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THE CLASSIFICATION OF G-SPACES

BY

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The Institute for Advanced Study

INTRODUCTION

If G is a compact Lie group then by a G-space we mean a completely regular space X together with a fixed action of G as a group of homeomorphisms of X. While one could probably point to results arbitrarily far back in the mathematical literature that could be interpreted as theorems about G-spaces, the degree of development and flavor of the theory has changed markedly in the past decade. Probably the most important single new influence that accounts for this is the theory of fiber bundles, particularly as developed in the 1949-50 Seminaire Henri Cartan [2]. The effect of this has been a fruitful tendency to regard G-spaces as generalized principal bundles. In view of the important role played by the cross sections of a principal bundle it was natural to try to develop an analogous concept for general G-spaces. The resulting notion of a slice in a G-space will play a central role in our development of the theory and perhaps a quick resume of its genesis is in order. In [4] (1950) A. M. Gleason proved the existence of a local cross section through each point of a G-principal bundle. The first definition of what finally, through successive modifications, became a slice is apparently found in [8] (1953) where J. L. Koszul in effect proved the existence of a slice through each point of a differentiable G-space. In [10] (1957) Montgomery and Yang gave the nearly final definition of a slice and proved the existence of a slice through any point of a G-space X which was a complete, separable metric space of finite dimension. Finally in [12] (1957) Mostow using an elegant and powerful generalization of Gleason's technique proved the existence of a slice in complete generality. As it will appear in the present work the definition of

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a slice has been simplified and generalized to some extent, and we feel that the notion has now perhaps found a final form.

Chapter I of this memoir will be devoted to giving an exposition of the general theory of G-spaces. Some of the theorems will appear unnatural for such an exposition and have been put in only because of a specific need for them in Chapter II. For the most part only the line of development is new, the results being more or less well-known. Exceptions perhaps are the discussions of compactifications of G-spaces and of G-AR's and G-ANR's although the latter are implicit in Mostow's paper [12] mentioned above. Also new is the notion of reduced join of G-spaces, an operation which plays a basic role in the second chapter when we come to the construction of universal G-spaces, and which we expect will see further service in the future. Although Chapter I does provide a reasonably complete introduction to the general properties of G-spaces, it covers only the surface of present knowledge. All the deep and beautiful results of a more specialized nature that have been proved by Smith, Montgomery and Zippin, Yang, Borel, Floyd, Conner, and Mostow, among others, have purposely not been mentioned. The latter theory, which deals mainly with G-spaces that are generalized manifolds and often makes special assumptions on the nature of G, has been treated in considerable detail in a seminar held at The Institute for Advanced Study in 1958-59 and it is expected that notes from this seminar will soon be published as an Annals of Mathematics Study.

Chapter II is devoted to a development of a classification theory for G-spaces that generalizes and parallels the well-known and powerful classification theory of principal bundles in terms of universal bundles and classifying spaces. To a large extent the theorems of Chapter II roughly parallel the theorems in Steenrod's book [13] or in [7] on the classification theory of principal bundles, and once the proper definitions are made even the proofs are often quite similar. There are two major exceptions though which take

Chapter II out of the realm of straightforward generalization. The proof of the analogue of the covering homotopy theorem (which, as in the principal bundle case, is the key which opens all doors) lies deeper in the general case, and secondly the construction of universal G-spaces is different, bearing however a certain resemblance to that used by J. Milnor in [9]. We have made no applications of the classification theory in this memoir, preferring to leave these to later papers. However there is a theoretical application which deserves mention. Just as in the principal bundle case, the classification theory makes possible the definition of characteristic classes for G-spaces and these we expect will prove quite useful in the future.

In closing I would like to thank the many members of the Institute for Advanced Study with whom I have discussed this work, in particular A. Borel and D. Montgomery. I would also like to thank the Institute itself for extending to me the privileges of membership during the preparation of this memoir and the National Science Foundation for their fellowship support during that period.

The Institute for Advanced Study

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THE CLASSIFICATION OF G-SPACES

1. G-SPACES

1.1. NOTATIONS AND BASIC THEOREMS.

Throughout both chapters G will denote a compact Lie group with identity e. We will write $H \subseteq G$ to denote that H is a closed subgroup of G and $H \subset G$ if in addition $H \neq G$. If X is a topological space an action of G on X is a continuous map $\Phi : G \times X \to X$ such that $\Phi(e, x) = x$ and $\Phi(g_1 g_2, x) = \Phi(g_1, \Phi(g_2, x))$. By a G-space we shall mean a completely regular space X together with a fixed action of G on X. We shall in general not explicitly name the action Φ and simply write gx for $\Phi(g, x)$. It is to be noted that for each $g \in G$ the map $x \to gx$ is a homeomorphism of X onto itself which we call an operation of G, and that the map which takes $g \in G$ into the operation $x \to gx$ is a homomorphism of G into the group of self homeomorphisms of X. If S is a subset of X and $g \in G$ we write gS for $\{gs \mid s \in S\}$, and if K is a subset of G then we write KS for $\{gs \mid g \in K \text{ and } s \in S\} = \bigcup_{g \in K} gS$. The following proposition is basic.

1.1.1. PROPOSITION. Let S be a subset of the G-space X and let K be a subset of G. Then:

(a) if S is open so is KS;

(b) if K is closed and S is compact, KS is compact;

(c) if K is closed and S is closed, KS is closed.

PROOF. If S is open then gS is open for each $g \in G$ (because each operation of G is a homeomorphism of X on itself) hence $KS = \bigcup_{g \in K} gS$ is open. If K is closed (and hence compact) and S is compact then KS being the continuous image of $K \times S$ under the map $(g, s) \to gs$ is also compact. Finally suppose K and S are closed and let x be adherent to KS. Choose a net $\{s_\alpha\}$ in S and $\{g_\alpha\}$ in K so that $g_\alpha s_\alpha \to x$. Since K is compact by passing to a subnet we can suppose $g_\alpha \to k \in K$. Then $\lim s_\alpha = \lim g_\alpha^{-1}(g_\alpha s_\alpha) = k^{-1}x$, so $k^{-1}x$ belongs to S since S is closed.

Hence $x \in kS \subseteq KS$. q.e.d.

Given a point x of a G-space X the set Gx is called the orbit of x. By the preceding result each orbit is compact. More generally given a subset S of X we call GS the saturation of S. Note that GS is the union of the orbits which intersect S. A subset S of X is called invariant if $GS = S$, so the saturation of S is just the smallest invariant set which includes S. As an immediate corollary of 1.1.1

1.1.2. PROPOSITION. If a subset S of a G-space X is open, closed, or compact then so is its saturation.

Given a point x of a G-space X we define the isotropy group at x, denoted by G_x, by $G_x = \{g \in G \mid gx = x\}$.

If X is a G-space then clearly two orbits of X are either disjoint or equal, in other words X is partitioned by its orbits. We denote the orbit space or set of orbits of X by X/G, and we denote by Π_X the natural map $x \to Gx$ of X onto X/G. We give X/G the usual identification space topology, namely the strongest topology making Π_X continuous, i.e. a subset S of X/G is open (closed) if and only if $\Pi_X^{-1}(S)$ is open (closed) in X. Now if $S \subseteq X$ then clearly $\Pi_X^{-1}(\Pi_X(S)) = GS$, the saturation of S, hence by 1.1.2

1.1.3. PROPOSITION. If X is a G-space then Π_X is an open and closed mapping of X onto X/G.

Now in general given a space X a set Y and a function f taking X onto Y there is at most one topology for Y which makes f both open and continuous, hence

1.1.4. PROPOSITION. If X is a G-space the topology for X/G is uniquely characterized by the conditions that it makes Π_X open and continuous.

A mapping f of a G-space X into a topological space Y is called invariant if $f(gx) = f(x)$ for all $g \in G$ and $x \in X$, i.e. if f is constant on each orbit. Clearly if $\tilde{f}$ is a mapping of X/G into Y then $\tilde{f} \circ \Pi_X$ is an invariant mapping. Conversely given an invariant mapping $f : X \to Y$, the function $\tilde{f} = f \circ \Pi_X^{-1}$ is well defined and, because Π_X is open, continuous. Since clearly $f = \tilde{f} \circ \Pi_X$

1.1.5. PROPOSITION. If X is a G-space and Y a topological space then $\tilde{f} \to \tilde{f} \circ \Pi_X$ is a one-to-one correspondence between maps of X/G into Y and invariant maps of X into Y.

Since each orbit of a G-space is closed (in fact compact) it follows that X/G is a T_1 space. Actually much better is true.

1.1.6. LEMMA. If f^* is a continuous real valued function on a G-space X then $f(x) = \int f(gx)dg$ (where dg represents the element of normalized Haar measure) is an invariant, continuous, real valued function on X. Moreover if S is an invariant subspace of X and $f^*|S$ has its range in some (open, half open, or closed) interval, then $f|S$ has its range in the same interval.

PROOF. Trivial.

1.1.7. LEMMA. Let X be a G-space and let K and F be disjoint invariant subspaces of X which are respectively compact and closed. Then there is an invariant map f of X into the unit interval such that $f|K \equiv 0$ and $f|F \equiv 1$. Moreover if X is normal it sufficies to assume that both K and F are closed.

PROOF. By Urysohn's lemma (applied, in case X is not normal, to K and the closure of F in the Čech compactification of X) we can find a continuous real valued function f^* mapping X into the unit interval such that $f^*|K \equiv 0$ and $f^*|F \equiv 1$. We then define f as in the preceding lemma.

1.1.8. PROPOSITION. If X is a G-space then X/G is completely regular and if X is normal then so is X/G.

PROOF. Let $\tilde{F}$ be a closed subset of X/G and $\tilde{x}$ a point of X/G not in $\tilde{F}$. Then $\tilde{x}$ is a compact invariant subset of X disjoint from the closed invariant subset $F = \Pi_X^{-1}(\tilde{F})$. By 1.1.7 we can find an invariant map $f : X \to [0,1]$ such that $f|\tilde{x} \equiv 0$ and $f|F \equiv 1$. By 1.1.5 $\tilde{f} = f \circ \Pi_X^{-1}$ is a continuous map of $X/G \to [0,1]$ and clearly $\tilde{f}|\tilde{F} \equiv 1$. If X is normal we can replace $\tilde{x}$ by any closed subset of X/G disjoint from $\tilde{F}$. q. e. d.

1.1.9. PROPOSITION. If X is a G-space then Π_X is a proper map of X onto X/G, i.e. if K is a compact subset of X/G then $\Pi_X^{-1}(K)$ is a compact subset of X.

PROOF. It is an easily proved general fact that a map of a Hausdorff space into a Hausdorff space is proper if it is closed and the inverse image of every point is compact.

1.1.10. COROLLARY. If X is a G-space then X/G is compact (respectively, locally compact), if and only if X is compact (respectively, locally compact).

PROOF. Immediate from the fact that Π_X is open and proper.

1.1.11. DEFINITION. If ρ is a metric for a G-space X then ρ is called <u>invariant</u> if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$, i.e. if each operation of G is an isometry.

1.1.12. PROPOSITION. If X is a metrizeable G-space there exists an invariant metric ρ for X. Moreover X/G is metrizeable and in fact $\tilde{\rho}(\tilde{x}, \tilde{y}) = \text{Inf}\{\rho(x, y) \mid x \in \tilde{x},\ y \in \tilde{y}\}$ is a metric for X/G. If X is in addition separable then so is X/G.

PROOF. Let ρ^* be any metric consistent with the topology of X and define $\rho(x, y) = \int \rho^*(gx, gy)dg$. It is easily verified that ρ is a metric defining the same topology and in fact the same uniform structure as ρ^* and clearly the invariance of Haar measure implies the invariance of ρ. It is also a matter of straightforward verification that $\tilde{\rho}$ is a metric function for the set X/G. Since Π_X is clearly distance decreasing relative to ρ and $\tilde{\rho}$ and maps the ϵ-sphere relative to ρ about x onto the ϵ-sphere about $\Pi_X(s)$ relative to $\tilde{\rho}$ it follows that Π_X is open and continuous relative to ρ and $\tilde{\rho}$, so that by 1.1.4 $\tilde{\rho}$ is consistent with the topology of X/G. If X is separable it is Lindelöf, hence $X/G = \Pi_X(X)$ is Lindelöf and therefore separable. q.e.d.

1.1.13. PROPOSITION. If X is a G-space and Y is an invariant subspace of X then $Y/G = \Pi_X(Y)$ with the topology induced from X/G.

PROOF. It is clear that as a set Y/G is just $\Pi_X(Y)$ and that $\Pi_Y = \Pi_X|Y$. Since $\Pi_X|Y$ is continuous when $\Pi_X(Y)$ is given the induced topology it will suffice, by 1.1.4, to show that it is open. But Y invariant means $Y = \Pi_X^{-1}(\Pi_X(Y))$ which implies $\Pi_X(Y \cap O) = \Pi_X(Y) \cap \Pi_X(O)$ for any subset O of X. This formula applied to open subsets O of X, together with the openness of Π_X gives the openness of $\Pi_X|Y$ onto $\Pi_X(Y)$. q.e.d.

1.1.14. PROPOSITION. If X is a G-space and J is an invariant subset of X then every neighborhood V of J includes an invariant neighborhood of J.

PROOF. We may assume that V is open. Then by 1.1.3 $\Pi_X(X-V)$

is closed in X/G. Moreover $\tilde{J} = \Pi_X(J)$ is disjoint from $\Pi_X(X-V)$ for if $\Pi_X(x) \in \tilde{J}$ where $x \in X - V$ then $x \in \Pi_X^{-1}(\tilde{J}) = J$ (by the invariance of J) which contradicts $J \subseteq V$. Hence $\tilde{U} = X/G - \Pi_X(X-V)$ is a neighborhood of $\tilde{J}$ so $U = \Pi_X^{-1}(\tilde{U})$ is a neighborhood of J which is clearly invariant and included in V.

1.1.15. DEFINITION. Let X and Y be G-spaces. A mapping $f : X \to Y$ is equivariant if $f(gx) = gf(x)$ for all $(g, x) \in G \times X$. If in addition f is one-to-one on each orbit of X then f is called an isovariant map. An equivariant homeomorphism of X onto Y is called an equivalence of X with Y. If there exists an equivalence of X with Y then X and Y are called equivalent.

1.1.16. PROPOSITION. *If X and Y are G-spaces and $f : X \to Y$ is equivariant, then for any $x \in X$ we have $G_x \subseteq G_{f(x)}$ and equality occurs if and only if f is one-to-one on Gx. In particular if f is isovariant then $G_x = G_{f(x)}$ for all $x \in X$.*

PROOF. That $G_x \subseteq G_{f(x)}$ is trivial. If f is one-to-one on Gx then $f|Gx$ is an equivalence of Gx with Gf(x) so $(f|Gx)^{-1}$ is an equivariant map of Gf(x) onto Gx. Since it carries f(x) into x we get the reverse inclusion $G_{f(x)} \subseteq G_x$. Conversely suppose $G_x = G_{f(x)}$. Then if $f(g_1x) = f(g_2x)$ we get $g_2^{-1}g_1f(x) = f(x)$ by equivariance, so $g_2^{-1}g_1 \in G_{f(x)} = G_x$ so $g_2^{-1}g_1x = x$ and $g_1x = g_2x$ proving that f is one-to-one in Gx.

1.1.17. PROPOSITION (AND DEFINITION). *If X and Y are G-spaces and $f : X \to Y$ is equivariant, then there is a unique map $\tilde{f} : X/G \to Y/G$ such that $\tilde{f} \circ \Pi_X = \Pi_Y \circ f$. This map is called the map induced by f.*

PROOF. The equivariance of f implies that f maps each orbit of X into a single orbit of Y from which the existence and uniqueness of a function $\tilde{f}$ satisfying the desired relation is immediate. The continuity of $\tilde{f}$ follows directly from the continuity of f and Π_Y and the openness of Π_X.

> 1.1.18. PROPOSITION. Let X and Y be G-spaces. If $T : X \to Y$ is an equivalence of X with Y then the induced map $\tilde{T}$ is a homeomorphism of X/G onto Y/G. If X is locally compact and $T : X \to Y$ is an isovariant map such that $\tilde{T}$ is a homeomorphism of X/G with Y/G then T is an equivalence of X with Y.

PROOF. The first statement is trivial and it is also clear that if $T : X \to Y$ is an isovariant map such that $\tilde{T}$ is a homeomorphism of X/G with Y/G then T is a one-to-one map of X onto Y. If X is locally compact and $x \in X$ let K be a compact invariant neighborhood of x (for example GV where V is any compact neighborhood of x). Then $\Pi_X(K)$ is a neighborhood of $\Pi_X(x)$ so $\tilde{T} \circ \Pi_X(K)$ is a neighborhood of $\tilde{T} \circ \Pi_X(x) = \Pi_Y T(x)$ hence $\Pi_Y^{-1}\tilde{T} \; \Pi_X(K) = T(K)$ is a neighborhood of Tx. Since $T|K$ is a homeomorphism of K onto T(K) it follows that T^{-1} is continuous at Tx. q. e. d.

1.1.19. DEFINITION. A G-space X is:
(a) <u>differentiable</u> if X is a paracompact differentiable ($=C^{\infty}$) manifold and the action of G on X is a differentiable map of $G \times X$ into X;
(b) <u>Riemannian</u> if X is a differentiable G-space with a differentiable Riemannian structure and if each operation of G on X is an isometry;
(c) <u>Euclidean</u> if X is a finite dimensional real vector space with an orthogonal structure and each operation of G on X is an orthogonal linear transformation of X.

> 1.1.20. PROPOSITION. Every differentiable G-space can be made into a Riemannian G-space by a proper

choice of Riemannian structure.

PROOF. Let T^* be any differentiable Riemannian tensor for the differentiable G-space X. For each $g \in G$ we denote by δg the differential of the operation of g on X, and for tangent vectors u, v at a point x of X we define $T(u, v) = \int T(\delta g(u), \delta g(v))dg$. The joint differentiability of $(g, x) \rightarrow gx$ implies that T is a differentiable tensor field on X (apply Euler's theorem on differentiating under an integral sign in local coordinates). Clearly T is a Riemannian tensor for X and as usual the invariance of Haar measure implies that each operation of G on X is an isometry relative to T. q. e. d.

1.1.21. DEFINITION. If $H \subseteq G$ we denote by G/H the space of left H-cosets in G made into a G-space by the operations $g(\gamma H) = g\gamma H$. We note that by [3] page 111, G/H is a differentiable G-space. By a local cross-section in G/H we mean a differentiable, non-singular map $\chi : U \rightarrow G$ where U is a neighborhood of H in G/H, $\chi(H) = e$, and $\chi(u) \in u$. The existence of a local cross-section follows from [3] page 109-110.

1.1.22. PROPOSITION. If X is a G-space and $x \in X$ then $T : gG_x \rightarrow gx$ is an equivalence of G/G_x with Gx. If X is differentiable then Gx is a compact submanifold of X and T is a non-singular differentiable imbedding of G/G_x onto Gx.

PROOF. Let F denote the map $g \rightarrow gx$ of G into X and Π the natural projection of G onto G/G_x. Since clearly $F = T \circ \Pi$ the continuity of T follows from the continuity of F and the openness of Π. That T is equivariant is clear, and since the isotropy group at the point G_x of G/G_x is just G_x it follows from 1.1.16 that T is one-to-one on the orbit of G_x which of course is all of G/G_x. Since G/G_x is compact T is an equivariant homeomorphism. Now suppose that X is differentiable. By homogeneity it will suffice to show that T is differentiable and non-singular at G_x. Let $\chi : U \to G$ be a local cross-section in G/G_x. Then we have $T|U = F \circ \chi$ so that T is differentiable because F and χ are. We finally must show that δT is non-singular at G_x. Since the image of the tangent space at G_x under $\delta\chi$ is clearly a linear complement to the tangent space to G_x in the full tangent space to G at e, it will suffice to show that the null space of δF at e is just the tangent space to G_x. Let Z be a tangent vector to G at e and let $\operatorname{Exp} tZ$ be the one-parameter group it generates. Then for each $y \in X$ $t \to (\operatorname{Exp} tZ)y$ is a differentiable curve in X with tangent vector Z_y^* at $t = 0$. From the relation $(\operatorname{Exp}(t_1 + t_2)Z)y = (\operatorname{Exp} t_1 Z)(\operatorname{Exp} t_2 Z)y$ it follows that $t \to (\operatorname{Exp} tZ)y$ is an integral curve of Z^* starting from y. By the uniqueness theorem of ordinary differential equations it follows that $(\operatorname{Exp} tZ)y \equiv y$ if and only if $Z_y^* = 0$. Now $F(\operatorname{Exp} tZ) = (\operatorname{Exp} tZ)x$ so $Z_x^* = \delta F(Z)$, hence $\delta F(Z) = 0$ if and only if $(\operatorname{Exp} tZ)x \equiv x$, i.e. if and only if $\operatorname{Exp} tZ$ is a one-parameter subgroup of G_x, i.e. if and only if Z is tangent to G_x.

q. e. d.

1.1.23. COROLLARY. Let X be a G-space, $x \in X$ and let $G_x \subseteq H \subseteq G$. Then the map $gx \to gH$ is an equivariant map of Gx onto G/H which is differentiable if X is differentiable and is an equivalence if $G_x = H$.

PROOF. In view of 1.1.22 it suffices to show that if $K \subseteq H \subseteq G$ then the map $F : gK \to gH$ is a differentiable, equivariant map of G/K onto G/H. Now if $\chi : U \to G$ is a local cross-section in G/K then $F|U = \Pi \circ \chi$ where Π is the natural projection of G on G/H. Since Π and χ are differentiable so is $F|U$ and by homogeneity F is everywhere differentiable. The equivariance of F is trivial.

q. e. d.

1.2. ORBIT TYPES.

1.2.1. DEFINITION. If $H \subseteq G$ we denote by (H) the collection of subgroups of G which are conjugate in G to H, $(H) = \{gHg^{-1} | g \in G\}$. A set of the form (H) will be called a G-<u>orbit type.</u> If X is a G-space and Ω is an orbit in X, say $\Omega = Gx$, then since clearly $G_{gx} = gG_xg^{-1}$ it follows that $\{G_\omega | \omega \in \Omega\} = (G_x)$ is a G-orbit type which we call the <u>orbit type of</u> Ω and denote by $[\Omega]$. For each closed subgroup H of G we denote by $X_{(H)}$ the union of all orbits of X that are of type (H) and we write $\tilde{X}_{(H)} =$ $\Pi_X(X_{(H)})$ for the set of orbits of X of type (H). We note that $\{X_{(H)}\}$ is a partitioning of X into invariant subsets indexed by the orbit types of G. W call this indexed partition of X the <u>orbit structure of</u> X. Similarly $\{\tilde{X}_{(H)}\}$ is a partition of X/G indexed by the orbit types of G which we call the <u>orbit structure of</u> X/G.

If (H) and (K) are two G-orbit types then we say that $(H) \leq (K)$ if some element of (H) is included in an element of (K). If $(H) \leq (K)$ and $(K) \leq (H)$ then clearly H and K have the same dimension and number of components and it follows easily that $(H) = (K)$, hence $\leq$ is a partial order ing of the G-orbit types (transitivity is clear) and we use $<$ for the associated strong partial ordering.

1.2.2. PROPOSITION. Let Ω and Ω' be orbits in perhaps different G-spaces. Then

(1) Ω and Ω' are equivalent as G-spaces if and only if $[\Omega] = [\Omega']$.

(2) There is an equivariant map $f: \Omega \to \Omega'$ if and only if $[\Omega] \leq [\Omega']$.

PROOF. Note that G/H is an orbit and clearly its type is (H). Since equivalent orbits clearly have the same type (e.g. by 1.1.16) and sin by 1.1.22 any orbit of type (H) is equivalent to G/H statement (1) is immediate. If there exists an equivariant map $f : \Omega \to \Omega'$ then choosing $x \in$

we have $G_x \subseteq G_{f(x)}$ by 1.1.16 so that $[\Omega] = (G_x) \leq (G_{f(x)}) = [\Omega']$. Conversely if $[\Omega] \leq [\Omega']$ then we can find $x \in \Omega$ and $y \in \Omega'$ with $G_x \subseteq G_y$ and then $gx \to gy$ is easily seen to be a well defined equivariant map of Ω into Ω'. Alternatively the existence of f given $[\Omega] \leq [\Omega']$ follows from 1.1.23. q. e. d.

1.2.3. COROLLARY. If X and Y are G-spaces and $f : X \to Y$ is equivariant (respectively, isovariant) then $[x] \leq [f(x)]$ (respectively $[x] = [f(x)]$) for all $x \in X/G$.

REMARK. This is also an immediate consequence of 1.1.16.

1.2.4. COROLLARY. If X and Y are G-spaces and if $f : X \to Y$ is isovariant then, for any $H \subseteq G$, $f(X_{(H)}) \subseteq Y_{(H)}$ and $\tilde{f}(\tilde{X}_{(H)}) \subseteq \tilde{Y}_{(H)}$.

There is one of the subsets $X_{(H)}$ of a G-space X which is of particular interest, namely $X_{(G)}$. This consists of the set of points x which are left fixed by every operation of G and is called the stationary subset of X. Since the elements left fixed by any one operation of G on X is clearly closed, $X_{(G)}$ is the intersection of closed sets and hence closed. We note that $\Pi_X | X_{(G)}$ is clearly a one-to-one map of $X_{(G)}$ onto $\tilde{X}_{(G)}$, in fact it simply takes each point of $X_{(G)}$ into its unit class. Since Π_X is a closed mapping and $X_{(G)}$ is closed, $\Pi_X | X_{(G)}$ is a closed mapping, hence

1.2.5. PROPOSITION. If X is a G-space then Π_X is a homeomorphism of $X_{(G)}$ onto $\tilde{X}_{(G)}$.

There is a question naturally suggested by 1.2.4 that is often of interest; namely suppose that X and Y are G-spaces and $\tilde{f} : X/G \to Y/G$ satisfies $\tilde{f}(\tilde{X}_{(H)}) \subseteq \tilde{Y}_{(H)}$ for every orbit type (H). When can we "lift" $\tilde{f}$, i.e., when can we find an isovariant map $f : X \to Y$ having $\tilde{f}$ as its induced map. More generally given an invariant subset U of X and an isovariant map $f^* : U \to Y$ whose induced map is $\tilde{f} | \Pi_X(U)$ when can we extend f^* to an isovariant map of X into Y with induced map $\tilde{f}$. We will now answer this question in two very special cases which will be of importance in Chapter II.

1.2.6. PROPOSITION. Let X and Y be G-spaces and let $\tilde{f} : X/G \to Y/G$ satisfy $\tilde{f}(\tilde{X}_{(H)}) \subseteq \tilde{Y}_{(H)}$ for all $H \subseteq G$. Then if $f^* : X - X_G \to Y$ is an isovariant map with induced map $\tilde{f}|(X/G - \tilde{X}_G)$ there is a unique extension f of f^* to an isovariant map of X into Y with induced map $\tilde{f}$.

PROOF. It is clear from 1.2.5 that if f exists it is given on $X_{(G)}$ by $f(x) = \Pi_Y^{-1} \circ \tilde{f} \circ \Pi_X(x)$ and it will suffice to check the continuity of this extension. Since $X_{(G)}$ is closed and f so defined is clearly continuous on $X_{(G)}$ it will suffice to show that if $\{x_\alpha\}$ is a net in $X - X_G$ converging to a point x of $X_{(G)}$, then $f^*(x_\alpha) \to f(x)$. Let V be any neighborhood of f(x). Since $f(x) \in \Pi_Y^{-1} \tilde{f} \circ \Pi_X(X_{(G)}) = \Pi_Y^{-1} \tilde{f}(X_{(G)}) \subseteq \Pi_Y^{-1}(\tilde{Y}_{(G)}) = Y_{(G)}$ it follows from 1.1.14 that there is an invariant neighborhood U of f(x) included in V. Then $U^* = \Pi_X^{-1} \circ \tilde{f}^{-1} \circ \Pi_Y(U)$ is a neighborhood of x so x_α is eventually in U^*. Then $\tilde{f} \circ \Pi_X(x_\alpha)$ is eventually in $\Pi_Y(U)$ so $f(x_\alpha) \in \Pi_Y^{-1} \circ \tilde{f} \circ \Pi_X(x_\alpha)$ is eventually in $\Pi_Y^{-1}(\Pi_Y(U))$. But since U is invariant $\Pi_Y^{-1}(\Pi_Y(U)) = U$. q.e.d

1.2.7. DEFINITION. Let X be a G-space. An <u>isogeny</u> of X is an isovariant map $T : X \to X$ whose induced map T is the identity map of X/G. We denote by $\mathcal{I}(X)$ the set of isogenies of X.

1.2.8. PROPOSITION. If X is a locally compact G-space then an isogeny of X is an equivalence of X with itself, hence $\mathcal{I}(X)$ is a group.

PROOF. Immediate from 1.1.19.

1.2.9. PROPOSITION. Let X and Y be G-spaces and let $f : X \to Y$ be an isovariant map. Then $T \to f \circ T$ is a one-to-one map of $\mathcal{J}(X)$ into the set of isovariant maps of X into Y with induced map $\tilde{f}$. If X is locally compact then this map is onto, i.e. every isovariant map of X into Y with induced map $\tilde{f}$ is of the form $f \circ T$ for a unique $T \in \mathcal{J}(X)$.

PROOF. It is clear that if $T \in \mathcal{J}(X)$ then $f \circ T$ is an isovariant map of X into Y with induced map $\tilde{f}$. If $f \circ T_1 = f \circ T_2$ then since f is one-to-one on each orbit while T_1 and T_2 are self-homeomorphisms of each orbit it follows that $T_1 = T_2$. Now suppose X is locally compact and that f' is an isovariant map of X into Y with induced map $\tilde{f}$. Given $\tilde{x} \in X/G$ $f|\tilde{x}$ and $f'|\tilde{x}$ are each equivalences of $\tilde{x}$ with $\tilde{f}(\tilde{x})$. We define $T : X \to X$ by $T|\tilde{x} = (f|\tilde{x})^{-1} \circ (f'|\tilde{x})$. Then clearly $f' = f \circ T$, $\Pi_X \circ T = \Pi_X$ and $T(gx) = gT(x)$, so it will suffice to prove that T is continuous. Let $\{x_\alpha\}$ be a net in X converging to x. We must show that $Tx_\alpha \to Tx$. Since X is locally compact we may assume that x_α is in a compact neighborhood V of x. Then Tx_α lies in GV which is also compact. If $\{Tx_\alpha\}$ did not converge to Tx we could suppose by passing to a subnet that $Tx_\alpha \to x' \neq Tx$. Now

$$\Pi_X(x') = \lim \Pi_X(Tx_\alpha) = \lim \Pi_X(x_\alpha) = \Pi_X(x) = \Pi_X(Tx)$$

so x' and Tx are on the same orbit. On the other hand $f(x') = \lim f(Tx_\alpha) = \lim f'(x_\alpha) = f'(x) = f(Tx)$. But since f is one-to-one each orbit it follows that $Tx = x'$ a contradiction.

q.e.d.

1.2.10. DEFINITION. If X is a G-space we denote by $X \times I$ the product of X with the unit interval made into a G-space by the operations $g(x, t) = (gx, t)$. If f is a map of $X \times I$ into a space Y then for each $t \in I$ we denote by f_t the map of X into Y defined by $f_t(x) = f(x, t)$. If Y is also a G-space and f is isovariant then we call f an <u>isovariant</u> homotopy, or more explicitly an isovariant homotopy of f_o (or an isovariant homotopy connecting f_o and f_1).

REMARK. It is clear that with a natural identification (which we shall always make) $(X \times I)/G = X/G \times I$ and that (by 1.2.4) if $f : X \times I \to Y$ is an isovariant homotopy, then $\tilde{f}(\tilde{X}_{(H)} \times I) \subseteq \tilde{Y}_{(H)}$ for every orbit type (H).

1.2.11. PROPOSITION. Let X be a normal G-space, C a closed invariant subspace of X and U an invariant neighborhood of C. If T^* is an isogeny of $U \times I$ satisfying $T^*(u, o) = u$ for all $u \in U$ then there exists an isogeny T of $X \times I$ such that $T(x, o) = x$ for all $x \in X$ and $T | C \times I = T^* | C \times I$.

PROOF. By 1.1.7 we can find an invariant map $f : X \to I$ such that $f | C \equiv 1$ and such that the support of f is included in U. We put $T(x, t) = T^*(x, f(x)t)$ for $x \in U$ and $T(x, t) = x$ for $x \in X - U$.

1.2.12. THEOREM. Let X be a locally compact G-space satisfying the second axiom of countability and let $f : X \times I \to Y$ be an isovariant homotopy. Let C be a closed invariant subspace of X and U and invariant neighborhood of C. Let $f^* : U \times I \to Y$ be an isovariant homotopy such that $f_o^* = f_o | U$ and $\tilde{f}^* = \tilde{f} | \Pi_X(U) \times I$. Then there exists an isovariant homotopy $f' : X \times I \to Y$ with induced map $\tilde{f}$ such that $f' | C \times I = f^* | C \times I$.

PROOF. By 1.2.9 $f^* = f | (U \times I) \; T^*$ where T^* is a uniquely determined isogeny of $U \times I$. Since $f_o^* = f_o | U$ we have $T^*(u, o) = u$ for $u \in U$. Now X being locally compact and second countable is metrizeable and hence normal, so by 1.2.11 we can find an isogeny T of $X \times I$ such that $T | C \times I = T^* | C \times I$. We put $f' = f \circ T$. That f' has the desired properties follows from 1.2.9.

1.3. PRODUCTS, JOINS, AND REDUCED JOINS OF G-SPACES.

1.3.1. DEFINITION. If $\{X_\alpha\}$ is an indexed family of G-spaces then we make their topological product $\prod_\alpha X_\alpha$ into a G-space by defining $g(x_\alpha) = (gx_\alpha)$.

REMARK. That the product so defined is a G-spa e follows from the fact that a product of completely regular spaces is completely regular and that convergence of a net in the product is equivalent to convergence of each component.

1.3.2. PROPOSITION. If $X_1, \ldots X_n$ is a finite set of G-spaces which are all differentiable, Riemannian, or Euclidean, then their product is also differentiable, Riemannian, or Euclidean.

PROOF. Trivial.

REMARK. If $X_1, \ldots X_n$ are Euclidean G-spaces then as usual we denote their product by $X_1 \oplus \ldots \oplus X_n$.

1.3.3. PROPOSITION. If $\{X_\alpha\}$ is an indexed family of G-spaces and (x_α) is a point of their product, then $G_{(x_\alpha)} = \bigcap_\alpha G_{x_\alpha}$.

PROOF. Trivial

We now define the join of a finite indexed set of G-spaces $X_1, \ldots X_n$. Let X^* be the invariant subspace of $(X_1 \times I) \times \ldots \times (X_n \times I)$ consisting of all points $((x_1, t_1), \ldots, (x_n, t_n))$ such that $\sum_{i=1}^{n} t_i = 1$. We define an equivalence relation in X^* by setting $((x_1, t_1), \ldots, (x_n, t_n)) \sim ((x_1', t_1'), \ldots, (x_n', t_n'))$ if and only if $t_i = t_i'$, and $t_i \neq 0 \Longrightarrow x_i = x_i'$; $i = 1, 2 \ldots n$. It is easily checked that the resulting identification space is completely regular and that, since the equivalence relation is preserved by the operations of G, that it is a

G-space. We denote the equivalence class of $((x_1, t_1), \ldots (x_n, t_n))$ by $(t_1x_1, \ldots, t_nx_n)$ so that $g(t_1x_1, \ldots, t_nx_n) = (t_1gx_1, \ldots, t_ngx_n)$.

1.3.4. DEFINITION. The G-space defined above is called the join of $X_1, \ldots, X_n$ and is denoted by $X_1 \circ \ldots \circ X_n$. If $X_i = X$ $i = 1, \ldots n$ then we denote the join by $X^{(on)}$.

REMARK. For the topological properties of joins we refer the reader to [9]. We note that if $\{i_1, \ldots i_k\}$ is a subset of $\{1, \ldots n\}$ and $\{j_1, \ldots j_{n-k}\}$ is the complementary subset, then $\{(t_1x_1, \ldots t_nx_n) | t_{i_1} = \ldots = t_{i_k} = 0\}$ is an invariant subspace of $X_1 \circ \ldots \circ X_n$ which is canonically equivalent to $X_{j_1} \circ \ldots \circ X_{j_{n-k}}$. In particular $\{(t_1x_1, \ldots, t_nx_n) | t_j = 1\}$ can be naturally identified with X_j and so we shall regard the X_j and their partial joins as being actual subspaces of $X_1 \circ \ldots \circ X_n$. It is clear that to within equivalence the join is a commutative and associative operation on G-spaces.

From the definition of the equivalence relation defining the join it is immediate that

$$(t_1x_1, \ldots, t_nx_n) = (t_1'x_1', \ldots t_n'x_n')$$

if and only if $t_i = t_i'$ and $t_i \neq 0 \Rightarrow x_i = x_i'$. It follows that

1.3.5. PROPOSITION. Let $X_1, \ldots, X_n$ be G-spaces and let $x = (t_1x_1, \ldots, t_nx_n)$ be a point of their join. Then $G_x = \bigcap \{G_{x_i} | t_i \neq 0 \; i = 1, \ldots, n\}$.

It is clear from this that the join of G-spaces $X_1, \ldots, X_n$ may have orbit types occurring in it that occur in none of the X_i. The reduced join which we define next is free from this often undesirable possibility.

1.3.6. DEFINITION. The reduced join of G-spaces $X_1, \ldots, X_n$, denoted by $X_1 * \ldots * X_n$, is the set of points $(t_1x_1, \ldots t_nx_n)$ of $X_1 \circ \ldots \circ$ such that among the G_{x_i} with $t_i \neq 0$ there is one that is included in all the others. If all the X_i are equal to X then we write $X^{(*n)}$ for their reduced join. We note that the reduced join is an invariant subspace of the join and hence a G-space. We note that the remarks following 1. 3. 4 hold equally well for the reduced join, that is each of the X_i and any partial reduced join is a subspace of the full reduced join and that the reduced join is a commutative and associative operation.

1. 3. 7. PROPOSITION. If $X_1, \ldots, X_n$ are G-spaces then an orbit type occurs in $X_1 * \ldots * X_n$ if and only if it occurs in at least one of the X_i.

PROOF. Immediate from 1. 3. 5 and the definition of the reduced join.

1. 3. 8. DEFINITION. Let $H \subseteq G$. If X is a G-space we define $X_H = \{x \in X | G_x = H\}$. We say that X is (H)-simple if X is the saturation of X_H or equivalently if $X = X_{(H)}$.

REMARK. The equivalence of the two conditions is immediate from the relation $G_{gx} = gG_x g^{-1}$.

1. 3. 9. PROPOSITION. Let $H \subseteq G$ and let $N(H)$ be the normalizer of H in G. If X is a G-space the X_H is an $N(H)$-space. Moreover since H acts trivially on X_H we may regard X_H as an $N(H)/H$ space, and as such the isotropy group at each point is the identity.

PROOF. Trivial.

1. 3. 10. PROPOSITION. Let $H \subseteq G$. If $X_1, \ldots, X_n$ are H-simple G-spaces then $X = X_1 * \ldots * X_n$ is also (H)-simple and $X_H = (X_1)_H \circ \ldots \circ (X_n)_H$.

PROOF. X is (H)-simple by 1. 3. 7. It is clear that $(X_1)_H \circ \ldots \circ (X_n)_H$

$\subseteq X_H$. If $x = (t_1x_1, \dots t_nx_n) \in X_H$ then the set of G_{x_i} with $t_i \neq 0$ all include H and are conjugate to H, hence equal H, so $x \in (X_1)_H \circ \dots \circ (X_n)_H$.

q.e.d.

1.3.11. PROPOSITION. Let X and $X_1, \dots, X_n$ be G-spaces and let $f : x \to (\phi_1(x)f_1(x), \dots, \phi_n(x)f_n(x))$ be an isovariant map of X into $X_1 * \dots * X_n$. Then

(1) The ϕ_i form an invariant partition of unity for X.

(2) f_i is an equivariant map of $\phi_i^{-1}((0,1])$ into X_i.

(3) For each $x \in X$ $\exists i$ such that $\phi_i(x) \neq 0$ and $G_{f_i(x)} = G_x$.

Conversely if $\phi_1, \dots, \phi_n$ and $f_1, \dots, f_n$ satisfy conditions (1) - (3) then the map f as defined above is an isovariant map of X into $X_1 * \dots * X_n$.

PROOF. Straightforward verification.

1.4. EUCLIDEAN G-SPACES

1.4.1. PROPOSITION. For each closed subgroup H of G there exists a non-singular, differentiable equivariant imbedding of G/H in some Euclidean G-space. Thus H occurs as an isotropy group in some Euclidean G-space or, what is the same, there is an orbit of type (H) in some Euclidean G-space.

PROOF. Let $L^2(G)$ be the Hilbert space of complex valued functions on G which are square summable with respect to Haar measure and make $L^2(G)$ into a G-space by defining gf for $f \in L^2(G)$ by $(gf)(\gamma) = f(\gamma g^{-1})$. Let $\bar{f}$ be a continuous real valued function on G/H which assumes the value one only at the coset H and define $f \in L^2(G)$ by $f(g) = \bar{f}(gH)$. It is clear that $G_f = H$. By the Peter-Weyl Theorem $L^2(G) = \bigoplus_{i=1}^{\infty} X_i$ where the X_i are finite dimensional, mutually orthogonal invariant linear subspaces and the direct sum is in the Hilbert space sense. Let f_i be the projection of f on X_i and let $H_i = G_{f_i}$. Then clearly $H = \bigcap_i H_i$. Now the closed subgroups of a compact Lie group satisfy the descending chain condition: at each step in a properly descending chain either the dimension or number of components must decrease. Hence we can find integers $i_1, \ldots, i_n$ such that $H = \bigcap_{j=1}^{n} H_{i_j}$. Then $X = X_{i_1} \oplus \ldots \oplus X_{i_n}$ is a Euclidean G-space and $x = f_{i_1} + \ldots + f_{i_n}$ is a point of X with $G_x = H$. That there exists a non-singular, differentiable, equivariant imbedding of G/H in X is now an immediate consequence of 1.1.22.

q.e.d.

1.4.2. PROPOSITION. If $H \subseteq G$ and X is a Euclidean H-space then there exists a Euclidean G-space which, considered as an H-space by restriction, has X as an invariant linear subspace.

PROOF. We can assume that X can be given the structure of a complex Hilbert space (consistent with its real vector space structure) in which the operations of G are unitary (otherwise let X^* be the complexification of X; since X^* has X as a real linear, invariant subspace, a Euclidean G-space Y which has X^* as an H-invariant linear subspace also has X as an H-invariant linear subspace). We can also assume that the representation of H in X is irreducible in this complex structure, for if we write X as a direct sum of complex irreducible subspaces X_i and we find a Y_i which works for each X_i, then the direct sum of the Y_i's works for X. Our proposition then follows directly from Proposition 1, page 211 of [3].

q.e.d.

We next come to one of the most basic results in the theory of G-spaces.

1.4.3. TIETZE-GLEASON THEOREM. Let X be a G-space (respectively, normal G-space) and let C be a compact (respectively, closed) invariant subspace of X. If f^* is an equivariant map of C into a Euclidean G-space Y there exists an extension of f^* to an equivariant map of X into Y.

PROOF. By Tietze's Extension Theorem (applied, when X is not normal, to the Čech compactification of X) there is an extension of f^* to a map f' of X into Y. We make f' equivariant by the usual process of averaging with respect to Haar measure. That is we define $f : X \to Y$ by $f(x) = \int g^{-1} f'(gx)dg$. It is clear that f is continuous, and if $x \in C$ then $g^{-1} f'(gx) = g^{-1} f^*(gx) = g^{-1} g f^*(x) = f^*(x)$ so $f(x) = f^*(x)$ and f is an extensio of f^*. For arbitrary x we have, by the invariance of Haar measure, $f(\gamma x) = \int g^{-1} f'(g\gamma s)dg = \int \gamma g^{-1} f'(gx)dg$. Since the operations of G on Y are linear we can pass γ through the integral sign to get $f(\gamma x) = \gamma f(x)$, so f is equivariant. q.e.d

1.4.4. PROPOSITION. If f is an equivariant map of a G-space X into a Euclidean G-space V, there exists an equivariant map f' of X into a Euclidean G-space such that $\|f'(x)\| \equiv 1$ and such that f' maps homeomorphically any subset of X which is mapped homeomorphically by f.

PROOF. Let W be a one-dimensional Euclidean G-space on whic G acts trivially and let ω be a non-zero vector in W. Clearly $f^* : x \to (f(x), \omega)$ is an equivariant map of X into the invariant set $J = \{(v, \omega) | v \in$ of $V \oplus W$ which is a homeomorphism on every subset of X on which f Since $T : u \to u/\|u\|$ is clearly an equivalence of J with a unit hemisphe of $V \oplus W$, $f' = T \circ f^*$ has the desired properties. q.e.

1.4.5. PROPOSITION. Let X be a G-space, $x \in X$ and F a closed (not necessarily invariant) subset of

X not containing x. Then there exists a bounded equivariant map f of X into a Euclidean G-space which is a homeomorphism on Gx such that $f(x) \notin \overline{f(F)}$.

PROOF. By 1.4.1 we can find an element v of a Euclidean G-space V_1 such that $G_x = G_v$, hence by 1.2.2 there is an equivariant homeomorphism of Gx onto Gv (namely $gx \to gv$) which by 1.4.3 can be extended to an equivariant map of X into V_1. By 1.4.4 we can find an equivariant map f^* of X into the unit sphere of a Euclidean G-space V which is a homeomorphism on Gx. Since $x \notin F$, $f^*(x) \notin f^*(Gx \cap F)$ hence, since $Gx \cap F$ is compact, we can find disjoint neighborhoods U_1 and U_2 of $f^*(x)$ and $f^*(Gx \cap F)$ in the unit sphere of V. We note that $I \cdot U_1$ and $I \cdot U_2$ have only the origin of V in common (where $I \cdot U_i = \{\alpha u \mid 0 \leq \alpha \leq 1,\ u \in U_i\}$). Now $f^{*-1}(U_2) \cup X - F$ is a neighborhood of Gx and so by 1.1.14 includes an invariant neighborhood W of Gx. By 1.1.7 we can find an invariant map $h : X \to I$ such that $h|Gx \equiv 1$ and $h|X - W \equiv 0$. We define $f : X \to V$ by $f(x') = h(x')f^*(x')$. It is clear that f is a bounded equivariant map of X into V and since it agrees with f^* on Gx it is a homeomorphism on Gx. Since f maps $F \cap (X-W)$ into the origin while $\|f(x)\| = 1$ it is clear that f(x) is not adherent to $f(F \cap (X-W))$. On the other hand $f(F \cap W) \subseteq f(F \cap (f^{*-1}(U_2) \cup (X-F))) \subseteq f(f^{*-1}(U_2)) \subseteq I \cdot f^*(f^{*-1}(U_2)) \subseteq I \cdot U_2$. Since $I \cdot U_1$ with the origin deleted is a neighborhood of $f(x) = f^*(x)$ in the closed unit ball of V which is disjoint from $I \cdot U_2$, it follows that f(x) is also not adherent to $f(F \cap W)$. Hence f(x) is not adherent to $f(F \cap (X-W)) \cup f(F \cap W) = f(F)$. q.e.d.

1.4.6. PROPOSITION (Mostow [12]). Let X be a metrizeable G-space, F a closed invariant subspace of X and suppose that each of F and X - F admit equivariant imbeddings in Euclidean G-spaces. Then X itself admits an equivariant imbedding in a Euclidean G-space.

PROOF. Let f_1 be an equivariant imbedding of F into a Euclidean G-space V, extended by 1.4.3 to an equivariant map of X into V. Let ρ be an invariant metric for X (1.1.12) and define $h(x) = \text{Inf}\,\{\rho(x,z) + \|f_1(x) - f_1(z)\| \mid z \in F\}$. Then h is continuous, invariant, positive on $X - F$ and $x_n \to x \in F \Rightarrow h(x_n) \to 0$. Now let f_2^* be an equivariant imbedding of $X - F$ in a Euclidean G-space W which, by 1.4.4 we can assume to satisfy $\|f_2^*(x)\| \equiv 1$. Define $f_2(x) = h(x)f_2^*(x)$. That f_2 is continuous, one-to-one, and equivariant is clear. If $f_2(x_n) \to f_2(x)$ then $h(x_n) = \|f_2(x_n)\| \to \|f_2(x)\| = h(x)$ so $f_2^*(x_n) \to f_2^*(x)$ and hence $x_n \to x$ so f_2 is an imbedding. Moreover we can extend f_2 to an equivariant map of X into W by setting $f_2(x) = 0$ for $x \in F$. We now define $f : X \to V \oplus W$ by $f(x) = (f_1(x), f_2(x))$. It is clear that f is continuous, equivariant and a homeomorphism on each of F and $X - F$. Since $f_2(x) = 0$ when $x \in F$ and $f_2(x) \neq 0$ when $x \in X - F$ it follows that f is one-to-one. Now suppose that $f(x_n) \to f(x)$. The proof will be complete if we show that $x_n \to x$. If $x \in X - F$ then $f_2(x) \neq 0$ so $f_2(x_n) \neq 0$ for large n, so $x_n \in X - F$ for large n and since f is a homeomorphism on $X - F$ we get $x_n \to x$. If $x \in F$ then by definition of h we can choose $z_n \in F$ so that $\rho(x_n, z_n) < 2h(x_n)$ and $\|f_1(x_n) - f_1(z_n)\| \leq 2h(x_n)$. Since $h(x_n) = \|f_2(x_n)\| \to \|f_2(x)\| = 0$ it follows that $\lim f_1(z_n) = \lim f_1(x_n) = f_1(x)$, and since f_1 is a homeomorphism on F we get $z_n \to x$. But since $\rho(x_n, z_n) \leq 2h(x_n) \to 0$ it follows that $x_n \to x$ also. q.e.d.

1.4.7. COROLLARY. Let X be a separable metric G-space of finite dimension. If $X - X_{(G)}$ admits an equivariant imbedding in a Euclidean G-space then so does X.

PROOF. By a standard result $X_{(G)}$ admits an imbedding in a Euclidean space Y. If we make Y into a Euclidean G-space by letting G act trivially then this imbedding is automatically equivariant. Since $X_{(G)}$ is closed in X the corollary follows.

1.4.8. COROLLARY. Let X be a G-space and let $O_1, \ldots, O_n$ be a covering of X by invariant open sets. If each O_i admits an equivariant imbedding in a Euclidean G-space then so does X.

PROOF. If $n = 1$ the corollary is trivial so we can proceed by induction and assume that $O_1 \ \ldots \ O_{n-1}$ admits an equivariant imbedding in a Euclidean G-space. Then $F = X - (O_1 \ \ldots \ O_{n-1})$ being included in O_n admits an equivariant imbedding in a Euclidean G-space, hence by 1.4.6 so does X.

1.5. COMPACTIFICATIONS OF G-SPACES.

Suppose X is a locally compact G-space and $X^* = X \cup \{\infty\}$ is the one-point compactification of X. Then it is clear that X^* becomes a G-space if we define $g(\infty) = \infty$ for all $g \in G$.

If X is not locally compact then we might try to compactify X with the Čech compactification $\beta(X)$ of X. Then each operation of G on X extends uniquely to a homeomorphism of $\beta(X)$ and the map which takes $g \in G$ into this extended operation is a homomorphism of G into the group of homeomorphisms of X. Unfortunately the natural map $(g, x) \rightarrow g(x)$ of $G \times \beta(X)$ into $\beta(X)$ is in general not continuous and in fact if we take $x^* \in \beta(X) - X$ the map $g \rightarrow g(x^*)$ is not even continuous in general. A simple example, pointed out by M. Jerison, is as follows: X = complex plane, G = unit circle, and the action of G on X is by multiplication. If R is the real axis considered as a net in $\beta(X)$ with its usual ordering and x^* is a limit point of R in $\beta(X)$ then $e^{i\theta}x^*$ will be a limit point of $e^{i\theta}R$. Let $\overline{K}$ be the closure of $K = \{z \in X \mid |\text{Im } z| \geq |\text{Re}(z)|^{-1}\}$ in $\beta(X)$. Then it is readily seen that $\overline{K}$ is disjoint from the closure of R in $\beta(X)$ (K and R can be separated by a bounded continuous real valued function f on X whose unique extension to $\beta(X)$ separates $\overline{K}$ and the closure of R). Hence $\beta(X) - \overline{K}$ is a neighborhood of x^*. On the other hand if $\theta \neq 0 \pmod{2\pi}$ the net

$e^{i\theta}R$ is eventually in K hence $e^{i\theta}x^*$ is in $\overline{K}$, so $e^{i\theta}x^*$ does not approach x^* as $\theta \to 0$.

We now set out to rectify this difficulty.

1.5.1. DEFINITION. Let X be a G-space. A G-space X^* will be called a G-compactification of X if X^* is compact and X is a dense invariant subspace of X^*.

1.5.2. DEFINITION. A compactification X^* of a G-space X will be called a Čech G-compactification of X if every equivariant map of X into a compact G-space Y admits an (automatically unique) extension to an equivariant map of X^* into Y.

1.5.3. PROPOSITION. If X^* and X^{**} are two Čech G-compactifications of the G-space X then the identity map of X extends (uniquely) to an equivalence of X^* with X^{**}. Hence there is essentially at most one Čech G-compactification of X.

PROOF. Trivial.

We now show that every G-space admits a Čech G-compactification. The construction is a fairly obvious modification of one of the classical methods of obtaining the ordinary Čech compactification, suitably modified so as to take the G-structure into account.

1.5.4. THEOREM. Every G-space admits a Čech G-compactification.

PROOF. Let F be the collection of all bounded equivariant maps of X into Euclidean G-spaces (to avoid logical difficulties the Euclidean G-spaces may be assumed to be restricted to the number spaces R^n). For each $f \in F$ let X_f be the closure of $f(X)$ in the ambient Euclidean space. Then $\{X_f\}_{f\in F}$ are compact G-spaces, hence so is their produce $\prod_{f\in F} X_f$. Now the map $e : X \to \prod_{f\in F} X_f$ defined by $(e(x))_f = f(x)$ is clearly equivariant and by 1.4.5 and Lemma 5, page 116 of [7] it follows that e maps X homeomorphically into $\prod_{f\in F} X_f$. We let X^* be the closure of e(X) in $\prod_{f\in F} X_f$ and identify $x \in X$ with its image e(x) in X^*. It is clear that X^* is a G-compactification of X. That it is a Cech G-compactification follows by an easy modification of the proof of Theorem 24 of [7], page 153. q.e.d.

For future purposes we will need only the following weak corollary.

1.5.5. COROLLARY. Every G-space is an invariant subspace of a compact G-space.

1.6. G-AR's and G-ANR's.

1.6.1. DEFINITION. A G-space X is a G-absolute retract (abbreviated G-AR) if given a normal G-space Y, a closed invariant subspace F of Y and an equivariant map $f : F \to X$ there is an extension of f to an equivariant map of Y into X.

A G-space X is a G-absolute neighborhood retract (G-ANR) if given a normal G-space Y and an equivariant map f of a closed invariant subspace F of Y into X there is an extension of f to an equivariant map of an invariant neighborhood of F into X.

1.6.2. PROPOSITION. A Euclidean G-space is a G-AR.

PROOF. This is just a restatement of the Tietze-Gleason Theorem (1.4.3).

1.6.3. PROPOSITION. Let Y be a G-space, F a compact invariant subspace of Y and f an equivariant map of F into a G-AR (respectively, G-ANR) X. Then there is an extension of f to an equivariant map of Y (respectively, an invariant neighborhood of F in Y) into X.

PROOF. By 1.5.5 there is a compact (and hence normal) G-space

Y^* with Y as an invariant subspace. Since F is closed in Y^* there is an extension $\tilde{f}$ of f to an equivariant map of Y^* (respectively, of an invariant neighborhood of F in Y^*) into X. Then $\tilde{f}|Y$ gives the desired extension.

q. e. d.

1.6.4. PROPOSITION. Let Y be an invariant subspace of a G-ANR X. If there exists an equivariant retraction of an invariant neighborhood of Y in X onto Y then Y is a G-ANR.

PROOF. Obvious.

1.6.5. THEOREM (Koszul [8] p. 138). Let M be a differentiable G-space and Σ a compact invariant submanifold of M. Then there is an open invariant neighborhood O of Σ in M and an equivariant differentiable retraction f of O onto Σ with the property that for each $\sigma \in \Sigma$ there is a coordinate system in M centered at σ such that G_σ acts linearly, and in fact orthogonally, relative to this coordinate system and $f^{-1}(\sigma)$ is an open disc centered at σ in an invariant linear subspace.

PROOF. By 1.1.20 we can assume that M is Riemannian. Let $N(\Sigma)$ be the normal bundle of Σ in M and put $N(\Sigma, \epsilon) = \{v \in N(\Sigma) \mid \|v\| < \epsilon\}$. Let E denote the restriction of the exponential map to $N(\Sigma, \epsilon)$. Then as is well known if ϵ is sufficiently small (as we henceforth assume) E is a diffeomorphism of $N(\Sigma, \epsilon)$ onto a neighborhood O of Σ in M. Moreover if δg denotes the differential of the operation of an element g of G on M then $(g, v) \rightarrow \delta g(v)$ defines an operation of G on $N(\Sigma, \epsilon)$ which makes $N(\Sigma, \epsilon)$ a differentiable G-space. Since the operations of G on M are isometries $E \circ \delta g = g \circ E$, i.e. E is an equivalence of $N(\Sigma, \epsilon)$ with O. Then the fibre projection of $N(\Sigma, \epsilon)$ onto the zero cross-section, carried over to O via E, gives the desired equivariant retraction of O onto Σ

and any Riemannian normal coordinates about σ has the required properties.

q. e. d.

1.6.6. PROPOSITION. If a compact differentiable G-space admits a non-singular, differentiable, equivariant imbedding in a Euclidean G-space it is a G-ANR.

PROOF. Immediate from 1.6.4 and 1.6.5 since a Euclidean G-space is a G-AR (1.6.2) and a fortiori a G-ANR.

REMARK. It is in fact the case that every compact differentiable G-space admits a non-singular, differentiable, equivariant imbedding in a Euclidean G-space, and hence is a G-ANR. This was proved independently by the author [14] and Mostow [12]. While both proofs are quite elementary we will not repeat either here, but rather content ourselves with the following special case, which is all we will require for further use.

1.6.7. COROLLARY. If $H \subseteq G$ then G/H is a G-ANR.

PROOF. 1.4.1 and 1.6.6.

1.7. KERNELS AND SLICES.

To avoid endless repetition we shall assume throughout this section that H is a closed subgroup of G.

1.7.1. DEFINITION. A subset S of a G-space X will be called an H-kernel (over $\Pi_X(S)$) if:

(1) S is closed in GS;

(2) S is H-invariant, i.e. $HS = S$;

(3) for each $g \in G$ not in H gS is disjoint from S.

An H-kernel S in X will be called an H-slice in X if $\Pi_X(S)$ is open, or equivalently if GS is open.

If $x \in X$ then by a slice at x we shall mean a G_x-slice in X which contains x.

REMARK. We note that the notion of an H-kernel is absolute while the notion of an H-slice is relative. In fact an H-kernel S is an H-slice in GS.

1.7.2. PROPOSITION. If S is an H-kernel (respectively, H-slice) in X over $\tilde{S}$ and $\tilde{O}$ is any subset (respectively, open subset) of $\tilde{S}$ then $S \quad \Pi_X^{-1}(\tilde{O})$ is an H-kernel (respectively, H-slice) in X over $\tilde{O}$.

PROOF. Direct verification.

1.7.3. PROPOSITION. If $\{\tilde{S}_\alpha\}$ is a disjoint collection of open sets in X/G and S_α is an H-slice over $\tilde{S}_\alpha$ then $S = \bigcup_\alpha S_\alpha$ is an H-slice over $\tilde{S} = \bigcup_\alpha \tilde{S}_\alpha$.

PROOF. The only fact that is perhaps not obvious is that S is closed in GS. But since GS is the disjoint union of open sets GS_α and $S_\alpha = S \cap GS_\alpha$ is closed in GS_α, this follows easily. q.e.d.

1.7.4. PROPOSITION. If S is an H-kernel in X and $s \in S$ then $G_s \subseteq H$ and hence $H_s = G_s$.

PROOF. Immediate from the third condition for an H-kernel.

Now let S be an H-kernel in the G-space X and let $K \subseteq H$. Regarding S as an H-space $S_{(K)}$ is the set of points $s \in S$ such that H_s is conjugate in H to K. On the other hand $X_{(K)} \cap S$ is the set of points $s \in S$ such that G_s (which, by 1.7.4, equals H_s) is conjugate in G to K. Clearly then $S_{(K)} \subseteq X_{(K)} \cap S$. The following additional information will be needed in Chapter II.

1.7.5. PROPOSITION. If S is an H-kernel in the G-space X and $K \subseteq H$ then $S_{(K)}$ is closed in $X_{(K)} \cap S$.

PROOF. Let $\{s_\alpha\}$ be a net in $S_{(K)}$ converging to a point $s \in X_{(K)}$ S. Then $H_{s_\alpha} = h_\alpha K h_\alpha^{-1}$, or $H_{h_\alpha s_\alpha} = K$ where $\{h_\alpha\}$ is a net in H. Since

H is compact by passing to a subset we can assume that $h_\alpha \to h \in H$, so $h_\alpha s_\alpha \to hs$. Now $H_{h_\alpha s_\alpha} = K$ clearly implies that $H_{hs} \subseteq K$ and so, by 1.7.4, $G_{hs} \subseteq K$ so $G_s \subseteq h^{-1}Kh$. On the other hand since $s \in X_{(K)}$ we have $G_s = gKg^{-1}$ so G_s and K have the same dimension and number of components. It follows that $H_s = G_s = h^{-1}Kh$ so $s \in S_{(K)}$. q. e. d.

1.7.6. PROPOSITION. If S is an H-kernel in the G-space X then $h: \Pi_S(s) \to \Pi_X(s)$ is a well defined homeomorphism of S/H onto $\Pi_X(S) = GS/G$, and for any $K \subseteq H$, $h(\tilde{S}_{(K)})$ is a closed subset of $\tilde{X}_{(K)}$.

PROOF. Since $\Pi_S(s) = \Pi_S(s')$ means $s' = hs$ for some $h \in H$, while $\Pi_X(s) = \Pi_X(s')$ means $s' = gs$ for some $g \in G$ it follows trivially that h is well-defined and, from the third condition for a slice, that h is one-to-one. In proving that h is a homeomorphism we may, because of 1.1.13, assume that $\Pi_X(S) = X/G$. Since Π_X is a closed map and S is closed in GS = S, $\Pi_X|S$ is a closed map. On the other hand Π_S is open, and since $\Pi_X|S = h \circ \Pi_S$ it follows that h is closed and h^{-1} open, so that h is bicontinuous. The final remark follows easily from 1.7.5 since $h(\tilde{S}_{(K)}) = h \circ \Pi_S(\tilde{S}_{(K)}) = (\Pi_X|S)(S_{(K)})$ while (since GS = X) $\tilde{X}_{(K)} = \Pi_X(X_{(K)}) = \Pi_X(X_{(K)} \cap S) = (\Pi_X|S)(X_{(K)} \cap S)$.

The following theorem, which could serve as an alternative definition of an H-kernel, will explain the choice of name.

1.7.7. THEOREM. Let X be a G-space and let $\tilde{S} \subseteq X/G$. Then $f \to f^{-1}(H)$ is a one-to-one correspondence between all equivariant maps of $\Pi_X^{-1}(\tilde{S})$ into G/H and all H-kernels in X over $\tilde{S}$. If S is any H-kernel in X over $\tilde{S}$ then the corresponding equivariant map f of $\Pi_X^{-1}(\tilde{S})$ into G/H is given by $f(gs) = gH$; $g \in G$, $s \in S$.

PROOF. With no loss of generality we can assume that $\tilde{S} = X/G$. It is a trivial matter to check that if $f : X \to G/H$ is equivariant then $f^{-1}(H)$ is an H-kernel over X/G. Now suppose S is an H-kernel over X/G. It is clear that if $f : X \to G/H$ is equivariant and $S = f^{-1}(H)$ then $f(gs) = gH$ for $g \in G$, $s \in S$. Now if $g_1 s_1 = g_2 s_2$; $g_1, g_2 \in G$, $s_1, s_2 \in S$ then $g_2^{-1} g_1 s_1 = s_2$ and hence by condition (3) for a kernel $g_2^{-1} g_1 \in H$ so $g_1 H = g_2 H$ so the above formula gives a well defined function f of X into G/H which clearly satisfies $f(gx) = gf(x)$ and it remains only to check that this function is continuous. Let $\tilde{K}$ be a closed subset of G/H and let K be the union of the cosets in $\tilde{K}$. By definition of the topology of G/K, K is closed in G and hence, since S is closed in X, KS is closed in X by 1.1.1. But clearly $KS = f^{-1}(\tilde{K})$, so f is continuous. q. e. d.

1.7.8. COROLLARY. Let X and Y be G-spaces and $f : X \to Y$ an equivariant map. If S is an H-kernel (respectively, H-slice) in Y over $\tilde{S}$ then $f^{-1}(S)$ is an H-kernel (respectively, H-slice) in X over $\tilde{f}^{-1}(\tilde{S})$.

PROOF. If $h : \Pi_Y^{-1}(S) \to G/H$ is equivariant with $S = h^{-1}(H)$ then $h \circ f : \Pi_X^{-1}(f^{-1}(S)) \to G/H$ is equivariant and $(h \circ f)^{-1}(H) = f^{-1}(S)$. q. e. d.

1.7.9. COROLLARY. Let S be an H-kernel in X and let $\chi : U \to G$ be a local cross-section in G/H. Then for any $g_o \in G$ the map $F : (u, s) \to g_o \chi(g_o^{-1} u)s$ is a homeomorphism of $g_o U \times S$ onto an open neighborhood of $g_o S$ in GS. Moreover if f is the equivariant map of GS into G/H with $S = f^{-1}(H)$ then $f(F(u, s)) \equiv u$.

PROOF. We recall that $f(gs) = gH$ for $g \in G$, $s \in S$, hence since $\chi(g_o^{-1}u)H = g_o^{-1}u$, because χ is a local cross-section, it follows that $f(F(u, s)) = g_o(\chi(g_o^{-1}u)H) = g_o g_o^{-1} u = u$. Thus $F(g_o U \times S) = f^{-1}(g_o U)$ which is an open neighborhood of g_oS in GS. The continuity of F is clear. Now suppose that $F(u_\alpha, s_\alpha) \to F(u, s)$. We will complete the proof by showing that $u_\alpha \to u$ and $s_\alpha \to s$. In fact $u_\alpha = f(F(u_\alpha, s_\alpha)) \to f(F(u, s)) = u$, so $\chi(g_o^{-1}u_\alpha)^{-1} \to \chi(g_ou)^{-1}$, and since $\chi(g_o^{-1}u_\alpha)s_\alpha = g_o^{-1}F(u_\alpha, s_\alpha) \to g_o^{-1}F(u, s) = \chi(g_o^{-1}u)^{-1}s$ it follows that $s_\alpha \to s$.

q.e.d.

1.7.10. THEOREM. Let S and S' be H-kernels in G-spaces X and Y respectively and let f_o be an H-equivariant map of S into S'. Then there is a uniquely determined G-equivariant map $f : GS \to GS'$ such that $f|S = f_o$, namely $f(gs) = gf_o(s)$, $g \in G$, $s \in S$. Moreover, if f_o is isovariant then so is f and if f_o is an imbedding of S into (respectively, an equivalence of S with) S' then f is an imbedding of GS into (respectively, an equivalence of GS with) GS'.

PROOF. It is clear that if f exists it is given by $f(gs) = gf_o(s)$, moreover since (1.7.4 and 1.1.16) $G_s \subseteq G_{f_o(s)}$ it follows that the above formula gives a well defined function of GS into GS' whose restriction to S is f_o and which clearly satisfies $f(gx) = gf(x)$. If f_o is isovariant then $G_s = H_s = H_{f_o(s)} = G_{f_o(s)} = G_{f(s)}$ for $s \in S$ so that (1.1.16) if f is an equivariant map it is isovariant. If f_o is one-to-one and $f(gs) = f(g's')$ then $gf_o(s) = g'f_o(s')$ so $g^{-1}g'f_o(s') = f_o(s)$. Since S' is an H-kernel and $f_o(s)$, $f_o(s') \in S'$ it follows from the third condition for a slice that $g^{-1}g' \in H$. Since f_o is H-equivariant $f_o(s) = f_o(g^{-1}g's')$ so $s = g^{-1}g's'$ and $gs = g's'$, so f is one-to-one. To complete the proof it remains only to check that f is continuous (if f_o^{-1} exists and is continuous then it will follow by symmetry that f^{-1} is continuous). Let $\chi: U \to G$ be a local cross-section in G/H and $g_o \in G$. By 1.7.9 $F : (u, s) \to g_o\chi(g_o^{-1}u)s$ is a homeomorphism of $g_oU \times S$ onto a neighborhood of g_oS in GS and similarly $F' : (u, s') \to$

$g_o\chi(g^{-1}u)s'$ is a homeomorphism of $g_oU \times S'$ onto a neighborhood of g_oS' in GS'. Since $f(F(u,s)) = f_o(g_o\chi(g_o^{-1}u)s) = g_o\chi(g_o^{-1}u)f_o(s) = F'(u, f_o(s))$ the continuity of f on g_oS is clear. Since g_o was arbitrary f is continuous on GS. q.e.d.

1.7.11. COROLLARY. Let S be an H-kernel in the G-space X. If S admits an equivariant imbedding in a Euclidean H-space then GS admits an equivariant imbedding in a Euclidean G-space.

PROOF. In view of 1.7.10 it will cearly suffice to prove the following lemma.

1.7.12. LEMMA. If V is a Euclidean H-space then there exists an H-equivariant imbedding of V onto an H-kernel in some Euclidean G-space.

PROOF. By 1.4.2 we can find a Euclidean G-space W which, as an H-space, includes V as an invariant linear subspace. By 1.4.1 we can find a Euclidean G-space U with a $u \in U$ such that $G_u = H$. Clearly $v \rightarrow (v, u)$ is an H-equivariant imbedding of V into $W \oplus U$ so it will suffice to that $S = \{(v,u) \mid v \in V\}$ is an H-kernel in $W \oplus U$. Clearly S is closed in $W \oplus U$ and is H-invariant. Moreover if $g \in G$ is not in H and $(v,u) \in S$ then $g(v,u) = (gv, gu)$ cannot be in S because $gu \neq u$. This shows that S is in fact an H-kernel. q.e.d.

We have seen above that if S is an H-kernel in the G-space X then $f : gs \rightarrow gH$ is an equivariant map of GS onto G/H. Moreover $f^{-1}(gH) = gS$ so that the inverse images of points are all homeomorphic to S. It is natural to guess that GS is a fiber bundle over G/H with fiber S and projection F. We shall see that this is in fact the case and moreover that this fiber bundle admits H/K as a structural group, where $K = \bigcap_{s \in S} H_s$ and that the associated principal bundle is G/K under the action $(hK)(gK) = gh^{-1}K$. First we give a quick review of the theory of principal bundles and their associated fiber bundles as developed in [2].

By definition a G-principal bundle is simply an (e)-simple G-space, i.e. a G-space X such that the isotropy group at each point of X is the identity. It is easily seen that if H is a closed subgroup of G and K a closed normal subgroup of H then the action defined above makes G/K a H/K principal bundle with orbit space G/H. Now let X be a G-principal bundle and Y any G-space. Note that the map $\Pi : (x, y) \to \Pi_X(x)$ is an invariant map of $X \times Y$ into X/G and hence induces a map f of $(X \times Y)/G$ into X/G. We write $X \times_G Y$ for $(X \times Y)/G$ and call the triple $(X \times_G Y, X/G, f)$ the fiber bundle with fiber Y associated with the principal bundle X. To proceed further we need the following proposition:

1.7.13. PROPOSITION. Let G and Γ be compact Lie groups and let X be a space which is simultaneously A G-space and a Γ-space in such a way that the operations of G and Γ on X commute. Then X/G is a Γ-space under the operation $\gamma \Pi_{(X,G)}(x) = \Pi_{(X,G)}(\gamma x)$, where $\Pi_{(X,G)}$ is the orbit map of $X \to X/G$.

PROOF. That these operations of Γ on X/G are well defined follows from the fact that the operations of G and Γ commute (for this implies that a operation of Γ carries G-orbits into G-orbits). Now we have commutativity in the diagram

$$\begin{array}{ccc} \Gamma \times X & \xrightarrow{\Phi} & X \\ {\scriptstyle id \times \Pi_{(X,G)}} \downarrow & & \downarrow {\scriptstyle \Pi_{(X,G)}} \\ \Gamma \times X/G & \longrightarrow & X/G \end{array}$$

and since $\Pi_{(X,G)}$ and Φ are continuous and $id \times \Pi_{(X,G)}$ is open, it follows that the action of Γ on X/G is continuous. q. e. d.

Now let us get back to our G-principal bundle X and G-space Y and suppose we have a compact Lie group Γ acting on X so that the operations of Γ on X commute with the operations of G. If we let Γ act trivially on Y then the operations of Γ on $X \times Y$ commute with the operations of G on $X \times Y$. Thus $(X \times Y)/G = X \times_G Y$ and X/G are both Γ-spaces and since the map $(x, y) \to x$ of $X \times Y \to X$ is clearly Γ equivariant it follows that the fiber map $f : X \times_G Y \to X/G$ is also.

Let us now specialize this process. Let as above K be a closed normal subgroup of the closed subgroup H of G and let S be an H-space such that $\bigcap_{s \in S} H_s = K$, so that we can regard S as an H/K space. As noted above G/K is an H/K principal bundle under the operations $(hK)(gK) = gh^{-1}K$ and the orbit space in G/H. On the other hand G/K is as usual a G-space under the action $g(g'K) = gg'K$. Since clearly the actions of G and of H/K on G/K commute we are in the situation discussed above (with H/K playing the role of G and G the role of Γ). In other words the fiber bundle $G/K \times_{H/K} S$ is a G-space and the fiber map f is an equivariant map of $G/K \times_{H/K} S$ onto G/H. It follows of course that $f^{-1}(H)$ is an H-slice in $G/K \times_{H/K} S$ (over the whole orbit space). Let us show that we can identify this slice with S. Let $\Pi : G/K \times S \to G/K \times_{H/K} S$ be the orbit map and define $k : S \to G/K \times_{H/K} S$ by $k(s) = \Pi(K, s)$. It is clear that k maps S into $f^{-1}(H)$ (for $f(\Pi(gK, s)) = gH$) and it is onto for if $z \in f^{-1}(H)$ then z is of the form $\Pi(hK, s)$. But $\Pi(hK, s) = \Pi(hK, h^{-1}hs) = \Pi((h^{-1}K)(K, hs)) = \Pi(K, hs) = k(hs)$. Moreover we see that k^{-1} is defined and continuous. Finally $k(hs) = \Pi(hK, s) = \Pi(h(K, s)) = h\Pi(K, s) = hk(s)$, so h is an equivalence of S onto $f^{-1}(H)$. Thus we have constructed a G-space X with S as a slice over X/G. If Y is any G-space with S as a slice over Y/G then by 1.7.10 the identity map of S extends uniquely to an equivalence of X onto Y. We collect these observations in the following theorem.

1.7.14. THEOREM. Let $H \subseteq G$ and let S be an

H-space. There exists a G-space X with S as an H-slice over X/G. Moreover if $K = \bigcap_{s \in S} H_s$ and f^* is the equivariant map of X onto G/H such that $S = f^{*-1}(H)$ then the triple $(X, G/H, f^*)$ is equivalent to the fiber bundle with fiber S associated with the H/K principal bundle G/K. If S occurs as an H-kernel in a G-space Y then the identity map of S extends uniquely to an equivalence of X with the subspace GS of Y.

We now consider under what conditions a point x of a G-space is contained in an H-slice.

1.7.15. PROPOSITION. If S is a compact (respectively, closed) H-kernel in a G-space (respectively, normal G-space) X then there is an H-slice $\tilde{S}$ in X such that $S = \tilde{S} \cap GS$.

PROOF. Let f be the equivariant map of GS into G/H such that $S = f^{-1}(H)$. Since GS is compact (closed) if S is (1.1.2) and G/H is a G-ANR (1.5.7) we can extend f to an equivariant map $\tilde{f}$ of an open invariant neighborhood O of GS in X into G/H (1.6.1, 1.6.3). Then $\tilde{S} = \tilde{f}^{-1}(H)$ is the required H-slice.

1.7.16. COROLLARY. If S' is an H-slice over X/G in the G-space X and if S' is an H-ANR then X is a G-ANR.

PROOF. Let Y be a normal G-space and f an equivariant map of a closed invariant subspace K of Y into X. Then by 1.7.8 $S = f^{-1}(s')$ is an H-kernel in Y over $\Pi_Y(K)$. By the preceding proposition we can find an H-slice $\tilde{S}$ in Y such that $S = \tilde{S} \cap K$. Now the closure of $\tilde{S}$ in Y is a normal H-space and $f|S$ is an H-equivariant map of the closed invariant subspace S into S', hence there exists an extension $\tilde{f}$ of $f|S$ to an H-equivariant map of a neighborhood V of S in $\tilde{S}$ into S'. By 1.7.10 there is a unique G-equivariant map f^* of GV into X whose restriction to V is $\tilde{f}$. Since GV is a neighborhood of $GS = K$ and clearly $f^*|K = f$ the proof is complete. q.e.d.

1.7.17. PROPOSITION. A necessary and sufficient condition that there exist an H-kernel in the G-space X which contains the point x is that $G_x \subseteq H$. If this condition is fulfilled then there is even a compact H-kernel in X containing x, namely Hx, and in fact Hx is a subset of every H-kernel containing x.

PROOF. It follows from 1.7.4 that the condition is necessary, and it is trivial that Hx is included in any H-kernel containing x. It remains to show that if $G_x \subseteq H$ then Hx is an H-kernel. Since Hx is compact and H-invariant it remains to show that if g is an element of G not in H then gHx is disjoint from Hx. In fact if $gh_1x = h_2xh_1$, $h_2 \in H$ then $h_2^{-1}gh_1x = x$ so $h_2^{-1}gh_1 \in H$ and $g \in h_2Hh_1^{-1} = H$. q.e.d.

1.7.18. THEOREM. A point x of a G-space X is contained in an H-slice in X if and only if $G_x \subseteq H$.

PROOF. Immediate from 1.7.15 and 1.7.17.

1.7.19. COROLLARY (Mostow [12]). For each point x of a G-space X there exists a slice at x.

PROOF. Recall (1.7.1) that a slice at x is a G_x-slice in X which contains x.

1.7.20. COROLLARY (Montgomery and Zippin [11]). If x is any point of the G-space X there is an invariant neighborhood V of x such that G_v is conjugate to a subgroup of G_x for all $v \in V$.

PROOF. Let S be a slice at x and let $V = GS$. If $v = gs \in V$ then $G_v = G_{gs} = gG_s g^{-1}$ and by 1.7.4 $G_s \subseteq G_x$.

1.7.21. COROLLARY. If X is a G-space and $H \subseteq G$ then $\bigcup\{X_{(K)} \mid (K) \leq (H)\}$ is an open invariant subset of X.

PROOF. Immediate from 1.7.20.

1.7.22. COROLLARY. If X is a G-space and $H \subseteq G$ then $\bigcup\{X_{(K)} \mid (K) < (H)\}$ is an open invariant subspace of X.

PROOF. This follows from 1.7.21 since $\bigcup\{X_{(K)} \mid (K) < (H)\} = \bigcup_{(K')<(H)} \bigcup\{X_{(K)} \mid (K) \leq (K')\}$.

1.7.23. COROLLARY. If X is a G-space and $H \subseteq G$ then $X_{(H)}$ is the intersection of an open invariant subspace of X and a closed invariant subspace of X. Hence if X is locally compact so is $X_{(H)}$ and if X is metrizeable then $X_{(H)}$ is an F_σ.

PROOF. $X_{(H)}$ is the intersection of $\bigcup\{X_{(K)} \mid (K) \leq (H)\}$ and the complement of $\bigcup\{X_{(K)} \mid (K) < (H)\}$.

REMARKS. The statements of 1.7.21, 1.7.22 and 1.7.23 remain valid if we replace $X_{(H)}$ by $\tilde{X}_{(H)}$ etc., for $\tilde{X}_{(H)} = \Pi_X(X_{(H)})$ and Π_X is open and closed.

For a differentiable G-space X we can strengthen 1.7.19 to say that at each point x of X there is a slice at x which is an open ball in a Euclidean G_x-space.

1.7.24. PROPOSITION (Koszul [8]). If M is a differentiable G-space and $\sigma \in M$ there exists a slice S at σ such that, in a suitable coordinate system centered at σ, G_σ acts orthogonally and S is an open ball in an invariant subspace.

PROOF. In 1.6.5 let $\Sigma = G\sigma$ and put $S = f^{-1}(\sigma)$. Since σ is a G_σ-kernel in Σ (1.7.17) and f is equivariant it follows from 1.7.8 (and the fact that O is open in M) that S is a slice at σ. The other properties are part of the statement of 1.6.5. q.e.d.

We now come to a very important result due to Yang [15].

1.7.25. THEOREM. If M is a differentiable G-space then the orbit structure of M is locally finite, i.e. for each $\sigma \in M$ there exists a neighborhood O of σ such that O meets only a finite number of different $M_{(H)}$. In particular if M is compact then it has finite orbit structure.

PROOF. If $\dim M = 0$ then M is discrete and the theorem is trivial. Hence we can proceed by induction and assume that the theorem holds for differentiable Γ-spaces N where Γ is any compact Lie group and $\dim N < \dim M$. Given $\sigma \in M$ let S be a slice at σ satisfying the conditions of 1.7.24 and let Σ be a sphere in S in a coordinate system centered at σ relative to which G_σ acts orthogonally. Since Σ is a compact differentiable G_σ-space and $\dim \Sigma < \dim M$ we can find subgroups $H_1, \ldots, H_k$ of G_σ such that if $\sigma' \in \Sigma$ then $G_{\sigma'}$ is conjugate in G_σ to one of the H_i (note that since S is a slice the isotropy group of σ' in G_σ is the same as its isotropy group $G_{\sigma'}$ in G by 1.7.4). Since the action of G_σ on S is linear it follows that the isotropy groups are constant along open rays, hence if B is the ball in S bounded by the sphere Σ then the isotropy groups at points of B are all conjugate in G_σ to one of $H_1, \ldots, H_k$

and G_σ itself. We now put $O = GB$. Then O is a neighborhood of σ in M and if $y = gb \in O$, $b \in B$, then $G_y = G_{gb} = gG_b g^{-1} = g\gamma H\gamma^{-1}g^{-1}$ where $\gamma \in G_\sigma$ and H is one of $H_1, \ldots H_k$, or G_σ. Hence O meets only the $M_{(H)}$ where H is one of the above groups.

q. e. d.

1. 7. 26. COROLLARY. If V is a Euclidean G-space then V has finite orbit structure.

PROOF. As remarked in the proof of 1. 7. 25 isotropy groups are clearly constant along open rays of V, from which it follows that if $(H_1), \ldots, (H_k)$ are the orbit types that occur in the unit sphere of V (there are only finitely many by the theorem) then the only other orbit type occurring in V can be the orbit type (G) of the origin.

1. 7. 27. COROLLARY. If G is a compact Lie group then G has at most countably many orbit types, i. e. there are at most countably many conjugate classes of closed subgroups of G.

PROOF. This follows from 1. 7. 26 and the well-known fact that to within equivalence there are only countably many finite dimensional representations of G.

1. 7. 28. COROLLARY. If X is a separable metric G-space then $\dim X = \text{Sup}\,\{\dim X_{(H)} \mid H \subseteq G\}$ and $\dim X/G = \text{Sup}\,\{\dim \tilde{X}_{(H)} \mid H \subseteq G\}$.

PROOF. We recall that by the sum theorem if a separable metric space Y is the union of countably many F_σ subsets Y_i then $\dim Y = \text{Sup}_i \dim Y_i$ ([6] Theorem III2 page 30). Now X is separable metric by assumption and X/G is by 1. 1. 12. Also the $X_{(H)}$ and $\tilde{X}_{(H)}$ are F_σ by 1. 7. 23 and the remark that follows it. Finally by 1. 7. 27 there are only countably many $X_{(H)}$'s and $\tilde{X}_{(H)}$'s.

q. e. d.

1.7.29. COROLLARY. If H and K are closed subgroups of G then the number of conjugate classes of closed subgroups of H that contain representatives of the form $H \cap gKg^{-1} (g \in G)$ is finite.

PROOF. Since G/K is a compact differentiable G-space, and hence by restriction a compact differentiable H-space it follows from 1.7.25 that the isotropy subgroups of G/K as an H-space fall into a finite number of conjugate classes. But clearly the isotropy group at gK is just $H \cap gKg^{-1}$.
q.e.d.

1.7.30. COROLLARY. If a G-space X has a finite orbit structure and $H \subseteq G$ then X has finite orbit structure as an H-space and any H-slice in X has finite orbit structure as an H-space.

PROOF. Let $(K_1), \ldots, (K_n)$ be the G-orbit types occurring in X. If $x \in X$ then the isotropy group at x when X is considered as an H-space is just $H \cap G_x = H \cap gK_i g^{-1}$. The first statement is now immediate from 1.7.29 and the second follows immediately since clearly an H-invariant subspace of an H-space with finite orbit structure itself has finite orbit structure.

1.7.31. PROPOSITION. If X is a separable metric G-space and $H \subseteq G$ then $\dim \tilde{X}_{(H)} = \dim X_{(H)} - \dim G/H$. More generally if $X_{(H)}$ has dimension n at a point x then $\tilde{X}_{(H)}$ has dimension $n - \dim G/H$ at $\tilde{x} = \Pi_X(x)$.

PROOF. Because of 1.1.13 there is no loss of generality in assuming that $X = X_{(H)}$. Choose $g \in G$ such that $G_{gx} = H$. Since the operation of g on X is a homeomorphism, X has dimension n at gx, so we can assume that $G_x = H$. Let S be a slice at x (1.7.19). Note that if $s \in S$ then $G_s \subseteq H$ (1.7.4) and since also G_s is conjugate to H it follows that $G_s = H$. Thus H acts trivially on S so S/H is homeomorphic to S. By 1.7.6 it follows that the dimension of $\Pi_X(S)$ at $\tilde{x}$ is the dimension of S at x. Since S is a slice in X, $\Pi_X(S)$ is open in $X/G = \tilde{X}_{(H)}$, hence the dimension of $\tilde{X}_{(H)}$ at $\tilde{x}$ is the dimension of S at x. Now by 1.7.9 X at x is locally homeomorphic to the product of S and the locally Euclidean space G/H. It follows from a theorem of Hurewicz [5] that the dimension of S at x is $n - \dim G/H$.

q. e. d.

1.7.32. COROLLARY. If X is a separable metric G-space then $\dim X/G = \operatorname{Sup}\{\dim \tilde{X}_{(H)} - \dim G/H \mid H \subseteq G\}$. In particular $\dim X/G \leq \dim X$.

PROOF. Immediate from 1.7.28 and 1.7.31.

If X is a G-space then in 1.3.8 we defined $X_H = \{x \in X \mid G_x = H\}$. Clearly $GX_H = X_{(H)}$. If x is adherent to X_H then of course $H \subseteq G_x$. If also x is in $X_{(H)}$ then G_x is conjugate to H and so has the same dimension and number of components as H, so $G_x = H$. In other words X_H is closed in its saturation $X_{(H)}$. Denoting the normalizer of H in G by $N(H)$ as previously it is clear that X_H is $N(H)$-invariant. Moreover if g is any element of G not in $N(H)$ and $x \in X_H$ then $G_{gx} = gG_xg^{-1} = gHg^{-1} \neq H$ so $gx \notin X_H$. In other words gX_H is disjoint from X_H. We have proved

1.7.33. PROPOSITION. If X is a G-space and $H \subseteq G$ then X_H is an $N(H)$-kernel in X over $\tilde{X}_{(H)}$.

REMARK. If $x \in X_{(H)}$ then G_x is conjugate in G to H and by elementary group theory $\{g \in G \mid gHg^{-1} = G_x\}$ is a well determined coset $f(x)$ of $G/N(H)$. It is easily seen that f is the equivariant map of $X_{(H)}$ onto $G/N(H)$ such that $X_H = f^{-1}(N(H))$.

We are now in a position to analyze a situation that was apparently first noted explicitly by A. Borel [1]; namely that $X_{(H)}$ is a fiber bundle with structure group $N(H)/H$ in two entirely different ways. In one case the fibering is by the orbits of $X_{(H)}$ under G, in the other it is by the sets

$X_{gHg^{-1}}$.

1.7.34. PROPOSITION. If X is a G-space and $H \subseteq G$ then $X_{(H)}$ is a fiber bundle over $G/N(H)$ with fiber X_H and structural group $N(H)/H$. The associated principal bundle is G/H and the fiber map is $x \to \{g \in G \mid gHg^{-1} = G_x\}$ so that the fiber over $gN(H)$ is just $X_{gHg^{-1}}$.

PROOF. This is immediate from 1.7.14 and 1.7.29.

We next note that, as was essentially remarked in 1.3.9, X_H itself is an $N(H)/H$ principal bundle. Moreover the orbit space $X_H/(N(H)/H) = X_H/N(H)$ can by 1.7.6 be naturally identified with $\tilde{X}_{(H)}$, because X_H is an $N(H)$-slice over $\tilde{X}_{(H)}$. Suppose now that we form the associated fiber bundle with fiber G/H, $(X_H \times G/H)/(N(H)/H)$. This is canonically homeomorphic to $(G/H \times X_H)/(N(H)/H)$ which is the fiber bundle with fiber X_H associated with the principal $N(H)/H$ bundle G/H. The latter in turn is by 1.7.34 canonically homeomorphic to $X_{(H)}$. We leave it to the reader to verify that under the combined homeomorphism of $(X_H \times G/H)/(N(H)/H)$ onto $X_{(H)}$ the fiber map goes into $\Pi_X | X_{(H)}$. The final result is

1.7.35. PROPOSITION. If X is a G-space and $H \subseteq G$ then $X_{(H)}$ is a fiber bundle over $\tilde{X}_{(H)}$ with fiber G/H and structural group $N(H)/H$. The associated principal bundle is X_H under the action defined by $(nH)x = nx$. The fiber projection is $\Pi_X | X_{(H)}$ so that the fibers are just the orbits of $X_{(H)}$ under G.

1.8. EQUIVARIANT IMBEDDINGS IN EUCLIDEAN G-SPACES.

In this section we will derive Mostow's remarkable necessary and sufficient conditions [12] for a G-space to admit an equivariant imbedding in some Euclidean G-space. The technique will be a combination of Mostow's and the independent proof by the author for the differentiable case found in [14].

We begin with a metatheorem which is often advantageous in proving a theorem about compact Lie groups. It replaces the awkward technique of doing a double induction on the dimension and number of components.

> 1.8.1. METATHEOREM. Let P be a statement valued function defined for all compact Lie groups. If whenever G is a compact Lie group the truth of $P(H)$ for all $H \subset G$ implies the truth of $P(G)$ then $P(G)$ is true for all compact Lie groups G. Hence in a proof that $P(G)$ is valid for all compact Lie groups G it suffices to prove $P(G)$ for an arbitrary compact Lie group G under the assumption that $P(H)$ is valid whenever $H \subset G$.

PROOF. If $P(G)$ were false for some compact Lie group G there would be at least integer n which was the dimension of a compact Lie group G for which $P(G)$ was false. Among all compact Lie groups G of dimension n for which $P(G)$ was false there would be one G^* with fewest components. But then clearly $P(H)$ is true for all $H \subset G^*$. q. e. d.

The following theorem is apparently due, at least in its present form, to J. Milnor.

> 1.8.2. THEOREM. Let X be a paracompact space with covering dimension n and let $\{U_\alpha\}$ be an open covering of X. Then there is an open covering $\{G_{i\beta}\}_{\beta \epsilon B_i}$ $i = 0, 1, \ldots n$, of X refining $\{U_\alpha\}$ such that $G_{i\beta} \cap G_{i\beta'}$ is empty if $\beta \neq \beta'$.

PROOF. By making an initial refinement of $\{U_\alpha\}$ we can assume that the order of the covering $\{U_\alpha\}$ is at most n. Let $\{\phi_\alpha\}$ be a locally finite partition of unity with support $\phi_\alpha \subseteq U_\alpha$. Given $i = 0, 1 \ldots n$ let B_i = the set of unordered $i+1$-tuples from the indexing set of the $\{U_\alpha\}$. Given $\beta = (\alpha_o, \ldots, \alpha_i) \in B_i$ set $G_{i\beta} = \{x \in X \mid \phi_{\alpha_j}(x) > 0$ and $\alpha \notin \beta \Longrightarrow \phi_\alpha(x) < \phi_{\alpha_j}(x)\, j = 0, 1, \ldots i\}$. Since in a neighborhood of any point x only a finite number of ϕ_α are not identically zero it follows that each $G_{i\beta}$ is open. Clearly $G_{i\beta}$ is disjoint from $G_{i\beta'}$ if $\beta \neq \beta'$, and $G_{i\beta} \subseteq \bigcap_{\alpha \in \beta} U_\alpha$ so $\{G_{i\beta}\}$ is a covering of X. Given $x \in X$ let $\alpha_o, \ldots, \alpha_m$ be the indices such that $\phi_\alpha(x) > 0$, so arranged that $\phi_o(x) = \phi_1(x) = \ldots = \phi_{\alpha_i}(x) > \phi_{\alpha_{i+1}}(x) \geq \ldots \geq \phi_{\alpha_m}(x)$. Since $x \in \bigcap_{j=o}^{m} \text{support } \phi_{\alpha_j} \subseteq \bigcap_{j=o}^{m} U_{\alpha_j}$ and $\{U_\alpha\}$ has order $\leq n$ it follows that m, and hence i, is $\leq n$, and clearly $x \in G_{i(\alpha_o, \ldots, \alpha_i)}$. q. e. d.

1.8.3. PROPOSITION. Let X be a separable metric G-space of dimension $n < \infty$ and let $H \subseteq G$. There exist $n+1$ H-slices $S_o, \ldots, S_n$ in X such that $\bigcup_{i=o}^{n} GS_i = \bigcup \{X_{(K)} \mid (K) \leq (H)\}$.

PROOF. Let $\{\tilde{S}_\alpha\}$ be the collection of subsets (automatically open) of X/G which are of the form $\tilde{S}_\alpha = \Pi_X(S_\alpha)$ where S_α is an H-slice in X. It follows from 1.7.18 that $\{\tilde{S}_\alpha\}$ is an open covering of $\bigcup\{\tilde{X}_{(K)} \mid (K) \leq (H)\}$. Now by 1.7.32 $\dim X/G \leq n$, so by the preceding theorem we can find an open covering $\{\tilde{G}_{i\beta}\}_{\beta \in B_i}$ $k = 0, \ldots n$ of $\bigcup\{\tilde{X}_{(K)} \mid (K) \leq (H)\}$ which refines $\{\tilde{S}_\alpha\}$ and is such that $\tilde{G}_{i\beta} \cap \tilde{G}_{i\beta'}$ is empty if $\beta \neq \beta'$. Now by 1.7.2 it is clear that an open refinement of $\{\tilde{S}_\alpha\}$ is actually a subset of $\{\tilde{S}_\alpha\}$, so that there is an H-slice $G_{i\beta}$ over $\tilde{G}_{i\beta}$. Then by 1.7.3 $S_i = \bigcup_{\beta \in B} G_{i\beta}$ is an H-slice over $\tilde{S}_i = \bigcup_{\beta \in B_i} G_{i\beta}$. Since $\bigcup_{i=o} \tilde{S}_i = \bigcup \tilde{G}_{i\beta} = \bigcup\{X_{(K)} \mid (K) \leq (H)\}$ it follows that $\bigcup_{i=o}^{n} GS_i = \Pi_X^{-1}(\bigcup_{i=o} \tilde{S}_i) = \bigcup\{X_{(K)} \mid (K) \leq (H)\}$. q. e. d.

1.8.4. THEOREM (Mostow [12]). Necessary and sufficient conditions that a G-space X admit an equivariant imbedding in a Euclidean G-space are that X be separable metric, finite dimensional, and have finite orbit structure.

PROOF. Necessity is obvious from 1.7.26. We note that in proving sufficiency we can assume that if $H \subset G$ then the theorem becomes true if in the statement we replace G by H (1.8.1). Let $(H_1), \ldots, (H_k)$ be the orbit types occurring in X. Because of 1.4.7 we can suppose that no $H_i =$ G so that each $H_i \subset G$. Now the sets $U_{(H_i)} = \bigcup \{X_{(H_j)} \mid (H_j) \leq (H_i)\}$ are open (1.7.21) and cover X, so by 1.4.8 it suffices to show that each $U_{(H_i)}$ admits an equivariant imbedding in a Euclidean G-space. By 1.8.3 there are a finite number of H_i-slices, say $S^i_o, \ldots, S^i_n$ such that $U_{(H_i)} = GS^i_o \cup \ldots \cup GS^i_n$. Since each GS^i_j is open it will suffice, again by 1.4.8, to show that each GS^i_j admits an equivariant imbedding in a Euclidean G-space. Now each S^i_j has finite orbit structure as an H_i-space by 1.7.30, and clearly each S^i_j is separable metric and of finite dimension. Since we may assume the theorem proved for H_i, because $H_i \subset G$, each S^i_j admits an H_i-equivariant imbedding in a Euclidean H_i-space. The desired consequence now follows from 1.7.11.

q.e.d.

2. THE CLASSIFICATION OF G-SPACES

In this chapter we will be interested in the following question. Suppose $\tilde{X}$ is a locally compact, second countable space and $\{\tilde{X}_{(H)}\}$ is a partition of $\tilde{X}$ into subsets indexed by a collection of orbit types of some compact Lie group G. How many different (in the sense of equivalence) G-spaces are there having $\tilde{X}$ as orbit space with $\{\tilde{X}_{(H)}\}$ as orbit structure, and how can these different G-spaces be constructed in a canonical way. The first section is devoted to making this question precise.

2.1. THE Σ-CATEGORY.

Let Σ be a collection of G-orbit types which we consider as fixed during the following discussion.

2.1.1. DEFINITION. A Σ-space is a locally compact, second countable space X together with a partition $\{X_{(H)}\}_{(H)\in\Sigma}$ of X indexed by Σ such that for each $(H)\in\Sigma$ $\bigcup\{X_{(K)} \mid (K)\leq(H)\}$ is open. If X and Y are Σ-spaces a Σ-map of X into Y is a map $f: X\to Y$ such that $f(X_{(H)})\subseteq Y_{(H)}$ for all $(H)\in\Sigma$. If such an f is an homeomorphism of X onto Y it is called a Σ-equivalence of X with Y. If I is the unit interval and X is any Σ-space we denote by $X\times I$ the Σ-space whose space is the product of X and I and partition $(X\times I)_{(H)} = X_{(H)}\times I$ for $(H)\in\Sigma$. If X and Y are Σ-spaces and f_o is a Σ-map of X into Y then a Σ-homotopy of f_o is a Σ-map f of $X\times I$ into Y such that $f_o(x)\equiv f(x,o)$. In general for $t\in I$ we write f_t for the Σ-map $f_t(x) = f(x,t)$ and we say that f_o is strongly Σ-homotopic to f_1. We say that two Σ-maps f_o and f_1 of X into Y are weakly Σ-homotopic if there is a Σ-equivalence h of X with itself such that f_o is strongly Σ-homotopic to $f_1\circ h$. We denote by $\mathfrak{M}(X,Y)$ the set of Σ-maps of X into Y, by $\tilde{\mathfrak{M}}(X,Y)$ the set of strong Σ-homotopy classes of Σ-maps of X into Y, and by $\mathfrak{M}^*(X,Y)$ the set of weak Σ-homotopy classes of Σ-maps of X into Y.

REMARK. If we give $\mathfrak{M}(X,Y)$ the compact-open topology then

$\tilde{\mathfrak{M}}(X, Y)$ is just its set of arc components. The group of Σ-equivalences of X induces a group of permutations of $\mathfrak{M}(X, Y)$ in an obvious way which, since it preserves arc components, in turn induces a group of permutations of $\tilde{\mathfrak{M}}(X, Y)$, and $\mathfrak{M}^*(X, Y)$ is just the orbit space of $\tilde{\mathfrak{M}}(X, Y)$ under this permutation group.

We note that a Σ-space being locally compact and second countable is separable metric. Using arguments entirely analogous to 1.7.22 et. seq. we see.

2.1.2. PROPOSITION. If X is a Σ-space and $(H) \in \Sigma$ then $\bigcup\{X_{(K)} \mid (K) < (H)\}$ is open in X. Each $X_{(H)}$, $(H) \in \Sigma$ is the intersection of an open and closed subset of X hence locally compact and an F_σ, so that (since Σ is at most countable) $\dim X = \text{SUP}\,\{\dim X_{(H)} \mid (H) \in \Sigma\}$ by the sum theorem.

2.2. G-SPACES OVER A Σ-SPACE.

In this section Σ is again some fixed collection of G-orbit types. By a G-space of type Σ we shall mean a locally compact, second countable G-space all of whose orbit types belong to Σ. The following proposition is immediate from 1.7.21 and 1.2.4.

2.2.1. PROPOSITION. If X is a G-space of type Σ then X and X/G with their respective orbit structures are Σ-spaces and Π_X is a Σ-map of X onto X/G. If Y is a second G-space of type Σ and $f : X \to Y$ is isovariant then f and $\tilde{f} : X/G \to Y/G$ are Σ-maps. if $f_o : X \to Y$ is isovariant and $f : X \times I \to Y$ is an isovariant homotopy of f_o then f is a Σ-homotopy of f_o and $\tilde{f}$ is a Σ-homotopy of $\tilde{f}_o$.

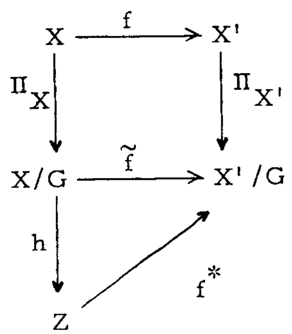
2.2.2. DEFINITION. Let Z be a Σ-space. A G-space over Z is a triple (X, Z, h) where X is a G-space of type Σ and h is a Σ-equivalence of X/G with Z. If (X', Z, h') is a second G-space over Z then an equivalence f of X with X' (as G-spaces) will be called a *weak equivalence* of (X, Z, h) with (X', Z, h'), and will be called a *strong equivalence* if $h' \circ f \circ h^{-1}$ (which in any case is a Σ-equivalence of Z with itself) is the identity map of Z. We will say that (X, Z, h) and (X', Z, h') are *strongly (weakly) equivalent* if there exists a strong (weak) equivalence of (X, Z, h) with (X', Z, h').

We can now state with more precision the problem whose solution (or reduction) will occupy the remainder of the chapter.

2.2.3. PROBLEM. Given a Σ-space Z describe a method of constructing a complete set of representatives of the weak and strong equivalence classes of G-spaces over Z.

2.3. INDUCED G-SPACES.

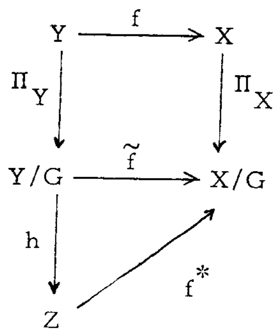
Let Z be a Σ-space, (X, Z, h) a G-space over Z and f an isovariant map of X into a G-space Y'. Then we have a commutative diagram



where the map f^* of Z into X'/G is the Σ-map $\tilde{f} \circ h^{-1}$. We would now like to point out that given any Σ-map $f^*: Z \to X'/G$ there is a G-space over Z, (X, Z, h), which is unique to within strong equivalence, for which it is possible to fill in the above diagram.

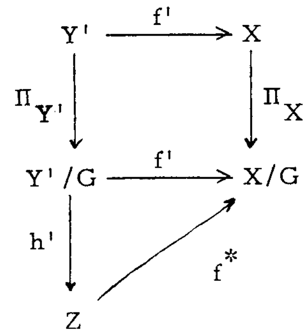
2.3.1. DEFINITION. Let Z be a Σ-space, X a G-space of type Σ and f^* a Σ-map of Z into X/G. Define $Y = \{(x, z) \in X \times Z \mid \Pi_X(x) = f^*(z)\}$ and make Y into a G-space by $g(x, z) = (gx, z)$. Define $f : Y \to Z$ by $f(x, z) = x$. Noting that the map $(x, z) \to z$ of Y into Z is invariant we define $h : Y/G \to Z$ by $h\ \Pi_Y(x, z) = z$. We call the triple (Y, Z, h) the *G-space over Z induced by* f^* and denote it by $f^{*-1}(X)$. The map $f : Y \to X$ is called the *canonical isovariant map of Y into X*.

2.3.2. PROPOSITION. If Z is a Σ-space, X a G-space and $f^* : Z \to X/G$ is a Σ-map then $(Y, Z, h) = f^{*-1}(X)$ is a G-space over Z and if $f : Y \to X$ is the canonical isovariant map of Y into X then f is in fact isovariant and the following diagram commutes:

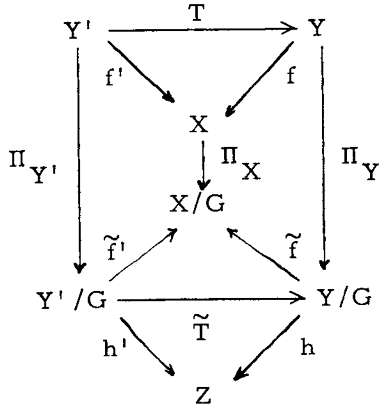


Moreover if (Y', Z, h') is any G-space over Z and $f' : Y' \to X$ is an isovariant map making the analogous diagram commute then there is a map $T : Y' \to Y$ which is a strong equivalence of (Y', Z, h') with (Y, Z, h).

PROOF. It is a matter of direct verification from the definition that f is isovariant and that the diagram commutes. It is also clear that h is continuous and one-to-one and, since the map $h \quad \Pi_Y : (x, z) \to z$ is an open map of Y onto Z, $h = (h \circ \Pi_Y) \circ \Pi_Y^{-1}$ is an open map of Y/G onto Z and so a homeomorphism. If $(H) \epsilon \Sigma$ then $\tilde{f}(\tilde{Y}_{(H)}) \subseteq \tilde{X}_{(H)}$ because f is isovariant, and since $f^{*-1}(X_{(H)}) = Z_{(H)}$ because f^* is a Σ-map, it follows that $h(\tilde{Y}_{(H)}) = f^{*-1}(\tilde{f}(\tilde{Y}_{(H)})) \subseteq Z_{(H)}$ from which it follows that $h : Y/G \to Z$ is a Σ-equivalence. This completes the proof that (Y, Z, h) is a G-space over Z and of the first part of the proposition. Now let (Y', Z, h') be a second G-space over Z and $f' : Y' \to X$ an isovariant map such that the following diagram commutes.



Then since $\Pi_X f'(y') = f^* h' \Pi_Y'(y')$ it follows that $T : y' \to (f'(y'), h' \circ \Pi_{Y'}(y'))$ is a map of Y' into Y (see definition of Y, 2.3.1). Moreover T is clearly isovariant and we have the following commutative diagram



The only non-obvious commutation relations are $f' = f \circ T$ and $h' \circ \Pi_{Y'} = h \circ \Pi_Y \circ T$ both of which follow directly from the definitions of T, f, and h.

Since h and h' are both homeomorphisms it follows that so is T and hence by 1.1.18 that T is an equivalence of Y' with Y. Moreover by the above diagram $hT = h'$ or hTh'^{-1} is the identity map of Z, so T is a strong equivalence of (Y', Z, h') with (Y, Z, h).

q. e. d.

We can now state the solution of the problem 2.2.3. If $\Sigma = ((H_1), \ldots, (H_k))$ is a finite collection of G-orbit-types and Z is a finite dimensional Σ-space we can find a G-space X depending only on $\dim Z_{(H_1)}, \ldots, \dim Z_{(H_k)}$ (and in fact we will give an explicit construction of X) which is "universal" in the following sense. Every G-space over Z is of the form $f^{*-1}(X)$ for some Σ-map $f^* : Z \to X/G$. Moreover $f_1^{*-1}(X)$ and $f_2^{*-1}(X)$ are strongly (weakly) equivalent if and only if f_1^* and f_2^* are strongly (weakly) Σ-homotopic. Thus the set of strong (weak) equivalence classes of G-spaces over Z is in natural one-to-one correspondence with $\tilde{\mathfrak{M}}(Z, X/G)(\mathfrak{M}^*(Z, X/G))$.

2.4. THE COVERING HOMOTOPY THEOREM.

The proof of the title theorem of this section is by far the hardest of this work. Fortunately however much of the spadework has been done in Chapter I.

> 2.4.1. COVERING HOMOTOPY THEOREM. Let X and Y be locally compact second countable G-spaces and let $f_o : X \to Y$ be an isovariant map. If $\tilde{f} : X/G \times I \to Y/G$ is any homotopy of the induced map $\tilde{f}_o$ such that $\tilde{f}(\tilde{X}_{(H)} \times I) \subseteq \tilde{Y}_{(H)}$ for all $H \subseteq G$ then there exists an isovariant homotopy $f : X \times I \to Y$ of f_o with induced map $\tilde{f}$.

The proof of this theorem will proceed by a sequence of lemmas. In all that follows all G-spaces are assumed to be locally compact and second countable. We first make a definition.

2.4.2. DEFINITION. Let X and Y be G-spaces. We shall say that Y is <u>admissible for</u> X if the conclusion of 2.4.1 holds for every $f_o : X \to Y$ and $\tilde{f} : X/G \times I \to Y/G$ which satisfy the hypotheses. We shall say that Y is <u>admissible</u> if it is admissible for each G-space X, and we shall say that a compact Lie group G is <u>admissible</u> if every G-space is admissible.

We now fix a particular Lie group G and note that 2.4.1 is equivalent to

2.4.3. COVERING HOMOTOPY THEOREM. G is admissible.

Because of 1.8.1

2.4.4. LEMMA. In proving 2.4.3 we can assume that if $H \subset G$ then H is admissible.

2.4.5. LEMMA. If $H \subset G$ and Y is a G-space such that there exists an H-slice S' in Y over Y/G then Y is admissible.

PROOF. Let X be a G-space and let $f_o : X \to Y$ and $\tilde{f} : X/G \times I \to Y/G$ be as in 2.4.1. Let $S = f_o^{-1}(S')$ so that by 1.7.8 S is an H-slice in X over X/G. Then clearly $S \times I$ is an H-slice in $X \times I$ over $X/G \times I$. It will suffice to show that there exists an H-isovariant homotopy $f^* : S \times I \to S'$ of $f_o|S$ such that $\Pi_Y(f^*(s,t)) = \tilde{f}(\Pi_X(s), t)$, for then by 1.7.10 $f(gs,t) = gf^*(s,t)$ will define an isovariant homotopy of $X \times I \to Y$ starting from f_o whose induced map is clearly $\tilde{f}$. Let us identify S/H with X/G and S'/H with Y/G via the maps $\Pi_S(s) \to \Pi_X(x)$ and $\Pi_{S'}(s') \to \Pi_Y(s')$ (see 1.7.6). Given $K \subseteq H$ it follows from 1.7.29 that we can choose closed subgroups $K = K_1, K_2, \ldots K_n$ of H which are conjugate in G such that each closed subgroup of H which is conjugate in G to K, is conjugate in H to precisely one of the K_i. Then under the above identification of S/H with X/G it follows from 1.7.6 that $\tilde{X}_{(K)}$ is the disjoint union of the closed subsets $\tilde{S}_{(K_1)}, \ldots, \tilde{S}_{(K_n)}$ and similarly $\tilde{Y}_{(K)}$ is the disjoint union of the closed subsets $\tilde{S}_{(K_1)}, \ldots, \tilde{S}'_{(K_n)}$. Now by 2.4.4 the existence of the desired map f^* will follow if we can show that $\tilde{f}(\tilde{S}_{(K)} \times I) \subseteq \tilde{S}'_{(K)}$. Given $\tilde{s} \in \tilde{S}_{(K)}$ we will show that $\tilde{f}(s \times I) \subseteq \tilde{S}'_{(K)}$. Since $\tilde{f}(\tilde{s} \times I) \subseteq \tilde{f}(\tilde{S}_{(K)} \times I) \subseteq \tilde{f}(\tilde{X}_{(K)} \times I) \subseteq \tilde{Y}_{(K)}$ and $\tilde{Y}_{(K)}$ is the union of disjoint relatively closed subsets $\tilde{S}'_{(K)} = \tilde{S}'_{(K_1)}, \ldots \tilde{S}'_{(K_n)}$ it will suffice in view of the connectivity of $\tilde{f}(\tilde{s} \times I)$, to show that $\tilde{f}(\tilde{s}, o) \in \tilde{S}'_{(K)}$. If we choose $s \in S$ with $\Pi_X(s) = \tilde{s}$ and $G_s = K$ then $\tilde{f}(\tilde{s}, o) = \tilde{f}_o(\Pi_X(s)) = \Pi_Y(f_o(s))$, and since f_o is isovariant $G_{f_o(s)} = G_s = K$, so $f_o(s) \in S'_{(K)}$ and $\tilde{f}(s, o) \in \Pi_Y(S'_{(K)}) = \tilde{S}'_{(K)}$.

q.e.d.

2.4.6. LEMMA. Let X and Y be G-spaces. If Y is admissible for $X - X_{(G)}$ then Y is admissible for X.

PROOF. Immediate from 1.2.6.

2.4.7. LEMMA. Let Y be admissible for X and let f_o and $\tilde{f}$ be as in 2.4.1. Let U be an open invariant subspace of X and C a closed invariant subspace of X included in U. Let $f^* : U \times I \to Y$ be an isovariant homotopy of $f_o | U$ with induced map $\tilde{f} | \Pi_X(U) \times I$. Then there exists an isovariant homotopy $f : X \times I \to Y$ of f_o with induced map $\tilde{f}$ such that $f | C \times I = f^* | C \times I$.

PROOF. Immediate from 1.2.12.

2.4.8. LEMMA. If Y is a G-space which is admissible for every compact G-space X with $X_{(G)} = \phi$

then Y is admissible.

PROOF. Let X be any G-space and let f_o, $\tilde{f}$ be as in 2.4.1. In constructing the desired f we may, by 2.4.6, suppose that $X_{(G)} = \phi$. Let $\{\tilde{X}_n\}$ be a sequence of compact subspaces of X/G with $\tilde{X}_n \subseteq$ interior $\tilde{X}_{n+1}$ and $X/G = \bigcup_n \tilde{X}_n$. Put $X_n = \Pi_X^{-1}(\tilde{X}_n)$ so that by 1.1.9 $\{X_n\}$ is a sequence of compact invariant subspaces of X with $X_n \subseteq$ interior X_{n+1} and $X = \bigcup_n X_n$. Moreover $(X_n)_{(G)} = X_n \cap X_{(G)} = \phi$. By hypothesis we can find an isovariant homotopy $f^2 : X_2 \times I \to Y$ of $f_o|X_2$ with induced map $\tilde{f}|\tilde{X}_2 \times I$ (we are using 1.1.13). Suppose that we have constructed maps $f^2, \ldots, f^k$ such that $f^j : X_j \times I \to Y$ is an isovariant homotopy of $f_o|X_j$ with induced map $\tilde{f}|\tilde{X}_j \times I$, and $f^j|X_{j-2} \times I = f^{j-1}|X_{j-2} \times I$. Then by hypothesis and 2.4.7 we can construct f^{k+1} so that $f^{k+1} : X_{k+1} \times I \to Y$ is an isovariant homotopy of $f_o|X_{k+1}$ with induced map $\tilde{f}|\tilde{X}_{k+1} \times I$ and such that $f^{k+1}|X_{k-1} \times I = f^k|X_{k-1} \times I$. It is now clear that $f = \lim f^k$ is an isovariant homotopy of f_o with induced map $\tilde{f}$. q.e.d.

PROOF OF COVERING HOMOTOPY THEOREM.

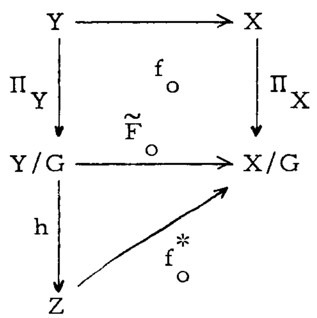
We now prove the Covering Homotopy Theorem in the form 2.4.3. Let Y be an arbitrary G-space. Let X, f_o, $\tilde{f}$ be as in 2.4.1. We must construct an isovariant homotopy f of f_o with induced map $\tilde{f}$. By 2.4.8 we can suppose that X is compact and $X_{(G)} = \phi$. Let $Y' = \Pi_Y^{-1}(\tilde{f}(X/G \times I))$. Then clearly it will suffice to show that Y' is admissible for X. Now $\tilde{f}(X/G \times I)$ is compact and hence by 1.1.9 so is Y', and $Y'_{(G)} = \Pi_Y^{-1}(\tilde{f}(\tilde{X}_{(G)} \times I))$ is empty. Thus we can assume that Y is compact and that $Y_{(G)} = \phi$. By 1.7.19 we can find a slice S at each point y of Y (i.e. a G_y-slice containing y). Since $G_y \subset G$ (because $Y_{(G)} = \phi$) it follows from 2.4.5 that GS is admissible. Since the set of such GS cover Y and since GS is open (by definition of a slice) it follows from the compactness of Y that we can find a covering of Y by invariant open sets $U_1, \ldots U_n$ such that each U_i

is admissible as a G-space. Let $\tilde{U}_i = \Pi_Y(U_i)$. By an obvious argument with Lebesgue numbers we can find compact sets $\tilde{V}_1, \ldots, \tilde{V}_m$ whose interiors cover X/G and an integer n such that for any choice of $i = 1, 2 \ldots m$ and $t \in I$ $\tilde{f}(\tilde{V}_i \times [t - \frac{1}{n}, t + \frac{1}{n}]) \subseteq$ some $\tilde{U}_j$. It will clearly suffice to construct $f^{(1)} : X \times [0, \frac{1}{n}] \to Y$ such that $f^{(1)}$ is an isovariant homotopy of f_o with induced map $\tilde{f}|X/G \times [0, \frac{1}{n}]$, for then we can in the same way construct $f^{(2)}|X \times [\frac{1}{n}, \frac{2}{n}] \to Y$ an isovariant homotopy of $f^{(1)}_{\frac{1}{n}}$ with induced map $\tilde{f}|X/G \times [\frac{1}{n}, \frac{2}{n}]$ and in n-such steps we will have the desired map $f : X \times I \to Y$ defined by $f(x, t) = f^{(i)}(x, t)$ $\frac{i-1}{n} \leq t \leq \frac{i}{n}$. In other words there is no loss of generality in assuming that $n = 1$. Hence for each $i = 1, 2, \ldots m$ we can assume there exists a well determined integer $j(i)$ such that $\tilde{f}(\tilde{V}_i \times I) \subseteq \tilde{U}_{j(i)}$. Let $\tilde{X}_1, \ldots, \tilde{X}_m$ be a covering of X/G by closed sets such that $\tilde{X}_i \subseteq$ interior $\tilde{V}_i$ and put $X_i = \Pi_X^{-1}(\tilde{X}_i)$, $V_i = \Pi_X^{-1}(\tilde{V}_i)$. We shall now construct inductively a sequence of functions $f^{(1)}, \ldots, f^{(n)}$ with the following properties: $f^{(i)} : W_i \times I \to Y$ is an isovariant homotopy of $f_o|W_i$ with induced map $\tilde{f}|\Pi_X(W_i) \times I$, where W_i is a compact invariant neighborhood of $X_1 \cup \ldots \cup X_i$. In fact since $\tilde{f}(\tilde{V}_1 \times I) \subseteq \tilde{U}_{j(1)}$ and $U_{j(1)}$ is admissible we can find $f^{(1)*} : V_1 \times I \to U_{j(1)} \subseteq Y$ an isovariant homotopy of $f_o|V_1$ with induced map $\tilde{f}|\tilde{V}_1 \times I$ and we put $f^{(1)} = f^{(1)*}$, $W_1 = V_1$. Now suppose $f^{(1)}, \ldots, f^{(i-1)}$ have been constructed. Since $\tilde{f}(\tilde{V}_i \times I) \subseteq \tilde{U}_{j(i)}$ and $U_{j(i)}$ is admissible we can find a map $f^{(i)*} : V_i \times I \to U_{j(i)} \subseteq Y$ which is an isovariant homotopy of $f_o|V_i$ with induced map $\tilde{f}|\tilde{V}_i \times I$. Moreover, by 2.4.7, if W'_{i-1} is any compact invariant neighborhood of $X_1 \cup \ldots \cup X_{i-1}$ included in the interior of W_{i-1} then we can suppose that $f^{(i)*}|(W'_{i-1} \cap V_i) \times I = f^{(i-1)}|(W'_{i-1} \cap V_i) \times I$. We now define $W_i = W'_{i-1} \cup V_i$ and $f^{(i)} = (f^{(i-1)}|W'_{i-1}) \cup f^{(i)*}$. Clearly $f^{(i)} : W_i \times I \to Y$ is an isovariant homotopy of $f_o|W_i$ with induced map $\tilde{f}|\Pi_X(W_i) \times I$ and W_i is a compact invariant neighborhood of $X_1 \cup \ldots X_i$. If we take $f = f^{(n)}$ then f is an isovariant homotopy of f_o with induced map $\tilde{f}$. q. e. d.

2.5. CONSEQUENCES OF THE COVERING HOMOTOPY THEOREM.

2.5.1. SECOND COVERING HOMOTOPY THEOREM. Let G be a compact Lie Group, Σ a collection of G-orbit types, and f_o a Σ-map of a Σ-space Z into a G-space X of type Σ. Let f_o^* be the Σ-map $\Pi_X \circ f_o$ of Z into X/G. Then if $f^* : Z \times I \to X/G$ is any Σ-homotopy of f_o^* there exists a Σ-homotopy $f : Z \times I \to X$ of f_o such that $f^* = \Pi_X \circ f$.

PROOF. Let $(Y, Z, h) = f_o^{*-1}(X)$ be the G-space over Z induced by f_o^* and let F_o be the canonical isovariant map of Y into X, so we have commutativity in the diagram

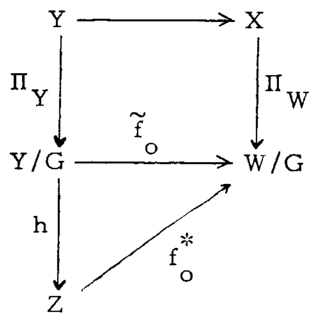


If we define $\tilde{F} = f^* \circ h$ then by 2.4.1 there exists an isovariant homotopy $F : Y \times I \to X$ of F_o with induced map $\tilde{F}$. Recall that $Y = \{(x, z) \in X \times Z \mid \Pi_X(x) = f_o^*(z)\}$ and that $h \circ \Pi_Y(x, z) = z$, $F_o(x, z) = x$. We see that $\phi(z) = (f_o(z), z)$ is a map of Z into Y such that $f_o = F_o \circ \phi$. Then we get the desired map f by letting $f(z, t) = F(\phi(z), t)$. q.e.d.

2.5.2. THEOREM. Let Σ be any collection of G-orbit types and Z a Σ-space. Then any G-space over $Z \times I$ is of the form $Y \times I$. More precisely if $(W, Z \times I, k)$ is a G-space over $Z \times I$ then it is strongly equivalent to $(Y \times I, Z \times I, h \times id)$ where

(Y, Z, h) is the G-space over Z induced by the map $z \to k^{-1}(z, o)$ of Z into W/G.

PROOF. Let $f^* = k^{-1}$ so that f^* is a Σ-equivalence of $Z \times I$ with W/G and (Y, Z, h) is the G-space over Z induced by the map $f_o^* : z \to f^*(z, o)$. Let f_o be the canonical isovariant map of Y into Z, so that we have commutativity in the diagram



Now $\tilde{f} = f^* \circ (h \times id)$ (i.e. the map $(y, t) \to f^*(h(y), t)$) is a Σ-equivalence of $Y/G \times I$ with W/G, and at the same time a Σ-homotopy of f_o. By 2.4.1 there is an isovariant homotopy $f : Y \times I \to W$ of f_o with induced map $\tilde{f}$, and by 1.1.18 f is an equivalence of $Y \times I$ with W. Since $k \circ f \circ (h \times id)^{-1} = f^{*-1} \circ f^* \circ (h \times id) \circ (h \times id)^{-1}$ = identity map of $Z \times I$, f is a strong equivalence of $(W, Z \times I, k)$ with $(Y \times I, Z \times I, h \times id)$. q.e.d.

Suppose Z is a Σ-space, Z' a closed subspace of Z (made into a Σ-space by $Z'_{(H)} = Z' \cap Z_{(H)}$), and (Y, Z, h) is a G-space over Z. Then if we put $Y' = \Pi_Y^{-1}(h^{-1}(Z'))$ and $h' = h | h^{-1}(Z')$ it is clear that (Y', Z', h') is a G-space over Z', which we call the part of (Y, Z, h) over Z'. Moreover if X is a G-space, $f^* : Z \to X/G$ is a Σ-map and $(Y, Z, h) = f^{*-1}(X)$ then it is also clear from the definitions of both sides that $(Y', Z', h') = f'^{*-1}(X)$ where $f' = f | Z'$. Now suppose $f^* : Z \times I \to X/G$ is a Σ-homotopy and let $(W, Z \times I, k) = f^{*-1}(X)$. Then by what we have just remarked it follows that, if we identify Z with $Z \times \{0\}$ and $Z \times \{1\}$ in the obvious way, then $f_o^{*-1}(X)$

and $f_1^{*-1}(X)$ are respectively the parts of $(W, Z \times I, k)$ over $Z \times \{0\}$ and $Z \times \{1\}$ respectively. On the other hand we know that we can write $(W, Z \times I, k)$ (to within strong equivalence) as $(Y \times I, Z \times I, h \times id)$ where (Y, Z, h) is a G-space over Z (2.5.2). Then $(y, 0) \rightarrow (y, 1)$ is clearly a strong equivalence of the part of $(W, Z \times I, k)$ over $Z \times \{0\}$ with the part over $Z \times \{1\}$. We have proved a basic fact for our classification theory, namely

2.5.3 THEOREM. If Z is a Σ-space and if f_o^* and f_1^* are strongly Σ-homotopic maps of Z into the orbit space X/G of a G-space X of type Σ then $f_o^{*-1}(X)$ and $f_1^{*-1}(X)$ are strongly equivalent G-spaces over Z.

2.5.4. LEMMA. Let k be a Σ-equivalence of a Σ-space Z with itself. If f^* is a Σ-map of Z into the orbit space of a G-space X of type Σ then $f^{*-1}(X)$ and $(f^* \circ k)^{-1}(X)$ are weakly equivalent G-spaces over X.

PROOF. If we put $f^{*-1}(X) = (Y, Z, h)$ and $(f^* \circ k)^{-1}(X) = (Y', Z, h')$ so that $Y = \{(x, z) \in X \times Z \mid f^*(z) = \Pi_X(x)\}$ and $Y' = \{(x, z) \in X \times Z \mid f^*(k(z)) = \Pi_X(x)\}$ then it is easily seen that $(x, z) \rightarrow (x, k^{-1}(z))$ sets up the desired equivalence of Y with Y'.

2.5.5. COROLLARY. If Z is a Σ-space and if f_o^* and f_1^* are weakly Σ-homotopic maps of Z into the orbit space of a G-space X of type Σ then $f_o^{*-1}(X)$ and $f_1^{*-1}(X)$ are weakly equivalent G-spaces over Z.

PROOF. By hypothesis there exists a Σ-equivalence k of Z with itself such that f_o^* and $f_1^* \circ k$ are strongly Σ-homotopic. By 2.5.3 $f_o^{*-1}(X)$ is strongly equivalent (and, a fortiori, weakly equivalent) to $(f_1^* \circ k)^{-1}(X)$ and by 2.5.4 the latter is in turn weakly equivalent to $f_1^{*-1}(X)$.

q. e. d.

2.6. UNIVERSAL G-SPACES AND CLASSIFYING SPACES.

Σ will again be a collection of G-orbit types. By a Σ-dimension function we mean simply a function d from Σ to the set of integers greater than or equal to -1. We denote by $1 + d$ the Σ-dimension function whose value at $(H) \in \Sigma$ is $1 + d((H))$. A Σ-space Z will be called a (Σ, d)-space if $\dim (Z_{(H)}) \leq d((H))$ for all $(H) \in \Sigma$. A G-space Y of type Σ will be said to be of type (Σ, d) if Y/G is a (Σ, d)-space. If Σ consists of a single orbit type (H) and $d((H)) = n$ then we shall write $((H), n)$ instead of (Σ, d).

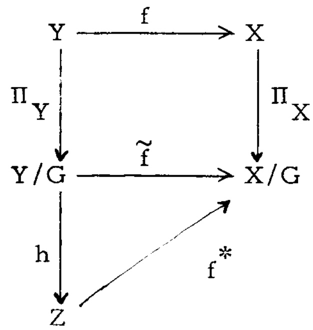
2.6.1. DEFINITION. Let d be a Σ-dimension function. A G-space X will be called (Σ, d)-universal if it is of type Σ and for every G-space Y of type (Σ, d) and any isovariant map f of a closed invariant subspace of Y into X there exists an extension of f to an isovariant map of Y into X (note that X is not required to be of type (Σ, d)). A Σ-space will be called (Σ, d)-classifying if it is the orbit space of a $(\Sigma, 1+d)$-universal G-space.

We now come to the main classification theorem for G-spaces. Note however that it only gains content when, in the next section, we show (by explicit construction) that at least if Σ is finite there always exist (Σ, d)-universal G-spaces.

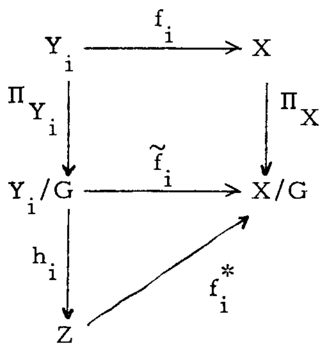
> 2.6.2. CLASSIFICATION THEOREM. Let X be a $(\Sigma, 1+d)$-universal G-space so that X/G is a (Σ, d)-classifying space. If Z is any (Σ, d)-space then the map which assigns to each strong Σ-homotopy class $[f^*]$ (respectively, each weak Σ-homotopy class (f^*)) of Σ-mappings of Z into X/G the strong (respectively, weak) equivalence class of G-spaces over Z which contains $f^{*-1}(X)$ is a one-to-one correspondence between $\tilde{\mathfrak{M}}(Z, X/G)$ (respectively $\mathfrak{M}^*(Z, X/G)$) and the set of all strong (respectively, weak) equivalence

classes of G-spaces over Z.

PROOF. It follows from 2.5.3 and 2.5.5 that the mappings in question are well defined. If (Y, Z, h) is any G-space over Z then the isovariant map of the empty set into X extends to an isovariant map f of Y into X, because X is $(\Sigma, 1+d)$-universal and Y is of type (Σ, d). Thus we have a commutative diagram



where f^* is by definition $f \circ h^{-1}$. It follows from 2.3.2 that (Y, Z, h) is strongly (and, a fortiori, weakly) equivalent to $f^{*-1}(X)$, which proves that the mappings in question are onto. It remains to show that they are one-to-one. Suppose then that $(Y_o, Z, h_o) = f_o^{*-1}(X)$ and $(Y_1, Z, h_1) = f_1^{*-1}(X)$ are strongly (weakly) equivalent G-spaces over Z where f_o^* and f_1^* are elements of $\mathfrak{M}(Z, X/G)$. We must show that f_o^* and f_1^* are strongly (weakly) Σ-homotopic. Let f_i be the canonical isovariant map of Y_i into X so that we have commutativity in the diagram



and let $k : Y_o \to Y_1$ set up the strong (weak) equivalence of (Y_o, Z, h_o) with (Y_1, Z, h_1) (i.e. k is an equivalence of Y_o with Y_1 and in the strong case $h_1 \circ k \circ h_o^{-1}$ is the identity map of Z). Let C be the closed invariant subspace $Y_o \times \{0\} \cup Y_o \times \{1\}$ of $Y_o \times I$ and define $F : C \to X$ by $F(y, o) = f_o(y)$ and $F(y, 1) = f_1(k(y))$. Now F is clearly isovariant and since Y_o is of type $(\Sigma, 1+d)$ and since X is $(\Sigma, 1+d)$-universal we can extend F to an isovariant map F' of $Y_o \times I$ into X. If F' is the induced map we define $f^* : Z \times I \to X/G$ by $f^*(z, t) = F'(h_o^{-1}(z), t)$. We will complete the proof by showing that f^* is a Σ-homotopy of f_o with $f_1^* \circ (h_1 \circ k \circ h_o^{-1})$ (and hence with f_1^* in the strong case). Given $z \in Z$ choose $y \in Y_o$ with $\Pi_{Y_o}(y) = h_o^{-1}(z)$. Then $f^*(z, 0) = F'(\Pi_Y(y), 0) = \Pi_X(F(y, 0)) = \Pi_X f_o(y) = f_o(\Pi_{Y_o}(y)) = f_o(h_o^{-1}(z)) = f_o^*(z)$ and $f^*(z, 1) = F'(\Pi_{Y_o}(y), 1) = \Pi_X(F(y, 1)) = \Pi_X(f_1(k(y))) = f_1^* \circ h_1 \circ \Pi_{Y_1}(k(y)) = f_1^* \circ h_1 \circ k(\Pi_{Y_o}(y)) = f_1^* \circ (h_1 \circ k \circ h_1^{-1})(z)$. q.e.d.

2.7. THE CONSTRUCTION OF UNIVERSAL G-SPACES.

In this final section we will show how to construct a (Σ, d)-universal G-space whenever Σ is a finite set of G-orbit types and d is an arbitrary Σ-dimension function. Our first goal is to reduce the problem to the construction of an $((H), n)$-universal G-space where (H) is an arbitrary G-orbit type and $n \geq -1$.

2.7.1. LEMMA. Let X be a locally compact second countable space, U an open subset of X, C a closed subset of X and F a relatively closed subset of U. If f is a continuous map of C into I which is positive on $C \cap F$ and vanishes on $C-U$ then f can be extended to a continuous map f^* of X into I which is positive on F and vanishes on $X-U$.

PROOF. Our extension f^* will be of the form $\min(1, f_1 + f_2)$ where

$f_1 : X \to I$ is an extension of f that vanishes on $X-U$ and $f_2 : X \to I$ vanishes on $C \cup (X-U)$ and is positive on $U-C$. To get f_1 first extend f to be identically zero on $X-U$ (this is continuous by assumption) and then extend to X by Tietze's Theorem. To get f_2 let $\{g_n\}$ be a locally finite partition of unity in $U-C$ with the support of each g_n compact. Then $\sum_n 2^{-n} g_n(x)$ is positive on $U-C$ and approaches zero as we approach the boundary of $U-C$. Then f_2 is this function extended to be zero on $X - (U-C) = C \cup (X-U)$.

q. e. d.

2.7.2. LEMMA. Let X be a locally compact, second countable G-space, U an open invariant subspace, C a closed invariant subspace, and F a relatively closed invariant subspace of U. If f is an invariant map of C into I which is positive on $C \cap F$ and vanishes on $C-U$ then f can be extended to an invariant map of X into I which is positive on F and vanishes on $X-U$.

PROOF. Recalling the relation between maps of $X/G \to I$ and invariant maps of X into I (1.1.5) this follows easily from the preceding lemma. Alternatively we may use the preceding lemma and 1.1.6.

2.7.3. LEMMA. Let X be an $((H), n)$-universal G-space which is a G-ANR, and let Y be a locally compact, second countable G-space with $\dim Y_{(H)} \leq n$. Let C be a closed invariant subspace of Y, $\phi : C \to I$ an invariant map which is positive on $C \cap Y_H$ and $f : \phi^{-1}((0,1]) \to X$ an equivariant map. Then there exists an extension ψ of ϕ to an invariant map of Y into I which is positive on $Y_{(H)}$ and an extension f^* of f to an equivariant map of $\psi^{-1}((0,1])$ into X.

PROOF. Let $O = \bigcup \{Y_{(K)} \mid (K) \leq (H)\}$, so that by 1.7.21 and 1.7.22

O is an open invariant subspace of Y and $Y_{(H)}$ is relatively closed in O. Let $D(f) = \phi^{-1}((0,1])$ be the domain of f. We note that $D(f) \subseteq O \cap C$, for if $c \in D(f)$ then, since f is equivariant $G_c \subseteq G_{f(c)}$ and, since $X = X_{(H)}$, $(G_c) \leq (G_{f(c)}) = (H)$. Since by assmuption ϕ is positive on $C \cap Y_{(H)}$ it follows that $D(f) \cap Y_{(H)} = C \cap Y_{(H)}$. Moreover since C is closed in X and $Y_{(H)}$ is closed in O it follows from $D(f) \subseteq O \cap C$ that (closure of D(f)) $\cap$ $Y_{(H)}$ and $D(f) \cap$ closure of $Y_{(H)}$ are both equal to $C \cap Y_{(H)}$. Hence if f' is an extension of $f|(C \cap Y_{(H)})$ to an equivariant map of $Y_{(H)}$ into X (which exists because X is ((H), n)-universal and $\dim Y_{(H)} \leq n$) then $f'' = f \cup f'$ is an extension of f to an equivariant map of $D(f) \cup Y_{(H)}$ into X. Since D(f) is open in C and C is closed in X, $(O-C) \cup D(f) = O - (C-D(f))$ is open in X and, since $C \cap Y_{(H)} \subseteq D(f)$, it is a neighborhood of $D(f) \cup Y_{(H)}$. Moreover since $Y_{(H)}$ is closed in O, $D(f) \subseteq O$, and closure $D(f) \subseteq C$ it is clear that $D(f) \cup Y_{(H)}$ is closed in $(O-C) \cup D(f)$. Since X is a G-ANR we can extend f'' to an isovariant map $f''' : U \to X$ where U is an invariant neighborhood of $D(f) \cup Y_{(H)}$ included in O. Since $Y_{(H)}$ is relatively closed in O it is relatively closed in U. If $y \in C - U$ then $y \in C - D(f)$ so $\phi(y) = 0$. It follows from 2.7.2 that we can extend ϕ to an isovariant map $\psi : Y \to I$ which is positive on $Y_{(H)}$ and vanishes on Y-U. Since $\psi^{-1}((0,1]) \subseteq U$ we can now define $f^* = f'''|\psi^{-1}((0,1])$. q.e.d.

2.7.4. THEOREM. Let $\Sigma = ((H_1), \ldots, (H_n))$ be a finite collection of G-orbit types and let d be a Σ-dimension function. Suppose that for each i = 1, 2, ... n X_i is a $((H_i), d((H_i)))$-universal G-space which is a G-ANR. Then $X = X_1 * \ldots * X_n$ is a (Σ, d)-universal G-space.

PROOF. Let Y be a G-space of type (Σ, d) and let C be a closed invariant subspace of Y and f and isovariant map of C into X. We must extend f to an isovariant map of Y into X. Now by 1.3.10 f(c) =

$(\phi_1(c)f_1(c), \ldots, \phi_n(c)f_n(c))$ where

(1) the ϕ_i form an invariant partition of unity in C

(2) f_i is an equivariant map of $\phi_i^{-1}((0,1])$ into X_i

(3) for each $c \in C$ we have $\phi_i(c) \neq 0$ and $G_c = G_{f_i(c)}$ for some i = 1, 2 . . . n.

If $c \in Y_{(H_i)} \cap C$ then let $\phi_j(c) \neq 0$ and $G_{f_j(c)} = G_c$. Since $(H_i) = (G_c) = (G_{f_j(c)}) = (H_j)$ (because $X_j = X_{j(H_j)}$) it follows that i = j, hence ϕ_i is positive on $Y_{(H_i)} \cap C$. By 2.7.3 we can extend ϕ_i to an invariant map $\psi_i : Y \to I$ which is positive on $Y_{(H_i)}$ and extend f_i to an equivariant map $f_i^* : \psi_i^{-1}((0,1]) \to X_i$. Let $\phi_i^*(y) = \psi_i(y)/\sum_{j=1}^{n} \psi_j(y)$. Then ϕ_i^* is an extension of ϕ_i to an invariant map of Y into I which is positive on $Y_{(H_i)}$ and $\sum_{i=1}^{n} \phi_i^*(y) \equiv 1$. Since $\phi_i^{*-1}((0,1])$, f_i^* is an equivariant map of $\phi_i^{*-1}((0,1])$ into X_i. If $y \in Y_{(H_i)}$ then $\phi_i^*(y) \neq 0$ and $G_y = G_{f_i(y)}$ (that $G_y \subseteq G_{f_i(y)}$ follows from 1.1.16, that we must have equality follow because $X_i = X_{i(H_i)}$ so that $(G_y) = (H_i) = (G_{f_i(y)})$). Then by 1.3.10 $y \to (\phi_i^*(y)f_1^*(y), \ldots, \phi_n^*(y)f_n^*(y))$ is the desired extension of f to an isovariant map of Y into X.

q. e. d.

2.7.5. LEMMA. Let (H) be a G-orbit type and let X and Y be (H)-simple G-spaces (1.3.8). Then $f \to f|X_H$ is a one-to-one correspondence between all isovariant maps of X into Y and all N(H)/H-equivariant maps of X_H into Y_H (where as usual N(H) is the normalizer of H in G).

PROOF. As remarked in 1.3.9 X_H is an N(H)/H-space (in fact an N(H)/H-principal bundle) under the action $(nH)x = nx$. If $f : X \to Y$ is isovariant then, for $x \in X_H$, $G_x = H$ so $G_{f(x)} = G_x = H$ so $f(x) \in Y_H$ so f maps X_H into Y_H and of course $f|X_H$ is N(H)/H-invariant. Conversely if $f^* : X_H \to Y_H$ is N(H)/H-invariant (and hence automatically isovariant because X_H and Y_H are N(H)/H-principal) then f^* is N(H)-isovariant and hence by 1.7.10 and 1.7.33 f^* is the restriction to X_H of a uniquely determined isovariant map of X into Y. This proves that the mapping is one-to-one and onto. q.e.d.

Recalling that by 1.7.6 and 1.7.33 if X is (H)-simple then X/G is homeomorphic to $X_H/N(H) = X_H/(N(H)/H)$, so that in particular they have the same dimension, it follows immediately from the above lemma and 1.7.16 that

2.7.6. THEOREM. A necessary and sufficient condition for an (H)-simple G-space X to be ((H), n)-universal is that for every N(H)/H-principal bundle Y with $\dim Y/(N(H)/H) \leq n$ and every equivariant map f of a closed invariant subspace of Y into X_H there exist an extension of f to an equivariant map of Y into X_H. Moreover if in addition X_H is an N(H)-ANR then X is a G-ANR.

2.7.7. THEOREM. Let X be an (H)-simple G-space such that X_H is n-connected (i.e. it is connected and its first n-homotopy groups are trivial), polyhedral, and an N(H)-ANR. Then X is ((H), n)-universal and a G-ANR.

PROOF. That the conditions of the preceding theorem are satisfied follows from [2] 1° proof of Théorème 6, VIII, 7.

We are now in a position to construct an ((H, n)-universal G-space for any $H \subseteq G$ and any non-negative integer n.

2.7.8. THEOREM. If $H \subseteq G$ and N(H)/H is m-connected then $(G/H)^{(*k)}$ (i.e. (1.3.6) the k-fold reduced join of G/H with itself) is an $((H), k(m+2)-2)$-universal G-space and a G-ANR.

PROOF. By 1.3.9 $(G/H)^{(*k)} = X$ is (H)-simple and (since clearly $(G/H)_H = N(H)/H)X_H = (N(H)/H)^{(ok)}$. Now by [9] Lemma 2.3 X_H is $k(m+2)-2$ connected. Since $N(H)/H$ is a compact differentiable manifold it is polyhedral, and clearly the join of polyhedral spaces is polyhedral, so X_H is polyhedral. Hence by 2.7.7 it remains only to show that X_H is an N(H)-ANR. Now $N(H)/H$ is an N(H)-ANR (1.6.7) and (1.4.1) admits an N(H)-equivariant imbedding in a Euclidean N(H)-space. The desired result then follows from

2.7.9. LEMMA. Let $X_1, \ldots, X_k$ be G-ANR's which admit equivariant imbeddings $f_i : X_i \to E_i$ in Euclidean G-spaces. Then $X = X_1 \circ \ldots \circ X_k$ is a G-ANR.

PROOF. Since $f_i(X_i)$ is a G-ANR we can find a neighborhood U_i of $f_i(X_i)$ in E_i which admits an equivariant retraction ρ_i onto $f_i(X_i)$. It is trivial that $f : (t_1 x_1, \ldots t_k x_k) \to \sum_{i=1}^{k} t_i f_i(x_i)$ is an equivariant imbedding of $X_1 \circ \ldots \circ X_k$ into $E_1 \oplus \ldots \oplus E_k$. Moreover $u_1 + \ldots + u_k \to \rho(u_1) + \ldots + \rho(u_k)$ is an equivariant retraction of $U = U_1 + \ldots + U_k$ onto $f(X_1 \circ \ldots \circ X_k)$. Since U is a neighborhood of $f(X_1 \ldots X_k)$ it follows from 1.6.4 and the fact (1.6.2) that $E_1 \oplus \ldots \oplus E_k$ is a G-ANR that $f(X_1 \circ \ldots \circ X_k)$ and hence $X_1 \circ \ldots \circ X_k$ is a G-ANR. q.e.d.

2.7.10. EXISTENCE THEOREM FOR UNIVERSAL G-SPACES. Let $\Sigma = ((H_1), \ldots, (H_n))$ be any finite collection of G-orbit types and let d be a Σ-dimension function. Let $N(H_i)/H_i$ be m_i-connected. If k_i is any positive integer greater than or equal to $d((H_i)) +$

$2/m_i + 2$ then

$$(G/H_1)^{(*k_1)} * \dots * (G/H_n)^{(*k_n)}$$

is a (Σ, d)-universal G-space.

PROOF. Immediate from 2.7.4 and 2.7.9.

List of Special Symbols

Symbol	Meaning	Page on which introduced
Gx	orbit of x	2
G_x	isotropy group at x	2
X/G	orbit space of X	2
Π_X	orbit map of $X \to X/G$	2
(H)	set of subgroups of G conjugate to H	10
$[\Omega]$	orbit type of $\Omega = (G_x)$ for any $x \in \Omega$	10
$X_{(H)}$	union of orbits of X of type (H)	10
$\tilde{X}_{(H)}$	set of orbits of X of type (H)	10
$\leq$	partial ordering of orbit types	10
$\mathcal{J}(X)$	isogenies of X	12
$X_1 \circ \ldots \circ X_k$	join of $X_1, \ldots, X_k$	16
$X^{(\circ n)}$	k-fold join of X	16
$X_1 * \ldots * X_k$	reduced join of $X_1, \ldots, X_k$	16
$X^{(*n)}$	k-fold reduced join of X	16
X_H	$\{x \in X \mid G_x = H\}$	17
$N(H)$	normalizer of H in G	17
$X \times_G Y$	associated fiber bundle of principal bundle X with fiber Y	33
Σ	a collection of G-orbit types	46
$\mathfrak{M}(X, Y)$	Σ-maps of X into Y	46

Symbol	Meaning	Page on which introduced
$\tilde{\mathfrak{m}}(X, Y)$	strong homotopy classes of Σ-maps of X into Y	46
$\mathfrak{m}^{*}(X, Y)$	weak homotopy classes of Σ-maps of X into Y	46
(X, Z, h)	G-space over a Σ-space Z	48
$f^{*-1}(X)$	G-space induced by $f^{*} : Z \to X/G$	49
(Σ, d) and $((H), n)$		59

Index of Terminology

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