THE METHODS OF ALGEBRAIC TOPOLOGY FROM THE VIEWPOINT OF COBORDISM THEORY

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Abstract. The goal of this work is the construction of the analogue to the Adams spectral sequence in cobordism theory, calculation of the ring of cohomology operations in this theory, and also a number of applications: to the problem of computing homotopy groups and the classical Adams spectral sequence, fixed points of transformations of period p, and others.

Introduction

In algebraic topology during the last few years the role of the so-called extraordinary homology and cohomology theories has started to become apparent; these theories satisfy all the Eilenberg–Steenrod axioms, except the axiom on the homology of a point. The merit of introducing such theories into topology and their first brilliant applications are due to Atiyah, Hirzebruch, Conner and Floyd, although in algebraic geometry the germs of such notions have appeared earlier (the Chow ring, the Grothendieck K-functor, etc.). Duality laws of Poincaré type, Thom isomorphisms, the construction of several important analogues of cohomology operations and characteristic classes, and also relations between different theories were quickly discovered and understood (cf. [2, 5, 8, 9, 11, 12]).

These ideas and notions gave rise to a series of brilliant results ([2]–[13]). In time there became manifest two important types of such theories: (1) theories of "K type" and (2) theories of "cobordism type" and their dual homology ("bordism") theories.

The present work is connected mainly with the theory of unitary cobordism. It is a detailed account and further development of the author's work [19]. The structure of the homology of a point in the unitary cobordism theory was first discovered by Milnor [15] and the author [17]; the most complete and systematic account together with the structure of the ring can be found in [18]. Moreover, in recent work of Stong [22] and Hattori important relations of unitary cobordism to K-theory were found. We freely use the results and methods of all these works later, and we refer the reader to the works [15, 17, 18, 22] for preliminary information.

Our basic aim is the development of new methods which allow us to compute stable homotopy invariants in a regular fashion with the help of extraordinary homology theories, by analogy with the method of Cartan–Serre–Adams in the usual classical Z_p-cohomology theory. We have succeeded in the complete computation of the analogue of the Steenrod algebra and the construction of a "spectral sequence

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of Adams type"[1] in some cohomology theories, of which the most important is the theory of U-cobordism, and we shall sketch some computations which permit us to obtain and comprehend from the same point of view a series of already known concrete results (Milnor, Kervaire, Adams, Conner–Floyd, and others), and some new results as well.

In the process of the work the author ran into a whole series of new and tempting algebraic and topological situations, analogues to which in the classical case are either completely lacking or strongly degenerate; many of them have not been considered in depth. All this leads us to express hope for the perspective of this circle of ideas and methods both for applications to known classical problems of homotopy theory, and for the formulation and solution of new problems from which one can expect the appearance of nontraditional algebraic connections and concepts.

The reader, naturally, is interested in the following question: to what extent is the program (of developing far-reaching algebraic-topological methods in extraordinary cohomology theory) able to resolve difficulties connected with the stable homotopy groups of spheres? In the author's opinion, it succeeds in showing some principal (and new) sides of this problem, which allow us to put forth arguments about the nearness of the problems to solution and the formulation of final answers. First of all, the question should be separated into two parts: (1) the correct selection of the theory of cobordism type as "leading" in this program, and why it is richer than cohomology and K-theory; (2) how to look at the problem of homotopy groups of spheres from the point of view of cobordism theory.

The answer to the first part of the question is not complicated. As is shown in Appendix 3, if we have any other "good" cohomology theory, then it has the form of cobordism with coefficients in an Ω-module. Besides, working as in §§ 9 and 12, it is possible to convince oneself that these give the best filtrations for homotopy groups (at any rate, for complexes without torsion; for $p = 2$ it may be that the appropriate substitute for MU is MSU). In this way, the other theories lead to the scheme of cobordism theory, and there their properties may be exploited in our program by means of homological algebra, as shown in many parts of the present work.

We now attempt to answer the second fundamental part of the question. Here we must initially formulate some notions and assertions. Let A_p^U $[A^U]$ be the ring of cohomology operations in U_p^*-theory $[U^*$, respectively], $\Lambda_p = U_p^*(P)$, $\Lambda = U^*(P)$, $P =$ point, $Q_p = p$-adic integers.[2] Note that $\Lambda \subset A^U$. The ring over Q_p, $\Lambda \otimes_Z Q_p \supset \Lambda_p$, lies in $A^U \otimes_Z Q_p \supset A_p^U$, and $\Lambda \otimes_Z Q_p$ is a local ring with maximal ideal $m \subset \Lambda \otimes_Z Q_p$, where $\Lambda \otimes_Z Q_p/m = Z_p$. Note that Λ_p is an A_p^U-module and A_p^U is also a left Λ_p-module.

[1]It may be shown that the Adams spectral sequence is the generalization specifically for S-categories (see § 1) of "the universal coefficient formula," and this is used in the proofs of Theorems 1 and 2 of Appendix 3.

[2]U_p^*-theory is a direct summand of the cohomology theory $U^* \otimes Q_p$, having spectrum M_p such that $H^*(M_p, Z_p) = A/(\beta A + A\beta)$ {where A is the Steenrod algebra over Z_p and β is the Bokšteĭn operator} (see §§ 1, 5, 11, 12).

Consider the following rings:

$$m_p = m \cap \Lambda_p, \quad \Lambda_p/m_p = Z_p,$$
$$\bar{\Lambda}_p = \sum_{i \geq 0} m_p^i / m_p^{i+1},$$
$$\bar{A} = \bar{A}_p^U = \sum_{i \geq 0} m_p^i A_p^U / m_p^{i+1} A_p^U,$$

where $\bar{\Lambda}_p$ is an $\bar{A}$-module.

In this situation arises as usual a spectral sequence $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$, where

$$\tilde{\tilde{E}}_r \searrow \mathrm{Ext}^{**}_{A^U}(\Lambda, \Lambda) \otimes_Z Q_p, \quad \tilde{\tilde{E}}_2 = \mathrm{Ext}^{***}_{\bar{A}_p^U}(\bar{\Lambda}_p, \bar{\Lambda}_p),$$

determined by the maximal ideal $m_p \subset \Lambda_p$ and the induced filtrations.

It turns out that for all $p > 2$ the following holds:

Theorem. *The ring $\mathrm{Ext}^{***}_{\bar{A}_p^U}(\bar{\Lambda}_p, \bar{\Lambda}_p)$ is isomorphic to $\mathrm{Ext}^{**}_{A}(Z_p, Z_p)$, and the algebraic spectral sequence $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ is associated with the "geometric" spectral sequence of Adams in the theory $H^*(\ , Z_p)$. Here $p > 2$ and A is the usual Steenrod algebra for Z_p-cohomology.*

We note that $\tilde{\tilde{E}}^{***}_\infty$ is associated with $\mathrm{Ext}^{***}_{A^U}(\Lambda, \Lambda) \otimes_Z Q_p$ (more precisely stated in § 12). *A priori* the spectral sequence $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ is cruder than the Adams spectral sequence in $H^*(\ , Z_p)$-theory and $\tilde{\tilde{E}}^{***}_\infty$ is bigger than the stable homotopy groups of spheres; on account of this, the Adams spectral sequence for cobordism theory constructed in this work can in principle be non-trivial, since $\tilde{\tilde{E}}_\infty$ is associated with $\mathrm{Ext}_{A^U}(\Lambda, \Lambda) \otimes_Z Q_p$.

We now recall the striking difference between the Steenrod algebra modulo 2 and modulo $p > 2$. As is shown in H. Cartan's well-known work, the Steenrod algebra for $p > 2$ in addition to the usual grading possesses a second grading ("the number of occurrences of the Bokšteĭn homomorphism") of a type which cannot be defined for $p = 2$ (it is only correct modulo 2 for $p = 2$). Therefore for $p > 2$ the cohomology $\mathrm{Ext}_A(Z_p, Z_p)$ has a triple grading in distinction to $p = 2$. In § 12 we show:

Lemma. *There is a canonical algebra isomorphism*

$$\tilde{\tilde{E}}^{***}_2 = \mathrm{Ext}^{***}_{\bar{A}_p^U}(\bar{\Lambda}_p, \bar{\Lambda}_p) = \mathrm{Ext}^{***}_{A}(Z_p, Z_p) \quad \textit{for } p > 2.$$

From this it follows that the algebra $\tilde{\tilde{E}}_2$ for the "algebraic Adams spectral sequence" $\tilde{\tilde{E}}_r$ is not associated, but is canonically isomorphic to the algebra $\mathrm{Ext}_A(Z_p, Z_p)$ which is the second term of the usual topological Adams spectral sequence.

If we assume that existence of the grading of Cartan type is not an accidental result of the algebraic computation of the Steenrod algebra A, but has a deeper geometric significance, then it is not out of the question that the whole Adams spectral sequence is not bigraded, but trigraded, as is the term

$$E_2 = \mathrm{Ext}^{***}_A(Z_p, Z_p), \quad p \neq 2.$$

From this, obviously, would follow the corollary: for $p > 2$ the algebraic Adams spectral sequence $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ coincides with the topological Adams spectral sequence (E_r, d_r), if the sequence (E_r, d_r) is trigraded by means of the Cartan grading, as is $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$. Therefore the orders $|\pi_{N+i}(S^N)|$ would coincide with $\left|\sum_{t-s=i} \operatorname{Ext}^{s,t}_{A^U}(\Lambda, \Lambda)\right|$ up to a factor of the form 2^h.

Moreover, this corollary would hold for all complexes without torsion (see § 12).

The case $p = 2$ is more complicated, although even there, there are clear algebraic rules for computing some differentials. This is indicated precisely in § 12.

In this way it is possible not only to prove the nonexistence of elements of Hopf invariant one by the methods of extraordinary cohomology theory as in [4] (see also §§ 9, 10), but also to calculate Adams differentials.

The content of this work are as follows: in §§ 1–3 we construct the Adams spectral sequence in different cohomology theories and discuss its general properties.

§§ 4, 5 are devoted to cohomology operations in cobordism theory. Here we adjoin Appendices 1 and 2. This is the most important part of the work.

§§ 6, 7 are largely devoted to the computations of $U^*(MSU)$ and $\operatorname{Ext}^{**U}_{A}(U^*(MSU), \Lambda)$.

§ 8 has an auxiliary character; in it we establish the facts from K-theory which we need.

§§ 10, 11 are devoted to computing $\operatorname{Ext}^{**}_{A^U}(\Lambda, \Lambda)$.

§§ 9, 12 were discussed above; they have a "theoretical" character.

Appendices 3 and 4 are connected with the problems of fixed points and the problem of connections between different homology theories from the point of view of homological algebra. Here the author only sketches the proofs.

The paper has been constructed as a systematic exposition of the fundamental theoretical questions connected with new methods and their first applications. The author tried to set down and in the simplest cases to clarify the most important theoretical questions, not making long calculations with the aim of concrete applications; this is explained by the hope mentioned earlier for the role of a similar circle of ideas in further developments of topology.

§ 1. The existence of the Adams spectral sequence in categories

Let S be an arbitrary additive category in which $\operatorname{Hom}(X, Y)$ are abelian groups for $X, Y \in S$, having the following properties:

1. There is a preferred class of sequences, called "short exact sequences" $(0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0)$, such that $f \cdot g = 0$ and also:

a) the sequence $(0 \to 0 \to 0 \to 0 \to 0)$ is short exact;

b) for commutative diagrams $\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \to & B' \end{array}$ or $\begin{array}{ccc} B & \to & C \\ \downarrow & & \downarrow \\ B' & \to & C' \end{array}$ there exists a unique map or short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

{extending the given square};

c) for any morphism $f\colon A \xrightarrow{f} B$ there exist unique short exact sequences $0 \to C' \to A \xrightarrow{f} B \to 0$ and $0 \to A \xrightarrow{f} B \to C \to 0$, where the objects C and C' are related by a short exact sequence $0 \to C' \to 0 \to C \to 0$ and C and C' determine each other.

We introduce an operator E in the category S by setting $C' = E^{-1}C$, or $C = EC'$, and we call E the suspension.

Let $\operatorname{Hom}^i(X,Y) = \operatorname{Hom}(X, E^iY)$ and $\operatorname{Hom}^*(X,Y) = \sum \operatorname{Hom}^i(X,Y)$.

2. For any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and any $T \in S$ there are uniquely defined exact sequences

$$\xrightarrow{\partial} \operatorname{Hom}^i(T,A) \xrightarrow{f_*} \operatorname{Hom}^i(T,B) \xrightarrow{g_*} \operatorname{Hom}^i(T,C) \xrightarrow{\partial} \operatorname{Hom}^{i+1}(T,A)$$

and

$$\xrightarrow{\delta} \operatorname{Hom}^i(C,T) \xrightarrow{g^*} \operatorname{Hom}^i(B,T) \xrightarrow{f_*} \operatorname{Hom}^i(A,T) \xrightarrow{\delta} \operatorname{Hom}^{i+1}(C,T),$$

which are functorial in T and in $(0 \to A \to B \to C \to 0)$. Here the homomorphisms f_*, g_*, f^*, g^* are the natural ones and the homomorphisms ∂, δ are induced by the projection $C \to EA$ in the short exact sequence $0 \to B \xrightarrow{g} C \to EA \to 0$ according to the above axiom 1.

3. In the category there exists a unique operation of direct sum with amalgamated subobjects: pairs $X, Y \in S$ and morphisms $Z \to X$, $Z \to Y$ define the sum $X +_Z Y$ and the natural maps $X \to X +_Z Y$ and $Y \to X +_Z Y$ such that the following sequences are exact:

$$0 \to X \to X +_Z Y \to C_1 \to 0,$$
$$0 \to Y \to X +_Z Y \to C_2 \to 0$$

(where C_1 and C_2 are defined by the exact sequences $0 \to Z \to X \to C_2 \to 0$ and $0 \to Z \to Y \to C_1 \to 0$). By definition we regard $X +_0 Y = X + Y$ {where 0 is the point object}.

Definition. We call two objects $X, Y \in S$ equivalent if there exists a third object $Z \in S$ and morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ inducing isomorphisms of the functor $\operatorname{Hom}^*(Z,\ )$ with $\operatorname{Hom}^*(X,\ )$ and $\operatorname{Hom}^*(Y,\ )$ and of the functor $\operatorname{Hom}^*(\ ,Z)$ with $\operatorname{Hom}^*(\ ,X)$ and $\operatorname{Hom}^*(\ ,Y)$. We call the maps f, g equivalences.

The transitivity of equivalences follows from the diagram

$$\begin{tikzcd}
X \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & H \arrow[dl] \\
& Z \arrow[dr] & & T \arrow[dl] & \\
& & Z +_y T & &
\end{tikzcd}$$

where all morphisms are equivalences (by virtue of the axiom on direct sums).

A spectrum in the category S is given by a sequence (X_n, f_n), where

$$f_n\colon EX_n \to X_{n+1} \quad \text{(direct spectrum)},$$
$$f_n\colon X_{n+1} \to EX_n \quad \text{(inverse spectrum)}.$$

By virtue of axioms 1 and 2 in the category S there is a canonical isomorphism

$$\operatorname{Hom}^*(X,Y) = \operatorname{Hom}^*(EX, EY).$$

Therefore for spectra there are defined the compositions

$$f_{n+k-1}\dots f_n\colon E^kX_n \to X_{n+k} \quad \text{(direct)}$$
$$f_n\dots f_{n+k-1}\colon X_{n+k} \to E^kX_n \quad \text{(inverse)}$$

which allow us to define passage to the cofinal parts of spectra.

For spectra $X = (X_n, f_n)$ and $Y = (Y_n, g_n)$ we define

$$\operatorname{Hom}^*(X,Y) = \varprojlim_n \varinjlim_m \operatorname{Hom}^*(X_n, Y_m)$$

in the case of direct spectra and

$$\operatorname{Hom}^*(X,Y) = \varprojlim_m \varinjlim_n \operatorname{Hom}^*(X_n, Y_m)$$

in the case of inverse spectra. Here, of course, let us keep in mind that in taking limits the grading in $\operatorname{Hom}^*(\ ,\ )$ is taken in the natural way. As usual, remember that the dimension of a morphism $E^\gamma T \to X_n$ is equal to $n + n_0 - \gamma$, where n_0 is a fixed integer, given together with the spectrum, defining the dimension of the mappings into X_n, and usually considered equal to zero. In addition, Hom and Ext here and later are understood in the sense of the natural topology generated by spectra.

Thus arise categories $\vec{S}$ (direct spectra over S) and $\overleftarrow{S}$ (inverse spectra). There are defined inclusions $S \to \vec{S}$ and $S \to \overleftarrow{S}$. We have the simple

Lemma 1.1. *In the categories $\vec{S}$ and $\overleftarrow{S}$ there exist short exact sequences $0 \to A \to B \to C \to 0$, where A, B, $C \in \vec{S}$ or A, B, $C \in \overleftarrow{S}$, satisfying axiom 1 of the category S and axiom 2 for the functor $\operatorname{Hom}^*(T,\ )$ if A, B, $C \in \vec{S}$ and $T \in \overleftarrow{S}$, and axiom 2 for $\operatorname{Hom}^*(\ ,T)$ if A, B, $C \in \overleftarrow{S}$ and $T \in \vec{S}$. In the categories $\vec{S}$ and $\overleftarrow{S}$ there exist direct sums with amalgamation satisfying axiom 3.*

Proof. The existence of direct sums with amalgamation in the categories $\vec{S}$ and $\overleftarrow{S}$ is proved immediately.

Let us construct short exact sequences in $\vec{S}$. Let A, $B \in \vec{S}$ and $f\colon A \to B$ be a morphism in $\vec{S}$. By definition, f is a spectrum of morphisms, hence is represented by a sequence $A_{n_k} \to B_{m_k}$ of maps. Consider the set of short exact sequences

$$(0 \to C_{n_k} \to A_{n_k} \to B_{m_k} \to 0) \quad \text{and} \quad (0 \to A_{n_k} \to B_{m_k} \to C'_{m_k} \to 0).$$

By axiom 1 of the category S we have spectra in $\vec{S}$, $C = (C_{n_k})$ and $C' = (C'_{m_k})$ and morphisms $C \to A$ and $B \to C'$. The corresponding sequences $0 \to C \to A \to B \to 0$ and $0 \to A \to B \to C' \to 0$ we call exact. Since passage to direct limit is exact, we have demonstrated the second statement of the lemma. For $\overleftarrow{S}$ analogously. Note that the spectra C and C' are defined only up to equivalences of the following form: in $\vec{S}$ the equivalence is an isomorphism of functors $\operatorname{Hom}^*(T,C)$ and $\operatorname{Hom}^*(T,C')$; in $\overleftarrow{S}$ an isomorphism of $\operatorname{Hom}^*(C,T)$ and $\operatorname{Hom}^*(C',T)$.

Obviously $C' = EC$. This completes the proof of the lemma. □

Definitions. a) Let $X \in \vec{S}$. The functor $\operatorname{Hom}^*(\ ,X)$ is called a "cohomology theory" and is denoted by X^*.

b) Let $X \in \overleftarrow{S}$. The functor $\operatorname{Hom}^*(X,\ )$ is called a "homology theory" and is denoted by X_*.

c) The ring $\mathrm{Hom}^*(X, X)$ for $X \in \vec{S}$ is called "the Steenrod ring" for the cohomology theory X^*. Analogously we obtain the Steenrod ring $\mathrm{Hom}^*(X, X)$ for $X \in \overleftarrow{S}$ (homology theory).

d) The Steenrod ring for the cohomology theory X^* is denoted by A^X, for the homology theory by A_X. They are graded topological rings with unity.

Note that an infinite direct sum $Z = \sum X_i$ of objects $X_i \in \vec{S}$ lies, by definition, in $\overleftrightarrow{S}$, if we let $Z_n = \sum\limits_{i \le n} X_i$ and $Z_n \to Z_{n-1}$ be the projection. Obviously, $X^*(\sum X_i)$ is an infinite-dimensional free A^X-module, being the limit of the direct spectrum

$$\mathrm{Hom}^*(Z_n, X) \to \mathrm{Hom}^*(Z_{n+1}, X),$$

where $X \in \vec{S} \subset \overleftrightarrow{S}$, all X_i are equivalent to the object X or $E^{\gamma_i} X$, and E is the suspension.

For an homology theory, if $X \in \vec{S}$, an infinite direct sum $\sum X_i$ is considered as the limit of the direct spectrum

$$\dots \to \sum_{i \le n} X_i \to \sum_{i \le n+1} X_i \to \dots,$$

where X_i is $E^{\gamma_i} X$, and therefore lies in $\overleftrightarrow{S}$, and the A_X-module $\mathrm{Hom}^*(X, \sum X_i)$ is free.

By X-free objects for $X \in \vec{S}$ we mean direct sums $\sum X_i$, where $X_i = E^{\gamma_i} X$ for arbitrary integers γ_i. Finite direct sums belong to $\vec{S}$.

There are simple properties which give the possibility of constructing the Adams spectral sequence by means of axioms 1–3 for the category S.

For any object $T \in \overleftarrow{S}$ and any X-free object $Z \in \vec{S}$ we have

$$\mathrm{Hom}^*(T, Z) = \mathrm{Hom}^*_{A^X}(X^*(Z), X^*(T)).$$

Let us give some definitions.

1) For an object $Y \in S$ we understand by a filtration in the category an arbitrary sequence of morphisms

$$Y = Y_{-1} \xleftarrow{f_0} Y_0 \leftarrow Y_1 \leftarrow \dots \xleftarrow{f_i} Y_i \leftarrow \dots.$$

2) The filtration will be called X-free for $X \in \vec{S}$ if $Z_i \in \vec{S}$ are X-free objects such that there are short exact sequences

$$0 \to Y_i \xrightarrow{f_i} Y_{i-1} \xrightarrow{g_i} Z_i \to 0, \quad Y_{-1} = Y.$$

3) By the complexes associated with the filtration, for any $T \in \overleftarrow{S}$, are meant the complexes (C_x, ∂_x) and (B_T, δ_T), where $(C_x)_i = X^*(Z_i)$ and $(B_T)^i = T_*(Z_i)$ and the differentials $\partial \colon (C_x)_i \to (C_x)_{i-1}$ and $\delta_T (B_T)_i \to (B_T)_{i+1}$ are the compositions

$$\partial_X \colon X^*(Z_i) \xrightarrow{g_i} X^*(Y_{i-1}) \xrightarrow{\delta} X^*(Z_{i-1})$$

and

$$\delta_T \colon T_*(Z_i) \xrightarrow{\partial} T_*(Y_i) \xrightarrow{{g_{i+1}}^*} T_*(Z_{i+1}).$$

4) An X-free filtration is called acyclic if (C_x, ∂_x) is acyclic in the sense that $H_0(C_x) = X^*(Y)$ and $H_i(C_x) = 0$ for $i > 0$.

From the properties (axioms 1 and 2) of the category S and Lemma 1.1 we obtain the obvious

Lemma 1.2. 1) *Each filtration* $(Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \dots)$ *defines a spectral sequence* (E_r, d_r) *with term* $E_1 = B_T$, $d_1 = \delta_T$, *associated with* $\mathrm{Hom}^*(T, Y)$ *in the sense that there are defined homomorphisms* $q_0\colon \mathrm{Hom}^*(T,Y) \to E_\infty^{0,*}$, $q_i\colon \mathrm{Ker}\, q_{i-1} \to E_\infty^{i,*}$, *where the filtration* $(\mathrm{Ker}\, q_i)$ *in* $T_*(Y) = \mathrm{Hom}^*(T,Y)$ *is defined by the images of compositions of filtration maps* $T_*(Y_i) \to T_*(Y)$.

2) *If the filtration is* X*-free, the complex* (B_T, δ_T) *is precisely* $\mathrm{Hom}^*_{A^X}(C_x, X^*(T))$ $\{$*with differential* $\mathrm{Hom}_{A^X}(\partial_x, 1)\}$.

3) *If the filtration is* X*-free and acyclic, then* E_2^{**} *in this spectral sequence coincides precisely with* $\mathrm{Ext}^{**}_{A^T}(X^*(Y), X^*(T))$.

Lemma 1.2 follows in the obvious way from axioms 1, 2 of the category S and Lemma 1.1.

However, the problem of the existence of X-free and acyclic filtrations is nontrivial. We shall give their construction in a special case, sufficient for our subsequent purposes.

Definition 1.1. The spectrum $X \in \vec{S}$ will be called stable if for any $T \in S$ and any j there exists an integer n such that $\mathrm{Hom}^s(T, X_m) = \mathrm{Hom}^m(T, X)$ for all $m \geq n$, $s \geq j$.

Definition 1.2. The cohomology theory X^*, $X \in \vec{S}$, defined by a stable spectrum X will be called Noetherian if for all $T \in S$ the A^X-module $X^*(T)$ is finitely generated over A^X.

We have

Lemma 1.3. *If* X^* *is a Noetherian cohomology and* $Y \in S$, *then there exists a filtration*

$$Y \leftarrow Y_0 \leftarrow \cdots \leftarrow Y_{i-1} \leftarrow Y_i \leftarrow \dots$$

such that $Z_i = Y_{i-1}/Y_i$ *is a direct sum* $Z_i = \sum\limits_j X_{n_j}$ *for large* n_j *and the complex* $C = \sum X^*(Z_i)$ *is acyclic through large dimensions. Here* $X = (X_n) \in \vec{S}$.

Proof. Take a large integer n and consider a map $Y \to \sum\limits_i X_n^{(i)}$ such that $X^*\left(\sum\limits_i X_n\right) \to X^*(Y)$ is an epimorphism, where X^* is a Noetherian cohomology theory.

By virtue of the stability of the spectrum X, for $Y \in S$ there is an integer n such that the map $Y \to \sum X_i$ factors into the composition $Y \xrightarrow{f_0} \sum\limits_i X_n \to \sum E^i X$, where $X_n \to X$ is the natural map. Therefore $X^*(\sum X_{n_i}) \to X^*(Y) \xrightarrow{X^*(f_0)} X^*(Y)$ is an epimorphism. Consider the short exact sequence

$$0 \to Y_0^{(n)} \to Y \xrightarrow{f_0} \sum_i X_n \to 0.$$

Obviously $X^*(Y_0^{(n)}) = \mathrm{Ker}\, X^*(f_0)$ and $Y_0^{(n)} \in S$. Now take a large number $n_1 \gg n$ and do the same to $Y_0^{(n)}$ as was done to Y, and so on. We obtain a filtration

$$Y \leftarrow Y_0^{(n)} \leftarrow Y_1^{(n,n_1)} \leftarrow Y_2^{(n,n_1,n_2)} \leftarrow \dots,$$

where the Z_i are sums of objects of the form $\sum X_{m_k}$, with m_k very large.

By definition, $C = \sum\limits_i X^*(Z_i)$ is an acyclic complex through large dimensions. $\square$

Definition 1.3. A stable spectrum $X = (X_n)$ in the category $\vec{S}$ is called acyclic if for each object $T \in S$ we have the equalities:

a) $\operatorname{Ext}^{i,t}_{A^X}(X^*(X_n), X^*(T)) = 0$, $i > 0$, $t - i < f_n(i)$, where $f_n(i) \to \infty$ as $n \to \infty$;

b) $\operatorname{Hom}^t_{A^X}(X^*(X_n),\, X^*(T)) = \operatorname{Hom}^t(T, X)$ for $t < f_n$, and $f_n \to \infty$ as $n \to \infty$.

The so-called Adams spectral sequence (E_r, d_r) with E_2-term $E_2 = \operatorname{Ext}^{**}_{A^X}(X^*(Y), X^*(T))$ arises in the following cases:

1. If in the category $\vec{S}$ there exists an X-free acyclic filtration $Y = Y_{-1} \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_{i-1} \leftarrow Y_i \ldots$, on the basis of Lemma 1.2. However, such a filtration does not always exist, since the theory X^* in the category $\vec{S}$ does not have the exactness property.

2. If $Y \in S$, $T \in S$ and the theory X^* is stable, Noetherian and acyclic, then, by virtue of Lemma 1.3, there exists a filtration

$$Y_{-1} = Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_i \leftarrow \ldots,$$

where the Y_i/Y_{i+1} are sums of objects X_n, for numbers n which may be taken as large as we want, with the filtration acyclic through large gradings. For such a filtration, the corresponding spectral sequence (E_r, d_r) has the term $E_2^{s,t} = \operatorname{Ext}_{A^{X^{s,t}}}(X^*(Y), X^*(T))$ through large gradings, by the definition of acyclicity for the theory X^*.

In this way we obtain:

Theorem 1.1. *For any stable Noetherian acyclic cohomology theory $X \in \vec{S}$ and objects $Y, T \in S$, one can construct an Adams spectral sequence (E_r, d_r), where $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ and the groups $\sum_{t-s=m} E_\infty^{s,t}$ are connected to $\operatorname{Hom}^m(t, Y)$ in the following way: there exist homomorphisms*

$$q_i\colon \operatorname{Ker} q_{i-1} \to E_\infty^{i,i+m}, \quad i \geq 0,$$

where

$$q_0\colon \operatorname{Hom}^m(T, Y) \to \operatorname{Hom}^m_{A^X}(X^*(Y), X^*(T))$$

is the natural homomorphism.

The Adams spectral sequence is functorial in T and Y.

Remark 1.1. The homomorphism $q_1\colon \operatorname{Ker} q_0 \to \operatorname{Ext}^{1,*}_{A^X}(X^*(Y), X^*(T))$ is called the "Hopf invariant."

Remark 1.2. For objects $T, Y \in S$ and a stable Noetherian acyclic homology theory $X \in \overleftarrow{S}$ one can also construct an Adams spectral sequence (E_r, d_r) such that $E_2 = \operatorname{Ext}^{**}_{A_X}(X_*(T), X_*(Y))$. In this spectral sequence, $d_r\colon E_r^{p,q} \to E_r^{p-r,q+r+1}$, and the homomorphisms q_i are such that

$$q_i\colon \operatorname{Ker} q_{i-1} \to E_\infty^{i,i+n},$$

where

$$q_0\colon \operatorname{Hom}^n(T, Y) \to \operatorname{Hom}^n_{A^X}(X^*(Y), X^*(T))$$

is the natural homomorphism and A_X is the Steenrod ring of the homology theory X_*.

The proof of Theorem 1.1 is a trivial consequence of Lemmas 1.1–1.3 and standard verifications of the functoriality of the spectral sequence in the case where the filtration is X-free and acyclic.

We shall be specially interested in those cases when the Adams spectral sequence converges exactly to $T_*(Y) = \mathrm{Hom}^*(T, Y)$. Let us formulate a simple criterion for convergence:

(A) If there exists an X-free filtration $Y_{-1} = Y \leftarrow Y_0 \leftarrow \cdots \leftarrow Y_i$ (not necessarily acyclic) such that for any j, l there exists a number $i > l$, depending on j and l, for which $\sum_{k \le j} \mathrm{Hom}^k(T, Y_i) = 0$, then the Adams spectral sequence converges exactly to $\mathrm{Hom}^*(T, Y)$. Criterion (A) does not appear to be the most powerful of those possible, but it will be fully sufficient for the purposes of the present work.

§ 2. The S-category of finite complexes with distinguished base points. Simplest operations in this category

The basic categories we shall be dealing with are the following:

1. The S-category of finite complexes and the categories $\vec{S}$ and $\overleftarrow{S}$ over it.

2. For any flat Z-module G (an abelian group such that $\otimes_Z G$ is an exact functor) we introduce the category $S \otimes_Z G$, in which we keep the old objects of S and let $\mathrm{Hom}(X, Y) \otimes_Z G$ be the group of morphisms of X to Y in the new category $S \otimes_Z G$. Important examples are: a) $G = Q$, b) $G = Q_p$ (p-adic integers). The respective categories will be denoted by S_0 for $G = Q$ and S_p for $G = Q_p$, p a prime.

3. In S (or S_p for $p > 0$) we single out the subcategory D (or $D_p \subset S_p$) consisting of complexes with torsion-free integral cohomology. It should be noted that the subcategories D and D_p are not closed with respect to the operations entering in axiom 1 for S-categories.

These subcategories, however, are closed with respect to the operations referred to, when the morphism $f\colon A \to B$ is such that $f^*\colon H^*(B, Z) \to H^*(A, Z)$ is an epimorphism.

Therefore the category D is closed under the construction of X-free acyclic resolutions (only acyclic), and it is possible to study the Adams spectral sequence only for $X, Y \in D$ (or D_p).

The following operations are well known in the S-category of spaces of the homotopy type of finite complexes (with distinguished base points):

1. The connected sum with amalgamated subcomplex $X +_Z Y$, becoming the wedge $X \vee Y$ if $Z = 0$ (a point).

2. Changing any map to an inclusion and to a projection (up to homotopy type): axiom 1 of § 1.

3. Exactness of the functors $\mathrm{Hom}^*(X,\ )$ and $\mathrm{Hom}^*(\ , X)$.

4. The tensor product $X \otimes Y = X \times Y / X \vee Y$.

5. The definition, for a pair $X, Y \in S$, of $X \otimes_Z Y$, given multiplications $X \otimes Z \to X$ and $Z \otimes Y \to Y$.

6. Existence of a "point"-pair $P = (S^0, *)$ such that $X \otimes P = X$ and $X \otimes_p Y = X \otimes Y$.

All these operations are carried over in a natural way into the categories S_0, S_p, $\vec{S}$, $\overleftarrow{S}$, $\vec{S}_p$ and $\overleftarrow{S}_p$.

The cohomology theory X^* will be said to be multiplicative if there is given a multiplication

$$X \otimes X \to X, \quad X \in \vec{S}.$$

The cohomology theory Y^* is said to act on the right [left] of the theory X^* if there is given a multiplication $X \otimes Y \to X$ or $Y \otimes X \to X$.

The previously mentioned theory P^*, generated by the point spectrum $P = (S^0, *)$, operates on all cohomology theories and is called "cohomotopy theory." Its spectrum, of course, consists of the spheres (S^n). It is obviously multiplicative, because $P \otimes P = P$.

We now describe an interesting operation constructed on a multiplicative cohomology theory $X = (X_n) \in \vec{S}$ of a (not necessarily stable) spectrum of spaces.

Let (H_n^i) be the spectrum of spaces of maps $H_n^i = \Omega^{n-i} X_n = \mathrm{Map}(S^{n-i}, X_n)$. Since X is multiplicative and $P \otimes P = P$, we have a multiplication

$$H_n^i \times H_m^j \to H_{m+n}^{i+j}.$$

Let now $i = j = 0$. Then

$$H_n^0 \times H_m^0 \to H_{m+n}^0.$$

Suppose that the cohomology ring $X^*(X)$ and all $X^*(K)$ have identities (the cohomology theory contains scalars with respect to multiplication $X \otimes X \to X$). Consider in the space H_n^0 the subspace $H_n \subset H_n^0 = \Omega^n X_n$ which is the connected component of the element $1 \in X^0(P)$. We have a multiplication

$$\begin{array}{ccccc} H_n & \times & H_n & \longrightarrow & H_n \\ \downarrow & & \downarrow & & \downarrow \\ H_n^0 & \times & H_n^0 & \longrightarrow & H_n^0 \end{array}$$

induced by the inclusion $H_n \subset H_n^0$.

Let $\pi(K, L)$ be the homotopy classes (ordinary, non-stable) of maps $K \to L$, and let $\Pi^{-1}(K) = \lim\limits_{n\to\infty} \pi(K, H_n)$. Obviously $\Pi^{-1}(K)$ is a semigroup with respect to the previously introduced multiplication. We have

Lemma 2.1. $\Pi^{-1}(K)$ *is a group, isomorphic to the multiplicative group of elements of the form* $\{1+x\} \in X^0(K)$*, where* x *ranges over the elements of the group* $X^0(K)$ *of filtration* > 0.

The proof of Lemma 2.1 easily follows from the definition of the multiplication $H_n \times H_m \to H_{m+n}$ by means of the multiplication in the spectrum X.

Therefore the spectrum (H_n) defines an "H-space" and the spectrum $BH = (BH_n)$ has often been defined. The set of homotopy classes $\pi(K, BH) = \varinjlim\limits_n \pi(K, BH_n)$ we denote by $\Pi^0(K)$, while $\Pi^0(EK) = \Pi^{-1}(K)$ by definition, where E is the suspension.

The following fact is. evident:

If $K = E^2 L$, then $\Pi^0(K) = X^1(K)$; therefore in the S-category $\Pi^0(K)$ is simply $X^1(K)$. As we have already seen by Lemma 2.1, this is not so for complexes which are only single suspensions, where $\Pi^0(EL)$ consists of all elements of the form $\{1+x\}$ in $X^0(L)$ under the multiplication in $X^0(L)$.

An important example. Let $X = P = (S^n, *)$. Then the spectrum H_n with multiplication $H_n \times H_n \to H_n$ is homotopic to the spectrum $\tilde{H}_n$ (maps of degree $+1$ of $S^n \to S^n$ with composition $\tilde{H}_n \times \tilde{H}_n \to \tilde{H}_n$).

The J-functor of Atiyah is the image of $\bar{K}(L) \to \Pi^0(L)$ in our case $X = P$. In particular, in an S-category $L = E^2 L'$ we have that $\Pi^0(L)$ is $P^*(L)$; in the case $L = EL'$, $\Pi^0(L)$ depends on the multiplication in $P^*(L')$.

Besides the enumerated facts relating to the S-category of finite complexes one should also mention the existence of an anti-automorphism $\sigma\colon S \to S$ of this S-category which associates to a complex X its S-dual complex (complement in a sphere of high dimension). The operator σ induces

$$\sigma\colon \vec{S} \to \overleftarrow{S}, \quad \sigma\colon \overleftarrow{S} \to \vec{S}, \quad \sigma^2 = 1.$$

Since $\operatorname{Hom}(X, Y) = \operatorname{Hom}(\sigma Y, \sigma X)$ and σX is a homology theory in $\overleftarrow{S}$ if X is a cohomology theory, then the duality law of Alexander–Pontrjagin is, obviously, the equality $X^*(K) = \sigma X_*(\sigma K)$, and by an X-homology manifold is meant a complex K such that $X^i(K) = \sigma K_{n-i}(K)$ in the presence of some natural identification of $\sigma X_*(K)$ with $\sigma X_*(\sigma K)$; for example, if K is a smooth manifold, then $\sigma(K)$ according to Atiyah [6] is the spectrum of the Thom complex of the normal bundle in a sphere. In the presence of a functorial Thom isomorphism in X^*-theory for some class of manifolds, we obtain Poincaré–Atiyah duality.

Let $X \in \vec{S}$, $Y \in S$, $T \in \overleftarrow{S}$. In § 1 we constructed the Adams spectral sequence with E_2 term equal to $\operatorname{Ext}^{**}_{A^X}(X^*(Y), X^*(T))$.

The law of duality for Adams spectral sequences reads:

The cohomology Adams spectral sequence (E_r, d_r) with term $E_2 = \operatorname{Ext}_{A^X}(X^*(Y), X^*(T))$ is canonically isomorphic to the homology Adams spectral sequence (E'_r, d'_r) with term $E'_2 = \operatorname{Ext}_{A_{\sigma X}}(\sigma X_*(\sigma Y), \sigma X_*(\sigma T))$. The homology Adams spectral sequence for $X = \sigma X = P$ was investigated by A. S. Miščenko [16].

Let us introduce the important notion of $(m-1)$-connected spectra.

Definition 2.1. The spectrum $(X_n, f_n) = X$ (direct) is called $(m-1)$-connected if each object X_n is $(n+m-1+n_0)$-connected, where the integer n_0 is defined in § 1. Analogously for inverse spectra.

Usually $n_0 = 0$ and X_n is $(n+m-1)$-connected, $f_n\colon EX_n \to X_{n+1}$ for direct spectra. Analogously for inverse.

Finally, we should formulate two obvious facts here, which will be used later.

Lemma 2.2. a) *If $X \in \vec{S}$, the cohomology theories EX and X have the same Adams spectral sequences for any Y and T for which the sequences exist* (*here $Y \in \vec{S}$, $T \in \overleftarrow{S}$*).

b) *Furthermore, if $\tilde{X} = \sum E^{\gamma_i} X$ is a direct sum, where $\gamma_i \to \infty$ for $i \to \infty$, then the theory $\tilde{X}^*$ defines the same Adams spectral sequence as the theory X^*.*

Proof. Since each $\tilde{X}$-free acyclic resolution is at the same time an X-free resolution, the lemma at once follows from the definitions. □

From the lemma follows

Corollary 2.1. *For any stable Noetherian acyclic cohomology theory $X \in \vec{S}$ and any $Y \in S$ and $T \in S$, all groups $\operatorname{Ext}^{st}_{A^X}(X^*(Y), X^*(T)) \otimes_Z Q = 0$ for $s > 0$.*

Proof. Since a stable spectrum X in the category $S_0 = S \otimes_Z Q$ is equivalent to a sum $\sum E^{\gamma_i} K(Z)$ of Eilenberg–MacLane spectra for $\pi = Z$, and since for $X' = K(Z)$ the ring $A^{X'} \otimes_Z Q$ is trivial, it follows that all $\operatorname{Ext}^s_{A^X}(\ ,\ ) \otimes_Z Q = 0$ for $s > 0$, since $\operatorname{Ext}^s_{A^{X'}} \otimes_Z Q(\ ,\ ) = 0$ for $s > 0$ and by virtue of Lemma 2.2. □

§ 3. Important examples of cohomology and homology theories. Convergence and some properties of Adams spectral sequences in cobordism theory

We list here the majority of the most interesting cohomology theories.

1. $X = K(\pi)$, where $X_n = K(\pi, n)$. This theory is multiplicative if π is a ring, and $X^* = H^*(\ ,\pi)$. The case $\pi = Z_p$ is well known, having been studied in many works [1, 9, 15, 17, 18]. The spectral sequence was constructed by Adams in [1], where its convergence was proved ($\pi = Z_p$). The ring A^X is the usual Steenrod algebra over Z_p. Here the commonly studied case is $p = 2$. The case $p > 2$ was first studied in [24].[3]

The criterion (A) for the convergence of the Adams spectral sequence applies easily in the category $S_p = S \otimes_Z Q_p$ under the condition that Y is a complex with $\pi_i^*(Y) \otimes_Z Q_p$ finite groups, in which case there is a nonacyclic resolution (the Postnikov system) which is X-free.

In the case $\pi = Z$, as is easy to see, the applicability of criterion (A) in the category S itself again easily follows from the properties of the usual contractible spaces and Postnikov systems (see, for example, [16]).

2. **Homotopy and cohomotopy theories**. Let P be the point in S, where $P_n = S^n$. The theory P_* is that of stable homotopy groups, and P^* that of stable cohomotopy groups. The (Eckmann–Hilton) dual of this spectrum is $K(Z)$ and the theory $H^*(\ , Z)$. Similarly, the spectra $P_{(m)} = P/mP$ (m an integer) are Eckmann–Hilton duals of the spectra $K(Z_m)$.

For the homology theory $P_*(X)$ the proof of convergence of the homology Adams spectral sequence with term $E_2 = \mathrm{Ext}^{**}_{A_P}$ is similar to the proof for the cohomology spectrum $K(Z)$ by virtue of Eckmann–Hilton duality and follows from criterion (A) of § 1.

The proof of convergence for the theory $P_{(m)*}$ analogously proceeds from the method of Adams for $K(Z_m)$. These theories were investigated in [16].

By virtue of the law of duality for the Adams spectral sequence (cf. § 2) and the fact that $\sigma P = P$ and $\sigma P_{(m)} = P_{(m)}$, we obtain convergence also in cohomotopy theory, where σ is the S-duality operator.

3. **Stable K-theory.**

a) Let $k = (k_n)$, where $\Omega^{2n} k_{2n} = BU \times Z$, and the complexes k_n are $(n-1)$-connected. Then k_{2n} is the $(2n-1)$-connected space over BU and the inclusion $x\colon k_{2n} \to k_{2n-2}$ is defined by virtue of Bott periodicity.

Here $k^i = K^i$ for $i \le 0$ for K^* the usual complex K-theory, and if $H^*(L, Z)$ has no torsion, then $k^{2i}(L)$ is the subgroup of $K^{2i}(L)$ consisting of elements of filtration $\ge i$.

b) Let $kO = (kO_n)$, where $\Omega^{8n} kO_{8n} = BO \times Z$, and all kO_n are $(n-1)$-connected. We have $kO^{[i]} = (kO_n^{[i]})$ where $\Omega^{8n} kO^{[i]}_{8n-i} = BO \times Z$, $kO^{[0]} = kO$ and the $kO_n^{[i]}$ are $(n-1)$-connected. Here i is to be taken $\mod 8$.

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[3] In Theorem 2 of the author's work [24] there are erroneous computations, not influencing the basic results. We note also the peculiar analogues, first discovered and applied in [24], to the Steenrod powers in the cohomology of a Hopf algebra with commutative diagonal. It turns out that for all $p > 2$ these "Steenrod powers" $St\,p^i$ are defined and nontrivial for $i \equiv 0, 1 \pmod{p-1}$, $i \ge 0$. These peculiar operations have never been noted in more recent literature on these questions, although they are of value; for example, they reflect on the multiplicative formulas of Theorem 2 in [24] for $p > 2$.

It is easy to show that in the category $S \otimes_Z Z[1/2]$ all spectra $kO^{[i]}$ coincide up to suspension, and the spectrum k is a sum of two spectra of the type $k = kO + E^2 kO^{[2]}$.

4. **Cobordism.** Let $G = (G_n)$ be a sequence of subgroups of the groups $O_{\alpha(n)}$ where $\alpha(n+1) > \alpha(n)$ and $\alpha(n) \to \infty$ for $n \to \infty$ with $G_n \subset G_{n+1}$ under the inclusion $O_{\alpha(n)} \subset O_{\alpha(n+1)}$. There arise natural homomorphisms $BG_n \to BG_{n+1}$ and a direct spectrum (not in the S-category) BG. With this spectrum BG is connected the spectrum of Thom complexes $MG = (MG_n)$ in the category $\vec{S}$.

Examples:

a) The spectrum $G = (e)$, $e \subset O_n$; then $MG = P$;

b) $G = O$, SO, Spin, U, SU, Sp; then $MG = MO$, MSO, $M\,\mathrm{Spin}$, MU, MSU, MSp have all been investigated. All of them are multiplicative spectra and the corresponding cohomology rings have commutative multiplication with identity. Let us mention the known facts:

$$1)\quad MO = \sum_i E^{\lambda_i} K(Z_2);$$

$$2)\quad MSO \otimes_Z Q_2 = \sum_j E^{\lambda_j} K(Z) + \sum_q E^{\mu_q} K(Z_2) \quad \text{(see [17, 18, 23]);}$$

$$3)\quad MG \otimes_Z Q_p = \sum_k E^{\lambda_k} M_{(p)},$$

where $H^*(M_{(p)}, Z_p) = A/\beta A + A\beta$, A is the Steenrod algebra over Z_p and β is the mod p Bokšteĭn homomorphism. This result holds for $G = SO$, U, Spin, Sp for $p > 2$, $G = U$ for $p \geq 2$, and $G = SU$ for $p > 2$ with reduction of the number of terms λ_k corresponding to certain partitions ω (see [15, 17, 18, 26])

$$4)\quad M\,\mathrm{Spin} \otimes_Z Q_2 = \sum_s E^{\lambda_s} K(Z_2) + \sum_q E^{\mu_q} kO + \sum_l E^{\delta_l} kO^{[2]}.$$

Facts (1) and (2) are known, and fact (4) is given in a recent result of Anderson–Brown–Peterson [10].

c) $G = T$, where $T^n = G_n \subset U_n \subset O_{2n}$ is the maximal torus. This leads to MG, again a multiplicative spectrum since $MT^{m+n} = MT^m \otimes MT^n$.

Let us mention the structure of the cohomology $M^*_{(p)}(P)$, where $P = (S^0, *)$ is a point, $M^*_{(p)}(P) = Q_p[x_1, \dots, x_i, \dots]$ (polynomials over Q_p) with $\dim x_i = -2_{p^i} + 2$ and $M^0_{(p)}(P)$ the scalars Q_p.

The ring $U^*(P)$ for $G = U$ (spectrum MU) is $Z[y_1, \dots, y_i, \dots]$, where $\dim y_i = -2i$.

For the spectra $M_{(p)} = X$ and $MU = X$ we have the important, simply derived

Lemma 3.1. *If $a \in A^X$ is some operation for $X = M_{(p)} \in \vec{S} \otimes_Z Q_p$ or $X = MU \in \vec{S}$ which operates trivially on the module $X^*(P)$, then the operation a is itself trivial.*

Proof. Since $a \in \mathrm{Hom}^*(X, X)$, the operation a is represented by a map $X \to E^\gamma X$. Since $\pi_*(X) \otimes_Z Q_p$ and $H^*(X, Q_p)$ for $X = M_{(p)}$ and $X = MU$ do not have torsion, it follows from obstruction theory in the usual fashion that the map $a\colon X \to E^\gamma X$ is completely determined by the map $a_*\colon \pi_*(X) \to \pi_*(X)$, which represents the operation a on $X^*(P)$, for $X^{-i}(P) = \pi_i^S(X)$. End of proof of lemma. $\square$

Since $MU \otimes_Z Q_p = \sum_k E^{\lambda_k} M(p)$, we have the following fact:

$$\mathrm{Ext}_{A^X}(X^*(K), X^*(L)) \otimes_Z Q_p = \mathrm{Ext}_{A^Y}(Y^*(K), Y^*(L)),$$

where $X = MU$, $X^* = U^*$, and $Y = M_{(p)}$, $Y^* = U^*_{(p)} = M^*_{(p)}$; we denote $M^*_{(p)}$ by $U^*_{(p)}$ and MU^* by U^*. Both are multiplicative theories. This fact, that the Ext terms and more generally the Adams spectral sequences coincide, follows from the fact that $MU \otimes_Z Q_p = \sum_k E^{\lambda_k} M_{(p)}$, as indicated in § 2, since $MU \otimes_Z Q_p$ is a sum of suspensions of a single theory $M_{(p)}$ and Q_p is a flat Z-module.

For any multiplicative cohomology theory X^* there is in the ring A^X the operation of multiplication by the cohomology of the spectrum P, since the spectrum P acts on every spectrum: $P \otimes X = X$. In this way there is defined a homomorphism $X^*(P) \to A^X$, where $X^*(P)$ acts by multiplication. From now on we denote the image of $X^*(P) \to A^X$ by $\Lambda \subset A^X$, the ring of "quasiscalars."

For spectra $X = M_{(p)}$, $X = MU$ we have the obvious

Lemma 3.2. *Let $Y \in \vec{D}_p$ be a stable spectrum. Then $X^*(Y)$ is a free Λ-module, where the minimal dimension of the Λ-free generators is equal to n, if $Y = (Y_m)$ is a spectrum of $(n+m)$-connected complexes Y_m.*

The lemma obviously follows from the fact that in the usual spectral sequence in which $E_2 = H^*(Y, X^*(P)) = H^*(Y, \Lambda)$ for $X = M_{(p)}, MU$ all differentials $d_r = 0$ for $r \geq 2$, and the sequence converges to $X^*(Y)$.

Now let Y satisfy the hypotheses of Lemma 3.2. We have

Lemma 3.3. *There exists an X-free acyclic resolution for $X = M_{(p)}, MU\colon Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_i \leftarrow \ldots$, where the stable spectra $Y_i \in D_p$ are $(m+2i-1)$-connected, if Y is a stable $(m-1)$-connected spectrum in $\vec{D}_p$. Furthermore, if $X = M_{(p)}$, the spectrum Y_i is $(m+2i(p-1)-1)$-connected.*

Proof. Since Y is an $(m-1)$-connected stable spectrum, the minimal Λ-free generator of the module $X^*(Y)$ has dimension m, and the set of m-dimensional Λ-free generators corresponds to the generators of the group $H^m(Y, Q_p)$.

Choose in correspondence with this system of Λ-free generators an X-free object C_0 and construct in a natural way a map $f_0\colon Y \to C_0$ such that

$$f_{0*}\colon H_{m+k}(Y, Q_p) \to H_{m+k}(C_0, Q_p)$$

is an isomorphism for $k \leq 1$. Obviously C_0 is also $(m-1)$-connected. Then the object Y_0 such that $0 \to Y_0 \to Y \to C_0 \to 0$ is a short exact sequence has the property that it is also a stable spectrum in $\vec{D}_p$. Furthermore, since f_{0*} is an isomorphism on the groups $H_{m+k}(Y, Q_p)$ for $k \leq 1$, the object Y_0 is m-connected in $\vec{D}_p$. If $X = M_{(p)}$, then it may be shown furthermore that in constructing C_0 in correspondence with Λ-free generators in $X^*(Y)$ the map $f_{0*}\colon H_j(Y, Q_p) \to H_j(C_0, Q_p)$ is an isomorphism for $j \leq m + 2p - 3$ and a monomorphism for $j = m + 2p - 2$.

Therefore Y_0 will be $(m+2p-3)$-connected if Y is $(m-1)$-connected. The result for $X = MU$ in the category D is obtained by substituting the minimal $p = 2$. This process we continue further, and obviously obtain the desired filtration. The lemma is proved. $\square$

Now let $T \in S$ be a finite complex. By virtue of Lemma 3.3 we have that $\mathrm{Hom}^j(T, Y_i) = 0$ for large i. Therefore the Adams spectral sequence converges to $\mathrm{Hom}^*(T, Y)$ by virtue of criterion (A) in § 1.

From these lemmas follows

Theorem 3.1. *For any stable $(m-1)$-connected spectrum $Y \in \vec{D} \subset S$, $X = MU$ and any finite complex $T \in S$ of dimension n, the Adams spectral sequence (E_r, d_r) with term $E_2 = \mathrm{Ext}^{**}_{A^X}(X^*(Y), X^*(T))$ exists and converges exactly to $\mathrm{Hom}^*(T, Y)$; moreover $\mathrm{Ext}^{s,t}_{A^X}(X^*(Y), X^*(T)) = 0$ for $t - s < s + m - n$. Furthermore, the p-primary part $\mathrm{Ext}^{s,t}_{A^X}(X^*(Y), X^*(T)) \otimes_Z Q_p = 0$ for $t < 2s(p-1) + m - n$.*

The proof follows immediately from the fact that if T is an n-dimensional complex and Y is a k-connected spectrum, then $\mathrm{Hom}^i_{A^X}(X^*(Y), X^*(T)) = 0$ for $i < k - n$ and from Lemma 3.3 for $X = MU$.

The statement about the p-components of the groups Ext follows from Lemma 3.3 for the spectrum $M_{(p)}$, since

$$MU \otimes_Z Q_p = \sum_k E^{\lambda_k} M(p).$$

The theorem is proved.

Note that for $X = MU, M_{(p)}$, stable spectra Y and finite complexes T, all groups $\mathrm{Ext}^{s,t}_{A^X}$ are torsion groups for $s > 0$, as derived in § 2.

Let $X = M_{(p)}$, $Y \in \vec{D}_p$ be a stable spectrum, and $T \in S \otimes_Z Q_p$, where the cohomology $H^*(Y, Q_p)$ and $H^*(T, Q_p)$ is different from zero only in dimensions of the form $2k(p-1)$.

Under these hypotheses we have

Theorem 3.2. a) *The groups $\mathrm{Hom}^i_{A^X}(X^*(Y), X^*(T))$ are different from zero only for $i \equiv 0 \mod 2p-2$;*

b) *A^X is a graded ring in which elements are non-zero only in dimensions of the form $2k(p-1)$;*

c) *The groups $\mathrm{Ext}^{s,t}_{A^X}(X^*(Y), X^*(T))$ are different from zero only for $t \equiv 0 \mod 2p-2$;*

d) *In the Adams spectral sequence (E_r, d_r) all differentials d_r are equal to zero for $r \not\equiv 1 \mod 2p-2$.*

Proof. Since the ring $X^*(P)$ (P a point) is nontrivial only in dimensions of the form $2k(p-1)$, statement (b) follows from Lemma 3.1. Statement (a) follows from (b) and the hypotheses on $X^*(T)$. From (b) it follows that it is possible to construct an A^x-free acyclic resolution for $X^*(Y)$ in which generators are all of dimensions divisible by $2p-2$. From this (c) follows. Statement (d) comes from (c) and the fact that $d_r(E^{s,t}_r) \subset E^{s+r,t+r-1}_r$. Q.E.D. □

Corollary 3.1. *For $X = MU$, $Y = P$, $T = P$ the groups $\mathrm{Ext}^{s,t}_{A^X}(X^*(P), X^*(P)) \otimes_Z Q_p = 0$ for $t < 2s(p-1)$ and for $t \not\equiv 0 \mod 2p-2$, and the differentials d_r on the groups $E_r \otimes_Z Q_p$ are equal to zero for $r \not\equiv 1 \mod 2p-2$.*

From now on we always denote the cohomology X^* for $X = MU$ by U^* and the Steenrod ring A^X by A^U. In the next section this ring will be completely calculated.

As for the question about the existence of the Adams spectral sequence in the theory U^* and category S, we have

Lemma 3.4. *The cohomology theory U^* is stable, Noetherian, and acyclic.*

Proof. The stability of the spectrum $MU = (MU_n)$ is obvious. Let T be a finite complex. We shall prove that $U^*(T)$ is finitely generated as a Λ-module, so of course as an A^U-module, where $\Lambda = U^*(P) \subset A^U$. Consider the spectral sequence (E_r, d_r) with term $E_2 = H^*(T, \Lambda)$, converging to $U^*(T)$. Since T is a finite complex, in this spectral sequence only a finite number of differentials $d_r, \dots, d_k$ are different from zero, $d_i = 0$ for $i > k$. Note that all d_r commute with Λ, and E_∞ as a Λ-module is associated with $U^*(T)$, where $E_\infty = E_k$. The generators of the Λ-module E_2 lie in $H^*(T, Z)$; $\Lambda^0 = Z$ and they are finite in number: $u_1^{(r)}, \dots, u_{l_r}^{(r)} \in E_2^{*,0}$. Note that $d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1}$. Denote by $\Lambda_N \subset \Lambda$ the subring of polynomials in generators of dimension $\leq 2N$, $\Lambda = U^*(P) = \Omega_U$. The ring Λ_N is Noetherian. Similarly, let $\Lambda^N \subset \Lambda$ be the subring of polynomials in generators of dimension $> 2N$. Obviously, $\Lambda = \Lambda_N \otimes_Z \Lambda^N$ and Λ has no torsion.

Assume, by induction, that the Λ-module E_r has a finite number of Λ-generators $u_1^{(r)}, \dots, u_{l_r}^{(r)}$ and there exists a number N_r such that $E_r = \tilde{E}_r \otimes_Z \Lambda^{N_r}$, where $\tilde{E}_r$ is a Λ_{N_r}-module with the finite number of generators $u_1^{(r)}, \dots, u_{l_r}^{(r)}$, above. Consider $d_r(u_j^{(r)}) = \sum_k \lambda_{kj}^{(r)} u_k^{(r)}$, where $\lambda_{kj}^{(r)} \in \Lambda$. Let $\dim \lambda_{kj}^{(r)} \leq \tilde{N}_r$ for all k, j. Set $N_{r+1} = \operatorname{Max}(\tilde{N}_r, N_r)$. Then $\lambda_{kj}^{(r)} \in \Lambda_{N_{r+1}}$. By virtue of the Noetherian property of the ring $\Lambda_{N_{r+1}}$, the module $H(\tilde{E}_r \otimes_Z \Lambda_{N_r}^{N_{r+1}}, d_r)$ is finitely generated, where Λ_{N_r} is generated by polynomial generators of dimension $N_r < k \leq N_{r+1}$ and $\Lambda_{N_r}^{N_{r+1}} \otimes_Z \Lambda^{N_{r+1}} = \Lambda^{N_r}$. Since

$$
\begin{aligned}
H(E_r, d_r) = E_{r+1} = H(\tilde{E}_r \otimes_Z \Lambda^{N_r}, d_r) &= H(\tilde{E}_r \otimes_Z \Lambda_{N_r}^{N_{r+1}} \otimes \Lambda^{N_{r+1}}, d_r) \\
&= H(\tilde{E}_r \otimes \Lambda_{N_r}^{N_{r+1}}, d_r) \otimes \Lambda^{N_{r+1}},
\end{aligned}
$$

if we set $\tilde{E}_{r+1} = H(\tilde{E}_r \otimes \Lambda_{N_r}^{N_{r+1}}, d_r)$, then E_{r+1} is a finitely generated $\Lambda_{N_{r+1}}$-module, and $E_{r+1} = \tilde{E}_{r+1} \otimes_Z \Lambda^{N_{r+1}}$.

Taking $N_2 = 0$, we complete the induction, since for some $k, E_k = E_\infty$ is a finitely generated Λ-module. Therefore the module $U^*(T)$ is finitely generated and the theory U^* is Noetherian.

Let us prove the acyclicity of the theory U^* in the sense of § 1. Since the $(4n-2)$-skeletons X_{2n} of the complexes MU_n do not have torsion, by virtue of the lemma for these complexes in the category D the spectral sequence exists; moreover, the module $U^*(X_{2n})$ is a cyclic A^U-module with generator of dimension $2n$ and with the single relation that all elements of filtration $\geq 2n$ in the ring A^U annihilate the generator. From this and the lemma it follows that $\operatorname{Ext}_{A^U}^{i,t}(U^*(X_{2n})) = 0$ for $t < 2n - \dim T$, and $\operatorname{Hom}_{A^U}^*(U^*(X_{2n}),\ ) = \operatorname{Hom}^*(\ , X_{2n}) = \operatorname{Hom}^*(\ , X)$ in the same dimensions. From this the lemma follows easily. ☐

Lemma 3.4 implies

Theorem 3.3. *For any $Y, T \in S$ there exists an Adams spectral sequence (E_r, d_r) with term $E_r = \operatorname{Ext}_{A^U}^{**}(U^*(Y), U^*(T))$.*

A. S. Miščenko proved the convergence of this spectral sequence to $\operatorname{Hom}^*(T, Y)$ (see [16]).

§ 4. O-cobordism and the ordinary Steenrod algebra modulo 2

As an illustration of our method of describing the Steenrod ring A^U (see §§ 5, 6) we exhibit it first in the simple case of the theory O^*, defined by the spectrum MO isomorphic to the direct sum $MO = \sum_\omega E^{\lambda_\omega} K(Z_2)$, where $\omega = (a_1, \dots, a_s)$, $\sum a_i = \lambda_\omega$, $a_i \neq 2^j - 1$, or $\omega = 0$. The Steenrod ring A^O is an algebra over the field Z_2. Let A be the ordinary Steenrod algebra. The simplest description of the algebra A^O is the following: $A^O = GL(A)$ consists of infinite matrices $a = (a_{\omega,\omega'})$, where ω, ω' are nondyadic partitions $(a_1, \dots, a_s), (a'_r, \dots, a'_s), a_{\omega,\omega'} \in A$ and $\dim a = \lambda_\omega - \lambda_{\omega'} + \dim a_{\omega,\omega'}$ is the dimension of the matrix. The ring $GL(A)$ is, by definition, a graded ring. This describes the ring A^O more generally for all spectra of the form $\sum E^{\lambda_\omega} K(Z_2)$.

In the ring $GL(A)$ we have a projection operator π such that $\pi A^O \pi = A$, $\pi^2 = 1$, $\pi \in A^O = GL(A)$.

Another description of the ring A^O is based on the existence of a multiplicative structure in $O^*(K, L)$. Let $\Lambda = O^*(P) \approx \Omega_O$ be the unoriented cobordism ring, $O^i(P) = \Omega_O^i$.

1. There is defined a multiplication operator

$$x \to \alpha x, \quad x \in O^*(K, L), \quad \alpha \in \Lambda = O^*(P).$$

This defines a monomorphism $\Lambda \to A^O$.

2. We define "Stiefel–Whitney characteristic classes" $\tilde{W}_i(\xi) \in O^i(X)$, where ξ is an O-bundle with base X:

a) for the canonical O_1-bundle ξ over RP^∞ we set:

$$\tilde{W}_i(\xi) = 0, \quad i \neq 0, 1,$$
$$\tilde{W}_0(\xi) = 1, \quad \tilde{W}_1(\xi) = DRP^{n-1} \subset O^{-1}(RP^n),$$

n large, D the Atiyah duality operator.

b) If $\eta = \xi_1 \oplus \xi_2$, then $\tilde{W}(\eta) = \tilde{W}(\xi_1)\tilde{W}(\xi_2)$, where $\tilde{W} = \sum \tilde{W}_i$.

These axioms uniquely define classes $\tilde{W}_i$ for all O-bundles.

As usual, the classes $\tilde{W}_i$ define classes $\tilde{W}_\omega$ for all $\omega = (a_1, \dots, a_s)$ such that $\tilde{W}_i = \tilde{W}_{1,\dots,1}$. In O-theory there is defined the Thom isomorphism $\phi\colon O^*(X) \to O^*(M\xi, *)$, where $M\xi$ is the Thom complex of ξ. Let $X = BO_n$, $M\xi = MO_n$. Let $u = \phi(1) \in O^*(MO_n)$. We define operations

$$\mathrm{Sq}^\omega\colon O^q(K, L) \to O^{q+d(\omega)}(K, L)$$

by setting $\mathrm{Sq}^\omega(u) = \phi(\tilde{W}_\omega)$, where $\tilde{W}_\omega \in O^*(BO_n)$.

Under the homomorphism $i^* \cdot j^*\colon O^*(MO_n) \to O^*(BO_n) \to O^*\left(\prod_{k=1}^n RP_k^\infty\right)$ the element $u = \phi(1)$ goes into $i^*j^*(u) = u_1 \dots u_n$, where $u_i \in O^1(RP_i^\infty)$ is the class $\tilde{W}_1(\xi_i), \xi_i$ the canonical O_1-bundle over RP_i^∞, defined above, and $\mathrm{Sq}^\omega(u_1 \dots u_n) = S^\omega(u_1, \dots, u_n) u_1 \dots u_n$, where S_ω is the symmetrized monomial $\sum u_1^{a_1} \dots u_s^{a_s}$, $s \leq n$.

There is defined the subset $\mathrm{Map}(X, MO_1) \subset O^1(X)$ and a (non-additive) map $\gamma\colon O^1(X) \to H^1(X, Z_2) \to \mathrm{Map}(X, MO_1)$, where $\varepsilon\colon O^* \to H^*(\ , Z_2)$ is the natural homomorphism {defined by the Thom class}. The operations Sq^ω have the following properties:

$$\text{a)}\quad \mathrm{Sq}^{\omega}(xy) = \sum_{(\omega_1,\omega_2)=\omega} \mathrm{Sq}^{\omega_1}(x)\,\mathrm{Sq}^{\omega_2}(y);$$

b) if $x = \gamma(x^1)$, then $\mathrm{Sq}^{\omega}(x) = 0$, $\omega \neq (k)$ and $\mathrm{Sq}^k(x) = x^{k+1}$;

c) the composition $\mathrm{Sq}^{\omega_1} \circ \mathrm{Sq}^{\omega_2}$ is a linear combination of the form $\sum \lambda_\omega \mathrm{Sq}^{\omega}$, $\lambda_\omega \in Z_2$, which can be calculated on $u = \phi(1) \in O^*(MO_n)$ or on $i^*j^*(u) = u_1 \dots u_n \in O^*(RP_1^\infty \times \dots \times RP_n^\infty), u_i \in \operatorname{Im}\gamma$;

d) there is an additive basis of the ring A^O of the form $\sum \lambda_i \alpha_i \mathrm{Sq}^{\omega_i}$, $\lambda_i \in Z_2$, α_i an additive basis of the ring $\Lambda = O^*(P) \approx \Omega_O$. Thus A^O is a topological ring with topological basis $\alpha_i \mathrm{Sq}^{\omega}$, or

$$A^O = (\Lambda \cdot S)^\wedge,$$

where $\wedge$ means completion and S is the ring spanned by all Sq^{ω}.

We note that the set of all Sq^{ω} such that $\omega = (a_1, \dots, a_s)$, where $a_1 = 2^j - 1$, is closed under composition and forms a subalgebra isomorphic to the Steenrod algebra $A \subset S \subset A^O$.

How does one compute a composition of the form $\mathrm{Sq}^{\omega} \circ \alpha$, where $\alpha \in \Lambda$? We shall indicate here without proof a formula for this (which will be basic in § 5, where the ring A^U is computed).

Let (X, ξ) be a pair (a closed manifold and a vector bundle ξ), considered up to cobordism of pairs, i.e. $(X, \xi) \in O^*(BO)$. In particular, if $\xi = -\tau_X$, where τ_X is the tangent bundle, then the pair $(X, \xi) \in \Omega_O = O_*(P)$.

We define operators ("differentiations")

$$W_\omega^* \colon O^*(BO) \to O^*(BO),$$
$$W_\omega^* \colon \Omega_O \to \Omega_O,$$

by setting $W_\omega^*(X, \xi) = (Y_\omega, f_\omega^*(\xi + \tau_X) - \tau_{Y_\omega})$, where $(Y_\omega, f_\omega \colon Y_\omega \to X)$ is $D\tilde{W}_\omega(\xi) \in O_*(X)$.

We also have multiplication operators

$$\alpha \colon O_*(BO) \to O_*(BO),$$
$$\alpha \colon \Omega_O \to \Omega_O,$$

where $(X, \xi) \to (X \times M, \xi \times (-\tau_M))$ and $(M, -\tau_M) \in \Omega_O$ represents $\alpha \in \Omega_O$.

In particular, we have the formula

$$W_\omega^* \cdot \alpha = \sum_{\omega=(\omega_1,\omega_2)} W_{\omega_1}^*(\alpha) \cdot W_{\omega_2}^*,$$

where $\alpha \in \Omega_O$, $W_{\omega_1}^*(\alpha) \in \Omega_O$.

It turns out that the following formula holds:

$$\mathrm{Sq}^{\omega} \cdot \alpha = \sum_{\omega=(\omega_1,\omega_2)} W_{\omega_1}^*(\alpha) \cdot \mathrm{Sq}^{\omega_2},$$

where $\alpha \in \Lambda = \Omega_O$.

We also have a diagonal

$$\Delta \colon A^O \to A^O \otimes_{\Omega_O} A^O,$$

where $\Delta(\alpha) = \alpha \otimes 1 = 1 \otimes \alpha$, and $\Delta\,\mathrm{Sq}^{\omega} = \sum_{\omega=(\omega_1,\omega_2)} \mathrm{Sq}^{\omega_1} \otimes \mathrm{Sq}^{\omega_2}$, so that $A^O \otimes_{\Omega_O} A^O$ may be considered as an A^O-module via Δ; $A^O \otimes_{\Omega_O} A^O = O^*(MO \otimes MO)$, and Δ arises from the multiplication in the spectrum, $MO \otimes MO \to MO$.

We note that the homomorphisms W^*_ω coincide with the Stiefel characteristic residues if $n = \dim \omega$.

We also note that any characteristic class $h \in O^*(BO)$ defines an operation $h \in A^O$, if we set $h(u) = \phi(h)$, where $u \in O^*(MO)$ is the Thom class and $\phi\colon O^*(BO) \to O^*(MO)$ is the Thom isomorphism.

In particular we consider the operations

$$\partial(u) = \varphi(h_1), \quad \text{where } h_1 = \gamma(\tilde{W}_1),$$
$$\Delta(u) = \varphi(h_2), \quad \text{where } h_2 = \gamma(\tilde{W}_1)^2.$$

It turns out that $\partial^2 = 0$, $\Delta\partial = 0$ and the condition $h_1(\xi) = 0$ defines an SO-bundle, since $h_1 = \gamma(\tilde{W}_1)$.

Further, it turns out that $O^*(MSO)$ is a cyclic A^O-module with a single generator $v \in O^*(MSO)$, given by the relations $\partial(v) = 0$, $\Delta(v) = 0$, and we have a resolution

$$(\cdots \to C_i \to \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} O^*(MSO) \to O) = C,$$

where $C_0 = A^O$ (generator u_0), $C_i = A^O + A^O$ (generators u_i, v_i, $i \geq 1$), and

$$d(u_i) = \partial u_{i-1}, \quad i \geq 1,$$
$$d(v_i) = \Delta u_{i-1}.$$

The homomorphisms ∂^* and $\Delta^*\colon \Omega_O \to \Omega_O$ coincide with the homomorphisms of Rohlin [20], [21] and Wall [23].

We consider the complex $\mathrm{Hom}^*_{A^O}(C, O^*(P))$ with differential d^* defined by the operators ∂^* and Δ^* on $O^*(P) \approx \Omega$. The homology of this complex is naturally isomorphic to $\mathrm{Ext}^{**}_{A^O}(O^*(MSO), O^*(P))$ or the E_2 term of the Adams spectral sequence.

It is possible to prove the following:

1) all Adams differentials are zero;

2) $\mathrm{Ext}^{0,*}_{A^O}(O^*(MSO), O^*(P)) = \Omega_{SO}/2\Omega_{SO} \subset \Omega_O$ where $\Omega_{SO}/2\Omega_{SO} = \mathrm{Ker}\,\partial^* \cap \mathrm{Ker}\,\Delta^*$ by definition of the complex C;

3) $\mathrm{Ext}^{i,i+s}_{A^O} = 0$, for $s \neq 4k$;

4) $\mathrm{Ext}^{i,i+4k}_{A^O}(O^*(MSO), O^*(P))$ is isomorphic to $Z_2 + \cdots + Z_2$, where the number of summands is equal to the number of partitions of k into positive summands $(k_1, \ldots, k_s)$, $\sum k_i = k$;

5) there exists an element $h_0 \in \mathrm{Ext}^{1,1}_{A^O}$ associated with multiplication by 2 in E_∞, such that $\mathrm{Ext}^{0,t}_{A^O} \xrightarrow{h_0} \mathrm{Ext}^{1,t+1}_{A^O}$ is an epimorphism, $t = 4k$, and $\mathrm{Ext}^{i,t}_A \xrightarrow{h_0} \mathrm{Ext}^{i+1,t+1}_A$ is an isomorphism, $i \geq 1$.

These facts actually are trivial since

$$\mathrm{Ext}_{A^O}(O^*(X), O(Y)) = \mathrm{Ext}_A(H^*(X, Z_2), H^*(Y, Z_2))$$

and $H^*(MSO, Z_2)$, as was shown by the author [17, 18] and by Wall [23], is $H^*(\sum E^j K(Z_2)) + H^*(\sum E^k K(Z))$, where there are as many summands of the form $K(Z)$ as would be necessary for (4) and (5).

We have mentioned these facts here in connection with the analogy later of MSO with MSU and the paper of Conner and Floyd [13].

In the study of $\mathrm{Ext}_{A^U}(U^*(MSU))$ all dimensions will be doubled, the groups $E^{i,8k+i}_\infty$ for $1 \leq i < 3$ will be constructed in an identical fashion, but the element

$h_0 \in \mathrm{Ext}^{1,1}$ will be replaced by an element $h \in \mathrm{Ext}^{1,2}_{A^U}$ and the Adams differential d_3 will be non-trivial (see §§ 6, 7).

We note that the construction described here gives us a natural representation of the ring $A^O = (\Lambda \circ S)^\Lambda$ on the ring Ω_O by means of the operators W^*_ω ("differentiation") and the multiplication on Λ.

In a certain sense the operators W^*_ω generalize the ordinary characteristic numbers. They can be calculated easily for $[RP^{2h}] \in \Omega_O$ and

$$W^*_\omega(\alpha\beta) = \sum_{(\omega_1,\omega_2)=\omega} W^*_{\omega_1}(\alpha) W_{\omega_2}(\beta)$$

(the Leibnitz formula). Completing their calculation would require that they be known also for "Dold manifolds."

It is interesting that the ring $A \subset A^O$, where $A \subset S$, is also represented monomorphically by the representation W^*_ω on Ω_O.

In conclusion, we note that the lack of rigor in this section is explained by the fact that O^*-theory will not be considered later and all assertions will be established in the more difficult situation of U^*-theory.

§ 5. Cohomology operations in the theory of U-cobordism

In this section we shall give the complete calculation of the ring A^U of cohomology operations in U^*-cohomology theory. We recall that for any smooth quasicomplex manifold (possibly with boundary) there is the Poincaré–Atiyah duality law

$$U^i(X) = U_{n-i}(X, \partial X) \quad \text{and} \quad U_i(X) = U^{n-i}(X, \partial X),$$

where quasicomplex means a complex structure in the stable tangent (or normal) bundle. Here there is also the Thom isomorphism $\phi\colon U^i(X) \to U^{2n+i}(M\xi, *)$ where ξ is a complex U_n-bundle of dimension $2n$, and $M\xi$ is its Thom complex. We denote the Poincaré–Atiyah duality operator by D. There is defined a natural homomorphism $\varepsilon\colon U_*(X) \to U_*(P)$, where P is a point and $\Omega_U = U_*(P) = Z[x_1, \dots, x_i, \dots]$, $\dim x_i = 2i$.

We consider the group $U_*(K)$ given by pairs (X, f), where X is a manifold and $f\colon X \to K$. Let α be arbitrary characteristic class, $\alpha \in U^*(BU)$. For any complex K in the category S, the class α defines an operator

$$\alpha\colon U_*(K) \to U_*(K),$$

if we set $\alpha(X, f) = (Y_\alpha, f \cdot f_\alpha)$, where $(Y_\alpha, f_\alpha) \in U_*(X)$ is the element having the form $D\alpha(-\tau_X)$, where $\tau_X \in K(X)$ is the stable tangent U-bundle of X. $\{D_\alpha(-\tau_X) = D((-\tau_X)^*(\alpha)).\}$

As we know, the operation of the class α on $U_*(K)$ can be defined in another way: since $U^*(MU) = U^*(BU)$ by virtue of the Thom isomorphism ϕ, we have $\phi(\alpha) = a \in U^*(MU) = A^U$. We consider the pair $L = (K \cup P, P)$ in the S-category; then $U_*(K) = \mathrm{Hom}^*(P, MU \otimes L)$ by definition, where P is the spectrum of a point. Every operation $a = \phi(\alpha)$ defines a morphism $\phi(\alpha)\colon MU \to MU$ and, of course, a morphism

$$\varphi(\alpha) \otimes 1\colon MU \otimes L \to MU \otimes L.$$

Hence there is defined a homomorphism $\bar{\alpha}^*\colon U_*(K) \to U_*(K)$ by means of $\phi(\alpha) \otimes 1$. We have the simple

Lemma 5.1. *The operators α^* and $\bar{\alpha}^*$ coincide on $U_*(K)$.*

The proof of this lemma follows easily from the usual considerations with Thom complexes, connected with t-regularity.

Thus there arises a natural representation of the ring A^U on $U_*(K)$ for any K, where $a \to [\phi^{-1}(a)]^* = \alpha^*$, $\phi\colon U^*(BU) \to U^*(MU)$.

We have

Lemma 5.2. *For $K = P$, the representation $a \to [\phi^{-1}(a)]^*$ of the Steenrod ring A^U in the ring of endomorphisms of $U_*(P) = \Omega_U$ is dual by Poincaré–Atiyah duality to the operation of the ring A^U on $U^*(P)$ and is a faithful representation.*

Proof. Since $K = P$ and $MU \otimes P = MU$, the operation of the ring A^U on $\mathrm{Hom}^*(P, MU)$ is dual to the ordinary operation, by definition. By virtue of Lemma 3.1 of § 3, this operation is a faithful representation of the ring A^U. The lemma is proved. □

We now consider the operation of the ring A^U on $U_*(P)$ and extend it to another operation on $U_*(BU)$. Let $x \in U_*(BU)$ be represented by the pair (X, ξ), $\xi \in K^0(X)$. We set

$$\tilde{a}(x) = \tilde{a}(X, \xi) = (Y_\alpha, f_\alpha^*(\xi + \tau_X) - \tau_{Y_\alpha}),$$

where $\alpha = \phi^{-1}(a)$, $a \in U^*(MU)$ and (Y_α, f_α) is the element of $U_*(X)$ equal to $D\alpha(\xi)$, $\alpha \in U_*(BU)$, and τ_M is the stable tangent U-bundle of M.

If $\xi = \tau_X$, then $f_\alpha^*(\xi + \tau_X) - \tau_{Y_\alpha} = -\tau_{Y_\alpha}$ and hence the pair $(X, -\tau_X)$ goes to $(Y_\alpha, -\tau_{Y_\alpha})$, i.e., the subgroup $U_*(P) \subset U_*(BU)$ is invariant under the transformation $\tilde{a}$.

We have the obvious

Lemma 5.3. *The representation $a \to \tilde{a}$ of the ring A^U on $U_*(BU)$ is well-defined and is faithful.*

Proof. The independence of the definition of $\tilde{a}$ from the choice of representative (X, ξ) of the class x follows from the standard arguments verifying invariance with respect to cobordism of pairs (X, ξ) and properties of Poincaré–Atiyah duality for manifolds with boundary.

The fidelity of the representation $\tilde{a}$ follows from the fact that it is already faithful on $U_*(P) \subset U_*(BU)$ by the preceding lemma, where $\tilde{a}$ coincides with $[\phi^{-1}(a)]^*$. The only thing that remains to be verified is that $\tilde{a}$ is a representation of the ring A^U and not of some extension of it. For this however, we note that the composition of transformations $\tilde{a}\tilde{b}$ is also induced by some characteristic class and hence has the form $\tilde{a}\tilde{b} = \tilde{c}$. Whence follows the lemma. □

Remark 5.1. It is easy to show that the transformation $\tilde{a}$ has the form $\phi^{-1}a_*\phi$, where $\phi\colon U_*(BU) \to U_*(MU)$ and $a_*\colon U_*(MU) \to U_*(MU)$ is the transformation induced by $a\colon MU \to MU$. In the future we shall use the geometric meaning of the transformation $\tilde{a} = \phi^{-1}a_*\phi$ and hence we have given the definition of $\tilde{a}$ in a geometric form.

The transformation $\tilde{a}$ induces a transformation $\alpha^*\colon \Omega_U \to \Omega_U = U_*(P)$, where $U_i(P) = \Omega_U^i = U^{-i}(P)$.

We shall also denote by α^* the dual transformation $U^*(P) \to U^*(P)$, $U^*(P) = \Lambda \approx \Omega_U$.

We shall now indicate the set of operations needed, from which we can construct all the operations of the Steenrod ring A^U.

1. **Multiplication operators.** For any element $a \in U^*(P) = \Lambda$ there is defined the multiplication operator $x \to ax$. Hence $\Lambda \subset A^U$. The corresponding transformation $\tilde{a}\colon U_*(BU) \to U_*(BU)$ has the form:

$$(X, \xi) \to (X \times Y_a, \xi \times (-\tau_{Y_a})),$$

where $(Y_a, -\tau_{Y_a})$ represents the element $Da \in U_*(P) = \Omega_U$.

2. **Chern classes and their corresponding cohomology operations.** As Conner and Floyd remarked in [11], if in the axioms for the ordinary Chern classes one replaces the fact that $c_1(\xi)$ for the canonical U_1-bundle over CP^N is the homology class dual to CP^{N-1}, by the fact that the "first Chern class" $\sigma_1(\xi)$ is the canonical cobordism class $\sigma_1 \in U^2(CP^N)$ which is dual, by Atiyah, to $[CP^{N-1}]$, then there arise classes $\sigma_i(\xi) \in U^{2i}(X)$ with the following properties:

1. $\sigma_i = 0$, $i < 0$; $\sigma_0 = 1$; $\sigma_i = 0$, $i > \dim_C \xi$;
2. $\sigma_i(\xi + \eta) = \sum\limits_{j+k=i} \sigma_i(\xi)\sigma_k(\eta)$;
3. $\sigma_1(\xi) \in \mathrm{Map}(X, MU) \subset U^2(X)$, if ξ is a U_1-bundle;
4. $\nu(\sigma_i) = c_i$, where $\nu\colon U^* \to H^*(\ , Z)$ is the map defined by the Thom class.

We note that in the usual way (by the symbolic generators of Wu) the characteristic classes σ_i determine classes $\sigma_\omega(\xi)$, $\omega = (k_1, \dots, k_s)$, such that $\sigma_\omega(\xi + \eta) = \sum\limits_{\omega=(\omega_1,\omega_2)} \sigma_{\omega_1}(\xi)\sigma_{\omega_2}(\eta)$, with $\sigma_{(1,\dots,1)} = \sigma_i$.

In the usual way the classes σ_ω determine elements $S_\omega = \phi\sigma_\omega \in U^*(MU)$ and, as was shown earlier, homomorphisms $\sigma_\omega^*\colon \Omega_U \to \Omega_U$ and $\tilde{S}_\omega\colon U_*(BU) \to U_*(BU)$.

We have the important

Lemma 5.4. *The following commutation formula is valid*:

$$S_\omega \cdot x = \sum_{\omega=(\omega_1,\omega_2)} \sigma_{\omega_1}^*(x) S_{\omega_2}, \quad x \in \Lambda = U^*(P) \subset A^U.$$

Proof. This formula can be established easily for the operation on $U_*(BU)$ by the faithful reprentation which we constructed earlier. Let (X, ξ) represent an element of $U_*(BU)$ and $(M, -\tau_M)$ represent an element x of Ω_U. We consider

$$\begin{aligned}\tilde{S}_\omega \circ \tilde{x}(X, \xi) = \tilde{S}_\omega[(X, \xi) \times (M, -\tau_M)] &= \sum_{\omega=(\omega_1,\omega_2)} \sigma_{\omega_1}^*(x)\tilde{\sigma}_{\omega_2}(X, \xi) \\ &= \sum_{\omega=(\omega_1,\omega_2)} (Y_{\omega_1}, f_{\omega_1}^*(\xi + \tau_X) - \tau_{Y_{\omega_1}}) \times (N_{\omega_2} - \tau_{N_{\omega_2}})\end{aligned}$$

by definition. Here $(Y_{\omega_1}, f_{\omega_1})$ represents the element $D\sigma_{\omega_1}(\xi)$, and similarly for N_{ω_2}. The lemma is proved. □

In order that the formula derived above be more effective, we shall indicate exactly the action of the operator σ_ω^* on the ring Ω_U.

It is known that by virtue of the Whitney formula the classes $\sigma_\omega(-\xi)$ are linear forms in the classes $\sigma_\omega(\xi)$ with coefficients which are independent of ξ. Let $\bar{\sigma}_\omega(\xi) = \sigma_\omega(-\xi)$ and let $\bar{\sigma}_\omega^*$ be the homomorphism associated with this linear form.

If $(X, -\tau_X)$ represents an element a of Ω_U, then the classes $\bar{\sigma}^*_\omega(a)$, represented by $\varepsilon D\bar{\sigma}_\omega(-\tau_X) \in \Omega_U$ {where ε is induced by $X \to P$} are the characteristic classes of the tangent bundle.

Let $X = CP^n$ and $u_\omega = \sum_{i_j \le n+1} t_{i_1}^{k_1} \dots t_{i_s}^{k_s}$ (the sum over all symmetrizations, $\omega = (k_1, \dots, k_s)$). Let λ_ω be the number of summands in the symmetrized monomial $u_\omega, k = \sum k_i$. We have the simple

Lemma 5.5. *If $X = [CP^n]$, then $\bar{\sigma}^*_\omega(X) = \lambda_\omega[CP^{n-k}]$ and*

$$\bar{\sigma}^*_\omega(ab) = \sum_{\omega=(\omega_1,\omega_2)} \bar{\sigma}^*_{\omega_1}(a)\bar{\sigma}^*_{\omega_2}(b), \quad a, b, \in \Omega_U.$$

Hence the above formula completely determines the action of the operators σ^*_ω and $\bar{\sigma}^*_\omega$ on the ring Ω_U.

Proof. Since for $X = [CP^n]$ we have that $\tau_X + 1 = (n+1)\xi$, where ξ is the canonical U_1-bundle, the Wu generators for τ_X are $u = t_1 = \dots = t_{n+1} = DCP^{n-1} \in U^2(X)$. Therefore $\bar{\sigma}^*_\omega[CP^n] = \lambda_\omega u^k$, where $k = \dim \omega$.

We note that by virtue of the structure of the intersection ring $U_*(CP^n)$ we have: $u^k = DCP^{n-k}$. Hence

$$\varepsilon D\bar{\sigma}^*_\omega[X] = \varepsilon\lambda_\omega CP^{n-k} = \lambda_\omega[CP^{n-k}] \in \Omega_U$$

{where $\varepsilon\colon U_*(CP^n) \to U_*(P)$ is the augmentation}. The Leibnitz formula for $\bar{\sigma}^*_\omega(ab)$ follows in the usual way from the Whitney formula. The lemma is proved. $\square$

We shall now describe the structure of the ring S generated by the operators S_ω.

We consider the natural inclusions

$$CP_1^\infty \times \dots \times CP_n^\infty \xrightarrow{i} BU_n \xrightarrow{j} MU_n$$

and homomorphisms

$$j^*\colon U^*(MU_n) \to U^*(BU_n),$$
$$i^*\colon U^*(BU_n) \to U^*(CP_1^\infty \times \dots \times CP_n^\infty).$$

We note that $U^*(CP_1^\infty \times \dots \times CP_n^\infty)$ has generators $u_i \in U^2(CP_i^\infty)$, and an additive basis of $U^*(CP_1^\infty \times \dots \times CP_n^\infty)$ has the form $\sum \lambda_q x_q P_q(u_1, \dots, u_n)$, where $x_q \in \Lambda = U^*(P)$, the λ_q are integers and P_q are polynomials. We have the following facts:

1. The image $\operatorname{Im} i^*$ consists of all sums of the form $\sum \lambda_q x_q P_q(u_1, \dots, u_n)$, where P_q is a symmetric polynomial and $\dim x_i P_i =$ constant (the series is taken in the graded ring).
2. The image $\operatorname{Im}(i^* j^*)$ consists of the principal ideal in $\operatorname{Im} i^*$ generated by the element $u_1 \dots u_n$.
3. The $i^*\sigma_q = \sigma_q(u_1, \dots, u_n)$ are the elementary symmetric polynomials, σ_q the characteristic classes.
4. For any $a \in U^*(BU_n)$ we have the usual formula $i^*(a)(u_1 \dots u_n) = i^* j^* \phi(a)$, where ϕ is the Thom isomorphism.

From these facts easily follows

Lemma 5.6. *The operations $S_\omega \in A^U$ have the following properties*:

1. *If $\alpha \in \operatorname{Map}(X, MU_1) \subset U^2(X)$, then $S_{(k)}\alpha = \alpha^{k+1}$ and $S_\omega(\alpha) = 0$ if $\omega \ne (k)$.*

2. $S_\omega(\alpha,\beta)=\sum\limits_{\omega=(\omega_1,\omega_2)} S_{\omega_1}(\alpha)S_{\omega_2}(\beta)$ *for all* $\alpha,\beta\in U^*$.

3. *If* $k^{(i)}<n$, $\omega_i=(k_1^{(i)},\dots,k_{s_i}^{(i)})$, $\sum\limits_j k_j^{(i)}=k^{(i)}$ *and* $a=\sum\lambda_i S_{\omega_i}$, *then* $a\phi(1)=au=0$ *is equivalent to* $a=0$.

4. *The composition of operations* $S_{\omega_1}\cdot S_{\omega_2}$ *is a linear combination of operations of the form* S_ω *with integral coefficients, so that an additive basis for the ring* S *consists of all* S_ω.

Proof. Let $X=BU_1=CP^\infty$. Since $MU_1=CP^\infty$, it is sufficient to prove property 1 for the element $u\in U^2(CP^\infty)$ equal to $\sigma_1(\xi)$ for the canonical U_1-bundle ξ. By definition, we have: $u=j^*\phi(1)\in U^2(CP^\infty)$ and $S_\omega(u)=j^*S_\omega\phi(1)=j^*u^{k+1}$ (if $\omega=(k)$) and $\sigma_\omega(\xi)=0$, if $\omega\neq(k)$, since $\sigma_i=0$, $i\geq 2$, for U_1-bundles ξ. This proves property 1.

Property 2 follows obviously from the Whitney formula for the classes σ_ω together with the remark that $\phi(1)\in U^*(MU_n)$ as $n\to\infty$ represents the universal element corresponding to the operation $1\in A^U$.

Property 3 is clear. Property 4 follows from the fact that on the basis of properties 1 and 2 it is possible to compute completely $S_{\omega_1}\cdot S_{\omega_2}(u)=\sum\lambda_\omega S_\omega(u)$ and then use property 3. Whence it will follow for large n that $S_{\omega_1}\circ S_{\omega_2}=\sum\lambda_\omega S_\omega$.

The lemma is proved. □

Further, we note the obvious circumstance: An additive topological basis of the ring A^U has the form x_iS_ω, where x_i is an additive homogeneous basis for $U^*(P)$, $U^i(P)=\Omega_U$.

The topology of A^U is defined by a filtration. This means that the finite linear combinations of the form $\sum\lambda_i x_i S_{\omega_i}$ are dense in A^U and the completion coincides with A^U, which thus consists of formal series of the form $\sum\lambda_i x_i S_{\omega_i}$, where the λ_i are integers and $\dim x_iS_{\omega_i}=\text{constant}$, since A^U is a graded ring.

Thus we have:

$$A^U=(\Lambda\cdot S)^\wedge,$$

where the sign $\wedge$ denotes completion. Here $\Lambda=Z[x_1,\dots,x_i,\dots]$, $\dim x_i=-2i$. The ring S is completely described by Lemma 5.6, and the commutation properties by Lemmas 5.4, 5.5.

We note that S is a Hopf ring with symmetric diagonal $\Delta\colon S\to S\times S$, where

$$\Delta(S_\omega)=\sum_{(\omega_1,\omega_2)=\omega} S_{\omega_1}\otimes S_{\omega_2}.$$

Since MU is a multiplicative spectrum $MU\otimes MU\to MU$, the ring A^U has a "diagonal"

$$\Delta\colon A^U\to A^U\otimes_\Lambda A^U,$$

where $\Delta(S_\omega)=\sum\limits_{\omega=(\omega_1,\omega_2)} S_{\omega_1}\otimes S_{\omega_2}$ and $xa\otimes b=a\otimes xb=x(a\otimes b)$ for $x\in\Omega_U=\Lambda$. The Künneth formula for K_1, $K_2\in D$ {complexes without torsion} has the form:

$$U^*(K_1\times K_2)=U^*(K_1)\otimes_\Lambda U^*(K_2),$$

and hence $A^U\otimes A^U$ is an A^U-module with respect to the diagonal Δ.

Moreover, we remark that A^U has a natural representation * on the ring Ω_U, where $\Omega_U^i=U^{-i}(P)$, under which the action of the ring Λ goes over to the multiplication operators $\Lambda\approx\Omega_U$ and the $S_\omega\to\sigma_\omega^*$.

We now define an important map $\gamma\colon U^2 \to U^2$ (nonadditive), such that $\nu\gamma(x) = \nu(x)$, $\nu\colon U^* \to H^*(\ ,Z)$ is defined by the Thom class, and $\gamma(x) \in \mathrm{Map}(X, MU_1) \subset U^2(X)$, for $x \in U^2(X)$.

We consider important examples of cohomology operations related to the class σ_1.

1. Let $\Delta_{(k_1,k_2)} \in A^U$ be the cohomology operation such that

$$\Delta_{(k_1,k_2)} = \varphi[\gamma(-\sigma_1)^{k_1}\gamma(\sigma_1)^{k_2}] \in U^*(MU),$$

where $\sigma_1 \in U^2(BU)$, $\gamma\colon U^2 \to U^2$.

In particular, $\Delta_{(1,0)}$ will be denoted by ∂ and $\Delta_{(1,1)}$ by Δ.

We shall describe the homomorphisms $\Delta^*_{(k_2,k_2)}$ and $\tilde{\Delta}_{(k_2,k_2)}$:

a) if (X,ξ) represents an element of $U_*(BU)$ and $i_1\colon Y_1 \to X$, $i_2\colon Y_2 \to X$ are submanifolds which realize the classes $Dc_1(\xi)$, $-Dc_1(\xi) \in H_{n-2}(X)$, then their normal bundles in X are equal respectively to ξ_1 and $\bar{\xi}_1$, where $c_1(\xi_1) = -\overline{c_1(\xi_1)} = -c_1(\bar{\xi}_1) = c_1(\xi)$.

Let

$$Y_{k_1,k_2} = \underbrace{Y_1 \dots Y_1}_{k_1} \cdot \underbrace{Y_2 \dots Y_2}_{k_2}$$

be the self-intersection in $U_*(X)$ with normal bundle

$$i^*(\underbrace{\xi_1 + \cdots + \xi_1}_{k_1} + \underbrace{\bar{\xi}_1 + \cdots + \bar{\xi}_1}_{k_2}) = W,$$

where $i\colon Y_{k_1,k_2} \to X$.

We set $\tilde{\Delta}_{(k_1,k_2)}(X,\xi) = (Y_{(k_1,k_2)}, i^*(\xi + W))$.

b) If $\xi = -\tau_X$, then the $\bar{\Delta}_{(k_1,k_2)}$ define homomorphisms $\Delta^*_{(k_1,k_2)}\colon \Omega_U \to \Omega_U$ for which the image of ∂^* consists only of SU-manifolds. The operations ∂^* and Δ^* on Ω_U were studied earlier in [13],

2. **The classes and operations** $\chi_{(k_1,k_2)}$. Just as was the case for the operations $\Delta_{(k_1,k_2)}$ and classes $\gamma(\sigma_1^{k_1})\gamma(-\sigma_1)^{k_2}$, the operations $\chi_{(k_1,k_2)}$ and the classes corresponding to them will be defined for a bundle ξ only as functions of $c_1(\xi)$ or of $\gamma(\sigma_1(\xi))$. We define these classes for one-dimensional bundles ξ over CP^n.

We consider the projectivization $P(\xi + k) \to CP^n$, where k is the trivial k-plane bundle.

It is obvious that $\tau(P(\xi + k)) = p^*\tau(CP^n) + \tau'$, where τ' consists of tangents to the fiber. Over $P(\xi + k)$ we have the following fibrations:

1) the Hopf fibration μ in each fiber;

2) The fibration $\xi' = p^*\xi$.

It is easy to see that the stable bundle τ' is equivalent to the sum

$$\tau' = \mu\bar{\xi}' + \overbrace{\mu + \cdots + \mu}^{k \text{ times}} \in K(P(\xi + k)).$$

We set {here $k_1 + k_2 = k$}

$$\tau'_{(k_1,k_2)} = \mu\bar{\xi}' + k_1\mu + k_2\bar{\mu},$$

which functorially introduces a U-structure into the bundle $\tau'_{(k_1,k_2)}$ such that $r\tau'_{(k_1,k_2)} = r\tau'$ where r is the realification of a complex bundle.

$P(\xi+k)$ has the induced U-structure $p^*\tau(CP^n)+\tau'_{(k_1,k_2)}$. We denote the result by $P^{(k_1,k_2)}(\xi+k)$. We denote the pair $(P^{(k_1,k_2)}(\xi+k),p)\in U_*(CP^n)$ by $D\chi_{(k_1,k_2)}$, where $\chi_{(k_1,k_2)}\in U^*(CP^n)$.

For any fibration ξ over X we set $\chi_{(k_1,k_2)}(\xi)=\chi_{(k_1,k_2)}(\xi_1)$, where $c_1(\xi)=c_1(\xi_1)$ and ξ_1 is a U_1-bundle.

There arise classes $\chi_{(k_1,k_2)}\in U^*(BU)$, operations $\phi\chi_{(k_1,k_2)}=\Psi_{(k_1,k_2)}$, and homomorphisms $\Psi^*_{(k_1,k_2)}$ and $\tilde{\Psi}_{(k_1,k_2)}$.

We note that $\chi_{(0,1)}=0$. We denote the operation $\chi_{(1,0)}$ by χ and the operation $\chi_{(1,1)}$ by Ψ.

The homomorphism $\Psi^*\colon \Omega_U\to\Omega_U$ was studied by Conner and Floyd (see [13]).

It is easy to establish the following equations:

a) $\Delta_{(k_1,k_2)}\circ\partial=0$ (in particular, $\partial^2=0$, $\Delta\partial=0$);

b) $\Delta\Psi=1$, $[\partial,\chi]=2$, $\chi\partial=x_1\circ\partial$, where $x_1=[CP^1]\in\Lambda\subset A^U$; $\partial\Psi=0$.

We shall prove these equations. Since $\operatorname{Im}\partial^*\subset\Omega_U$ is represented by SU-manifolds, $\Delta^*_{(k_1,k_2)}\circ\partial^*=0$ by definition; since $*$ is a faithful representation of the ring A^U by virtue of Lemma 3.1, $\Delta_{(k_1,k_2)}\circ\partial=0$, where $\partial=\Delta_{(1,0)}$.

The equations $\Delta^*\Psi^*=1$, $\partial^*\Psi^*=0$ were proved by direct calculation in [13]. Hence $\Delta\Psi=1$ in A^U. Since $\operatorname{Im}\partial^*$ consists of SU-manifolds, it is easy to see that $\chi^*\partial^*=x_1\circ\partial^*$. This means that $\chi\partial=x_1\partial$. The equation $[\chi,\partial]=2$ follows easily from the fact that for one-dimensional bundles ξ over X such that $c_1(\xi)=-c_1(X)$, we have:

$$c_1(P(\nu+1))=-2c_1(\mu)=-2c,$$

and the class DC is realized by the submanifold $X=P(\xi)\subset P(\xi+1)$.

Remark 5.2. Equations of the type $[a,b]=\lambda\circ 1$ arise frequently in the ring A^U. For example, if $a_k=S_k$ and $b_k=[CP^k]$, then $[a_k,b_k]=(k+1)\circ 1$ by Lemma 5.5.

Remark 5.3. The operation $\pi=[\Delta,\Psi]=1-\Psi\Delta$ is the "projector of Conner–Floyd" $\pi^2=\pi$. (Conner and Floyd studied π^*.)

This projector has the property that it allows the complete decomposition of the cohomology theory U^* into a sum of theories π_jU^*, where $\sum\pi_j=1$, $\pi_j\in A^U$, with $\pi_0=1-\Psi\Delta$ and $\pi_j=\Psi^j\Delta^j-\Psi^{j+1}\Delta^{j+1}$. Later on we shall meet other projectors of this same sort.

3. We consider still another important example of a cohomology operation in U^*-theory, connected with the following question:

Let ξ,η be U-bundles. How does one compute the class $\sigma_1(\xi\otimes\eta)$?

We have

Lemma 5.7. a) *For any U_n-bundle ξ there is a cohomology operation $\gamma_{n-1}\in A^U$ such that $\sigma_1(\lambda_{-1}(\xi))=\gamma_{n-1}(\sigma_n(\xi))$, where $\lambda_{-1}=\sum(-1)^i\Lambda^i$ and the Λ^i are the exterior powers.*

b) *If $u_1,\dots,u_n\in U^2(X)$ are elements in the subset $\operatorname{Im}\gamma=\operatorname{Map}(X,MU_1)\subset U^2(X)$, then we have the equation*

$$\gamma_{n-1}(u_1\dots u_n)=\gamma_1(u_1\cdot\gamma_1(u_2\cdot\gamma_1(\cdots\gamma_1(u_{n-1}\cdot u_n))\dots)),$$

where γ_1 is such that for a pair of U_1-bundles ξ,η we have the formula

$$\sigma_1(\xi\otimes\eta)=\sigma_1(\xi)+\sigma_1(\eta)+\gamma_1(\sigma_1(\xi)\sigma_1(\eta)).$$

The proof of this lemma follows from the definition of the operation γ_1. Let $X = CP^\infty \times CP^\infty$ and let ξ, η be the canonical U_1-bundles over the factors. Since $\nu\sigma_1(\xi \otimes \eta) = c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$ and $\sigma_1 \in \mathrm{Map}(X, MU_1)$ it is possible to calculate the class $\sigma_1(\xi \otimes \eta)$ completely as a function of $\sigma_1(\xi)$ and $\sigma_1(\eta)$. Namely:

$$-\sigma_1(\xi) - \sigma_1(\eta) + \sigma_1(\xi \otimes \eta) = \sum_{i \geq 1, j \geq 1} x_{i,j} \sigma_1^i(\xi) \sigma_1^j(\eta), \quad x_{i,j} \in \Lambda.$$

Since the bundle $\lambda_1(\xi + \eta)$ lies in a natural way in $K^0(MU_2)$ and $\lambda_1(\xi + \eta) = \xi \otimes \eta - \xi - \eta + 1$, the difference $-\sigma_1(\xi) - \sigma_1(\eta) + \sigma_1(\xi \otimes \eta) + 1$ has the form $\gamma_1 u$, where $u \in U^*(MU_2)$ is the fundamental class $u = \phi(1)$.

The operation γ_1 can be written in the form

$$\gamma_1 u = \sum x_{i,j} S_{(i,j)}(u_1 u_2), \quad u = u_1 u_2,$$

where $u_1 = \sigma_1(\xi)$, $u_2 = \sigma_1(\eta)$.

Let $\omega = (k_1, \dots, k_s)$, where $s > 2$. Then $S_\omega(u) = 0$. Hence γ_1 is uniquely defined (mod $x_\omega S_\omega$).

We set $\gamma_{n-1} = \gamma_1(u_1 \dots \gamma_1(u_{n-1} u_n) \dots)$ on the element $u = u_1 \dots u_n = \phi(1) \in U^*(MU_n)$. The operation γ_{n-1} is well defined $\mod x_\omega S_\omega$, where $\omega = (k_1, \dots, k_s)$, $s > n$. By definition, we have the formula $\sigma_1 \lambda_{-1}(\xi) = \gamma_{n-1} \sigma_n(\xi)$ for a U_n-bundle ξ.

The lemma is proved.

Remark 5.4. It would be very useful, if it were possible, to define exactly an operation $\gamma_1 \in A^U \otimes Q$ so as to satisfy the equations $\gamma_1^i = \gamma_i$. The meaning of this will be clarified later in § 8.

We now consider analogues of the Adams operations and the Chern character in the theory of U-cobordism which are important for our purposes.

We have already considered above how the class $\sigma_1(\xi \otimes \eta)$ is related to the classes $\sigma_1(\xi)$ and $\sigma_1(\eta)$ for U_1-bundles ξ, η. Namely

$$\sigma_1(\xi \otimes \eta) = u + v + \gamma_1(uv),$$

where $u = \sigma_1(\xi)$, $v = \sigma_1(\eta)$ and

$$\gamma_1(uv) = \sum_{\substack{i \geq 0 \\ j \geq 0 \\ i \neq j}} x_{ij}(u^{i+1} v^{j+1} + u^{j+1} v^{i+1}) + \sum_{i \geq 0} x_{i,i} u^{i+1} v^{i+1}, \quad x_{i,j} \in \Omega_U.$$

We set $u + v + \gamma_1(u, v) = f(u, v)$. Then we have the "law of composition" $u \oplus v = f(u, v)$ for $u, v \in \operatorname{Im} \gamma_1 = \mathrm{Map}(X, MU_1)$, which turns $\mathrm{Map}(X, MU_1)$ into a formal one-dimensional commutative group with coefficients in the graded ring Ω_U, while $\dim u, v, f(u, v) = 2$. As A. S. Miščenko has shown, if we make the change of variables with rational coefficients

$$g(u) = \sum_{i \geq 0} \frac{x_i}{i+1} u^i, \quad x_i = [CP^i],$$

where $[CP^i] \in \Omega_U^{2i} = \Lambda^{-2i}$, then the composition law becomes additive:

$$g(u \oplus v) = g(f(u, v)) = g(u) + g(v)$$

(see Appendix 1). This allows the introduction of the "Chern character":

a) We set $\sigma h(\xi) = e^{g(u)}$, where $u = \sigma_1(\xi)$ for U_1-bundles ξ;

b) if $\xi = \xi_1 + \xi_2$, then $\sigma h(\xi) = \sigma h(\xi) = \sigma h(\xi_1) + \sigma h(\xi_2)$;

c) if $\xi = \xi_1 \otimes \xi_2$, then $\sigma h(\xi) = \sigma h(\xi_1) \sigma h(\xi_2)$.

Thus, we have a ring homomorphism

$$\sigma h\colon K(X) \to U^*(X) \otimes Q.$$

We now consider an operation $a \in A^U$ such that

$$\Delta a = a \otimes a \in A^U \otimes_\Lambda A^U.$$

We already know some examples of such operations:

$$1) \quad a = \sum_\omega S_\omega,$$

$$2) \quad a = \sum_{i \geq 0} S_{\underbrace{(1,\dots,1)}_{i \text{ times}}}.$$

The Chern character gives a new example of such an operation $a \in A^U \otimes Q$: We consider the "Riemann–Roch" transformation $\lambda_{-1}^{(n)}\colon U^{2n} \to K$ which is defined by the element $\lambda_{-1}^{(n)} \in K(MU_n)$, and let $\lambda = (\lambda_{-1}^{(n)})$, $n \to \infty$. Let $\Phi^{(n)} = \sigma h^n \circ \lambda_{-1}^{(n)}$, and $\Phi = (\Phi^{(n)})$. The operation Φ obviously has the property that $\Delta\Phi = \Phi \otimes \Phi$ since σh and λ_{-1} are multiplicative, and if the element $\xi \in K(X)$ has filtration m and the element η has filtration n, then $\sigma h^{m+n}(\xi \otimes \eta) = \sigma h^m(\xi)\sigma h^n(\eta)$. It is easy to verify that the operator Φ has the following properties:

1) $\Phi^2 = \Phi$,
2) $\Phi^*(1) = 1$,
3) $\Phi^*(x) = 0$, $\dim x < 0$, $x \in \Lambda$, $\Lambda = U^*(P)$, $\Phi^*\colon \Lambda \to \Lambda$, where

$$\Phi = \sum_\omega \frac{x_{i_1}}{i_1 + 1} \dots \frac{x_{i_s}}{i_s + 1} S_\omega, \quad \omega = (i_1, \dots, i_s).$$

Hence, the operation Φ associated with the Chern character σh defines a projection operator, which selects in the theory $U^* \otimes Q$ the theory $H^*(\ , Q) = \Phi(U^* \otimes Q)$.

A multiplicative operation $a \in A^U$ is uniquely defined, obviously, by its value $a(u) \in U^*(CP^\infty)$, where $u \in \mathrm{Map}(CP^\infty, MU_1)$ is the canonical generator, $a(u) = u(1 + \dots)$.

Conversely, the element $a(u) \in U^*(CP^\infty)$ can be chosen completely arbitrarily. For example, for $a = \sum_\omega S_\omega$, $a(u) = \frac{u}{1-u}$; for $a_k = \sum_{\omega=(k,\dots,k)} S_\omega$, $a(u) = u(1 + u^k)$.

For our subsequent purposes the following operations will be important:

1) The analogues of Adams operations $\Psi_U^p \in A^U \otimes_Z Z[1/p]$.
2) Projection operators which preserve the multiplicative structure.

All these operations are given by series $a(u) \in U^*(CP^\infty)$, since $\Delta a = a \otimes a$.

We define the Adams operations Ψ_U^k, which arise from the requirements:

1) $\Psi_U^k(xy) = \Psi_U^k(x)\Psi_U(y)$, $x, y \in U^*$;
2) $\Psi_U^k \cdot x = k^i x \cdot \Psi_U^k$, where $x \in \Lambda^{-2i} = \Omega_U^{2i}$;
3) $\Psi_U^k(u) = \dfrac{u \oplus \dots \oplus u}{k}$ (k times), where $u \in U^2(CP^\infty)$ is the canonical element and $\oplus$ is composition in $\mathrm{Map}(X, MU_1) \subset U^2(X)$.

Lemma 5.8. a) *The series $\Psi_U^k(u)$ has the form*

$$\Psi_U^k(u) = \frac{u \oplus \dots \oplus u}{k} = \frac{1}{k} f(u, f(u, \dots, f(u, u \dots)));$$

b) $\Psi_U^{k*}(x) = k^i x$, $x \in \Lambda^{-2i} = U^{-2i}(P) = \Omega_U^{2i}$;
c) $\Delta\Psi_U^k = \Psi_U^k \otimes \Psi_U^k$, $\Delta\colon A^U \to A^U \otimes_\Lambda A^U$;

d) $\Psi_U^k\Psi_U^l = \Psi_U^{kl} = \Psi_U^l\Psi_U^k$;
e) *for a prime p, all $\lambda_i, \lambda_i \in \Lambda^{-2i}$, such that*

$$\Psi_U^p(u) = u + \lambda_2 u^2 + \cdots + \lambda_i u^i + \dots, \quad u \in \mathrm{Map}(X, MU_1),$$

are integral for $i < p$. Hence the element $p^n\Psi_U^p(u_1 \dots u_n)$, where $u_1 \dots u_n \in U^{2n}(MU_n)$ is a universal element, is integral, and so the operation $p^n\Psi_U^n$ for elements of dimension $2n$ is "integral," if the dimension of the complex is $< 2pn$. (See Appendix 2 for proof.)

4. We now consider the projection operators. The condition defining a projection operator $\pi \in A^U$ is obviously $\pi^2 = \pi$, or $\pi^{*2} = \pi^*$, where $\pi^* \colon \Lambda \to \Lambda$ is the natural representation. We shall consider only those π for which $\pi(xy) = \pi(x)\pi(y)$ and $\pi(u) = \sum_{i \geq 1} \lambda_i u^i \in U^*(CP^\infty)$. Let

$$x_i = [CP^i], \quad \pi(u) = \left(1 + \sum_{i \geq 1} \lambda_i u^i\right) u,$$

where the $\lambda_i \in \Lambda \otimes Q$ are polynomials in x_i with rational coefficients, $\dim \lambda_i = -2i$. It is easy to show that $\pi^*(\lambda_i) = 0$, since $\pi^2 = \pi$.

We shall be especially interested in the case when there exists a complete system of orthogonal projectors (π_j), $\pi_j\pi_k = 0$, $j \neq k$, which split the cohomology theory U^* into a direct sum of identical theories.

Let $y \in \Lambda$ and $\Delta_y \in A^U \otimes Q$ be the "operator of division by y," which has the following properties:
1) $\Delta_y(ab) = \Delta_y(a)b + a\Delta_y(b) - y\Delta_y(a)\Delta_y(b)$,
2) $\Delta_y^*(y) = 1$.

Let $\Phi_y = y\Delta_y$, $\Psi_y = 1 - \Phi_y \in A^U \otimes Q$. It is easy to see that $\Phi_y^2 = \Phi_y$, $\Psi_y^2 = \Psi_y$, and $\Phi_y \circ \Psi_y = 0$. Moreover, the collection of projectors $\pi_i = y^i\Delta_y^i - y^{i+1}$ is such that $\pi_j\pi_k = 0$, $j \neq k$, and it decomposes the theory $U^* \otimes Q$ into a sum of identical theories.

Let $y_i \in \Lambda^{-2j} = \Omega_U^{2j}$ be a system of polynomial generators, and $\Phi_i = y_i\Delta_{y_i}$. We note that $\Phi_i^*(y_i) = 0$ for $j < i$. Let $\tilde{y}_k = y_k$ for $k \leq j$ and $\tilde{y}_k = (1-\phi_i)^* y_k = \Psi_i^*(y_k)$. Obviously, $\Phi_i^*(\tilde{y}_k) = 0$ for $k \neq i$ and $\Phi_i^*(\tilde{y}_i) = y_i = \tilde{y}_i$.

Since $(1 - \Phi_i)^*(y_j) = y_j$ for $j \leq i$ and $y_j - y_i\Delta_{y_i}(y_j) = (1 - \phi_i)^* y_j$ for $j > i$, the collection of elements $\tilde{y}_k$ is a system of polynomial generators.

The projectors $\pi_j = y_i^j\Delta_{y_i}^j - y_i^{j+1}\Delta_{y_i}^{j+1}$ clearly are such that $\pi^* f \colon \Lambda \to \Lambda$ carries monomials of the form $y_i^j\tilde{y}_{i_1}, \dots, \tilde{y}_{i_s}$, $j > 0$, into themselves for $i_1, \dots, i_s \neq i$, and all other monomials into zero. This means that

$$\operatorname{Im}\pi_j^* = y_i^j Q(\tilde{y}_1, \dots, \hat{\tilde{y}}_i, \dots)$$

and

$$\operatorname{Ker}\pi_j^* = \bigcup_{s \neq j} y^s Q(\tilde{y}_1, \dots, \hat{\tilde{y}}_i, \dots).$$

In particular, $1 - \sum_j \pi_j$ and $\pi_{j+1} = y_i\pi_j\Delta_{y_i}$. Hence $\Delta_{y_i}\pi_{j+1}(x) = \pi_j(y_i x)$ for all $x \in U^*$, and all theories $\pi_j(U \otimes Q)$ are isomorphic.

The projector $\pi_0 = 1 - y_i\Delta_{y_i}$ has the following properties:
a) $\pi_0(xy) = \pi_0(x)\pi_0(y)$, i.e., π_0 is multiplicative.

b) The cohomology ring of a point for the theory $\pi_0(U^* \otimes Q)$ has the form $Q(\tilde{y}_1, \dots, \hat{\tilde{y}}_i, \dots)$, where $\pi_0^*(\bar{y}_j) = \bar{y}_j$ for $j \neq i$.

c) All theories $\pi_s(U^* \otimes Q)$ are canonically isomorphic to the theory $\pi_0(U^* \otimes Q)$ by means of the operator of multiplication by y_i^s, and their defining spectra differ only by suspension.

Examples of operators Δ_y: if $\dim y = 2k$, i.e., $y \in \Omega_U^{2k}(P) = U^{-2k}(P)$, and $\sigma_{(k)}^* y = -\lambda \neq 0$, then we set

$$\Delta_y = \sum_{q \geq 1} \frac{(-1)^q y^{q-1}}{\lambda^q} S_{\underbrace{(k, \dots, k)}_{q \text{ times}}}.$$

For the generators $y_i \in \Omega_U^{2i}$ we have $|\lambda| = 1$ for $i \neq p^j - 1$ and $|\lambda| = p$ for $i = p^i - 1$ for any prime p. Hence

$$\Delta_{y_i} = \sum_{q \geq 1} y_i^{q-1} S_{(i, \dots, i)} \quad (i \neq p^j - 1)$$

and

$$\Delta_{y_i} = \sum_{q \geq 1} \frac{y_i^{q-1}}{p^q} S_{(i, \dots, i)} \quad (i = p^j - 1).$$

It is easy to see that for $i + 1 \neq p^j$ for given $p, \Delta_{y_i} \in A^U \otimes_Z Q_p$; for $i + 1 = p^j$ and $p \geq 2$, $\Delta_{y_i} \in A^U \otimes Q$, where Q_p is the p-adic integers.

Now let y_i be a collection of polynomial generators of Ω_U and let p be prime. We consider all numbers $i_k \neq p^j - 1$ in the natural order, $i_1 < i_2 < \cdots < i_k < \dots$. Let $\Phi_k = (1 - y_{i_k} \Delta_{y_{i_k}})$, where k is some sufficiently large integer. The projector Φ_k is such that the ring $\Phi_k^* \Lambda \subset \Lambda$ has as a system of polynomial generators all $\tilde{y}_i$ for $i \neq i_k$, and $\Phi_k^* y_{i_k} = 0$.

Obviously, the operator Φ_k commutes with the operator of multiplication by y_j for $j \leq i_k$ since $\Phi_k = 1 - y_{i_k} \Delta_{y_{i_k}}$, and $\Delta_{y_{i_k}}$ commutes with $y_j, j \leq i_k$.

We consider the operator $\Phi_k \Delta_{y_{i_{k-1}}} \Phi = \tilde{\Delta}_{k-1}$. Since Φ_k is multiplicative, $\tilde{\Delta}_{k-1}$ is the operator of division by $\Phi_k y_{i_{k-1}} \Phi_k$. Hence in the cohomology theory $\Phi_k(U^*)$ the operator $\tilde{\Delta}_{k-1}$ has all the properties making $1 - y_{i_{k-1}} \circ \tilde{\Delta}_{k-1} = \Phi_{k-1}$ a multiplicative projector, and $\Phi_{(k-1)}^{(j)} = y_{i_{k-1}}^j \tilde{\Delta}_{k-1}^j - y_{i_{k-1}}^{j+1} \tilde{\Delta}_{k-1}^{j+1}$ forms a complete system of orthogonal projectors.

Thus, $\tilde{\Phi}_{k-1} = \Phi_k - y_{i_{k-1}} \Phi_k \circ \Delta_{y_{i_{k-1}}} \circ \Phi_k = \Phi_k (1 - y_{i_{k-1}} \Delta_{y_{i_{k-1}}}) \Phi_k$ and $\tilde{\Phi}_{k-1} \Phi_k = \tilde{\Phi}_{k-1}$, while

$$\tilde{\Phi}_{k-1} = \Phi_k \Phi_{k-1} \Phi_k,$$

where $\Phi_k = 1 - y_{i_k} \Delta_{y_{i_k}}$. If $\Phi_s = 1 - y_{i_s} \Delta_{y_{i_s}}$, then we set:

$$\tilde{\Phi}_2^{[k]} \Phi_1 \tilde{\Phi}_2^{[k]} = \tilde{\Phi}^{[k]},$$

where $\tilde{\Phi}_k^{[k]} = \Phi_k$, or:

$$\tilde{\Phi}_{(k-1)}^{[k]} = \tilde{\Phi}_k^{[k]} \Phi_{k-1} \tilde{\Phi}_k^{[k]}, \ \dots, \ \tilde{\Phi}_i^{[k]} = \tilde{\Phi}_{i+1}^{[k]} \Phi_i \tilde{\Phi}_{i+1}^{[k]}, \ \dots, \ \tilde{\Phi}^{[k]} = \tilde{\Phi}_2^{[k]} \Phi_1 \tilde{\Phi}_2^{[k]}.$$

The projector $\Phi^{[k]}$ is obviously such that

$$\text{a)}\quad \Phi^{[k]^*}(y_s) = \begin{cases} 0, & s+1 \neq p^j, \\ \tilde{y}_s, & s+1 = p^j \text{ for } s \leq i_k, \end{cases}$$

$$\text{b)}\quad \Phi^{[k]} \in A^U \otimes_Z Q_p.$$

The collection of $\Phi^{[k]}$ with $k \to \infty$ is such that $\Phi^{[k]_*}$ is independent of k when it operates on Ω_U and hence the sequence $\Phi^{[k]}$ as $k \to \infty$, or the series $\sum_{k\geq 1}(\Phi^{[k+1]} - \Phi^{[k]}) = \Phi$ defines a projector $\Phi \in A^U \otimes_Z Q_p$ which is multiplicative and such that:

$$\text{a)}\quad \Phi^*(y_s) = \begin{cases} 0, & s \neq p^j - 1, \\ \tilde{y}_s, & s = p^j - 1, \end{cases}$$

$$\text{b)}\quad \Phi^2 = \Phi,$$

c) the theory $U^* \otimes_Z Q_p$ splits into a sum of identical theories of the form $\Phi(U^* \otimes_Z Q_p)$ up to a shift of grading (suspension).

We note that the elements $\tilde{y}_s = \Phi^*(y_s)$ for $s = p^j - 1$ have the property that all $\sigma^*_\omega(\tilde{y}_s) \equiv 0 \mod p$ for all ω, $\dim \omega = 2s$.

The cohomology theory $\Phi(U^* \otimes_Z Q_p)$ is given by a spectrum $M_{(p)}$, where $H^*(M_{(p)}, Z_p) = A/\beta A + A\beta$, A the Steenrod algebra and β the Bokšteĭn homomorphism.

Thus, we have shown

Lemma 5.9. a) *There exists a multiplicative projector $\Phi \in A^U \otimes_Z Q_p$ such that the cohomology theory $\Phi(U \otimes_Z Q_p)$ is given by a spectrum $M_{(p)}$, where $H^*(M_{(p)}, Z_p) = A/A\beta + \beta A$, and the homomorphism $\Phi^*\colon \Lambda \to \Lambda$ annihilates all polynomial generators of the ring $\Lambda = U^*(P) \approx \Omega_U$ of dimension different from $p^j - 1$.*

b) *The theory $U^* \otimes_Z Q_p$ decomposes into a direct sum of theories of the form $M^*_{(p)} = U^*_p$ and their suspensions.*

§ 6. The A^U-modules of cohomology of the most important spaces

In this section we shall give the structure of the module $U^*(X)$ for the most important spectra $X = P$ (a point), $X = CP^n$, $X = RP^{2n}$, $X = RP^{2n-1}$, $X = MSU$, $X = S^{2n-1}/Z_p$, $X = BG$, $G = Z_p$.

1. Let $X = P$. The A^U-module $U^*(P)$ is given by one generator $u \in U^0(P)$ and the relations $S_\omega(u) = 0$ for all $\omega > 0$. An additive basis for $U^*(P)$ is given by the fact that $U^*(P)$ is a free one-dimensional Λ-module, where $\Lambda \approx \Omega_U$. We shall denote the module $U^*(P)$ by Λ.

Clearly, we have:

$$\mathrm{Hom}^*_{A^U}(A^U, \Lambda) = U_*(P) = \Omega_U.$$

If $d\colon A^U \to A^U$ is a map such that $d(1) = a \in A^U$, then it is easy to see that $d^*(h_x) = h_{a^*(x)}$, where $h_x \in \mathrm{Hom}_{A^U}(A^U, \Lambda)$, $x \in \Lambda$, and h_x is such that $h_x(1) = x$. In particular, for $a = S_\omega$ we have $a^* = \sigma^*_\omega$, and for $a = \partial, \Delta$ we have $a^* = \partial^*$ or Δ^*, the known homomorphisms of the ring Ω_U.

These remarks are essential for computing

$$\mathrm{Ext}_{A^U}(\ , U^*(P)) = \mathrm{Ext}_{A^U}(\ , \Lambda)$$

2. Let $X = CP^n = (E^k CP^n) \in S$. It is easy to see that $U^*(X)$ is a cyclic module with generator $u \in U^2(X)$ satisfying the relations:

a) $S_\omega(u) = 0$, $\omega \neq (k)$,
b) $S_{(k)}(u) = 0$, $k \geq n$.
These results are easily derived from the properties of the ring $U^*(CP^n)$ and the properties of the operations S_ω given in Lemma 5.6.

3. $X_k^{(n)} = S^{2n+1}/Z_k = (E^k S^{2n+1}/Z_k) \in S$, $X_2^{(n)} = RP^{2n+1}$. $U^*(X)$ has two generators $u \in U^2(X_k^{(n)})$, $v \in U^{2n+1}(X_k)$, satisfying the relations:
a) $S_\omega(u) = 0$, $\omega \neq (q)$,
b) $S_{(q)}(u) = 0$, $q \geq n$,
c) $(k\Psi_U^k)(u) = 0$, $u \in \operatorname{Map}(X_k, MU_1)$,
d) $S_\omega(v) = 0$, $\omega > 0$.
These results follow from [7] for $K^*(BG)$, $G = Z_p$, and the $\sigma_1\colon K^0 \to U^2$ and the ring $U^*(BG)$.

4. For $X = RP^{2n}$, BG, the module $U^*(X)$ is described as follows:
a) $U^*(RP^{2n}) = U^*(RP^{2n+1})/v$.
b) $U^*(BZ_k) = \lim[U^*(X_k^{(n)})]$.

5. We now consider the case $X = MSU$. Since $U^*(MSU) = \phi U^*(BSU)$ and SU-bundles are distinguished by the condition $c_1 = 0$, which is equivalent to the condition $\gamma\sigma_1 = 0$, we have $U^*(MSU) = U^*(MU)/\phi J(\gamma\sigma_1)$, where J is the ideal spanned by $(\gamma\sigma_1)$, $J \subset U^*(BU)$.

The natural map $U^*(MU) \to U^*(MSU)$ is an epimorphism. Hence $U^*(MSU)$ is a cyclic A^U-module with generator $u \in U^0(MSU)$ and $au = 0$ if and only if $a \in \phi J(\gamma\sigma_1)$.

In particular, $au = 0$ for $a = \Delta_{(k_1,k_2)}$.

We have the important

Theorem 6.1. a) *The module $U^*(MSU)$ is completely described by the relations $\partial(u) = 0$, $\Delta(u) = 0$.*

b) *The left annihilator of the operation ∂ consists of all operations of the form $a\partial + b\Delta$, $a, b \in A^U$.*

Proof. We consider the module $N = A^U/A^U\Delta + A^U\partial$ and the natural map $f\colon N \to U^*(MSU)$. We shall show that this map is an isomorphism. Since for the operation Δ there exists a right inverse Ψ such that $\Delta\Psi = 1$ and $\partial\Psi = 0$, the module $A^U\Delta$ is free, and it is not possible to have a relation of the form $a\Delta + b\partial$ if $a \neq 0$ or $b\partial \neq 0$.

We now consider $A^U\partial$. We shall establish the following facts:

1) The left annihilator of the operation ∂ consists precisely of the operations of the form $\phi J(\gamma\sigma_1) \subset A^U$.

2) The operations of the form $A^U\partial$ form a direct summand in the free abelian group of operations A^U under addition.

We consider the representation $a \to \tilde{a}$ on $U_*(BU)$. Let ξ be an SU-bundle. It is easy to see that we have the equation

$$(X, \xi) = \tilde{\partial}[(X, \xi) \otimes (CP^1, \eta)],$$

where $c_1(\eta)$ is the basic element of $H^2(CP^1)$. It is also obvious that $\operatorname{Im}\partial$ consists only of pairs $(X, \xi) \in U_*(BU)$, where ξ is an SU-bundle. Hence $\operatorname{Im}\partial$ is precisely $U_*(BSU)$. Whence follows fact (1).

For the proof of (2) we note that $U_*(BSU)$ is a direct summand in $U_*(BU)$. We decompose $U^*(BU)$ into a direct sum $U^*(BU) = U^*(BSU) + J(\gamma\sigma_1)$. Then $U^*(MU) = A^U$ decomposes into a direct sum $A + B$, where B is the annihilator of

$U_*(BSU)$ with respect to the representation $\tilde{a}$. Obviously, $A^U\partial = (B+A)\partial = A\partial$. If the operation $a \in A$ is such that $a\partial$ is divisible by the integer λ, then $\tilde{a}\tilde{\partial}$ is divisible by λ, and hence for all SU-bundles ξ the characteristic class $\phi^{-1}\bar{a}(\xi)$ is divisible by λ. Hence this class is a λ-multiple class in $U^*(BSU)$ and (up to $J(\gamma\sigma_1)$) a λ-multiple class in $U^*(BU)$. Whence follows fact (2).

We deduce from (1) and (2) that the map $f\colon N \to U^*(MSU)$ is a monomorphism. Since $N = A^U/A^U\Delta + A^U\partial$, it follows from (1) and (2) that the kernel $\operatorname{Ker} f$ is a direct summand. Since $A^U\Delta$ is a free module and $A^U\partial$ is a module isomorphic to $U^*(MSU)$ with shifted dimension (see (1)), the equation $\operatorname{Ker} f = 0$ follows from the calculation of ranks in the groups

$$(A^U\Delta \otimes_\Lambda Z)^k = H^{(k-4)}(MU, Z), \quad (A^U\partial \otimes_\Lambda Z)^k = H^{k-2}(MSU, Z),$$
$$(U^*(MSU) \otimes Z)^k = H^k(MSU, Z).$$

Thus, $U^*(MSU) = A^U/A^U\Delta + A^U\partial$. Since the left annihilator of the operation ∂ is precisely the left annihilator of the element $u \in U^0(MSU)$, it follows by what was proved for $U^*(MSU)$ that this left annihilator is precisely $A^U\Delta + A^U\partial$. The theorem is proved. □

§ 7. Calculation of the Adams spectral sequence for $U^*(MSU)$

In this section we shall compute the ring

$$\operatorname{Ext}^{**}_{A^U}(U^*(MSU), \Lambda)$$

and all differentials d_i of the Adams spectral sequence (E_r, d_r), where $E_2 = \operatorname{Ext}^{**}_{A^U}(U^*(MSU), \Lambda)$. In particular, it turns out that $d_i = 0$ for $i \neq 3$, $d_3 \neq 0$, and $E_\infty^{i,*} = E_4^{i,*} = 0$ for $i \geq 3$.

For the calculation of $\operatorname{Ext}^{**}_{A^U}$ we consider the complex of A^U-modules

$$C = (U^*(MSU) \xleftarrow{\varepsilon} C_0 \xleftarrow{d} C_1 \xleftarrow{d} \cdots \leftarrow C_i \ldots),$$

where the generators are denoted by $u_i \in C_i$ for $i \geq 0$ and $v_i \in C_i$ for $i \geq 1$, $C_0 = A^U$ and $C_i = A^U + A^U$ for $i \geq 1$. We set $d(u_i) = \partial u_{i-1}$ and $d(v_i) = \Delta u_{i-1}$. Since $\partial^2 = 0$ and $\Delta\partial = 0$, $d^2 = 0$. It follows from the theorem above that C is an acyclic resolution of the module $U^*(MSU) = H_0(C)$.

We now consider the complex $\operatorname{Hom}^*_{A^U}(C, \Lambda)$, where $\Lambda = U^*(P)$. Since $\operatorname{Hom}^*_{A^U}(A^U, \Lambda) = \Omega_U$, we obtain the complex

$$\operatorname{Hom}^*_{A^U}(C, \Lambda) = (\Omega_U \xrightarrow{d^*} \Omega_U + \Omega_U \xrightarrow{d^*} \Omega_U + \Omega_U \xrightarrow{d^*} \ldots),$$

where $d^* = \partial^* + \Delta^*\colon \Omega_U \to \Omega_U + \Omega_U$.

Since Δ^* is an epimorphism, the complex $\operatorname{Hom}^*_{A^U}(C, \Lambda)$ reduces to the following:

$$\tilde{W} = (W \xrightarrow{\partial^*} W \xrightarrow{\partial^*} W \xrightarrow{\partial^*} \ldots),$$

where $W = \operatorname{Ker}\Delta^* \subset \Omega_U$

From this we deduce the following assertion.

Lemma 7.1. a) *For all $s \geq 1$, we have isomorphisms*

$$\operatorname{Ext}^{s,t}_{A^U}(U^*(MSU), \Lambda) = H_{t-2s}(W, \partial^*).$$

b) $\operatorname{Ext}^{0,*}_{A^U}(U^*(MSU), \Lambda) = \operatorname{Ker}\partial^* \cap \operatorname{Ker}\Delta^* \subset \Omega_U$.

c) *If* $h \in \mathrm{Ext}^{1,2}_{A^U}(U^*(MSU),\Lambda) = Z_2$ *is the nonzero element, then the homomorphism* $\alpha \to h\alpha\colon \mathrm{Ext}^{i,*}_{A^U} \xrightarrow{h} \mathrm{Ext}^{i+1,*}_{A^U}$ *is an epimorphism with kernel* $\operatorname{Im}\partial^*$ *for* $i=0$*, and an isomorphism for* $i \geq 1$ *(we recall that the spectrum* MSU *is multiplicative).*

Proof. Statements (a), (b) of the lemma obviously follow from the structure of the complex W, in which the grading of each term is shifted by 2 from the one before by the construction. For the proof of (c), we note that $h = \frac{1}{2}\partial^*(x_1)$, where $x_1 = [CP^1] \in \Omega_U$, $x_1 \in W$, and $\partial^*(x_1) = -2$. Further, we note that $\partial^*(x_1 y) = -2y$ if $\partial^*(y) = 0$. Hence the element hy is represented by the element $\frac{1}{2}\partial^*(yx_1)$ for a representative of $y \in H^*(\tilde{W},\partial^*)$. But since $\frac{1}{2}\partial^*(yx_1) = y$ under the condition $\partial^* y = 0$, statement (c) is proved, and therewith the lemma. □

We consider the element $K = 9x_1^2 - 8x_2 \in \Omega^4_U$, where $x_1 = [CP^1]$, $x_2 = [CP^2]$. Clearly, $\partial^* K = \Delta^* K = 0$. The element K is a generator of the group

$$\mathrm{Ext}^{0,4}_{A^U}(U^*(MSU),\Lambda) = \operatorname{Ker}\partial^* \cap \operatorname{Ker}\Lambda^* = Z.$$

Since $A[K] = \pm 1$, where $A = e^{-c_1/2}T$ and T is the Todd genus, by virtue of the Riemann–Roch theorem there is an i such that $d_i(K) \neq 0$ in the Adams spectral sequence, since for all 4-dimensional SU-manifolds the A-genus is even (see [20]).

It follows from dimensional considerations that $d_2(K) = 0$ and $d_3(K) = h^3$.

We note that from dimensional considerations it follows trivially that $d_{2k} = 0$ (see theorem in § 2). Consider the differential

$$d_3\colon E_3^{p,q} \to E_3^{p+3,q+2},$$

where $d_3(K) = h^3$. We have

Lemma 7.2. *If* $\alpha \in E_3^{p,q}$ *for* $p \geq 3$*, and* $d_3(\alpha) = 0$*, then* $\alpha = d_3(\beta)$*. Hence* $E_4^{p,q} = 0$ *for* $p \geq 3$*, and* $E_\infty = E_4$*.*

Proof. Let $d_3(\alpha) = 0$; since $\alpha = \widetilde{h^3\beta}$ from Lemma 7.1, $d_3(\alpha) = d_3(\widetilde{h^3\beta}) = 0$. Hence $\widetilde{d^3(\beta)} = 0$ since multiplication by $h\colon E_3^{p,*} \to E_3^{p+1,*}$ is a monomorphism for $p > 0$. This means that $\alpha = d_3(\widetilde{K\beta})$. Since

$$\sum_{p \geq 3} E_3^{p,*} = \sum_{p \geq 3} \mathrm{Ext}^{p,*}_{A^U}(U^*(MSU),\Lambda)$$

is the ideal generated by the element h, we have $E_4^{p,q} = 0$ for $p \geq 3$.

From dimensional considerations it follows that $E_4^{**} = E_\infty^{**}$.

Since $E_\infty^{**} = E_\infty^{0,*} + E_\infty^{1,*} + E_\infty^{2,*}$ is associated with $\Omega_{SU} = \pi_*(MSU)$, and $E_\infty^{1,*} = hE_\infty^{0,*}, E_\infty^{2,*} = hE_\infty^{1,*} = h^2 E_\infty^{0,*}$ we obtain □

Corollary 7.1. a) $\Omega^{2k+1}_{SU} = h\Omega^{2k}_{SU}$; b) $h^2\Omega^{2k}_{SU} = \operatorname{Tor}\Omega^{2k+1}_{SU}$.

The equality (a) was first established in [18] by other methods, and (b) in [12].

Corollary 7.2. a) *The image of* Ω_{SU} *in* Ω_U *is singled out within the intersection* $\operatorname{Ker}\partial^* \cap \ker\Delta^*$ *by setting equal to zero a certain collection of linear forms* mod 2*, generated by the homomorphism*

$$\begin{aligned} d_3\colon (\operatorname{Ker}\partial^* \cap \operatorname{Ker}\Delta^*)^{2k} &= E_3^{0,2k} \to E_3^{3,2k+2} \\ &= h^3(\operatorname{Ker}\partial^* \cap \operatorname{Ker}\Delta^*)^{2k-4} = h^3 E_3^{0,2k-4} = \sum Z_2. \end{aligned}$$

b) *The group* $\operatorname{Ext}^{1,2k}_{A^U} = H(W,\partial^*)$ *is isomorphic to the direct sum* $\Omega^{2k-1}_{SU} + \Omega^{2k-5}_{SU}$, *and this isomorphism comes from the differential* d_3:

$$0 \to \Omega^{2k-1}_{SU} \to \operatorname{Ext}^{1,2k}_{A^U} \xrightarrow{h^{-3}d_3} \Omega^{2k-5}_{SU} \to 0,$$

where $\operatorname{Ext}^{1,2k}_{A^U} = E^{1,2k}_2 = E^{2k-1}_3 \supset \Omega^{2k-1}_{SU} = \operatorname{Ker} d_3$, $h^{-3}d_3$ *is well-defined since* h^3 *is a monomorphism on* $\operatorname{Ext}^{1,2k-4}_{A^U}$, *and the image* $\operatorname{Im} h^{-3}d_3 = \operatorname{Ker} d_3 = \Omega^{2k-5}_{SU} \subset \operatorname{Ext}^{1,2k-4}_{A^U}$.

Corollary 7.2 follows from Lemma 7.2.

Remark 7.1. Part (b) of the corollary explains the meaning of the "Conner–Floyd exact sequence" (see [13])

$$0 \to \Omega^{2k-1}_{SU} \to H_{2k-2}(W,\partial^*) \to \Omega^{2k-5}_{SU} \to 0,$$

since $H_{2k-2}(W,\partial^*) = \operatorname{Ext}^{1,2k}_{A^U}(U^*(MSU),\Lambda)$.

We note now that the groups $H_*(W,\partial^*)$ are computed in [13]: namely, $H_{8k}(W) = H_{8k+4}(W) = Z_2 + \cdots + Z_2$ (the number of summands is equal to the number of partitions of the integer k), $H_i(W) = 0$, $i \neq 8k$, $8k+4$. Whence we have:

$$\operatorname{Ext}^{1,8k+2}_{A^U}(U^*(MSU),\Lambda) = \operatorname{Ext}^{1,8k+6}_{A^U}(U^*(MSU),\Lambda)$$

and

$$\operatorname{Ext}^{1,i}_{A^U}(U^*(MSU),\Lambda) = 0, \quad i \neq 8k+2, 8k+6.$$

We have

Lemma 7.3. a) $\operatorname{Ext}^{1,8k+6}_{A^U} = K \operatorname{Ext}^{1,8k+2}_{A^U}$, *where* $K \in \operatorname{Ext}^{0,4}_{A^U}(U^*(MSU),\Lambda)$.

b) $d_3(\operatorname{Ext}^{0,i}_{A^U}) = 0$ *for* $i \neq 8k+4$, *and* $d_3(\operatorname{Ext}^{0,8k+4}_{A^U}) = \operatorname{Ext}^{3,8k+2}_{A^U}$ *is defined by the condition* $d_3(K) = h_3$.

Proof. Suppose both parts of the lemma proved for $k \leq k_0 - 1$. We show that $d_3(\operatorname{Ext}^{0,8k_0}_{A^U}) = 0$. In fact, by the induction hypothesis on the groups $\operatorname{Ext}^{3,8k+2}_{A^U}$ the differential d_3 is a monomorphism. Hence $\operatorname{Ext}^{0,8k_0}_{A^U} \xrightarrow{d_3} 0$.

We now consider $d_3(K \operatorname{Ext}^{0,8k_0}_{A^U}) = h^3 \operatorname{Ext}^{0,8k_0}_{A^U}$. We see that $d_3(K \operatorname{Ext}^{0,8k_0}_{A^U})$ is an epimorphism onto $\operatorname{Ext}^{3,8k_0+6}_{A^U}$. Whence parts (a) and (b) of the lemma follow; on the group $\operatorname{Ext}^{3,8k+6}$ the differential is trivial, and on the group $\operatorname{Ext}^{3,8k+2} \supset \operatorname{Ker} d_3 = 0$.

The lemma is proved. ☐

Thus, we obtain

Corollary 7.3. a) *The image* $\Omega^i_{SU}/\operatorname{Tor} \subset \Omega_U$ *coincides with* $\operatorname{Ker}\partial^* \cap \operatorname{Ker}\Delta^*$ *for* $i \neq 8k+4$.

b) *For* $i = 8k+4$ *the image* $\Omega^{8k+4}_{A^U}/\operatorname{Tor} \subset \Omega^{8k+4}_U$ *is picked out precisely by the requirement of the "Riemann–Roch Theorem"*:

$$\operatorname{ch}(c\xi)A(X)[X] \equiv 0 \pmod 2,$$

where X *is an SU-manifold,* $\xi \in kO(X)$.

We note that (a) follows immediately from the lemma. As to (b), we note that $A[K] = 1$. In [9], "Pontrjagin classes" $\pi_l \in kO^*[X]$ are introduced in kO-theory. Consider the classes $\pi_{2l} \in \widetilde{kO}(X)$; let $\pi_{2l} = \kappa_l$. Now consider the numbers $\operatorname{ch}(c\kappa_{l_1} \ldots c\kappa_{l_k})A(X)[X]$ for $X \in \Omega^{8k}_U/\operatorname{Tor} \subset \Omega^{8k}_U$. These numbers are different from zero $\bmod 2$ if and only if $hX \neq 0$ in Ω^{8k+1}_{SU}. Hence the condition

$d_3(KX) = h^3X \neq 0$ in $E_3^{3,*}$ is equivalent to the fact that one of the numbers $\mathrm{ch}(c(\kappa_{l_1},\dots,\kappa_{l_k}))A(X)[X] \not\equiv 0 \pmod 2$. All such numbers are in 1-1 correspondence with partitions of $8k$ into summands $(8l_1,\dots,8l_k)$ (these facts are easily deduced from [9]).

Since $\mathrm{ch}(c\kappa_{l_1}\dots c\kappa_{l_k}\otimes 1)A(X\times K)[X\times K] = \mathrm{ch}(c\kappa_{l_1}\dots c\kappa_{l_k})A(X)[X]\circ A[K]$, $A[K]=1$, we have found elements $\kappa_{l_1}\dots\kappa_{l_k}\otimes 1 \in kO(X\times K)$ which do not satisfy the Riemann–Roch theorem, and they determine $\pi(k)$ linearly independent forms mod 2, where $\pi(k)$ is the number of partitions of k. From this part (b) of the corollary follows.

The results of the lemmas and corollaries of this section together completely describe the Adams spectral sequence for $U^*(MSU)$.

§ 8. k-theory in the category of complexes without torsion

Here we shall consider the cohomology theory k^*, defined by the spectrum $k = (k_n)$, where $\pi_i(k_n) = 0$, $i < n$, and $\Omega^{2n}k_{2n} = BU\times Z$. The spectrum k is such that the cohomology module $H^*(k, Z_2)$ is a cyclic module over the Steenrod algebra, with a generator $u \in H^0(k, Z_2)$, satisfying the relations $\mathrm{Sq}^1(u) = \mathrm{Sq}^3(u) = 0$. Hence the spectrum k does not lie in the category D of complexes without torsion.

There is defined the "Bott operator" $x\colon k_{2n}\to k_{2n-2}$ by virtue of the Bott periodicity $\Omega^2 k_{2n} = k_{2n-2}$, and k_{2n} is a connective fiber of BU. Since $k^0(X) = K^0(X)$, we have on k^0 the Adams operations (see [2])

$$\Psi^k\colon K^0(X)\to K^0(X),$$

defined by morphisms $\Psi^k\colon BU\to BU$ such that $\Psi^k_*\colon \pi_{2n}(BU)\to\pi_{2n}(BU)$ is the operator of multiplication by the integer k^n (see [2] concerning the operation of Ψ^k on $K^0(S^{2n}) = \pi_{2n}(BU)$). By virtue of this, the operators Ψ^k can be extended to the whole theory $K^*\otimes Q$, starting from the identity

$$kx\Psi^k = \Psi^k x,$$

where $x\colon K^i\to K^{i-2}$ is Bott periodicity.

In the category D of complexes without torsion the operator $x^n\colon k^{2n}(X)\to k^0(X)$ is such that its image consists precisely of all elements in $k^0(X) = K^0(X)$ whose filtration is $\geq 2n$; moreover, x is a monomorphism.

In the category D we define an operation $(k^n\Psi^k)$ by setting

$$(k^n\Psi^k) = x^{-n}\Psi^k x^n,$$

where $(k^n\Psi^k)\colon k^{2n}(X)\to k^{2n}(X)$.

It is easy to see that this is well defined and gives rise to an unstable operation $(k^n\Psi^k)$ such that $(k^n\Psi^k)$ can be considered as a map $k_{2n}\to k_{2n}$ for which $(k^n\Psi^k)_*\colon \pi_{2n+2j}(k_{2n})\to\pi_{2n+2j}(k_{2n})$ is multiplication by k^{n+j}.

Let $a_n = \sum_k \lambda_k^{(n)}(k^n\Psi^k)$, where the $\lambda_k^{(n)}$ are integers, be an unstable cohomology operation and $a_n^{(j)*}$ multiplication by $\sum_k \lambda_k^{(n)}k^{n+j}$.

Definition 8.1. The sequence $a = (a_n)$ will be called a stable operation if for any j there is a number n such that for all $N\geq n$ the number $a_N^{(j)} = \sum_k \lambda_k^{(N)}k^{N+j}$ is independent of N.

Definition 8.2. If the stable operation a has a zero of order q in the sense that $a_*^{(j)} = 0$ for $j \le q$, then we also call $b = (x^{-q}a)$ a stable operation, where $a = x^q b$, $b\colon k^i(X) \to k^{i+2q}(X)$, $X \in D$.

We consider the ring generated by the operations so constructed and the operation x by means of composition, taking into account the facts that $kx\Psi^k = \Psi^k x$ and $\Psi^k\Psi^l = \Psi^{kl}$. The resulting uniquely defined ring, which we denote by A_Ψ^k, is a ring of operations acting in the category D. In it lies the subring of operations generated by the operations indicated in Definition 8.1 together with the periodicity x. This ring we denote by $B_\Psi^k \subset A_\Psi^k$. There is defined the inclusion $B_\Psi^k \to A_\Psi^k$.

We shall exhibit a basis for the ring A_Ψ^k. It is easy to see that it is possible to construct operations $\delta_i \in A_\Psi^k$ of dimension $2i$, where $\delta_0 = 1$, such that the elements $x^k\delta_i$ give an additive topological basis for the ring A_Ψ^k, and all elements of A_Ψ^k can be described as formal series $\sum \lambda_k x^k \delta_{k-i}$, where the λ_k are integers. The choice of such elements δ_i is of course unique only $\mod xA_\Psi^k$ (elements of higher filtration).

We construct these elements δ_i in a canonical fashion: it suffices to define operations $\gamma_i = x_i\delta_i$ of dimension 0. Let $\delta_0 = 1$. Let $\gamma_{1*}^{(0)} = 0$ and $\gamma_{1*}^{(1)}$ be multiplication by 2. By definition, we shall take $\gamma_{i*}^{(j)} = 0$ for $j < i$ and $\gamma_{i*}^{(i)}$ to be multiplication by a number $\tilde\gamma_i$ which is a linear combination $\tilde\gamma_i = \sum\limits_k \mu_k^{(i)} k^{n+i}$, where the numbers $\mu_k^{(i)}$ are such that $\sum\limits_k \mu_k^{(i)} k^{n+i} = 0$ for $j < i$. We require in addition that $\tilde\gamma_i$ be the smallest positive integer of all linear combinations of the form $\sum\limits_k \mu_k^{(i)} k^{n+i}$ under the conditions:

$$\sum_k \mu_k^{(i)} k^{n+i} = 0, \quad j < i.$$

We consider the operation $a_{in} = \sum\limits_k \mu_k^{(i)}(k^n\Psi^k)$. Here n is very large compared with i. It is easy to see that the number $\tilde\gamma_i$ does not depend on n for large $n \to \infty$. Hence the operation is well-defined.

Consider the operations a_{in} for $n \to \infty$; we shall successively construct the δ_i from them. We have $a_{0n} = 1$; let $b_{mn} = a_{1n} + \kappa_1 a_{2n} + \cdots + \kappa_m a_{mn}$ be linear combinations such that the homomorphisms $(b_{mn})_*^{(j)}$ for $j \le m \ll n$ are multiplications by integers $\tilde\gamma_{i,j}$, where $0 \le \tilde\gamma_{1,j} < \tilde\gamma_j$. Clearly the numbers $\tilde\gamma_{1,j}$ are uniquely defined. Let $m \to \infty$, $n \to \infty$; then in the limit, the sequence (b_{mn}) gives an operation which we denote by $\gamma_1 = x_1\delta_1$. It is uniquely defined by the properties that $\gamma_{1*}^{(0)} = 0$, $\gamma_{1*}^{(1)} = 2$, and $0 \le \gamma_{1*}^{(i)} < \tilde\gamma_i$, $\gamma_{1*}^{(i)} = \tilde\gamma_{1,i}$.

The operations γ_i are constructed in a similar fashion, and are uniquely determined by the conditions $\gamma_{i*}^{(j)} = 0$, $j < i$, $\gamma_{i*}^{(i)} = \tilde\gamma_i$, and $0 \le \gamma_{i*}^{(k)} < \tilde\gamma_k$ for $k > i$.

We exhibit a table of the integers $\gamma_{i*}^{(j)} = \tilde\gamma_{ij}$ in low dimensions:

	γ_0	γ_1	γ_2	
0	1	0	0	...
2	1	2	0	...
4	1	0	24	...
	....	....	....	....

By definition, $\delta_i = x^{-i}\gamma_i$. It is clear that the operations γ_i commute. Since $\pi_{2i}(BU)$ is Z, the rings A_Ψ^k and B_Ψ^k are represented as operators on $k^*(P) = Z[x]$ in a natural way, in particular, the operations of dimension 0 by diagonal operators with integral characteristic values; the operation x is represented by the translation operator (or multiplication by x in $k^*(P)$). It is easy to show that we have a transformation $*\colon A^k_\Psi \to A^k_\Psi$ such that $*(B^k_\Psi) \subset B^k_\Psi$ and $ax = xa^*$. This transformation $*$ is completely determined by the condition that in $k^* \otimes Q$-theory we have $kx\Psi^k = \Psi^k x$ and ${}^*\Psi^k = k\Psi^k$.

We also indicate the following simple fact.

Lemma 8.1. *The greatest common divisor of the integers $\gamma^{(q)}_{i*} = (x^i\delta_i)^{(q)}_*$ for all $i > 0$, for a fixed integer q, coincides precisely with the greatest common divisor of the numbers $k^N(k^q - 1)$ for all k. There exist operations $a_{k,n} \in B^k_\Psi$ such that $a^{(j)}_{k,n*} = k^{n+j}$ for $j \leq f(n)$, where $f(n) \to \infty$ as $n \to \infty$.*

The proof of this consists of the fact that the operations $x^i\delta_i = \gamma_i$ are obtained as linear combinations of the operations $k^n(\Psi^k - 1)$ by virtue of the condition $\gamma^{(0)}_{i*} = 0$ for $i > 0$, where n is large, and the determinant of the transition from the $k^n(\Psi^k - 1)$ to the $x^i\delta_i$ is equal to 1. In fact, the process described above for constructing $(x^i\delta_i)$ is the process of reduction of the set of transformations $k^n(\Psi^k - 1)$ to the set γ_i of "triangular type" on $Z[x] = k^*(P)$. More exactly: let n be sufficiently large that $\gamma^{(j)}_{i,n*} = 0$ for $j < i$ and $\gamma^{(i)}_{i,n*} = \tilde{\gamma}_i$ for $i < f(n)$, where $f(n) \to \infty$ as $n \to \infty$ and $\gamma_{i,n} = \sum \lambda^{(n)}_{k,i}(k^n\Psi^k)$. Under the condition $\sum \lambda^{(n)}_{k,i} k^n = 0$, one can write all these operations in the form $\sum \mu^{(n)}_{k,i} k^n(\Psi^k - 1)$ and then apply to the set $k^n(\Psi^k - 1)$ the process of reduction to "triangular form" described above for constructing the operations $(x^i\delta_i)$ up to high dimensions. We assert that the passage from $\{k^n(\Psi^k - 1)\}$ to $\{\gamma_{i,n}\}$ is invertible. Indeed, any operation of the form $\sum \lambda_k k^n \Psi^k$ has the form $\mu_1\gamma_{1,n} + b_1$, where $b^{(0)}_{1*} = b^{(1)}_{1*} = 0$. Hence the operation b_1 has the form $b_1 = \mu_2\gamma_{2,n} + b_2$, where $b^{(0)}_{2*} = b^{(1)}_{2*} = b^{(2)}_{2*} = 0$, etc.

Consequently, $a = \sum\limits_{i \leq f(n)} \mu_i\gamma_i + b_{f(n)}$ where $b^{(j)}_{f(n),*} = 0$, $j \leq f_n$. If $n \to \infty$, then $f(n) \to \infty$ and the coefficients μ_i stabilize, while $a_n = \sum \mu_i x^i\delta_i + b_{f(n)}$ if $a = (a_n) \in B^k_\Psi$. Since the greatest common divisor of the homomorphisms $a^{(j)}_*$, for all $a \in B^k_\Psi$ such that $a^{(0)}_* = 0$, is invariant and this invariant can be calculated with respect to any basis of operations in B^k such that $a^{(0)}_* = 0$, we have that for the basis $(x_i\delta_i) = (\gamma_i)$ it coincides with the greatest common divisor for the basis $(k^n(\Psi^k - 1)) = b_k$, where $b^{(j)}_{k*} = k^n(k^j - 1)$. We note that the operations $(k^n\Psi^k)$ are nonstable, but, by virtue of what has been said, there exist operations $a_{k,n}$ such that $a^{(j)}_{k,n*} = k^{n+j}$ for $j \leq f(n)$, where $f(n) \to \infty$ as $n \to \infty$. These operations are obtained by the transformation from $(x^i\delta_i)$ to $(k^n\Psi^k)$ inverse to that described above.

The lemma is proved.

Remark 8.1. The same operations Ψ^k in $k^* \otimes Q$ are obtained as formal sums of the form $\sum \mu_i x^i\delta_i = \Psi^k$, where $\mu_i \in Q$ and $k^n\mu_i \in Z$ for large n and $i < f(n)$.

Example 1. Let $X = P \in D$ be the point spectrum. Then $k^*(P)$ has a single generator t as an A^k_Ψ module and is given by the relations $\delta_i(t) = 0$, $i > 0$. The module $k^*(P)$, as a B^k_Ψ-module, has a single generator t and is given by the relations

$(p^n\Psi^p)(t) = p^n t$ for all primes p (n large). (Or: all operations $a \in B^k_\Psi$ which have zeros of order one are such that $at = 0$.)

Example 2. Let $X = MU_n$. Then $k^*(MU_n)$ can be described by the ideal in the ring of symmetric polynomials in the ring $\Lambda[u_1, \dots, u_n]$, $\dim u_i = 2i$, generated by $u = u_1 \dots u_n$. Let $v_i = xu_i$, $\Psi^k(v_i^l) = ((v_i + 1)^k - 1)^l$ and $\Psi^k(xy) = \Psi^k(x)\Psi^k(y)$. The elements of $k^*(MU_n)$ have the form $\sum \lambda_{i,s} x^s d_s$, where $d_s = f(u_1, \dots, u_n)$ is an element of the symmetric ideal in $Z[u_1, \dots, u_n]$ generated by $u = u_1 \dots u_n$, and x is the Bott operator. This uniquely determines $k^*(MU_n)$ and $k^*(MU)$ as A^k_Ψ- and B^k_Ψ-modules.

We have the following

Lemma 8.2. *The ring $B^k_\Psi \subset A^k_\Psi$ coincides exactly with the subring of A^k_Ψ consisting of operations of dimension ≤ 0.*

The only thing which must be proved is that B^k_Ψ contains all operations of dimension ≤ 0. For a pair $a_1 \in B^k_\Psi$, $a_2 \in B^k_\Psi$ of operations which have zeros of order q_1, q_2 respectively, we introduce the operations $x^{-q_1}a_1 = b_1$ and $x^{-q_2}a_2 = b_2$ and the composition $b_1 \circ b_2$ in A^k_Ψ. We shall show that $x^{q_1+q_2} b_1 \circ b_2$ lies in B^k_Ψ if $x^{q_1} b_1 \in B^k_\Psi$. Let $a_{1n} = \sum\limits_k \lambda_k^{(n)} k^n \Psi^k$ and $a_{2n} = \sum\limits_k \mu_k^{(n)} k^n \Psi^k$. We consider

$$\begin{aligned} x^{q_1+q_2} b_{1n} \cdot b_{2n} &= x_1^{q_1+q_2} x^{-q_1} a_{1n} x^{-q_2} a_{2n} \\ &= \left(\sum_k \lambda_k^{(n)} k^{(n-q_2)} \Psi^k \right) \left(\sum_k \mu_k^{(n)} k^n \Psi^k \right) \end{aligned}$$

$\{$using $k^{q_2} x^{q_2} \Psi^k = \Psi^k x^{q_2}\}$. We shall assume that n is very large, $n \to \infty$, q_1 and q_2 are fixed. We set $m = n - q_2$. Then

$$\begin{aligned} &\left(\sum_k \lambda_k^{(n)} k^m \Psi^k \right) \left(\sum_k \mu_k^{(n)} k^{q_2} k^m \Psi^k \right) \\ &= \left(\sum_k \bar{\lambda}_k^{(m)} k^m \Psi^k \right) \left(\sum_k \bar{\mu}_k^{(m)} k^m \Psi^k \right), \end{aligned}$$

where $\lambda_k^{(m)} = \lambda_k^{(n)}$ and $\bar{\mu}_k^{(m)} = k^{q^2} \mu_k^{(n)}$. Clearly, as $m \to \infty$ we have a composition of operations in B^k_Ψ which lies in B^k_Ψ.

The lemma is proved.

By virtue of the lemma, the rings B^k_Ψ and A^k_Ψ contain operations which coincide up to dimensions $f(n) \to \infty$ (as $n \to \infty$) with the operations $(k^n \Psi^k)$ in the sense that $a^{(j)}_{k,n,*} = k^{n+j}$ for $j \leq (n)$.

This remark allows us to use (up to any dimension) the ring B^k_Ψ as if it were the ring generated by $(p^n\Psi^p)$, with p prime, and by $x \in B^k_\Psi$ where $(p^n\Psi^p)x = px(p^n\Psi^p)$ and $\gamma_p = (p^n\Psi^p)$ are polynomial generators. Thus, a (topological) basis here is $x^k P(\gamma_2, \gamma_3, \dots)$, where P is a polynomial.

We consider the B^k_Ψ-module $k^*(P)$. We have

Lemma 8.3. *The torsion part of the group* $\operatorname{Ext}^{1,2i}_{B^k_\Psi}(k^*(P), k^*(P))$ *is a cyclic group, whose order is equal to $\{p^n(p^i - 1)\}_p$, where n is large, p is prime, and $\{\ \}_p$ means the greatest common divisor of the sequence of integers.*

Proof. We construct a B^k_Ψ-free resolution of the module $k^*(P)$. Let n be large. Then the module $k^*(P)$ is given by the relations $(\gamma_p - p^n)t = 0$. We choose generators $\kappa_p = (\gamma_p - p^n)$ and 1. Then the κ_p are polynomial elements,

$$\dots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} k^*(P),$$
$$C_0 = B^k_\Psi, \quad C_1 = \sum_p B^k_{\Psi,p},$$

while $du = t$ and $du_p = \kappa_p(u)$, where u, u_p are free generators of the modules $B^k_\Psi = C_0$, $B^k_{\Psi,p} \subset C_1$, respectively.

We consider the complex $\mathrm{Hom}^*_{B^k_\Psi}(C, k^*(P))$.

Let $h_i \in \mathrm{Hom}^{2i}_{B^k_\Psi}(C_0, k^*(P))$ be elements such that $h_i(u) = x^i(t)$ and $h_i^{(p)} \in \mathrm{Hom}^*(C_1, k^*(P))$ be such that $h_i^{(p)}(u_p) = x^i(t)$ and $0 = h_i^{(p)}(u_{p'})$ for $p' \neq p$.

Obviously, we have

$$(d^* h_i, u_p) = (h_i, \kappa_p u) = \kappa_p x^i(t)$$
$$= p^n(\Psi^p - 1)x^i(t) = p^n(p^i - 1)x^i(t).$$

Hence, $d^* h_i = \sum\limits_p p^n(p^i - 1)h_i^{(p)}$. Thus, $\frac{d^*(h_i)}{\{p^n(p^i-1)\}_p}$ is a 1-cocycle for the operator d^*. Since $\mathrm{Hom}^*(\ , k^*(P))$ is a free abelian group, the element $\frac{d^*(h_i)}{\{p^n(p^i-1)\}_p}$ is the unique element of finite order equal to d_i in the group $\mathrm{Ext}^{1,2i}_{B^k_\Psi}(k^*(P), k^*(P))$, $d_i = \{p^n(p^i - 1)\}_p$, $n \to \infty$.

The lemma is proved. □

Note that the computation of $\mathrm{Ext}^{**}_{B^k_\Psi}(k^*(P), k^*(P))$ presents no difficulties, since the module $k^*(P)$ has a B^k_Ψ-free resolution which coincides with the complex for the polynomial algebra $Z[\gamma_2, \dots, \gamma_p, \dots]$, as long as the operator x acts freely on $k^*(P)$, B^k_Ψ.

We have

Theorem 8.1. *The groups* $\mathrm{Ext}^{1,2i}_{A^k_\Psi}(k^*(P), k^*(P))$ *are cyclic groups of order* $d_i = \{p^n(p^i - 1)\}_p$, *where n is large.*

Proof. It is easy to see that the algebra $A^k_\Psi \otimes Q$ is precisely the algebra of operations in k-theory $k^* \otimes Q$. Hence, by virtue of § 2, we have:

$$\mathrm{Ext}^{s,*}_{A^k_\Psi \otimes Q}(k^*(P) \otimes Q, k^*(P) \otimes Q) = \mathrm{Ext}^{s,*}_{A^k_\Psi}(k(P), k(P) \otimes Q) = 0$$

for $s > 0$. Hence the groups $\mathrm{Ext}^{1,2i}_{A^k_\Psi}$ are all torsion. We consider the resolution

$$\left(\dots \xrightarrow{d} \sum_i A^k_{\Psi,i} \xrightarrow{d} A^k_\Psi \xrightarrow{\varepsilon} k^*(P)\right) = C,$$

where $d(u_i) = \delta_i(u)$, and u_i, u are free generators of C_1 and C_0.

We consider the nonacyclic complex

$$\left(\dots \xrightarrow{\bar{d}} \sum_i A^k_{\Psi,i} \xrightarrow{\bar{d}} A^k_\Psi \xrightarrow{\varepsilon} k^*(P)\right) = \bar{C},$$

where $\bar{d}(v_i) = x^i\delta_i(v)$ and v_i, v are free generators. We shall show that the complex $\bar{C}$ is such that in the group

$$H_1(\mathrm{Hom}^*_{A^k_\Psi}(\bar{C}, k^*(P)), \bar{d}^*)$$

the torsion part is exactly the same as in the group

$$H_1(\mathrm{Hom}^*_{A^k_\Psi}(C, k^*(P)), d^*) = \mathrm{Ext}^{1,*}_{A^k_\Psi}.$$

In fact, if $h_j \in \mathrm{Hom}_{A^k_\Psi}(\bar{C}, k^*(P))$, where $h_j(v) = x^j(t)$, then

$$\begin{aligned}(\bar{d}^* h_j, v_i) &= h_j(x^i\delta_i(t)) = x^i\delta_i h_j(t)\\ &= x^i\delta_i x^j(t) = x^i(\delta_i x^j(t)).\end{aligned}$$

Thus, if $d^* h_j = \sum \mu_i^{(j)} h_{j-i}^{(i)}$, where $h_{j-i}^{(i)}(u_i) = x^{j-i}(t)$, then

$$\bar{d}^* h_j = \sum \mu_i^{(j)} h_i^{(j)},$$

where $h_j^{(i)}(v_i) = x^j(t)$ and the numbers $\mu_i^{(j)}$ are the same. We note that the order of the group $\mathrm{Ext}^{1,2j}_{A^k_\Psi}$ is precisely the greatest common divisor of the numbers $\mu_i^{(j)}$ as i varies, and a generator is $d^*(h_j)/\{\mu_i^{(j)}\}_i$. Since the elements $(x^i\delta_i)$ give a system of relations in $k^*(P)$ over the ring $B^k_\Psi \subset A^k_\Psi$, the same integers $\mu_i^{(j)}$ give the torsion part of $\mathrm{Ext}^{1,2j}_{B^k_\Psi}$, since the complex $\bar{C}$ over A^k_Ψ is a segment of a B^k_Ψ-resolution of the module $k^*(P)$. By virtue of the lemma, we get the required result. The theorem is proved. □

We now pass to the module $k^*(MU)$.

We have

Theorem 8.2. *For any complex $X \in D$ there is a canonical isomorphism*

$$\mathrm{Hom}^*_{A^k_\Psi}(k^*(MU), k^*(X)) = U^*(X).$$

The proof of this assertion is essentially a straightforward consequence of the result of [22] concerning the fact that the Riemann–Roch theorem on the integrality of the number $\mathrm{ch}\,\xi T(X)[X]$ gives a complete set of congruence relations on Chern numbers in Ω_U. More precisely: if $[X] \in \Omega_U$ indivisible element, then there exists $\xi \in K(X)$ such that $\mathrm{ch}\,\xi T(X)[X] = 1$. By virtue of the properties of the Thom isomorphism in K-theory, this assertion is equivalent to the following: for any indivisible element $\alpha \in \pi_*(MU)$, there exists $\xi \in K^0(MU)$ such that $(\mathrm{ch}\,\xi, H\alpha) = 1$, where $H\colon \pi_* \to H_*$ is the Hurewicz homomorphism. Let $\beta \in \mathrm{Hom}^*_{A^k_\Psi}(k^*(MU), k^*(P))$; then the number $(\mathrm{ch}\,\xi, \beta)$ is also an integer by virtue of Bott periodicity. Both groups $\mathrm{Hom}^*_{A^k_\Psi}(k^*(MU), k^*(P))$ and $\pi_*(MU)$ have no torsion. {Note that $\mathrm{Hom}_{A^k_\Psi}(k^*(MU), k^*(P)) \subset k^*(P)$, for $k^*(MU)$ is cyclic on u_n.} Hence $\pi_*(MU) \subset \mathrm{Hom}^*_{A^k_\Psi}$. By virtue of what was said about the indivisibility of the numbers $(\mathrm{ch}\,\xi, H\alpha)\alpha \in \pi_*(MU)$, the group $\pi_*(MU)$ is indivisible in $\mathrm{Hom}^*_{A^k_\Psi}$. Since the ranks of these groups coincide, the groups coincide. Thus the assertion is proved for the point spectrum.

Let $X \in D$, $X_1, X_2 \in D$, with X_1 a skeleton of X, $X_2 = X/X_1$; we have exact sequences:

$$0 \to U^*(X_2) \to U^*(X) \xrightarrow{i^*} U^*(X_1) \to 0,$$
$$0 \to k^*(X_2) \to k^*(X) \to k^*(X_1) \to 0.$$

We assume by induction that the theorem has been proved for X_1 and X_2 (we do induction on the rank of the group $H_*(X, Z)$). Then we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & U^*(X_2) \arrow[r] \arrow[d, "\approx"] & U^*(X) \arrow[r] \arrow[d, "\nu"', "\approx"] & U^*(X_1) \arrow[r] \arrow[d, "\gamma"', "\approx"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^*_{A^k_\Psi} \arrow[r] & \mathrm{Hom}^*_{A^k_\Psi} \arrow[r] & \mathrm{Hom}^*_{A^k_\Psi} \arrow[r, "\delta"] & \mathrm{Ext}^{1,*}_{A^k_\Psi}(k^*(MU), k^*(X_2)).
\end{tikzcd}$$

However, by virtue of the commutativity of the diagram we have that the homomorphism

$$\mathrm{Hom}^*_{A^k_\Psi}(k^*(MU), k^*(X)) \to \mathrm{Hom}^*_{A^k_\Psi}(k^*(MU), k^*(X_2))$$

is an epimorphism, since i^* is an epimorphism and γ is an isomorphism. Hence the homomorphism δ is trivial, and hence by the 5-lemma the homomorphism ν is an isomorphism. The theorem is proved.

Remark 8.2. In what follows it will become clear that the groups $\mathrm{Ext}^{i,*}_{A^k_\Psi}(k^*(MU), k^*(P))$ are nontrivial even for $i = 1$, and the question of their computation is extraordinarily important (see § 9, 11).

By analogy with the rings A^k_Ψ and B^k_Ψ it is possible to construct analogous rings A^{kO}_Ψ and B^{kO}_Ψ. Let kO^* be the theory defined by the spectrum kO such that $\Omega^{8n} kO_n = BO \times Z$ (see § 3). The cohomology ring of a point $kO^*(P) = \Lambda_O$ is described as follows: generators $1 \in \Lambda_O^0$, $h \in \Lambda_O^{-1}$, $v \in \Lambda_O^{-4}$, $w \in \Lambda_O^{-8}$; relations $2h = 0$, $h^3 = 0$, $hv = 0$, $v^2 = 4w$.

We have the "complexification" operator

$$c\colon kO^* \to k^*$$

such that $c(h) = 0$, $c(v) = 2x^2$, $c(w) = x^4$, where x is the Bott periodicity operator.

In the theory kO^* it is possible by analogy with the theory k^* to introduce operations $(k^n \Psi^k)$ and their combinations $a = (a_n)$, $a_n = \sum \lambda_k^{(n)} (k^n \Psi^k)$, where $a_{n*}^{(j)}$ does not depend on n. The ring of such operators is identical to the analogous ring for k^*-theory which lies in B^k_Ψ. The ring B^{kO}_Ψ is composed, in a fashion identical to that for the ring B^k_Ψ, from such operators $a = (a_n)$ constructed from Ψ^k and from the multiplication operators on $\Lambda_O = kO^*(P)$, keeping in mind the following commutativity relations: $\Psi^k h = kh\Psi^k$; $\Psi^k v = k^2 v \Psi^k$; $\Psi^k w = k^4 w \Psi^k$. We denote the resulting ring by B^{kO}_Ψ. Similarly, it is possible to construct a ring A^{kO}_Ψ also, but we shall not consider this ring in what follows.

We consider the category $B \subset D \subset S$.

1) The spectral sequence $(E_n, d_r) \downarrow kO^*$ is trivial in B; in B there is a subcategory B' such that:

2) the operation of the ring B^{kO}_Ψ is well-defined in B'. As is easy to see, the spheres S^n (their spectra in S) lie in B' by definition, since $kO^*(S^n) \approx kO^*(P)$.

If $f\colon S^{n+k} \to S^n$ is a mapping, then a necessary and sufficient condition for the complex $D^{n+k+1} \cup_f S^n$ to belong to B' is that $f^* = 0$, $f^*\colon kO^*(S^n) \to kO^*(S^{n+k})$.

In the category $B \supset B'$ the operation of the ring $\tilde{B}_\Psi^{kO}$ is well-defined, the latter being *a priori* an extension $B_\Psi^{kO} \to B_\Psi^k$, since in view of the presence of torsion in $A_O = kO^*(P)$ the operation is not defined by its own representation on $kO^*(P)$, in contrast to k^*-theory in the category D.

There is defined a homomorphism (epimorphism):

$$\operatorname{Ext}^{1,*}_{B_\Psi^{kO}}(kO^*(P), kO^*(P)) \to \operatorname{Ext}^{1,*}_{\tilde{B}_\Psi^{kO}}(kO^*(P), kO^*(P))$$

and a Hopf invariant

$$q_1\colon \operatorname{Ker} q_0 \to \operatorname{Ext}^{1,*}_{\tilde{S}_\Psi^{kO}}(kO^*(P), kO^*(P)).$$

It is easy to see that the complexification $c\colon kO^* \to k^*$ is an algebraic functor (see Definition 9.1) from the category of $\tilde{B}_\Psi^{kO}$-modules to the category of B_Ψ^{kO}-modules.

It is also easy to show that

$$\operatorname{Ext}^{1,4k}_{B_\Psi^{kO}}(kO^*(P), kO^*(P)) = Z_{d_k}, \quad d_k = \{p^n(p^{2k} - 1)\}_p$$

and

$$\operatorname{Ext}^{0\,s}_{B_\Psi^{kO}}(\Lambda_O, \Lambda_O) = Z_2 \quad \text{for} \quad s = 8k+1, 8k+2.$$

We have a natural ring homomorphism $\tau\colon B_\Psi^{kO} \to B_\Psi^k$ generated by the homomorphism $c\colon kO^*(P) \to k^*(P)$, and consequently a homomorphism

$$\begin{gathered} \tilde{c}\colon \operatorname{Ext}^{1,4k}_{B_\Psi^{kO}}(\Lambda_O, \Lambda_O) \to \operatorname{Ext}^{1,4k}_{B_\Psi^k}(\Lambda, \Lambda), \\ \Lambda = k^*(P), \quad \Lambda_O = kO^*(P), \end{gathered}$$

whose image has, as is easy to see, index 1 for $k = 2l$ and index 2 for $k = 2l+1$, in consequence of the fact that the image of the homomorphism $c\colon kO^t \to k^t$ has index 1 for $t = 8l$ and 2 for $t = 8l+4$. Later, in § 9, this homomorphism will be considered from another point of view.

There is defined an element $h \in \operatorname{Ext}^{0,1}_{B_\Psi^{kO}}(\Lambda_O, \Lambda_O)$ such that $2h = 0$, $h^3 = 0$, while multiplication by h

$$\operatorname{Ext}^{0,8k+1}_{B_\Psi^{kO}} \xrightarrow{h} \operatorname{Ext}^{0,8k+2}_{B_\Psi^{kO}} \text{ and } \operatorname{Ext}^{1,s}_{B_\Psi^{kO}} \to \operatorname{Ext}^{1,s+1}_{B_\Psi^{kO}}$$

is a monomorphism for $s = 8k$, $8k+1$.

The images of the homomorphisms

$$q_0\colon \pi_*(S^n) \to \operatorname{Ext}^{0,*}_{B_\Psi^{kO}}(\Lambda_O, \Lambda_O)$$

and

$$q_1\colon \pi_*(S^n) \to \operatorname{Ext}^{1,*}_{B_\Psi^{kO}}(\Lambda_O, \Lambda_O)$$

are easy to study: namely, q_0 is an epimorphism (see [9]), and the image $\operatorname{Im} q_1$ is realized by the image of $q_1 \circ J$, where $J\colon \pi_*(SO) \to \pi_*(S^n)$, and is nontrivial in dimensions $(1, 4k)$, $(1, 8k+1)$, $(1, 8k+2)$.

§ 9. Relations between different cohomology theories. Generalized Hopf invariant. U-cobordism, k-theory, Z_p-cohomology

Let $X \in \vec{S}$ be a cohomology theory. Suppose given a subcategory $B \in \vec{S}$. We define the notion of the "Steenrod ring" A_B^X of the theory X^* in the subcategory B: the ring A_B^X is the set of transformations $\theta_K\colon X^*(K) \to X^*(K)$ which commute with the morphisms of the category B (according to Serre). The ring A_B^X contains the factor-ring $A^X/J(B)$, where $J(B)$ consists of all operations which vanish on the category B.

We now define "the generalized Hopf invariant:" let

$$g\colon K_1 \to K_2$$

be a morphism in B such that the object $CK_1 \cup_g K_2$ ($= 0 +_{K_1} K_2$ in the notation of § 1, i.e., the sum with respect to the inclusions $K_1 \xrightarrow{g}$ and $K_1 \xrightarrow{g} K_2$) also lies in B.

We have an exact sequence

$$\begin{tikzcd} & X^*(CK_1 \cup_g K_2) \arrow[dl] & \\ X^*(K_2) \arrow[rr, "g^*"] & & X^*(EK_1) \arrow[ul] \end{tikzcd}$$

If the homomorphism $g^* = q_0(g)\colon X^*(K_2) \to (EK_1)$ is trivial, then we have

$$0 \to X^*(K_1) \to X^*(CK_1 \cup_g K_2) \to X^*(K_2) \to 0,$$

where $X^*(K_i)$, $X^*(CK_1 \cup_g K_2)$ are modules, and our short exact sequence determines a unique element

$$q_1(g) \in \operatorname{Ext}^{1,*}_{A_B^X}(X^*(K_1), X^*(K_2)).$$

We thus obtain a mapping

$$q_1\colon \operatorname{Ker} q_0^{(B)} \to \operatorname{Ext}^{1,*}_{A_B^X}(X^*(K_1), X^*(K_2)),$$

where $q_0\colon \operatorname{Hom}^*(K_1, K_2) \to \operatorname{Hom}^*_{A_B^X}(X^*(K_2), X^*(K_1))$, $K_1, K_2 \in B$ and $g \in \operatorname{Ker} q_0^{(B)}$, provided $CK_1 \cup_g K_2 \in B$. This map is "generalized Hopf invariant."

General problem: which elements of $\operatorname{Ext}^{1,*}_{A_B^X}(X^*(K_2), X^*(K_1))$ ate realized geometrically as images $q_1(\operatorname{Ker} q_0^{(B)})$?

If $\bar{A}_B^X \in A_B^X$ is an arbitrary subring, then there is defined the usual homomorphism:

$$i\colon \operatorname{Ext}^{**}_{A_B^X}(X^*(K_2), X^*(K_1)) \to \operatorname{Ext}^{**}_{\bar{A}_B^X}(X^*(K_2), X^*(K_1))$$

and we set $\bar{q}_1 = iq_1$, where $\bar{q}_1$ is the "Hopf invariant" of the subring $\bar{A}_B^X \subset A_B^X$.

Examples.

1. If B consists of a single object K, then $A_B^X = \operatorname{End} X^*(K)$ and there is no Hopf invariant.

2. If B consists of objects $K_1, K_2, L = CK_1 \cup_g K_2$ and morphisms $g\colon K_1 \to K_2$, $\beta\colon L \to EK_1$, $\alpha\colon K_2 \to L$, where $g^*\colon X^*(K_2) \to X^*(K_1)$ is the trivial homomorphism, then the ring A_B^X consists of all endomorphisms of $X^*(L)$ which preserve the image $\beta^* X^*(K_1) \subset X^*(L)$. In this case, the Hopf invariant $q_1^{(B)}(g) \in \operatorname{Ext}^1_{A_B^{X^*}}(X^*(K_1, X^*(K_2))$ is defined, and is equal to zero if and only if $X^*(L) =$

$X^*(K_1) + X^*(K_2)$ (as groups). Of course, examples 1 and 2 are uninteresting. We go on now to the examples which interest us.

3. Let $B = D$ (complexes with no torsion) and $X^* = H^*(\ , Z_p)$. In this case $A_B^X = A/(\beta A + A\beta)$, where β is the Bokšteĭn homomorphism and A is the Steenrod algebra (over Z_p).

There is a canonical isomorphism

$$\operatorname{Ext}_A^{1,t}(Z_p, Z_p) = \operatorname{Ext}_{A/\beta A + A\beta}^{1,t}(Z_p, Z_p)$$

for $t > 1$, where $Z_p = H^*(P, Z_p)$, P is a point, and the Hopf invariant $q_1(D)$ coincides with the Hopf invariant q_1 for $K_1 = K_2 = P$.

4. Let $B = D$ and $X^* = k^*$. In this case $A_B^X \supset A_\Psi^k$, and the latter ring contains the ring $A^X/J(B)$ but apparently does not coincide with it. The Hopf invariant in this theory will be discussed later; the $\operatorname{Ext}_{A^k}^{1,*}(k^*(P), k^*(P))$ were computed in § 8. In § 8 we considered the subring $B_\Psi^k \subset A^k$ and

$$\operatorname{Ext}_{A_\Psi^k}^{1,*}(k^*(P), k^*(P)) = \operatorname{tor} \operatorname{Ext}_{B_\Psi^k}^{1,*}(k^*(P), k^*(P)).$$

5. For the theory $X^* = U^*$ we shall also consider the category $B = D$ and the Hopf invariant for the whole ring A^U.

The groups $\operatorname{Ext}_{A^U}^{1,*}$ will be computed later (for $K_2 = MSU$; see § 6).

6. In § 2 it was indicated that for complexes $K = E^2L$ the homomorphism $J\colon K^0(X) \to J(X)$ can be considered as a homorphism $J\colon K^0(X) \to P^*(X)$, where P is the point spectrum or cohomotopy theory. A lower bound for the groups $\tilde{J}(X)$ can be computed in any cohomology theory Y^*, if we consider the composition

$$q_1^{(Y)} \cdot J\colon K^0(X) \to P^*(X) \to \operatorname{Ext}_{A^Y}^{1,*}(Y^*(P), Y^*(X)),$$

where $P^*(X) = \operatorname{Hom}^*(X, P)$, defined on elements such that $q_0^{(Y)} \cdot J = 0$.

If $K = EL$, then in this case the computation can also be carried out by means of $\operatorname{Ext}_{A^Y}^{**}(Y^*(P), Y^*(X))$, but here the multiplicative structure in $\operatorname{Ext}_{A^Y}^{**}$ y enters by virtue of Lemma 2.1 of § 2.

We now consider two cohomology theories $X^*, Y^* \in \vec{S}$, a subcategory $B \subset S$ and a transformation $\alpha\colon X^* \to Y^*$ of the cohomology functors in the subcategory B. Let subrings $\bar{A}_B^X \subset A_B^X$, $\bar{A}_B^Y \subset A_B^Y$ be chosen.

Definition 9.1. We call the transformation $\alpha\colon X^* \to Y^*$ algebraic with respect to the subrings $\bar{A}_B^X$, $\bar{A}_B^Y$, if it induces a functor $\bar{\alpha}$ from the category of $\bar{A}_B^X$-modules to the category of $\bar{A}_B^Y$-modules. When $\bar{A}_B^X = A_B^X$, and $\bar{A}_B^Y = A_B^Y$ we call the transformation α algebraic.

Examples.

1. Let X^*, Y^* be arbitrary cohomology theories. An arbitrary element $\alpha \in Y^*(X)$ determines a transformation of theories

$$\alpha\colon X^* \to Y^*.$$

2. If the theory X^* is such that $X^i(P) = 0$ for $i > 0$ and $X^0(P) = \pi$, then there arises an augmentation functor

$$\nu\colon X^* \to H^*(Y, \pi)$$

and hence for any group G a functor

$$\nu_G\colon X^* \to H^*(\ , \pi \otimes G).$$

For example, for $G = Z_p$ we have $\nu_p\colon X^* \to H^*(\ ,\pi \otimes Z_p)$. In the cases of interest to us, $\pi = Z$ and $\pi \otimes Z_p = Z_p$.

3. **The Riemann–Roch functor.** Let $X^* = U^*$ and $Y^* = k^*$; we consider the Atiyah–Hirzebruch-Grothendieck element $\lambda_1^{(n)} \in K^0(MU_n)$. It defines a map

$$\lambda_{-1}\colon U^* \to K^*$$

and $\lambda\colon U^* \to k^*$, where $\lambda = (\lambda^{(n)}), \lambda^{(n)} \in k^{2n}(MU_n)$, is the element (uniquely defined) such that $x^n\lambda^{(n)} = 1 \in K^0(MU_n)$, where x is the Bott operator.

For the theory $X^* = U^*$, the augmentation functors ν, ν_p and the Riemann–Roch functor λ preserve the ring structure of the theory.

Later it will be shown that these functors are algebraic in the category D.

Now let $\alpha\colon X^* \to Y^*$ be an algebraic transformation of theories in the category $B \subset S$ with respect to the subrings $\bar{A}_B^X, \bar{A}_B^Y$. What is the connection between the "Hopf invariants" $q_1^{(B)}$ in the theories X^* and Y^*?

Since $\alpha\colon X^* \to Y^*$ leads to a functor in the category of modules, the trivial morphism $g_X^*\colon X^*(K_2) \to X^*(K_1)$ corresponds to the trivial morphism $g_Y^*\colon Y^*(K_2) \to Y^*(K_1)$ for $K_1, K_2, g \in B$. Hence we have the inclusion $\operatorname{Ker} q_{0X}^{(B)} \subset \operatorname{Ker} q_{0Y}^{(B)}$, and the domain of definition of the Hopf invariant $\bar{q}_{1X}^{(B)}$ is contained in the domain of definition of $\bar{q}_{1Y}^{(B)}$.

Now let $\bar{\alpha}$ be a right exact functor in the category of modules. We consider a resolution C_x of the module $M = X^*(K_2)$ and the following (commutative) diagram:

$$\begin{tikzcd}
C_Y \arrow[d] & \tilde{C}_Y \arrow[l, "\beta_2"'] \arrow[r, "\beta_1"] \arrow[d] & \bar{\alpha}C_X \arrow[d] \\
Y^*(K_2) \arrow[r, phantom, "="] & Y^*(K_2) \arrow[r, phantom, "="] & \bar{\alpha}M,
\end{tikzcd}$$

where C_Y is an acyclic $\bar{A}_B^Y$-free resolution of the module $\bar{\alpha}M = Y^*(K_2), \tilde{C}_Y$ is a free complex such that $H^0(\tilde{C}_Y) = \bar{\alpha}M$. Let $N = X^*(K_1)$, $\bar{\alpha}N = Y^*(K_1)$,

By definition we have: $H^*(\operatorname{Hom}^*_{\bar{A}_B^Y}(\bar{\alpha}C_X, \bar{\alpha}N)) = R^*G_N(M)$, where $R^* = \sum\limits_q R^q$ and $G_N = \operatorname{Hom}^*_{\bar{A}_B^Y}(\ ,\bar{\alpha}N) \circ \bar{\alpha}$ is the composite functor, R^qG is the q-th right derived functor. There is defined a natural homomorphism

$$r_q\colon \operatorname{Ext}^{q,*}_{\bar{A}_B^X}(M,N) \to R^qG_N(M), \quad r = \sum_q r_q,$$

and homomorphisms

$$\begin{tikzcd}
\beta_1^*\colon R^qG_N(M) \arrow[r] & H^{q,*}(\operatorname{Hom}^*_{\bar{A}_B^Y}(\tilde{C}_Y, \bar{\alpha}N)) \\
& \operatorname{Ext}^{q,*}_{\bar{A}_B^Y}(\bar{\alpha}M, \bar{\alpha}N), \arrow[u, "\beta_2^*"']
\end{tikzcd}$$

where $\operatorname{Ker}\beta_2^* = 0$.

We have the composite map

$$\tilde{\alpha} = (\beta_2^*)^{-1}\beta_1^* r_1\colon E_1(\bar{\alpha}) \to \operatorname{Ext}^1_{\bar{A}_B^Y}(Y^*(K_2), Y^*(K_1)),$$

where

$$E_1(\bar{\alpha}) \subset \mathrm{Ext}^1_{\bar{A}^X_B}(X^*(K_2), X^*(K_1)),$$
$$E_1(\bar{\alpha}) = r_1^{-1} \circ \beta_1^{*^{-1}} \beta_2^*(\mathrm{Ext}^1_{\bar{A}^Y_B}(Y^*(K_2), Y^*(K_1)).$$

In the following cases the group $E_1(\bar{\alpha})$ coincides with the whole group Ext:

a) $\bar{\alpha}$ is an exact functor; here $H^i(\bar{\alpha} C_X) = 0$, $i > 0$, and one can assume that $\tilde{C}_Y = C_Y$, $\beta_2 = 1$.

b) If in addition $\bar{\alpha}$ is such that $\mathrm{Ext}^i_{\bar{A}^Y_B}(\ , \varepsilon N) \circ \varepsilon = 0$ for $i > 0$, then $\bar{\alpha} C_X = C_Y$ and an isomorphism is generated:

$$\mathrm{Ext}^{**}_{\bar{A}^Y_B}(\bar{\alpha} M, \bar{\alpha} N) = R^* G_N(M).$$

In case (a) ($\bar{\alpha}$ is an exact functor) there arises a spectral sequence (E_r, d_r), where $E_2^{p,q} = \sum_{p,q} R^p G_{q,N}(M)$, which converges to $\mathrm{Ext}^{**}(\bar{\alpha} M, \bar{\alpha} N)$, and $G_{q,N}(M) = \mathrm{Ext}^{q,*}(\ , \varepsilon N) \circ \varepsilon$. From this spectral sequence it follows immediately that the homomorphism

$$\beta_1^*\colon R^1 G_N(M) \to \mathrm{Ext}^1(\bar{\alpha} M, \bar{\alpha} N)$$

is a monomorphism.

The basic examples which we shall consider are the subcategory D of torsion-free complexes, the theories $U^*, k^*, H^*(\ , Z_p)$, the Riemann–Roch functor $\lambda\colon U^* \to k^*$ and the augmentations $\nu_p\colon U^* \to H^*(\ , Z_p)$. We have

Lemma 9.1. a) *The functors $\lambda\colon U^* \to k^*$ and $\nu_p\colon U^* \to H^*(\ , Z_p)$ are algebraic in the subcategory D;*

b) *The functors λ and ν_p are exact in this category.*

c) *The functor λ is such that $R^q G_N(M) = \mathrm{Ext}^q_{A^U}(M, N)$, where $M = U^*(K_2)$, $N = U^*(K_1)$, $M, N \in D$, $C_N = \mathrm{Hom}_{A^k}(\ , \lambda N) \circ \lambda$, $\lambda N = k^*(K_1)$.*

d) *The functor ν_p is such that $R^q G_N(M) = \mathrm{Ext}^q_{A/\beta A + A\beta}(\nu_p M, \nu_p N)$.*

Proof. The category of A^U-modules corresponding to the category D is the category of Λ-free modules, where $\Lambda = U^*(P) \approx \Omega_U$. On the cohomology of a point Λ the functor λ is such that $\Lambda \xrightarrow{\lambda} Z[x]$ and $\lambda(y) = T(y) x^i$, where $y \in \Omega_U^{2i} = U^{-2i}(P)$ and T is the Todd genus.

From the group point of view we have $\lambda M = M \otimes_\Lambda Z[x]$, where M is Λ-free. There follows the exactness of the functor λ and $R^q \lambda = 0$, $q > 0$. For ν_p we have $\nu_p U^*(P) = Z_p$, and in the category D, $\nu_p M = M \otimes_\Lambda Z_p$; since in the category D all groups $U^*(K)$ and $H^*(K)$ are free abelian, the functor ν_p is exact in this category. This proves part (d). Part (c) follows immediately from the theorem in § 7. Part (b) follows from the well-known fact that $H^*(MU, Z_p)$ is a free $(A/\beta A + A\beta)$-module. We shall now prove the fundamental part (a).

Consider first the functor λ. We recall that in § 5 we constructed operations $\Psi_U^k \in A^U \otimes Q$. Let

$$\Psi^k(\lambda \tilde{x}) = \lambda \Psi_U^k(\tilde{x}), \quad \tilde{x} \in U^*(K),$$

where $K \in D$ is a complex with no torsion. Since λ is an epimorphism and $\lambda(y) = T(y) x^i$ where x is the Bott operator, the desired formula follows easily from the construction of the Adams operations Ψ^k in K-theory and of the operations Ψ_U^k in § 5. The operations $(k^n \Psi^k)$ have the form $k^n \lambda \Psi_U^k$ and are “integral” for large

n. Thus, the action of the operators $(k^n\Psi^k)$ and multiplication by x in k^*-theory are calculated by A^U and λ. This proves part (a) of the lemma for the functor λ.

Now let $\alpha = \nu_p \colon U^* \to H^*(\ , Z_p)$. In § 5 we constructed a projector $\Phi \in A^U \otimes_Z Q_p$ of the theory U^* onto a smaller theory having the cohomology of a point $\Lambda_p = Q_p[x_1, \dots, x_i, \dots]$, $\dim x_i = -2(p^i - 1)$.

We set

$$P^k(\nu_p \tilde{x}) = \nu_p \Phi S_\omega \Phi(\tilde{x}),$$

where $\omega = (p-1, \dots, p-1)$ (k times) and the P^k are the Steenrod powers. The correctness of this formula follows from the fact that all homomorphisms $(\Phi S_\omega \Phi)^*(y) \equiv 0 \mod p$ if $\dim \omega = \dim y$, i.e., $(\Phi S_\omega \Phi)^*(y) \in \Omega_U^0 = Z$. The lemma is proved. □

Corollary 9.1. *For any $K_1, K_2 \in D$ the homomorphism*

$$\tilde{\alpha} = \tilde{\lambda} \colon \operatorname{Ext}^{1,*}_{A^U}(U^*(K_2), U^*(K_1)) \to \operatorname{Ext}^{1,*}_{A^k_\Psi}(k^*(K_2), k^*(K_1))$$

is a monomorphism.

Proof. As was established in Theorem 8.2, the homomorphism r_1 is an isomorphism; the homomorphism β_1^* is a monomorphism, as was shown above, while $\beta_2^* = 1$, since $R^q\lambda = 0$, $q > 0$. Hence, $\beta_1^* r_1 = \tilde{\lambda}$ is a monomorphism. □

Corollary 9.2. *For any complex $K = E^2 L$ the lower bound of the J-functor*

$$q^{(D)}_{1k} \cdot J(K^0(X)) \in \operatorname{Ext}^{1,*}_{A^k_\Psi}(k^*(P), k^*(X))$$

coincides with the bound

$$q^{(D)}_{1U} \cdot J(K^0(X)) \in \operatorname{Ext}^{1,*}_{A^U}(U^*(P), U^*(X))$$

Corollary 9.2 follows from Corollary 9.1.

Corollary 9.3. *The groups* $\operatorname{Ext}^{1,2i}_{A^U}(U^*(P), U^*(P))$ *are cyclic groups – subgroups of cyclic groups of order equal to the greatest common divisor of the integers $\{k^n(k^i - 1)\}_k$ for all k, for large n.*

Proof. Since the groups $\operatorname{Ext}^{1,2i}_{A^k_\Psi}(k^*(P), k^*(P))$ by virtue of the theorem are cyclic of the asserted orders, Corollary 9.3 follows from Corollary 9.1. We shall indicate a simple fact about the connection between the Hopf invariants in different cohomology theories X^*, Y^* in the presence of an algebraic transformation $\alpha \colon X^* \to Y^*$ with respect to the rings $\bar{A}^X_B$, $\tilde{A}^Y_B$ in the subcategory $B \subset S$. □

Lemma 9.2. *We have the equality*

$$\bar{q}^{(B)}_{1Y} = \tilde{\alpha} \cdot \bar{q}^{(B)}_{1X}$$

on $\operatorname{Ker} q^{(B)}_{0X}$*, the group* $q^{(B)}_{1X}(\operatorname{Ker} q^{(B)}_{0X})$ *being contained in $E_1(\alpha)$, the domain of definition of the homomorphism $\tilde{\alpha} = (\beta_1^{*-1} \cdot \beta_1^* \cdot r_1)$.*

The proof of this lemma follows immediately from the fact that by construction of the generalized Hopf invariants $\bar{q}^{(B)}_{1X}$ and $\bar{q}^{(B)}_{1Y}$ we can compute both quantities $\bar{q}^{(B)}_{1X}(a)$ and $\bar{q}^{(B)}_{1Y}(a)$ for any $a \in \operatorname{Ker} \bar{q}^{(B)}_{0X} \subset \operatorname{Ker} \bar{q}^{(B)}_{0X}$.

As is easy to see, the equality

$$\beta_2^* q^{(B)}_{1Y}(a) = \beta_1^* \cdot r_1 \cdot \bar{q}^{(B)}_{1X}(a)$$

is true. This equation is equivalent to everything asserted by the lemma. The lemma is proved.

Corollary 9.4. a) *If the element* $\gamma \in \mathrm{Ext}^{1,*}_{\bar{A}^X_B}$ *does not belong to* $E_1(\alpha)$ *for any algebraic* $\alpha\colon X^* \to Y^*$*, then the element* γ *is not realized as the Hopf invariant of any element of* $\mathrm{Hom}^*(K_1, K_2)$.

b) *If* $\gamma \in \mathrm{Ext}^{1,*}_{\bar{A}^Y_B}$ *does not belong to the image of the homomorphism* $\tilde{\alpha}(\mathrm{Ext}^1_{\bar{A}^X_B})$ *and* $\mathrm{Ker}\, q^{(B)}_{0X} = \mathrm{Ker}\, q^{(B)}_{0Y}$*, then the element* γ *is not realized as the Hopf invariant of any element of* $\mathrm{Hom}^*(K_1, K_2)$.

§ 10. Computation of $\mathrm{Ext}^1_{A^U}(U^*(P), U^*(P))$. Computation of Hopf invariants in certain theories

In the preceding section the monomorphicity of the mapping

$$\mathrm{Ext}^1_{A^U}(U^*(P), U^*(P)) \to \mathrm{Ext}^1_{A^k_\Psi}(k^*(P), k^*(P))$$

was established.

We shall now bound the order of the groups $\mathrm{Ext}^{1,2i}_{A^U}$ from below. We consider the resolution

$$(\dots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} U^*(P) = \Lambda) = C,$$

where $C_0 = A^U$ (generated by u) and $C_1 = \sum_\omega A^U_\omega$ with generators $u_\omega, du_\omega = S_\omega(u)$, $\dim \omega > 0$. We consider the differential

$$d^*: \Omega^{2i}_U \to \sum_{\omega > 0} \Omega^{2i - 2\dim\omega}_U$$

where

$$d^*(x) = \sum_\omega \gamma^*_\omega, \quad x \in \Omega^{2i}_U = \mathrm{Hom}^{2i}_{A^U}(C_0, \Lambda)$$

and

$$\sigma^*_\omega(x) \in \mathrm{Hom}^{2i}_{A^U}(C_1, \Lambda) = \sum_\omega \Omega^{2i-2\dim\omega}_U,$$

where $\sigma^*_\omega(x)[u_{\omega'}] = 0$ if $\omega \neq \omega'$, and $\sigma^*_\omega(x)[u_\omega] = \sigma^*_\omega(x) \in \Lambda$. These facts follow from § 5.

Now let i be odd. We consider the element x_1^i, where $x_1 = [CP^1] \in \Omega^2_U$. Since $\sigma^*_1(x_1) = \pm 2$, all $\sigma^*_\omega(x_1^i) \equiv 0 \mod 2$, $\omega > 0$, from the properties of the homomorphisms σ^*_ω described in § 5. Hence the cokernel $\mathrm{Coker}\, d^*$ always contains an element of order 2. Since the homomorphism $\mathrm{Ext}^{1,4l+2}_{A^U} \to \mathrm{Ext}^{1,4l+2}_{A^k_\Psi}$ is monomorphic and $\{k^n(k^i - 1)\}_k = 2$, $i \equiv 1 \pmod 2$, we have

$$\mathrm{Ext}^{1,4l+2}_{A^U}(U^*(P), U^*(P)) = \mathrm{Ext}^{1,4l+2}_{A^k_\Psi}(k^*(P), k^*(P)) = Z_2$$

for $4l + 2 = 2i$, $i \equiv 1 \pmod 2$.

Thus, we have proved

Theorem 10.1. *The groups* $\mathrm{Ext}^{1,2i}_{A^U}(\Lambda, \Lambda)$ *are isomorphic to* Z_2 *for* $i = 2l + 1$.

We now study the case of even $i = 2l$. Let $y_i \in \Omega^{2i}_U$ be an indivisible element such that some multiple λy_i, $\lambda \neq 0$, represents an almost-parallelizable manifold M^{2i}, whose tangent bundle τ is a multiple of the basic element κ_i of the group $K^0(S^{2i})$, $\tau = \mu_i \bar{\kappa}_i$, μ_i integral, where $\bar{\kappa}_i = f^* \bar{\kappa}_i$, $f\colon M^{2i} \to S^{2i}$ a projection of degree ± 1. From the requirement of the integrality of the Todd genus and the fact that $(\operatorname{ch}\bar{\kappa}_1, M^{2i}) = 1$, it follows easily that all $\sigma^*_\omega y_i$ for all ω are divisible by the denominator of the number $(B_l/a_l \cdot 2l)$, where $a_{2S+1} = 1$ and $a_{2S} = 2$, B_l is the Bernoulli number entering into the Todd genus, $i = 2l$ (see [14]). Hence Coker d^* contains a group of order of equal of the denominator of the number $(B_l/a_l \cdot 2l)$. Since this number is only half of the number $\{k^n(k^i - 1)\}_k (a_l = 1)$, the image $\tilde{\lambda}\operatorname{Ext}^{1,2i}_{A^U} \subset \operatorname{Ext}^{1,2i}_{A^k_\Psi}$ coincides with $\operatorname{Ext}^{1,2i}_{A^k_\Psi}$ for $l = 2s$ and has index 2 in $\operatorname{Ext}^{1,2i}_{A^k_\Psi}$ for $l = 2s + 1$.

From this, for the case $a_l = 2$, $l = 2s$, follows the

Theorem 10.2. *The groups* $\operatorname{Ext}^{1,8k}_{A^U}(\Lambda, \Lambda)$ *are isomorphic to the groups* $\operatorname{Ext}^{1,8k}_{A^k_\Psi}(k^*(P), k^*(P))$.

In the case $a_l = 1$ there arises an uncertainty: do the groups $\operatorname{Ext}^{1,8k+4}_{A_U}$ coincide with the groups $\operatorname{Ext}^{1,8k+4}_{A^k_\Psi}$ or do they have index 2 in them?

Hence, we have the weaker

Theorem 10.3. *The groups* $\operatorname{Ext}^{1,8k+4}_{A_U}(\Lambda, \Lambda)$ *are cyclic groups whose order is equal to either the denominator of the number* $B_{2k+1}/(4k + 2)$ *or the denominator of the number* $B_{2k+1}/(8k + 4)$.

Remark 10.1. In what follows it will be established that this order is in fact equal to the denominator of $B_{2k+1}/(8k + 4)$ for $k \geq 1$ (however, for $k = 0$ it is easy to see that $\operatorname{Ext}^{1,4}_{A^U}(\Lambda, \Lambda) = Z_{12}$). The basis element u_k of the group $\operatorname{Ext}^{1,8k+4}_{A^U}$ is such that

$$d_3(u_k) = h^3 \operatorname{Ext}^{1,8k}_{A^U}(\Lambda, \Lambda), \quad h \in \operatorname{Ext}^{1,2}_{A^U}(\Lambda, \Lambda) = Z_2.$$

We now study the question of the relations among different cohomology theories and the question of the existence of elements in the homotopy groups of spheres with given Hopf invariant

$$\gamma \in \operatorname{Ext}^{1,*}_{\tilde{A}^X_D}(X^*(P), X^*(P))$$

for the cases $X^* = U^*$, k^*, kO^*, $H^*(\ , Z_p)$, with the help of the functors $\alpha = \lambda$, $\alpha = c$, $\alpha = \nu_p$ relating these theories.

1. The first question which we consider here is the complexification

$$c \colon kO^* \to k^*$$

with respect to the rings B^{kO}_Ψ and B^k_Ψ. The structure of the groups

$$\operatorname{Ext}^{s,*}_{B^{kO}_\Psi}(kO^*(P), kO^*(P)),$$

where $s = 0, 1$ is known to us, namely:

$$\text{a)}\quad \operatorname{Ext}^{0,t}_{B^{kO}_\Psi}(kO^*(P), kO^*(P)) = Z_2, \quad t = 8k + 1, 8k + 2,\ k \geq 0,$$

$$\operatorname{Ext}^{0\,t}_{B^{kO}_\Psi}(kO^*(P), kO^*(P)) = 0, \quad t \neq 8k + 1, 8k + 2;$$

$$\text{b)}\quad \operatorname{Ext}^{1,4}_{B^{kO}_\Psi}(kO^*(P), kO^*(P)) = Z_{\{k^n(k^q-1)\}_k} + \dots, \quad n \to \infty$$

$$\operatorname{Ext}^{1,8k+t}_{B^{kO}_\Psi}(kO^*(P), kO^*(P)) = Z_2 + \dots$$

for $t = 1, 2$;

c) the homomorphism $q_0 \colon \pi^S_*(P) \to \operatorname{Hom}^*_{B^{kO}_\Psi}(kO^*(P), kO(P))$ is an epimorphism (result of Brown–Peterson–Anderson [9]);

d) the homomorphism $q_1 \cdot J\colon \widetilde{kO^t}(P) \to \mathrm{Ext}^{1,t}_{B^{kO}_\Psi}(kO^*(P), kO^*(P))$ is an epimorphism. This last fact follows from the work of Adams [3] for the groups $\mathrm{Ext}^{1,4k}_{B^{kO}_\Psi}(\Lambda_O, \Lambda_O)$; since

$$\mathrm{Ext}^{1,8k+t}_{B^{kO}_\Psi} = h^t\, \mathrm{Ext}^{1,8k}_{B^{kO}_\Psi} \quad (t = 1, 2), \quad h \in \mathrm{Ext}^{0,1}_{B^{kO}_\Psi},$$

the required fact follows for the groups $\mathrm{Ext}^{1,8k+t}_{B^{kO}_\Psi}(\Lambda_O, \Lambda_O)$.

We now consider the complexification c, defining homomorphisms $\tilde{c}$, $\tilde{c}'$:

$$\begin{tikzcd}
\mathrm{Ext}^{1,*}_{\tilde{B}^{kO}_\Psi}(\Lambda_O, \Lambda_O) \arrow[rr, "\tilde{c}"] & & \mathrm{Ext}^{1,*}_{B^{k}_\Psi}(\Lambda, \Lambda) = \mathrm{Ext}^{1,*}_{A^k_\Psi}, \\
& \mathrm{Ext}^{1,*}_{B^{kO}_\Psi}(\Lambda_O, \Lambda_O) \arrow[ul] \arrow[ur, "\tilde{c}'"'] &
\end{tikzcd}$$

Since in the groups $k^t(P)$ the image of the homomorphism c has index 2 for $t = 8k+4$, index 1 for $t = 8k$ and is equal to zero for $t \neq 8k$, $8k+4$, we can draw from this the conclusion that the image group $\mathrm{Im}\,\tilde{c} \subset \mathrm{Ext}^{1,t}_{A^k_\Psi}(\Lambda, \Lambda)$ has index 2 for $t = 8k+4$, index 1 for $t = 8k$ and is equal to zero for $t \neq 8k$, $8k+4$, since $\mathrm{Im}\,\tilde{c} = \mathrm{Im}\,\tilde{c}'$.

Consider the groups $\pi_{n+4k-1}(S^n)$ and the Hopf invariants in kO^*- and k^*-theories. These invariants are always defined since $\mathrm{Ext}^{0,4k-1}_{\tilde{B}^{kO}_\Psi} = 0$. We have thus the

Conclusion. The image of the Hopf invariant

$$q_{1,k}\colon \pi_{n+4k-1}(S^n) \to \mathrm{Ext}^{1,4k}_{A^k_\Psi}(k^*(P), k^*(P)), \quad n \to \infty,$$

has index 2 for $k = 2l+1$ and index 1 for $k = 2l$. Moreover, the image $q_{1,k}(\pi_{n+4k-1}(S^n))$ coincides with the image $q_{1,k} \cdot J\pi_{4k-1}(SO)$.

2. We now consider the Riemann–Roch functor $\lambda\colon U^* \to k^*$ and the corresponding homomorphism $\tilde{\lambda}\colon \mathrm{Ext}^{1,t}_{A^U}(\Lambda, \Lambda) \to \mathrm{Ext}^{1,t}_{A^k_\Psi}$. Since $\tilde{\lambda}$ is a monomorphism, we get from item 1 on complexification the following conclusion:

The Hopf invariant $q_{1U}\colon \pi_{n+4k-1}(S^n) \to \mathrm{Ext}^{1,4k}_{A^U}$ is always defined, and its image $\mathrm{Im}\, q_{1U}$ coincides with $q_{1U}(J\pi_{2k-1}(SO))$; it coincides with $\mathrm{Ext}^{1,4k}_{A^U}(\Lambda, \Lambda)$ for $k = 1$, $k = 2l$ and has index 2 in the group $\mathrm{Ext}^{1,8l+4}_{A^U}$ for $l \geq 1$.

Later we shall study $\mathrm{Ext}^{1,8k+2}_{A^U}$ and $\mathrm{Ext}^{1,8k+6}_{A^U}$.

3. We now consider the functor $\nu_p\colon U^* \to H^*(\ , Z_p)$ and the corresponding Hopf invariant

$$\begin{tikzcd}
& \pi_{n+i-1}(S^n), \quad n \to \infty, \quad i > 1 \arrow[dl, "q_{1U}"'] \arrow[dr, "q_{1H}"] & \\
\mathrm{Ext}^{1,i}_{A^U}(\Lambda, \Lambda) \arrow[rr, "\tilde{\lambda}"] & & \mathrm{Ext}^{1,i}_{A/\beta A + A\beta}(Z_p, Z_p) = \mathrm{Ext}^{1,i}_{A}(Z_p, Z_p).
\end{tikzcd}$$

Since $\mathrm{Ext}^{1,i}_{A/\beta A + A\beta}(Z_p, Z_p) = \mathrm{Ext}_A(Z_p, Z_p)$ for $i > 1$ this becomes the usual Hopf invariant. Since the homomorphism

$$q_{1U}\colon J\pi_{8k-1}(SO) \to \mathrm{Ext}^{1,8k}_{A^U}(\Lambda, \Lambda)$$

is an epimorphism, the question of the existence of elements with ordinary Hopf invariant equal to 1 reduces to the calculation of this invariant on the group $J\pi_{4k-1}(SO)$. For example, let $p = 2$, and let $h_i \in \mathrm{Ext}_A^{1,2^i}(Z_2, Z_2)$ be basis elements. Since h_1, h_2, h_3 are cycles for all Adams differentials and represent elements in the groups $J\pi_*(SO)$, it follows that, in view of the fact that $\mathrm{Im}\, J$ is closed under composition, $h_i \cdot h_2 \in \mathrm{Ext}_A^{2,*}(Z_2, Z_2)$ must represent an element of $q_2 J\pi_*(SO)$ if h_i represents an element of $q_1 J\pi_*(SO)$. Moreover, since $q_2 J\pi_{4k-2}(SO) = 0$, we have $h_2 h_i = 0$ if $h_i \in q_1 J\pi_*(SO)$, since $h_2 \in q_1 J\pi_*(SO)$.

However, $h_i \cdot h_1 \neq 0$ for $i \geq 4$. We have thus the

Conclusion. For $i \geq 4$ the elements $h_i \in \mathrm{Ext}_A^{1,2^i}(Z_2, Z_2)$ do not belong to the image of the homomorphism

$$\nu_2\colon\ \mathrm{Ext}_{A^U}^{1,2^i}(\Lambda, \Lambda) \to \mathrm{Ext}_A^{1,2^i}(Z_2, Z_2).$$

The case $p > 2$ is considered analogously.

In fact, we have the purely algebraic

Theorem 10.4. *The image of the homomorphism*

$$\nu_p\colon\ \mathrm{Ext}_{A^U}^{1,2p^i(p-1)} \to \mathrm{Ext}_{A/A\beta+\beta A}^{1,2p^i(p-1)}(Z_p, Z_p)$$

is nontrivial only for $i = 0, 1, 2$ $(p = 2)$ and for $i = 0$ $(p > 2)$.

4. We now consider the homomorphism

$$\delta\colon\ \mathrm{Ext}_{A^U}^{**}(U^*(P), U^*(P)) \to \mathrm{Ext}_{A^U}^{**}(U^*(MSU), U^*(P)).$$

We assume that $K \in \mathrm{Ext}_{A^U}^{0,4}(U^*(MSU), \Lambda)$, $y \in \mathrm{Ext}_{A^U}^{0,8}(U^*(MSU), \Lambda)$, and $h \in \mathrm{Ext}_{A^U}^{1,2}(U^*(MSU), \Lambda)$ are elements such that $d_3(K) = h^3$, and $y \in \Omega_U^8$ is represented by an almost-parallelizable manifold. We have

Lemma 10.1. *All elements of the form $h^{n+1} \cdot K^\varepsilon \cdot y^m$, $n \geq 0$, $m \geq 0$, $\varepsilon = 0, 1$, belong to* $\mathrm{Im}\,\delta$.

Proof. Since $h \in \mathrm{Im}\,\delta$, it suffices to show that $K^\varepsilon \cdot y^m \cdot h$ belongs to $\mathrm{Im}\,\delta$. For this it suffices to establish that all homomorphisms $\sigma_\omega^*(x_1 \cdot K^\varepsilon \cdot y^m)$ are divisible by 2. It is easy to verify that $\sigma_\omega^*(x_1)$, $\sigma_\omega^*(K)$ and $\sigma_\omega^*(y^m)$ are divisible by 2. The general result follows from the Leibnitz formula

$$\sigma_\omega^*(ab) = \sum_{\omega=(\omega_1,\omega_2)} \sigma_{\omega_1}^*(a)\sigma_{\omega_2}^*(b).$$

The lemma is proved. □

As was shown in § 6, in the Adams spectral sequence for $U^*(MSU)$ we have:

a) $d_3(hKy^n) = h^4 y^m \neq 0$,

b) $d_i(hy^m) = 0$, $i \geq 2$.

Moreover, Brown–Peterson–Anderson showed in [9] that elements of the form $hy^m \in \Omega_{SU}^{8m+1}$ belong to the image of the homomorphism $\pi_*(S^n) \to \pi_*(MSU_n)$ by a direct construction of the elements.

We have thus the

Theorem 10.5. a) *The groups $\mathrm{Ext}_{A^U}^{1,8k+2}(\Lambda, \Lambda) = Z_2$ are cycles for all Adams differentials and belong to the image of the Hopf invariant*

$$q_{1U}\colon \pi_{n+8k+1}(S^n) \to \mathrm{Ext}_{A^U}^{1,8k+2}(\Lambda, \Lambda), \quad n \to \infty.$$

b) *The groups* $\mathrm{Ext}^{1,8k+6}_{A^U}(\Lambda,\Lambda)=Z_2$ *are not cycles for the differential* d_3.

Remark 10.2. Since $\mathrm{Ext}^{1,4i+2}_{A^U}=\mathrm{Ext}^{1,4i+2}_{A^k_\Psi}$, the analogous facts hold also for k-theory, although basis elements here are not related to the J-functor, in contrast to $\mathrm{Ext}^{1,4k}_{A^k_\Psi}$ (here, the elements go into hy^m under the homomorphism $\Omega_e\to\Omega_{SU}$).

We summarize the results of this section:

1) The groups $\mathrm{Ext}^{1,*}_{A^U}(\Lambda,\Lambda)$ were considered and also the associated homomorphisms

$$\begin{tikzcd}
 & \mathrm{Ext}^{1,*}_{A^U}(\Lambda,\Lambda) \arrow[dl, "\tilde\nu_p"'] \arrow[dr, "\tilde\lambda"] & & \mathrm{Ext}^{1,*}_{\tilde B^{kO}_\Psi}(kO^*(P),kO^*(P)) \arrow[dl, "\tilde c"'] \\
\mathrm{Ext}^{1,*}_{A}(Z_p,Z_p) & & \mathrm{Ext}^{1,*}_{A^k_\Psi}(k^*(P),k^*(P)) & \\
 & \pi_*(S^n) \arrow[ul, "q_{1H}"] \arrow[uu, "q_{1U}"] \arrow[ur, "q_{1k}"'] \arrow[uurr, bend right, "q_{1,kO}"'] & & \\
 & \pi_*(SO) \arrow[u, "J"] & &
\end{tikzcd}$$

where q_{1H} is the classical Hopf invariant, J is the Whitehead homomorphism, λ is the "Riemann–Roch" functor, c is complexification, and ν_p is the augmentation of U^*-theory into Z_p-cohomology theory.

2) The homomorphism $\mathrm{Ext}^1_{A^U}(\Lambda,\Lambda)\to\mathrm{Ext}^1_{A^U}(U^*(MSU),\Lambda)$ was studied.

3) It was established which elements of all these groups Ext^1 are realized as images of the Hopf invariant q_1. In particular, for the groups $\mathrm{Ext}^{1,2t}_{A^U}(\Lambda,\Lambda)$ this image $\mathrm{Im}\,q_{1U}$ is trivial for $t=4k+3$; q_{1U} is an epimorphism for $t=4k-1,4k$; for $t=4k+2$ $(k\geq 1)$ and $t=4k+3$ $(k\geq 0)$ the Adams differential

$$d_3\colon \mathrm{Ext}^{1,2t}_{A^U}(\Lambda,\Lambda)\to\mathrm{Ext}^{4,2t+2}_{A^U}(\Lambda,\Lambda)$$

is nontrivial; it can be shown that $d_3(E_3^{1,2t})=h^3E_3^{1,2t-4}$ for $t=4k+2$ $(k\geq 1)$ and $t=4k+3$ $(k\geq 0)$ (see § 11).

4) The nonexistence of elements with classical Hopf invariant 1 is a consequence of the fact that $\tilde\nu_2(\mathrm{Ext}^{1,2^i}_{A^U})=0$ for $i\geq 4$. Analogously for $p>2$ (see § 12).

5. For $t\neq 8k+4$, the fact of the following group isomorphism was established:

$$\mathrm{Ext}^{1,t}_{A^U}(\Lambda,\Lambda)\overset{\tilde\lambda}{=}\mathrm{Ext}^{1,t}_{A^k_\Psi}(k^*(P),k^*(P));$$

for $t=4$ this fact is false. For $t=8k+4$, $k\geq 1$, it is true and will be proved later (see § 11).

§ 11. Cobordism theory in the category $S\otimes_Z Q_p$

Earlier, in § 5, it was proved that in the algebra $A^U\otimes_Z Q_p$ there exists a projector $\Phi\in A^U\otimes_Z Q_p$ such that $\Phi(x,y)=\Phi(x)\Phi(y)$ and $\mathrm{Im}\,\Phi^*\subset\Lambda$ is the ring of polynomials in generators $y_1,\dots,y_i,\dots$, $\dim y_i=2p^i-2$, where the y_i are polynomial generators of the ring $\Lambda=\Omega_U\otimes_Z Q_p$ such that the numbers $\sigma^*_\varepsilon(y)\in Q_p$ are divisible by p and $\sigma^*_k(y_i)=p$, $k=p^i-1$. Moreover, a complete system of orthogonal projectors $\Phi^{(i)}$ was constructed, $\sum \Phi^{(i)} = 1$, $\Phi^{(i)} \cdot \Phi^{(j)} = 0$, $i \neq j$, where the $\Phi^{(i)}(U^* \otimes_Z Q_p)$ are isomorphic theories up to shift of dimensions. Hence, in the category $S \otimes_Z Q_p$ the spectrum MU is equal to the sum $MU \approx \sum_{\omega} E^{2d(\omega)} M_p$, where ω is not p-adic {i.e., $\omega = (i_1, \dots, i_k)$, all $i_q \neq p^r - 1$ for any r, and $d(\omega) = \sum_{q=1}^{k} i_q$}. If A_p^U is the Steenrod ring of the spectrum M_p, where $A_p^U = \Phi \cdot A^U \cdot \Phi$, then we have:

1) $A^U \otimes_Z Q_p = GL(A_p^U)$ is the appropriately graded ring of infinite matrices of the form $(a_{\omega_i \omega_j})$, $a_{\omega_i \omega_j} \in A_p^U$, ω_i not p-adic and $\dim(a_{\omega_i \omega_j}) = 2d(\omega_j) - 2d(\omega_j) + \dim a_{\omega_i \omega_j}$ {i.e., the right-hand side is a constant for the whole matrix and defines the degree of the matrix}.

2) $\operatorname{Ext}^{s,t}_{A^U}(U^*(K), U^*(L)) \otimes_Z Q_p = \operatorname{Ext}^{s,t}_{A_p^U}(U_p^*(K), U_p^*(L))$, where $U_p^* = \Phi(U^* \otimes_Z Q_p)$ is the theory defined by the spectrum M_p.

3) The Adams spectral sequences $(E_r \otimes Q_p, d_r \otimes Q_p)$ in U-theory and $(\tilde{E}_r, \tilde{d}_r)$ in U_p-theory coincide. These facts follow from §§ 1–3.

We note that the polynomial generators of the ring $\Lambda_p = U_p^*(P) = \Phi^* U^*(P)$ can be chosen to be polynomials with rational coefficients in the elements $x_i = [CP^{p^i - 1}] \in \Omega_U$, where the polynomial generator can be identified with $[CP^{p-1}] = x_1 = y_1$ in the first nontrivial dimension, equal to $p - 1$.

We consider the ring $\Lambda_{p,i} \subset \Lambda$, generated by the first i polynomial generators $y_1, \dots, y_i \in \Lambda_p$. This ring $\Lambda_{p,i}$ does not depend on the choice of generators.

The following fact is clear: the subring $\Lambda_{p,i} \subset \Lambda_p$ is invariant with respect to the action of all operations $\Phi \cdot S_2 \cdot \Phi$ on the ring Λ_p. The proof follows from the fact that the subring $\Lambda_{(j)} \subset \Lambda = \Omega_U$, generated by all generators of dimension $\leq 2j$, is invariant with respect to S_ω and with respect to Φ, while $\Phi(\Lambda_{p^i - 1}) = \Lambda_{p,i}$.

We consider the projection operator $\Phi_i \in A_p^U \otimes_{Q_p} Q$ such that $\Phi_i^* \colon \Lambda_p \to \Lambda_{p,i}, \Phi_i | \Lambda_{p,i} = 1$ and $\Phi_i(y_j) = 0$ for $j > i$. The ring $\Phi_i A_p^U \Phi_i$ will be denoted by $A_{p,i}$. It is generated by the operators of multiplication by elements of $\Lambda_{p,i} \subset A_{p,i}$ and by operators of the form $\Phi_i \cdot \Phi \cdot S_\omega \cdot \Phi \cdot \Phi_i$, where it is sufficient to take only partitions $\omega = (k_1, \dots, k_s)$, $k_j = p^{qi} - 1$, while $q_j \leq i$.

We have the following general fact.

The ring A_p^U is generated by operators of the form $\Phi \cdot S_\omega \cdot \Phi$ for $\omega(k_1, \dots, k_s)$, $k_j = p^{q_j} - 1$.

This fact follows easily from properties of the projector Φ and the structure of the spectrum M_p.

However, if $\omega = (p^{q_1} - 1, \dots, p^{q_s} - 1)$ and at least one $q_j > i$, then clearly $\sigma_\omega(\Lambda_{p,i}) = 0$. Hence in the ring $\Phi_i A_p^U \Phi_i$ it suffices to consider only $\Phi_i \cdot \Phi \cdot S_\omega \cdot \Phi \cdot \Phi_i$ for $\omega = (p^{p_1} - 1, \dots, p^{q_s} - 1)$, where all $q_j \leq i$.

Additive bases for the rings A_p^U and $A_{p,i}$:

a) $A_p^U = (\Lambda_p \cdot S_\omega)^\wedge$, where ω is p-adic and $\wedge$ denotes completion (by formal series).

b) $A_{p,i} = (\Lambda_{p,i} \cdot S_\omega)^\wedge$, $\omega = (p^{j1} - 1, \dots, p^{js} - 1)$, $j_k \leq i$.

We consider the operations $e_{i,k} = S_{(p^i - 1, \dots, p^i - 1)}$ (k times), regarded as elements of the ring $A_{p,i}$, i.e., $e_{i,k} = \Phi_i \Phi S_\omega \Phi \Phi_i$. Clearly, we have:

1) $\Delta(e_{i,k}) = \sum_{k=l+s} e_{is} \otimes e_{i,l}$ (the projectors Φ_i and Φ preserve the diagonal);

2) $e_{i,k}^*(\Lambda_{p,i-1}) = 0$;

3) $e^*_{i,1}(y_i) = p$, where $y_i \in \Lambda_{p,i} \subset \Lambda_p$ is the polynomial generator of dimension $p^i - 1$.

We denote by $\mathcal{E}_i \subset A_{p,i}$ the subring generated by the elements $(e_{i,k})$, $k \leq 1$.

We denote by D_i the subring of $A_{p,i}$ spanned by $\mathcal{E}_i$ and the operator of multiplication by the generator y_i, i.e., $D_i = Q_p[y_i]\mathcal{E}_i$.

We have the following

Lemma 11.1. a) *The subring $\mathcal{E}_1$ commutes with all operators of multiplication $\Lambda_{p,i-1} \subset \Lambda_p$ and all operators $\Phi_i \Phi S_\omega \Phi \Phi_i$ for all $\omega = (p^{j_1} - 1, \dots, p^{j_s} - 1)$, where $j_k \leq i - 1$.*

b) *In the ring $\mathcal{E}_i$ we have the relations*

$$e_{i,k} \cdot e_{i,s} = \binom{k+s}{s} \cdot e_{i,k+s},$$
$$e_{i,k} \cdot y_i^q = \sum_{s+m=i} e^*_{i,m}(y^q) \cdot e_{i,s},$$

where $e^*_{i,m}(y^q) = \binom{q}{m} p^m y^{q-m}$.

c) *The ring $A_{p,i-1}$ is obtained from the ring $A_{p,i}$ by discarding the polynomial generator y_i and then factoring the remaining subring $B_{p,i} \subset A_{p,i}$ by the ideal spanned by the central subalgebra $\mathcal{E}_i$ of $B_{p,i}$, where*

$$B_{p,i} = \{\Lambda_{p,i-1} \cdot (\Phi_i \Phi S_\omega \Phi \Phi_i)\}^\wedge$$

for all p-adic ω.

The proof of all parts of Lemma 11.1 follows easily from what has preceded.

Thus, the ring $A_{p,i}$ is obtained from the ring $A_{p,i-1}$ in the following way (in two steps):

Step 1. Without altering the "ring of scalars" $\Lambda_{p,i-1}$, we make a central extension of $\mathcal{E}_{i+1}$ by $A_{p,i}$:

$$0 \to \mathcal{E}_i \to B_{p,i} \to A_{p,i-1} \to 0,$$

with $\mathcal{E}_i$ acting trivially on $\Lambda_{p,i-1}$.

Step 2. We adjoin to the ring of scalars $\Lambda_{p,i-1}$ a polynomial generator y_i of dimension $p^i - 1$, setting $e^*_{i,k}(y_i) = \binom{q}{k} p^k y^{q-k}$ with all the consequences derived from this.

The ring $Q_p[y_i] \cdot B_{p,i}$ coincides with $A_{p,i}$, while the commutation rules for y_i and $\Phi_i S_\omega \Phi_i$ are derived from part (b) of Lemma 11.1.

In particular, the action of the operators $\Phi_i S_\omega \Phi_i$ for $\omega = (p^{j_1} - 1, \dots, p^{j_k} - 1)$ and for $j_k < i$ can also turn out to be nontrivial.

We shall denote $\Phi S_\omega \Phi$ by P^k when $\omega = (p-1, \dots, p-1)$ (k times).

We denote $\Phi_i P^k \Phi_i$ by P^k. For $p = 2$ we set $P^k = \mathrm{Sq}^k$.

As in the ordinary Steenrod algebra $\bmod p$, we have here the following fact: the operations P^k together with Λ_p generate the entire ring A^U_p (it suffices to take P^{p^s}). This follows easily from the fact that for the ring $A^U_p \otimes Z_p$ it is easily derived from the properties of the ordinary Steenrod algebra. Hence, it suffices to determine only the action of the operators P^k on the generators y_i (and even only of the P^{p^s}).

We now consider the ring D_i, operating on the module $Q_p[y_i]$, and the groups $\mathrm{Ext}^{s,t}_{D_i}(Q_p[y_i], Q_p[y_i])$. We set

$$\Gamma^{s,-t} = \mathrm{Ext}^{s,t}_{D_i}(Q_p[y_i], Q_p[y_i]).$$

We consider the groups $\Gamma^{s,t} \otimes \Lambda_{p,i-1}$. We have

Lemma 11.2. a) *There is a well-defined graded action of the ring $A_{p,i-1}$ on $\sum \Gamma^{s,t} \otimes \Lambda_{p,i-1}$ such that*:

1) $\lambda(x \otimes \mu) = x \otimes \lambda\mu$, $\lambda, \mu \in \Lambda_{p,i-1} \subset A_{p,i-1}$;

2) *if* $e_{i,\omega} = \Phi_i S_\omega \Phi_i$,

$$e_{i,\omega} \in A_{p,i-1}, \quad \omega = (p^i - 1, \dots, p^{j_s} - 1), \quad j_j < i,$$

then

$$e^*_{i,\omega}(x \otimes \mu) = \sum_{\omega=(\omega_1,\omega_2)} e^*_{i,\omega}(x) \otimes e^*_{i,\omega_2}(\mu),$$

where $e^*_{i,\omega_2}(\Lambda_{p,i-1})$ is the ordinary action and $e^*_{i,\omega_2}(x) \in \Gamma^s \otimes \Lambda_{p,i-1}$ for $x \in \Gamma^s$, $\mu \in \Lambda_{p,i-1}$;

b) *we have the equality*

$$\begin{aligned}
&\mathrm{Hom}^*_{A_{p,i-1}}(A_{p,i-1}, \Gamma^{s,t} \otimes \Lambda_{p,i-1}) \\
&\qquad\qquad = \mathrm{Hom}_{A_{p,i-1}}(A_{p,i-1}, \Lambda_{p,i-1}) \otimes \mathrm{Ext}^{s_1-t}_{D_i}(Q_p[y_i], Q_p[y_i]).
\end{aligned}$$

Proof. Part (b) is obvious. To construct the action of $A_{p,i-1}$ on $\Gamma^s \otimes \Lambda_{p,i-1}$ we note that the ring B_i acts on $\Lambda_{p,i} = Q_p[y] \otimes \Lambda_{p,i-1}$ naturally, while the action is trivial on $\Lambda_{p,i-1}$. From this follows the natural action of the factor-ring $A_{p,i-1}$ on the groups

$$\mathrm{Ext}_{D_i}(\Lambda_{p,i}, Q_p[y_i]) = \mathrm{Ext}_{D_i}(Q_p[y], Q_p[y]) \otimes \Lambda_{p,i-1},$$

where $D_i = Q_p[y] \cdot \mathcal{E}_1$. It is now easy to derive part (a). □

We note that the ring B_i is a free right module over $\mathcal{E}_i$.

We have the following

Theorem 11.1. *There exists a spectral sequence (E_2, d_r), where*:

a) *E_∞ is associated with* $\mathrm{Ext}_{A_{p,i}}(\Lambda_{p,i}, \Lambda_{p,i})$;

b) *$E_2^{p,q}$ coincides with $\mathrm{Ext}^p_{A_{p,i-1}}(\Lambda_{p,i-1}, \Gamma^q \otimes \Lambda_{p,i-1})$, where $\Gamma^q \otimes \Lambda_{p,i-1}$ is a $\Lambda_{p,i-1}$-module by virtue of Lemma* 11.2;

c) *$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$; all differentials d_r preserve the dimension of elements induced by the dimension of rings and modules*;

d) $E_2^{p,0} = \mathrm{Ext}^p_{A_{p,i-1}}(\Lambda_{p,i-1}, \Lambda_{p,i-1})$;

e) *the spectral sequence (E_r, d_r) is a spectral sequence of rings, where the multiplicative structure is induced by the diagonal Δ of the ring $A_{p,i}$.*

The proof of this theorem is more or less standard and is constructed by starting from the double complex corresponding to the central extension B_i of the rings $\mathcal{E}_i, A_{p,i-1}$. We shall not give it here.

For what follows it will be useful to us to compute $\mathrm{Ext}^{**}_{D_i}(Q_p[y], Q_p[y])$. We note that $A_{p,1} = D_1$, and the calculation of these groups gives certain information about the ring

$$\mathrm{Ext}_{A^U}(U^*(P), U^*(P)) \otimes_Z Q_p.$$

Lemma 11.3. *Let C be a bigraded differential ring over Q_p, which is associative and is generated by elements*

$$x \in C^{0,2p^i-2}, \quad h_j \in C^{1,2j(p^i-1)}, \quad j \geq 1,$$

such that:

$$1)\quad ph_{j+1} = [x, h_j] = xh_j - h_jx;$$

$$2)\quad d(x) = ph_1;$$

$$3)\quad d(h_1) = 0, \quad d(h_{j+1}) = \sum_{k=1}^{j} \binom{j+1}{k} h_{j+1-k} \cdot h_k, \quad j \geq 1;$$

4) $d(uv) = (du)v + (-1)^i u(dv)$ *where* $u \in C^i$. *Here, d is the differential in the ring* C^{**}.

Then the cohomology ring $H^{**}(C)$ *is canonically isomorphic to the ring* $\mathrm{Ext}^{**}_{D_i}(Q_p[y_i], Q_p[y_i])$.

The proof of the lemma consists in constructing a D_i-free acyclic resolution $\tilde{F}$ of the module $Q_p[y_{i+1}]$, having the form $Q_p[y] \cdot F$, where F is a standard resolution over $\mathcal{E}_i$ of the trivial module $Q_p = \mathcal{E}_i/\bar{\mathcal{E}}_i$, where $\bar{\mathcal{E}}_i$ is the set of elements of positive dimension and $\mathcal{E}_i$ is described in Lemma 11.1. The ring $\mathcal{E}_i$ has a diagonal, as do D_i and $Q_p[y]$. Hence the complex $\mathrm{Hom}^{**}_{D_i}(\bar{F}, Q_p[y])$ is a ring, which coincides exactly, as is easy to verify, with the ring C together with the differential operator d. Whence the lemma follows.

From Lemma 11.3 it is easy to derive

Lemma 11.4. a) *For $p = 2$ the cohomology ring $H^{**}(C \otimes Z_2)$ is isomorphic to the polynomial algebra $Z_2[x] \otimes Z_2[h_1, h_2, \dots, h_{2^k}]$ with Bokšteĭn homomorphism β of the following form*:

1) $\beta(x) = h_1$;

2) $\beta(h_{2^k}) = h^2_{2^{k-1}}$, $k \geq 1$, $x \in H^{0,2(2^i-1)}, h_{2^k} \in H^{2^{k+1}(2^i-1)}(C \otimes Z_2)$.

b) *For $p > 2$ the ring $H^{**}(C \otimes Z_p)$ is isomorphic to the ring*

$$Z_p[x] \otimes \Lambda[h_1, h_p, \dots, h_{p^k}, \dots] \otimes Z_p(\gamma_2, \dots, \gamma_k, \dots),$$

where

$$h_{p^k} \in H^{1,2p^k(p^i-1)}(C_i \otimes Z_p),$$

$$\gamma_k \in H^{2,2p^k(p^i-1)}(C_i \otimes Z_p),$$

$$x \in H^{0,2p^i-2}(C_i \otimes Z_p)$$

and the Bokšteĭn homomorphism β has the following:

1) $\beta(x) = h_1$;

2) $\beta(h_{p^k}) = \gamma_k$, $k \geq 1$.

c) *The group $\mathrm{Ext}^{1,t}_{D_i}(Q_p[y_i], Q_p[y_i])$ is nontrivial for $t = 2p(p^i - 2)$, $q \geq 1$, and is isomorphic to the cyclic group $Z_{f(q)}$, where $f(q) - 1$ is equal to the largest power of p which divides q. We shall denote the generator of the group $\mathrm{Ext}^{1,2q(p^i-1)}_{D_i}(Q_p[y], Q_p[y])$ by δ_q.*

d) *The image of the homomorphism of "reduction modulo p,"*

$$\alpha\colon \mathrm{Ext}^{1,2q(p^i-1)}(Q_p[y_i], Q_p[y_i]) \to H^{1,2q(p^i-1)}(C_i \otimes Z_p)$$

is generated by the following elements:

1) h_1x^{q-1} *for $p > 2$ and all q,*

2) h_1x^{q-1} *for $q = 2$ and $q \equiv 1 \mod 2$,*

3) $h_1x^{q-1} + h_2x^{q-2}$ *for $p = 2$ and $q \equiv 0 \mod 2$.*

e) *For all $t > 1$, in the groups $H^{t,*}(C_i \otimes Z_2)$ the kernel* $\operatorname{Ker}\beta$ *coincides with the image* $\operatorname{Im}\beta$. *Hence, the homomorphism of reduction modulo p,*

$$\alpha_p\colon \operatorname{Ext}^{t,*}_{D_i}(Q_p[y_i], Q_p[y_i]) \to H^{t,*}(C_i \otimes Z_p)$$

is an isomorphism on the kernel $\operatorname{Ker}\beta = \operatorname{Im}\beta$ *and none of these groups has elements of order p^2.*

The proof of (a) and (b) follows easily from the form of the ring C — in particular, from the fact that $C \otimes Z_p$ is commutative, C is obtained from the standard $\mathcal{E}_i$-resolution and $\mathcal{E}_i \otimes Z_p$ has a system of generators $\{l_{i,p^j}\}$, while

$$H^{**}(C_i \otimes Z_p) = Z_p[x] \otimes \operatorname{Ext}_{\mathcal{E}_i \otimes Z_p}(Z_p, Z_p).$$

The structure of the Bokšteĭn homomorphism β is derived immediately from Lemma 11.3.

Part (c) follows from the fact that $e^*_{i,k}(y_i^q) = \sum\limits_{k\geq 1} \binom{q}{k} p^k x^{q-k}$, as was shown in Lemma 11.1, and from the construction of the standard $\mathcal{E}_i$-resolution F for the module $\mathcal{E}_i/\bar{\mathcal{E}}_i = Q_p$ and the differential d^* in the complex $\operatorname{Hom}^*(Q_p[y_i]\cdot F, Q_p[y_i])$. Namely, we have:

$$d^*(x^q) = \sum_{k\geq 1} \binom{q}{k} p^k h_k x^{q-k}.$$

Part (d) is derived from the fact that $\frac{1}{p^{f(q)}} d(x^q) \mod p$ is equal to $h_1 x^{q-1}$ for $p > 2$ or $p = 2$, $q = 2s+1$, and is equal to $h_1 x^{q-1} + h_2 x^{q-2}$ for $p = 2$, $q = 2s$.

We shall now prove part (e). Since the homomorphism β is a differential operator, it suffices to show that $H^t(H^*(C_i \otimes Z_p), \beta) = 0$ for $t > 1$. The structure of the homomorphism β was determined in parts (a) and (b) of Lemma 11.4, and the required fact is easily derived from the usual homological arguments. The lemma is proved.

1. The ring structure in $\operatorname{Ext}^{**}_{D_i}(Q_p[y], Q_p[y])$ completely follows from Lemma 11.4, since the homomorphism of reduction modulo p,

$$\alpha_p\colon \operatorname{Ext}^{**}_{D_i}(Q_p[y], Q_p[y]) \to H^{**}(C_i \otimes Z_p),$$

is a monomorphism on $\operatorname{Ker}\beta$ and in dimensions ≥ 2; hence, from $\alpha_p(xy) = 0$ it follows that $xy = 0$ for elements x, y of positive dimension. The image of the homomorphism $\alpha_p(\operatorname{Ext}^{**}_{D_i})$ coincides with $\operatorname{Ker}\beta$ in all dimensions ≥ 1, although $\operatorname{Ker}\alpha_p$ is nontrivial in dimension 1 [see parts (c) and (d)].

2. The product $\operatorname{Ext}^{1,*}_{D_i} \otimes \operatorname{Ext}^{1,*}_{D_i}(Q_p[y], Q_p[y])$ is identically equal to zero for $p > 2$.

3. A basis for the group $\operatorname{Ext}^{2,*}_{D_i}(Q_p[y], Q_p[y])$ is completely given by the set of elements:

a) $\alpha_k^{(m)} = \beta(h_k x^m)$, $\quad k \geq 1,\ m \geq 0$ where $p > 2$, where

$$\beta(h_{p^k} x^m) = (\gamma_k x^m - m h_{p^k} h_1) \in \operatorname{Ext}^{2,(p^k+2m)(p^i-1)}_{D_i},$$

b) $\alpha_k^m = \beta(h_{2^k} x^m) = (h^2_{2^{k-1}} x^m + m h_{2^k} h_1 x^{m-1})$ where $p = 2, k \geq 2, m \geq 0$.

4. For $p = 2$ the product $\operatorname{Ext}^{1,*}_{D_i} \otimes \operatorname{Ext}^{1,*}_{D_i} \to \operatorname{Ext}^{2,*}_{D_i}$ is defined by the formulas:

$$\text{a)}\quad \delta_{2q+1}\cdot\delta_{2l+1} = \alpha_1^{(2q+2l)},$$

$$\text{b)}\quad \delta_{2q+1}\cdot\delta_{2l} = \alpha_1^{(2q+2l-1)},$$

$$\text{c)}\quad \delta_{2l}\cdot\delta_{2m} = \alpha_1^{(2q+2l-2)} + \alpha_1^{(2q+2l-4)}.$$

In particular, we shall denote the element δ_1 by $h \in \mathrm{Ext}^{1,2}$. Hence, from (a) and (b) it follows that

$$\delta_{2q+1} \cdot \delta_m = h\delta_{2q+m} \text{ for all } q, m.$$

We note that $D_1 = A_{p,1}$ and there is defined a natural homomorphism

$$\mathrm{Ext}^{t,*}_{D_1}(Q_p[y_1], Q_p[y_1]) \xrightarrow{\gamma^{(t)}} \mathrm{Ext}^{t,*}_{A^U}(U^*(P), U^*(P)) \otimes_Z Q_p.$$

From Lemma 11.4 and the results of §§ 7, 8 is derived the following

Theorem 11.2. a) *For $t = 1$ the homomorphism $\gamma_p^{(1)}$ is a monomorphism.*

b) *For all $p > 2$ the homomorphism $\gamma_p^{(1)}$ is an isomorphism.*

c) *For $p = 2$ the homomorphism $\gamma_p^{(1)}$ is an isomorphism on the groups $\mathrm{Ext}^{1,2q}$ for $q = 2$ and for q odd; for $q = 2s$, $s \geq 2$, the image of the homomorphism $\gamma_2^{(1)}$ nas index 1 or 2 in $\mathrm{Ext}^{1,2q}_{A^U} \otimes_Z Q_2$ and in fact index 2 for all $q = 4s$, $s \geq 1$.*

d) *For all $q = 4s + 1$ and $q = 4s + 2$ the image $\mathrm{Im}\,\gamma_2^{(1)}$ coincides with the image of the Hopf invariant $q_1^U(\pi_*(S^n)) = q_1^U(J\pi_*(SO))$. For $p > 2$ the image $\mathrm{Im}\,\gamma_p^{(1)}$ coincides with the image of the Hopf invariant $q_1^U(\pi_*(S^n)) = q_1^U(J\pi_*(SO))$.*

In the formulation of Theorem 11.2 the calculation of the group $\mathrm{Ext}^{1,8k+4}_{A^U} \otimes_Z Q_p$ not complete — is the homomorphism $\gamma_2^{(1)}$ an epimorphism or does $\mathrm{Im}\,\gamma_2^1$ have index 2?

For the study of this question we shall use the spectral sequence (E_r, d_r) described in Theorem 11.1, which converges to the groups $\mathrm{Ext}_{A_{2,2}}(\Lambda_{2,2}, \Lambda_{2,2})$. Namely, we must compute the groups $E_2^{0,1}$ and the differential

$$d_2 \colon E_2^{0,1} \to E_2^{2,0} \approx \mathrm{Ext}^2_{A_{2,1}}(\Lambda_{2,1}, \Lambda_{2,1}) = \mathrm{Ext}^2_{D_1}(Q_2[y_1], Q_2[y_1]).$$

The groups $\mathrm{Ext}^2_{D_1}$ were computed in Lemma 11.4 for all $p \geq 2$. We may assume that $y_1 = [CP^{p-1}] \in \Omega_U$ and

$$y_2 = \frac{1}{p}([C_p^{p^2-1}] + \lambda[C_p^{p-1}]^{p+1}) = \frac{1}{p}(x_2 + \lambda x_1^{p+1}),$$

where $x_i = [CP^{p^i-1}]$. Moreover, by the integrality of the Todd genus we can set $\lambda = p - 1$ and

$$y_2 = \frac{1}{p}(x_2 + (p-1)x_2^{p+1}), \quad y_1 = x_1.$$

We have:

$$\Lambda_{p,1} = Q_p[y_1], \quad \Lambda_{p,2} = Q_p[y_1, y_2].$$

The action of the operation $\Phi \cdot P^k \cdot \Phi$ on $\Lambda_{p,1}$ and $\Lambda_{p,2}$ is given by the formulas:

$$\Phi \cdot P^k \cdot \Phi(x_1^q) = \binom{q}{k} p^k x_1^{q-k},$$

$$\Phi \cdot P^k \cdot \Phi(x_2) = \begin{cases} 0, & k \neq p,\ p+1, \\ \dbinom{p^2}{p} x_1, & k = p, \\ \dbinom{p^2}{p+1}, & k = p+1. \end{cases}$$

As a consequence of this we have

Lemma 11.5. *The action of the operators P^k on the generators y_2^q of the ring $\Lambda_{p,2}$ is given by the following formula*:

$$(\Phi P^k \Phi)^*(y_2^q) = \sum_{k\geq 1}\left[y_2^{q-k} \otimes \left(\sum_{\substack{l_1>0\\ l_k>0\\ \sum l_i=k}} P^{l_1}(y_2)\circ\cdots\circ P^{l_k}(y_2)\right)\circ \binom{q}{k}\right],$$

where $(l_2,\ldots,l_k)$ is an ordered partition of the integer k and

a) $P^l(y_2) = \dfrac{1}{p}(P^l(x_2)+(p-1)P^l(x_1^{p+1})) = (p-1)\dbinom{p+1}{l}p^l x_1^{p+1-l}$ *for $l\neq p$, $p+1$,*

b) $P^p(y_2) = \dfrac{1}{p}\left(\dbinom{p^2}{p} + \dbinom{p+1}{p}p^p\right)x_1,$

c) $P^{p+1}(y_2) = \dfrac{1}{p}\left(\dbinom{p^2}{p+1} + p^{p+1}\right).$

We note that $P^1(y_2)$ is divisible by p for $1\neq p$ and $P^p(y_2)$ is not divisible by p.

Now we can describe the action of the ring $A_{p,1}$ on $\Gamma^1\otimes\Lambda_{p,1}$, where $\Gamma^{t,s} = \mathrm{Ext}_{D_2}^{t,-s}(Q_p[y_2],Q_p[y_2])$ and $\lambda_{p,1} = Q_p[y_1] = Q_p[x_1]$, $x_1 = [CP^{p-1}]$. The groups $\Gamma^{1,*}$ were computed in Lemma 11.4, part (c). The generator of the group $\mathrm{Ext}^{1,q(p^2-1)}(Q_p[y_2],Q_p[y_2])$ is obtained as $d^*(x^q)/p^{f(q)}$ in the complex $\mathrm{Hom}^*_{D_2}(F,Q_p[\gamma_2])$ where F is a D_2-free acyclic resolution of the module $Q_p[y_2], x^q\in\mathrm{Hom}^*(D_2,Q_p[y_2])$ is an element such that $x^q(1) = y_2^q$, $f(q)-1$ is the maximal power of p which divides q, and d^* is the differential in the complex $\mathrm{Hom}^*_{D_2}(F,Q_p[y_2])$.

We set

$$P^k(y_2^q) = \sum_{k=1}^{q} y_2^{q-k}\otimes a_k,$$

where

$$a_k = \sum_{\substack{\sum l_i=k\\ l_i>0}} P^l(y_2)\ldots P^{l_s}(y_2)\binom{q}{k}$$

by virtue of Lemma 11.5 and $a_k\in Q_p[y_1]$, $a_k = \lambda^k y_1^{s_k}$. From what has been said it is easy to derive

Lemma 11.6. *The action of the ring $A_{p,1} = D_1$ on $\Gamma^1\otimes Q_p[y_2]$ is described in the following fashion*:

$$P^k(\alpha_q) = \sum_{k=1}^{q-1}\alpha_{q-k}\otimes p^{f(q-k)-f(q)}\circ a_k,$$

where

$$\alpha_q\in\Gamma^{1,-2q(p^2-1)} = \mathrm{Ext}_{D_2}^{1,2q(p^2-1)}(Q_p[y_2],Q_p[y_2])$$

are generators (*their orders are $p^{f(q)}$*) *and $a_k\in Q_p[y_1]$ is described in Lemma* 11.4.

Lemma 11.6 follows easily from Lemma 11.4 and the definition of the generators $\alpha_q = d^*x^q/p^{f(q)}$, where $x^q\in\mathrm{Hom}^*(F,Q_p[y_2])$ is such that $x^q(1) = y_2^q\in Q_p[y_2]$. Further, we compute $\mathrm{Hom}^*_{A_{p,1}}(\Lambda_{p,2},\Gamma^1\otimes\Lambda_{p,1}) = E_2^{0,1}$ in the spectral sequence (E_2,d_2) of Theorem 11.1, which converges to $\mathrm{Ext}^{**}_{A_{p,2}}(\Lambda_{p,2},\Lambda_{p,2})$; here $A_{p,1} = D_1$ and $\Lambda_{p,1} = Q_p[y_1]$.

We have the following

Lemma 11.7. *The groups* $\operatorname{Hom}^t_{A_{p,1}}(\Lambda_{p,1}, \Gamma^1 \otimes \Lambda_{p,1})$ *are spanned by generators* $\kappa_{i,q}$ *of dimension* $2p^i(p^2-1)+2q(p-1)$ *for all* $i \geq 0$, $q \geq 0$, *where the order of the generator* $\kappa_{i,q}$ *is* p.

The proof of the lemma follows easily from Lemma 11.5 and 11.6 by direct calculation.

Since $\operatorname{Hom}^*_{A_{p,1}}(\Lambda_{p,1}, \Gamma^1 \otimes \Lambda_{p,1}) = E_2^{0,1}$, our problem is to calculate $d_2 \colon E_2^{0,1} \to E_2^{2,0} = \operatorname{Ext}^2_{A_{p,1}}(\Lambda_{p,1}, \Lambda_{p,1})$, where the latter groups are computed in Lemma 11.4 and in the conclusions drawn from it.

Direct calculation proves

Lemma 11.8. *The differential* $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ *of the spectral sequence* (E_r, d_r) *converging to* $\operatorname{Ext}_{A_{p,2}}(\Lambda_{p,2}, \Lambda_{p,2})$ *is given by the following formula*:

$$d_2(\kappa_{i,t}) = \beta(h_{p^{i+1}} x^{p^i+t} + h_{p^i} x^{p^i+t}), \quad i \geq 0, \quad t \geq 0,$$

where h_{p^i} *and* x *are in the notation of Lemma* 11.4, *and* β *is the Bokšteĭn homomorphism* $H^{**}(C) \to \operatorname{Ext}_{D_1}(Q_p[x_1])$ *described in Lemma* 11.4.

From Lemma 11.8 follows the important

Corollary 11.1. a) *For* $p > 2$, *the kernel* $\operatorname{Ker} d_2 | E_2^{0,1}$ *is trivial*;

b) *For* $p = 2$, *the kernel* $\operatorname{Ker} d_2 | E_2^{0,1}$ *is generated by elements*

$$\kappa_{0,2t+1} \in \operatorname{Hom}^{4t+8}_{A_{p,1}}(\Lambda_{p,1}; \Gamma^1 \otimes \Lambda_{p,1}), \quad t \geq 0.$$

Hence, the image of the homomorphism

$$\operatorname{Ext}^{1,4t+8}_{A_{p,1}}(\Lambda_{p,1}; \Lambda_{p,1}) \to \operatorname{Ext}^{1,4t+8}_{p,2}(\Lambda_{p,2}; \Lambda_{p,2})$$

has index 2 *for all* $t \geq 0$.

Parts (a) and (b) of the corollary are derived in an obvious way from the structure of the homomorphism β, which was completely described in Lemma 11.4. The sharp distinction between the cases $p = 2$ and $p > 2$ is explained by the fact that for $p > 2$ we have $h_1^2 = 0$ and $\beta(h_1 x^s) = 0$ for all $s \geq 0$, while for $p = 2$, $\beta(h_1 x^{2s+1}) \neq 0$.

Comparing part (b) of Corollary 11.1 with Theorem 11.2, we obtain the following result.

Theorem 11.3. a) *In all dimensions* $t \neq 4$, *the order of the cyclic group* $\operatorname{Ext}^{1,t}_{A^U}(U^*(P), U^*(P))$ *coincides exactly with the order of the group* $\operatorname{Ext}^{1,t}_{A^k_\Psi}(K^*(P), K^*(P))$, *and this isomorphism is induced by the Riemann–Roch functor* λ.

b) *The Hopf invariant*

$$q_1 \colon \pi_{n+t-1}(S^n) \to \operatorname{Ext}^{1,t}_{A^U}(U^*(P), U^*(P))$$

is an epimorphism for $t = 8k$, $t = 8k+2$ *and* $t = 4$, *and the image* $\operatorname{Im} q_1$ *has index* 2 *in* $\operatorname{Ext}^{1,t}_{A^U}$, *for* $t = 8k+6$, $k \geq 0$, *and* $t = 8k+4$, $k \geq 1$.

Corollary 11.2. *The generators* α_q *of the groups* $\operatorname{Ext}^{1,2q}_{A^U}(U^*(P), U^*(P))$ *are cycles for all Adams differentials* d_i *for* $q = 4s$, $4s+1$, $s \geq 0$, *and* $q = 2$, *and are not* *cycles for all differentials for* $q = 4s-1,\ 4s+2,\ s \geq 1$ (*the elements* $2\alpha_q$ *are cycles for all differentials*)[4].

Supplementary remark. It is possible in all dimensions to prove the formula $d_3(\alpha_q) = h^3 \cdot \alpha_{q-2}$ for $q = 4s-1,\ 4s+2,\ s \geq 1$, where $h = \alpha_1 \in \operatorname{Ext}^{1,2}_{A^U}$. In particular, for $q = 4s+2$ this follows from the fact that $h^3\alpha_{q-2} \neq 0$ in $\operatorname{Ext}^4_{A^U}$, while at the same time α_{q-2} is realized by the image of the J-homomorphism, and we must have in E_∞ that $h^3\alpha_{q-2} = 0$.

§ 12. The Adams spectral sequence and double complexes. Comparison of different cohomology theories

We assume that there is given a complex $Y = Y_{-1} \in \vec{S}$ and a filtration

$$Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \dots \leftarrow Y_i \leftarrow \dots,$$

where the complex of A^X-modules $\{X^*(Y_i, Y_{i+1}) = M_i\}$

$$M = \{M_0 \overset{d}{\leftarrow} M_1 \leftarrow M_2 \leftarrow \dots \overset{d}{\leftarrow} M_i \leftarrow \dots\}$$

is acyclic in the sense that $H_i(M) = 0$, $i > 0$, and $H_0(M) = X^*(Y)$. The modules M_i are not assumed to be projective. In the usual way a double complex of A^X-free modules $N = (N_{ij})$ is constructed.

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & & & & & \\
N_{2,0} & \longleftarrow & N_{2,1} & \longleftarrow & N_{2,2} & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
N_{1,0} & \overset{d_2}{\longleftarrow} & N_{1,1} & \overset{d_2}{\longleftarrow} & N_{1,2} & \longleftarrow & \dots \\
\downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \\
N_{0,0} & \overset{d_2}{\longleftarrow} & N_{0,1} & \overset{d_2}{\longleftarrow} & N_{0,2} & \longleftarrow & \dots
\end{array}$$

such that (a) $d_1 d_2 = -d_2 d_1$; (b) $\{\to \dots \to N_{ij} \xrightarrow{d_1} N_{i-1,j} \to \dots\}$ for all j is an A^X-free acyclic resolution of the module M_j; (c) if $Q_k = \sum\limits_{j+i=k} N_{ij}$ and $d = d_1 + d_2 \colon Q_k \to Q_{k-1}$, then the complex $\{Q_k \xrightarrow{d} Q_{k-1} \to \dots\}$ is an A^X-free acyclic resolution of the module $X^*(Y)$; (d) the complex $N_i = \{\to N_{i,j} \xrightarrow{d_2} N_{i,j-1} \to \dots\}$ is such that $H_k(N_i) = 0$ for $k > 0$, $H_0(N_i)$ is a free A^X-module and the complex $\{\dots H_0(N_k) \xrightarrow{d_1} H_0(N_{k-1}) \to \dots\}$ represents an A^X-free acyclic resolution of the module $X^*(Y)$.

As usual, there arises a spectral sequence of the double complex $(E_r^{t,q}, d_r)$, where

$$d_r \colon E_r^{t,q} \to E_r^{t+r,q-r+1}$$

and

$$E_2^{t,q} = \operatorname{Ext}^t_{A^X}(M_q, L),$$

[4]We take this opportunity to note the small computational error in parts (3) and (4) of Theorem 5 of the author's paper [19], which is completely corrected in Theorem 11.3 and Corollary 11.2 of the present paper.

with L an arbitrary A^X-module; this spectral sequence converges to $\operatorname{Ext}_{A^X}(X^*(Y), L)$.

Definition 12.1. By a geometric realization of the double complex $N = (N_{ij})$ in the category $\vec{S}$ or $\vec{S} \otimes_Z Q_p$ is meant a set of objects (Z_{ij}), $i \geq -1$, $j \geq -1$, and morphisms

$$\begin{array}{ccccccc}
Z_{-1,-1} & \longleftarrow & Z_{-1,0} & \longleftarrow & Z_{-1,1} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
Z_{0,-1} & \longleftarrow & Z_{0,0} & \longleftarrow & Z_{0,1} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
Z_{1,-1} & \longleftarrow & Z_{1,0} & \longleftarrow & Z_{1,1} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

with the following properties:

a) $Z_{-1,-1} = Y$, $Y_i = Z_{-1,i}$ and the filtration $Z_{-1,-1} \leftarrow Z_{-1,0} \dots$ coincides with the filtration $Y \leftarrow Y_0 \leftarrow \dots \leftarrow Y_i \leftarrow \dots$

b) The filtration

$$Y_i/Y_{i+1} = Z_{-1,i}/Z_{-1,i+1} \leftarrow Z_{0,i}/Z_{0,i+1} \leftarrow Z_{1,i}/Z_{1,i+1} \leftarrow \dots$$

represents a geometric realization of the A^X-free resolution of the module $X^*(Y_i/Y_{i+1}) = M_i$,

$$\{M_i \xleftarrow{\varepsilon} N_{0,i} \xleftarrow{d_1} N_{1,i} \xleftarrow{d_1} \dots\}$$

and hence $X^*(Z_{k,i}/Z_{k,i+1} \cup Z_{k+1,i}) = N_{k,i}$.

c) The differentials $d_1 \colon N_{k,i} \to N_{k-1,i}$ and $d_2 \colon N_{k,i} \to N_{k,i-1}$ coincide with the natural homomorphisms

$$X^*(Z_{k,i}/Z_{k,i+1} \cup Z_{k+1,i}) \xrightarrow{d_1} X^*(Z_{k-1,i}/Z_{k,i} \cup Z_{k-1,i+1}),$$

$$X^*(Z_{k,i}/Z_{k,i+1} \cup Z_{k+1,i}) \xrightarrow{d_2} X^*(Z_{k,i-1}/Z_{k,i} \cup Z_{k+1,i-1}).$$

We make some deductions fromi the properties of the geometric realization of a double complex:

1. The filtration $Z_{-1,-1} = Y \leftarrow Z_{0,-1} \leftarrow Z_{1,-1} \leftarrow \dots \leftarrow Z_{i,-1} \leftarrow \dots$ represents the geometric realization of the A^X-free resolution $\{H^*(N_0) \xleftarrow{d_2} H^*(N_1) \leftarrow \dots\}$.

2. The filtration $Y \leftarrow Z_{-1,0} \cup Z_{0,-1} \leftarrow \cdots \bigcup_{i+j=k-1} Z_{i,j} \leftarrow \dots$ represents the geometric realization of the A^X-free resolution

$$X^*(Y) \xleftarrow{\varepsilon} N_{0,0} \xleftarrow{d} N_{0,1} + N_{1,0} \leftarrow \dots \leftarrow \sum_{i+j=k} N_{i,j},$$

where $d = d_1 + d_2$.

3. The double complex (Z) defines two Adams spectral sequences:

a) the Adams spectral sequence $E_{r,X}$ in the theory X^*, induced by the filtration

$$Y \leftarrow Z_{0,-1} \cup Z_{-1,0} \leftarrow \dots \leftarrow \bigcup_{i+j=k-1} Z_{i,j} \leftarrow \dots$$

with term $E_2^k = \operatorname{Ext}^k_{A^X}(X^*(Y), X^*(K))$ for any $K \in S$;

b) the spectral sequence $\bar{E}_r$ of the filtration

$$Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_i \leftarrow \dots$$

with term $\bar{E}_1 = \{\mathrm{Hom}^*(K, Y_i/Y_{i+1})\}$.

In view of the presence of the double filtration (Z_{ij}) of the complex Y in all terms of both Adams spectral sequences there arises yet another filtration: in the first case it is equal to $\phi(x)$, $x \in E_r^k$, where $\phi(x)$ coincides in E_2^k with the filtration in $\mathrm{Ext}^k_{A^X}(X^*(Y), X^*(K))$ induced by the non-free resolution $X^*(Y) \leftarrow M_0 \leftarrow M_1 \leftarrow \dots$, and in E_∞^k is induced by the geometric filtration

$$\bigcup_{\substack{i+j=k-1\\ i\le k}} Z_{i,j} \supset \cdots \supset \bigcup_{\substack{i+j=k-1\\ i\le k-\phi(x)}} Z_{i,j} \supset \cdots \supset Z_{-1,k}.$$

c) For the second Adams spectral sequence the filtration in $\bar{E}_r^k$ and $\bar{E}_\infty^k$ is induced by the geometric filtration

$$Z_{-1,k} \supset Z_{0,k} \supset \cdots \supset Z_{s,k} \supset \dots .$$

We shall denote it by $\Psi(y)$, $y \in \bar{E}_r$.

In addition, each of the indicated spectral sequences defines in the groups of homotopy classes of mappings $\mathrm{Hom}^*(K, Y)$ the usual filtration $i(x)$, whose corresponding index i is such that the element $x \in \mathrm{Hom}^*(K, Y)$ is nontrivial in E_∞^i and trivial in E_∞^j for $j > i$. For the Adams spectral sequence of the theory X^* we shall denote this filtration by i_X. We have the double filtration $[i_X(x), \phi(x)]$ where $x \in \mathrm{Hom}^*(K, Y)$, $\phi(x) \le i_x(x)$.

The second Adams spectral sequence for $\mathrm{Hom}^*(K, Y)$, induced by the filtration

$$Y \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_1 \leftarrow \dots$$

also induces a double filtration in $\mathrm{Hom}^*(K, Y)$: $[i(x), \Psi(x)]$.

From the construction of the double complex it is obvious that we have

Lemma 12.1. *The filtrations described above are related by*

$$i(x) \le \phi(x) \le i_X(x) \le i(x) + \Psi(x)$$

for all $x \in \mathrm{Hom}^*(K, Y)$ in the presence of a geometric realization of the double complex defining both Adams spectral sequences.

By standard methods one proves

Lemma 12.2. *If X^* is the theory of Z_p-cohomology, then for any acyclic filtration $Y = Y_{-1} \leftarrow Y_0 \leftarrow Y_1 \leftarrow \dots$ there exists a geometrically realizable A-free double complex (Z), where A is the ordinary Steenrod algebra.*

The proof of this lemma is obtained easily by the methods of [1].

The most important example which we consider here is the theory of cobordism in the category $S \otimes_Z Q_p$:

a) $Y \in D_p$, i.e., $H^*(Y, Q_p)$ has no torsion.

b) $X = H^*(\ , Z_p)$, $A^X = A$.

c) The filtration $Y \supset Y_0 \supset Y_1 \supset \dots$ is an acyclic free filtration in the theory $U^* \otimes_Z Q_p$ or in the theory $U_p^* \subset U^* \otimes_Z Q_p$. By virtue of the exactness of the functor $U_p^* \to H^*(\ , Z_p)$ in the category D_p, the filtration $Y \supset Y_0 \supset \dots$ is also acyclic (although not free) in the theory $X = H^*(\ , Z_p)$.

In this example, the filtration $i(x)$ is a homotopy invariant, with $i(x) = i_{U_p}(x)$, where U_p^* is cobordism theory. Moreover, we have

Lemma 12.3. a) *All the filtrations i_X, i_{U_p}, ϕ, Ψ, for $X^* = H^*(\ ,Z_p)$, a U_p^*-free acyclic filtration $Y \supset Y_0 \supset Y_1 \supset \dots$ in the category $S \otimes_Z Q_p$ and any X^*-free acyclic double complex (Z) are homotopy invariants of Y, and we have the following inequalities*:

$$i_{U_p^*}(x) \le \phi(x) \le i_{H^*(\ ,Z_p)}(x) \le i_{U_p^*}(x) + \Psi(x),$$

where $i = i_{U_p^*}$, $i_X = i_{H^*(\ ,Z)}$ are the respective filtrations in the theories $U^* \otimes_Z Q_p$ and $H^*(\ ,Z_p)$.

b) *The second Adams spectral sequence E_r coincides in this case with the Adams spectral sequence in the theory $U^* \otimes_Z Q_p$ for $r \ge 2$.*

c) *Both Adams spectral sequences E_r in the theories $X^* = H^*(\ ,Z_p)$ and $U^* \otimes_Z Q_p$ (or U_p^*) in our case preserve, respectively, the filtrations ϕ and Φ.*

d) *The Adams spectral sequence in the theory $H^*(\ ,Z_p)$ is such that each differential d_r for $r \ge 2$ raises the filtration ϕ at least by 1, i.e.,*

$$\phi(d_r y) \ge \phi(y) + 1, \quad y \in E_r.$$

For the proof of (a) we note that the U_p^*-filtration $Y \supset Y_0 \supset Y_1 \supset \dots$ depends functorially on the U_p-free resolution and is uniquely determined by it. For a fixed U_p-filtration the same thing is true with respect to the double complex N and the double filtration (Z) defined by it. Parts (b) and (c) are obvious. Part (d) follows immediately from the fact that the complex Y_i/Y_{i+1} is a direct sum of spectra M_p of the theory U_p^* up to suspension. For such objects the Adams spectral sequence has zero differentials for $r \ge 2$, as was proved by Milnor and the author [15, 17, 18].

The lemma is proved.

We now consider the graded ring $\Lambda_p \subset \Omega_U \otimes_Z Q_p$, where $\Lambda_p = Q_p[x_1, \dots, x_i, \dots]$, $\dim x_i = 2p^i - 2$. The ring Λ_p is a local ring: it has a unique maximal ideal $m \subset \Lambda_p$ such that $\Lambda_p/m = Z_p$. Hence the bigraded ring $\bar{\Lambda}_p = \sum_{i=0} m^i/m^{i+1}$ is an algebra over Z_p, and $\bar{\Lambda}_p = Z_p[h_0, h_1, \dots, h_i, \dots]$, where h_0 is associated with multiplication by p and $\dim h_i = (1, 2p^i - 1)$, i.e., $h_i \in m/m^2$. Clearly $\bigcap_i m^i = 0$ and, by [15, 17, 18], we have:

$$\bar{\Lambda}_p = \operatorname{Ext}_A^{**}(H^*(M_p, Z_p), Z_p).$$

As was established in § 11, the action of the ring A_p^U on $\Lambda_p = U_p^*(P)$ preserves the filtration generated by the maximal ideal m. Hence it defines an action on $\bar{\Lambda}_p$, which is described as follows:

1) the action of Λ_p on $\bar{\Lambda}_p$ is defined by multiplication;

2) the action of P^k on $\bar{\Lambda}_p$ is defined so that

$$P^{p^i}(h_i) = h_{i-1} \text{ and } P^j(h_0) = 0, \quad j \ge 1, \quad P^k(ab) = \sum_{l+s=k} P^l(a)P^s(b).$$

We consider the ring $\bar{A}$ associated to A_p^U by the filtration $A_p^U \supset mA_p^U \supset \dots \supset m^i A_p^U \supset \dots$. We note that in the ordinary Steenrod algebra A there is a normal (exterior) subalgebra $Q \subset A$, $Q = \Lambda(Q_0, \dots, Q_i, \dots)$, $\dim Q_i = 2p^i - 1$, such that $A//Q$ is isomorphic to the quotient $A/\beta A \cup A\beta$ and $\operatorname{Ext}_A(H^*(M_p, Z_p)) = \operatorname{Ext}_Q(Z_p, Z_p) = \bar{\Lambda}_p = Z_p[h_0, h_1, \dots, h_i, \dots]$.

From the results of § 11 and the structure of the Steenrod algebra A follows

Lemma 12.4. *The algebra $\bar{A}$ associated to the ring A_p^U is isomorphic to $(\bar{\Lambda}_p \cdot A//Q)^\wedge$, where the commutation law $ah = \sum_i \bar{a}_i^*(h)\bar{\bar{a}}_i$ is given by the action of*

$A//Q$ on $\bar{\Lambda}_p$ defined by the formulas $P^{p^r}(h_r) = h_{r-1}$, $r \geq 1$, $P^k(h_0) = 0$ for $k > 0$, and $\Delta a = \sum_i \bar{a}_i \otimes \bar{\bar{a}}_i$, where $\Delta\colon A//Q \to A//Q \otimes A//Q$ is the diagonal and P^k is the ordinary Steenrod power.

We note now the following identity:

$$\mathrm{Ext}^s_{\bar{A}}(\bar{\Lambda}_p, \bar{\Lambda}^t_p) = \mathrm{Ext}^s_{A//Q}(Z_p, \bar{\Lambda}^t_p) = \mathrm{Ext}^s_{A//Q}(Z_p, \mathrm{Ext}^t_Q(Z_p, Z_p))$$

(here, t is the dimension in $\bar{\Lambda}_p$ defined by the filtration $\bar{\Lambda}^t_p = m^t/m^{t+1}$). Moreover, if $Y \subset D_p$, then for $L^p = U^*_p(Y)$ and $M = H^*(Y, Z_p) = L/mL$ we have:

a) M is an $A//D$-module;

b) there exists the identity

$$\mathrm{Ext}^s_{\bar{A}}(\bar{L}, \bar{\Lambda}^t_p) = \mathrm{Ext}^s_{A//Q}(M, \mathrm{Ext}^t_Q(Z_p, Z_p)),$$

where $\bar{L} = \sum m^i L/m^{i+1}L$ is an $\bar{A}$-module and, clearly, a $\bar{\Lambda}_p$-free module.

Two spectral sequences $(\tilde{E}_r)$, $(\tilde{\tilde{E}}_r)$ arise, both with the term

$$\tilde{E}_2 = \tilde{\tilde{E}}_2 = \mathrm{Ext}_{A//Q}(M, \mathrm{Ext}_Q(Z_p, Z_p)).$$

These sequences have the following properties:

1) In the first, which converges to $\mathrm{Ext}_A(M, Z_p)$, we have

$$\tilde{d}_r\colon \tilde{E}^{s,t}_r \to \tilde{E}^{s+r,t-r+1}_r.$$

2) In the second, which is induced by the filtrations in Λ_p, A^U_p, L and which converges to $\mathrm{Ext}_{A^U_p}(L, \Lambda_p)$, we have:

$$\tilde{\tilde{d}}_r\colon \tilde{\tilde{E}}^{s,t}_r \to \tilde{\tilde{E}}^{s+1,t+r-1}_r.$$

3) $\tilde{d}_1 = \tilde{\tilde{d}}_1$ and $\tilde{E}^{s,t}_2 = \tilde{\tilde{E}}^{s,t}_2 = \mathrm{Ext}^s_{A//Q}(M, \mathrm{Ext}^t_Q(Z_p, Z_p))$.

4) In both spectral sequences there is yet another grading $\tilde{E}^{s,t}_r = \sum_q \tilde{E}^{s,t,q}_r$ and $\tilde{\tilde{E}}^{s,t}_r = \sum_q \tilde{\tilde{E}}^{s,t,q}_r$, induced by the dimensions in all modules and algebras which appear, and connected to the spectral sequences as follows:

a) the third grading q is preserved by all differentials $\tilde{d}_r$ of the spectral sequence $\tilde{E}_r$ which converges to $\mathrm{Ext}_A(M, Z_p)$;

b) since $\sum_{t-q=m} \bar{\Lambda}^{t,q}_p$ is associated to Λ^m_p, the third grading q in the second spectral sequence $\tilde{\tilde{E}}_r$, which converges to $\mathrm{Ext}_{A^U_p}(L, \Lambda_p)$, is increased by $r-1$ by the differential $\tilde{\tilde{d}}_r$:

$$\tilde{d}_r\colon \tilde{E}^{s,t,q}_r \to \tilde{E}^{s+r,t-r+1,q}_r,$$

$$\tilde{\tilde{d}}_r\colon \tilde{\tilde{E}}^{s,t,q}_r \to \tilde{\tilde{E}}^{s+1,t+r-1,q+r-1}_r.$$

5). a) The group $\sum_{s+t=m} \tilde{E}^{s,t,q}_\infty$ is associated with

$$\mathrm{Ext}^{m,q}_A(M, Z_p) = \mathrm{Ext}^{m,q}_A(H^*(Y, Z_p), Z_p).$$

b) The group $\sum_{q-t=l} \tilde{\tilde{E}}^{s,t,q}_\infty$ is associated with

$$\mathrm{Ext}^{s,t}_{A^U_p}(L, \Lambda_p) = \mathrm{Ext}^{s,l}_{A^U}(U^*(Y), U^*(P)) \otimes_Z Q_p,$$

where $L = U^*(Y)$, $\Lambda_p = U^*_p(P)$.

Thus, in the groups

$$E_2^{s,t,q} = \mathrm{Ext}^{s,q}_{A//Q}(M, \mathrm{Ext}^t_Q(Z_p, Z_p))$$

we have two "dimensions": $(m, q) = (s+t, q)$ is the "cohomological" and $(s, q-t) = (s, l)$ is the "unitary" (in U-cobordism). The "geometric" dimension (of the homotopy groups) is equal to $q - m = l - s = q - s - t$.

We note the important fact: the dimension of the element $\tilde{d}_r(y)$ for the element y of "unitary" dimension (s, l) is equal to $(s + r, l + r - 1)$, where $l = q - t$; and, conversely, $\tilde{\tilde{d}}_r(y)$ of an element of "cohomological" dimension (m, q) has "cohomological" dimension $(m+r, q+r-1)$, $m = s+t$. This means that both these spectral sequences have the form of the Adams spectral sequence, although they are defined purely algebraically by the ring A^U_p.

Up to this point there has been no difference between $p = 2$ and $p > 2$, if we speak of the results if this section. However, the following theorem shows the comparative simplicity of the case $p > 2$.

Theorem 12.1. *For any $p > 2$ and complex $Y \subset D_p$, the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ has all differentials $\tilde{d}_r = 0$ for $r \geq 2$. The groups*

$$\sum_{s+t=m} \tilde{E}_2^{s,t,q} = \sum_{s+t=m} \mathrm{Ext}^{s,q}_{A//Q}(M, \mathrm{Ext}^t_Q(Z_p, Z_p))$$

are isomorphic to $\mathrm{Ext}^{m,q}_A(M, Z_p)$, where $M = H^*(Y, Z_p)$,

$$\mathrm{Ext}^*_Q(Z_p, Z_p) = Z_p[h_0, \dots, h_i, \dots], \quad \dim h_i = (1, 2p^i - 1),$$

and the algebra $A//Q$ generated by the Steenrod powers P^{p^i} acts on $\mathrm{Ext}_Q(Z_p, Z_p)$ in the following way: $P^{p^i}(h_{i+1}) = h_i$, $P^k(h_0) = 0$ for $k > 0$, and $P^k(xy) = \sum\limits_{i+j=k} P^i(x)P^j(y)$.

From Theorem 12.1 follows

Corollary 12.1. *For any complex $Y \in D_p$, where $p > 2$, there is defined an "algebraic Adams spectral sequence" $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$, where $\tilde{\tilde{E}}_2^{s,t,q} = \mathrm{Ext}^{s,q}_{A//Q}(M, \mathrm{Ext}^t_Q(Z_p, Z_p))$, the group $\sum\limits_{s+t=m} \tilde{\tilde{E}}_2^{s,t,q} = \tilde{\tilde{E}}_2^{m,q}$ is associated to $\mathrm{Ext}^{m,q}_A(M, Z_p)$, $\tilde{d}_r \colon \tilde{\tilde{E}}_r^{s,t,q} \to \tilde{\tilde{E}}_r^{s+t,t+r-1,q+r-1}$, and the group $\sum\limits_{t-q=l} \tilde{\tilde{E}}^{\infty}_{s,t,q}$ is associated to $\mathrm{Ext}^{s,1}_{A^U_p}(U^*_p(Y), U^*_p(P))$, $M = H^*(Y, Z_p)$.*

We prove Theorem 12.1. In the Steenrod algebra A for $p > 2$ there is defined a second grading — the so-called "type in the sense of Cartan," equal to the number of occurrences of the homomorphism β in the iteration. We shall denote by $\tau(a) \geq 0$ the type of the operation $a \in A$, with $A = \sum\limits_{\tau} A^\tau$, where τ is the type and $A^{\tau_1} \cdot A^{\tau_2} \subset A^{\tau_1+\tau_2}$. By the same token, for any $Y \in D_p$ there is an extra grading — the type τ — in the groups $\mathrm{Ext}_A(M, Z_p)$, and

$$\mathrm{Ext}^{s,l}_A(M, Z_p) = \sum_{\tau \geq 0} \mathrm{Ext}^{s,l-\tau,\tau}_A(M, Z_p),$$

where $l-\tau \equiv 0 \mod 2p-2$. We note that for $Q \subset A$, $\tau(Q_r)=1$ and $\tau(P^k)=0$. It is also obvious that $\tau(h_i)=1$, $h_i \in \operatorname{Ext}^1_Q(Z_p, Z_z)$, and the type is an invariant of the spectral sequence $(\tilde{d}_r, \tilde{E}_r)$ for $r \geq 1$.

Since the type is trivial on the ring $A//Q$, and $A//Q \subset A$, all $\tilde{d}_r = 0$ for $r \geq 2$, since on the groups $\operatorname{Ext}^{s,q}_{A//Q}(Z_p, \operatorname{Ext}^t_Q(Z_p, Z_p))$ the type $\tau = t$ and $\tau(d_r y) = \tau(y)$ for $r \geq 1$.

This implies the isomorphism

$$\operatorname{Ext}^{m,q}_A(M, Z_p) = \sum_{s+t=m} \operatorname{Ext}^{s,q}_{A//Q}(M, \operatorname{Ext}^t_Q(Z_p, Z_p))$$

and $\tilde{E}_2 = \tilde{E}_\infty$. The theorem is proved.

From the proof of Theorem 12.1 follows

Corollary 12.2. *The second term of the "algebraic Adams spectral sequence" $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ of Corollary 12.1 is canonically isomorphic to the sum $\sum \tilde{\tilde{E}}_2^{s,t,q}$, where $\tilde{\tilde{E}}_2^{s,t,q} = \operatorname{Ext}^{s,t,q}_A(M, Z_p)$, t is the Cartan type, $M = H^*(Y, Z_p)$ for $Y \in D_p$, and $\sum\limits_{t+s=m} \operatorname{Ext}^{s,t,q}_A(M, Z_p) = \operatorname{Ext}^{m,q}_A(M, Z_p)$.*

In this spectral sequence

$$\tilde{\tilde{d}}_r \colon \tilde{\tilde{E}}_r^{s,t,q} \to \tilde{\tilde{E}}_r^{s+1,t+r-1,q+r-1}$$

and the group $\sum\limits_{t-q=l} \tilde{\tilde{E}}_\infty^{r,t,q}$ is associated to $\operatorname{Ext}^{s,l}_{A^U_p}(U^*_p(Y), U^*_p(P))$.

From the geometric realization of double complexes as defined above, Theorem 12.1 and Corollaries 12.1, 12.2, there follows

Theorem 12.2. *The "algebraic Adams spectral sequence" $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ is associated to the Adams spectral sequence (E_r, d_r) in $H^*(\ , Z_p)$-cohomology theory for all $p > 2$ in the following sense:*

1) $E_2^{m,q} = \sum\limits_{s+t=m} \tilde{\tilde{E}}_2^{s,t,q} = \operatorname{Ext}^{m,q}_A(M, Z_p)$;

2) *if for some $y \in E_2^{s,t,q}$ we have $\tilde{\tilde{d}}_i(y) = 0$ for $i < k$ and $\tilde{\tilde{d}}_k(y) \neq 0$, then there is a $\tilde{y}$ such that $\phi(y-\tilde{y}) \geq \phi(\tilde{y})+1$, $d_i(\tilde{y}) = 0$ for $i < k$, and $d_k(\tilde{y}) \neq 0$, and moreover $\phi(d_k\tilde{y}) = \phi(\tilde{y})+1$, where $\phi(\tilde{y}) = \phi(y) = t$ and $\phi(d_k\tilde{y} - d_k y) > \phi(\tilde{y})+1$;*

3) *if $\tilde{y} \in \operatorname{Ext}^{m,q}_A(M, Z_p)$ is such that $d_i(\tilde{y}) = 0$ for $i < k$ and $\phi(d_k\tilde{y}) > \phi(\tilde{y})+1$, then for the projection y of the element $\tilde{y}$ in $\operatorname{Ext}^{m-\phi(\tilde{y}),\phi(\tilde{y}),q}_A(M, Z_p)$ we have the equation $\tilde{\tilde{d}}_i(y) = 0$ for $i \leq k$ (we note that for elements $y \in \sum\limits_{t \geq a} \operatorname{Ext}^{s,t,q}_A(M, Z_p), \varphi(y) \geq \alpha$).*

The groups $\operatorname{Ext}^{1,s}_{A^U_p}(U^*_p(P), U^*_p(P))$ were computed in previous sections; they are cyclic for $s = 2k(p-1)$ of order $P^{f(k)}$, where $f(k)-1$ is the exponent of the greatest power of p which divides k.

Corollary 12.3. *The generator α_k of the group $\operatorname{Ext}^{1,2k(p-1)}_{A^U_p}(\Lambda_p, \Lambda_p)$ has filtration $(1, k-f(k))$ or, in other words, $\phi(\alpha_k) = k - f(k)$ in the term E_∞ of the "algebraic Adams spectral sequence" $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ for $p > 2$. Since $\operatorname{Ext}^{1,*}_{A^U_p}(\Lambda_p, \Lambda_p)$ consists of cycles for all Adams differentials in U^*_p-theory, $d_i(\alpha_k) = 0$, $i \geq 2$, and there is an* *associated element* $\tilde{\alpha}_k \in \pi_*(S^n)$; *we have* $\phi(\tilde{\alpha}_k) = k - f(k)$, $i_U(\tilde{\alpha}_k) = 1$, $i_H(\tilde{\alpha}_k) \leq k - f(k) + 1$.

Proof. As was shown in § 11, the homomorphism $\operatorname{Ext}^1_{D_1}(\Lambda_{p,1}, \Lambda_{p,1}) \to \operatorname{Ext}^1_{A^U_p}(\Lambda_p, \Lambda_p)$ is an epimorphism for $p > 2$. For the ring D_1 and the module $\Lambda_{p,1} = Q_p[x_1]$ the ring $C = \operatorname{Hom}^*(F, \Lambda_{p,1})$ was determined (see Lemma 11.4), where $\phi(x) = 1$, $\phi(h_j) = 0$, $\phi(p) = 1$ and $d(x^k) \sum \binom{k}{j} p^j x^{k-j} h_j$. The element α_k was represented by $\alpha_k = (1/p^{f(k)}) d(x^k)$.

From this we have:

$$\varphi(\alpha_k) = \min j \left[\varphi \binom{k}{j} + j + k - j - f(k) \right] = k - f(k)$$

Thus, the filtration ϕ of the element α_k is equal to $k - f(k)$, since the filtration ϕ is induced by the filtration in the ring Λ_p. The Corollary is proved. □

As is known, the groups $\operatorname{Ext}^{1,s}_A(Z_p, Z_p)$ are equal to Z_p for $s = 1$ or $s = 2p^j(p-1)$ and are generated by elements u_j, $j \geq 0$, of type 0 for $s = 2p^j(p-1)$ and $h_0 \in \operatorname{Ext}^{0,1,1}_A$ of type 1 in the sense of Cartan.

Hence, $u_i \in \operatorname{Ext}^{1,0,2p^i(p-1)}_A(Z_p, Z_p)$ and $h_0 \in \operatorname{Ext}^{0,1,1}_A(Z_p, Z_p)$, where $\operatorname{Ext}^{m,q}_A = \sum\limits_{s+t=m} \operatorname{Ext}^{s,t,q}_A$ and t is the type. In the groups $\operatorname{Ext}^{2,2p^i(p-1)}_A$ there are nonzero elements $y_i, i \geq 1$, having type 0.

Corollary 12.4. *In the "algebraic Adams spectral sequence" we have the equation* $\tilde{\tilde{d}}_2(u_i) = h_0 \gamma_i$, *for* $i \geq 1$.

The proof, by analogy with the proof of Corollary 12.3, follows easily from the structure of the homomorphism β in $H(C \otimes Z_p)$, where $\beta(h_p i) = \gamma_i$ for $i \geq 1$ (see Lemma 11.4).

Thus, we see that with the help of the "algebraic Adams spectral sequence" it is not only possible to prove the absence of elements with Hopf–Steenrod invariant 1, but also to compute (ordinary) Adams differentials by purely algebraic methods which come from the ring A^U.

Conjecture. For $p > 2$ the "algebraic Adams spectral sequence," which converges to $\operatorname{Ext}_{A^U}(U^*(P), U^*(P)) \otimes_Z Q_p$, coincides with the "real" Adams spectral sequence, and the homotopy groups of spheres $\pi_*(S^n) \otimes_Z Q_p$ are associated to $\operatorname{Ext}_{A^U}(U^*(P), U^*(P)) \otimes_Z Q_p$. {Equivalently: all differentials d_r, $r \geq 2$, are zero in the Adams spectral sequence over U^*_p.}

We now consider $p = 2$. As was indicated earlier, here there are two spectral sequences $(\tilde{E}_r, \tilde{d}_r)$ and $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$, where $\tilde{\tilde{E}}_2 = \tilde{E}_2 = \operatorname{Ext}_{A//Q}(M, \bar{\Lambda}_2)$, $M = H^*(X, Z_2)$, and $\bar{\Lambda}_2 = \operatorname{Ext}^{**}_Q(Z_2, Z_2)$ is associated to $U^*_2(P) = \Lambda_2$. The sequence $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ converges to $\operatorname{Ext}_{A^U_2}(U^*_2(X), \Lambda_2)$ and $(\tilde{E}_r, \tilde{d}_r)$ converges to $\operatorname{Ext}_A(M, Z_2)$.

By analogy with Theorem 12.2 for $p > 2$, here we have

Theorem 12.3. *The differentials* $\tilde{d}_r$ *are associated with the Adams differentials in Cobordism theory on the group* $\tilde{\tilde{E}}_\infty$ *associated with* $\operatorname{Ext}_{A^U_2}(U^*_2(X), \Lambda_2)$. *The differentials* $\tilde{\tilde{d}}_r$ *are associated with the Adams differentials in* $H^*(\ , Z_2)$*-theory on the groups* $\tilde{E}_\infty$ *associated with* $\operatorname{Ext}_A(H^*(X, Z_2), Z_2)$, *where* $X \in D$.

The proof of Theorem 12.3, as of 12.2, follows immediately from the properties of the geometric realization of the double complex.

Thus, for $p = 2$, it is possible to compute the Adams differentials in $H^*(\ ,Z_2)$-theory, starting from cobordism, and conversely.

Question. Do the algebraic Adams spectral sequences $\tilde{E}_r$ and $\tilde{\tilde{E}}_r$ define the real Adams spectral sequences in both theories?

In any case in all examples known to the author all Adams differentials are subsumed under this scheme.

Example. Let $X = MSU$. We consider $\mathrm{Ext}_{A//Q}(M, \bar{\Lambda})$, where $M = H^*(X, Z_2)$. We write an $A//Q$-resolution of the module M:

$$(\cdots \to C_i \xrightarrow{d} C_{i-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} M) = C.$$

We recall that $M = F + \sum_\omega M_\omega$, where F is $A//Q$-free and $M\omega$ has one generator u_ω for all $\omega = (4k_1, \dots, 4k_s)$ and is given by the relations $\mathrm{Sq}^2 u_\omega = 0$ over $A//Q$, where $\dim u_\omega = 8 \sum k_j$. Hence one can assume that $C = C(F) + \sum C(M_\omega)$, where $C(F) = F$ and $C(M_\omega)$ has the form:

$$C(M_\omega) = (\to \dots \xrightarrow{d} A//Q \xrightarrow{d} A//Q \to \dots \xrightarrow{d} A//Q \xrightarrow{\varepsilon} M_\omega),$$

where u_i is a generator of $C_i(M_\omega)$ and $du_i = \mathrm{Sq}^2 u_{i-1}$. The action of Sq^2 on $\bar{\Lambda}_2$ was indicated earlier: $\bar{\Lambda}_2 = Z_2[h_0, \dots, h_i, \dots]$, $\dim h_i = (1, 2^{i+1} - 1)$, $i \geq 0$, while $\mathrm{Sq}^2 h_1 = h_0$.

There follows straightforwardly (by direct calculation)

Lemma 12.5. $\mathrm{Ext}^{***}_{A//Q}(M_\omega, \bar{\Lambda}_2)$ *for* $\omega = (0)$ *has a system of multiplicative generators:*

$$h_0 \in \mathrm{Ext}^{0,1,1}, \quad x_1 \in \mathrm{Ext}^{1,0,2}, \quad h_i \in \mathrm{Ext}^{0,1,2^{i+1}-1}, \quad i \geq 2, \quad y \in \mathrm{Ext}^{0,2,6}$$

and is given by the relation $h_0 x_1 = 0$.

We note that the dimension of $\mathrm{Ext}^{s,t,q}$ in $H^*(\ ,Z_2)$ is equal to $(s+t, q)$ and the dimension in U_2^*-theory is equal to $(s, q-t)$ (see above).

We now describe the spectral sequences $\tilde{E}_r \searrow \mathrm{Ext}_A$ and $\tilde{\tilde{E}}_r \searrow \mathrm{Ext}_{A_2^U}$.

Lemma 12.6. a) *The spectral sequence* $(\tilde{E}_r, \tilde{d}_r)$ *is such that*:

$$\tilde{d}_3(y) = x_1^3, \quad \tilde{d}_3(h_0) = \tilde{d}_3(x_1) = \tilde{d}_3(h_i) = \tilde{d}_3(v_\omega = 0),$$
$$\tilde{d}_3 | \mathrm{Hom}_{A//Q}(F, \bar{\Lambda}_2) = 0$$

and all $\tilde{d}_r = 0$ *for* $r = 3$.

b) *The spectral sequence* $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$ *is such that*:

$$\tilde{\tilde{d}}_2(v_{(2^i)}) = x_1 h_{i+2}, \quad i \geq 0,$$
$$\tilde{\tilde{d}}_2(x_1) = \tilde{\tilde{d}}_2(h_i) = \tilde{\tilde{d}}_2(y) = \tilde{\tilde{d}}_2(v_{(k)}) = 0, \quad k \neq 2^i,$$
$$\tilde{\tilde{d}}_2(\mathrm{Hom}_{A//Q}(F, \Lambda_2)) = 0,$$
$$M = F + \sum_\omega M_\omega,$$

where F is $A//Q$-free, $\tilde{\tilde{d}}_r = 0$ for $r \leq 2$ (we note that v_ω is conjugate to the generator u_ω of the module M_ω, $\omega = (k_1, \dots, k_s)$, $\dim u_\omega = 8(\sum k_j)$ and $v_\omega v_{\omega_1} = v_{(\omega,\omega_1)}$ by virtue of the diagonal in the module M).

The proof of Lemma 12.6 for $\tilde{E}_r$ follows easily from the calculations [18] for $\mathrm{Ext}_A(M, Z_2)$. For the case $(\tilde{\tilde{E}}_r, \tilde{\tilde{d}}_r)$, part (b) of Lemma 12.6 follows from the fact that the elements x, h_{i+2} must be zero in $\mathrm{Ext}_{A_2^U}(U_2^*(U_2^*(MSU), \Lambda_2)$ on the basis of § 7.

Corollary 12.5. *For MSU, the Adams spectral sequence (in U-cobordlsm and $H^*(\ , Z_2)$-theory) determined by the algebraic spectral sequences $\tilde{E}_r$ and $\tilde{\tilde{E}}_r$.*

In analogous fashion it can be shown that all known Adams differentials for $X = P$ in both homology theories (the case of the homotopy groups of spheres) are also determined by $\tilde{E}_r$, $\tilde{\tilde{E}}_r$ and $\tilde{d}_r$, $\tilde{\tilde{d}}_r$.

By analogy with the case $p > 2$, bounds can be determined here also for the filtrations of elements $\mathrm{Ext}^1_{A^U}$ (see Corollary 12.3).

Appendix 1. On the formal group of "geometric" cobordism (Theorem of A. S. Miščenko)

We consider an arbitrary complex X, the group $U^*(X)$ and its subgroup $\mathrm{Map}(X, MU_1) \subset U^2(X)$. In what follows we shall denote $\mathrm{Map}(X, MU_1)$ by $V(X)$. Since $MU_1 = CP^\infty$ is an H-space, $V(X)$ becomes a group, which is communicative, and with respect to this law of multiplication we obviously have:

$$V(X) \approx H^2(X, Z).$$

How is this multiplication in $V(X)$ connected with operations in $U^*(X) \supset V(X)$?

As was already indicated in § 5, we have

Lemma 1. a) *If $u, v \in V(X)$ and $\oplus$ is the product in $V(X)$, then the law of multiplication $u \oplus v = f(u, v)$ has the form*

$$u \otimes v = u + v + \sum_{\substack{i \geq 1 \\ j \geq 1}} x_{i,j} u^i v^i,$$

where $x_{ij} \in \Lambda^{-1(i+j-1)} = \Omega_U^{2(i+j-1)}$ are coefficients independent of u, v,

b) *$u \oplus v = v \oplus u$,*

c) *$(u \oplus v) \oplus w = u \oplus (v \oplus w)$,*

d) *there exists an inverse element $\bar{u}$, where $\bar{u} \oplus u = 0$.*

The proof of this lemma follows in an obvious way from the fact that $V(X) \approx H^2(X, Z)$ and the possibility of computing all the coefficients on the universal example $X = CP^\infty$. We note that $x_{1,1} = [CP^1]$.

Thus, we have a commutative formal group with graded ring of coefficients Λ, and $\dim u = \dim v = 2$. As is known, the structure of such a group is completely determined by a change of variables g over the ring $\Lambda \otimes_Z Q, u \to g(u) = \sum_{i \geq 0} y_i u^{i+1}$, $y_0 = 1$, such that

$$g(u \oplus u) = g(u) + g(v).$$

We have the following

Theorem (A. S. Miščenko)**.** *The change of variables $u \to g(u)$, where $g(u) = \sum_{n\geq 0} \frac{x_n}{n+1} u^{n+1}$, $x_n = [CP^n] \in \Lambda^{-2n}$, reduces the formal group $V \otimes_Z Q$ to linear form $g(u \oplus v) = g(u) + g(v)$. Hence, the change $u \to g(u)$ reduces to linear form the formal group $V(X) \otimes Q$ for all X and uniquely determines the structure of the one-dimensional formal group V over the ring Λ.*

Proof. We consider the ring $U^*(CP^\infty) = \Lambda[[u]]$ and the multiplication $CP^\infty \times CP^\infty \to CP^\infty$, sending the one-dimensional canonical U_1-bundle ξ over CP^∞ into $\xi_1 \otimes \xi_2$, where ξ_1, ξ_2 are canonical bundles over $CP \times CP$. This multiplication induces a diagonal $\Delta\colon U^*(CP^\infty) \to U^*(CP^\infty) \otimes_\Lambda U^*(CP^\infty)$, which gives the multiplication in $V(CP^\infty)$.

Let $u' = g(u) \sum \lambda_i u^i$, where $\Delta(u') = u' \otimes 1 + 1 \otimes u'$. Then g is the desired change of variables.

We compute the coefficients λ_i. Let $S_{(k)} \in A^U$ (see § 5).

We have the easy

Lemma 2. *The operations $S_{(k)}$ form a system of multiplicative generators for the ring $S \otimes Q$. If $\sigma_k^*(x) = 0$ for all $k, x \in \Lambda$, then $x = 0$.*

Proof. We order the partitions ω naturally (by length) and consider

$$S_{(k)} S_\omega(u_1 \dots u_n) = S_{(k)} \sum u_i^{k_1+1} \dots u_s^{k_s+1} u_{s+1} \circ \dots \circ u_n,$$
$$\omega = (k_1, \dots, k_s),$$
$$S_{(k)} S_\omega(u_1 \dots u_n) = \sum a_i S_{\omega_i}(u_1 \dots u_n) + a_0 S_{(k,\omega)}(u_1 \dots u_n),$$

where $a_0 \neq 0$, $\omega_i = (k_1, \dots, k_i + k, k_{i+1}, \dots, k_s)$. Since by the induction hypothesis all S_{ω_i} can be expressed by the $S_{(k_j)}$, the same is true for $S_{(\omega,k)}$. Since all S_ω can be expressed by the $S_{(k)}$, the lemma is proved. $\square$

We note the following equation:

$$S_{(k)} u^i = \sum_i S_{(k)}(\lambda_i u^i) = \sum_i (\sigma_k^*(\lambda_i) u^i + i \lambda_i u^{i+k})$$
$$= \sum_i (\sigma_{(k)}^*(\lambda_i) + (i-k)\lambda_{i-k}) u^i.$$

We set

$$u'^k = \sum_i \lambda_i^{(k)} u^i, \quad u^k = \sum_i \mu_i^{(k)} u'^i, \quad \sum_i \lambda_i^{(k)} \mu_j^{(i)} = \delta_j^k.$$

Obviously, $S_{(k)} \Delta u' = \Delta S_{(k)} u'$, since $\Delta u' = u' \otimes 1 + 1 \otimes u'$. Since

$$S_{(k)} u' = \sum_i (\sigma_{(k)}^*(\lambda_i) + (i-k)\lambda_{i-k}) u^i = \sum_i \sum_j (\sigma_{(k)}^*(\lambda_i) + (i-k)\lambda_{i-k}) \mu_j^{(i)} u'^j,$$

we have

$$\Delta S_{(k)}u' = S_{(k)}\Delta u' = \sum_i \sum_j (\sigma^*_{(k)}(\lambda_i) + (i-k)\lambda_{i-k})\mu_j^{(i)}(u'\otimes 1 + 1\otimes u')^j$$

$$= \sum_j \sum_i \left[(\sigma^*_{(k)}(\lambda_i) + (i-k)\lambda_{i-k})\mu_j^{(i)} \left(\sum_{\alpha+\beta=j} \binom{\alpha+\beta}{\alpha} u'^{\alpha}\otimes u'^{\beta} \right)\right]$$
$$= S_{(k)}(u'\otimes 1 + 1\otimes u').$$

It obviously follows that for $\alpha \neq 0$, $\beta \neq 0$ we have:

$$\sum_i (\sigma^*_{(k)}\lambda_i + (i-k)\lambda_{i-k})\mu_j^{(i)} = 0, \quad j = \alpha+\beta \geq 2.$$

Since $\mu_1^{(i)} = 0$ for all $i \geq 2$, $\mu_1^{(i)} = 1$, $\lambda_1 = 1$ and $\sigma^*_{(k)} + (i-k)\lambda_{i-k} = 0$, $k \geq 1$, we have

$$\sum_i (\sigma^*_{(k)}\lambda_i + (i-k)\lambda_{i-k})\mu_j^{(i)} = 0$$

for all $j \geq 1$, and since $\sum_j \mu_j^{(i)}\lambda_s^{(j)} = \delta_j^i$, we have

$$\sum_j \sum_i (\sigma^*_{(k)}\lambda_i + (i-k)\lambda_{i-k})\mu_j^{(i)}\lambda_s^{(j)} = \sigma^*_{(k)}\lambda_i + (i-k)\lambda_{i-k} = 0.$$

Hence,

$$\sigma^*_{(k)}\lambda_i = -(i-k)\lambda_{i-k}.$$

Further, since $\sigma^*_{(k)}[CP^n] = -(n+1)[CP^{n-k}]$ (see § 5, Lemma 5), it follows that $\bar{\lambda}_i = x_{i-1}/i$, $x_j = [CP^j] \in \Lambda^{-2j}$ satisfies the condition $\sigma^*_{(k)}\bar{\lambda}_i = -(i-k)\bar{\lambda}_{i-k}$ for all i, k. By Lemma 2, $\bar{\lambda}_i = \lambda_i$, and the theorem is proved. □

Remark. For a quasicomplex manifold X, the group $V(X)$ is isomorphic to $H_{2n-2}(X)$ and the meaning of the sum $u \oplus v$ is such that the homology class $\nu(u)\nu(v)$ is realized by the inclusion of the submanifold $V_1 \otimes V_2$, where $u \in U_{2n-2}(X)$, $v \in U_{2n-2}$ are realized by the submanifolds $V_1, V_2 \subset X$. Then the series

$$u \oplus v = u + v + \cdots = f(u, v)$$

must be considered in the intersection ring $U_*(X)$.

Appendix 2. On analogues of the Adams operations in U^*-theory

Analogues of the Adams operations $\Psi_U^k \in A^U \otimes_Z Z[(1/k)]$ were defined in § 5 in the following way:

a) $\Psi_U^k(xy) = \Psi_U^k(x)\Psi_U^k(\gamma)$,

b) $k\Psi_U^k(x) = x \oplus \cdots \oplus x$ (k times), where $x \in V(X)$.

Thus, the series Ψ_U^k has the form:

$$k\Psi_U^k(x) = g^{-1}(kg(x)) = f(x, f(x, \ldots, f(x,x), \ldots)),$$

where $f(u, v)$ is the law of addition in the formal group $V(X)$ and

$$g(x) = \sum_{k\geq 0} \frac{x_k}{k+1} x^{k+1}, \quad x_j = [CP^j], \quad x \in V(X),$$

the basis of Appendix 1, $g^{-1}(g(x)) = x$.

From the associativity of the law of multiplication in $V(X)$ follows the equation:

$$\Psi_U^k(\Psi_U^l(x)) = \Psi_U^{kl}(x).$$

Hence, always $\Psi_U^k \circ \Psi_U^l = \Psi_U^{kl}$ in $A^U \otimes_Z Q$, since for any $n \to \infty$ and $u = u_1 \dots u_n$ we have $\Psi_U^k \Psi_U^k(u) = \Psi_U^{kl}$ by virtue of properties (a) and (b).

Of the assertions in Lemma 5.8, only part (d) is nontrivial, and it asserts that $\Psi_U^{k,*}(y) = k^i y$, $y \in \Lambda^{-2i} = \Omega_U^{2i}$.

Theorem 1. [5] *If $a \in A^U$ is an arbitrary cohomology operation of dimension $2m$, then we have the following commutation law*:

$$a\Psi_U^k = k^m \Psi_U^k \circ a.$$

Proof. Let $a_m = S(m) \in A^U$ and $u \in V(CP^\infty) \subset U^2(CP^\infty)$. Then

$$a_m(u) = u^{m+1},$$

$$\begin{aligned}\Psi_U^k(a_m u) = \Psi_U^k(u^{m+1}) = \Psi_U^k(u)^{m+1} &= \frac{1}{k^{m+1}}(u \oplus \dots \oplus u)^{m+1} \\ = \frac{1}{k^m} a_m \left(\frac{u \oplus \dots \oplus u}{k}\right) &= \frac{1}{k^m} a_m \Psi_U^k(u),\end{aligned}$$

since $u \oplus \dots \oplus u \in V(CP^\infty)$. Hence, for the operations $a_{(m)} = S_{(m)}$ the theorem is proved. From this Theorem 1 follows for all operations $S_{(\omega)}$, since by Lemma 2 of Appendix 1 the ring $S \otimes_Z Q$ is generated by the operations $S_{(k)}$.

Now let $a \in \Lambda^{-2m} = U^{-2m}(P)$. We assume by induction that for all operations in Λ^{-2j}, $j < m$, the theorem is proved. This means that for $b \in \Lambda^{-2j}$, $j < m$, we have:

$$\Psi_U^{k,*}(b) = k^j b.$$

In view of the fact that $\Psi_U^{k,*}(b_1 b_2) = \Psi_U^{k,*}(b_1)\Psi_U^{k,*}(b_2)$, the theorem is also proved for all decomposable elements of Λ^{-2m}. Let $a \in \Lambda^{-2m}$ be an indecomposable element. We consider $\Psi_U^{k,*}\sigma_\omega^*(a) = k^{m-\dim\omega}\sigma_\omega^*(a)$ by induction, for $\omega \neq (0)$. Since

$$\Psi_U^k S_\omega = k^{-\dim\omega} S_\omega \Psi_U^k,$$

we have

$$\Psi_U^{k,*}\sigma_\omega^*(a) = \sigma_\omega^*(k^m a).$$

Hence, $\Psi_U^{k,*}(a) = k^m a$, since $\bigcap\limits_{\omega>0} \operatorname{Ker} \sigma_\omega^* = 0$. □

Since Theorem 1 is proved for Λ and S, it is also proved for $A^U = (\Lambda S)^\wedge$.

Thus, all assertions of Lemma 5.8 are proved.

We now consider an arbitrary ring K, the group of units $U_k \subset K$ and $A^U \otimes_Z K$. We define the following semigroups in $A^U \otimes_Z K$:

1. The semigroup of multiplicative operations $a \in A^U \otimes_Z K$, where $\Delta a = a \otimes a \in A^U \otimes_\Lambda A^U \otimes_Z K$.
2. The semigroup of multiplicative operations of dimension 0,

$$A_K^0 \subset A_K \subset A^U \otimes_Z K.$$

3. The center $Z_K \subset A_K^0$ of the semigroup A_K.

[5]From Theorem 1 it follows easily that all operations Ψ_U^k are well-defined over the integers on $U^0(X)$, as in K-theory.

4. The "Adams operations" $\Psi_U^q \in A_K^0$, where $q \in U_k$ (the group of units), defined by the requirements of Theorem 1:

$$\Psi_U^q(a) = q^{-m} a \Psi_U^q, \quad \dim a = 2m,$$
$$\Psi_U^{q_1}\Psi_U^{q_2} = \Psi_U^{q_1 q_2}.$$

Just as earlier, a multiplicative operation $a \in A_K$ is defined by a series $a(u), u \in U^2(CP^\infty) \otimes_Z K$ the canonical element, $a(u) \in \Lambda_K[[u]]$, $\Lambda_K = \Lambda \otimes_Z K$.

We now consider the question of defining the Adams operation. Let $K = Q[t]$, $\Lambda_K = \Lambda \otimes_Z K$. We consider for all integral values t the series $t\Psi_U^t(u) \in U^*(CP^\infty)$, defining the series $t\Psi_U(u) \in U^*(CP^\infty) \otimes_Z K$.

Remark. If K is an algebra over Q, then the Adams operations $\Psi_U^\alpha \in A^U \otimes_Z K$ are always defined, since the series $t\Psi_U^t$ is divisible by t and $\Psi_U^t(u) \in U^*(CP) \otimes_Z K$.

We have the following

Theorem 2. a) *For any algebra K over Q without zero divisors and for $K = Q_p, Z$, the "Adams operations" $\Psi_U^\alpha \in A^U \otimes_Z K$ are defined, where $\alpha \in K^*$ in the Q-algebra case and $\alpha \in U_p$ in the case $K = Q_p$ $\{$i.e., $U_p = U_{Q_p}\}$, $\alpha = \pm 1$ in the case $K = Z$, such that*:

1) $\Psi_U^{\alpha_1}\Psi_U^{\alpha_2} = \Psi_U^{\alpha_1\alpha_2}$.

2) $\Psi_U^{\alpha,*} \colon \Lambda_K^{-2i} \to \Lambda_K^{-2i}$ *is multiplication by* α^i.

3) $\Psi_U^\alpha \circ a = \alpha^{-i} a \Psi_U^\alpha$, *where $a \in A^U \otimes_Z K$ is of dimension* $2i$.

4) *The series $\alpha\Psi_U^\alpha(u)$ for $u \in V(CP^\infty)$ makes the operation of raising to the power α, $\alpha \in K^*$, well-defined in the formal group V.*

b) *The collection of all Adams operations forms a semigroup $K^* \approx \Psi(K)$ for a Q-algebra K, $\Psi(K) \approx U_p$ for $K = Q_p, \Psi(Z) = Z_2$, which coincides precisely with the center Z_K of the semigroup A_K^0 of multiplicative operations of dimension 0 in the ring $A_U \otimes K$ for $K = Q_p, Z$, while for a Q-algebra K the center consists of $\Psi(K)$ and the operator Φ, where $\Phi(u) = g(u)$.*

Remark. Although $a \in A_K$ is such that $\Delta a = a \otimes a$ and is given by a formal series beginning with 1, where $a(u) = u + \dots$, still the coefficients of the series lie in Λ or $\Lambda \otimes K$, while the law of super-position of series $a_1 \cdot a_2(u)$ takes into account the representation of $A^U \otimes K$ on $\Lambda \otimes K$. Hence A_K is not a group (as usual in formal series of this kind), but a semigroup. An example of a "noninvertible" element $a \in A^0 + K$ is given by the series

$$\Phi(u) = \sum \frac{[CP^i]}{i+1} u^{i+1} = g(u),$$

where $\Phi^2 = \Phi$ and $\Phi^*(y) = 0$, $y \in \Lambda^{2j}$ for $j > 0$.

We prove Theorem 2. Part (a) was essentially already proved above. In order to establish that $\Psi(K) = Z_K$, we consider an arbitrary element $a \in Z_K$ and we shall show that $a \in \Psi(K)$. Since the series $a(u) = u + \dots$, we have $a^*|\Lambda^0 = 1$ and $a^*|\Lambda^2$ is multiplication by a number $\alpha \in K$. If $a^*|\Lambda^{-2j} = 0$ for all $j > 0$, then it follows that $a^* = \Phi^*$ and hence $a = \Phi$, while $\Phi \notin A^U \otimes_Z Q_p$. It will be assumed that for some j, $a^*|\Lambda^{-2j} \neq 0$, $j > 0$. If $a^*|\Lambda^{-2j}$ is the operator of multiplication by a number k_j, then it is easy to see that $k_j = k_1^j$ and $a = \Psi_U^{k_1}$, where $k_1 \in K^*$ or $k_1 \in U_p$. We shall show that for all j the operator a^* is multiplication by a number k_j. If j_0 is the first number for which $a^*|\Lambda^{-2j_0}$ is not multiplication by a number, then, nevertheless, on the decomposable elements $\bar{\Lambda}^{-2j_0} \subset \Lambda^{-2j_0}, a^*$ is multiplication by a number in view of the fact that $a^*(xy) = a^*(x)a^*(y)$. If $y \in \Lambda^{-2j_0}$ is an indecomposable element then $a^*(y) = \lambda y + \bar{y}$, $\bar{y} \subset \tilde{\Lambda}$ and $\bar{y} = 0$. Let $b \in A_K^0$ be such that $b^*(y) = \mu y + \bar{\bar{y}}$, where $\bar{\bar{y}} \in \Lambda$, $\bar{\bar{y}} \neq 0$. Then $b^*a^* \neq a^*b^*$ on Λ^{-2j_0}, which is impossible. The theorem is proved.

Appendix 3. Cell complexes in extraordinary cohomology theory, U-cobordism and k-theory

Let X be a homology theory with a multiplicative stable spectrum, $X \otimes X^* \to X$, and let $\Lambda^* = X^*(P)$ be the cohomology ring of a point. We require that Λ^* be a ring with identity. We note that $\Lambda = X_*(P)$ is also a ring, and we have the formulas $\Lambda = \mathrm{Hom}^*_{\Lambda^*}(\Lambda^*, \Lambda^*)$ and $\Lambda^* = \mathrm{Hom}^*_{\Lambda}(\Lambda, \Lambda)$. Obviously, the rings Λ and Λ^* are isomorphic and $\Lambda^i = \Lambda^{*-i}$, $\Lambda^i = 0$, $i < 0$.

Let K be a cell complex and $K^i \subset K$ be its skeleton of dimension i. We construct a "cell complex of Λ-modules" $S_X(K)$:

a) if $\dim K = 0$, then $S_X(K)$ is a free complex $\sum_P \Lambda(P)$, where the P are the vertices of K and $\Lambda(P)$ is a one-dimensional free module with generator u_P: we set $\partial u_P = 0$.

b) Suppose that for all K^j, $j < i$, $S_X(K^j)$ has been constructed so that $\partial\lambda = \lambda\partial$, $\lambda \in \Lambda$, and the generators of $S_X(K^j)$ are in one-one correspondence with the cells of K^j.

We consider the pair (K^j, K^{i-1}), where K^i/K^{i-1} is a bouquet of spheres $S_1^i \vee \cdots \vee S_{q_i}^i$. We adjoin to $S_X(K^{i-1})$ free generators $u_1, \dots, u_{q_i}$ of dimension i. A differential in the complex $S_X(K^{i-1}) + \Lambda(u_1) + \cdots + \Lambda(u_{q_i})$ is introduced as follows:

1) $\partial\lambda = \lambda\delta$, $\lambda \in \Lambda$;

2) $\partial u_j = z_j \in S_X(K^{i-1})$, where z_j is such that $\partial z_j = 0$ in $S_X(K^{i-1})$ and the homology class $[z_j] \in X_*(K^{i-1})$ is represented by the element equal to ∂u_j, where $\partial\colon X_*(S_1^i \vee \cdots \vee S_{q_i}^i) \to X_*(K^{i-1})$ is the boundary homomorphism of the pair (K^i, K^{i-1}) and $u_j \in X_*(K^i/K^{i-1})$ corresponds to the sphere S_j^i.

Thus, a complex $S_X(K)$ of free modules arises.

Lemma 1. *The complex $S_X(K)$ is uniquely defined up to the choice of the system of generators, and the differential ∂ in S_X coincides up to higher filtration with the homology one. Obviously, $H(S_X(K), \partial) = X_*(K)$ as Λ-modules.*

For a cellular map $Y_1 \to Y_2$, there is defined analogously a morphism of free complexes $S_X(Y_2) \to S_X(Y_2)$, also unique.

Let $Y = Y_1 \times Y_2$ with the natural cellular subdivision.

Question. When is there defined a pairing

$$S_X(Y_1) \otimes_\Lambda S_X(Y_2) \to S_X(Y_1 \times Y_2),$$

which is an isomorphism of complexes?

Now let $X = U$.

Conjecture. *For a pair Y_1, Y_2, the complex $S_U(Y_1 \times Y_2)$ is homo topic ally equivalent to the tensor product*

$$S_U(Y_1) \otimes_{\Omega_U} S_U(Y_2).$$

Let A be an arbitrary Ω_U-module. The homology of the complex $S_U \otimes \Omega_U A$ we shall denote by $U_*(Y, A)$, and the homology of the complex $\operatorname{Hom}^*_{\Omega_U}(S_U, A)$ by $U^*(Y, A)$ (cohomology with coefficients in A).

We shall indicate important examples:

1. $A = \Omega_U$ is a one-dimensional free module.

2. $A = Z[x]$, $\dim x = 2$, and the action of Ω_U on A is such that $y(x^m) = T(y)x^{m+i}$, where $y \in \Omega_U$, $2i = \dim y$, and $T(y)$ is the Todd genus. Here A is a ring and there is defined a homomorphism $\lambda\colon \Omega_U \to Z[x]$, such that $\lambda(y) = T(y)x^i$.

3. $A = Z[x, x^{-1}]$, where $\dim x = 2$, $\dim x^{-1} = -2$ and $xx^{-1} = 1$. Here A is a ring, while Ω_U acts on A just as in example 2: $y(x^m) = T(y)x^{m+i}$, $-\infty < m < \infty$.

4. $A = Z$, where $Z = \Omega_U/\Omega_U^+, \Omega_U^+$ is the kernel of the augmentation $\Omega_U \to Z$ and the action of Ω_U on Z is the natural one.

Conjecture. *For the Ω_U-modules $A = \Omega_U, Z[x], Z[x, x_{-1}], Z$, the corresponding cohomology groups $U^*(\ , A)$ are isomorphic, respectively, to the cobordism theory U^* for $A = \Omega_U$, to stable k^*-theory for $A = Z[x]$, to unstable K-theory K^* for $A = Z[x, x^{-1}]$ and to the theory $H^*(\ , Z)$ for $A = Z$. The homology theories $U_*(\ , A)$ for $A = \Omega_U$, $Z[x]$, $Z[x, x^{-1}]$, Z, are isomorphic, respectively, to U_*, k_*, K_* and $H_*(\ , Z)$.*

Theorem 1. *Since the complex $(S_U(Y), \partial)$ is a complex of free Ω_U-modules, there exists a spectral sequence with term $E_2 = \operatorname{Ext}^{**}_{\Omega_U}(U_*(Y), A)$ which converges to $U^*(Y, A)$, and there exists a spectral sequence with term $E_2 = \operatorname{Tor}^{**}_{\Omega_U}(U_*(Y), A)$ which converges to $U_*(Y, A)$.*

Theorem 2. *Since the complexes $S_U(Y) \otimes_{\Omega_U} Z[x] = S_k(Y)$ and $S_U(Y) \otimes_{\Omega_U} Z[x, x^{-1}] = S_k(Y)$ are complexes of free A-modules for $A = Z[x]$, $Z[x, x^{-1}]$, and the ring $Z[x]$ is homologically onedimensional, we have the following universal coefficient formulas*:

1) $0 \to \operatorname{Ext}^{2,*}_{Z[x]}(k_*, Z[x]) \to k^* \to \operatorname{Hom}^*_{Z[x]}(k_*, Z[x]) \to 0$,

2) $0 \to k_* \otimes_{Z[x]} Z[x, x^{-1}] \to K_* \to \operatorname{Tor}^{1,*}_{Z[x]}(k_*, Z[x, x^{-1}]) \to 0$,

3) $0 \to k_* \otimes_{Z[x]} Z \to H_*(\ , Z) \to \operatorname{Tor}^{1,x}_{Z[x]}(k_*, Z)$,

where in formula 1) *k_* and k^* are connected, in formula* 2) *k_* and K_*, since $Z[x, x^{-1}]$ is a $Z[x]$-module, and in formula* 3) *k_* and H_*, since Z is a $Z[x]$-module.*[6]

It is possible to find a number of other formulas connecting k_*, k^*, K_*, K^*, H_*, H^* and also Künneth formulas for the direct product $Y_1 \times Y_2$, starting with the complex $S_U \otimes_\Omega Z[x]$ as a $Z[x]$-module and the fact that $Z[x]$ is one-dimensional, as is Z.

We note also that $\operatorname{Hom}^*_{Z[x]}(Z[x], Z[x]) = Z[y]$, where $\dim y = -2$.

In all the formulas of Corollary 2 one can start from the complex $\operatorname{Hom}^*_{\Omega_U}(S_U, Z[x])$, which is a complex of free $Z[y]$-modules for k^*-theory.

With the help of the complex $S_U(Y)$ it is possible to introduce, in addition to the cohomological multiplication, also the "Čech operation" $\cap$ such that $(a \cap b, c) = (a, bc)$, where $c, a \in U^*$, $b \in U_*$ and $a \cap b \in U_*$, while $(a \cap b, c) \in \Omega_U$. Analogously for k_*- and k^*-, K^*- and K_*-theories.

The Poincaré–Atiyah duality law, of course, is treated in the usual way by means of the fundamental cycle and the Čech operation.

[6]The author has available a derivation of Theorems 1 and 2 from the Adams spectral sequence in cobordism theory, and hence Theorems 1 and 2 do not depend on the preceding conjectures.

We note that the homomorphisms σ_ω^* introduced by the author on the ring Ω_U represent "characteristic numbers" with values in Ω_U, since the scalar product lies in Ω_U.

Appendix 4. U_*- and k_*-theory for BG, where $G = Z_m$. Fixed points of transformations

In this appendix we shall consider the following questions:

1. What are the cell complexes $S_U(BG)$ and $S_k(BG)$ for $G = Z_m$? What are the Λ-modules $U_*(BG)$ and $k_*(BG)$, where $\Lambda = \Omega_U$ and $\Lambda = Z[x]$?

2. How to compute in $U_*(BG)$ the following elements: let the group Z_m act on C^n linearly, and without fixed points on $C^n\setminus 0$, i.e., by means of diagonal matrices (a_{ij}), where $a_{ij} = \exp(2\pi i x_j/m)$ and x_j is a unit in the ring Z_m. Then an action of Z_m on S^{2n-1} is defined, and by the same token a map $f_{x_1,\dots,x_n}\colon S^{2n-1}/Z_m \to BG$, which represents an element of $U_{2n-1}(BG)$. This element we denote by $\alpha_n(x_1,\dots,x_n) \in U_{2n-1}(BG)$. It is trivial to find $\alpha_n(1,\dots,1)$ ("geometric bordism") and to show that $\alpha_n(x_1,\dots,x_n) \neq 0$ for all (invertible) $x_1,\dots,x_n \in Z_m$ (see [11]),

$$\nu\alpha_n(x_1,\dots,x_n) \neq 0, \quad \nu\colon U_* \to H_*(\ ,Z).$$

This question arises in connection with the Conner–Floyd approach to the study of fixed points (see [11]).

We consider the question of computing the cell complexes $S_U(BG)$, $S_k(BG)$ and $S_K(BG)$ (see Appendix 3).

We recall the well-known result of Atiyah [7] that $K^1(BG) = 0$ and $K^0(BG) = R_U G^\wedge$, where $R_U(G)$ is the ring of unitary representations. For $G = Z_m$, the basic unitary representations $\rho_0 = 1$, $\rho_1 = \{l^{2\pi i/m}\},\dots,\rho_k = \{l^{2\pi i k/m}\},\dots,\rho_{m-1}$ are one-dimensional, while as a ring a generator is $\rho_1 = \rho$ with the relation $\rho^m = 1$. By virtue of this we can choose in $K^0(BG)$ an element t, corresponding to $\rho - 1$, with the relation $\Psi^m(t) = 0$, where Ψ_*^m is the Adams operator.

We consider the ring $k^*(P) = \operatorname{Hom}^*_{Z[x]}(Z[x], Z[x]) = Z[y]$, $\dim y = -2$. We have

Lemma 1. *The $Z[y]$-module $k^*(BG)$ for $G = Z_m$ is described as follows*:

a) $k^{2j+1} = 0$.

b) *$k^{2j}(BG)$ is isomorphic to the subgroup of $k^0(BG)$ consisting of elements of filtration $\geq 2j$, an this isomorphism is established by the Bott operator y^j*:

$$k^{2j}(BG) \to K^0(BG).$$

c) *The action of the rings B_Ψ^k and A_Ψ^k is well defined on $k^*(BG)$.*

d) *There exists a natural generator $u \in k^2(BG)$ such that every element of $k^*(BG)$ has the form $\sum_j y^{s_j}u^{q_j}$ and there is the relation $(m\Psi^m)(u) = 0$, or $\Psi^m(yu) = 0$, where $yu \in k^0 = K^0$ is the canonical generator $t \in K^0(BG)$; and we have the equation*

$$(m\Psi^m)(u) = \sum_{k\geq 1}\binom{m}{k}(-y)^{k-1}u^k.$$

The proof of the lemma follows easily from the results mentioned about $K^0(BG)$ and the discussion of the spectral sequence with term $E_2 = H^*(BG, Z[y])$ which converges to $k^*(BG)$.

We denote the expression $m\Psi^m(u)$ by $F_m(u) = \sum \binom{m}{k}(-y)^{k-1}u^k$. From Lemma 1 follows

Lemma 2. *The cell complex $S_k^*(BG) = \mathrm{Hom}^*_{Z[x]}(S_k, Z[x])$ of modules over $Z[y]$ is a ring with multiplicative generators (over $Z[y]$) v (of dimension (1)) and u (of dimension (2)) and additive basis $\{y^s u^n, y^q v u^l\}$. The differential d in this complex satisfies the Leibnitz formula, commutes with multiplication by y and has the form:*

$$du = 0, \quad dv = F_m(u).$$

The cell complex of k-theory $S_k(BG)$ for $G = Z_m$ in the natural cellular subdivision has the form $S_k(BG) = \mathrm{Hom}^*_{Z[y]}(S_k^*, Z[y])$, while $S_k(BG)$ over $Z[x], Z[x] = \mathrm{Hom}^*_{Z[y]}(Z[y], Z[x])$, is a complex of free modules.

Lemma 2 follows easily from Lemma 1 and Appendix.

We turn now to U_*- and U^*-theories. For the element $u \in V(X) = \mathrm{Map}(X, MU_1) \subset U^2(X)$, the series $m\Psi_U^m(u) = g^{-1}(mg(u))$ (see Appendix 2), where $g(u) = \sum\limits_{n\geq 0}[CP^n]u^{n+1}/(n+1)$ is the "Miščenko series" (see Appendix 1). We denote the series $m\Psi_U^m(u)$ by $F_{m,U}(u)$. Let

$$\Lambda = U^*(P) = \mathrm{Hom}^*_{\Omega_U}(\Omega_U, \Omega_U),$$

and let $S_U^*(BG)$ be the cell complex in U^*-theory which is a complex of Λ-modules, with $\Lambda^{-2i} = \Omega_U^{2i}$.

With the help of the Conner–Floyd homomorphism $\sigma_1\colon k^0 \to U^2$, $k^0 = K^0$, we obtain from Lemma 2

Theorem 1. *The cell complex (in the natural cellular subdivision)*

$$S_U^*(BG) = \mathrm{Hom}^*_{\Omega_U}(S_U, \Omega_U),$$

which is a complex of free Λ-modules, $\Lambda = U^*(P)$, with differential d, is a ring with multiplicative generators v (of dimension (1)) and u (of dimension (2)) over Λ, given in the following way:

$$v^2 = 0, \quad d(v) = F_{m,U}(u), \quad d(u) = 0.$$

The complex $S_U(BG)$ is isomorphic to $\mathrm{Hom}^*_\Lambda(S_U, \Lambda)$, $G = Z_m$, where $\Omega_U = \mathrm{Hom}^*_\Lambda(\Lambda, \Lambda)$. The complexes $S_U^* \otimes_\Omega Z[x]$ and $S^* \otimes_\Omega Z[x, x^{-1}]$ are isomorphic, respectively, to the complexes $S_k^*(BG)$ and $S_k^*(BG)$ in k- and K-theories.

We pass now to the automorphisms of the complex $BG \to BG$. Such automorphisms for $G = Z_m$ are completely determined by automorphisms of the group $Z_m \to Z_m$, which are multiplication by k, where k is a unit in Z_m.

There arise automorphisms

$$\lambda_k\colon BG \to BG,$$
$$\lambda_k^*\colon S_U^*(BG) \to S_U^*(BG),$$

where λ_k^* is completely determined by the images

$$\lambda_k^*(v) \in S_U^*(BG), \quad \lambda_k^*(u) \in S_U^*(BG),$$

since λ_k^* is a ring homomorphism which commutes with the action of Λ.

We have

Theorem 2. [7] *The homomorphism of multiplication by* k

$$\lambda_k^*\colon S_U^*(BG) \to S_U^*(BG)$$

for $G = Z_m$ *and* $(k, m) = 1$ *is a ring homomorphism which commutes with* d *and multiplication by* Λ *and is defined by the following formulas*:

a) $\lambda_k^*(u) = F_{k,U}(u)$,

b) $\lambda_k^*(v) = \dfrac{F_{km,U}(u)}{F_{m,U}(u)} \cdot v$.

The proof of Theorem 2 is obtained from the fact that under λ_k^* ("geometric cobordism") $u \in U^2(BG)$ must go into $k\Psi^k(u)$ by definition of the operator Ψ^k. This implies part (a). Part (b) follows from the fact that $d\lambda_k^*(v) = \lambda_k^* dv = D(u)v$, where $D(u)$ is a series of dimension 0 with coefficients in Λ.

We now pass to the question of fixed points of transformations Z_m. Let $Z_m^* \subset Z_m$ be the multiplicative group of units, $x_1, \ldots, x_n \in Z_m^*$ and $\alpha_n(x_1, \ldots, x_n) \in U_{2n-1}(BG)$ the element defined by the linear action of the group Z_m on $S^{2n-1} \subset C^n \setminus 0$ by means of multiplication of the j-th coordinate by $\exp(2\pi i x_j/m)$, $x_j \in Z_m^*$. There arises a function

$$\alpha_n\colon Z_m^* \times \cdots \times Z_m^* \to U_{2n-1}(BG).$$

Let $m = p^n$, p a prime and $m_1 = p^{n-1}$. Then $Z_{m_1} \subset Z_m$ and there is defined a homomorphism $U_*(BZ_{m_1}) \to U_*(BZ_m)$. We have

Lemma 3. *Given a quasicomplex transformation* $T\colon M^n \to M^n$ *of order* m *which has only isolated fixed points* $P_1, \ldots, P_q$, *we have the equation*

$$\sum_{j=1}^{q} \alpha_n(x_{1j}, \ldots, x_{nj}) \equiv 0 \mod U_*(BZ_{m_1}),$$

where the x_{ij} *are the orders of the linear representation of the group* Z_m *at the point* P_j *(clearly,* $x_{ij} \in Z_m^*$*).*

This lemma for prime $m = p$ was found by Conner–Floyd [11] (here, $m_1 = 0$), and it is trivial to go over to $m = p^h$.

It is easy to show that for any $(x_1, \ldots, x_n) \in Z_m^* \times \cdots \times Z_m^*$

$$\alpha_n(x_1, \ldots, x_n) \not\equiv 0 \mod U_*(BZ_{m_1}),$$

whence follows the theorem of Conner–Floyd–Atiyah: there does not exist a transformation T with one fixed point. For $p > 2$ this is also true for real transformations T, as can be seen from the analogous application of the theory $SO_* \otimes Z[1/2]$.

We now pass to the question of calculating the function $\alpha_n(x_1, \ldots, x_n) \in U_{2n-1}(BZ_m)$. We denote by $v_{2n-1} \in U_{2n-1}(BG)$ the so-called "geometric bordism" $\alpha_n(1, \ldots, 1)$. In the complex $S_U(BG)$ described in Theorem 1, the element v_{2n-1} is adjoint to $vu^{n-1} \in S_U^*(BG)$, i.e., $(v_{2n-1}, vu^{n-1}) = 1$, $(v_{2n-1}, vu^{n-1+k}) = 0$ for $k > 0$, where $x \in \Lambda^*$.

We shall calculate the function $\alpha_n(x_1, \ldots, x_n)$ by the following scheme:

1) Clearly, $\alpha_1(x) = xv_1 \in U_1(BG) = Z_m$.

---

[7] All homological deductions from Theorems 1 and 2 of this appendix can be justified, without the complexes S_U, merely from Theorems 1 and 2 of Appendix 3.

2) If $\sum\limits_{j=1}^{m} \alpha_k(x_{1j}, \dots, x_{kj}) \equiv 0$, $\sum\limits_{j=1}^{l} \alpha_{n-k}(y_{k+1,j}, \dots, y_{n,j}) \equiv 0$, then we have the equation:

$$\sum_{j,s} \alpha_n(x_{1j}, \dots, x_{kj}, y_{k+1,s}, \dots, y_{n,s}) \equiv 0 \mod BZ_{m_1}.$$

This follows in an obvious way from the fact that transformations $T_1 \colon M^k \to M^k$ and $T_2 \colon M^l \to M^l$ induce $(T_1, T_2) \colon M^k \times M^{n-k}$, where fixed points (and their orders) correspond to each other.

3) If $\lambda_x \colon BG \to BG$ is induced by multiplication by $x \in Z_m^*$, then $\alpha_n(x, \dots, x) = \lambda_{x,*}(v_{2n-1})$, where the structure of $\lambda_{x,*}$ is described in Theorem 2.

As examples of the application of this scheme we shall indicate the following simple results:

Lemma 4. *If $\nu \colon U_* \to H_*(\ ,Z)$ is the natural homomorphism, then we have the equation*

$$\nu\alpha_n(x_1, \dots, x_n) = (x_1, \dots x_n)\nu(v_{2n-1}),$$

where $\nu(v_{2n-1}) \in H_{2n-1}(BZ_m) = Z_m$ is the basis element.

Lemma 5. *For $n = 1, 2, 3$ we have the formulas*:

$$\lambda_{x,*}(v_1) = xv_1,$$
$$\lambda_{x,*}(v_2) = x^2 v_3,$$
$$\lambda_{x,*}(v_5) = x^3 v_5 + \frac{x^3 - x^2}{2}[CP^1]v_3.$$

From Lemma 2, in an obvious way, follows the corollary on the impossibility of one fixed point.

Now let $m = p$, where $p > 2$ for $n = 2$ and $p > 3$ for $n = 3$. Under these conditions, by the scheme indicated above, one easily obtains from Lemmas 2 and 3

Theorem 3. *The functions $\alpha_n(x_1, x_2, \dots, x_n)$ for $n \leq 3$ has the following form*:

$$\alpha_1(x) = xv_1 \ (\textit{obviously});$$
$$\alpha_2(x_1, x_2) = (x_1 x_2)v_3;$$
$$\alpha_3(x_1, x_2, x_3) = (x_1 x_2 x_3)v_5 + \frac{x_1 x_2 x_3 - \dfrac{x_1 x_2 + x_1 x_3 + x_2 x_3}{3}}{2}[CP^1]v_3.$$

Suppose given a group of quasicomplex transformations $Z_p \colon M^n \to M^n$ with isolated fixed points $P_1, \dots, P_q$ at which the generator $T \in Z_p$ has orders $x_{1j}, \dots, x_{nj} \in Z_p$, $j = 1, \dots, q$, where $x_{kj} \in Z_p^*$. We consider the point $(x_{1,1}, \dots, x_{kj}, \dots, x_{nq}) \in Z_p^{qn}$ up to a factor $\mu \in Z_p^*$, $\mu \neq 0$. Thus, $(x_{1,1}, \dots, x_{nq}) \in P^{qn-1}$. The group $S_n \times S_q$, where S_k is the group of permutations of k elements, acts on P^{qn-1}.

Definition. By the type of the action of the group Z_p on M^n with isolated fixed points we shall mean the set of orders of $(x_{1,1}, \dots, x_{kj}, \dots, x_{nq})$, of any generator $T \in Z_p$, considered in the projective space P^{qn-1}, factored by the actions of the group S_n of permutations of orders of each point and the group S_q of permutations of points.

From Theorem 3 follows the

Corollary. *For $p > n$ and for $n = 2, 3$, the set of types of actions of Z_p on M^n is given in P^{qn-1} by the set of equations*:

$$\sum_{P_j} \sigma_k(x_{1j}, \dots, x_{nj}) = 0, \quad x_{js} \neq 0, \quad k = 2, 3, \dots, n,$$

where the x_{sj} are the orders at the point P_j and $\sigma_k(x_{1j}, \dots, x_{nj})$ is the elementary symmetric polynomial.

Appendix 5. The conjecture on the bigradation of algebraic functors in S-topology for all primes $p > 2$

In the Introduction and also in § 12 the possibility was already discussed of the appearance of a new categorical invariant — an additional grading, connected with the Cartan type, in the Adams spectral sequence for ordinary cohomology mod p, $p > 2$, from which it would follow (see the Introduction) that the homotopy groups in the category of torsion-free complexes could be formally computed algebraically by the theory of unitary cobordism. We shall formulate here more exactly the corresponding conjecture.

First of all, we shall go to the question of the category $S \otimes_Z Q_p$ for $p > 2$. Let $K(\pi) \in S$ be the spectrum $K(\pi, n)$. The following fact is known (H. Cartan): the Steenrod algebra $A = H^*(K(Z_p), Z_p)$ is bigraded: $A = \sum A^{k,\beta}$, where $\dim = k + \beta$ and β is the type.

Conjecture: I) Let the bigradings $H(X, Z_p) = \sum H^{k,\beta}$ and $H(Y, Z_p) = \sum H^{k,\beta}$ be well defined, and let the morphism $f\colon X \to Y$ in the S-category preserve the bigrading. Then in the exact sequences

$$0 \to X \xrightarrow{f} Y \to Z \to 0$$

and

$$0 \to Z' \to X \xrightarrow{f} Y \to 0$$

for the objects $Z, Z' \in S$, the bigradings of the functors $H^{**}(Z, Z_p)$ and $H^{**}(Z', Z_p)$ are well defined, and the exact sequence of the triple (X, Y, Z) is

$$\dots \to H^{k,\beta}(X) \xrightarrow{\delta} H^{k,\beta+1}(Z) \to H^{k,\beta+1}(Y) \xrightarrow{f^*} H^{k,\beta+1}(X) \to \dots$$

II) For $X = K(Z_p)$ the bigrading coincides with that of Cartan.

III) The cohomology A-module $H^*(X, Z_p)$ is bigraded, if in $H^*(X, Z_p)$ the bigrading is well defined.

IV) All these properties are fulfilled in the subcategory $S_{gr} \subset S \otimes_Z Q_p$ obtained from $K(Z_p)$ inductively by means of bigraded morphisms and passage to "kernels" and "cokernels"

$$0 \to X \xrightarrow{f} Y \to Z \to 0, \quad 0 \to Z' \to X \xrightarrow{f} Y \to 0;$$

here Q_p is the p-adic integers.

Assertion. 1. If the conjecture is true, then the spectra of points (spheres) P and complexes without p-torsion in homology belong to the category S_{gr}.

2. If the analogous conjecture of bigradation for other functors (for example, homotopy groups) is true, then the entire classical Adams spectral sequence and the stable homotopy groups of spheres for $p > 2$ can be completely calculated by means

of the theory of unitary cobordism by the scheme described in the Introduction and in § 12. In particular, we should have the equation:

$$\pi_{**}^{(p)}(S^N) \approx \mathrm{Ext}_{A^U}^{**}(\Lambda, \Lambda) \otimes_Z Q_p, \quad p > 2.$$

This means two things: a) the triviality of the Adams spectral sequence constructed by the author in the theory of U-cobordism; b) the absence of extensions in the term $E_\infty = E_2$.

3. For $p = 2$, the conjecture in such a simple form is trivially false. {In the spectral sequence for the stable groups of spheres, all powers $\eta^k \neq 0$ for an element η representing the Hopf map in $\pi_1(S)$, hence η^k for $k \geq 4$ must be killed off by differentials.}

4. The classical Adams spectral sequence with second term $E_2^{s,k,\beta} = \mathrm{Ext}_A^{s,k,\beta}(H^{**}(X), Z_p)$ for $X \in S_{gr}$ is arranged as follows:

$$d_r\colon E_r^{s,k,\beta} \to E_r^{s+r,k,\beta+r-1}.$$

We note that $h_0 \in \mathrm{Ext}_A^{1,0,1}(Z_p, Z_p)$ is associated with multiplication by p (ordinarily we have $h_0 \in \mathrm{Ext}_A^{1,1}$). Here, the dimension differs slightly from that described in § 12 by a simple linear substitution.

Examples of bigradation (the simplest). Let $X = K(Z) + EK(Z)$ and $Y = K(Z_{p^q})$. From the ordinary point of view we have:

$$H^*(X, Z_p) = H^*(Y, Z_p) = A/A\beta(u) + A/A\beta(v),$$

where $\dim u = 0$ and $\dim v = 1$. However, for X the ordinary Adams spectral sequence is zero, but for Y we have: $d_q(v^*) = h_0^q u^*$, where $u^* \in \mathrm{Ext}_A^{0,0}$ and $v^* \in \mathrm{Ext}_A^{0,1}$, since $\pi_*(Y) = Z_{p^q}$.

From our point of view the situation is thus:

a) $H^{**}(X, Z_p) = A/A\beta(u) + A/A\beta(v)$, where $u \in H^{0,0}$, $v \in H^{0,1}$. Hence $u^* \in \mathrm{Ext}_A^{0,0,0}$, $v^* \in \mathrm{Ext}_A^{0,1,0}$ and $h_0^q u^* \in \mathrm{Ext}^{q,q,0}$; by dimensional considerations, $d_q(v^*) \in \mathrm{Ext}_A^{q,1,q-1}$, and $\mathrm{Ext}_a^{q,1,q-1} = 0$.

b) $H^{**}(Y, Z_p) = A/A\beta(u) + A/A\beta(v)$, $u \in H^{0,0}$, $v \in H^{0,1}$, then $u^* \in \mathrm{Ext}_A^{0,0,0}$ and $v^* \in \mathrm{Ext}_A^{0,0,1}$, $d_i v^* \neq 0$ for $i = q$.

Besides the facts indicated earlier, there are subtler circumstances which corroborate the conjecture:

1. From the results of the author's series of papers on the J-homomorphism $J_* \subset \pi_*(S^N)$ and the results of the present paper, it follows that $\mathrm{Ext}_{A^U}^{1,*}(\Lambda, \Lambda) \otimes_Z Q_p$ consists (for $p > 2$) of cycles for all differentials, while elements of $\mathrm{Ext}_{A^U}^{1,*}$ are realized by elements of $\pi_*^{(p)}(S^N)$ of the same order; moreover, $\pi_*^{(p)}(S^N) = \mathrm{Ext}_{A^U}^{1,*} + \dots$, where $\mathrm{Ext}^{1,*} = J_* \otimes_Z Q_p$.

2. The Adams spectral sequence in U-theory would not have to be trivial from dimensional considerations (obviously, only d_i is zero for $i - 1 \equiv 0 \mod 2p-2$). There first appears an element $x \in \mathrm{Ext}_{A^U}^{2,2p^2(p-1)}$ where $d_{2p-1}(x) = ?$, since $\mathrm{Ext}_{A^U}^{2p+1,2p^2(p-1)+2p-2} \neq 0$. In reality, these elements in U-theory are "inherited" from ordinary cohomology theory $H^*(\ , Z_p)$ together with the question about $d_{2p-1}(x)$. A few years ago L. N. Ivanovskiĭ informed the author that with the help of partial operations of Adams type he had succeeded in showing that $d_{2p-1}(x) = 0$ for $p > 3$ (?). However, neither Ivanovskiĭ nor the author were able to verify this calculation, and hence this fact remained obscure. Recently Peterson informed the

author that it has only recently been proved by the young American topologist Cohen [25] for all $p \geq 3$ (more exactly, the analogue of this question in the classical theory, from which, of course, it follows).

3. The fact that the "algebraic" Adams spectral sequence associated with the "topological" one, which begins with $E_2 = \mathrm{Ext}_A^{****}(Z_p, Z_p)$ and converges to $\mathrm{Ext}_{A^U}^{**}(\Lambda, \Lambda) \otimes_Z Q_p$ (see § 12), is algebraically well-defined, is non-trivial a priori. The situation here is that if for some spectral sequence (E_r, d_r) we consider the complementary filtration in E_2 and define on all the $\bar{E}_r$ associated differentials $\bar{d}_r$, then very often the $\bar{d}_r$ are not included in a well-defined spectral sequence (of algebras). Hence the fact of such a well-defined inclusion is in our case an extra geometric argument for the existence of an invariant second grading in the subcategory $S_{gr} \subset S \otimes_Z Q_p$.

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