# Distributivity for a monad and a comonad

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**Abstract**

We give a systematic treatment of distributivity for a monad and a comonad as arises in incorporating category theoretic accounts of operational and denotational semantics, and in giving an intensional denotational semantics. We do this axiomatically, in terms of a monad and a comonad in a 2-category, giving accounts of the Eilenberg-Moore and Kleisli constructions. We analyse the eight possible relationships, deducing that two pairs are isomorphic, but that the other pairs are all distinct. We develop those 2-categorical definitions necessary to support this analysis.

## 1 Introduction

In recent years, there has been an ongoing attempt to incorporate operational semantics into a category theoretic treatment of denotational semantics. The denotational semantics is given by starting with a signature $\Sigma$ for a language without variable binding, and considering the category $\Sigma$-*Alg* of $\Sigma$-algebras [4]. The programs of the language form the initial $\Sigma$-algebra. For operational semantics, one starts with a behaviour functor $B$ and considers the category $B$-*Coalg* of $B$-coalgebras [5,7]. By combining these two, one can consider the combination of denotational and operational semantics [10,12]. Under size

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conditions, the functor $\Sigma$ gives rise to a free monad $T$ on it, the functor $B$ gives rise to a cofree comonad $D$ on it, and the fundamental structure one needs to consider is a distributive law of $T$ over $D$, i.e., a natural transformation $\lambda : TD \Rightarrow DT$ subject to four axioms; and one builds the category $\lambda$-*Bialg* from it, a $\lambda$-bialgebra being an object $X$ of the base category together with a $T$-structure and a $D$-structure on $X$, subject to one evident coherence axiom. This phenomenon was the subject of Turi and Plotkin's [12], with leading example given by an idealised parallel language, with operational semantics given by labelled transition systems. In fact the work of this paper sprang from discussions between one of the authors and Plotkin, whom we acknowledge gratefully.

As a separate piece of work, Brookes and Geva [2] have also proposed the study of a monad and a comonad in combination. For them, the Kleisli category for the comonad gives an intensional semantics, with maps to be regarded as algorithms. They add a monad in the spirit of Moggi to model what has been called a notion of computation [9]. They then propose to study the category for which an arrow is a map of the form $DX \longrightarrow TY$ in the base category, where $T$ is the monad and $D$ is the comonad. In order for this to form a category, one needs a distributive law of $D$ over $T$, i.e., a natural transformation $\lambda : DT \Rightarrow TD$ subject to four coherence axioms. Observe that this distributive law allowing one to make a two-sided version of a Kleisli construction is in the opposite direction to that required to build a category of bialgebras.

Motivated by these two examples, in particular the former, we seek an account of distributive laws between a monad and a comonad, with a treatment of Eilenberg-Moore and Kleisli constructions. That is the topic of this paper. The answer is not trivial. It is not just a matter of considering the situation for a distributive law between two monads and taking a dual of one of them, as there are fundamental differences. For instance, to give a pair of monads $T$ and $T'$ and a distributive law of $T$ over $T'$ is equivalent to giving a monad structure on $T'T$ [1] with appropriate coherence, but nothing like that is the case for a distributive law of a monad $T$ over a comonad $D$. To give a distributive laws of $T$ over $T'$ is also equivalent to giving a lifting of the monad $T$ to $T'$-*Alg*, but not a lifting of $T'$ to $T$-*Alg*. However, to give a distributive law of a monad $T$ over a comonad $D$ is equivalent to lifting $T$ to $D$-*Coalg* and also to lifting $D$ to $T$-*Alg*. Dual remarks, with the Kleisli construction replacing the Eilenberg-Moore construction, apply to distributive laws of comonads over monads. So we need an analysis specifically of distributive laws between a monad and a comonad, and that does not amount to a mild variant of the situation for two monads.

In principle, when one includes an analysis of maps between distributive laws, one has eight choices here: given $(T, D, \lambda)$ on a category $C$ and $(T', D', \lambda')$ on $C'$ and a functor $J : C \longrightarrow C'$, one could consider natural transformations $t : T'J \Rightarrow JT$ and $d : JD \Rightarrow D'J$, or the other three alternatives given by dualisation; and one could dualise by reversing the directions of $\lambda$ and $\lambda'$. But not all of these possibilities have equal status. Two of them each arise in two different ways, reflecting the fact that a category $(T, D)$-*Bialg* of bialgebras for a monad $T$ and a comonad $D$ may be seen as both the category of algebras for a monad on $D$-*Coalg* and as a category of coalgebras for a comonad on $T$-*Alg*. And two of the eight possibilities do not correspond to applying an Eilenberg-Moore or Kleisli construction to an Eilenberg-Moore or Kleisli construction at all. We investigate the possibilities in Section 4.

We make our investigations in terms of an arbitrary 2-category $K$. The reason is that although the study of operational and denotational semantics in [12] was done in terms of ordinary categories, i.e., modulo size, in the 2-category *Cat*, it was done without a direct analysis of recursion, for which one would pass to the 2-category of $O$-categories, i.e., categories for which the homsets are equipped with $\omega$-cpo structure, with maps respecting such structure. More generally, that work should and probably soon will be incorporated into axiomatic domain theory, requiring study of the 2-category $V$-*Cat* for a symmetric monoidal closed $V$ subject to some domain-theoretic conditions [3]. Moreover, our definitions and analysis naturally live at the level of 2-categories, so that level of generality makes the choices clearest and the proofs simplest. Mathematically, this puts our analysis exactly at the level of generality of the study of monads by Street in [11], but see also Johnstone's [6] for an analysis of adjoint lifting that extends to this setting.

Formally, we recall the definition of 2-category in Section 2, define the notion of a monad in a 2-category, and characterise the Eilenberg-Moore construction in those terms. We also explain a dual, yielding the Kleisli construction. This is all essentially in Street's paper [11]. In Section 3, we give another dual, yielding accounts of the Eilenberg-Moore and Kleisli constructions for comonads. Then, in Section 4 lies the heart of the paper, in which we consider the eight possible combinations of monads and comonads, characterise two of them, explain how they arise in two different ways, and show that there are precisely six distinct possible combinations. We also give universal properties for a category of bialgebras and a Kleisli category, and explain why, from the former universal property, it follows that there is precisely one natural way to induce a functor between categories of bialgebras.

## 2 Monads in 2-categories

In this section, we define the notion of 2-category and supplementary notions. We then define the notion of a monad in a 2-category, and characterise the Eilenberg-Moore and Kleisli constructions in terms of the existence of adjoints to diagonal 2-functors [11].

**Definition 2.1** A 2-*category* $K$ consists of

- a set of 0-*cells* or *objects*

- for each pair of 0-cells $X$ and $Y$, a category $K(X,Y)$ called the *homcategory from $X$ to $Y$*
- for each triple of 0-cells $X$, $Y$ and $Z$, a *composition* functor
$$\circ : K(Y,Z) \times K(X,Y) \longrightarrow K(X,Z)$$
- for each 0-cell $X$, an object $id_X$ of $K(X,X)$, or equivalently, a functor $id_X : 1 \longrightarrow K(X,X)$, called the *identity on $X$*

such that the following diagrams of functors commute

$$\begin{array}{ccc} K(Z,W)\times K(Y,Z)\times K(X,Y) & \xrightarrow{\circ\times K(X,Y)} & K(Y,W)\times K(X,Y) \\ {\scriptstyle K(Z,W)\times\circ}\Big\downarrow & & \Big\downarrow{\scriptstyle\circ} \\ K(Z,W)\times K(X,Z) & \xrightarrow[\circ]{\qquad\qquad} & K(X,W) \end{array}$$

$$\begin{array}{ccccc} K(X,Y)\times K(X,X) & \xleftarrow{K(X,Y)\times id_X} & K(X,Y) & \xrightarrow{id_Y\times K(X,Y)} & K(Y,Y)\times K(X,Y) \\ & {\scriptstyle\circ}\searrow & \Big\downarrow{\scriptstyle =} & \swarrow{\scriptstyle\circ} & \\ & & K(X,Y) & & \end{array}$$

In the definition of a 2-category, the objects of each $K(X,Y)$ are often called 1-*cells* and the arrows of each $K(X,Y)$ are often called 2-*cells*. We typically abbreviate the composition functors by juxtaposition and use $\cdot$ to represent composition within a homcategory.

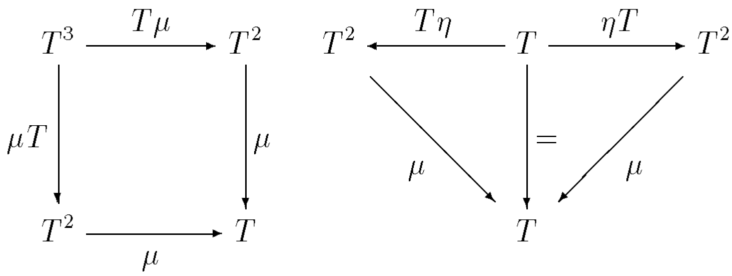
Obviously, the definition of 2-category is reminiscent of the definition of category: if one takes the definition of category and replaces homsets by homcategories, composition functions by composition functors, and the axioms by essentially the same axioms but asserting that pairs of functors are equal, then one has exactly the definition of a 2-category.

**Example 2.2** The leading example of a 2-category is $Cat$, in which the 0-cells are small categories and $Cat(C,D)$ is defined to be the functor category $[C,D]$. More generally, for any symmetric monoidal closed category $V$, one has a 2-category $V$-$Cat$, whose objects are small $V$-categories, and with homcategories given by $V$-functors and $V$-natural transformations. In particular, there is a 2-category $LocOrd$ of small locally ordered categories, locally ordered functors, and natural transformations. As another instance of $V$-$Cat$, there is a 2-category of small $O$-categories, $O$-functors, and natural transformations, where $O$ is the cartesian closed category of $\omega$-cpo's.

Each 2-category $K$ has an underlying ordinary category $K_0$ given by the 0-cells and 1-cells of $K$. A 2-*functor* between 2-categories $K$ and $L$ is a functor from $K_0$ to $L_0$ that respects the 2-cell structure. A 2-*natural transformation* between 2-functors is an ordinary natural transformation that respects the 2-cell structure. Given a 2-functor $U : K \longrightarrow L$, these definitions give rise to the notion of a left 2-*adjoint*, which is a left adjoint that respects the 2-cells. More details and equivalent versions of these definitions appear and are analysed in [8].

Now we have the definition of 2-category, we can define the notion of a monad in any 2-category $K$, generalising the definition of monad on a small category, which amounts to the case of $K = Cat$.

**Definition 2.3** A *monad* in a 2-category $K$ consists of a 0-cell $C$, a 1-cell $T : C \longrightarrow C$, and 2-cells $\mu : T^2 \Rightarrow T$ and $\eta : Id \Rightarrow T$ subject to commutativity of the following diagrams in the homcategory $K(C, C)$

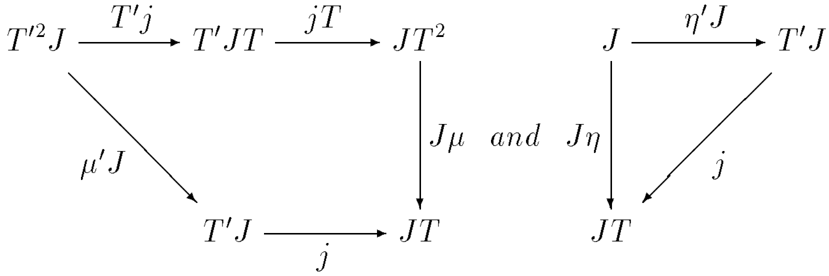



For example, if one lets $K = Cat$, then a monad in $K$ as we have just defined it amounts exactly to a small category with a monad on it. More generally, if $K = V\text{-}Cat$, then a monad in $K$ amounts exactly to a small $V$-category together with a $V$-monad on it. So, for instance, a monad in $O\text{-}Cat$ amounts to a small $O$-category together with a monad on it, such that the monad respects the $\omega$-cpo structure of the homs.

For any 2-category $K$, one can construct a 2-category of monads in $K$.

**Proposition 2.4** *For any 2-category $K$, the following data forms a 2-category $Mnd(K)$:*

- *0-cells are monads in $K$.*
- *A 1-cell in $Mnd(K)$ from $(C, T, \mu, \eta)$ to $(C', T', \mu', \eta')$ is a 1-cell $J : C \longrightarrow C'$ in $K$, together with a 2-cell $j : T'J \Rightarrow JT$ in $K$, subject to commutativity in $K(C, C')$ of*



- *A 2-cell in $Mnd(K)$ from $(J, j)$ to $(H, h)$ is a 2-cell in $K$ from $J$ to $H$ subject to the evident axiom expressing coherence with respect to $j$ and $h$.*

There is a forgetful 2-functor $U : Mnd(K) \longrightarrow K$ sending a monad $(C, T, \mu, \eta)$ in $K$ to the underlying object $C$. This 2-functor has a right 2-adjoint given by the 2-functor $Inc : K \longrightarrow Mnd(K)$ sending an object $X$ of $K$ to $(X, id, id, id)$, i.e., to $X$ together with the identity monad on it. The definition of $Mnd(K)$ and analysis of it are the central topics of study of [11], a summary of which appears in [8].

**Definition 2.5** A 2-category $K$ *admits Eilenberg-Moore constructions for monads* if the 2-functor $Inc : K \longrightarrow Mnd(K)$ has a right 2-adjoint.

**Example 2.6** $Cat$ admits Eilenberg-Moore constructions, and the right 2-adjoint gives exactly the usual Eilenberg-Moore construction for any monad on any small category. Note here what the universal property says: it says that for any small category $D$ and any small category $C$ with a monad $T$ on it, there is a natural isomorphism of categories between $[D, T\text{-}Alg]$ and the category for which an object is a functor $J : D \longrightarrow C$ together with a natural transformation $TJ \Rightarrow J$ subject to two coherence conditions generalising those in the definition of $T$-algebra. This is a stronger condition than the assertion that every adjunction gives rise to a unique functor into the category of algebras of the induced monad.

**Example 2.7** If $V$ has equalisers, then $V\text{-}Cat$ admits the Eilenberg-Moore construction for monads, and again, the construction is exactly as one expects. This is a fundamental observation underlying [11].

Given an arbitrary 2-category $K$, we have constructed the 2-category $Mnd(K)$ of monads in $K$. Modulo size, this construction can itself be made 2-functorial, yielding a 2-functor $Mnd : 2\text{-}Cat \longrightarrow 2\text{-}Cat$, taking a small 2-category $K$ to $Mnd(K)$, with a 2-functor $G : K \longrightarrow L$ sent to a 2-functor $Mnd(G) : Mnd(K) \longrightarrow Mnd(L)$ and similarly for a 2-natural transformation. It follows that, given a 2-adjunction $F \dashv U : K \longrightarrow L$, one obtains another 2-adjunction $Mnd(F) \dashv Mnd(U) : Mnd(K) \longrightarrow Mnd(L)$. We shall use this fact later.

Finally in this section, we mention a dual construction. For any 2-category $K$, one may consider the opposite 2-category $K^{op}$, which has the same 0-cells as $K$ but $K^{op}(X, Y) = K(Y, X)$, with composition induced by that of $K$. This allows us to make a different construction of a 2-category of monads in $K$, as we could say

**Definition 2.8** For a 2-category $K$, define $Mnd^*(K) = Mnd(K^{op})^{op}$.

Analysing the definition, a 0-cell of $Mnd^*(K)$ is a monad in $K$; a 1-cell from $(C, T, \mu, \eta)$ to $(C', T', \mu', \eta')$ is a 1-cell $J : C \longrightarrow C'$ in $K$, together with a 2-cell $j : JT \Rightarrow T'J$ in $K$, subject to two coherence axioms, expressing coherence between $\mu$ and $\mu'$ and between $\eta$ and $\eta'$; and a 2-cell from $(J, j)$ to $(H, h)$ is a 2-cell in $K$ from $J$ to $H$ subject to one axiom expressing coherence with respect to $j$ and $h$. The central difference between $Mnd(K)$ and $Mnd^*(K)$ is in the 1-cells, because $j$ is in the opposite direction.

By putting $L = K^{op}$, we can deduce results about $Mnd^*(K)$ from results about $Mnd(K)$. In particular, we have

**Proposition 2.9** (i) *The construction $Mnd^*(K)$ yields a 2-functor $Mnd^* : 2\text{-}Cat \longrightarrow 2\text{-}Cat$.*

(ii) *The forgetful 2-functor $U : Mnd^*(K) \longrightarrow K$ has a left 2-adjoint given by $Inc : K \longrightarrow Mnd^*(K)$, sending an object $X$ of $K$ to the identity monad on $X$.*

**Proposition 2.10** *If $K = Cat$, then $Inc : Cat \longrightarrow Mnd^*(Cat)$ has a left 2-adjoint given by Kleisli construction for a monad on small category.*

**Proof.** Let $(C, T, \mu, \eta)$ be a monad in $Cat$. We have a functor $J : C \longrightarrow Kl(T)$ as usual, and it is routine to verify that $J$ is part of a monad morphism $(J, j) : (C, T, \mu, \eta) \longrightarrow Inc(Kl(T))$, where $j : JT \Rightarrow J$ has $a$-component given by $id : Ta \longrightarrow Ta$. We must show that $(J, j)$ is the unit for a left 2-adjoint to $Inc : Cat \longrightarrow Mnd^*(Cat)$.

Given a category $C'$ and given a map $(H, h) : (C, T, \mu, \eta) \longrightarrow Inc(C')$ in $Mnd^*(Cat)$, define $\bar{H} : Kl(T) \longrightarrow C'$ on objects by putting $\bar{H}a = Ha$, and on arrows by sending $f : a \longrightarrow Tb$ to the composite $h_b Hf$. The coherence axioms force $\bar{H}$ to be a functor such that $\bar{H}J = H$ and $\bar{H}j = h$.

For unicity, the unicity of objects is immediate, and for arrows, observe that every arrow $f : a \longrightarrow b$ in $Kl(T)$ is a composite in $Kl(T)$ of the image of the arrow $\hat{f} : a \longrightarrow Tb$ in $C$ with the $b$-component of $j$. This fact together with the coherence conditions yields the unicity.

It it routine to verify the 2-dimensional property that every coherent natural transformation $\alpha : (H, h) \Rightarrow (K, k)$ extends uniquely to a natural transformation $\hat{\alpha} : \hat{H} \Rightarrow \hat{K}$: the components are already given, and the extended naturality holds by the coherence. □

Spelling out the action of the 2-functor $Kl : Mnd^*(Cat) \to Cat$ on 1-cells and 2-cells, a 1-cell $(J, j) : (C, T, \mu, \eta) \to (C', T', \mu', \eta')$ is sent to the functor $Kl(J, j) : Kl(T) \to Kl(T')$, which sends an object $a$ of $Kl(T)$ to the object $Ja$ of $Kl(T')$, and an arrow $f : a \to b$ of $Kl(T)$, i.e., an arrow $\hat{f} : a \to Tb$ of $C$, to the arrow of $Kl(T')$ given by $j_b \circ J\hat{f} : Ja \to T'Jb$. A 2-cell $\alpha : (J, j) \Rightarrow (H, h)$ is sent to the natural transformation $Kl(\alpha) : Kl(J, j) \Rightarrow Kl(H, h)$ whose $a$ component is given by ${\eta'}_{Ha} \circ \alpha_a : Ja \to T'Ha$.

The above construction and proof extend readily to the case of $K = V\text{-}Cat$.

In light of these result, we say

**Definition 2.11** A 2-category $K$ *admits Kleisli constructions for monads* if the 2-functor $Inc : K \longrightarrow Mnd^*(K)$ has a left 2-adjoint.

## 3 Comonads in 2-categories

We now turn from monads to comonads. The results we seek about comonads follow from those about monads by consideration of another duality applied to an arbitrary 2-category. Given a 2-category $K$, one may consider two distinct duals: $K^{op}$ as in the previous section and $K^{co}$. The 2-category $K^{co}$ is defined to have the same 0-cells as $K$ but with $K^{co}(X,Y)$ defined to be $K(X,Y)^{op}$.

In $K^{op}$, the 1-cells are reversed, but the 2-cells are not, whereas in $K^{co}$, the 2-cells are reversed but the 1-cells are not. One can of course reverse both 1-cells and 2-cells, yielding $K^{coop}$, or isomorphically, $K^{opco}$.

**Definition 3.1** A *comonad* in $K$ is defined to be a monad in $K^{co}$, i.e., a 0-cell $C$, a 1-cell $D : C \longrightarrow C$, and 2-cells $\delta : D \Rightarrow D^2$ and $\epsilon : D \Rightarrow Id$, subject to the duals of the three coherence conditions in the definition of monad.

Taking $K = Cat$, a comonad in $K$ as we have just defined it is exactly a small category together with a comonad on it.

One requires a little care in defining $Cmd(K)$, the 2-category of comonads in $K$. If one tries to define $Cmd(K)$ to be $Mnd(K^{co})$, then there is no forgetful 2-functor from $Cmd(K)$ to $K$. So we do something a little more subtle.

**Definition 3.2** For a 2-category $K$, define $Cmd(K)$ to be $Mnd(K^{co})^{co}$.

Explicitly, a 0-cell in $Cmd(K)$ is a comonad in $K$. A 1-cell in $Cmd(K)$ from $(C,D,\delta,\epsilon)$ to $(C',D',\delta',\epsilon')$ is a 1-cell $J : C \longrightarrow C'$ in $K$ together with a 2-cell $j : JD \Rightarrow D'J$ subject to two coherence conditions, one relating $\delta$ and $\delta'$, the other relating $\epsilon$ and $\epsilon'$. A 2-cell from $(J,j)$ to $(H,h)$ is a 2-cell in $K$ from $J$ to $H$ subject to one coherence condition relating $j$ and $h$.

Note carefully the definition of a 1-cell in $Cmd(K)$. It consists of a 1-cell and a 2-cell in $K$; of those, the 1-cell goes in the same direction as that in the definition of $Mnd(K)$, but the 2-cell goes in the opposite direction.

Again, there is an underlying 2-functor $U : Cmd(K) \longrightarrow K$, which has a right 2-adjoint given by $Inc : K \longrightarrow Cmd(K)$, sending an object $X$ to the identity comonad on $X$; and again, one may say

**Definition 3.3** A 2-category $K$ *admits Eilenberg-Moore constructions for comonads* if $Inc : K \longrightarrow Cmd(K)$ has a right 2-adjoint.

Although not stated explicitly in [11], it follows routinely that the 2-category $Cat$ admits Eilenberg-Moore constructions for comonads, and they are given by the usual Eilenberg-Moore construction. Again here, the construction $Cmd(K)$ yields a 2-functor $Cmd : 2\text{-}Cat \longrightarrow 2\text{-}Cat$. Also, one may define $Cmd^*(K) = Cmd(K^{op})^{op}$. Since the operations $(\ )^{op}$ and $(\ )^{co}$ commute, we have

**Proposition 3.4** *For any 2-category $K$, $Cmd^*(K) = Mnd^*(K^{co})^{co}$.*

As before, we say

**Definition 3.5** A 2-category $K$ *admits Kleisli constructions for comonads* if $Inc : K \longrightarrow Cmd^*(K)$ has a left 2-adjoint.

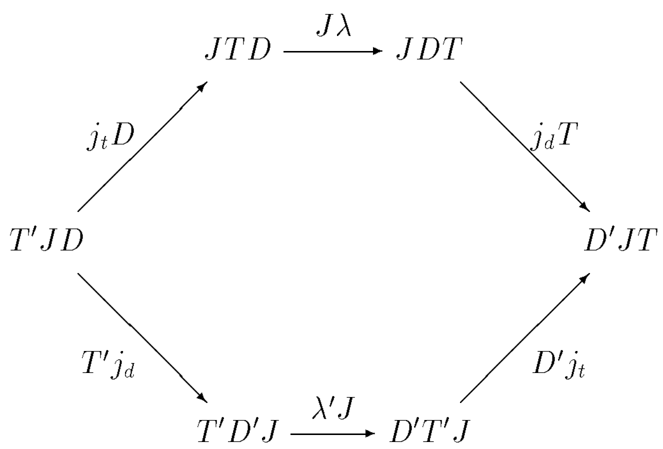
## 4 Distributive laws

In previous sections, we have defined 2-functors $Mnd$, $Mnd^*$, $Cmd$ and $Cmd^*$. So in principle, one might guess that there are eight possible ways of combining a monad and a comonad as there are three dualities: start with the monad or start with the comonad; taking $(\,)^*$ on the monad or not; and likewise for the comonad. In fact, as we shall see, there are precisely six. First we analyse the 2-functor $Cmd\,Mnd$. In order to do that, we give the definition of a distributive law of a monad over a comonad in a 2-category.

**Definition 4.1** Given a monad $(T, \mu, \eta)$ and a comonad $(D, \delta, \epsilon)$ on an object $C$ of a 2-category $K$, a *distributive law of $T$ over $D$* is a 2-cell $\lambda : TD \Rightarrow DT$ such that the four elementary diagrams involving each of $\mu$, $\eta$, $\delta$ and $\epsilon$ commutes.

**Proposition 4.2** *For any 2-category $K$, the 2-category $Cmd\,Mnd(K)$ is isomorphic to the 2-category $Dist(K)$ as follows:*

- *A 0-cell consists of a 0-cell $C$ of $K$, a monad $T$ on it, a comonad $D$ on it, and a distributive law $\lambda : TD \Rightarrow DT$.*
- *A 1-cell $(J, j_t, j_d) : (C, T, D, \lambda) \longrightarrow (C', T', D', \lambda')$ consists of a 1-cell $J : C \longrightarrow C'$ in $K$ together with a 2-cell $j_t : T'J \Rightarrow JT$ subject to the monad laws, together with a 2-cell in $K$ of the form $j_d : JD \Rightarrow D'J$ subject to the comonad laws, all subject to one coherence condition given by a hexagon*




- *A 2-cell from $(J, j_t, j_d)$ to $(H, h_t, h_d)$ consists of a 2-cell from $J$ to $H$ in $K$ subject to two conditions expressing coherence with respect to $j_t$ and $h_t$ and coherence with respect to $j_d$ and $h_d$.*

Thus Proposition 4.2 gives as 0-cells exactly the data considered by Turi and Plotkin [12]. Turi and Plotkin did not, in that paper, address the 1-cells of

Proposition 4.2, but they propose to do so in future. The 0-cells provide them with a combined operational and denotational semantics for a language; the 1-cells allow them to account for the interpretation of one language presented in such a way into another language thus presented. In fact, it was in response to Plotkin's specific proposal about how to do that that much of the work of this paper was done. For a very simple example, one might have a monad and comonad on the category $Set$, and embed it into the category of $\omega$-cpo's in order to add an account of recursion.

**Example 4.3** We give an example of a distributive law for a monad over a comonad. Let $(T, \mu, \eta)$ be the monad on $Set$ sending a set $X$ to the set $X^*$ of finite lists, and let $(D, \delta, \epsilon)$ be the comonad that sends a set $X$ to the set of streams $X^\omega$. Consider the natural transformation $\lambda : TD \Rightarrow DT$ whose $X$ component sends a finite list of streams $\bar{a_1}\bar{a_2}\cdots\bar{a_n}$ with $\bar{a_i} = a_{i1}a_{i2}a_{i3}\cdots, (1 \leq i \leq n)$ to the stream of finite lists $(a_{11}a_{21}\cdots a_{n1})(a_{12}a_{22}\cdots a_{n2})(a_{13}a_{23}\cdots a_{n3})\cdots$. This natural transformation satisfies the axioms for a distributive law of a monad over a comonad. Hence these data give an example of a 0-cell of $CmdMnd(Cat)$ and $Cmd^*Mnd(Cat)$ and $Mnd^*Cmd(Cat)$.

**Corollary 4.4** *$Cmd\,Mnd(K)$ is isomorphic to $Mnd\,Cmd(K)$.*

**Proof.** $Cmd(K)$ is defined to be $Mnd(K^{co})^{co}$ and $Mnd(K) = Cmd(K^{co})^{co}$. So $Mnd\,Cmd(K) = (Cmd\,Mnd(K^{co}))^{co}$. So we may apply Proposition 4.2 to $L = K^{co}$ and apply $(\ )^{co}$ to the result of that, then observe that the corollary follows directly. □

**Theorem 4.5** *Suppose $K$ admits Eilenberg-Moore constructions for monads and comonads. Then, $Inc : K \longrightarrow Cmd\,Mnd(K)$ has a right 2-adjoint.*

**Proof.** Since $K$ admits Eilenberg-Moore constructions for monads, $Inc : K \longrightarrow Mnd(K)$ has a right 2-adjoint. Since $Cmd$ : 2-$Cat \longrightarrow$ 2-$Cat$ is a 2-functor, it sends adjunctions to adjunctions, so $Cmd(Inc) : Cmd(K) \longrightarrow Cmd\,Mnd(K)$ has a right 2-adjoint. Since $K$ admits Eilenberg-Moore constructions for comonads, $Inc : K \longrightarrow Cmd(K)$ has a right adjoint. Composing the right adjoints gives the result. □

This result gives us a universal property for the construction of the category of $(T, D)$-bialgebras, given a monad $T$, a comonad $D$, and a distributive law of $T$ over $D$. In this precise sense, one may see the construction of a category of bialgebras as a generalised Eilenberg-Moore construction.

Using Proposition 4.2 and Corollary 4.4, we may characterise the right 2-adjoint in three ways, giving

**Corollary 4.6** *If $K$ admits Eilenberg-Moore constructions for monads and comonads, then given a distributive law of a monad $(T, \mu, \eta)$ over a comonad $(D, \delta, \epsilon)$, the following are equivalent:*

- *$(T, D) - Bialg$ determined directly by the universal property of a right 2-adjoint to the inclusion $Inc : K \longrightarrow Dist(K)$ sending $X$ to the identity* *distributive law on* $X$

- *the Eilenberg-Moore object for the lifting of* $T$ *to* $D$-*Coalg*
- *the Eilenberg-Moore object for the lifting of* $D$ *to* $T$-*Alg*

By the universal property, the right 2-adjoint ( )-*Bialg* inherits an action on 1-cells and 2-cells. The behaviour of the right 2-adjoint on 0-cells gives exactly the construction ( )-*Bialg* studied by Turi and Plotkin [12]. Its behaviour on 1-cells will be fundamental to their later development as outlined above.

In the example $K = Cat$, the universal property means that the notion of map of distributive laws is determined uniquely by Proposition 4.2 and Corollary 4.4, and the universal property of a right 2-adjoint uniquely induces a functor between the categories of bialgebras.

Turning to the other possibilities, one can deduce from Corollary 4.4

**Corollary 4.7** *$Mnd^* Cmd^*(K)$ is isomorphic to $Cmd^* Mnd^*(K)$.*

Moreover, one can deduce an equivalent result to Proposition 4.2: this yields that the isomorphic 2-categories of Corollary 4.7 amount to giving the opposite distributive law to that given by $Cmd$ and $Mnd$, and hence give an account of Kleisli constructions lifting along Kleisli constructions. The left 2-adjoint to $Inc : K \longrightarrow Mnd^* Cmd^*(K)$ can again be characterised in three ways:

**Corollary 4.8** *If $K$ admits Kleisli constructions for monads and comonads, then given a distributive law of comonad $(D, \delta, \epsilon)$ over a monad $(T, \mu, \eta)$, the following are equivalent:*

- *$Kl(D,T)$ determined directly by the universal property of the inclusion $Inc : K \longrightarrow Dist^*(K)$ sending $X$ to the identity distributive law on $X$*
- *the Kleisli object for the lifting of $T$ to $Kl(D)$*
- *the Kleisli object for the lifting of $D$ to $Kl(T)$*

This is the construction proposed by Brookes and Geva [2] for giving intensional denotational semantics.

The fundamental step in the proof here lies in the use of the proof of Theorem 4.5, and that proof relies upon the following: some mild conditions on $K$ hold of all our leading examples, allowing us to deduce that $K$ admits Eilenberg-Moore and Kleisli constructions for monads and comonads; and each of the constructions $Mnd$, $Mnd^*$, $Cmd$ and $Cmd^*$ is 2-functorial on 2-$Cat$, so preserves adjunctions.

**Proposition 4.9** *When $K = Cat$, the Kleisli construction for monads and comonads exists and is given as follows. Let $(D, \delta, \epsilon)$ be a comonad and $(T, \mu, \eta)$ be a monad on $C$ and $\lambda : DT \to TD$ be a distributive law on $D$ and $T$. Then the objects of $Kl(D,T)$ are the those of $C$. An arrow from $x$ to $y$ in $Kl(D,T)$ is given by an arrow $f : Dx \to Ty$ in $C$. For each object*

*$x$, the identity is given by $\eta_x \circ \epsilon_x$. The composition of arrows $f : x \to y$ and $g : y \to z$ in $Kl(D,T)$ seen as arrows $\hat{f} : Dx \to Ty$ and $\hat{g} : Dy \to Tz$ in $C$ is given by the composite $\mu_z(T\hat{g})\lambda_y(D\hat{f})\delta_x$ in $C$.*

**Proof.** We need only write the image under the left 2-adjoint to $Inc : Cat \to Cmd^* Mnd^*(Cat)$. This left adjoint is given by composing the two left 2-adjoints as in Theorem 4.4, the Kleisli construction of $T$ sent by $Cmd^*$ and the Kleisli construction of $D$. □

Applying those facts to the remaining four possible combinations of a monad with a comonad, $Cmd\,Mnd^*(K)$ gives an account of the Kleisli construction for a monad interacting with coalgebras for a comonad, and similarly for the other examples. So how do $Cmd\,Mnd^*(K)$ and its duals compare? $Cmd\,Mnd^*(K)$ and $Mnd^*\,Cmd(K)$ are different because they have different 0-cells: a 0-cell of the latter is a 0-cell of $Mnd\,Cmd(K) = Cmd\,Mnd(K)$, but a 0-cell of the former is not. So dually, $Cmd^*\,Mnd(K)$ and $Mnd\,Cmd^*(K)$ are different. Similarly, $Cmd^*\,Mnd(K) \neq Cmd\,Mnd^*(K)$, and $Mnd^*\,Cmd(K) \neq Mnd\,Cmd^*(K)$. Moreover, $Mnd\,Cmd^*(K) \neq Cmd\,Mnd^*(K)$ because they have different 1-cells although the same 0-cells. $Mnd^*\,Cmd(K) \neq Cmd^*\,Mnd(K)$ because although they have the same 0-cells, their 1-cells differ. This may be surprising as one might guess that $Cmd\,Mnd^*(K)$ might equal $Mnd^*\,Cmd(K)$, as one gives an account of a coalgebra construction for a comonad applied to a Kleisli construction for a monad, while the latter does the same in the opposite order. The upshot of all this is that there are precisely six, not eight, ways of combining monads and comonads.

We find this result surprising, because in considering the possible ways of combining a pair of categories each with a monad and a comonad, there appear to be three possible independent dualities:

- $TD \Rightarrow DT$ or the dual
- $JT \Rightarrow T'J$ or the dual
- $D'J \Rightarrow JD$ or the dual.

This gives eight possibilities. There are two sorts of 0-cell: given by a monad, a comonad, and one of two possible distributive laws. For each, there are four possible sorts of 1-cell, but not all of these arise from the above mixtures of $Mnd$, $Cmd$, and $(\,)^*$, because some of the eight possibilities there agree, e.g., $Mnd\,Cmd = Cmd\,Mnd$.

Of the eight possibilities, the two that do not arise are

$$TD \Rightarrow DT \;\; JT \Rightarrow T'J \;\; D'J \Rightarrow JD$$

and the complete dual, dualising all three items,

$$DT \Rightarrow TD \;\; T'J \Rightarrow JT \;\; JD \Rightarrow D'J.$$

A complete list appears in Table 1.

The six possibilities yield the six possibilities for liftings: along the Kleisli

Table 1
Distributive laws

| | | | |
|---|---|---|---|
| $Cmd\,Mnd = Mnd\,Cmd$ | $TD \Rightarrow DT$ | $T'J \Rightarrow JT$ | $JD \Rightarrow D'J$ |
| $Cmd^* \, Mnd$ | $TD \Rightarrow DT$ | $T'J \Rightarrow JT$ | $D'J \Rightarrow JD$ |
| $Mnd^* \, Cmd$ | $TD \Rightarrow DT$ | $JT \Rightarrow T'J$ | $JD \Rightarrow D'J$ |
| $Cmd\,Mnd^*$ | $DT \Rightarrow TD$ | $JT \Rightarrow T'J$ | $JD \Rightarrow D'J$ |
| $Mnd\,Cmd^*$ | $DT \Rightarrow TD$ | $T'J \Rightarrow JT$ | $D'J \Rightarrow JD$ |
| $Cmd^*Mnd^* = Mnd^*Cmd^*$ | $DT \Rightarrow TD$ | $JT \Rightarrow T'J$ | $D'J \Rightarrow JD$ |

or the Eilenberg-Moore construction. You can see there that many possibilities disagree as they do not have the same objects. For instance, applying the Eilenberg-Moore construction to the Kleisli construction will always have different objects to doing so in the reverse order.

## References


[1] M. Barr and C. Wells, Toposes, Triples and Theories, Grundlagen der mathematischen Wissenschaften Vol 278, Springer-Verlag, 1985

[2] S. Brookes and S. Geva, Computational comonads and intensional semantics, Proc Durham Conf. Categories in Computer Science 1991, (1993)

[3] M.P. Fiore, Axiomatic domain theory in categories of partial maps, Cambridge University Press Distinguished Dissertations in Computer Science, 1996

[4] J.A. Goguen, J.W. Thatcher, and E.G. Wagner, An initial algebra approach to the specification, correctness and implementation of abstract data types, Current Trends in Programming Methodology Vol 4, R.T. Yet (ed.) Prentice Hall (1978), 80–149

[5] B. Jacobs and J. Rutten, A tutorial on (Co)Algebras and (Co)Induction, EATCS Bulletin 62 (1997), 222–259

[6] P.T. Johnstone, Adjoint lifting theorems for categories of algebras, Bull. London Math. Soc. 7 (1975), 294–297

[7] P.T. Johnstone, A.J. Power, T. Tsujishita, H. Watanabe, and J. Worrell, Proc LICS 98 (1998), 207–213

[8] G.M. Kelly and R. Street, Review of the elements of 2-categories, Sydney Category Seminar, Lecture Notes in Mathematics 420 (1974), 75–103

[9] E. Moggi, Notions of computation and monads, Information and Computation 93 (1991), 55–92

[10] J. Rutten and D. Turi, Initial algebra and final coalgebra semantics for concurrency, Proc. REX workshop A Decade of Concurrency – Reflections and Perspectives, Lecture Notes in Comp Science 803, J. de Bakker et al (eds.) Springer-Verlag (1994), 530–582

[11] R. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (1972), 149–168

[12] D. Turi and G. Plotkin, Towards a mathematical operational semantics, Proc. LICS 97 (1997), 280–291