Enriched categories and models for spaces of dipaths.

A discussion document and overview of some techniques

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Abstract. Partially ordered sets, causets, partially ordered spaces and their local counterparts are now often used to model systems in computer science and theoretical physics. The order models 'time' which is often not globally given. In this setting directed paths are important objects of study as they correspond to an evolving state or particle traversing the system. Many physical problems rely on the analysis of models of the path space of a space-time manifold. Many problems in concurrent systems use 'spaces' of paths in a system. We review some ideas from algebraic topology and discrete differential geometry that suggest how to model the dipath space of a pospace by an enriched category. Much of the earlier material is 'well known', but, coming from different areas, is dispersed in the literature.

1 Introduction

Partially ordered sets are frequently used to model systems in both computer science and physics. The order models 'time', or 'use of resources', and often can not be globally given. For instance, in models for the temporal modal logic S4, the models are partial orders (or more generally preorders) but the time dependency is merely 'before'; there is no clock. Similarly in the theory of causal sets, which are 'locally finite' or 'discrete' partial orders, 'causality' is represented by '$\leq$' and again no global clock is given.

Many physical systems are analysed by models of an evolving state space, or, almost equivalently, a space of 'evolving states'. In the study of 'space-time' manifolds, the evolving states are modelled by 'time-like' paths.

Here we will review some ideas from algebraic topology that suggest approaches on how to model spaces of directed (hence 'time-like') paths in a directed space using enriched, and, in particular, simplicially enriched, categories. We will point out some of the possible constructions, but also, where possible, their inadequacies for the task of using simplicially enriched category theory in a useful way for the study of 'spaces of dipaths'. We will explore various constructions, and their interpretation in an attempt to identify the way in which 'topology change' might be detected in directed homotopy and related dynamical systems.

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The idea of constructing a simplicially enriched category using the paths in a partially ordered *set* or small category seems to have occurred first in work by Leitch, [1]. At about the same time, Boardman and Vogt, [2], used a closely related construction for topological categories. This was pushed forward in Vogt's paper, [3], and later exploited by Cordier, [4]. We will describe this and briefly look at related issues of simplicially enriched functors from such a gadget to various target categories, or structured coefficients.

Order enriched categories have been quite often used in theoretical computer science as have category enriched ones (2-categories), for instance in rewriting theory. Both these can be subsumed within the simplicial enrichment setting.

Having passed from various situation to simplicially enriched categories, we explore what information they give you and how it is 'packaged'. It is clear that as a simplicial set has various invariants modelling parts of their homotopy type, one can, with care, pass via these homotopy or homology models to other enriched settings such as chain complexes. Of particular note however are the analogues of the 'cochain-cohomology' group of constructions, as these are nearer to the invariants used to explore the geometry, rather than the homotopy of a space. We show how to pass from some simplicially enriched categories to dg-categories via a cobar construction and briefly explore some of the consequences.

Another theme that will emerge is the search for 'evolving bundles'. The theory of fibre bundles on spaces is highly dependant on the 'symmetry group', 'gauge group', 'reversible path' technologies, so will need a new approach if it is to be transformed into something that is optimised for the 'directed path' paradigm. We propose a possible set of analogues, but will not be able to develop the theory that far here.

The section on differential graded categories of paths has benefited enormously from joint work with Jonathan Gratus, and the construction there owes a lot to some unpublished joint work with him.

2 Path spaces

Given a space, X, the usual 'classical' homotopy invariants such as its homotopy groups are closely linked to the space of *loops* on X. This works well for arc-wise connected spaces and ordinary maps. For some problems however, the space of free paths is needed. This is X^I, usually considered with the compact open topology. Again this is fine for standard topological situations, but when, for instance, non-compact spaces are involved it leads to inadequate information, (see, for instance, the survey article, [5]), as asymptotic information of the behaviour out towards the 'ends' of the space cannot be included. Similarly in directed homotopy, paths are there not reversible and homotopies cannot 'undo' what has already been done, yet the (directed) paths are what are of most interest as they correspond to *evolving states* with 'time' as the variable in the paths.

Returning to the case of paths in a space, X, the usual path space is X^I, that is, the space of continuous functions, $a : I \to X$, from the unit interval,

$I = [0,1]$, to X. This has certain nice structure that is fairly obviously 'of use', but shows some difficulties as well. There are two continuous maps

$$e_0, e_1 : X^I \to X,$$

where $e_0(a) = a(0)$, and $e_1(a) = a(1)$. There is also a continuous map $s : X \to X^I$, where is $x \in X$, $s(x)$ is the constant path, $s(x)(t) = x$ for all $t \in [0,1]$. There is also almost a composition on paths, but if a is a path from x to x' in X (so $e_0(a) = x$ and $e_1(a) = x'$), and b is one from x' to x'', then the natural composite $a \star b$ is of 'length' 2, i.e. $a \star b : [0,2] \to X$ with

$$a \star b(t) = \begin{cases} a(t) & \text{if } 0 \leq t \leq 1, \\ b(t-1) & \text{if } 1 \leq t \leq 2. \end{cases}$$

This almost looks like a category structure, but as well as the 'composition' not quite working, the constant path does not quite act as an identity! We could rescale the composition, but then we loose associativity, and dividing by enough 'homotopy' to fix the identities results in the destruction of much of the sense that the parameter in the path corresponds to time. It is possible to build a 'homotopy coherent' category in this way, but it is technically quite difficult to do so in detail.

From the point of view of standard topology, this would not matter and, of course, the above forms part of the construction of the fundamental groupoid of a space, (cf. Brown, [6]). From the point of view of modelling physical phenomena or concurrent distributed systems, the 'faults' noted above are important, but here we do not need to invert paths as we are not heading for a groupoid, rather our aim is a category. There is, of course, an alternative used in standard homotopy theory that gets around some of the difficulties in a neat way and is closer to the intuition of time as the variable. It is the *space of Moore paths* of X.

For this we replace X^I by a much larger space, namely

$$Paths(X) = \bigsqcup_{r \geq 0} X^{[0,r]}.$$

This has a beautiful structure of a spatially enriched category. For points $x, x' \in X$, we consider the subspace $Paths(X)(x,x')$ of all those $a \in Paths(X)$, $a : [0,r] \to X$ for some r, such that $a(0) = x$ and $a(r) = x'$. Composition of paths, adapted in an obvious way from the above, gives, for $x, x', x'' \in X$,

$$Paths(X)(x,x') \times Paths(X)(x',x'') \to Paths(X)(x,x''),$$

which is continuous, and associative and has identities, since $Paths(X)(x,x)$ always contains the identity path of length 0 at x, something we did not have available before. Of course, paths of length r compose with paths of length s to give paths of length $r+s$ and later this will be a cause of some problems as it inhibits rescaling of paths. The category $Paths(X)$ is enriched over the category of topological spaces with monoidal structure given by the Cartesian product. (For generalities on enriched categories, see the entry in Wikipedia, (`http://en.wikipedia.org/wiki/Enriched_category`). For more detailed information, consult the references there.) Later we will use simplicially enriched categories quite a lot and some brief discussion of them is given in an appendix.

This is fine if X is just a topological space, but if it is, say, a smooth manifold then the paths need to be piecewise smooth for $Paths(X)$ to have a hope of reflecting more than just the structure of the underlying space. This is not the only problem. It is often desirable to perform analogues of the operations of calculus on such spaces of paths, but they are infinite dimensional even when the space X is a nice finite dimensional smooth manifold. If X has other structure such as that of a space-time, then similarly operations such as integration seem to be needed for the study of this context. (This leads, for instance, to the theory of iterated integrals due to Chen.) The hope has been to replace the 'hom-sets', $Paths(X)(x, x')$, etc. by discrete, combinatorial models. We will discuss one such in particular, namely replacing them by simplicial sets, as this seems to be relevant in many contexts and has obvious extensions to areas of more general interest.

For the moment we will work with $Paths(X)$ and the topological situation. One possibility, then, is to replace each $Paths(X)(x, x')$ by its singular complex, i.e. by a simplicial set made up of singular simplices in $Paths(X)(x, x')$. To set this up properly we need to digress for a short while.

Recall, (cf. Curtis [7]), $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the topological n-simplex given by

$$\underline{t} = (t_0, \ldots, t_n) \in \Delta^n \Longleftrightarrow \sum t_i = 1 \text{ and all } t_i \geq 0.$$

The simplices of adjacent dimensions are related by *coface* and *codegeneracy* maps:

$$\delta_i : \Delta^{n-1} \to \Delta^n \qquad 0 \leq i \leq n$$
$$\delta_i(\underline{t}) = (t_0, \ldots, 0, \ldots, t_{n-1}),$$

so, in $\delta_i(\underline{t})$, a 0 is inserted in position i, and the later coordinates are shifted right; whilst

$$\sigma_i : \Delta^{n+1} \to \Delta^n, \qquad 0 \leq i \leq n,$$
$$\sigma_i(\underline{t}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}),$$

so adds the i^{th} and $(i+1)^{st}$ coordinates together.

Let $\mathbf{\Delta}$ be the skeletal category of finite ordinals, $[n] = \{0 < 1 < \ldots < n\}$. A simplicial set is a presheaf on $\mathbf{\Delta}$, and so is a functor

$$K : \mathbf{\Delta}^{op} \to Sets.$$

There are generating maps in $\mathbf{\Delta}$,

$$\delta_i : [n-1] \to [n]$$

and

$$\sigma_i : [n+1] \to [n]$$

corresponding to the topological ones considered above. The usual convention is that, if K is a simplicial set, we write $K_n := K[n]$, for the set of n-simplices of K, $d_i := K(\delta_i) : K_n \to K_{n-1}$ and $s_i := K(\sigma_i) : K_n \to K_{n+1}$, these maps being called the *face* and *degeneracy* maps respectively.

(Good classical introductions to simplicial sets can be found in Curtis, [7], and May, [8], whilst Gabriel and Zisman's treatment in [9] is more categorical. The theory is also explored in Kamps and Porter, [10], and in numerous other sources.)

The category, $Sets^{\mathbf{\Delta}^{op}}$, is called the category of simplicial sets and will be denoted $\mathcal{S}$.

Given a space Y, we can define a simplicial set, $Sing(Y)$, by setting $Sing(Y)_n = Top(\Delta^n, Y)$, the set of continuous maps from Δ^n to Y, (so called *singular n-simplices*). This simplicial set, $Sing(Y)$, is the *singular complex* of Y. Of course, $Sing(Y)_1 = Top(I, Y)$, so consists of paths of 'length' 1 in Y. We note

$$Sing(Y \times Z) \cong Sing(Y) \times Sing(Z),$$

so, for a space X, we can obtain a simplicially enriched category from $Paths(X)$ by specifying

$$\mathsf{Paths}(X)(x, x') := Sing(Paths(X)(x, x')).$$

The fact that $Sing$ preserves products means that if the composition in $\mathsf{Paths}(X)$, is taken to be that induced from the one in the Top-enriched case, then it works well at this simplicial level.

One motivation for working with $\mathsf{Paths}(X)$ rather than $Paths(X)$ is that it is more suited for generalisation to non-topological contexts. Before we investigate that, however, we need to rework our description of $\mathsf{Paths}(X)(x, x')$.

We have

$$Paths(X)(x, x') = \coprod_{r \geq 0} X^{[0,r]}(x, x'),$$

where $X^{[0,r]}(x, x') = \{a : [0, r] \to X \mid a(0) = x, a(r) = x'\} \subset X^{[0,r]}$. Thus

$$\begin{aligned} Sing(Path(X)(x, x'))_n &= \coprod_{r \geq 0} Sing(X^{[0,r]}(x, x'))_n \\ &\cong \coprod_{[0,r]} Top(\Delta^n, X^{[0,r]}(x, x')) \\ &\subset \coprod_{[0,r]} Top([0, r] \times \Delta^n, X), \end{aligned}$$

i.e. a subspace of the set of singular prisms in X. The maps in this subspace are those which squash the two ends of the prism, sending them to x and x' respectively.

Composition, identities etc. make just as much sense in this description, and, of course, we get the same $\mathcal{S}$-enriched category, $\mathsf{Paths}(X)$, as before with now a nice geometric interpretation of the arrows in terms of singular prisms.

Now to return to our problem of rescaling, when working with the fundamental groupoid and its higher analogues, based on a unit interval as the only domain of a path, then the operations of rescaling, reversion and the associativity homotopy are all important (see, for instance, the papers by Hardie, Kamps and Kieboom, [11,12] for their use in defining a fundamental 2-groupoid or bi-groupoid of a space, or Fahrenberg and Raussen [13] for an in-depth discussion of reparametrisation relevant to our overall topic of directed paths). In our setting we have associativity for free, and we do not want reverses (which are only needed for inverses), but we do need rescaling for some of the interpretations in the geometric semantics. What we will do is to consider any two constant prisms at the same point to be the same, so we allow rescaling of a constant path of length r to be 'the same' as one of length 0 at the same point. This will be sufficient to allow us to normalise prisms to have whatever positive length we need. Note that it does not disturb associativity, nor identities and we still have a (simplicially enriched) category.

For future reference and motivation we note

- $\mathsf{Paths}(X)_0$ consists of paths in X;
- $\mathsf{Paths}(X)_1$ consists of fixed end-point homotopies between paths in X;
- $\mathsf{Paths}(X)_2$ consists of fixed end-point homotopies between fixed end-point homotopies between paths in X,
- and so on.

Each set in fact forms the arrows of a category with object set the set of points of X, in which source and target maps are the end points of paths, and composition is in the direction of the paths. Face and degeneracy maps are functors that are fixed on objects.

We can now adapt this to the context of directed homotopy, causets, and space-time.

3 Causets

Definition: A *causal set* or *causet* $\mathcal{C}$ is a discrete partially ordered set.

By discrete here, we mean that for each pair p, q of points in $\mathcal{C}$,

$$\mathcal{C}(p,q) = \{r \in \mathcal{C} \mid p \leq r \leq q\}$$

is finite. (Of course, if $p \not\leq q$, it is empty.)

The notion, which is also known as a '*locally finite poset*', occurs in models of space-time (cf. [14,15]). A nice categorical and logical gloss on their use in physics can be found in Markopoulou, [16].

The basics of enriched category theory require the input of a symmetric monoidal category and the category of posets, *Poset*, is one such. The monoidal structure is given by product, just as in the two earlier examples of enrichment that we have met in this paper, namely with Top and $\mathcal{S}$. If $\mathcal{C}$ is a partially ordered set, then for each pair of elements $a, b \in \mathcal{C}$, the 'interval' hom-set, $\mathcal{C}(a, b)$, as above, is a partially ordered set. The obvious composition is : if $\mathcal{C}(a,b)$ is non-empty (so $a \leq b$) and $\mathcal{C}(b,c)$ is non-empty (so $b \leq c$) then $a \leq c$ so $\mathcal{C}(a,c)$ is non-empty, but this does not correspond to an order preserving function

$$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c).$$

This, however, is forgetting the motivation and intuition behind the study of possible enrichments. If $\mathcal{C}$ is a causet, or more generally, any poset, we can consider $Paths(\mathcal{C})$, and the structure of the set of all paths in $\mathcal{C}$. We would expect a categorical structure corresponding to concatenation of paths, but is there more structure around? (A basic reference for this is Cordier's paper from 1982, [4]. This was followed by work which analysed the use of this in conjunction with other simplicially enriched categories in the theory of homotopy coherence, see Cordier and Porter, [17].) We first take a 'geometric' viewpoint.

First some necessary standard definitions and notation: we will write $[0,r]$ for the poset $[r] = \{0 < 1 < \ldots < r\}$, when we are considering it more as a subdivided line of length r rather than as a simplex. (Of course, it is both, but a line has a 'start' and 'end' more clearly than a simplex!)

Definition: A *path a of length r* in a poset, $\mathcal{C}$, is a morphism

$$a : [0,r] \to \mathcal{C}.$$

The *source* of a is $a(0)$ and its *target* is $a(r)$. We write $Paths(\mathcal{C})$ for the set of all paths in $\mathcal{C}$. If $x, x' \in \mathcal{C}$, $Paths(\mathcal{C})(x,x')$ will denote the subset of those paths starting at x and ending at x'. Of course,

$$Paths(\mathcal{C}) = \coprod_{r \geq 0} \mathcal{C}^{[0,r]},$$

and so on, ... just as before, and $\mathcal{C}^{[0,r]}$ has a natural poset structure given by pointwise comparison, and such that the source and target maps, e_0, and e_1, defined in the evident way, are order preserving. There is a well defined 'composition',

$$\mathcal{C}^{[0,r]}(x,x') \times \mathcal{C}^{[0,s]}(x',x'') \to \mathcal{C}^{[0,r+s]}(x,x''),$$

with identities given by the zero length paths, and this induces a category structure on $Paths(\mathcal{C})$. What needs noting is that $Paths(\mathcal{C})(x,x')$ is a poset, but paths of different lengths are incomparable as they are in different parts of the disjoint union.

Once again we have an enriched category, this time *Poset*-enriched. (Such categories are also called *locally ordered categories*.) To emphasise the similar intuitions involved (and for numerous other reasons), we will use an $\mathcal{S}$-enriched version of this.

Any poset $\mathcal{P} = (P, \leq)$ yields a simplicial set, $Ner(\mathcal{P})$, with

$$Ner(\mathcal{P})_n = Poset([n], \mathcal{P}),$$

and the face and degeneracy maps induced by the δ_i and σ_i. This is the analogue of the singular complex for posets. It is the well known *nerve* construction and, of course, an n-simplex $\tau \in Ner(\mathcal{P})_n$ is just a chain of length n,

$$\tau = (p_0 \leq \ldots \leq p_n).$$

If the chain is not strict (i.e. if it has repeats, so say $p_i = p_{i+1}$), then it will be *degenerate.*

We leave it to the reader to check that if $\mathcal{P}$, $\mathcal{Q}$ are posets then

$$Ner(\mathcal{P} \times \mathcal{Q}) \cong Ner(\mathcal{P}) \times Ner(\mathcal{Q}),$$

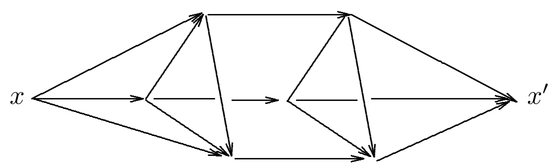
hence from $Paths(\mathcal{C})$, we can obtain a simplicially enriched category $\mathsf{Paths}(\mathcal{C})$, where, for $x, x' \in \mathcal{C}$,

$$\mathsf{Paths}(\mathcal{C})(x, x') = Ner(Paths(\mathcal{C})(x, x')).$$

As in the topological example, we have a description of the n-simplices of this simplicial set, $\mathsf{Paths}(\mathcal{C})(x, x')$, as 'singular' prisms

$$a : [0, r] \times [n] \to \mathcal{C}$$

with $a(0, k) = x$ for all $0 \leq k \leq n$ and $a(r, k) = x'$, similarly:



The condition of 'discreteness' or 'local finiteness' on a causet corresponds to ensuring that each $\mathsf{Paths}(\mathcal{C})(x, x')$ has only finitely many non-degenerate simplices.

4 Pospaces and directed homotopy

We can combine the two previous case studies to look at the category of partially ordered spaces. (As references for this, see work by Grandis, [18,19,20,21] in addition to the papers of Fajstrup, Goubault, Haucourt and Raussen, for instance, [22,23,24,25]. The terminology used here, however, will not necessarily be identical to that used in those papers.)

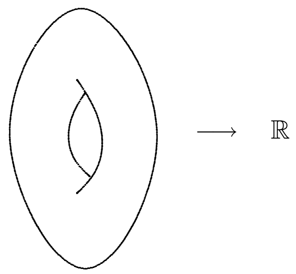
Definition: A *partially ordered space* or *pospace*, X, is a topological space with a (globally defined) closed partial order, $\leq$, so considering $\leq$ as a subset of $X \times X$, it is a closed subset.

A *dimap* $f : X \to Y$ between two pospaces, X and Y, is a continuous map that respects the partial order,

$$x \leq x' \Rightarrow f(x) \leq f(x').$$

Examples:

1. Give the unit interval $I = [0, 1]$, the usual order. This gives it the structure of a pospace that we will denote by $\overrightarrow{I}$. A related similar pospace is the closed interval $[0, r]$ of length $r \geq 0$ with its usual order. This will be denoted $\overrightarrow{[0, r]}$.
2. Let M be a compact differentiable manifold and $f : M \to \mathbb{R}$ a Morse function, so that f is smooth with no degenerate critical points. (As a simple example, take a torus "on end" with f a height function,



then f has 4 critical points, one is a minimum, one a maximum and there are two saddle points. This example is put forward as it shows some of the structure found in the case of $d + 1$ cobordisms in topological quantum field theory.)
Define a pospace structure on M by

$$x \leq x' \Longleftrightarrow x = x' \text{ or } f(x) < f(x').$$

(The idea is to make $t = f(x)$ into a 'time-like variable', in such a way that the space-like slices are the level sets $f^{-1}(t)$.)
3. The 'Swiss flag' and other examples, well known from the work of Fajstrup, Goubault, Raussen and others (see [22,23,24,25], as before), have a pospace structure derived from the product $(\overrightarrow{I})^n$ after carving out some cubical or hypercubical 'forbidden' regions. This occurs in models of PV languages and for situations involving 'mutual exclusion', cf. [26].

Definition: A *dipath* a in a pospace X is a dimap $a : \overrightarrow{[0, r]} \to X$. (The usual terminology will apply to the ends of a.)

If $a, b : \overrightarrow{[0, r]} \to X$ are dipaths with the same ends, so $a(0) = b(0)$ and $a(r) = b(r)$, then a (fixed end-point) *homotopy* between them is a map

$$h : [0, r] \to X$$

such that

(i) $h(0,t) = a(0)$ and $h(r,t) = a(r)$ for all $t \in I$;
(ii) $h(-,t) : \overrightarrow{[0,r]} \to X$ is a dipath for each $t \in I$;
(iii) $h(-,0) = a$ and $h(-,1) = b$.

The terminology we are using differs from that sometimes used. We think of this as a continuously varying family of dipaths, but that family, itself, is 'unordered', so 'homotopy of dipaths' seems appropriate. We will reserve the term 'dihomotopy' as a diminutive of 'directed homotopy', following more closely the terminology of Grandis in this (cf. Grandis [18,19,20,21]). (This choice of abbreviation is partially a question of taste. 'Dihomotopic' is also less awkward to say than 'directed homotopic'.)

Definition: A *directed homotopy* between a and b (as above) is a dimap

$$h : \overrightarrow{[0,r]} \times \vec{I} \to X,$$

where $\overrightarrow{[0,r]} \times \vec{I}$ is given the product partial order. We say a and b are *dihomotopic*, the relation being called *directed homotopy*.

Directed homotopy is not reversible, hence is not symmetric, but is transitive and reflexive.

The two notions, homotopy and dihomotopy, are closely related, but distinct. It is often the case that if two dipaths are homotopic, then they are connected by a zig-zag of dihomotopies, whilst clearly any two dihomotopic dipaths are homotopic.

Both of these notions yield simplicially enriched categories of paths. The first requires less preparation so is easier to give.

Definition: Let X be a pospace. For $x, x' \in X$, let $\mathsf{diPaths}(X)_n(x,x')$ be the set of dimaps $a : \overrightarrow{[0,r]} \times \Delta^n \to X$, for any $r \geq 0$ and where Δ^n is given the trivial partial order, such that $a(0,t)$ and $a(r,t)$ are constant with respect to t. This gives a simplicial set $\mathsf{diPaths}(X)(x,x')$ and there is an obvious concatenation composition

$$\mathsf{diPaths}(X)(x,x') \times \mathsf{diPaths}(X)(x',x'') \to \mathsf{diPaths}(X)(x,x'')$$

and identities yielding a simplicially enriched category, $\mathsf{diPaths}(X)$.

The second construction requires a partially ordered version of the simplices such that all the coface and codegeneracy maps are dimaps. The usual topological models of simplices do not give this immediately, so we will use a slightly different model.

Consider the subset $\mathsf{D}^n \subset I^n$ given by

$$\underline{x} \in \mathsf{D}^n \Leftrightarrow \underline{x} = (x_1, \ldots, x_n) \text{ with } x_1 \leq x_2 \leq \ldots \leq x_n.$$

Thus, for $n = 2$, D^2 is the upper triangle of the unit square subdivided by the $x_1 = x_2$ diagonal. In general, D^n is an n-simplex. The correspondence between this and the earlier description of Δ^n, which we give in dimension 2 only for convenience, is that (x_1, x_2) corresponds to $(1 - x_2, x_2 - x_1, x_1)$ or conversely (t_0, t_1, t_2) to $(t_2, t_1 + t_2)$. We leave the reader the task of generalising this to n-dimensions.

The coface maps are by insertion of 1 on the right, 0 on the left, or repeating x_i for the i^{th} coface with $0 < i < n$. With the identification of D^n with Δ^n, the codegeneracy maps are now simple to write down. For example, again for $n = 2$, the two codegeneracy maps from D^1 to D^2 are induced by the two projections from I^2 to I^1.

We give D^n an induced order from $\overrightarrow{I^n}$, but will write the result as $\overrightarrow{\Delta^n}$. The following is now the obvious thing to do.

Definition: Let X be a pospace. For $x, x' \in X$, let $\mathsf{DiPaths}(X)_n(x, x')$ be the set of dimaps $a : \overrightarrow{[0, r]} \times \overrightarrow{\Delta^n} \to X$, etc.

Of course, $\mathsf{DiPaths}(X)$ gives a simplicially enriched category. We will refer to the element of $\mathsf{DiPaths}(X)_n(x, x')$ as *singular n-prisms* from x to x'.

These simplicially enriched categories have a slight disadvantage. They are intended to mirror the properties of the pospace X, but for any point $x \in X$, $\mathsf{DiPaths}(X)_0(x, x)$, and similarly $\mathsf{diPaths}(X)_0(x, x)$, have constant paths of all lengths in them, and similarly in higher dimensions. The sort of interpretation which is sought for categorical invariants of pospaces, would prefer there to be no loops other than the constant paths at each x (of length zero). In those interpretations of dipaths it is customary to consider the variable as being 'time', yet in non-synchronised systems there is no 'global clock'. It is thus usual to normalise paths so as to have 'length' or 'duration' 1, but here in $\mathsf{DiPaths}(X)$ and $\mathsf{diPaths}(X)$, we have paths of arbitrary duration. For comparison with the normalised theories, it would be useful to be able to rescale paths. To do this we adapt the comment on constant paths made in an earlier section. In fact this problem is the same as that of constant loops that we have just discussed. We will consider a variant of these $\mathcal{S}$-categories in which all constant paths, of any duration, are considered to be equivalent. More formally:

Define a relation, $\sim$, on $\mathsf{DiPaths}(X)_n(x, x')$, resp. $\mathsf{diPaths}(X)_n(x, x')$ by : if $a : [0, r] \times \Delta^n \to X$ is a singular n-prism from x to x' and $\mathsf{const}^n(s, y) : [0, s] \times \Delta^n \to X$ denotes the constant n-prism of duration s at a point y of X, then

$$a \sim a * \mathsf{const}^n(s, a(r)),$$

and

$$a \sim \mathsf{const}^n(s, a(0)) * a.$$

We will also denote by $\sim$ the congruence generated by this primitive $\sim$, so, for instance,

$$a * b \sim a * \mathsf{const}^n(s, a(r)) * b.$$

The following helps explain the usefulness of this.

Proposition 1. *Suppose $a : [0, r_1] \to X$ with $r_1 > 0$, is a dipath, and let $r_2 > r_1$. Define a dipath $b : [0, r_2] \to X$ by rescaling a, so*

$$b(t) = a\Big(\frac{r_1}{r_2}t\Big) \textit{ for } t \in [0, r_2],$$

then there is a directed homotopy

$$h : a * \mathsf{const}^1(r_2 - r_1, a(r_1)) \xrightarrow{\simeq} b.$$

□

The similar result with a a singular n-prism also holds.

The directed homotopy is fairly easy to construct explicitly. Of course, this means that combining directed homotopies with identifying all constant paths does yield a well behaved rescaling operation. (The combination of this with the insights in Fahrenberg and Raussen's paper, [13], have yet to be explored.) We will usually continue to work with $\mathsf{DiPaths}(X)$ and $\mathsf{diPaths}(X)$ as constructed, but if an application or interpretation needs normalising or rescaling, then we note the following: denoting by $\mathsf{DiPaths}^{\sim}(X)$ and $\mathsf{diPaths}^{\sim}(X)$, the result of dividing these two $\mathcal{S}$-categories by the congruence $\sim$, then

Proposition 2. *The two structures $\mathsf{DiPaths}^{\sim}(X)$ and $\mathsf{diPaths}^{\sim}(X)$ have well defined compositions making them into $\mathcal{S}$-categories.* □

In fact, the construction of the congruence makes this almost tautologous. Of course, by dividing out by $\sim$ we get rid of the difficulty of non-trivial 'loops' in these categories. In a later section we will look at the free $\mathcal{S}$-category on a small category, again using the analogue of paths, this time in a directed graph, and there also it is necessary to avoid non-identity 'constant' loops. In both case there is an aspect that relates to rewriting, although we cannot explore that here.

It is to be noted that when calculating the component categories of these examples using the methods derived from [23] and [27] (see below, section 8.1), any $\mathsf{const}^0(s, x)$ yields an arrow which is weakly invertible, so will be killed off in that process.

Variants of this construction should be moderately easy to manufacture. For instance, if X is a smooth manifold, piecewise smooth order preserving singular simplices may work and if, say, hyperbolic structures are given on X, then using hyperbolic versions of simplices (cf., for instance, Bridson and Haefliger, [28]) would perhaps work, although these are normally of constant curvature.

The basic objects we have used here and earlier are prisms $[0, r] \times \Delta^n$ in ordered or unordered variants. As $[0, r]$ is like the path $0 \to 1 \to \ldots \to r$ in

Δ, this may be linked to $\Delta^r \times \Delta^n$ and hence to viewing these $\mathcal{S}$-categories as being related to bisimplicial sets. Why stop there? In general, it may be useful to consider order variants of 'hyperprisms' $\overrightarrow{\Delta}^{r_1} \times \ldots \times \overrightarrow{\Delta}^{r_k}$ and to interrelate them generalising the approach to weak category theory due to Simson and Tamsamani, see [29].

5 $\mathcal{S}$-groupoids from simplicial sets.

Simplicial sets, as such, have some attributes that resemble partially ordered sets as well as others that look spatial. The theory described in this section relates to the ordinary homotopy of simplicial sets and gives simplicially enriched *groupoids*, so might seem out of place here. However, it allows, to some extent, a comparison of the directed theory with a fairly standard construction from standard algebraic topology, so it seems worth while to see this.

Let K be a simplicial set. Near the start of simplicial homotopy theory, Kan showed how, if K was reduced (that is, if K_0 was a singleton), then the free group functor applied to K in a subtle way, gave a simplicial group whose homotopy groups were those of K, with a shift of dimension. They modelled loops on K in some sense that we shall not explore here. With Dwyer in [30], he gave the necessary variant of that construction to enable it to apply to the non-reduced case. This gives a 'simplicial groupoid' $G(K)$ as follows:

The object set of all the groupoids, $G(K)_n$, will be in bijective correspondence with the set of vertices K_0 of K. Explicitly this object set will be written $\{\overline{x} \mid x \in K_0\}$.

The groupoid $G(K)_n$ is generated by edges

$$\overline{y} : \overline{d_1 d_2 \ldots d_{n+1} y} \to \overline{d_0 d_2 \ldots d_{n+1} y} \quad \text{for } y \in K_{n+1}$$

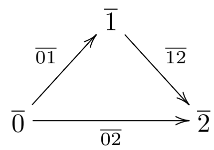
with relations $\overline{s_0 x} = id_{\overline{d_1 d_2 \ldots d_n x}}$. Note since these relations just 'kill' some of the generating edges, the resulting groupoid $G(K)_n$ is still a free groupoid.

Define $\sigma_i \overline{x} = \overline{s_{i+1} x}$ for $i \geq 0$, and, for $i > 0$, $\delta_i \overline{x} = \overline{d_{i+1} x}$, but for $i = 0$, $\delta_0 \overline{x} = (\overline{d_1 x})(\overline{d_0 x})^{-1}$.

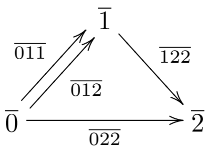
(We use here σ for the degeneracy morphisms and δ for the faces, since the formulae for them already us the more usual s and d, and a double usage may increase the risk of confusion.)

These definitions yield a simplicial groupoid as is easily checked and, as is clear, its simplicial set of objects is constant, so it also yields a simplicially enriched groupoid, $G(K)$, see Proposition 3 below.

It is instructive to compute some examples and we will look at $G(\Delta[2])$ and $G(\Delta[3])$. These simplicially enriched groupoids are free groupoids in each simplicial dimension, and their structure can be clearly seen from the generating graphs. For instance, $G(\varDelta[2])_0$ is the free groupoid on the graph

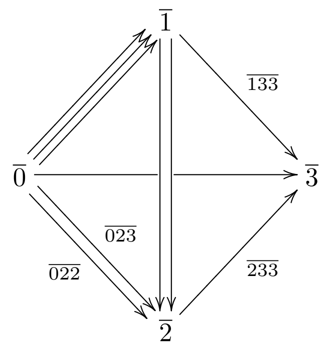


whilst $G(\varDelta[2])_1$ is the free groupoid on the graph



Here it is worth noting that $\delta_0(\overline{012}) = (\overline{02}).(\overline{12})^{-1}$. Higher dimensions do not have any non-degenerate generators.

Again with $G(\varDelta[3])$, in dimension 0, we have the free groupoid on the directed graph give by the 1-skelton of $\varDelta[3]$. In dimension 2, the generating directed graph is



Here only a few of the arrow labels have been given. Others are easy to provide (but moderately horrible to typeset in a sensible way!). Those from $\overline{0}$ to $\overline{1}$ are $\overline{012}$, $\overline{011}$ and $\overline{013}$; those from $\overline{1}$ to $\overline{2}$ are $\overline{122}$ and $\overline{123}$, and finally from $\overline{0}$ to $\overline{3}$, we have $\overline{033}$.

The next dimension is only a little more complicated. It has extra degenerate arrows such as $\overline{0112}$ and $\overline{0122}$ from $\overline{0}$ to $\overline{1}$, but also between these two vertices has $\overline{0123}$, coming from the non-degenerate 3-simplex of $\varDelta[3]$. The full diagram is easy to draw (and again a bit tricky to typeset in a neat way), and is therefore left 'as an exercise'.

The functor G has a right adjoint $\overline{W}$ and the unit $K \to \overline{W}G(K)$ is a weak equivalence of simplicial sets. This is part of the result that shows that simplicially enriched groupoids model all homotopy types, for which see the original paper, [30]. Whether any analogue of this result in the directed case is feasible seems not to have been examined. Its importance is that, for G a simplicial group, any simplicial map $K \xrightarrow{\tau} \overline{W}G$ gives a 'twisting function' and induces a principal G-bundle on K, see [7] for the basic theory. Such maps also correspond to morphisms of simplicial groupoids from $G(K)$ to G. We will see a related construction later.

The way that the twisting function $\tau : K \to G$ works is worth spelling out in a bit more detail. (For an excellent 'classical' exposition of these ideas consult Curtis' survey article, [7].) We will assume given a slightly more general situation. Suppose Y is a 'fibre' and we want 'fibre bundles' on K with fibre Y. We have a trivial bundle $K \times Y$ with the obvious face and degeneracies, and a simplicial group oif automorphisms of Y, $G = \mathsf{aut}(Y)$, (see Curtis for how this is constructed). We have a twisting function $\tau : K \to G$, which we can convert either to a simplicial map $\tau : K \to \overline{W}G$ or to a simplicial group morphism $\tau : G(K) \to G$, depending on whichever suites us better as they are completely equivalent, and we form $K \times_\tau Y$ by $(K \times_\tau Y)_n = (K \times Y)_n$ with all d_i, $i > 0$ and all s_i just as in $K \times Y$, that is $d_i(k, y) = (d_i k, d_i y)$, but with $d_0(k, y) = (d_0 k, \tau(k)(d_0 y))$, i.e. we twist the 0-face of the cartesian product.

We will see a similar twisting later with 'twisting cochains' and the twisted tensor product. If K is connected, the twisting function can also be specified by a $\mathcal{S}$-enriched functor $F : G(K) \to \mathcal{S}$ and in this interpretation, $K \times_\tau Y$ is the homotopy colimit of F. This is essentially encoding a fibre bundle on K as a $\mathcal{S}$-functor to simplicial sets, a viewpoint that may be useful for future development.

6 From simplicial resolutions to $\mathcal{S}$-cats.

There is an abstract way of generating a simplicially enriched category from a small category using simplicial resolutions. This views 'paths' as sequences of edges or arrows or perhaps transitions, and so uses the free category on a directed graph as a basic tool.

The forgetful functor $U : Cat \to DGrph_0$ has a left adjoint, F. Here $DGrph_0$ denotes the category of directed graphs with 'identity loops', so U forgets just the composition within each small category but remembers that certain loops are special 'identity loops'. These directed graphs are sometimes also called *quivers* and later we will look at an enriched version of these as well. The free category functor here takes, between any two objects, all strings of composable *non-identity* arrows that start at the first object and end at the second, that is, all paths from the first to the second. One can think of F identifying the old identity arrow at an object x with the empty string at x.

This adjoint pair gives a comonad on Cat in the usual way, and hence a functorial simplicial resolution, which we will denote $S(\mathsf{A}) \to \mathsf{A}$ for A a small category. In more detail, we write $T = FU$ for the functor part of the comonad, the unit of the adjunction $\eta : Id_{DGrph_0} \to UF$ gives the comultiplication $F\eta U : T \to T^2$ and the counit of the adjunction gives $\varepsilon : FU \to Id_{Cat}$, that is, $\varepsilon : T \to Id$. Now for A a small category, set $S(\mathsf{A})_n = T^{n+1}(\mathsf{A})$ with face maps $d_i : T^{n+1}(\mathsf{A}) \to T^n(\mathsf{A})$ given by $d_i = T^{n-i}\varepsilon T^i$, and similarly for the degeneracies which use the comultiplication in an analogous formula. (Note that there are two conventions possible here. The other will use $d_i = T^i \varepsilon T^{n-i}$. The only effect of such a change is to reverse the direction of certain 'arrows' in diagrams later on. The two simplicial structures are 'dual' to each other.)

This $S(\mathsf{A})$ is a simplicial object in Cat, $S(\mathsf{A}) : \mathbf{\Delta}^{op} \to Cat$, so does not immediately give us a simplicially enriched category, however its simplicial set of objects is constant because U and F took note of the identity loops.

In more detail, let $ob : Cat \to Sets$ be the functor that picks out the set of objects of a small category, then $ob(S(\mathsf{A})) : \mathbf{\Delta}^{op} \to Sets$ is a constant functor with value the set $ob(\mathsf{A})$ of objects of A. More exactly it is a discrete simplicial set, since all its face and degeneracy maps are bijections. Using those bijections to identify the possible different sets of objects, yields a constant simplicial set where all the face and degeneracy maps are identity maps, i.e. we do have a *constant* simplicial set.

Proposition 3. *Let $\mathcal{B} : \mathbf{\Delta}^{op} \to Cat$ be a simplicial object in Cat such that $ob(\mathcal{B})$ is a constant simplicial set with value B_0, say. For each pair $(x, y) \in B_0 \times B_0$, let*

$$\mathcal{B}(x, y)_n = \{\sigma \in \mathcal{B}_n | \ \mathrm{dom}(\sigma) = x, \mathrm{codom}(\sigma) = y\},$$

where, of course, dom *refers to the domain function in $\mathcal{B}_n$, similarly for* codom.

(i) The collection $\{\mathcal{B}(x, y)_n | \ n \in \mathbb{N}\}$ has the structure of a simplicial set, $\mathcal{B}(x, y)$, with face and degeneracies induced from those of $\mathcal{B}$.

(ii) The composition in each level of $\mathcal{B}$ induces

$$\mathcal{B}(x, y) \times \mathcal{B}(y, z) \to \mathcal{B}(x, z).$$

Similarly the identity map in $\mathcal{B}(x, x)$ is defined as id_x, the identity at x in the category $\mathcal{B}_0$.

(iii) The resulting structure is an $\mathcal{S}$-enriched category. □

The proof is easy. In particular, this shows that $S(\mathsf{A})$ is a simplicially enriched category. The description of the simplices in each dimension of $S(\mathsf{A})$ that start at a and end at b is intuitively quite simple. The arrows in the category, $T(\mathsf{A})$, correspond to strings of symbols representing non-identity arrows in A itself, those strings being 'composable' in as much as the domain of the i^{th} arrow must be the codomain of the $(i-1)^{th}$ one and so on. Because of this we have:
$S(\mathsf{A})_0$ consists exactly of such composable chains of maps in A, none of which is the identity;
$S(\mathsf{A})_1$ consists of such composable chains of maps in A, none of which is the identity, together with a choice of bracketing;
$S(\mathsf{A})_2$ consists of such composable chains of maps in A, none of which is the identity, together with a choice of two levels of bracketing;
and so on.

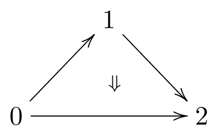
Face and degeneracy maps remove or insert brackets, but care must be taken when removing innermost brackets as the compositions that can then take place can result in chains with identities and these identities then need removing, see

[4]. This is why the comonadic description is so much simpler, as it manages all that itself.

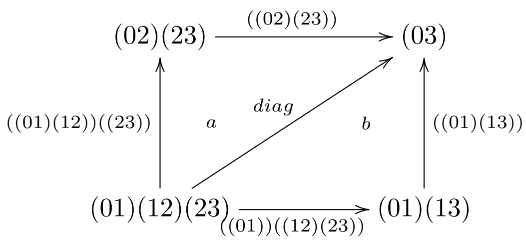
To understand $S(\mathsf{A})$ in general, it pays to examine the simplest few cases. The key cases are when $\mathsf{A} = [n]$, the ordinal $\{0 < \ldots < n\}$ considered as a category in the usual way. The cases $n = 0$ and $n = 1$ give no surprises. $S[0]$ has one object 0 and $S[0](0,0)$ is isomorphic to $\Delta[0]$, as the only simplices are degenerate copies of the identity. $S[1]$ likewise has a trivial simplicial structure, being just the category [1] considered as an $\mathcal{S}$-category. Things do get more interesting at $n = 2$. The key here is the identification of $S[2](0,2)$. There are two non-degenerate strings or paths that lead from 0 to 2, so $S[2](0,2)$ will have two vertices. The bracketted string ((01)(12)) on removing inner brackets gives (02) and outer brackets, (01)(12), so represents a 1-simplex

$$(01)(12) \xrightarrow{((01)(12))} (02)$$

Other simplicial homs are all $\Delta[0]$ or empty. It thus is possible to visualise $S[2]$ as a copy of [2] with a 2-cell going towards the bottom:



The next case $n = 3$ is even more interesting. $S[3](i,j)$ will be
(i) empty if $j < i$,
(ii) isomorphic to $\Delta[0]$ if $i = j$ or $i = j - 1$,
(iii) isomorphic to $\Delta[1]$ by the same reasoning as we just saw for $j = i + 2$,
and that leaves $S[3](0,3)$. This is a square, $\Delta[1]^2$, as follows:



where the diagonal $diag$ = ((01)(12)(23)), a = (((01)(12))((23))) and b = (((01))((12)(23))).

The case of $S[4]$ is worth doing. It is left to the reader, but as might be expected $S[4](0,4)$ is a cube. All higher $S[n](0,n)$ are $(n-1)$-cubes for good combinatorial reasons, which we will not go into here.

These $S[n]$ are all subcategories of a $\mathcal{S}$-category, $\mathbb{S}$, that has been called *the generic homotopy coherent ω-path.* This $\mathcal{S}$-category is studied by Verity in [31] as a precursor for other instances in which the simplicial enrichments are constrained to carry more structure, mimicking that of weak infinity categories. He gives some useful categorical characterisations of it, and its relation with locally ordered categories. The definition of this $\mathcal{S}$-category, $\mathbb{S}$, is that it has $\mathbb{N}$ as its set of objects and $\mathbb{S}(r,s) = \Delta[1]^{(s-r)}$. It is *the generic model for paths* in many contexts.

The S-construction given above for small categories can be extended to small $\mathcal{S}$-categories. If A is a small $\mathcal{S}$-category, we form up for each n the $\mathcal{S}$-category $S(\mathsf{A}_n)$, this gives a category enriched over bisimplicial sets. Taking the diagonal of each of these gives us a $\mathcal{S}$-enriched category. The process of removing all brackets then gives an $\mathcal{S}$-functor, $S(\mathsf{A}) \to \mathsf{A}$, called the *evaluation* or *augmentation* map.

We saw earlier the way in which a simplicial fibre bundle on a (connected) simplicial set K, corresponded to a simplicially enriched functor

$$F : G(K) \to \mathcal{S}.$$

The interpretation was that $F(x)$ is the fibre over $x \in K_0$, whilst the edges etc. of K give the ways in which 'transitions' between base points yield transition functions between the fibres. There is more than a superficial link between $G(K)$ and $S(\mathsf{A})$ in terms of their construction, so what would be the interpretation of a simplicially enriched functor

$$F : S(\mathsf{A}) \to \mathcal{S}.$$

This was explored by Cordier in the paper already mentioned earlier, [4]. Such a $\mathcal{S}$-functor corresponds to a *homotopy coherent diagram* of 'shape' A within $\mathcal{S}$. Some idea of what that is can be gleaned from the case $\mathsf{A} = [3]$ above. If $F : S[3] \to \mathcal{S}$ is a $\mathcal{S}$-functor, it gives four simplicial sets, $F(i)$, $i = 0, 1, 2, 3$, and a tetrahedral diagram of maps between them. The triangular faces are 'filled' with homotopies, specified by F, for instance $F((01)(13))$ is a homotopy from the composite $F(13)F(01)$ to $F(03)$. These homotopies compose according to the diagram (of a square) above , and the two specified 2-homotopies $F(a)$ and $F(b)$ handle the non-commutativity of the result. (This is discussed in more detail in Kamps-Porter, [10] at a fairly elementary level, or see [32] or Cordier-Porter, [17] for how the theory fits in to other geometric considerations.)

If a 'total space' for such a fibre bundle' is desired then the homotopy colimit, *hocolimF*, can be used, but beware, it will work best when each $F(x)$ is a Kan complex. (The theory in the case where each $F(x)$ is just a quasi-category is really what would be needed in our directed setting, but is not yet fully developed; see forthcoming ideas of Joyal and to a minimal extent some comments later here.)

7 Dwyer-Kan Hammock Localisation: more simplicially enriched categories.

(In this section we will need to assume more than a basic knowledge of abstract and simplicial homotopy theory.)

There is another construction that gives simplicially enriched categories from a 'combinatorial' situation, and again it involves prism-like diagrams (although the intuition of prisms is replaced by that of hammocks!) First some background: in his original contribution, [33], to abstract homotopy theory, Quillen introduced the notion of a *model category*. Such a context is a category, $\mathcal{C}$, together with three classes of maps: weak equivalences, $\mathcal{W} = \mathcal{C}_{w.e.}$; fibrations, $fib = \mathcal{C}_{fib}$; and cofibrations, $cofib = \mathcal{C}_{cofib}$, satisfying certain axioms so as to give a general framework for 'doing homotopy theory'. One of the constructions he used was a categorical localisation already well known from Gabriel's thesis and the work of the French school of algebraic geometers, (Grothendieck, Verdier, etc.) and, concurrently with the publication of [33], studied in some depth by Gabriel and Zisman, [9]. The main point was that the analogues of homotopy equivalences, in important instances of homotopical or homological algebra, were only 'weak equivalences' so, whilst, with a homotopy equivalence between two spaces, you are given two maps, one in each direction, plus of course some homotopies, when you have, for instance, a quasi-isomorphism between two chain complexes, you only have one map in one direction: $f : C \to D$, together with the knowledge that the induced map $f_* : H_*(C) \to H_*(D)$ is an isomorphism. The partial solution used by Verdier, Gabriel, Zisman and Quillen, was to go to the 'homotopy category' by formally inverting the weak equivalences/quasi-isomorphisms, thus getting *formal* maps going in the opposite direction! (This may look like cheating, but really is no worse than introducing fractions into the integers, so as to be able to solve certain equations, and, of course, the detailed construction is closely related!) We thus end up with a category $\mathcal{C}[\mathcal{W}^{-1}]$.

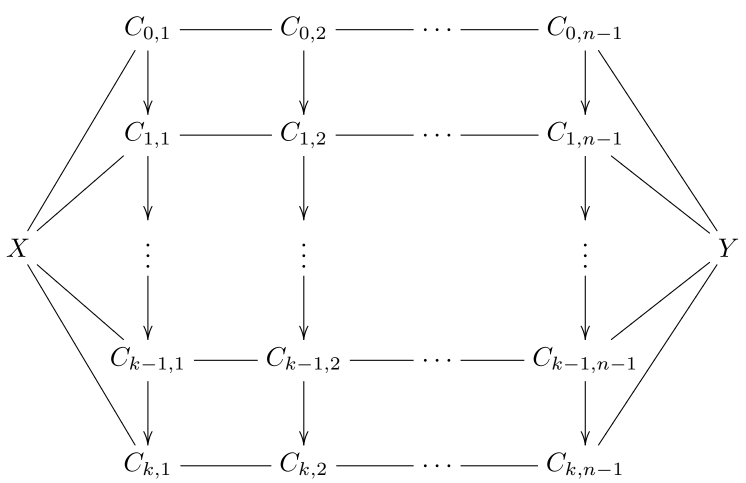
This construction is very useful, but this homotopy category does *not* capture the *higher order homotopy information* implicit in $\mathcal{C}$. In a series of articles [34,35,36] published in 1980, Dwyer and Kan proposed a neat solution to this problem, simplicial localisations. We will limit ourselves to one of the two versions here, the hammock localisation.

7.1 Hammocks

Given a category $\mathcal{C}$, and a subcategory $\mathcal{W}$, having the same class of objects, construct a $\mathcal{S}$-category, $L^H(\mathcal{C}, \mathcal{W})$, or $L^H\mathcal{C}$ for short, the *hammock localisation of* $\mathcal{C}$ *with respect to* $\mathcal{W}$, as follows:

The objects of $L^H\mathcal{C}$ are the same as those of $\mathcal{C}$.

Given two objects X and Y, the k-simplices of $L^H\mathcal{C}(X,Y)$ will be the "reduced hammocks of width k and any length" between X and Y. Such a thing is a commutative diagram of form



in which
(i) the length n of the hammock can be any integer ≥ 0,
(ii) all the vertical maps are in $\mathcal{W}$,
(iii) in each column of horizontal maps, all maps go in the same direction; if they go left, then they have to be in $\mathcal{W}$;
plus two reduction conditions,
(iv) the maps in adjacent columns go in different directions,
and
(v) no column contains only identity maps.

(In manipulating hammocks, these last two conditions often become violated, but then it is simple to reduce the hammock by, for example, composing adjacent columns if they point in the same direction or by removing a column of identities. Repeated use of the reductions may be needed. One reduction may create a need for another one. It is often useful to work with unreduced hammocks and then to reduce.)

The face and degeneracy maps are defined in the obvious way, (remember the vertices of such a simplex are the 'zigzags' from X to Y), however they may result in a non-reduced hammock.

Composition is by concatenation followed by reduction:

$$L^H\mathcal{C}(X,Y) \times L^H\mathcal{C}(Y,Z) \to L^H\mathcal{C}(X,Z),$$

expanding the intervening Y node into a vertical line with identities and then reducing if need be.

Each $L^H\mathcal{C}(X,Y)$ is the direct limit of nerves of small categories in an obvious way, i.e. increasing the length n of the hammocks, and so is itself a quasi-category in the sense of Joyal, [37]. Given our earlier discussion, the similarity of this construction with the corresponding diagrams for 'prisms' is striking. One possible adaptation of the prismatic approach is to allow from the start some collection of Yoneda invertible maps in the sense examined later on and to apply the construction to them. For the case of a calculus of fractions, this was already done by Dwyer and Kan, as we will see in the next section.

7.2 Hammocks in the presence of a calculus of left fractions.

If the pair $(\mathcal{C}, \mathcal{W})$ satisfies any of the usual 'calculus of fractions' type conditions, then the homotopy type of those nerves already stabilises early on in the process (i.e. for small n). The argument given in [35] is indirect, so let us briefly see why one of these claims is true. Suppose that $(\mathcal{C}, \mathcal{W})$ satisfies a calculus of left fractions, thus
(i) whenever there is a diagram $X' \stackrel{u}{\leftarrow} X \stackrel{f}{\rightarrow} Y$ in $\mathcal{C}$ with $u \in \mathcal{W}$, there is a diagram $X' \stackrel{f'}{\rightarrow} Y' \stackrel{v}{\leftarrow} Y$ so that $v \in \mathcal{W}$ and $vf = f'u$,
and similarly
(ii) if $f, g : X \to Y \in \mathcal{C}$ and $u : X \to X' \in \mathcal{W}$ is such that $fu = gu$, then there is a $v \in \mathcal{W}$ such that $vf = vg$.
By this means any word in arrows of $\mathcal{C}$ and $\mathcal{W}^{-1}$ can be rewritten to get all the occurrences of arrows from $\mathcal{W}^{-1}$ to the left of those 'ordinary' arrows from $\mathcal{C}$. Each of the two substrings, those formed from $\mathcal{W}^{-1}$ and those from $\mathcal{C}$, can then be composed to reduce the word to one of the form $w^{-1}c$, i.e. a *left fraction.* To understand how this reacts with hammocks, consider a simple case where the chosen vertex of the hammock, $L^H\mathcal{C}(X,Y)$, is simply the following vertex (zig-zag) (*):

$$X \xleftarrow{w} C \xrightarrow{c} Y \xleftarrow{id} Y$$

with $w \in \mathcal{W}$. We construct a new diagram, using the left fractions rule (i), giving a 1-simplex with the given vertex at one end:

$$\begin{array}{ccccccc} X & \xleftarrow{w} & C & \xrightarrow{c} & Y & \xleftarrow{id} & Y \\ \| & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \| \\ X & \xleftarrow{id} & X & \xrightarrow[c']{} & C' & \xleftarrow{w'} & Y \end{array},$$

so our zigzag (*) was homotopic to a 'left biased' hammock $((w')^{-1}, c')$.

Of course, if the length of the hammock had been greater then the chain of 'moves' to link it to the 'left biased ' form would be longer. Again of course, although combinatorially feasible, a detailed proof that the left baissed hammocks with vertices of the form

$$X \to C \leftarrow Y$$

provide a deformation retract of $L^H\mathcal{C}(X,Y)$ is technically quite messy.

Even with a better knowledge of what the $L^H\mathcal{C}(X,Y)$ looks like, there is still the problem of composition. Two left biased hammocks compose by concatenation to give a more general form of hammock that then gets reduced by the left fractions rules, but these rules do *not* give a normal form for the composite. Much as in the composite of arrows in a quasi-category, cf. Joyal, [37], the composite here is only defined up to homotopy.

Suppose we let $L^1(X,Y)$ be the simplicial set of such left biased hammocks, then it is a deformation retract of $L^H\mathcal{C}(X,Y)$. After composition we reduce to get a diagram

$$\begin{array}{ccc} L^1(X,Y)\times L^1(Y,Z) & \longrightarrow & L^1(X,Z) \\ \simeq \big\downarrow & \overset{concat}{\searrow} & reduce \uparrow\big\downarrow \simeq \\ L^H\mathcal{C}(X,Y)\times L^H\mathcal{C}(Y,Z) & \longrightarrow & L^H\mathcal{C}(X,Z) \end{array}$$

This looks as if it should work well, but if we look at the associativity axiom, it is represented by a commutative diagram, and we have replaced each of the nodes of that diagram by a homotopy equivalent object, so we risk getting a homotopy coherent diagram, not a commutative one. This is happening inside $L^H\mathcal{C}$, so this does not matter so much. We see that although attempting to cut down the size of the 'hom-sets' does allow us more control over some aspects of the situation, it also has its downside.

The solution is to study the homotopy theory of $\mathcal{S}$-categories as such. This will lead us towards the Segal maps (see below) as well as interacting with homotopy coherence. Both of these areas would seem to have their importance for our study, but we will only give a brief discussion of the first of them here.

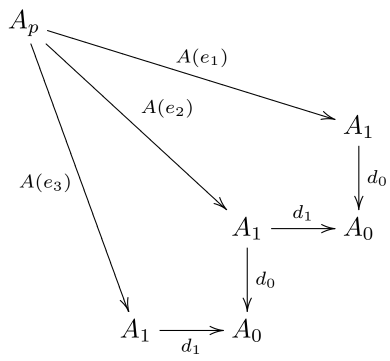
For a short time, for the purpose of exposition, we will restrict ourselves to small $\mathcal{S}$-categories with a fixed set of objects, O, say, and $\mathcal{S}$-functors will be the identity on objects. We will denote the category of such things by $\mathcal{S}-Cat/O$. (The material here is adapted from [38].) This category has a closed simplicial model category structure in which the simplicial structure is more or less obvious, in which a map $D \to D'$ is a weak equivalence (resp. a fibration), whenever, for every pair of objects, $x, y \in O$, the restricted map

$$D(x,y) \to D'(x,y)$$

is a weak equivalence (resp. fibration). (Note, (i) that several of the constructions we have been looking at gave us weak equivalences in this sense, for instance, the augmentation/evaluation map, $S(\mathsf{A}) \to \mathsf{A}$ is one such, and (ii) that the fibrant objects are the 'locally Kan' $\mathcal{S}$-categories over O.)

Now, as we know, any of the categories, $\mathcal{S}-Cat/O$, forms a subcategory of the category of simplicial categories, $Cat^{\mathbf{\Delta}^{op}}$. This latter category also has a closed simplicial model category structure in the sense of Quillen, [33], and the nerve and categorical realisation functors induce an equivalence of homotopy categories (even of the simplicial localisations if you want) between $Cat^{\mathbf{\Delta}^{op}}$ and the category of bisimplicial sets, $\mathcal{S}^{\mathbf{\Delta}^{op}}$. Within $Cat^{\mathbf{\Delta}^{op}}$ we are used to considering $\mathcal{S}$–Cat as a full subcategory via our earlier proposition. Related to the problem of reducing the size of the $L^H\mathcal{C}(X,Y)$s is the question of determining the result of restricting the induced nerve functor to $\mathcal{S}$–Cat. The solution is rather surprising. We first introduce the notion of Segal maps.

Let $p > 0$, and consider the increasing maps $e_i : [1] \to [p]$ given by $e_i(0) = i$ and $e_i(1) = i + 1$. For any simplicial set A considered as a functor $A : \mathbf{\Delta}^{op} \to Sets$, we can evaluate A on these e_i and, noting that $e_i(1) = e_{i+1}(0)$, we get a family of functions $A_p \to A_1$, which yield a cone diagram, for instance, for $p = 3$:



and in general, thus yield a map

$$\delta[p] : A_p \to A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1.$$

The maps, $\delta[p]$, have been called the *Segal maps*.

Lemma 1. *If $A = Ner(\mathcal{C})$ for some small category $\mathcal{C}$, then for A, the Segal maps are bijections.*

Proof: A simplex $\sigma \in Ner(\mathcal{C})_p$ corresponds uniquely to a composable p-chain of arrows in $\mathcal{C}$, and hence exactly to its image under the relevant Segal map. $\square$

Better than this is true:

Proposition 4.
If A is a simplicial set such that the Segal maps are bijections then there is a category structure on the directed graph

$$A_1 \rightrightarrows A_0 \quad (\text{with } A_0 \to A_1)$$

making it a category whose nerve is isomorphic to the given A.

Proof: To get composition you use

$$A_1 \times_{A_0} A_1 \stackrel{\cong}{\to} A_2 \stackrel{d_1}{\to} A_1.$$

Associativity is given by A_3. The other laws are easy, but illuminating, to check. $\square$

Now consider the full subcategory of $\mathcal{S}^{\mathbf{\Delta}^{op}}$ determined by those objects X such that (i) $X[0]$ is a discrete simplicial set (cf. the condition on the object simplicial set in an $\mathcal{S}$-category);

and
(ii) for every integer $p \geq 2$, the Segal map

$$\delta[p] : X[p] \to X[1] \times_{X[0]} X[1] \times_{X[0]} \ldots \times_{X[0]} X[1]$$

is a weak equivalence of simplical sets.

These objects are called *Segal categories* or sometimes *Segal 1-categories*. Of course, there is a notion of Segal 0-categories, but these are just nerves of ordinary categories. We will denote the category of these Segal 1-categories by $Segal-Cat$. The result of Dwyer, Kan and Smith, [38], is that the nerve from $Cat^{\mathbf{\Delta}^{op}}$ to $S^{\mathbf{\Delta}^{op}}$, restricts to given an equivalence of homotopy categories between $\mathcal{S}-Cat$ and $Segal-Cat$. In particular this says that any Segal category is weakly equivalent to a bisimplicial set that is a nerve of a simplicially enriched category. Segal categories are weakened simplicial versions of the algebraic structures given by the categorical axioms, so this is in many ways a coherence theorem for Segal categories rather like the MacLane-type coherence theorems for bicategories, etc.

We have gone from constructions involving directed paths in pospaces, etc., to some relatively technical constructions from homotopy theory. The reason for going so far is that some of the earlier constructions of $\mathcal{S}$-categories that we have given do look to be imposing equivalences on arrows, or, alternatively, extra conditions on arrows too early in the development. Examination of ideas such as Segal-categories, quasi-categories and complicial sets would seem to provide some additional technical ways around such slightly artificial constraints. They thus suggest ways forward to encode the structure of spaces of dipaths that are, perhaps, closer to the physical or computational 'reality' that the models seek to mirror. The coherence results then state that given such models one *can* reduce to the $\mathcal{S}$-categorical model without fear of destroying important aspects of the model.

8 Now we have it, what can we do with it?

8.1 Fundamental categories

Given any $\mathcal{S}$-category, $\mathcal{C}$, we can use the fact that the connected component functor, π_0, preserves products to obtain a category, $\overrightarrow{\pi_0}(\mathcal{C})$. Explicitly this has

$$\overrightarrow{\pi_0}(\mathcal{C})(x, x') = \pi_0(\mathcal{C}(x, x'))$$

with the induced composition.

For the case of a pospace X and $\mathcal{C} = \mathsf{diPaths}(X)$ or $\mathsf{DiPaths}(X)$, these would seem to be the fundamental category of X studied by Fajstrup, Goubault, Haucourt and Raussen, [23] for the case with homotopies and, with directed homotopies, by Grandis, see [21] for instance.

These are quite difficult to handle. Just like the fundamental groupoid on a space, they have the set of points of X as their set of objects. The methods developed in [23] and pushed further in [27,39], develop ways of replacing them by small categories without loops (scwols).

Some idea about what needs to be done can be gleaned from the classical situation of the fundamental groupoid of a non-connected space, X. This has as many objects as X has points. To get a manageable algebraic object you can 'pick' a basepoint in each connected component. This results in a disjoint union of groups. Of course, picking things is non-canonical so we can form an alternative by 'quotienting' out by the equivalence relation underlying the groupoid. Doing this however is quite delicate. One way is to pick a tree in each component, then kill this off, proving, eventually, that you get the same answer independently of the tree chosen. Here we have a category not a groupoid, and in some sense that makes what we have to do easier. Along some 'inessential' arrows the future and past behaviour of the category (i.e. $\mathcal{C}(x,-)$ and $\mathcal{C}(-,x)$) does not really change. If we formally invert some such 'inessential' arrows to obtain a 'compressed' category of 'components' then the result will be much smaller yet contain the same essential combinatorial/geometric information as the original. The only problems are to decide what does 'inessential' mean and how to form a quotient in this sense. We will recall this in the case of the 'fundmental category' $\vec{\pi_1}(X)$ of a pospace, X. This is defined as

$$\vec{\pi_1}(X) := \vec{\pi_0}(\mathsf{diPaths}(X)).$$

For the mutual exclusion models considered in the geometric analysis of PV languages, this is the same as $\vec{\pi_0}(\mathsf{DiPaths}(X))$.

The 'inessential arrows' may be determined in various ways. We will briefly mention [23], but note that in subsequent work presented in [27,39], Goubault and Haucourt would seem to have a neater approach to the same basic idea. 'Inessential' is taken to mean 'weakly invertible' or 'Yoneda invertible'.

Definition: Given a small category, $\mathcal{C}$, we say $\mathcal{C}$ is without loops if each non-identity arrow in $\mathcal{C}$ has distinct source and target. We say $\mathcal{C}$ is a *scwol* (small category without loops).

The notion is discussed in Bridson and Haefliger, [28]. (Some of the other constructions and ideas in that source may, eventually, be useful in other parts of this area of models for spaces of directed paths.) Our fundamental categories, $\vec{\pi_1}(X)$, are examples of scwols.

Definition: Given a scwol $\mathcal{C}$, we say an arrow $\sigma : x \to y$ is *weakly invertible* if the following conditions are satisfied

1. for each object z of $\mathcal{C}$ such that $\mathcal{C}(y,z) \neq \emptyset$,

   $$\mathcal{C}(\sigma, z) : \mathcal{C}(y,z) \to \mathcal{C}(x,z)$$

   is a bijection, i.e. σ is *future weakly invertible* and

2. for each object z of $\mathcal{C}$ such that $\mathcal{C}(z, x) \neq \emptyset$,

$$\mathcal{C}(z, \sigma) : \mathcal{C}(z, x) \to \mathcal{C}(z, y)$$

is a bijection, so σ is also *past weakly invertible*.

The condition '$\mathcal{C}(y, z) \neq \emptyset$' is a guard condition to avoid silly situations, since there may be z reachable from x, but not from y, yet not essentially different from either. For instance, if σ factors as $x \to z \to y$ with both $x \to z$ and $z \to y$ weakly invertible, we would expect $\mathcal{C}(y, z)$ to be empty, whilst $\mathcal{C}(x, z)$ is not, so for such a z, $\mathcal{C}(\sigma, z)$ cannot be a bijection.

Although the idea is simple, there are still technical problems to solve, and we refer the reader to the papers and notes previously cited for a much fuller discussion.

In a causet, $\mathcal{C}(-, x)$ measures the past of x and $\mathcal{C}(x, -)$ its future, so weak invertibility corresponds to 'no large topology change along σ'. The significance of weak invertibility for the case of $\mathcal{C} = \vec{\pi_1}(X)$ for a pospace X is discussed in [23], so we will not explore it much here. By factoring out by the weakly invertible arrows, $\vec{\pi_1}(X)$, can be reduced in size considerably. Two objects x and x' will be identified if there is a directed path, a, from x to x' along which the 'components' of the past and future of the point $a(t)$ do not change.

In the case of a pospace derived from a Morse function $f : M \to \mathbb{R}$, there is a well known construction, the Reeb graph. This is a quotient of $M \times \mathbb{R}$ by an equivalence relation where $(x_1, f(x_1)) \cong (x_2, f(x_2))$ if and only if $f(x_1) = f(x_2)$ and x_1 and x_2 are in the same component of $f^{-1}f(x_1)$, the level set of $f(x_1)$.

Although of a similar nature, this graph encodes less about M and f than does the component category of the pospaces. For instance, even in the example of the torus, as illustrated earlier, each side tube contributes one edge to the Reeb graph, but with directed paths we can find examples that wind their way around the tube as many times as we like, corresponding to the fact that the cross section is a circle, S^1, and the standard fundamental group $\pi_1(S^1)$ is infinite cyclic. The point is that the Reeb graph uses only the geodesic curves or gradient flow lines to join representatives of each 'component'.

The study of the component category construction is still in its infancy and some of its complexities are still very mysterious.

8.2 Fundamental 2-categories

Given any $\mathcal{S}$-category, $\mathcal{C}$, we have found a small category $\vec{\pi_0}(\mathcal{C})$, which in our motivating examples will often be a scwol. Within that, we have defined weakly invertible arrows, at least in the 'scwol' case. As is clear from the definition, this notion can be split into two parts, the first being 'σ induces an equivalence of the futures of x and y' and, of course, the second is a dual asking for past equivalences. (The splitting of this into two separate notions is closely related to the ideas considered in Raussen, [40], and Grandis, [20], but is also related to the view of future and past 'internally' within a category, cf. Markopoulou,

[16] and Bell, [41].) For simplicity of exposition we will restrict attention to the future, ... not dwelling on the past!

In terms of the original $\mathcal{S}$-category, an arrow $\sigma : x \to y$ gives a *future weakly invertible arrow* of $\overrightarrow{\pi_0}(\mathcal{C})$ if, for each z such that $\pi_0(\mathcal{C}(y,z)) \neq \emptyset$, $\pi_0(\mathcal{C}(y,z)) \to \pi_0(\mathcal{C}(x,z))$ is a bijection, etc., thus $\mathcal{C}(\sigma, z)$ is a 0-equivalence of simplicial sets, (i.e. it induces a bijection after application of π_0). This is clearly just the first of a sequence of variants of 'future weakly invertible'. For instance, $\sigma : x \to y$ is 'future weakly 1-invertible' if each $\pi_0(\mathcal{C}(\sigma, z))$ and $\Pi_1(\mathcal{C}(\sigma, z))$ are isomorphisms, where $\Pi_1 K$ indicates the fundamental groupoid of the simplicial set K. (We note that the guard condition about non-emptiness would still be required here to avoid silly situations.)

This idea is related to the fundamental 2-category, or more exactly, groupoid-enriched category, of a pospace. This just applies the fundamental groupoid functor to each $\mathcal{C}(x,y)$ of a $\mathcal{S}$-category $\mathcal{C}$, so can be applied to $\mathsf{DiPaths}(X)$ or $\mathsf{diPaths}(X)$. It needs to be noted that it inverts the 1-simplices of $\mathcal{C}(x,y)$, so does not observe '2-directional' information. (In any case, at the present level of knowledge and understanding, the exact meaning of such 2-directional information is not at all clear.)

Again conjecturally, there should be a component 2-category, derivable by this means, for any pospace, X. It would monitor the topology change at the second level, that is, the way the 1-type of the view of the space at time t varied with t. There is no reason to stop there as 2-groupoid enrichment is also possible, see, for instance, [42]. Beyond that the situation gets more obscure, but other derived enrichments are possible.

The usefulness, or otherwise, of this encoding of the structure of the original pospace, X, will depend, to some extent, on the structure of the simplicial sets, $\mathsf{DiPaths}(X)(x,x')$, and $\mathsf{diPaths}(X)(x,x')$. The first would seem to be a Kan complex, whilst the second is a 'weak Kan complex' or 'quasi-category', the idea that we have mentioned several times earlier. We will not explore this further here except to note once more the papers by Joyal, [37], on quasi-categories, and Verity on complicial sets [43,44,31], which are models for weak infinity categories. (An introduction to some of the types of weak infinity category including quasicategories and further information on the weakening of $\mathcal{S}$-categories that we met briefly earlier, the *Segal categories*, the reader is refered to the notes, [45].) Another link with another type of weak infinity categories occurs via the constructions in the next section.

9 Differential graded categories of Paths

In this and the following sections, we will continue to explore how to exploit these $\mathcal{S}$-categorical models, but by following a route suggested more by cohomology than by homotopy. This also gives a tantalising possible link with aspects of string theory and a set of possible tools for a 'discrete' differential geometry in these contexts, including bundle-like structures.

9.1 Differential graded categories

The category of simplicial sets is not the only well structured monoidal category that is useful for analysing 'spaces' of paths. Simplicial sets have a beautiful combinatorial structure coming from the different basic ways of combining simplices. That structure is, however, non-commutative and computational techniques for handling it are more complex than for, say, simplicial vector spaces where processes adapted from numerical linear algebra can be used.

The basic structures for these enriched categories are outlined in the sections of the appendix (sections 12 and 13). They include the following, which for convenience will be briefly given here. We will be working over a fixed field $\mathbb{K}$, which will usually be thought of as $\mathbb{R}$ or $\mathbb{C}$, (but this restriction is not at all necessary).

- *pre-graded vector space* (*pre-gvs*): $V = \bigoplus_{p\in\mathbb{Z}} V_p$. The elements of V_p are said to be homogeneous of degree p. If $x \in V_p$, we write $\mid x \mid = p$.
- *graded vector space* (*gvs*) : V is a pre-gvs which is non-negatively or non-positively graded, that is, with $V = \bigoplus_{p\geq 0} V_p$ so $V_p = 0$ if $p < 0$, or $V = \bigoplus_{p\leq 0} V_p$ so $V_p = 0$ if $p > 0$. The non-negatively graded case tends to be written with a superfix, i.e. $V^p = V_{-p}$ for $p \geq 0$.
- *degree*: if $f : V \to W$ is a $\mathbb{K}$-linear map of pre-gvs, it is *of degree* p if $f(V_q) \subseteq W_{p+q}$ for all q. A *morphism* of pre-gvs is of degree 0.
- $Hom_p(V, W)$ denotes the set of linear maps of degree p from V to W and

$$Hom(V, W) = \bigoplus_p Hom_p(V, W)$$

  is a pre-gvs.
- *r-suspension* of V, $s^r(V)_n = V_{n-r}$. We mostly need s and s^{-1}. If $v \in V_p$, the corresponding element in $s^r(V)_{r+p}$ will be denoted $s^r v$.
- *duals*: thinking of $\mathbb{K}$ as a gvs concentrated in degree 0,

$$\#(V) = Hom(V, \mathbb{K}),$$

  so $\#V_p \simeq V^{-p}$ if V is of finite type, i.e. $dim\ (V_p) < \infty$ for all p.
- the *tensor product* of two pre-gvs, V and W,

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q.$$

  On morphisms we get

$$(f \otimes g)(v \otimes w) = (-1)^{|g||f|}(f(v) \otimes g(w))$$

  and is of degree $|f| + |g|$.

Example: given a simplicial set, K, set $\mathbb{K}(K)_p = span_{\mathbf{K}}(K_p)$ to get a non-negatively graded $\mathbb{K}$-vector space. The dual of $\mathbb{K}(K)$ is a non-positively graded gvs. If $f : K \to L$ is a morphism of simplicial sets, we get $f_* : \mathbb{K}(K) \to \mathbb{K}(L)$, a morphism of gvs, and its dual / transpose, $f^* = {}^t f_* : \#\mathbb{K}(L) \to \#\mathbb{K}(K)$.

The key definition is that of a *differential graded vector space* or *dgvs*:

Definition A *dgvs*, (V, ∂), consists of a gvs V and a linear map

$$\partial \in Hom_{-1}(V, V)$$

such that $\partial \circ \partial = 0$. This endomorphism of degree -1 is called the *differential* or *boundary operator* of the dgvs.

Morphisms of dgvs both preserve the grading (so are of degree 0) and are compatible with the differential: $f : V \to W$ must satisfy $\partial^W f = f\partial^V$. The category of dgvs will be denoted dgvs.

The terminology 'chain complex (of vector spaces)' is usually considered to be synonymous with 'non-negatively graded dgvs', whilst a cochain complex is a 'non-positively graded dgvs'. The notation used earlier extends so if (V, ∂) is a cochain complex, $\partial : V^p \to V^{p+1}$.

Example continued: If K is a simplicial set, $C(K)$ will denote the simplicial vector space, with the obvious structure, $C(K)_p = \mathbb{K}(K)_p$, but also the dgvs with the same vector spaces in each dimension but with a differential given by: for $\sigma \in K_p$,

$$\partial(\sigma) = \sum_{i=0}^{p} (-1)^i d_i(\sigma).$$

Dualising we will write $C(K)^* = \#(C(K))$ with differential given by the transpose of the original ∂.

Of importance for the use we will make of these ideas is the following: for simplicial sets K and L,

$$C(K \times L) \cong C(K) \otimes C(L),$$

as simplicial vector spaces, see Curtis, [7], for instance. The key result here is the Eilenberg-Zilber Theorem, (see MacLane, [46], p.238). For simplicial Abelian groups (or, more generally, simplicial modules or vector spaces), A and B, this relates the dg-module, $(A\otimes B, \partial)$, with the tensor product, $(A, \partial)\otimes(B, \partial)$. There are morphisms

$$(i) \quad \nabla : (A, \partial) \otimes (B, \partial) \to (A \otimes B, \partial),$$

given by a 'shuffle' formula:

$$\nabla(a \otimes b) = \sum \pm(s_\beta a \otimes s_\alpha b)$$

where $a \in A_p$, $b \in B_q$, $p + q = n$, and (α, β) is a (p, q)-shuffle of $\{0, \dots, n-1\}$ (again see MacLane [46] or many other books on homological algebra), and (ii) the Alexander-Whitney map,

$$f : (A \otimes B, \partial) \to (A, \partial) \otimes (B, \partial),$$

where

$$f(a\otimes b)=\sum_{p+q=n} d_{q+1}\dots d_{n-1}d_n a\otimes d_0^q b.$$

(The Alexander-Whitney map is an 'approximation to the diagonal' if $A=B$; see MacLane, [46], p.242.)

It is worth noting that for any simplicial module, A, there is not only the differential graded module, (A,∂) with ∂ given by the alternating sum of the face maps, but also a normalised version, where the degenerate elements are equated to zero. The two maps above induce maps on the normalised versions and, there, the composite $f\nabla$ is the identity; see again MacLane for a discussion. In general the Alexander-Whitney map is 'associative' in as much as, for A, B, C, simplicial modules, the two ways of getting

$$(A\otimes B\otimes C,\partial)\to(A,\partial)\otimes(B,\partial)\otimes(C,\partial)$$

agree (up to the usual coherence isomorphisms between tensors). We will be using this in its non-positively graded / cochain complex dual form as well.

– *Homs of dgvs*: if (V,∂), and (V',∂') are two pre-dgvs,

$$Hom(V,V')=\bigoplus_{p\in\mathbb{Z}} Hom_p(V,W)$$

is a pre-dgvs if it is given the differential

$$Df=\partial' f-(-1)^{|f|}f\partial,$$

for f homogeneous.

We are now ready to start converting a simplicially enriched category, $\mathcal{C}$, into a differential graded category, that is a category enriched over dgvs (usually non-positively graded).

First we note the somewhat less useful, non-negatively graded construction. In this we are given an $\mathcal{S}$-category, $\mathcal{A}$, and we take for each pair x, y of objects, the chain complex $C(\mathcal{A}(x,y))$ to be our $C(\mathcal{A})(x,y)$. The composition is induced directly from that of $\mathcal{A}$ and causes no problem, giving a chain complex enriched category, $C(\mathcal{A})$.

Of more interest and potentially of more use is the 'non-positively graded' or 'cochain complex' construction. This is the analogue for the many object case, i.e. 'paths' rather than 'loops', of the cobar construction, which is well known from differential homological algebra. It normally gives a differential graded algebra from a differential Hopf algebra or more general coalgebra, (cf. Tanré, [47], for instance). Here it leads to a differential graded *category* (dg-category).

The theory of dg-categories extends that of dg-algebras. This means that it has the potential to extend constructions such as that of the de Rham complex of a differential manifold. This way some ideas from differential geometry can be introduced and adapted to this context. This leads to the so called *discrete* *differential calculus* and *discrete differential geometry*, see, for instance, Forgy and Schreiber, [48] or Raptis and Zapatrin, [49]. There is a considerable literature on dg-categories and their generalisations, A_∞-categories. These latter objects are to dg-categories as Segal-categories are to $\mathcal{S}$-categories, i.e. composition is associative up to higher coherence, etc. We note Keller's survey article, [50], and also [51] or Lazaroiu's paper, [52], which gives some indications of links with string theory.

9.2 Cobar constructions for many object settings

Our aim here is to give the many object version of the cobar construction. (That such a construction exists follows from more general categorical considerations on operads, but the precise explicit formulations seem difficult to find in the literature, so we will reproduce them here.)

Given a small simplicially enriched category, $\mathcal{C}$, we get for each pair of objects x, y of $\mathcal{C}$, a simplicial set $\mathcal{C}(x,y)$ and hence a dgvs, $C(\mathcal{C}(x,y))^*$. This thus is a *differential graded* $\mathbb{K}$*-quiver* in the terminology of, for instance, Lyubashenko and Manzynk, [53], or, if we write $\mathcal{O}$ for the set of objects of $\mathcal{C}$, and dgvs for the category of differential graded vector spaces, it is an $\mathcal{O}$-graph in dgvs in the terminology, say, of May, [54]. We therefore will continue the development with $\{C(x,y) \mid x,y \in \mathcal{O}\}$ being a general dg-quiver. Of course, we need analogues of some of the above constructions in this many object setting. These are fairly obvious, but do need specifying:

- *Tensor product of dg-quivers*, $C \otimes D$:

$$(C \otimes D)(x,y) = \bigoplus_{z \in \mathcal{O}} (C(x,z) \otimes D(z,y));$$

- *Tensor powers*, $T^nC = C^{\otimes n}$, giving

$$T^nC(x,y) = \bigoplus_{x=x_0,x_1,\ldots,x_n=y} C(x_0,x_1) \otimes \ldots \otimes C(x_{n-1},x_n)$$

  with, by convention, $T^0C(x,y) = \begin{cases} \mathbb{K} & \text{if } x=y \\ 0 & \text{otherwise.} \end{cases}$
- *Tensor cocategory*: $TC = \bigoplus_{n\geq 0} T^nC$.

The 'cocategory' structure comes from the 'cut' cocomposition

$$\Delta : TC \to TC \otimes TC,$$

$$\Delta : TC(x,y) \to \bigoplus_{z \in \mathcal{O}} TC(x,z) \otimes TC(z,y)$$

with

$$\Delta(h_1 \otimes h_2 \otimes \ldots \otimes h_n) = \sum_{k=0}^{n} (h_1 \otimes \ldots \otimes h_k) \otimes (h_{k+1} \otimes \ldots \otimes h_n)$$

together with the counit

$$(\varepsilon : TC \to \mathbb{K}) = (TC \stackrel{proj}{\to} T^0C = \mathbb{K}).$$

We adopt the notation of, for instance, [53], and write $\mathbb{K}$ for the dg-quiver concentrated in dimension 0 and at the 'objects', so

$$\mathbb{K}(x, y)^p = \begin{cases} \mathbb{K} & \text{if } x = y \text{ and } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is worth noting that Δ decomposes an element into its parts in all possible ways, and that elements in this tensor cocategory look like weighted labelled paths through the quiver. Of course, in the case of interest to us, C will be best behaved when each original $\mathcal{C}(x, y)_n$ is finite, as then all the vector spaces will be finite dimensional. Duality will work nicely and well behaved inner products are available if needed. This is likely to be the case with situations coming from causets, for instance, since these are 'locally finite', but in general other tools may be needed.

In the single object case with a gvs V, TV has a natural 'free' algebra structure, the *tensor algebra on* V, given by concatenation of the tensors. In this slightly more general case of a quiver, we get, of course, a free graded category structure in exactly the same way.

We next abstract further from this 'tensor cocategory', which is the 'free' construction from a given dg-quiver, to consider an arbitrary dg-cocategory, i.e. a dg-$\mathbb{K}$-quiver, C, together with given structure

$$\Delta : C \to C \otimes C,$$

$$\varepsilon : C \to \mathbb{K},$$

that is, a diagonal or cocomposition

$$\Delta : C(x, y) \to \bigoplus_{z \in \mathcal{O}} C(x, z) \otimes C(z, y),$$

and a counit

$$\varepsilon : C(x, y) \to \begin{cases} \mathbb{K} & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

with the 'obvious' diagrams being commutative.

Of course, our main example is when $C = C(\mathcal{C})^*$ and we will usually impose a 'local finiteness' condition that any non-zero f in any $\mathcal{C}(x, y)$ can only be decomposed in finitely many ways as $f = gh$, g in some $\mathcal{C}(z, y)$ and h in the corresponding $\mathcal{C}(x, z)$. If this condition is satisfied, then C gives a cocategory with

$$\Delta f = \sum \{f_1 \otimes f_2 \mid f_2 f_1 = f\}.$$

We also assume our dg-cocategory C is coaugmented, i.e. we have given a coaugmentation

$$\eta : \mathbb{K} \to C$$

picking out 'the identity' in each $C(x,x)$. If C is as in our main example, this is quite literally true, $\eta(1) = Id_x$.

Assuming, as we have, that $\mathbb{K}$ is a field,

$$Coker\,\eta \cong Ker\,\varepsilon = \overline{C},$$

the dg-quiver of non-identity elements of C.

The reduced diagonal $\overline{\Delta}$ is defined by

$$\Delta a = 1 \otimes a + a \otimes 1 + \overline{\Delta} a,$$

so picks out the non-trivial decompositions. The *quiver of primitives*, $P(C)$, is the kernel of $\overline{\Delta}$, so $a \in P(C)(x,y)$ if and only if it has only the trivial decompositions. (We will not be going deeply enough into the theory of the cobar construction here to need to use $P(C)$ very much, if at all, but its usefulness should be clear from its definition and the intuitions behind it, so we have included its definition.)

The 'obvious' thing to do in order to model paths in the quiver C would now be to form $T(C)$, however if C is concentrated in degree 0, the resulting tensor dg-category will itself also be concentrated there and there will be no link between the degree of an element and the length of the 'path' it represents, so in the cobar construction, which was originally developed to model loop spaces in topology, the tensor cocategory construction is applied to the 'desuspension', $s^{-1}\overline{C}$, not to C itself. We therefore form $T(s^{-1}\overline{C})$, so

$$T(s^{-1}\overline{C})(x,y) = \bigoplus_{n\geq 0} T^n C(x,y)_{\bullet+1},$$

e.g. if, for some quiver / directed graph $\mathcal{A}$, $C(x,y)_n = span_{\mathbb{K}}\mathcal{A}(x,y)$ if $n = 0$ and is 0 in all other degrees, then, for $x \neq y$,

$$s^{-1}\overline{C}(x,y)_n = \begin{cases} 0 & \text{if } n = 0 \\ span_{\mathbb{K}}\mathcal{A}(x,y) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2, \end{cases}$$

and $(T^n(s^{-1}\overline{C})(x,y))_p = 0$ unless $p = n$, in which case it is isomorphic to $\bigoplus span_{\mathbb{K}}(\mathcal{A}(x_0,x_1) \times \ldots \times \mathcal{A}(x_{n-1},x_n))$, the sum being over all $(x_0,\ldots,x_n) \in \mathcal{O}^{n+1}$ with $x_0 = x$, $x_n = y$. For $x = y$, as $T^o(s^{-1}\overline{C})(x,x) = \mathbb{K}$, we get extra terms.

Aside: This use of the shift suspension is completely analogous to the shift in dimensions of the generating simplices $x \in K_{n+1}$ for $G(K)_n$ in section 5. It is closely related to the use of the décalage functors $Dec : \mathcal{S} \to \mathcal{S}$, which strips off the zeroth face map and zeroth degeneracy map of a simplicial set, then shifts dimension (so $Dec(K)_n = K_{n+1}$) and shifts indices on the structural maps down by 1. This is a beautifully structured functor and yields yet another way in which paths can be modelled; (see Duskin's AMS memoir, [55]).

We now have a differential graded cocategory $T(s^{-1}\overline{C})$, but have not completely specified the differential. There is clearly a differential inherited from that of the dg-quiver, but there is also one coming from the 'conerve' of the 'cocategory' structure. The total differential is thus made up of two types of term. The first comes from the tensor product being of differential objects: we have:

$$\partial_I(s^{-1}c_1\otimes \ldots s^{-1}c_n) = -\sum_{i=1}^{n} o(i-1)s^{-1}c_1\otimes\ldots\otimes s^{-1}c_{i-1}\otimes s^{-1}\partial c_i\otimes\ldots\otimes s^{-1}c_n,$$

where $o(i) = (-1)^{\sum_{k=1}^{i}|s^{-1}c_k|}$.

For instance, any tensor square $D\otimes D$ for a dg-quiver D has

$$\begin{aligned}(D\otimes D)(x,y)_n &= \oplus_z(D(x,z)\otimes D(z,y))_n\\ &= \oplus_z \oplus_{p+q=n} D(x,z)_p\otimes D(z,y)_q\end{aligned}$$

and each homogeneous $a\otimes b$, with $a\in D(x,z)_p$ and $b\in D(z,y)_q$ has 'boundary' determined by the Leibniz rule, $\partial a\otimes b+(-1)^p a\otimes\partial b$, with a $\pm$ sign determined by the degrees of a and b. In our example, in which $D = s^{-1}C$ and $C = span_{\mathbb{K}}(\mathcal{A})$, we have $a = s^{-1}c_1$ and $b = s^{-1}c_2$, $|c_1| = |c_2| = 0$, so $|s^{-1}c_1| = |s^{-1}c_2| = 1$, and

$$\partial_I(s^{-1}c_1\otimes s^{-1}c_2) = -s^{-1}\partial c_1\otimes s^{-1}c_2 + s^{-1}c_1\otimes s^{-1}\partial c_2.$$

Of course, when C is concentrated in a single degree, it will have zero differential and this type of term will be trivial.

Lemma 2. *∂_I is a differential on $T(s^{-1}C)$.* □

This is well known and standard in the single object case and the proof extends easily. A trial evaluation shows to some extent 'why it is true':

$$\begin{aligned}&\partial_I\partial_I(s^{-1}c_1\otimes s^{-1}c_2) = \partial_I(-s^{-1}\partial c_1\otimes s^{-1}c_2) + \partial(s^{-1}c_1\otimes s^{-1}\partial c_2)\\ &= s^{-1}\partial^2 c_1\otimes s^{-1}c_2 - s^{-1}\partial c_1\otimes s^{-1}\partial c_2 + s^{-1}\partial c_1\otimes s^{-1}\partial c_2 - s^{-1}\partial^2 c_1\otimes s^{-1}\partial^2 c_2\end{aligned}$$

and as $\partial^2 = 0$, the first and last terms are trivial, whilst the middle terms cancel. (This indicates the importance of the signs of the terms in the expressions.)

The second differential reflects the 'path structure' in the quiver or more exactly, the cocategory structure:

$\partial_E(s^{-1}c_1\otimes\ldots\otimes s^{-1}c_n)$
$= -\sum_{i=1}^{n} o(i-1)\sum_\mu(-1)^{|c_{i\mu}|+1}(s^{-1}c_1\otimes\ldots\otimes s^{-1}c'_{i\mu}\otimes s^{-1}c''_{i\mu}\otimes\ldots\otimes s^{-1}c_n)$,
where $\overline{\Delta}c_i = \sum_\mu c'_{i\mu}\otimes c''_{i\mu}$ decomposes c_i.

Whilst ∂_I stayed within the same part of the direct sum decomposition of $T(s^{-1}C)(x,y)$, ∂_E changes the index, so checking it is a differential involves more properties of the diagonal/cocomposition structure and we will not attempt to give it in any generality here. Again in the single object case, it is well known.

It is clear that $\partial_I \partial_E = \partial_E \partial_I$, so $\partial = \partial_I + \partial_E$ is a differential on $T(s^{-1}\overline{C})$ and it is then easy to check that the classical proofs of compatibility with multiplication extend from the single object case to this many object one with respect to the (categorical) composition. We have therefore a cobar construction from dg-cocategories to dg-categories and, hence, combining this with the functor from the base $\mathcal{S}$ to dgvs, we get a dg-category from any (locally finite) $\mathcal{S}$-category. If C is a dg-cocategory, we will denote the corresponding dg-category by $\Omega(C) := (T(s^{-1}\overline{C}), \partial)$. (The notation suggests that, in some sense, Ω acts a bit like the analogue of the de Rham complex of differential forms on a manifold. Collapsing the objects to a point does give a variant of the discrete differential manifold algebras used by some researchers in quantum cosmology, cf. [49], for instance. Classically it also recalls the notation ΩX for the loops on a space, X.)

Not all the classical theory generalises, however, from the 'single object' case. If V is a graded vector space, $T(V)$ is a commutative dg-algebra for the shuffle product. Of course, $T(V)$ is a graded algebra for the usual 'tensor' algebra product, corresponding to concatenation, and that generalises, as we noted, to the many object case. The shuffle product on $T(V)$ is given by

$$(v_1 \otimes \ldots v_p) *_{\mathsf{shuff}} (v_{p+1} \otimes \ldots \otimes v_n) = \sum_{\sigma} \varepsilon(\sigma) v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)},$$

where the sum is over all $(p, n-p)$-shuffles, i.e. permutations, σ, of n-elements retaining the original order on the two parts $(1, \ldots, p)$ and $(p+1, \ldots, n)$ into which n is partitioned, and $\varepsilon(\sigma)$ is the Koszul sign of the permutation σ. This gives a Hopf algebra structure to $T(V)$, but depends on being able to form the product on the right of that expression and the analogue of this in the many object case is not at all clear, although it would seem likely that some analogues may exist in special cases.

9.3 Twisting cochains

The cobar construction applied to coalgebras has a significant role to play in 'classifying' twisting cochains. These are the analogue of the twisting functions, $\tau : K \to G$, from a simplicial set to a simplicial group. These correspond either to a simplicial map $K \to \overline{W}G$ or equivalently to $G(K) \to G$, a morphism of simplicial groupoids. Recall, for any simplicial set, Y with an action of G on it, we get a twisted Cartesian product $K \times_\tau Y$, together with a natural map $K \times_\tau Y \to K$ which is a simplicial fibre bundle. We are now operating in the dual dg-category setting, so we can expect a somewhat dual theory.

Let, therefore, $\mathcal{C}$ be a coaugmented dg-cocategory, considered as a dg-quiver on an object set $\mathcal{O}$ and let $\mathcal{A}$ be an augmented dg-category, which, for simplicity, we will assume is also defined on $\mathcal{O}$. (The general case where $\mathcal{A}$ is defined on a different object set can be reduced to this one by means of a pullback construction.) Consider the complex $Hom^*(\mathcal{C}, \mathcal{A})$, whose n^{th} component consists of the homogeneous $\mathbb{K}$-linear maps, f, of degree n, of the *underlying dg-quivers* from $\mathcal{C}$ to $\mathcal{A}$. The differential in $Hom^*(\mathcal{C}, \mathcal{A})$ is the usual one on Hom-complexes, i.e.

that from homological algebra, cf. page 30 above, so if $f : \mathcal{C} \to \mathcal{A}$ with $|f| = n$, then

$$Df = \partial^{\mathcal{A}} f - (-1)^n f \partial^{\mathcal{C}}.$$

If $f, g : \mathcal{C} \to \mathcal{A}$ are two such maps, then we can form a composite

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f \otimes g} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A},$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the composition in the dg-category $\mathcal{A}$. This composite is called the convolution of f and g and will be denotes $f * g$.

Definition A homogeneous $\mathbb{K}$-linear map $\tau : \mathcal{C} \to \mathcal{A}$ is called a *twisting cochain* if it is homogeneous of degree -1 and satisfies the *Maurer-Cartan equation*,

$$D(\tau) + \tau * \tau = 0$$

and the composite

$$\mathbb{K} \to \mathcal{C} \xrightarrow{\tau} \mathcal{A} \to \mathbb{K}$$

is the zero map. Here the first map is the coaugmentation of $\mathcal{C}$, whilst the third map is the augmentation of $\mathcal{A}$. Let $\mathsf{Tw}(\mathcal{C}, \mathcal{A})$ denote the set of twisting cochains. (It is functorial in both $\mathcal{C}$ and $\mathcal{A}$, but we will be looking mostly at a fixed $\mathcal{C}$.)

Proposition 5. *The functor* $\mathsf{Tw}(\mathcal{C}, -)$ *is representable, being represented by the dg-category* $\Omega(\mathcal{C})$*, so there is a natural isomorphism*

$$\mathsf{Tw}(\mathcal{C}, \mathcal{A}) \cong \mathsf{dg-Cat}(\Omega(\mathcal{C}), \mathcal{A}.$$

□

The proof is fairly routine, generalising that in the single object case. The only problem is the question of the 'signs'. As different sources in the literature may use different sign conventions, it is better to try to use 'elementless' arguments wherever possible. This can be helped by the following observation.

Corollary 1. *(i) The universal twisting cochain in* $\mathsf{Tw}(\mathcal{C}, \Omega(\mathcal{C}))$ *is given by*

$$s^{-1} : \mathcal{C} \to s^{-1}\mathcal{C} \to T(s^{-1}\mathcal{C}).$$

(ii) The second differential ∂_E *of* $\Omega(\mathcal{C})$ *is* $-\mu(s^{-1} \otimes s^{-1})\Delta$*, i.e.* $-s^{-1} * s^{-1}$.

Proof: The second statement is a consequence of the representability as

$$\mathsf{Tw}(\mathcal{C}, \Omega(\mathcal{C})) \cong \mathsf{dg-Cat}(\Omega(\mathcal{C}), \Omega(\mathcal{C}))$$

with the universal twisting cochain corresponding to the identity dg-functor. Given any twisting cochain $\tau : \mathcal{C} \to \mathcal{A}$, the corresponding dg-morphism $\overline{\tau} : \Omega(\mathcal{C}) \to \mathcal{A}$ satisfies $\overline{\tau}(s^{-1}c) = \tau(c)$, (what else could it be?), so in the case where $\overline{\tau}$ is the identity, $\tau(c) = s^{-1}c$. From this it follows that $\partial^{\Omega(\mathcal{C})}(s^{-1}c) + s^{-1}\partial^{\mathcal{C}}c + s^{-1} * s^{-1} = 0$, which gives the value of $\partial^{\Omega(\mathcal{C})}$ on generators, since $s^{-1}\partial^{\mathcal{C}}c = -\partial_I c$. (Miraculously the signs do all agree!) The result follows. □

We thus do have a neat elementless description of ∂_E as $(-1)s^{-1} * s^{-1}$ and this could have been used in the definition, but it also needs unpacking in the form we initially gave it in order to see what it is doing. For the single object case, this is, of course, well known, and a definition of the differential of the cobar in this form is given by Baues, [56].

9.4 'Directed' vector bundles, modules and comodules

It is well known and 'classical' that in the correspondence between manifolds and the function algebras defined on them, a vector bundle on X corresponds to a module over the algebra of continuous (real or complex valued) functions on X. For the situation we have with evolving spaces, pospaces, etc., the analogue of bundles has yet to be investigated in any detail, but within the dg-category and dg-cocategory settings modules and comodules are easily defined.

Definition: Let $\mathcal{A}$ be a dg-category on the object set $\mathcal{O}$. A *right* $\mathcal{A}$*-module*, $\mathcal{M}$ is an $\mathcal{O}$-indexed family of differential graded vector spaces, $\{\mathcal{M}(x) : x \in \mathcal{O}\}$, together with $\mathbb{K}$-linear maps

$$\mathcal{M}(x) \otimes \mathcal{A}(x,y) \xrightarrow{\mu} \mathcal{M}(y)$$

satisfying the analogues of the usual module axioms, for instance,

– (associativity) for all $x, y, z \in \mathcal{O}$,

$$\begin{array}{ccc} \mathcal{M}(x) \otimes \mathcal{A}(x,y) \otimes \mathcal{A}(y,z) & \xrightarrow{\mu \otimes \mathcal{A}} & \mathcal{M}(y) \otimes \mathcal{A}(y,z) \\ \downarrow{\scriptstyle \mathcal{M}\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ \mathcal{M}(x) \otimes \mathcal{A}(x,z) & \xrightarrow[\mu]{} & \mathcal{M}(z) \end{array}$$

commutes (where indices have been left off the maps for simplicity);

– an identity axiom:

$$\mathcal{M} \cong \mathcal{M}(x) \otimes \mathbb{K} \overset{\mathcal{M}\otimes\eta}{\to} \mathcal{M}(x) \otimes \mathcal{A}(x,x) \xrightarrow{\mu} \mathcal{M}(x)$$

is the identity.

Extending our previous notation, we will usually write $\mathcal{M} \otimes \mathcal{A}$ for the family $\{\oplus_x \mathcal{M}(x) \otimes \mathcal{A}(x,y) : y \in \mathcal{O}\}$, so $\mu : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M}$.

It should be fairly clear that this version of the definition of module can be rephrased as a dg-functor $\mathcal{M}$ from $\mathcal{A}$ to the dg-category dgvs. We give it in this form as it makes it clear what a comodule over a dg-cocategory must be:

Definition: Let $\mathcal{C}$ be a dg-cocategory. A *right comodule*, $\mathcal{M}$, over $\mathcal{C}$ is given by a family $\{\mathcal{M}(x) : x \in \mathcal{O}\}$ of differential graded vector spaces together with a coaction

$$\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{C},$$

thus, for each $x \in \mathcal{O}$, we have

$$\mathcal{M}(x) \stackrel{\Delta}{\to} \oplus_w \mathcal{M}(w) \otimes \mathcal{C}(w,x),$$

so that if $x \in \mathcal{O}$, the diagram

$$\begin{array}{ccc} \mathcal{M}(x) & \xrightarrow{\quad\Delta\quad} & \oplus_w \mathcal{M}(w) \otimes \mathcal{C}(w,x) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathcal{C}} \\ \oplus_v \mathcal{M}(v) \otimes \mathcal{C}(v,x) & \xrightarrow{\mathcal{M}\otimes\Delta} & \oplus_{v,w} \mathcal{M}(v) \otimes \mathcal{C}(v,w) \otimes \mathcal{C}(w,x) \end{array}$$

is commutative, and if $\eta : \mathcal{C} \to \mathbb{K}$ denotes the coidentity then

$$\mathcal{M}(x) \stackrel{\Delta}{\to} \oplus_w \mathcal{M}(w) \otimes \mathcal{C}(w,x) \to \mathcal{M}(x) \otimes \mathbb{K}$$

is the natural isomorphism.

Now assume $\mathcal{L}$ is a right module for an augmented dg-category $\mathcal{A}$ and $\tau : \mathcal{C} \to \mathcal{A}$ is a twisting cochain, (so we need $\mathcal{C}$ to be coaugmented). We examine the family

$$\mathcal{L} \otimes \mathcal{C} = \{(\mathcal{L} \otimes \mathcal{C})(x) : x \in \mathcal{O}\} = \{\oplus_v \mathcal{L}(v) \otimes \mathcal{C}(v,x) : x \in \mathcal{O}\}.$$

This has a natural $\mathcal{C}$-comodule structure in which the coaction

$$\Delta : (\mathcal{L} \otimes \mathcal{C})(x) \to \oplus_w (\mathcal{L} \otimes \mathcal{C})(w) \otimes \mathcal{C}(w,x)$$

is just

$$\oplus_v \mathcal{L}(v) \otimes \mathcal{C}(v,x) \stackrel{\mathcal{L}\otimes\Delta}{\to} \oplus_{v,w} \mathcal{L}(v) \otimes \mathcal{C}(v,w) \otimes \mathcal{C}(w,x)$$

and so is the obvious map induced by the cocomposition on $\mathcal{C}$.

This comodule, of course, comes with a usual differential namely $\partial_{\mathcal{L}} \otimes \mathcal{C} + \mathcal{L} \otimes \partial_{\mathcal{C}}$, but we can 'deform' or 'twist' this using the twisting cochain $\tau : \mathcal{C} \to \mathcal{A}$, by using the composite

$$\mathcal{L} \otimes \mathcal{C} \stackrel{\mathcal{L}\otimes\Delta}{\to} \mathcal{L} \otimes \mathcal{C} \otimes \mathcal{C} \stackrel{\mathcal{L}\otimes\tau\otimes\mathcal{C}}{\to} \mathcal{L} \otimes \mathcal{A} \otimes \mathcal{C} \stackrel{\mu\otimes\mathcal{C}}{\to} \mathcal{L} \otimes \mathcal{C},$$

which we will denote by ∂_τ and we set

$$\partial = \partial_{\mathcal{L}} \otimes \mathcal{C} + \mathcal{L} \otimes \partial_{\mathcal{C}} + \partial_\tau.$$

We write $\mathcal{L} \otimes_\tau \mathcal{C}$ for $\mathcal{L} \otimes \mathcal{C}$ with this differential.

Lemma 3. *$\mathcal{L} \otimes_\tau \mathcal{C}$ is a dg-comodule over $\mathcal{C}$.*

Proof: Again this is a straightforward generalisation of the single object case. The important thing to note is that it is the Maurer-Cartan equation that guarantees that $\partial^2 = 0$. □

That construction used the twisting cochain to go from $\mathcal{A}$-modules to $\mathcal{C}$-comodules. Suppose instead we are given a $\mathcal{C}$-comodule, $\mathcal{M} = \{\mathcal{M}(x) : x \in \mathcal{O}\}$ with coaction

$$\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{C}.$$

We can form a family, $\mathcal{M} \otimes \mathcal{A}$, in the obvious way by taking

$$(\mathcal{M} \otimes \mathcal{A})(x) = \oplus_v \mathcal{M}(v) \otimes \mathcal{A}(v, x)$$

and not surprisingly we get an $\mathcal{A}$-module structure on it using

$$\begin{aligned}(\mathcal{M} \otimes \mathcal{A})(x) \otimes \mathcal{A}(x, y) &= \oplus_v \mathcal{M}(v) \otimes \mathcal{A}(v, x) \otimes \mathcal{A}(x, y) \\ &\stackrel{\mathcal{M}\otimes\mu}{\to} \oplus_v \mathcal{M}(v) \otimes \mathcal{A}(v, y) = (\mathcal{M} \otimes \mathcal{A})(y)\end{aligned}$$

This $\mathcal{A}$-module comes, of course, with a differential much as in the dual construction: $\partial_{\mathcal{M}} \otimes \mathcal{A} + \mathcal{M} \otimes \partial_{\mathcal{A}}$, but also has a twisted term

$$\partial_\tau = (\mathcal{M} \otimes \mu)(\mathcal{M} \otimes \tau \otimes \mathcal{A})(\Delta \otimes \mathcal{A})$$

i.e. the composite

$$\mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \otimes \mathcal{C} \otimes \mathcal{A} \to \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{M} \otimes \mathcal{A}.$$

Thus given $m_v \otimes a_{vx}$ with $\Delta m_n = \sum m_u \otimes c_{uv}$, then

$$\partial_\tau(m_v \otimes a_{vx}) = \sum m_u \otimes \tau(c_{uv}) a_{vx}.$$

Again we have that $\partial = \partial_{\mathcal{M}} \otimes \mathcal{A} + M \otimes \partial_{\mathcal{A}} + \partial_\tau$ deforms the basic differential of $\mathcal{M} \otimes \mathcal{A}$ yielding an $\mathcal{A}$-module, $\mathcal{M} \otimes_\tau \mathcal{A}$, the *twisted tensor product of* $\mathcal{M}$ *and* $\mathcal{A}$.

We will not use this construction below since, as yet, its applications are still not clear and it is included mainly to point out that the classical 'undirected' theory does generalise easily. To clarify applications, we will need a good reserve of examples of modules and/or comodules. To this end we look at alternative ways of defining them.

The above approach is not the only way to introduce modules and comodules in this setting. Suppose $\mathcal{M} = \{\mathcal{M}(x) \mid x \in \mathcal{O}\}$ is an $\mathcal{O}$-indexed family of differential graded vector spaces. Now if $\mathcal{M}$ and $\mathcal{N}$ are two such, we set, for $a, b, \in \mathcal{O}$,

$$Hom(\mathcal{M}, \mathcal{N})(a.b) = Hom(\mathcal{M}(a), \mathcal{N}(b)).$$

This gives a dg-quiver $Hom(\mathcal{M}, \mathcal{N})$ and we set $End(\mathcal{M}) = Hom(\mathcal{M}, \mathcal{M})$ to get a dg-category on $\mathcal{O}$ with composition

$$Hom(\mathcal{M}(a), \mathcal{M}(b)) \otimes Hom(\mathcal{M}(b), \mathcal{M}(c)) \to Hom(\mathcal{M}(a), \mathcal{M}(c))$$

given in the obvious way. If $\mathcal{A}$ is a dg-category, then an $\mathcal{A}$-module structure on the family $\mathcal{M}$ corresponds to a morphism of dg-categories

$$\mathsf{act}_{\mathcal{M}} : \mathcal{A} \to End(\mathcal{M}).$$

Of course, this is getting very close to being a dg-functor from $\mathcal{A}$ to dgvs and that link could be explored further - but will not be here.

Example: We will look at an obvious type of module on $\mathcal{A}$, namely, a representable one, so for an object $a \in \mathcal{A}$, consider the functor $\mathcal{A}(a,-) : \mathcal{A} \to \mathsf{dgvs}$. The corresponding family is, of course, $\{\mathcal{A}(a,y) \mid y \in \mathcal{O}\}$ and the action is given by the composition in $\mathcal{A}$. More generally take a finite direct sum of such modules, i.e. pick a finite set $\{a_i : i = 1,\ldots,k\}$ of objects of $\mathcal{O}$ and define $\mathcal{M}(y) = \bigoplus_{i=1}^{k} \mathcal{A}(a_i, y)$ with the obvious action.

In the case of a generating $\mathcal{S}$-category, $\mathcal{C}$ with $\mathcal{A} = \Omega(\mathcal{C})$, i.e. the cobar construction applied to the cocategory $C(\mathcal{C})^*$, the module $\mathcal{M}$ with $\mathcal{M}(y) = \mathcal{A}(a,y)$ is generated by the basic future tangent directions at a If we need to consider an embedded 'space', then we can restrict to specifying a single such tangent direction for a subset of the objects of $\mathcal{C}$. It is interesting to see that something along these lines has been put forward in the work of Lazaroiu, [52]. He studies a slightly more specialised form of dg-category, but then looks at the situation where a set $S \subset \mathcal{O}$ is given together with a set of degree one elements $q_{ab} \in \mathcal{A}(a,b)_1$, for $a, b, \in S$. (This can, of course, be also viewed as a family, $\{q_{ab} \mid a, b \in \mathcal{O}\}$ by setting $q_{ab} = 0$ if either a or b is not in S.) Such a situation is considered in [52] with the, for us, very interesting extra 'tadpole condition' $\partial q_{ab} + \sum q_{ac} q_{cb} = 0$. As Lazaroiu points out, this is just the Maurer-Cartan condition in this setting. It would seem fairly clear that this defines not only a deformation of the basic theory represented by the dg-category $\mathcal{A}$ as discussed in [52], but also a twisting cochain in the sense we have discussed above. (I have not checked this in detail, nor attempted, as yet, to explore what consequences beyond the most elementary ones this observation gives us, but it is very suggestive of other constructions within discrete differential geometry which have interpretations that may be useful in our search for tools for handling evolving spatial contexts using $\mathcal{S}$-categorical machinery in both the physical 'space-time' setting and the pospace one.

It is feasible to define two sided modules and comodules, to consider derivations and to relate them to intuitions of vector fields and even, to some extent, to mimic Lie theory in this context, but as that research is still far from being in anything like in its 'definitive' presentable form and its relevance to directed space theory is still to be investigated, we will not pursue this further.

10 Conclusion

The aim of this paper was to suggest that the machinery of $\mathcal{S}$-category theory may provide a useful addition to the tools available for the study of such contexts as pospaces, evolving spaces and related contexts from physics. We have developed a reasonable amount of algebraic topological machinery in this context with fundamental group analogues, etc., and have sketched the development of a discrete differential geometry for this setting using a variant of the cobar construction. There is a lot left to do, initially to evaluate the cobar construction and its relationships to other constructions such as those used by Raptis and Zapatrin, [49], and to interpret these constructions back in the directed space context, but many of the intuitions of directed spaces, space-times etc. do seem to have a useful model in this $\mathcal{S}$-enriched, or dg-enriched, settings.

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Appendix:

In this appendix, we collect up some background material for the convenience of the reader.

11 $\mathcal{S}$-categories

We assume we have a category $\mathcal{A}$ whose objects will be denoted by lower case letter, x,y,z, ..., at least in the generic case, and for each pair of such objects, (x, y), a simplicial set $\mathcal{A}(x, y)$ is given; for each triple x, y, z of objects of $\mathcal{A}$, we have a simplicial map, called *composition*

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \longrightarrow \mathcal{A}(x, z);$$

and for each object x a map,

$$\Delta[0] \to \mathcal{A}(x, x),$$

that 'names' or 'picks out' the 'identity arrow at x' in the set of 0-simplices of $\mathcal{A}(x,x)$. This data is to satisfy the obvious axioms, associativity and identity, suitably adapted to this situation. Such a set up will be called a *simplicially enriched category* or more simply *an $\mathcal{S}$-category.* Enriched category theory is a well established branch of category theory, see Kelly, [57] for a detailed technical treatment.

WARNING: Some authors use the term simplicial category for what we have termed a simplicially enriched category. There is a close link with the notion of simplicial category that is consistent with usage in simplicial theory *per se*, since any simplicially enriched category can be thought of as a simplicial object in the 'category of categories', but a simplicially enriched category is not just a simplicial object in the 'category of categories' and not all such simplicial objects correspond to such enriched categories. That being said that usage need not cause problems provided the reader is aware of the usage in the paper to which reference is being made.

Examples: (i) $\mathcal{S}$, the category of simplicial sets:
here we take, for simplicial sets, K, L, $\mathcal{S}(K,L)$ to be the simplicial set with

$$\mathcal{S}(K,L)_n := S(\Delta[n]\times K, L)$$

and face and degeneracy maps induced from their duals between the $\Delta[n]$s. Composition : for $f\in\mathcal{S}(K,L)_n$, $g\in\mathcal{S}(L,M)_n$, so $f:\Delta[n]\times K\to L$, $g:\Delta[n]\times L\to M$,

$$g\circ f := (\Delta[n]\times K\stackrel{diag\times K}{\longrightarrow}\Delta[n]\times\Delta[n]\times K\stackrel{\Delta[n]\times f}{\longrightarrow}\Delta[n]\times L\stackrel{g}{\to}M);$$

Identity : $id_K : \Delta[0]\times K\stackrel{\cong}{\to}K$,

(ii) $\mathcal{T}op$, 'the' category of spaces (of course, there are numerous variants but you can almost pick whichever one you like as long as the constructions work): $\mathcal{T}op(X,Y)$ is the simplicial set with

$$\mathcal{T}op(X,Y)_n := Top(\Delta^n\times X, Y).$$

Composition and identities are defined analogously to those in (i).

(iii) For each X, $Y\in Cat$, the category of small categories, then we similarly get $\mathcal{C}at(X,Y)$,

$$\mathcal{C}at(X,Y)_n = Cat([n]\times X, Y).$$

We leave the other structure up to the reader.

In general any category of simplicial objects in a 'nice enough' category has a simplicial enrichment, although the general argument that gives the construction does not always make the structure as transparent as it might be without a deal of 'unpacking'.

There is an evident notion of $\mathcal{S}$-enriched functor, so we get a category of 'small' $\mathcal{S}$-categories, denoted $\mathcal{S}-Cat$. Of course, none of the above examples are 'small' unlike those in the body of this paper.

12 Graded and Differential Graded Vector Spaces

Here we will gather together some of the basic ideas and terminology of graded and differential graded algebras and their many object analogues. We will work over a fixed field, $\mathbb{K}$, which we usually think of as being $\mathbb{R}$ or $\mathbb{C}$. Many of the ideas would work over a commutative ring. We start by repeating, and expanding on, some of the definitions from earlier, so as to have them immediately available here.

Definition:

(i) A *pre-$\mathbb{Z}$-graded vector space* (sometimes abbreviated to *pre-gvs*) is a direct sum $V = \bigoplus_{p\in\mathbb{Z}} V_p$ of vector spaces. The elements of V_p are said to be *homogeneous of degree* p. If $x \in V_p$, write $|x| = p$. Sometimes it may be convenient to write $\bar{x} = (-1)^{|x|}x$ and $V_+ = \bigoplus_{p>0} V_p$. Another very useful piece of notation is $V^p = V_{-p}$.

(ii) A *graded vector space* (often abbreviated to *gvs*) is a positively or negatively graded pre-graded vector space, that is, either $V = \sum_{p\geq 0} V_p$ or $V = \sum_{p\leq 0} V^p$.

(iii) We consider the field $\mathbb{K}$ to be a pre-gvs with $(\mathbb{K})_0 = \mathbb{K}$, and $(\mathbb{K})_p = 0$ if $p \neq 0$. We say a gvs,V, is of *finite type* if $dim(V_p) < \infty$ for all p.

(iv) A linear map $f : V \to W$ between pre-gvs is *of degree* p if $f(V_q) \subseteq W_{p+q}$ for all q. (Note this may also occur as $f(V^q) \subseteq W^{q-p}$.)

(v) A *morphism* $f : V \to W$ is a linear map of degree zero.

(vi) Pregraded vector spaces and the morphisms between them define the category $\mathsf{pre-gvs}$. More importantly we have subcategories of graded vector spaces, denoted gvs.

(vii) The set of all linear maps of degree p from V to W will be denoted $Hom_p(V, W)$ and we set

$$Hom(V, W) = \bigoplus_p Hom_p(V, W).$$

Of course, we now have two notations for the same object, $\mathsf{pre-gvs}(V, W) = Hom_0(V, W)$.

Duals:

The dual of a (pre-)gvs V is $\#V$ defined by

$$\begin{aligned}(\#V)_p &:= Hom_p(V, \mathbb{K}) \\ &\cong \mathsf{Vect}(V_{-p}, \mathbb{K}) \\ &\cong \#(V_{-p}) \\ &= \#(V^p).\end{aligned}$$

If $f : V \to W$ is of degree $|f|$, then

$$^t f : \#W \to \#V$$

is given by

$$({}^t f)(\psi)(x) = (-1)^{|f||\psi|}\psi f(x),$$

for $\psi \in \#W$ and $x \in V$. Thus if $V \xrightarrow{f} W \xrightarrow{g} X$, then

$$\boxed{{}^t(g \circ f) = (-1)^{|f||g|}({}^t f \circ {}^t g).}$$

In particular, for f an isomorphism

$$({}^t f)^{-1} = (-1)^{|f|}\ {}^t(f^{-1}).$$

Duality:

Let V be a gvs, by convention in the duality

$$\langle\quad;\quad\rangle : V \leftrightarrow \#V,$$

we will usually assume V is non-negatively graded (so $V = \bigoplus_{p\geq 0} V_p$), whilst the right hand side is non-positively graded.

If V is of finite type then $\#\#V \cong V$, of course. The suspension of the dual $s(\#V)$ can be identified with $\#(s^{-1}V)$ and similarly $s^{-1}(\#V) = \#s(V)$. These identifications are via the rules:

$$\boxed{\begin{aligned} \langle s^{-1}z; su\rangle &= \ \ (-1)^{|z|}\langle z; u\rangle, \\ \langle sz; s^{-1}u\rangle &= (-1)^{|z|+1}\langle z; u\rangle. \end{aligned}}$$

This sign convention is needed to ensure that $ss^{-1} = id$.

Tensor products:

The *tensor product* of two pre-gvs, V and W, is $V \otimes W$, where

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q.$$

On morphisms

$$\boxed{(f \otimes g)(v \otimes w) = (-1)^{|g||f|}(f(v) \otimes g(w))}$$

and is of degree $|f|+|g|$. In particular there is a natural injection $(\#V)\otimes(\#W) \to \#(V \otimes W)$, and this is an isomorphism if either V or W is of finite type.

Differential (pre-)graded vector spaces:

Definition: A *differential (pre-)graded vector space*, (dgvs), is a pair (V, ∂), where V is a (pre-)graded vector space and $\partial \in Hom_{-1}(V, V)$ satisfies $\partial \circ \partial = 0$. This endomorphism, ∂, of degree -1 is called the *differential* or sometimes the *boundary operator* of the dgvs.

Given any dgvs, $H(V, \partial)$, a gvs defined by

$$H(V, \partial)_q = \frac{Ker(\partial : V_q \to V_{q-1})}{Im(\partial : V_{q+1} \to V_q)}$$

in the usual way.

Let (V, ∂), (V', ∂') be two pre-dgvs

$$Hom(V, V') = \bigoplus_{p \in \mathbb{Z}} Hom_p(V, V')$$

is a pre-dgvs with differential

$$Df = \partial' \circ f - (-1)^{|f|} f \circ \partial$$

for f homogeneous. A degree r linear morphism f is *compatible with the differentials* if it is a cycle for this differential D, i.e., $Df = 0$ or $\partial' f = (-1)^r f\partial$.

A *morphism between pre-dgvs* is a linear morphism of degree 0 that is compatible with the differentials:

$$f : (V, \partial) \to (V', \partial').$$

This induces $H(f) : H(V, \partial) \to H(V', \partial')$.

We get a category pre - dgvs and, of course, a subcategory dgvs of differential graded vector spaces, then H is a functor H : pre−dgvs → pre−gvs.

Chains and cochains: terminology. If (V, ∂) is a pre-dgvs with 'lower grading' that is the summands are written V_p, then (V, ∂) may be called a *chain complex* and terms such as *cycle*, *boundary*, *homology* are used with the usual meanings. If (V, ∂) is presented with the 'upper grading', so V^p, then the corresponding words will have a 'co' as prefix, cochain complex, cocycle, etc. There is no real distinction between the two cases in the abstract, but in applications there is often a fixed 'dimensional' interpretation and then the 'natural' and 'geometric' aspects determine which is more appropriate or useful. (Baues has suggested using the terminology 'chain algebra' for positively graded differential algebras (see below) and 'cochain algebras' for the negatively graded ones. This is a good convention but I have not used it here as I have, in general, been following Tanré, [47] for notation and teminology.)

13 Differential graded algebras

Pre-graded algebras: A *pre-graded algebra* (pre-ga) or $\mathbb{Z}$*-graded algebra* is a pre-gvs, A, together with an algebra multiplication satisfying $A_p.A_q \subseteq A_{p+q}$ for any p, q. The relevant morphisms are pre-gvs morphisms which respect the multiplication. This gives a category pre-ga. There are also graded algebras corresponding to graded vector spaces, of course. All the definitions below work in both pre-graded and graded versions.

An *augmentation* of a pre-ga, A, is a homomorphism $\varepsilon : A \to \mathbb{K}$. The *augmentation ideal* of (A, ε) is $Ker\ \varepsilon$ and will also be denoted $\bar{A}$. The pair (A, ε) is called an *augmented pre-ga*. A morphism $f : (A, \varepsilon) \to (A', \varepsilon')$ of augmented pre-gas is a homomorphism $f : A \to A'$ (thus of degree zero) such that $\varepsilon = \varepsilon' f$. The resulting category will be written pre-εga.

Tensor product: If A, A' are two pre-gas, then $A \otimes A'$ is a pre-ga with

$$(a \otimes a')(b \otimes b') = (-1)^{|a'||b|} ab \otimes a'b'$$

for homogeneous $a, b \in A$, $a', b' \in A'$.

If ε, ε are augmentations of A and A' respectively, then $\varepsilon \otimes \varepsilon'$ is an augmentation of $A \otimes A'$.

Derivations: (These have not been used in the main text but are included here to suggest a generalisation whose details have yet to be fully worked out.)

Let A be a pre-ga. An *(algebra) derivation* of degree $p \in \mathbb{Z}$ is a linear map $\theta \in Hom_p(A, A)$ such that

$$\theta(ab) = \theta(a)b + (-1)^{p|a|} a\theta(b)$$

for homogeneous $a, b \in A$.

A derivation θ of an augmented algebra, (A, ε), is an algebra derivation which, in addition, satisfies $\varepsilon\theta = 0$.

Let $Der_p(A)$ be the vector space of derivations of degree p of A, then $Der(A) = \bigoplus_p Der_p(A)$ is a pre-gvs.

N.B. In the case of upper gradings, an element of $Der_p(A)$ sends A^n into A^{n-p}.

Pre-DGAs: and DGAs

A *differential* ∂ on an (augmented) pre-ga (ga) is a derivation of the (augmented) algebra of degree -1 such that $\partial \circ \partial = 0$. The pair (A, ∂) is called a *pre-differential graded algebra* (pre-dga). If A is augmented, then (A, ∂) will be called an *augmented pre-dga* (pre-εdga).

If (A, ∂) and (A', ∂') are pre-dgas, then $(A, \partial) \otimes (A', \partial')$, with the conventions already noted, is one as well.

A morphism of pre-dgas (or pre-εdgas) is a morphism which is both of pre-gdvs and of pre-gas (with ε as well if used). This gives categories pre-DGA and pre-εDGA.

14 Differential graded categories

It is standard that $\mathbb{K}$-linear categories are the 'many object analogue' of $\mathbb{K}$-algebras, or put more precisely a $\mathbb{K}$-linear category having only one object is 'the same as' a $\mathbb{K}$-algebra. The same is true for differential graded algebras and differential graded categories.

Definition: A graded ($\mathbb{K}$-)category $\mathcal{A}$, is a category enriched over the category of graded vector spaces, with the tensor product giving the monoidal structure. We thus have that the $\mathcal{A}(x, y)$ are graded ($\mathbb{K}$-)vector spaces and the compositions

$$\mu^y_{x,z} : \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z),$$

are degree zero maps. Alternatively, these compositions can be specified as bilinear maps,

$$\mathcal{A}(y,z) \times \mathcal{A}(x,y) \to \mathcal{A}(x,z).$$

Using the tensor product of quivers introduced earlier the composition is a map

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

obeying associativity and identity axioms, of course.

There is also an identity map made up of a family $\eta(x) : \mathbb{K} \to \mathcal{A}(x,x)$, or merely $\eta : \mathbb{K} \to \mathcal{A}$, where as we will often do, we indicate families rather than the individual components.

(NOTE: As most of the sources that we have used themselves use functional composition order, we have adopted the same convention in these contexts.)

Any homogeneous $u \in \mathcal{A}(x,y)$ has a grade $\mid u \mid \in \mathbb{Z}$ and for compositions

$$\mid uv \mid = \mid u \mid + \mid v \mid .$$

Definition: A *differential graded* (or *dg*) category $\mathcal{A}$ is one enriched over dgvs, so the $\mathcal{A}(x,y)$ now, in addition, have a differential ∂ of degree 1, $\partial\partial = 0$, and for the composition, the Leibnitz rule

$$\partial(uv) = \partial u.v + (-1)^{|u|} u.\partial v,$$

holds.

We have been considering mainly negatively graded dg-categories.