\check{L})] \ =\ H^\bullet\ \text{Hom}[S_A(\check{L}) \underset{A}{\otimes} Ae^* \xrightarrow{s\otimes e^* \mapsto se^*} S_A(\check{L}),\ S_A(\check{L})]$$

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[6]From the point of view of algebraic geometry, Berezian is by definition the "relative dualizing sheaf" for $0\hookrightarrow V$. Its computation above is essentially the computation of the dualizing sheaf on V since $\omega_V \overset{\text{def}}{=} (V \to pt)^! \Bbbk$ has form $\omega_V = \mathcal{O}_V \otimes_\Bbbk \Omega$ and $\Bbbk = \omega_0 = (0\hookrightarrow V)^! \omega_V = (0\hookrightarrow V)^! (\mathcal{O}_V \otimes_\Bbbk \Omega) = \Omega \otimes_\Bbbk \overset{top}{\wedge} V\ [-\dim(V)]$ gives $\Omega = \overset{top}{\wedge} V^*\ [\dim(V)]$.

[7]The coming calculation of the Berezian determinant can be bi viewed as a description of the functoriality of $[e_1, ..., e_n]$ in $e_1, ..., e_n$. Roughly, even ones are covariant and odd ones contravariant, so we may write $[e_1, ..., e_p; e_{p+1}, ..., e_{p+q}]$

[8]More generally, in the *case* $p|q = p|0$, the resolution is

$$S^\bullet_A(\check{L}) \underset{A}{\otimes} \overset{p}{\underset{A}{\wedge}} \check{L} \to S^\bullet_A(\check{L}) \underset{A}{\otimes} \overset{p-1}{\underset{A}{\wedge}} \check{L} \to \cdots \to S^\bullet_A(\check{L}) \underset{A}{\otimes} \overset{1}{\underset{A}{\wedge}} \check{L} \to\ A \to 0.$$

$$= H^\bullet[\, S_A^\bullet(\check{L}) \xrightarrow{s\mapsto\ se^*\otimes e} S_A^\bullet(\check{L})\underset{A}{\otimes}Ae] = Ae\,[-1] = \overset{top}{\underset{A}{\wedge}}L\,[-1].$$

(2) *The case* $p|q = 0|q$. If $L = A^{0|q} = \oplus_1^q A\theta_i$ for odd θ_i's, then $S_A^\bullet(\check{L}) = A[\theta_1^*,...,\theta_q^*]$ is a Frobenius algebra, hence it is an injective module over itself. So,

$$\mathrm{Ext}^\bullet_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})] = \mathrm{Hom}_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})] = \mathrm{Hom}_{A[\theta_1^*,...,\theta_q^*]}(A,\ A[\theta_1^*,...,\theta_q^*])$$
$$= \cap\ Ker(\theta_i^* :\ A[\theta_1^*,...,\theta_q^*] \longrightarrow A[\theta_1^*,...,\theta_q^*]) \ = \ A\theta_1^*\cdots\theta_q^*.$$

□

Now (b) and (c) follow. Let $L = \oplus Ae_i \oplus \oplus A\theta_j$ with e_i even and θ_j odd. The factorization from **(0)**,

$$Ber(L) \cong \otimes_i \ Ber(Ae_i) \otimes Ber(\oplus A\theta_j)$$

is canonical since Ae_i's are even. According to **(1)** and **(2)** it provides a basis of $Ber(L)$ of the form $e_1\otimes\cdots\otimes e_p\otimes\theta_1\cdots\theta_q^*$ which depends on the choice of order of e_i and θ_j's.

More precisely, it is a tensor product of basis $e_1\wedge\cdots\wedge e_p$ of $\Lambda^{\mathrm{top}}\ L_0$ (if one calculates $Ber(L_0)$ in one step, using the Koszul resolution of the $S_A(L_0^*)$-module A), and $\theta_1\cdots\theta_q^*$ of $A\theta_1\cdots\theta_q^* \subseteq S^{\mathrm{top}}(A\theta_1^*\oplus\cdots\oplus A\theta_q^*)$.

1.6.3. *Remarks.* (1) A basis $e_1,...,e_p,\theta_1,...,\theta_q$ of L gives a basis $[e_1\cdots e_p\theta_1^*\cdots\theta_q^*]$ of $Ber(L)$. This gives a more elementary approach to Berezinians – a free module with a basis $[e_1\cdots e_p\theta_1^*\cdots\theta_q^*]$, with a given rule on how this basis element transforms under a change of basis of L.

(2) We will remember that $Ber(L)$ is in degree p, i.e., we will consider it as an object of the category of graded A-modules.

Corollary. A short exact sequence of free modules $0 \to L' \to L \to L'' \to 0$, gives a canonical isomorphism

$$Ber(L) \cong Ber(L')\otimes Ber(L'').$$

1.6.4. *Berezinian of a map (= "determinant").* For an isomorphism $T : L \longrightarrow M$ of free A-modules,

$$Ber(T) \in \mathrm{Hom}[Ber(L), Ber(M)] \ \overset{\mathrm{def}}{=} \ \text{the induced isomorphism of Berezinians.}$$

In particular, for an automorphism T of a free A-module L,

$$Ber(T) \in (A_0)^* \text{ is the action of } T \text{ on } Ber(L).$$

Observe that in order for $T : L \to L$ to act on $Ber(L) = \mathrm{Ext}^\bullet_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})]$,

- T needs to be even and
- T needs to be invertible.

Lemma. (a) (Berezinian in matrix terms.) If $T : L \to M$ is invertible, then so are the diagonal componentsT_{++} and T_{--} of $T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix}$. Then

$$T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} 1 & T_{+-}T_{--}{}^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} T_{++} - T_{+-}\cdot(T_{--})^{-1}\cdot T_{-+} & 0 \\ T_{-+} & T_{--} \end{pmatrix},$$

and

$$Ber(T) = \det(T_{++} - T_{+-}\cdot(T_{--})^{-1})\cdot \det(T_{--})^{-1}.$$

(b) (Berezinian in terms of algebraic super groups.) Berezinian is characterized as a map of groups $GL(p|q, A) \to GL(1|0, A)$ which satisfies

$$Ber(1 + \varepsilon T) = 1 + \varepsilon T$$

when ε is even of square zero.

Remarks. (0) Clearly the appearance of inversion in $Ber \begin{pmatrix} T_{++} & 0 \\ 0 & T_{--} \end{pmatrix} = \det(T_{++})\cdot \det(T_{--})^{-1}$ corresponds to subtraction in the trace formula.

(1) If $A = \mathbb{k}$ is even then the automorphism group of $\mathbb{k}^{p|q}$ is $GL_p(\mathbb{k}) \times GL_q(\mathbb{k})$, and $Ber = det_p/det_q : \mathrm{Aut}(\mathbb{k}^{p|q}) \to \mathbb{k}^*$.

(2) As usual, one important application of super determinants (over non-trivial super commutative algebras) comes from (local) isomorphisms of super manifolds $F : M \to N$. Then $dF : \mathcal{T}_M \to \mathbb{F}^*\mathcal{T}_N$ is an isomorphism of locally free $\mathcal{O}_M$-modules, hence one has $Ber(dF) : Ber(\mathcal{T}_M) \to F^*Ber(\mathcal{T}_N)$. We will use this for change of variables in integrals.

1.7. **The automatic extension of algebraic concepts to the super setting ("Even rules").** This is Bernstein's? idea to reduce the sign calculations to super-commutative algebras. A super vector space V defines a functor from super-commutative algebras to ordinary vector spaces, $B \mapsto V(B) \stackrel{\text{def}}{=} (V_B)_0$ for $V_B \stackrel{\text{def}}{=} B \otimes V$.

For instance, a super *Lie algebra* structure on V is the same as an ordinary *Lie algebra* structure on the functor $B \mapsto V(B)$. What this means is that for any super commutative algebra B, on $V(B)$ one is given a Lie algebra structure over B_0, which is functorial (natural) in B. This principle allows one to calculate the defining relations for super Lie algebras (instead of inventing the signs in these relations).

The same works if we replace the *Lie algebra* by any other algebraic structure.

2. **Manifolds**

2.0.1. *Definition of super manifolds as ringed spaces.* By definition a super manifold is a topological space $|M|$ with the sheaf $\mathcal{O}_M$ of supercommutative algebras which is locally the same as some $(\mathbb{R}^{p,q}, C^\infty_{\mathbb{R}^p}\otimes\wedge^*\mathbb{R}^q)$. This is analogous to one of characterizations of smooth manifolds N as a topological space $|N|$ with the sheaf $\mathcal{O}_N$ of commutative algebras which is locally the same as some $(\mathbb{R}^p, C^\infty_{\mathbb{R}^p})$. A way to make this *less* abstract is provided by

2.0.2. *The set theoretical view on a supermanifold M as the functor* $\mathrm{Hom}(-,M)$ *of* **points** *of M.* In differential geometry we visualize a manifold N as a set of points and we would like to do the same in super geometry. The categorical way of thinking of a set of points of N is $\mathrm{Hom}_{manifolds}(\mathbb{R}^0, N)$. The same construction in super manifolds does not notice enough information as $\mathrm{Hom}_{super\ manifolds}(\mathbb{R}^0, M) = \ |M|$ is the same as the set of points of the underlying ordinary manifold. The problem is that it is not at all sufficient to probe a super manifold with an even object $\mathbb{R}^0$. It works better if one probes with all super points, the collection of sets $\mathrm{Hom}_{super\ manifolds}(\mathbb{R}^{0|q}, M),\ q\geq 0$, contains more information. In the end, as emphasized by Grothendieck, to restore the set theoretic point of view on M one should look not at a single set but at the functor

$$\mathrm{Hom}(-,M):\ SuperManifolds^o \longrightarrow Sets.$$

One says that $\mathrm{Hom}(X.M)$ is the *set of X-points of M.*

2.0.3. *Example.* As an example, the super manifold $GL(p|q)$ has underlying ordinary manifold $GL_p{\times}GL_q$, and the two are the same on the level of ordinary points. However for a super manifold X with a supercommutative algebra of functions $A=\mathcal{O}(X)$, the set of X-points of $GL(p|q)$ is more interesting – this is the set of automorphisms of the free A-module $A^{p|q}$.

2.1. **Super manifolds – definitions.** A super manifold M is a ringed topological space $(|M|,\mathcal{O}_M)$ locally isomorphic to some

$$\mathbb{R}^{p|q}\overset{\mathrm{def}}{=}(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1,...,\psi^q]),$$

where

$$\mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1,...,\psi^q])\ \overset{\mathrm{def}}{=}\ \oplus_{I=\{I_1<\cdot<I_k\}}\ \mathcal{C}^\infty_{\mathbb{R}^p}{\cdot}\psi^{I_1}\cdot\cdot\cdot\psi^{I_k}$$

is a super-commutative algebra freely generated over the smooth functions $\mathcal{C}^\infty_{\mathbb{R}^p}$, by odd generators $\psi^1,...,\psi^q$.

So, the functions on $\mathbb{R}^{p|q}$ are $\mathcal{C}^\infty_{\mathbb{R}^p}\underset{\Bbbk}{\otimes}\ S(W)$ for a q-dimensional odd vector space W.

2.1.1. *Maps.* A map of ringed spaces $f:(|M|,\mathcal{O}_M)\longrightarrow(|N|,\mathcal{O}_N)$ is by definition a pair of a map $|f|:|M|\longrightarrow|N|$ of topological spaces and a map of sheaves of rings $f^!:\mathcal{O}_M\longrightarrow|f|^*\mathcal{O}_N$.

2.1.2. *The associated ordinary manifold M_{red}.* The *reduced manifold of M* is the ringed space $M_{red} \overset{\text{def}}{=} (|M|, \mathcal{O}_M/J_M)$ where J_M is the ideal generated by odd functions. This is a $\mathcal{C}^\infty$-manifold since $\mathbb{R}^{p|q}_{red}$ is clearly $\mathbb{R}^p$. Notice that

(1) $\mathcal{O}_M \twoheadrightarrow \mathcal{O}_{M_{red}}$ corresponds to the canonical *closed inclusion* of super manifolds $M_{red} \hookrightarrow M$.
(2) M is not a fiber bundle over M_{red} since there is *no canonical map $M \to M_{red}$.*

Let me mention another (less useful) passage to ordinary world – where one simply keeps the even functions. This gives a (usually nonreduced) scheme $M_{even} \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_{M,0})$, which fits into

$$\mathcal{O}_{M_{even}} = (\mathcal{O}_M)_0 \subseteq \mathcal{O}_M \twoheadrightarrow \mathcal{O}_M/J_M, \quad \text{hence} \quad M_{even} \twoheadleftarrow M \hookleftarrow M_{red}.$$

The composition $M_{red} \to M_{even}$ makes M_{even} into an infinitesimal extension of M_{red}.

2.1.3. ∞ *dimensional super manifolds.* Douady noticed that in ∞ dimension one needs local charts – the approach through the sheaf of functions is not enough to define manifolds.

∞-dimensional super manifolds appear as various spaces of fields $\mathcal{F}$ in QFT, for instance the map spaces such as $Map(S^{1|1}, M)$. Deligne-Freed treat such spaces only as functors. So they do not spell the structure of a super manifold on $\mathcal{F}$, but only the functor of points $Map(-, \mathcal{F})$ defined on finite dimensional super manifolds.

2.1.4. *Lemma.* (a) Any super manifold M is locally a product of the underlying even manifold $|M|$ and a super point $\mathbb{R}^{0|q}$.

(b) All super manifolds are of the form $\mathcal{O}_M = \boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)$ for a vector bundle $\mathcal{V}$ over a smooth manifold $|M|$, however non-canonically.

Proof. (a) is just the definition of super-manifolds. (b) Part (a) shows that locally $\mathcal{O}_M = \boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V})$ for a trivial vector bundle $\mathcal{V} = |M| \times \mathbb{R}^q$ of rank q.

2.1.5. *Remarks.* (a) Operation $\mathcal{V} \mapsto \text{Spec}[\boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)]$ will later be called the parity change of a vector bundle $\mathcal{V}$ and denoted $\Pi(\mathcal{V})$.

(b) For a super manifold M, the choice $\mathcal{V}$ is an additional rigidification of M, observe that it lifts a super-manifold M to graded manifold. The functor from "manifolds with a vector bundle" to super manifolds: $(|M|, \mathcal{V}) \mapsto M$, is surjective on objects but not an equivalence.

2.1.6. *Remark.* Super C^r-manifolds do not make sense.

2.1.7. *Super-schemes.* A super space $M = (|M|, \mathcal{O}_M)$ is a ringed space (topological space $|M|$ with a sheaf of super-rings $\mathcal{O}_M$), such that

- the structure sheaf $\mathcal{O}_M$ is super-commutative and
- the stalks are local rings.

A super scheme is a super space M such that the even part $M_0 \stackrel{\text{def}}{=} (|M|, \mathcal{O}_{M,0})$ is a scheme, and that the odd part $\mathcal{O}_{M,1}$ is a coherent module for the even part $\mathcal{O}_{M,0}$. Some examples:

(1) The algebraic versions of $\mathbb{R}^{p|q}$'s are affine (super) schemes $\mathbb{A}^{n|m}$ over a ground ring $\Bbbk$, given by the super-commutative algebras of polynomial functions on these spaces

$$\mathcal{O}(\mathbb{A}^{n|m}) = \Bbbk[x^1, ..., x^n, \psi^1, ..., \psi^m] = S^\bullet(\Bbbk x^1 \oplus \cdots \oplus \mathbb{R}x^n) \otimes \overset{*}{\wedge} \mathcal{F}(\Bbbk\psi^1 \oplus \cdots \oplus \Bbbk\psi^m),$$

with x^μ's even and ψ^k's odd. It is a product of the affine space $\mathbb{A}^n = \mathbb{A}^{n|0}$ and a super point $\mathbb{A}^{0|m}$. The functions on a super point $\mathcal{O}(\mathbb{A}^{0|m})$ have a finite basis of monomials $\psi^{i_1 < \cdots < i_k} \stackrel{\text{def}}{=} \psi^{i_1} \cdots \psi^{i_k}$.

(2) A vector bundle $\mathcal{V}$ over an ordinary scheme X. defines a super scheme $\Pi(\mathcal{V}) \stackrel{\text{def}}{=} (|X|, \overset{\bullet}{\wedge}_{\mathcal{O}_X} \mathcal{V}^*)$. The first infinitesimal neighborhood of X in $\Pi(\mathcal{V})$ is $\mathcal{N} \stackrel{\text{def}}{=} (|X|, \mathcal{O}_X \oplus \mathcal{V}^*)$, given by imposing $\mathcal{V}^* \wedge \mathcal{V}^* = 0$. If X is smooth so is $\Pi(\mathcal{V})$, but $\mathcal{N}$ is not smooth.

2.1.8. *The value of a function $f \in \mathcal{O}_M$ at a point $x \in |M|$.* This is defined as the unique number c such that $f - c$ is not invertible in any neighborhood of x.

This leads to the observation that $\mathcal{O}_M$ is a sheaf of local rings, the maximal ideal in the stalk at x is $\mathfrak{m}_x = \{f;\ f(x) = 0\}$.

2.1.9. *Coordinates.* On $\mathbb{R}^{p|q}$ one has x^μ's and ψ^i's.

A map $M \to \mathbb{R}^{p|q}$ is the same as p even and q odd functions on M.

2.1.10. M can be thought of as $|M|$ plus fuzz.

2.2. **Super-manifold as a functor.** We consider a super-manifold M as a functor $S \mapsto M(S)$ from super manifolds to sets. For instance,

- (0) $Map(\mathbb{R}^{0|0}, M) = |M|$.
- (1) $Map(\mathbb{R}^{0|1}, M)$ is the moduli of pairs of a point $x \in M$ and an odd derivation of the local ring at x, i.e., these are the odd tangent vectors.
- (2) $Map(S, \mathbb{R}^{1|0}) = \Gamma\mathcal{O}_{M,0}$ and $Map(S, \mathbb{R}^{0|1}) = \Gamma(\mathcal{O}_M)_1$.
- The *universal point of M* is the M-point $M \xrightarrow{1_M} M$.

The idea is that $\mathbb{A}^{0|1}$ has only one $\Bbbk$-point so in this respect it looks like $\mathbb{A}^{0|0}$; however, S-points of $\mathbb{A}^{0|1}$ are numerous – the same as odd functions on S.

2.2.1. *Fiber products.* Fiber products $S'\underset{S}{\times} M$ exist for maps $S'\to S$ that are locally projections of the form $S' = \mathbb{R}^{p|q}\times S\to S$.

2.2.2. *The use of a base S.* A super space M/S (also called an S-super space M, or a super space M with a base S), means simply a map of super spaces $M \to S$. The basic idea is that one studies such relative super space M/S using all super spaces $T/$ with base S – one studies the the set of T/S-points of M/S, defined by $(M/S)(T/S)\overset{\text{def}}{=}Hom_S(T,M)$.

In order to be able to construct maps $M \xrightarrow{\phi} N$ in terms of the corresponding maps of functors, one has to systematically use families of manifolds $M\to S$ (?). In particular, it means that one should with each family $M\to S$ also consider all families obtained by the base changes $M' = M\underset{S}{\times}S'$ (under projection-like maps $S'\to S$).

2.2.3. *"Functions are determined by their values on S-points".* A function f on M gives for any S-point $S \xrightarrow{\sigma} M$ a function $f\sigma$ on S, which we can think of as the value of f on the S-point σ. So,tautologically, f is the same as its value on the M-point $M \xrightarrow{id} M$.

2.3. **The functor of maps between two super spaces.** It is defined by

$$\underline{\text{Hom}}(M,N)\ (S)\overset{\text{def}}{=} Map_S(M\times S, N\times S) = Map(M\times S, N),$$

i.e., the S-points of this functor are simply the S-families of maps form M to N. We leave out the question of what kind of a space would represent this functor (in nice cases it is give by a super manifold, possibly infinite dimensional).

2.4. **Lie groups and algebraic groups.**

2.4.1. *Super Lie groups.* $\mathbb{R}^{p|q}$ is a group (contrary to the intuition from commutative schemes where: group $\Rightarrow$ smooth $\Rightarrow$ reduced).

2.5. **Sheaves.** The sheaves on M are by definition sheaves on the topological space $|M|$ (this is the only topological space around). The sheaves on $|M|$ that are related to its structure of a super manifold are the sheaves of $\mathcal{O}_M$-modules. Two notions of Hom give two relevant notions of global sections of an $\mathcal{O}_M$-module $\mathcal{A}$:

- The inner notion $\underline{\Gamma}$ gives a super module $\underline{\Gamma}(\mathcal{A})$ over the super-commutative algebra $\underline{\Gamma}(\mathcal{O}_M)$, by

$$\underline{\Gamma}(M,\mathcal{A})\overset{\text{def}}{=} \underline{\mathcal{H}\text{om}}_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{A}) = \Gamma(|M|,\mathcal{A}) = \Gamma(|M|,\mathcal{A}_0)\oplus \Gamma(|M|,\mathcal{A}_1)\ .$$

- The notion given by the category of $\mathcal{O}_M$-modules is $\Gamma(\mathcal{A}) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}_M}(\mathcal{O}_M, \mathcal{A})$,

$$\Gamma(M, \mathcal{A}) \stackrel{\text{def}}{=} \mathcal{H}\text{om}_{\mathcal{O}_M}(\mathcal{O}_M, \mathcal{A}) = \Gamma(|M|, \mathcal{A}_0).$$

So, $\Gamma(\mathcal{A}) = [\underline{\Gamma}(\mathcal{A})]_0 = \Gamma(|M|, \mathcal{A}_0)$ and one can recover the odd part of the functor Γ from the even part by

$$\underline{\Gamma}(M, \mathcal{A})_1 = \Gamma(|M|, \mathcal{A}_1) = \Pi\ \Gamma(M, \Pi\mathcal{A}).$$

3. **Differential Geometry**

As usual, on a super manifold M one can identify vector bundles with locally free sheaves. One consequence is that the operation of change of parity which is defined obviously for locally free sheaves is quite unexpected for vector bundles.

3.1. **Vector bundles.** A vector bundle V over M is a fiber bundle $V \to M$ which is (i) locally $\mathbb{A}^{p|q} \times M \to M$, (ii) has a structure group $GL(p|q)$ (i.e.the transition functions lie in that group).

On the other hand, one can consider locally free $\mathcal{O}_M$-modules $\mathcal{V}$ of dimension $p|q$.

3.1.1. *Super-vector space V gives a "super-vector space $\underline{V}$ in the category of super-manifolds".* One has $|\boldsymbol{V}| = V_0$ and $\mathcal{O}_{\boldsymbol{V}} = \mathcal{C}^\infty(V_0) \otimes S^\bullet(V_1^*)$, or in algebraic geometry simply, $\mathcal{O}(\boldsymbol{V}) = S^\bullet(V^*)$. The structure of a "super-vector space in the category of super-manifolds" is clear. This is the way we have obtained our basic manifolds $\mathbb{A}^{p|q}$.

In the opposite direction, a "super-vector space in the category of super-manifolds" $\boldsymbol{V}$, defines a super-vector space $V = [\mathcal{O}_{lin}(\boldsymbol{V})]^*$, the dual of linear functions. In the infinitesimal language, $V = T_0(\boldsymbol{V})$. Similarly, $V = Map(\mathbb{A}^{0|1}, \boldsymbol{V})$ with $V_0 = Map(\mathbb{A}^{0|0}, \boldsymbol{V})$ (the maps that factor thru the point $\mathbb{A}^{0|0}$) and $V_1 = Map[(\mathbb{A}^{0|1}, 0), (\boldsymbol{V}, 0)]$ (maps that send the point $\mathbb{A}^{0|0} = \mathbb{A}^{0|1}_{red} \subseteq \mathbb{A}^{0|1}$ to $0 \in \boldsymbol{V}$).

3.1.2. *Sheaf of sections.* For the equivalence of the two notions, a vector bundle V gives a a sheaf

$$\mathcal{V} \stackrel{\text{def}}{=} \mathcal{O}^*_{V,lin} = \mathcal{H}\text{om}_{\mathcal{O}_M}(\mathcal{O}_{V,lin},\ \mathcal{O}_M) = \mathcal{M}ap_M(M \times \mathbb{A}^{0|1}, V) = \mathcal{T}_{V \to M}.$$

The last interpretation is as vertical vector fields, the one with $\mathbb{A}^{0|1}$ is the closest to the idea of a "sheaf of sections".

3.1.3. *The underlying vector bundle.* In the opposite direction, $\mathcal{V}$ defines a functor

$S \mapsto \{(f, v),\ f:\ S \to M$ is an S-point of M and $v \in \Gamma(S, [f^*\mathcal{V}]_0)$ is an even section of $\mathcal{V}$ over $f\ \}$,

which is represented by $V \stackrel{\text{def}}{=} \text{Spec}[S^\bullet_{\mathcal{O}_M}(\mathcal{V}^*)]$.

3.1.4. *The underlying topological space* $|V|$. The restriction of $\mathcal{V}$ to M_{red}, $\mathcal{V}|M_{red} = \mathcal{O}_{M_{red}} \underset{\mathcal{O}_M}{\otimes} \mathcal{V} = \mathcal{V}/J_M{\cdot}\mathcal{V}$, is a super vector bundle over the even manifold M_{red}. Say, if $\mathcal{V} = \mathcal{O}_M^{p|q} = \oplus_i\ \mathcal{O}_M e_i \oplus \oplus_j\ \mathcal{O}_M \theta_j$, then $\mathcal{V}|M_{red} = \mathcal{O}_{M_{red}}^{p|q} = \oplus_i\ \mathcal{O}_{M_{red}} e_i \oplus \oplus_j\ \mathcal{O}_{M_{red}} \theta_j$.

The underlying topological space $|V|$ is the vector bundle $|V| \to |M|$, corresponding to the sheaf $(\mathcal{V}|M_{red})_0$. For instance if $V = M \times \mathbb{A}^{p|q}$, then $|V| = |M| \times \mathbb{A}^p$, while $\mathcal{V} = \mathcal{O}_M^{p|q}$ gives $(\mathcal{V}|M_{red})_0 = \oplus_i\ \mathcal{O}_{M_{red}} e_i = \mathcal{O}_{M_{red}}^p$.

3.1.5. *Change of parity of a vector bundle.* It is defined on locally free sheaves, hence also on vector bundles. From the point of view of locally free sheaves the change seems simple and formal, but on vector bundles it is drastic. Locally, $V \cong M \times \mathbb{A}^{p|q}$ and $\Pi(V) \cong M \times \Pi(\mathbb{A}^{p|q}) \cong M \times \mathbb{A}^{q|p}$. Observe, that the parity change on vector bundles changes the underlying topological space: while locally $|V| \cong |M| \times \mathbb{A}^p$ corresponds to the sheaf $(\mathcal{V}|M_{red})_0 \cong \oplus_i\ \mathcal{O}_{M_{red}} e_i = \mathcal{O}_{M_{red}}^p$, $|\Pi(V)| \cong |M| \times \mathbb{A}^q$ corresponds to $(\mathcal{V}|M_{red})_1 \cong \oplus_i\ \mathcal{O}_{M_{red}} \pi\theta_i \cong \mathcal{O}_{M_{red}}^q$.

However, this operation is still elementary. For instance, suppose that V is an ordinary vector bundle over an ordinary manifold M. One can describe the super manifold $\Pi(V)$ as a pair $(M, \mathcal{O}_{\Pi(V)})$, i.e., the underlying manifold is the base M of the vector bundle V and the algebra of functions is $\mathcal{O}_{\Pi(V)} \overset{\text{def}}{=} \overset{*}{\wedge}\ \mathcal{V}^*$.

3.2. **(Co)tangent bundles.** A vector field means a derivative of the algebra of functions, so the vector fields on $\mathbb{A}^{n|m}$ are all $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^k \frac{\partial}{\partial \psi^k}$, where $\frac{\partial}{\partial \psi^k}$ has the usual properties that it kills x^μ's and $\frac{\partial}{\partial \psi^k} p^j = \delta_{jk}$, but

$$\frac{\partial}{\partial \psi^k}\ (fg) = (\frac{\partial}{\partial \psi^k} f)\ g + (-1)^{p_f}\ f(\frac{\partial}{\partial \psi^k} g).$$

A possible confusion regarding the $\mathbb{Z}_2$-grading: one could say that $\mathcal{T}_M^{(0)} = \oplus\ \mathcal{O}_M \partial_{x^\mu}$ are "even" vector fields and $\mathcal{T}_M^{(1)} = \oplus\ \mathcal{O}_M \partial_{\psi^i}$ are "odd", while the correct parity is $(\mathcal{T}_M)_0 = \oplus\ \mathcal{O}_{M,0} \partial_{x^\mu} \oplus \oplus \oplus\ \mathcal{O}_{M,1} \partial_{\psi^i}$.

The differential $df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial \psi^k} d\psi^k$ is of parity 0, so it satisfies $d(fg) = df{\cdot}g + f{\cdot}dg$.

3.2.1. *Differential forms.* Define Ω_M^1 as the dual of $\mathcal{T}_M$ and $\Omega_M^\bullet \overset{\text{def}}{=} \overset{\bullet}{\wedge}\ \Omega_M^1$. Then $\Omega_M^\bullet$ is a graded object in the category of (sheaves of) super vector spaces, the associated super vector space $\boldsymbol{s}\Omega_M^\bullet$ combines the parity of the grading and the parity of Ω_M^1. Say, if M were even then Ω_M^1 would be even, but $(\boldsymbol{s}\Omega_M^\bullet)^1$ would be odd.

Observe that if M is not even, there are no highest degree forms: $\Omega_{\mathbb{A}^{p|q}}^n = \oplus_{r+s=n}\ \Omega_{\mathbb{A}^p}^r \otimes \overset{s}{\wedge}(\oplus\ \Bbbk\psi^i)$.

Lemma. (a) Differential $d:\ \mathcal{O}_M\to\ \Omega^1_M$ extends to the De Rham differential on $\Omega^\bullet_M$.

(b) (Poincare lemma) $(\Omega^\bullet_M, d)$ is a resolution of $\mathbb{R}_{|M|}$.

(c) $H^\bullet(\Omega^\bullet_M, d) =\ H^\bullet(|M|,\mathbb{R})$.

Proof. (b) reduces to the local setting $M =\ \mathbb{R}^{p|q}\times\ \mathbb{R}^p\times\ \mathbb{R}^{0|q}$ and then to factors $\mathbb{R}^p$ (standard Poincare lemma) and $\mathbb{R}^{0|q}$, or even $\mathbb{R}^{0|1}$ (Koszul complex).

3.3. **Parity change of the tangent bundle.** The most natural appearance of super commutative algebras are the algebras of differential forms Ω^*_M on an ordinary manifold M. This is the algebra of functions on a super manifold which has two natural interpretations as either (1) the moduli of maps of the super point $\mathbb{A}^{0|1}$ into M, or (2) the super manifold obtained from the tangent vector bundle TM by parity change.

3.3.1. *Lemma.* (a) $\mathrm{Spec}(\boldsymbol{s}\Omega^\bullet_M) =\ \Pi(TM)$.

(b) $\Pi(TM)$ represents the functor $\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)$ defined by

$$\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)\ (S) \stackrel{\text{def}}{=} Map_S(\mathbb{A}^{0|1}\times\ S,\ M\times\ S) =\ Map(\mathbb{A}^{0|1}\times\ S,\ M),$$

i.e., S-points are "S-families of odd tangent vectors on M".

Proof. In (a)

$$\mathcal{O}_{\Pi\ TM/M} =\ S^\bullet_{\mathcal{O}_M}([\Pi\otimes\ \mathcal{T}_M]^*) =\ S^\bullet_{\mathcal{O}_M}(\Pi\otimes\ \mathcal{T}^*M) =\ \oplus_k\ \ \Pi^{\otimes k}\otimes \overset{k}{\underset{\mathcal{O}_M}{\wedge}} (\ \Omega^1_M) =\ \boldsymbol{s}(\Omega^\bullet_M).$$

In (b), let S be the spectrum of a commutative super algebra A. Then

$$\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)\ (S) =\ Map(\mathbb{A}^{0|1}\times\ S,\ M) =\ \mathrm{Hom}_{\Bbbk-alg}[\mathcal{O}(M),\ (\Bbbk\oplus\psi\Bbbk)\underset{\Bbbk}{\otimes}\ A] =\ \mathrm{Hom}_{\Bbbk-alg}(\mathcal{O}(M),\ A\oplus\ \psi A).$$

An element is a map $\phi =\ \alpha+\psi\beta:\ \mathcal{O}(M)\to\ A\oplus\ \psi A$ with

$$\alpha(fg)+\psi\beta(fg) =\ (\alpha(f)+\psi\beta(f))\cdot(\alpha(g)+\psi\beta(g)) =\ \alpha(f)\alpha(g)+\ \psi\beta(f)\alpha(g)+\alpha(f)\psi\beta(g)) =$$

$$\alpha(f)\alpha(g)+\ \ \psi[\beta(f)\alpha(g)+\ \ (-1)^{p_f}\alpha(f)\beta(g))].$$

So $\alpha:\ \ \mathcal{O}(M)\to A$ is a morphism of algebras and $\beta:\ \ \mathcal{O}(M)\to A$ is an odd α-derivative. So ϕ consists of a map $\alpha:\ \ S\to M$ and $\beta\in\ \Gamma[S,\ (\alpha^*TM)]_1$.

On the other hand, an element ϕ of $\mathrm{Hom}(S,\Pi\otimes\ TM) =\ \mathrm{Hom}_{k-alg}[\mathcal{O}(\Pi\otimes\ TM),\ A] = \mathrm{Hom}_{k-alg}[\boldsymbol{s}(S^\bullet_{\mathcal{O}(M)}\Omega^1_M),\ A]$, consists of a map of algebras $\alpha:\ \mathcal{O}(M)\to A$ (the restriction of ϕ to $\mathcal{O}(M)$), and a map of $\mathcal{O}(M)$-modules $\beta:\ \ [\boldsymbol{s}(S^\bullet_{\mathcal{O}(M)}\Omega^1_M)]^1\to\ A$, i.e., $\beta:\ \Pi\otimes\ \Omega^1_M\to\ A$. Now, a map of $\mathcal{O}(M)$-modules $\Omega^1_M\to\ \mathcal{O}(M)$ is the same as a section of $(TM)_0$ (an even vector field on M), a map of $\mathcal{O}(M)$-modules $\Omega^1_M\to\ \mathcal{O}(S)$ is a section of $(\alpha^*TM)_0$, and so $\beta:\ \Pi\otimes\ \Omega^1_M\to\ A$ is a section of $(\alpha^*TM)_1$.

Remarks. (1) The underlying topological space $|\Pi(TM)|$ is just $|M|$. If M and S were even then $\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)(S)$ are just the maps from S to M.

(2) The statement (b) is just the odd version of the standard description of TM as the moduli of all maps from a "double point" (or a "point with a tangent vector") to M, i.e., description of TM as the space that represents the functor $\underline{\mathrm{Hom}}(\mathrm{Spec}(D), M)$ for the algebra of dual numbers $D = \mathbb{k}[\varepsilon]/\varepsilon^2$.

4. **Integration on super affine spaces**

Integrals of functions on super manifolds will be (ordinary) numbers. Integrals on affine spaces will be defined "by hand". In general the objects one can integrate are called *densities*, the correct replacements of top differential forms.

Our basic example of an integral is the fermionic Gaussian integral. We motivate Gaussian integrals as the simplest case of path integrals. Gaussian integral on a super point $\mathbb{R}^{0|q}$ turns out to be the Pffafian of the quadratic form on an odd vector space (if q is even, otherwise it is zero).

4.0.2. *SUSY (supersymmetry).* Supersymmetry of a function f on a super manifold is a vector field δ that kills it (i.e., f is constant on the flow lines). The interesting case is when δ is odd, i.e., so it mixes even and odd stuff. If such δ can be interpreted as one of the coordinate vector fields then the integral of f is zero. If this can be done generically – say everywhere except on some submanifolds C_i of $|M|$ – then the integral will be given by contributions from submanifolds C_i.

Example. Our example will be the integral $\int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P(x)^2}$ which has a super interpretation

$$\int_{\mathbb{R}^{1|2}} Dx\ d\psi^2\ d\psi^1\ \ e^{-\frac{1}{2}P(x)^2+P'(x)\psi^1\psi^2}.$$

It has a super-symmetry δ which is a part of a coordinate system as long as one stays away from zeros of P. So the integral is a sum of contributions from zeros b of $P(x)$. The contributions are $sign(P'(b))$, and then the integral is the degree of P as a map from S^1 to itself. Actually this integral can be easily calculated by a substitution – what we got from the super picture is a localization of the integral around few critical points.

4.1. **Integration on affine spaces.**

4.1.1. *Integration on super points.* Integration of functions on a super point $\mathbb{A}^{0|m}$ is defined by using successively the formula

$$\int d\psi\ a + b\psi \overset{\text{def}}{=} b.$$

So, in general the integral just takes the highest degree coefficient (with a correct sign)

$$\int d\psi^n \cdots d\psi^1 \sum_{I=\ \{i_1<\cdots<i_k\}} c_I \cdot \psi^I \stackrel{\text{def}}{=} c_{12\cdots m}.$$

We also denote it

$$\int d\psi^n \cdots d\psi^1\ f \stackrel{\text{def}}{=} [f : \psi^1 \cdots \psi^m].$$

4.1.2. *Covariance.* (1) Formula $\int\ d\psi\ a_0 + a_1\psi\ = a_1$ is somewhat reminiscent of integrals of holomorphic functions over a circle. There, $\int_{S^1}\ \sum_n\ b_n z^n\ dz =\ b_{-1}$ is the coefficient of z^{-1} in the Laurent series expansion.

(2) However, only one of these formulas can work: a change $z = cu$ does not affect $\int_{S^1}\ z^{-1}\ dz$, but $\psi = c\phi$ seems to give nonsense: $\int\ d\psi\ \psi =\ c^2 \cdot \int\ d\phi\ \phi$. The reason is that $d\psi$ is really contravariant

$$d(c\psi)\ =\ c^{-1}\ d\psi.$$

We will deal with this in the next section when we tackle change of variable.(9)

4.1.3. *Integration on super affine spaces.* Integrals on $\mathbb{A}^{n|m}$ are evaluated so that one first integrates over the fermionic variables and then we are left with an ordinary integral. For example if $S[x, \psi^1, \psi^2] =\ U(x) + V(x)\psi^1\psi^2$ then

$$\int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-S[x,\psi^1,\psi^2]} = \int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-U(x)} \sum_k \frac{(-1)^k}{k!} V(x)^k (\psi^1\psi^2)^k$$

$$= -\int_{\mathbb{A}^{1|0}} dx\ e^{-U(x)} \int_{\mathbb{A}^{0|2}} d\psi^2\ d\psi^1\ V(x)\psi^1\psi^2 =\ -\int_{\mathbb{A}^{1|0}} dx\ V(x) e^{-U(x)}.$$

4.2. **Gaussian integrals.**

4.2.1. *Even Gaussian integrals.* On an ordinary real vector space M we consider a quadratic form $S[x]$ and a choice of coordinates x^i.

We normalize the Lebesgue measure on M with respect to coordinates x^i

$$Dx \stackrel{\text{def}}{=} \prod \frac{dx^i}{\sqrt{2\pi}},$$

and we consider the Gaussian integral

$$\int_M Dx\ e^{-\frac{1}{2}S(x)}.$$

[9] We will denote $d\psi =\ (\boldsymbol{d}\psi)^{-1}$, where quantity with $\boldsymbol{d}\psi$ is covariant.

Lemma. When we write S in terms of the coordinates $S(x) = x^i A_{ij} x^j$, the integral is

$$\int_M Dx \; e^{-\frac{1}{2}(x^i A_{ij} x^j)} = (\det A)^{-\frac{1}{2}}.$$

4.2.2. *σ-model interpretation.* Consider a Σ-model, i.e., the moduli of maps from Σ to M. In the simplest case when Σ is just a point (hence a 0-dimensional manifold), the moduli of maps from Σ to M is just M. A positive definite quadratic form $S(x)$ on M gives a path integral which is the above Gaussian integral.

4.2.3. *Fermionic Gaussian integrals give Pffafian.* Now let M be a super point $(0, m)$. Quadratic functions on M are function of the form $S[x] = \psi^i B_{ij} \psi^j$ for an anti-symmetric B (so $\frac{1}{2}S[x] = \sum_{i<j} \psi^i B_{ij} \psi^j$). A fermionic Gaussian integral is

$$\int_M d\psi^m \cdots d\psi^1 \; e^{\frac{1}{2} \; S[x]} = \int_M d\psi^m \cdots d\psi^1 \; e^{\sum_{i<j} \; \psi^i B_{ij} \psi^j}.$$

4.2.4. *Odd Feynman amplitudes.* Let $\mathcal{P}(m)$ be the set of all *pairings* of the set $\{1, ..., m\}$, i.e., all partitions γ of A into 2-element subsets. To a pairing γ one assigns the sign σ_γ as the sign of any permutation $i_1, j_1, ..., i_q, j_q$ that one obtains by choosing an ordering $\{i_1 < j_1\}, ..., \{i_q < j_q\}$ on γ.

The γ-amplitude of a quadratic form $B(x) = \psi^i B_{ij} \psi^j$ is

$$F_\gamma(B) \stackrel{\text{def}}{=} \sigma_\gamma \cdot \prod_{\{i<j\} \in \gamma} B_{ij}.$$

Notice that the difference from the even case is that there is a sign σ_γ attached to a Feynman graph γ.

4.2.5. *Fermionic Gaussian integrals give Pffafian.*

Lemma. The odd Gaussian integral on $\mathbb{R}^{0|m}$ is a sum over all pairings

$$\int_M d\psi^m \cdots d\psi^1 \; e^{\frac{1}{2}\sum \; \psi^i B_{ij} \psi^j} = \sum_{\gamma \in \mathcal{P}(\{1,...,m\}}} \sigma_\gamma \prod_{\{i<j\} \in \gamma} B_{ij} = \sum_{\gamma \in \mathcal{P}(\{m\})} F_\gamma(B).$$

Proof. The exponential power series is a finite sum

$$\int_M d\psi^m \cdots d\psi^1 \; \sum_k \frac{1}{k!} (\sum_{i<j} B_{ij} \psi^i \psi^j)^k.$$

Since we get only the even degree terms, this is zero if m is odd. If $m = 2q$ is even, this is

$$\int_M d\psi^m \cdots d\psi^1 \; \frac{1}{q!} \; (\sum_{i<j} B_{ij} \psi^i \psi^j)^q,$$

and we get a contribution $\frac{1}{q!}\ \sigma_{i_1j_1\dots i_qj_q}B_{i_1j_1}\cdots B_{i_qj_q}$, whenever $i_1,j_1,...,i_q,j_q$ is a permutation of $1,...,m$, such that $i_k<j_k$. Therefore, the fermionic Gaussian integral is a sum over all such permutations

$$\int_M\ d\psi^m\cdots d\psi^1\ e^{-\frac{1}{2}\ \psi^iB_{ij}\psi^j}=\ \frac{1}{q!}\sum_{(i_1j_1\dots i_nj_n)\in\ S_m,\ i_k<j_k}\ \sigma_{i_1j_1\dots i_nj_n}\cdot B_{i_1j_1}\cdots B_{i_nj_n}.$$

However, $\sigma_{i_1j_1\dots i_nj_n}$ and $B_{i_1j_1}\cdots B_{i_nj_n}$ only depend on the associated pairing $\gamma=\{\{i_1<j_1\},...,\{i_q<j_q\}\}$. Moreover, all permutations $(i_1,j_1,...,i_n,j_n)$ over one pairing γ form a S_q-torsor. so the RHS simplifies to the claim of the lemma.

Remarks. (0) The sum $\sum_{\gamma\in\mathcal{P}(\{1,...,m\}}\ \sigma_\gamma\ \prod_{\{i<j\}\in\gamma}\ B_{ij}$ is called the Pffafian $Pf(B)$ of the anti-symmetric matrix B.[10]

(1) The integral is a Feynman sum. So, Pffafian may be the first appearance of Feynman sums.[11]

(2) The Pffafian of an antisymmetric matrix of even size is a square root of its determinant:

$$Pf(B)^2=\ \det(B).$$

This square root is normalized by $Pf=1$ on $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ (and on block diagonal matrices with such blocks on the diagonal).

(3) Again we find that the odd part gives a contribution in the opposite direction since $(\det(A)^{-\frac{1}{2}}$ is replaced by $(\det(B)^{\frac{1}{2}}=Pf(B)$.

4.2.6. *Question.* Use integrals to prove (i) $Pf^2=\det$ and (ii) $Pf(ABA^{-1})=Pf(B)$ for orthogonal A.

A linear operator A on an ordinary vector space V gives a symmetric bilinear form $\mathcal{A}$ on the odd vector space $T^*(\Pi V)=\Pi[V\oplus V^*]$ by

$$\mathcal{A}(u\oplus\lambda,v\oplus\mu)\ \stackrel{\text{def}}{=}\ .$$

Notice that the space $T^*(\Pi V)=\Pi[V\oplus V^*]$ comes with a canonical volume element (5.1.2).

Corollary. The determinant of a linear operator A on an even vector space V can be calculated as the Gaussian integral for the form $\mathcal{A}$ (using the canonical volume element).

4.3. **Wick's theorem.**

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[10]Here $Pf(B)$ is attached to a symmetric form B on an odd vector space M and a system of coordinates $\psi^1,...,\psi^m$ on M, i.e., precisely to the associated matrix (B_{ij}). However, one needs less – a volume form $dv=\ d\psi^1\cdots d\psi^m$ on M rather then a system of coordinates.

[11]This raises a question of whether the even Gaussian $(\det(A)^{-\frac{1}{2}}$ is a Feynman sum in some way. This could be interesting for infinite dimensional spaces.