HILBERT SCHEMES OF POINTS AND THE ADHM CONSTRUCTION

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Overview

In this talk, I discuss a how a special case of the Instanton Moduli space on $\mathbb{R}^4$ can be described as a Hilbert Scheme of points on $\mathbb{C}^2$. First, I will (very briefly) review the ADHM construction of instantons on $\mathbb{R}^4$. The next topic will be to define Hilbert Schemes of points and look at the particular case of $(\mathbb{A}^2)^{[n]}$. After going through some examples for small n, I will introduce the framed moduli space of torsion free sheaves on $\mathbb{P}^2_{\mathbb{C}}$ and show that in a specific case it is isomorphic to $(\mathbb{C}^2)^{[n]}$. Almost all the material follows Hiraku Nakajima's book *Lectures on Hilbert Schemes of Points on Surfaces.*

0. Instantons on $\mathbb{R}^4$ and the ADHM Construction

This section will serve as motivation for the remainder of the talk. We would like to consider the ADHM moduli space, so we begin by considering the space

$$\mathbf{M} = \{(B_1, B_2, i, j) \mid B_1, B_2 \in \mathrm{Hom}(V, V)\; i \in \mathrm{Hom}(W, V)\; j \in \mathrm{Hom}(V, W)\}$$

where V is a rank r $\mathbb{C}$-vector space and W is a rank n $\mathbb{C}$-vector space. In order to form the ADHM moduli space beginning with the space $\mathbf{M}$, we would like to impose certain compatibility conditions on the linear maps (B_1, B_2, i, j). In particular, we define

$$\mu_1(B_1, B_2, i, j) = \frac{i}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j)$$

and

$$\mu_{\mathbb{C}}(B_1, B_2, i, j) = [B_1, B_2] + ij.$$

We can further consider the $U(V)$-action on $\mathbf{M}$ given by

$$(B_1, B_2, i, j) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

Then, we consider the quotient $\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)/U(V)$, which we will denote by $\mathcal{M}_0(r, n)$. This quotient will be singular along the locus where the $U(V)$ action fails to be free, we denote the non-singular locus $\mathcal{M}_0^{\mathrm{reg}}(r, n)$. This is called the ADHM moduli space. It is of interest because of the following theorem

Theorem 0.0.1 (A.D.H.M.)**.** *There is a bijective correspondence between the framed moduli space of anti-self-dual connections on a 4-dimensional Hyper Kähler manifold X with unitary vector bundle E and $\mathcal{M}_0^{\mathrm{reg}}(r, m)$ where r is the rank of E and $n = c_2(E)$.*

We won't discuss this theorem here (hopefully the construction of anti-self-dual connections from ADHM data will be covered next time). However, this theorem gives some context for the remaining discussion of Hilbert Schemes of points.

1. Hilbert Schemes of Points

1.1. **General Definition.** The idea of the Hilbert Scheme of points comes from configuration space. Specifically, we may have the goal of describing the space of configurations of n points on X, (for the purposes of this talk, we will always take X to be a smooth variety over $k = \bar{k}$, though these topics can be discussed in a more general setting). A first attempt to write down what this space is could be the product

$$P^n X = \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

It usually makes sense to take the quotient by the natural action of the symmetric group, since we do not distinguish between points. This gives the symmetric product

$$S^n X = \underbrace{X \times \cdots \times X}_{n \text{ times}} / \Sigma_n.$$

We write points in $S^n X$ as formal sums $\sum n_i[x_i]$ for nom-negative integers n_i and $x_i \in X$. This quotient will not always produce a non-singular variety. For example, when $n = 2$, the group action is not free along the diagonal $\Delta \subset X \times X$, and this results in a singular locus along the diagonal in $S^2 X$. In terms of our initial goal of parameterizing configurations of points in X, it is not surprising that something bad happens along the diagonal. Approaching the diagonal corresponds to the two points approaching each other and eventually overlapping. At the point where they overlap, the system has somehow lost one degree of freedom. From this point of view, a possible solution would be to keep track of the direction the two points approach each other along, i.e. gluing in a copy of $\mathbb{P}^{\dim X - 1}$ along the diagonal in $S^2 X$. In other words, we can consider the blow up $\mathrm{Bl}_\Delta S^2 X$ of $S^2 X$ along the diagonal.

This blow up actually goes by a name, the Hilbert Scheme of 2 points on X, denoted $X^{[2]}$. We now turn to describing this for a general number of points, e.g. defining the Hilbert Scheme of n points, $X^{[n]}$. While there is a more general definition of this object as a scheme, we settle here for just defining its closed points (the real definition in terms of representable functors will not be needed in our discussion here).

Definition 1.1.1. *The Hilbert Scheme of n points on X has closed points given by zero-dimensional closed subschemes $Z \subset X$ which satisfy $\dim_k \mathcal{O}_X / \mathcal{J} = n$ where $\mathcal{J}$ is the ideal sheaf of Z.*

Going back to the $n = 2$ case, we can describe how this definition of $X^{[2]}$ agrees with the previous one. There are two cases to consider. In the first case, we take two distinct points $x \neq y \in X$ and set $\mathcal{J} = \{f \in \mathcal{O}_X \mid f(x) = f(y) = 0\}$. We see that $\dim_k \mathcal{O}_X / \mathcal{J} = 2$, in fact, we can describe the structure sheaf of Z in this case: $\mathcal{O}_Z = \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,y}$ where $\mathcal{O}_{X,x}$ denotes the skyscraper sheaf at x. In the second case, we may consider an ideal of the form $\mathcal{J} = \{f \in \mathcal{O}_X \mid f(x) = 0, df(v) = 0\}$ for $x \in X$ and $v \in T_x X$. This case represents two points coincident at $x \in X$ with "infinitesimal vector of separation" v. We will see both of these explicitly when we consider the example of $X = \mathbb{A}^2$.

Since $X^{[2]}$ is a resolution of the singular locus of $S^2 X$, there is a natural morphism $X^{[2]} \to S^2 X$. In general, there is always a morphism $\pi : X^{[n]} \to S^n X$ called the

Hilbert-Chow morphism. It is defined by

$$\pi(Z) = \sum_{x\in X} \left(\dim_k \left(\mathcal{O}_X/\mathcal{J}\right)_x\right)[x] = \sum_{x\in X} \left(\dim_k \mathcal{O}_{X,x}/\mathcal{J}_x\right)[x].$$

Note that since $\dim Z = 0$ the sum in the above definition is finite. As an example, the two ideals given above have images under the Hilbert Chow morphism $[x]+[y]$ and $2[x]$ respectively.

1.2. **Main Example.** We now look at the specific case of $X = \mathbb{A}^2$. Here we have a nice explicit description of $(\mathbb{A}^2)^{[n]}$ as a smooth variety. The construction is as follows. Denote by $\widetilde{H}$ the set of "stable triples"

$$\{(B_1, B_2, i) \mid [B_1, B_2] = 0\}$$

where $B_1, B_2 \in \mathrm{Hom}(k^n, k^n)$, $i \in \mathrm{Hom}(k, k^n)$, and by stable we mean that there is no proper subset $k \subset k^n$ such that $B_i(S) \subset S$ and $\mathrm{im}(i) \subset S$. This set admits an action of $GL_n(k)$ given by $(B_1, B_2, i) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gi)$. We denote the quotient of $\widetilde{H}$ by this action by H.

Theorem 1.2.1. *H is a non-singular scheme and is isomorphic to $(\mathbb{A}^2)^{[n]}$ as a scheme.*

We will give the idea behind the bijection of closed points and how to show smoothness here. In this case, we have $\mathcal{O}_{\mathbb{A}^2} \cong k[z_1, z_2]$ and closed points in $(\mathbb{A}^2)^{[n]}$ correspond to ideals $I \subset k[z_1, z_2]$ such that $\dim_k k[z_1, z_2]/I = n$. We let V denote the n-dimensional vector space $k[z_1, z_2]/I$ and see that there are two endomorphisms of V given by multiplication by z_1 and z_2. We let B_i be multiplication by z_i and i be defined by sending $1 \in k \subset k[z_1, z_2]$ to the image of $1 \in V$. We can then see that $[B_1, B_2] = 0$ and the stability condition holds since $B_1^{m_1}B_2^{m_2}i(1)$ for $m_1, m_2 \geq 0$ spans all of $k[z_1, z_2]$. Conversely, given $(B_1, B_2, i) \in H$, we can consider the map $\varphi : k[z_1, z_2] \to k^n$ given by $\varphi(f) = f(B_1, B_2)i(1)$. We then see that the image of this map is invariant under B_i and contains the image of i, so it must be all of k^n. We then define $k[z_1, z_2] \supset I = \ker\varphi$. It is clear that $\dim_k(k[z_1, z_2]/I) = n$, so this reppresents a closed point in $(\mathbb{A}^2)^{[n]}$. Moreover, these maps are mutually inverse.

To see that H is smooth, we first show that $\widetilde{H}$ is smooth, then smoothness of H follows from the freeness of the $GL_n(k)$ action. We first note that the space $\widetilde{H}$ is defined by the equation $[B_1, B_2] = 0$. The derivative of this map at the point (B_1, B_2, i) is given by $[B_1, C_2] + [C_1, B_2]$ for $C_1, C_2 \in \mathrm{Hom}(k^n, k^n)$, thought of as the tangent space to $\widetilde{H}$ at (B_1, B_2, i). Thus, the cokernel of this map is given by $\xi \in \mathrm{Hom}(k^n, k^n)$ such that $\mathrm{tr}(\xi([B_1, C_2] + [C_1, B_2])) = 0$ for all C_1, C_2. This is just the set $\{\xi \mid [B_1, \xi] = [B_2, \xi] = 0\}$. We then define the maps from the cokernel to k^n and from k^n to the cokernel by $\xi \mapsto \xi(i(1))$ and $k^n \ni v \mapsto \xi$ defined by $\xi(B_1^n B_2^m i(1)) = B_1^n B_2^m v$. The stability condition implies that this defines ξ on all of k^n. Moreover, these are mutually inverse since ξ commutes with B_i. This shows that the cokernel of derivative map has constant rank, hence $\widetilde{H}$ is non-singular. Similarly, the stabilizer of the $GL_n(k)$ action is trivial by the stability condition.

It is also worth noting that we can describe the tangent space of H at (B_1, B_2, i) can be written down as follows. Consider the complex

$$\mathrm{Hom}(k^n, k^n) \xrightarrow{d_1} \mathrm{Hom}(k^n, k^n) \oplus \mathrm{Hom}(k^n, k^n) \oplus k^n \xrightarrow{d_2} \mathrm{Hom}(k^n, k^n)$$

where d_1 is the differential of the $GL_n(k)$ action which sends $\xi \mapsto ([\xi, B_1], [\xi, B_2], \xi i)$ and d_2 is the differential of the map $(B_1, B_2, i) \mapsto [B_1, B_2]$ described earlier. Then d_1 is injective, the cokernel of d_2 has dimension n, and the tangent space at (B_1, B_2, i) is given by the middle homology of this complex.

Now we can look at some explicit examples. For $n = 1$, we should have $(\mathbb{A}^2)^{[1]} \cong \mathbb{A}^2$. Indeed, B_1 and B_2 will just act by scalar multiplication and we may use the $GL_1(k)$ action to set $i = 1$. Then we see that $(\lambda, \mu, 1)$ corresponds to

$$I = \{f \in k[z_1, z_2] \mid f(\lambda, \mu) = 0\}$$

which defines that point $(\lambda, \mu) \in \mathbb{A}^2$.

The story for $n = 2$ is more interesting. There are two cases. If one of B_1 or B_2 has distinct eigenvalues then we may write $B_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $B_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$ since B_1 and B_2 may be simultaneously diagonalized by some $g \in GL_2(k)$. The stability condition implies that $i(1)$ must be of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ and so we may choose $g \in GL_2(k)$ so that $i(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then we see that

$$I = \{f \in k[z_1, z_2] \mid f(\lambda_1, \mu_1) = f(\lambda_2, \mu_2) = 0\}$$

which corresponds to the distinct points (λ_1, μ_1) and (λ_2, μ_2).

On the other hand, if both B_1 and B_2 have only one eigenvalue, then we can at least put both into upper-triangular form so that $B_1 = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}$ and $B_2 = \begin{pmatrix} \mu & b \\ 0 & \mu \end{pmatrix}$. The stability condition here tells us that we can't have $a = b = 0$. Furthermore, we can choose $i(1)$ to be of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Calculating B_i^n we see that

$$B_1^n = \begin{pmatrix} \lambda^n & a\frac{\partial}{\partial \lambda}\lambda^n \\ 0 & \lambda^n \end{pmatrix}$$

and similarly for B_2. This shows that

$$f(B_1, B_2)i(1) = \begin{pmatrix} a\frac{\partial f}{\partial z_1}(\lambda, \mu) + b\frac{\partial f}{\partial z_2}(\lambda, \mu) \\ f(\lambda, \mu) \end{pmatrix}$$

and so we see that

$$I = \left\{ f \in k[z_1, z_2] \,\middle|\, f(\lambda, \mu) = \left(a\frac{\partial f}{\partial z_1} + b\frac{\partial f}{\partial z_2} \right)(\lambda, \mu) = 0 \right\}.$$

Here, we see that (λ, μ) corresponds to two overlapping points in $\mathbb{A}^2$ and the vector (a, b) is the direction of infinitesimal separation between the points. This vector lives in $\mathbb{P}^1$, so this description of $(\mathbb{A}^2)^{[2]}$ agrees with the previous statement that $X^{[2]} \cong \mathrm{Bl}_\Delta S^2 X$.

Finally, we can give a description of the Hilbert Chow morphism in this case. For any n and $(B_1, B_2, i) \in (\mathbb{A}^2)^{[n]}$ we can simultaneously put B_1, B_2 in upper-triangular form with diagonal entries $\lambda_1, ..., \lambda_n$ and $\mu_1, ..., \mu_n$ respectively. When all the points (λ_i, μ_i) are distinct π is given by

$$\pi(B_1, B_2, i) = \{[(\lambda_1, \mu_1)], ..., [(\lambda_n, \mu_n)]\}.$$

When this is not the case (as in the example above) we will have

$$\pi(B_1, B_2, i) = \{m_1[(\lambda_1, \mu_1)], ..., m_r[(\lambda_r, \mu_r)]\}$$

where $(\lambda_1, \mu_1), ..., (\lambda_r, \mu_r)$ are the distinct pairs of eigenvalues of B_1 and B_2.

2. Framed Moduli Space of Torsion Free Sheaves

2.1. **General Case.** We now consider a different kind of moduli space. We will see that this space has connections to both the ADHM moduli space and the Hilbert Scheme of n points on $\mathbb{C}^2$.

In this setting, $k = \mathbb{C}$ and $\mathcal{M}(r, n)$ denotes the framed moduli space of torsion-free sheaves on $\mathbb{P}^2$ with rank r and second Chern class n up to isomorphism (the framing implies that the first Chern class is trivial, so c_2 carries all of the topological information about this sheaf). Unpacking this, elements of $\mathcal{M}(r, n)$ are equivalence classes of pairs (E, Φ) with E a torsion-free sheaf on $\mathbb{P}^2$ with $c_2(E) = n$, $\text{rank}(E) = n$, and $\Phi : E_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}}^{\oplus r}$ is the framing of E at the line at infinity $l_\infty = \{[0 : z_1 : z_2]\} \cong \mathbb{P} \subset \mathbb{P}^2$. We would like to show that there is another description of $\mathcal{M}(r, n)$ which allows us to relate it to the other moduli spaces we have talked about so far.

Let $\widetilde{H}_r$ be the space of stable quadruples (B_1, B_2, i, j) where $B_i \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, $i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ which satisfy $[B_1, B_2] + ij = 0$ (here, stable is exactly the same definition given above). Then, we may take the quotient of this space by the $GL_n(\mathbb{C})$ action given by $(B_1, B_2, i, j) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$, which we will call H_r.

Theorem 2.1.1 (Barth)**.** *There is a bijection* $\mathcal{M}(r, n) \cong H_r$.

The proof of this theorem is somewhat technical, so we only recall some of the main themes here. The main object of importance is the monad description of torsion free sheaves on $\mathbb{P}^2$. A monad is a complex of sheaves (in our case, on $\mathbb{P}^2$)

$$\mathcal{G} \to \mathcal{F} \to \mathcal{G}'$$

such that the first map is injective and the second map is surjective, i.e. the outer homology vanishes. The middle homology, however, can (and in our case will) be non-trivial. In fact, it defines a new sheaf on $\mathbb{P}^2$. Calling the first map a and the second map b, the idea behind this construction is that we can fix $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{G}'$ and still construct a wide variety of sheaves as $\ker(a)/\text{coker}(b)$ by varying a and b.

In our case, the monad we will want to consider is of the form

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to (V \otimes \mathcal{O}_{\mathbb{P}^2})^{\oplus 2} \oplus W \otimes \mathcal{O}_{\mathbb{P}^2} \to V \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

The result we need is that we can exhibit E as $\ker(a)/\text{coker}(b)$ for certain a and b in the above monad. Here, the vector spaces V, W arise as the cohomology of $E(a)$ (possibly tensored with the cotangent bundle over $\mathbb{P}^2$). The important fact we will need is that $\dim V = n$ and $\dim W = r$. The point is that, given the data (B_1, B_2, i, j), we can construct a sheaf E on $\mathbb{P}^2$, provided we choose a framing Φ of E at l_∞. The condition on B_1, B_2, i, and j will correspond to the map a being injective and the stability condition will correspond to b being surjective.

The description of H_r is reminiscent of the ADHM data we wrote down earlier. There are two slight differences, which turn out not to be differences at all. In particular, in the ADHM data, we required the quadruples satisfy an additional constraint, $[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j = 0$ and the quotient was only taken over the unitary group $U(n) \cong U(V)$ instead of the general linear group $GL_n(\mathbb{C})$. However, we may decompose a general linear matrix into its unitary and non-unitary pieces (there are many decompositions of this sort) and it turns out that quotienting out the non-unitary part will be equivalent to requiring the additional condition from the ADHM data.

2.2. **Rank 1 Case.** Now we would like to show how $\mathcal{M}(r, n)$ is equivalent to $(\mathbb{C}^2)^{[n]}$ when $r = 1$. The descriptions H and H_1 already look similar, save for the appearance of j in H_1. Before we show that these spaces are in bijection, we may already see that points in $\mathcal{M}(1, n)$ can be associated with points in $(\mathbb{C}^2)^{[n]}$. This is true because we have the inclusion $E \hookrightarrow E^{\vee\vee}$ and $E^{\vee\vee} \cong \mathcal{O}_{\mathbb{P}^2}$ since $E^{\vee\vee}$ is locally free of rank 1 with vanishing first Chern class. Then, since E is locally trivial on l_∞ we can consider the quotient sheaf $\mathcal{O}_{\mathbb{P}^2}/E$ defined on $\mathbb{C}^2$ and we see that this is n dimensional as a $\mathbb{C}$ vector space (since $c_2(E) = n$) and so it represents a point in $(\mathbb{C}^2)^{[n]}$.

In order to align the descriptions of H and H_1, we must deal with the existence of j. It turns out that the stability condition will imply that this term vanishes. To see this, we will define a subspace $S \subset \mathbb{C}^n$ and show that $j|_S = 0$ and that S contains the image of i and is mapped to itself by B_i. Namely, we take S to be the subspace spanned by $\sum B_{i_1} B_{i_2} \cdots B_{i_k} i(\mathbb{C})$, i.e. sums of products of the B_i's acting on the image of i. We can show, using identities of the trace of products and commutators of matrices, that $j\widehat{B}i = 0$ where $\widehat{B}$ is some product of B_i's, which implies that $j|_S = 0$. It is clear that the image of i is contained in S and that S is invariant under B_1, B_2.

Furthermore, the above description of E shows that the sheaf realized as the middle homology of the monad description corresponds to the ideal $I \subset \mathbb{C}[z_1, z_2]$ from the discussion of the correspondence between H and $(\mathbb{C}^2)^{[n]}$. This can also be shown explicitly using the concrete descriptions of a and b in the monad description.