NOTES ON COMBINATORIAL MODEL CATEGORIES

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These notes were written for a series of lectures on combinatorial model categories at the University of Stuttgart in July 2014 (slightly revised and expanded in 2020). The purpose of the notes is to give an overview of the theory of combinatorial model categories and discuss some of their applications. We assume that the reader is familiar with model categories (see, for example, [11, Chapters 1–2]) and simplicial homotopy theory.

1. Cofibrantly generated model categories

A useful and general way of constructing factorizations of morphisms in homotopical algebra, and consequently also of constructing derived functors, is given by Quillen's *small object argument* (used in [16]). This powerful method is formalized in the following statement.

Theorem 1.1. *Let $\mathcal{C}$ be a cocomplete category and I a set of morphisms whose domains are small relative to* $\mathrm{cell}(I)$*. Then there is a functorial factorization of every morphism in $\mathcal{C}$ into a morphism in* $\mathrm{cell}(I)$ *followed by a morphism in* $\mathrm{inj}(I)$*.*

Proof. See [11, Theorem 2.1.14]. □

If the set of morphisms I satisfies the condition of the theorem, then we say that *I permits the small object argument.* Under this smallness condition on I, it can be shown that the class of morphisms that have the left lifting property with respect to $\mathrm{inj}(I)$ is exactly the cofibrant closure $\mathrm{cof}(I)$ of I, i.e., the closure of I under pushouts, transfinite compositions and retracts. This is a consequence of the small object argument combined with the 'retract argument' [11, Lemma 1.1.9].

The small object argument is used to show the existence of model category structures if the candidate classes of cofibrations and trivial cofibrations are generated by suitable sets of morphisms. Indeed, this often turns out to be case since the classes of trivial fibrations and fibrations are often described by right lifting properties with respect to sets of morphisms which permit the small object argument. This situation is formalized in the notion of a *cofibrantly generated model category.*

Definition 1.2. A model category $\mathcal{M}$ is called *cofibrantly generated* if there exist sets of morphisms I and J such that the following hold:

(i) I and J permit the small object argument.
(ii) $\mathrm{inj}(J)$ is the class of fibrations; $\mathrm{inj}(I)$ is the class of trivial fibrations.

The set I is a called a set of *generating cofibrations* and J a set of *generating trivial cofibrations.*

Equivalently, a model category is cofibrantly generated if the classes of cofibrations and trivial cofibrations are cofibrantly generated (i.e. they are the cofibrant closures of sets of morphisms) and the required smallness condition (i) of the definition is satisfied.

We have the following useful and widely applied *recognition theorem for cofibrantly generated model structures.*

Theorem 1.3. *Let $\mathcal{C}$ be a category which admits all small limits and colimits, $\mathcal{W} \subseteq \mathcal{C}$ a subcategory, and let I and J be two sets of morphisms in $\mathcal{C}$. Suppose that the following conditions are satisfied:*

1. *$\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts.*
2. *I and J permit the small object argument.*
3. $\mathrm{cell}(J) \subseteq \mathrm{cof}(I) \cap \mathcal{W}$.
4. $\mathrm{inj}(I) \subseteq \mathrm{inj}(J) \cap \mathcal{W}$.
5. *Either* $\mathrm{cof}(I) \cap \mathcal{W} \subseteq \mathrm{cof}(J)$ *or* $\mathrm{inj}(J) \cap \mathcal{W} \subseteq \mathrm{inj}(I)$.

Then there is a cofibrantly generated model category structure on $\mathcal{C}$ with weak equivalences $\mathcal{W}$, and I and J as the sets of generating cofibrations and trivial cofibrations respectively.

Proof. See [11, Theorem 2.1.19]. □

An important application of this theorem is the construction of model structures on diagram categories when the underlying model category is cofibrantly generated.

Example 1.4. (Projective model structure) Let $\mathcal{M}$ be a cofibrantly generated model category and let C be a small category. It is easy to see that the product model category $\mathcal{M}^{\mathrm{Ob}(C)}$ is cofibrantly generated. Sets of generating cofibrations $I_{\mathrm{Ob}(C)}$ and generating trivial cofibrations $J_{\mathrm{Ob}(C)}$ can be given by placing the morphisms of I and J in each one of the components of the product while the rest of the components have trivial entries. There is an adjunction:

$$u_!\colon \mathcal{M}^{\mathrm{Ob}(C)} \leftrightarrows \mathcal{M}^C : u^*$$

where u^* is the restriction functor along the inclusion $u\colon \mathrm{Ob}(C) \to C$. Then the recognition theorem applies easily to endow $\mathcal{M}^C$ with a cofibrantly generated model structure where the weak equivalences are defined pointwise and $I_C\colon = u_!(I_{\mathrm{Ob}(C)})$ and $J_C\colon = u_!(J_{\mathrm{Ob}(C)})$ are sets of generating cofibrations and trivial cofibrations respectively. By adjunction, it follows that the fibrations in $\mathcal{M}^C$ are also defined pointwise.

With respect to this model structure, every functor $u\colon C \to D$ gives rise to a Quillen adjunction

$$u_!\colon \mathcal{M}^C \leftrightarrows \mathcal{M}^D : u^*.$$

In particular, this shows the existence of the left derived homotopy Kan extension $\mathbb{L}u_!\colon \mathrm{Ho}(\mathcal{M}^C) \to \mathrm{Ho}(\mathcal{M}^D)$ along any functor u.

A general difficulty which arises when one tries to apply the recognition theorem is that a set of generating trivial cofibrations J is often difficult to specify. This happens, for example, in the construction of left Bousfield localizations of model categories. The analogue of the recognition theorem for combinatorial model categories avoids the assumption of an explicit set J and its existence is formally deduced from special accessibility properties.

The notion of a combinatorial model category introduces the following additional assumptions: (a) every object in $\mathcal{M}$ is assumed to be presentable, and as consequence, every set of morphisms permits the small object argument, and (b) the underlying category is generated in a suitable sense by a small subcategory. These assumptions are practically convenient and widely applicable – most model categories of interest are combinatorial (up to Quillen equivalence). The theory of combinatorial model categories rests crucially on fundamental results about accessible and locally presentable categories, while at the same time the theory extends these results from ordinary category theory to homotopical algebra.

2. Accessible and locally presentable model categories

We review some basic definitions and facts about locally presentable categories. The concept was introduced by Gabriel and Ulmer in [8]. See also [1] for a detailed account of the theory.

Let λ be a regular cardinal, i.e. an infinite cardinal which is not a sum of a smaller number of smaller cardinals. For example, $\aleph_0$ and $\aleph_1$ are regular. A poset is called λ-*directed* if every subset of cardinality smaller than λ has an upper bound.

2.1. Basic definitions and examples.

Let $\mathcal{C}$ be a (locally small) category. An object X of $\mathcal{C}$ is called λ-*presentable* if the corepresentable functor

$$\mathcal{C}(X, -) : \mathcal{C} \to \mathcal{S}et$$

preserves λ-directed colimits. An object X is called *presentable* if it is λ-presentable for some λ. If X is λ-presentable and $\lambda' > \lambda$, then X is also λ'-presentable. The colimit of a diagram $F : I \to \mathcal{C}$ such that I is λ-small ($= |\mathrm{Mor}(I)| < \lambda$), and $F(i)$ is λ-presentable for all $i \in \mathrm{Ob}(I)$, is also λ-presentable [1, 1.16].

Example 2.1. In the category of sets, X is λ-presentable if its cardinality is less than λ. In the category of groups, a group G is $\aleph_0$-presentable (= finitely presentable) if it admits a finite presentation. In the category of presheaves $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$, the representable functors are finitely (= $\aleph_0$-)presentable objects. In the category of topological spaces, only the discrete spaces are presentable.

Let $\mathcal{C}$ be a category and $\mathcal{A}$ a small full subcategory. For every object X in $\mathcal{C}$, there is a comma category $\mathcal{A} \downarrow X$. Its objects are the morphisms $f : A \to X$ where $A \in \mathrm{Ob}(\mathcal{A})$ and a morphism from $(f : A \to X)$ to $(f' : A' \to X)$ is given by a morphism $g : A \to A'$ such that $f = f'g$. The diagram

$$\underline{X} : \mathcal{A} \downarrow X \to \mathcal{C}$$

that forgets the morphism to X is called the *canonical diagram* of X with respect to $\mathcal{A}$. X is a *canonical colimit of $\mathcal{A}$-objects* if the canonical diagram has a colimit with colimit-object X and colimit-maps $f : \underline{X}(f : A \to X) \to X$. A small full subcategory $\mathcal{A}$ of $\mathcal{C}$ is called *dense* if every object X is a canonical colimit of $\mathcal{A}$-objects. The objects of a dense subcategory form a *strong generator*[1] of $\mathcal{C}$. $\mathcal{C}$ is called *bounded* if it has a dense subcategory. (We use the convention that a dense subcategory is assumed to be small by definition.)

Definition 2.2. A category $\mathcal{C}$ is called *λ-accessible* if it is closed under λ-directed colimits and has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{A}$. A category $\mathcal{C}$ is called *accessible* if it is λ-accessible for some λ.

In a λ-accessible category, there exists only a set of λ-presentable objects, up to isomorphism, and this defines a dense subcategory [1, 2.2(4) and 2.8].

Definition 2.3. A category $\mathcal{C}$ is called *locally λ-presentable* if it is cocomplete and λ-accessible. A category $\mathcal{C}$ is called *locally presentable* if it is locally λ-presentable for some λ.

An equivalent characterization is that $\mathcal{C}$ is cocomplete and has a strong generator consisting of λ-presentable objects [1, 1.20].

The canonical diagram of $X \in \mathcal{C}$ in a locally λ-presentable category $\mathcal{C}$ with respect to the (essentially) small full subcategory of λ-presentable objects is λ-filtered, rather than λ-directed. For our purposes, the difference between λ-filtered and λ-directed is not significant (see the discussion in [1, 1.21] for more details).

Example 2.4. The categories of sets, R-modules, and presheaves (of sets) are locally finitely presentable.

Grothendieck topoi are locally presentable. The idea of the proof is as follows: By Giraud's theorem, a Grothendieck topos is a full reflective subcategory of a presheaf category where the left adjoint (= sheafification functor) preserves finite limits. It follows that the corresponding subcategory of sheaves is closed under λ-filtered colimits for some λ (indeed the sheaf condition can be stated as an orthogonality condition in terms of the sieves of the Grothendieck topology). Then the sheafification functor preserves λ-presentable objects. In particular, the sheaves associated to the representable presheaves are λ-presentable and form a strong generator – since they do so in the presheaf category.

Grothendieck abelian categories (i.e. cocomplete abelian categories with a generator satisfying AB5) are also locally presentable. They may be regarded as additive analogues of Grothendieck topoi and an analogous argument as above shows that they are locally presentable – using the Gabriel-Popescu theorem instead of Giraud's theorem (see [3, Proposition 3.10]).

[1] A set of objects S in a category $\mathcal{C}$ defines a strong generator if the corepresentable functors $\mathcal{C}(A, -) \colon \mathcal{C} \to \mathcal{S}et$, $A \in S$, are jointly faithful and jointly isomorphism–reflecting.

2.2. **Structural properties.** We would like to understand better how a locally λ-presentable category is generated by its (essentially) small full subcategory of λ-presentable objects. This will lead us to a nice characterization of locally presentable categories.

Let $\mathcal{A}$ be a small category. The category $\mathcal{S}et^{\mathcal{A}^{op}}$ of presheaves on $\mathcal{A}$, together with the Yoneda embedding $Y : \mathcal{A} \to \mathcal{S}et^{\mathcal{A}^{op}}$, is characterized by the following property: $\mathcal{S}et^{\mathcal{A}^{op}}$ is cocomplete and for every cocomplete category $\mathcal{K}$, the Yoneda embedding induces an equivalence between the category of cocontinuous functors $\mathcal{S}et^{\mathcal{A}^{op}} \to \mathcal{K}$ and the category of functors $\mathcal{A} \to \mathcal{K}$. That is, for every functor $F : \mathcal{A} \to \mathcal{K}$ to a cocomplete category $\mathcal{K}$, there exists a canonical cocontinuous extension $F^* : \mathcal{S}et^{\mathcal{A}^{op}} \to \mathcal{K}$ which is unique up to natural isomorphism. This is essentially a reformulation of the fact that every object in $\mathcal{S}et^{\mathcal{A}^{op}}$ is a canonical colimit of representable functors. Thus, the category of presheaves on $\mathcal{A}$ can be regarded as the cocomplete category which is obtained by adding freely all small colimits to $\mathcal{A}$.

Let $\mathrm{Cont}_\lambda(\mathcal{A})$ denote the full subcategory of $\mathcal{S}et^{\mathcal{A}^{op}}$ with objects the set-valued presheaves on $\mathcal{A}$ that preserve all λ-small colimits that exist in $\mathcal{A}$. Note that the representable presheaves satisfy this property. Moreover, the Yoneda embedding $\mathcal{A} \to \mathrm{Cont}_\lambda(\mathcal{A})$ satisfies the following universal property of λ-*free cocompletion*: every functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is cocomplete and F preserves λ-small colimits, has a cocontinuous extension $F^\lambda : \mathrm{Cont}_\lambda(\mathcal{A}) \to \mathcal{B}$, unique up to natural isomorphism (see [1, 1.45]).

Theorem 2.5. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(a) *$\mathcal{C}$ is a locally λ-presentable category.*
(b) *$\mathcal{C}$ is cocomplete and has a dense subcategory consisting of λ-presentable objects.*
(c) *$\mathcal{C}$ is a full reflective subcategory of a presheaf category $\mathcal{S}et^{\mathcal{A}^{op}}$ closed under λ-directed colimits, for some small category $\mathcal{A}$.*
(d) *$\mathcal{C}$ is equivalent to $\mathrm{Cont}_\lambda(\mathcal{A})$ for some small category $\mathcal{A}$.*

Proof. See [1, 1.46]. (a) $\Rightarrow$ (b): it is easy to see that the (essentially) small subcategory of λ-presentable objects is dense (see [1, 1.22]). (b) $\Rightarrow$ (c): let $\mathcal{A}$ be a dense small subcategory of λ-presentable objects. There is a functor $\mathcal{Y}_\mathcal{A} : \mathcal{C} \to \mathcal{S}et^{\mathcal{A}^{op}}$ defined on objects by

$$C \mapsto (A \mapsto Hom_\mathcal{C}(A, C))$$

Notice that $\mathcal{A}$ is dense if and only if $\mathcal{Y}_\mathcal{A}$ is full and faithful [1, 1.26]. Furthermore, $\mathcal{Y}_\mathcal{A}$ preserves λ-directed colimits if and only if every object in $\mathcal{A}$ is λ-presentable in $\mathcal{C}$. $\mathcal{Y}_\mathcal{A}$ has a left adjoint which is the cocontinuous extension of the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$. (c) $\Rightarrow$ (a) is left as an exercise. (a) $\Rightarrow$ (d): for $\mathcal{A}$ the (essentially) small dense subcategory of λ-presentable objects, the essential image of the full embedding $\mathcal{Y}_\mathcal{A} : \mathcal{C} \to \mathcal{S}et^{\mathcal{A}^{op}}$ is the full subcategory $\mathrm{Cont}_\lambda(\mathcal{A})$ [1, 1.46]. (d) $\Rightarrow$ (c): $\mathrm{Cont}_\lambda(\mathcal{A})$ can be described as a λ-orthogonality class in $\mathcal{S}et^{\mathcal{A}^{op}}$ (see [1, 1.46]). These define full reflective subcategories by [1, 1.39]. ∎

Corollary 2.6. *Every locally presentable category is complete and well-powered.*

Proof. This follows from Theorem 2.5(c). □

2.3. **Accessible functors.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called λ-*accessible* if $\mathcal{C}$ and $\mathcal{D}$ are λ-accessible and F preserves λ-directed colimits. A functor F is called *accessible* if it is λ-accessible for some regular cardinal λ. According to the *Uniformization Theorem* (see [1, 2.19]), for every accessible functor $F : \mathcal{C} \to \mathcal{D}$, there is a regular cardinal λ such that F is λ-accessible and preserves λ-presentable objects.

A full subcategory $\mathcal{B}$ of an accessible category $\mathcal{C}$ is called *accessibly embedded* if it is closed under λ-directed colimits in $\mathcal{C}$ for some λ. The following fact is often very useful.

Proposition 2.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be an accessible functor and $\mathcal{D}'$ an accessible, accessibly embedded subcategory of $\mathcal{D}$. Then $F^{-1}(\mathcal{D}')$ is accessible and accessibly embedded in $\mathcal{C}$.*

Proof. See [1, 2.50]. □

It is difficult in general to determine the rank of accessibility of $F^{-1}(\mathcal{D}')$, even when the accessibility ranks of F and of the inclusion functor $\mathcal{D}' \hookrightarrow \mathcal{D}$ are known. An estimate for this accessibility rank is provided by the following fundamental theorem – this theorem can be used to give a proof of Proposition 2.7.

Theorem 2.8 (Pseudopullback Theorem)**.** *Let λ be a regular cardinal and let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be λ-accessible categories which admit κ-filtered colimits for some $\kappa < \lambda$. Let $F : \mathcal{K} \to \mathcal{L}$ and $G : \mathcal{M} \to \mathcal{L}$ be functors which preserve κ-filtered colimits and λ-presentable objects. Consider the pseudopullback*

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{K} & \xrightarrow{F} & \mathcal{L} \end{array}$$

Then $\mathcal{P}$ is λ-accessible and has κ-filtered colimits.

Proof. The statement can be found in [18]. See [1, 14] for more details about limits of accessible categories. □

The following result relates accessibility with the solution–set condition. This connection is used in the theory of combinatorial model categories in order to assert the existence of generating sets.

Theorem 2.9 (Hu–Makkai)**.** *Let $\mathcal{C}$ be an accessible category and $\mathcal{B}$ an accessibly embedded subcategory. Then $\mathcal{B}$ is accessible if and only if it is cone–reflective in $\mathcal{C}$.*

Proof. See [1, 2.53]. □

We recall that *cone–reflective* here means that for every object C of $\mathcal{C}$, there is a set of morphisms $g_i : C \to B_i$, with B_i in $\mathcal{B}$, such that every morphism $g : C \to B$, with B in $\mathcal{B}$, factors as the composition of some g_i followed by a morphism in $\mathcal{B}$.

2.4. **Vopěnka's principle.** Vopěnka's large–cardinal principle has many strong connections with the theory of locally presentable categories. They are investigated in detail in [1, Chapter 6]. Among the many equivalent formulations of this axiom here is perhaps one of the most basic ones. Recall that a *graph* is simply a set endowed with a binary relation and a morphism of graphs is a function (of sets) that preserves the binary relation.

Vopěnka's principle. For every large collection of graphs there is a non-identity morphism between two graphs in the collection.

We refer to [1, Appendix] for the set–theoretical status of this axiom and its relation to other large–cardinal axioms. We mention the following implications of Vopěnka's principle in the theory of locally presentable and accessible categories:

(1) a category $\mathcal{C}$ is locally presentable if and only if it is cocomplete and bounded (= has a dense subcategory) [1, 6.14],
(2) a full subcategory $\mathcal{A}$ of an accessible category $\mathcal{C}$ is accessible if and only if it is accessibly embedded in $\mathcal{C}$ [1, 6.17],
(3) every full subcategory $\mathcal{A}$ of a locally presentable category $\mathcal{C}$, that is closed under limits in $\mathcal{C}$, is reflective in $\mathcal{C}$ [1, 6.22],
(4) every full subcategory $\mathcal{A}$ of a locally presentable category $\mathcal{C}$, that is closed under colimits in $\mathcal{C}$, is coreflective in $\mathcal{C}$ [1, 6.28].

Moreover, each one of the statements (1), (2), and (4) is actually equivalent to Vopěnka's principle – (3) is equivalent to *weak Vopěnka's principle.*

3. Combinatorial model categories

Definition 3.1 (Smith)**.** A model category $\mathcal{M}$ is called *combinatorial* if it is cofibrantly generated and the underlying category of $\mathcal{M}$ is locally presentable.

Example 3.2. The projective model category $\mathcal{SS}et^C$ (Example 1.4) is combinatorial.

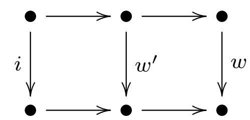
3.1. **The recognition theorem and other properties.** The following is a useful recognition theorem for combinatorial model categories.

Theorem 3.3 (Smith)**.** *Let $\mathcal{C}$ be a locally presentable category, $\mathcal{W} \subseteq \mathcal{C}$ a subcategory, and let I be a set of morphisms in $\mathcal{C}$. Suppose that the following conditions are satisfied:*

1. *$\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts.*
2. *$\mathcal{W}$ satisfies the solution-set condition at I, i.e., for each $i \in I$, there is a set of morphisms $\mathcal{W}_i \subseteq \mathcal{W}$ such that for every commutative square*

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w} \\ \bullet & \longrightarrow & \bullet \end{array}$$

where $w \in \mathcal{W}$, there is a factorization

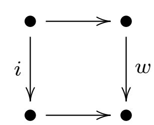


where $w' \in \mathcal{W}_i$.

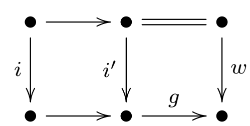
3. $\mathrm{cof}(I) \cap \mathcal{W}$ *is closed under pushouts and transfinite compositions.*
4. $\mathrm{inj}(I) \subseteq \mathcal{W}$.

Then $(\mathcal{C}, \mathcal{W}, \mathrm{cof}(I), \mathrm{inj}(\mathrm{cof}(I) \cap \mathcal{W}))$ is a combinatorial model category.

Proof. A proof can be found in [3, Theorem 1.7]. This proof can be simplified slightly as follows. First, following [3, Lemma 1.9], we find a set of morphisms $J \subset \mathrm{cof}(I) \cap \mathcal{W}$ which has the same property as the set of morphisms $\mathcal{W}_i$ in condition (2). This set J consists of morphisms which are obtained from commutative squares

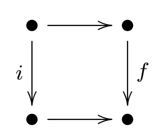


with $i \in I$ and $w \in \mathcal{W}_i$, in the following way: let i' be the pushout of i,

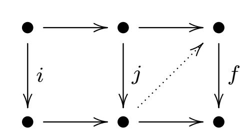


and let $g = qh$ be a factorization into $h \in \mathrm{cof}(I)$ and $q \in \mathrm{inj}(I)$ – using the small object argument (Theorem 1.1). Then $q \in \mathcal{W}$ – using (4) – and hence also $j := hi' \in \mathrm{cof}(I) \cap \mathcal{W}$ – using (1). The morphisms j that arise in this way define the set J. This is our candidate set of generating trivial cofibrations.

Then we apply the recognition theorem for cofibrantly generated model categories (Theorem 1.3). As most of the conditions of Theorem 1.3 are easy to verify, it suffices to show that $\mathrm{inj}(J) \cap \mathcal{W} \subseteq \mathrm{inj}(I)$. Let $f \in \mathrm{inj}(J) \cap \mathcal{W}$ and consider a lifting problem



where $i \in I$. Then there is a factorization



where $j \in J$ and the dotted arrow exists because $f \in \mathrm{inj}(J)$. Thus, the initial square also admits a lift. □

Condition (2) of Theorem 3.3 is the most mysterious and difficult to verify in practice. From earlier results (Theorem 2.9), we know that

> if the full subcategory $\mathcal{W} \subseteq \mathcal{C}^{\to}$ is accessible and accessibly embedded, then $\mathcal{W}$ is cone-reflective.

In particular, in this case, $\mathcal{W}$ satisfies the solution-set condition at every morphism. This means that *Condition (2) can be replaced by the following condition:*

> 2^*. The full subcategory $\mathcal{W} \subseteq \mathcal{C}^{\to}$, spanned by the morphisms in $\mathcal{W}$ (*weak equivalences*), is accessible and accessibly embedded in $\mathcal{C}^{\to}$.

In this case, $\mathcal{W}$ is automatically closed under retracts. This stronger condition (2^*) turns out to be also necessary for combinatorial model categories.

Theorem 3.4. *Let $\mathcal{M}$ be a combinatorial model category. Then the full subcategory of weak equivalences $\mathcal{W} \subseteq \mathcal{M}^{\to}$ is accessible and accessibly embedded in $\mathcal{M}^{\to}$. In particular, $\mathcal{W}$ satisfies the solution-set condition at every morphism of $\mathcal{M}$.*

Proof. Proofs can be found in [13, Corollary A.2.6.8], [19, Theorem 4.1], [17], and [18]. The idea is to express $\mathcal{W}$ as a pullback of accessible categories and apply the Pseudopullback Theorem (Theorem 2.8). For example, the following is a (pseudo)pullback:

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{F} \cap \mathcal{W} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\to} & \xrightarrow{R} & \mathcal{M}^{\to} \end{array}$$

where $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{M}^{\to}$ denotes the full subcategory of trivial fibrations, and $R : \mathcal{M}^{\to} \to \mathcal{M}^{\to}$ is the functor which replaces a morphism by a fibration using the small object argument. The subcategory $\mathcal{F} \cap \mathcal{W}$ is accessible and accessibly embedded (see [18, Section 2] or [13, Corollary A.2.6.6]) and R is accessible. □

Remark 3.5. Moreover, for every combinatorial model category $\mathcal{M}$, there is an accessible functor $F : \mathcal{M} \to \mathcal{SS}et$ such that a morphism f in $\mathcal{M}$ is a weak equivalence if and only if $F(f)$ is a weak equivalence of simplicial sets [17], [18, Remark 2.4]. This also shows the accessibility of the weak equivalences of $\mathcal{M}$ using the accessibility of the weak equivalences of simplicial sets (see [3], [17]).

The problem of finding a good estimate for the accessibility rank of weak equivalences was studied in [18]. We recall that a model category $\mathcal{M}$ is called λ-*combinatorial* if it has generating sets I and J between λ-presentable objects and its underlying category is locally λ-presentable. A *Cisinski model category* is a combinatorial model category whose underlying category is a Grothendieck topos and the cofibrations are exactly the monomorphisms.

Theorem 3.6. *Let $\mathcal{M}$ be a Cisinski model category. Suppose that $\mathcal{M}$ is a κ-combinatorial model category and there is a cylinder functor which preserves κ-presentable objects. Then the full subcategory of weak equivalences $\mathcal{W} \subseteq \mathcal{M}^{\to}$ is κ-accessible.*

Proof. This is a special case of [18, Theorem 4.2]. We refer to [18] for further and stronger results in this direction. □

Corollary 3.7. *The full subcategory of weak equivalences $\mathcal{W} \subset \mathcal{SS}et^{\rightarrow}$ is finitely accessible.*

Remark 3.8. We do not know whether the class of quasi-isomorphisms of chain complexes is finitely accessible in general (cf. [18, Corollary 5.3]).

Applications 3.9. (1) Homological localizations of spaces. Let $h_* : \mathcal{SS}et \to \mathcal{A}b^{\mathbb{Z}}$ be a generalized homology theory satisfying the limit axiom. Then h_* is accessible and therefore the class of h_*-equivalences is accessible and accessibly embedded in $\mathcal{SS}et^{\rightarrow}$. The recognition theorem for combinatorial model categories (Theorem 3.3) can be applied to show that $\mathcal{SS}et$ carries a combinatorial model structure where the cofibrations are the monomorphisms and the weak equivalences are the h_*-equivalences. This model structure was first constructed by Bousfield [4].

(2) (Beke [3, Proposition 3.13]) Let $\mathcal{A}$ be a Grothendieck abelian category. The category of unbounded chain complexes $Ch(\mathcal{A})$ is (again) locally presentable. The homology functor $H_* : Ch(\mathcal{A}) \to \mathcal{A}b^{\mathbb{Z}}$ is accessible, hence the class of quasi-isomorphisms is accessible and accessibly embedded in $Ch(\mathcal{A})^{\rightarrow}$. The class of monomorphisms in $Ch(\mathcal{A})$ is cofibrantly generated (see [3, Proposition 1.12] for a general method). Then the recognition theorem for combinatorial model categories (Theorem 3.3) and basic homological algebra show there is a (injective) combinatorial model structure on $Ch(\mathcal{A})$ where the cofibrations are the monomorphisms and the weak equivalences are the quasi-isomorphisms.

(3) (Injective model category) Let $\mathcal{M}$ be a combinatorial model category and C a small category. The class of pointwise weak equivalences $\mathcal{W}_C$ of C-diagrams in $\mathcal{M}$ is accessible and accessibly embedded. (This can be shown by applying Theorem 3.4 to the projective model structure of Example 1.4.) Therefore for every set of pointwise cofibrations I which is large enough so that $\mathrm{inj}(I) \subseteq \mathcal{W}_C$, there is a combinatorial model structure on $\mathcal{M}^C$ with $\mathcal{W}_C$ as the class of weak equivalences and I as a set of generating cofibrations. We note that the class of pointwise cofibratios is indeed cofibrantly generated (see [13, Lemma A.2.8.3]).

See also [2] (e.g., [2, Theorem 4.38]) for further interesting applications and examples of combinatorial model categories.

4. Small Presentations of Combinatorial Model Categories

4.1. **Bousfield localization.** Let $\mathcal{M}$ be a model category and S a class of morphisms in $\mathcal{M}$. An object X is called *S-local* if it is fibrant and for every $f : A \to B$ in S, the induced map

$$f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is a weak equivalence of simplicial sets. Here $\mathrm{Map}(B, X)$ denotes the homotopy function complex (or derived mapping space), which can be defined functorially in every model category – not just a simplicial one. We recall that one way of defining $\mathrm{Map}(B, X)$ is by choosing a (functorial) cosimplicial resolution $\widetilde{B}^\bullet$ of B (= a cofibrant replacement of the constant cosimplicial object at B in the Reedy model category of cosimplicial objects in $\mathcal{M}$) . Then $\mathrm{Map}(B, X)$ is the simplicial set whose n-simplices are the morphisms in $\mathcal{M}$ from $\widetilde{B}^n$ to (the fibrant object) X. The connected components of this mapping spaces correspond to the morphisms $B \to X$ in the homotopy category of $\mathcal{M}$. We refer to [10] for a detailed account of the construction and general properties of homotopy function complexes.

Example 4.1. Let $\mathcal{T}op_*$ be the model category of based topological spaces, and $S = \{q : S^n \to *\}$. A based space X is S-local (or q-local) if and only if $\pi_k(X) = 0$ for all $k \geqslant n$.

A morphism $g : X \to Y$ is called an *S-local equivalence* if for every S-local object Z, the induced map

$$g^* : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a weak equivalence of simplicial sets.

Definition 4.2. The (left) Bousfield localisation of $\mathcal{M}$ at S is a model category $L_S\mathcal{M}$ with the following properties.

(1) It has the same underlying category as $\mathcal{M}$.
(2) The weak equivalences $\mathcal{W}_S$ of $L_S\mathcal{M}$ are the S-local equivalences,
(3) The cofibrations of $L_S\mathcal{M}$ are the cofibrations of $\mathcal{M}$. These are also called S-cofibrations.
(4) The fibrations of $L_S\mathcal{M}$ are the morphisms that have the right lifting property with respect to the trivial S-cofibrations (= morphisms that are both cofibrations and S-local equivalences). These are called S-fibrations.

Proofs of the following statements can be found in [10].

Proposition 4.3 (basic properties of $\mathcal{W}_S$)**.** Every weak equivalence of $\mathcal{M}$ is also a weak equivalence of $L_S\mathcal{M}$. The class of S-local equivalences has the "2-out-of-3" property and is closed under retracts. An S-local equivalence between S-local objects is a weak equivalence. If $g : C \to D$ is a cofibration and an S-local equivalence, then so is any pushout of g assuming that $\mathcal{M}$ is left proper.

Proposition 4.4 (basic properties of $L_S\mathcal{M}$)**.** The identity functor $\mathrm{Id}_\mathcal{M} : \mathcal{M} \to L_S\mathcal{M}$ is a left Quillen functor. The Quillen adjunction $\mathrm{Id}_\mathcal{M} : \mathcal{M} \rightleftarrows L_S\mathcal{M} : \mathrm{Id}_\mathcal{M}$ induces a reflection on the homotopy categories. Moreover, the Bousfield localization (when it exists) has the following universal property: for every left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ that maps a cofibrant replacement of every map in S to a weak equivalence, the functor $F : L_S\mathcal{M} \to \mathcal{N}$ is also left Quillen.

We now discuss the problem of the existence of Bousfield localizations. Axioms (M1–M3) are obvious. Since fibrations in $L_S\mathcal{M}$ are defined implicitly as the morphisms that have a certain right lifting property, half of (M4) is obviously satisfied.

The factorizations into (cofibration, trivial fibration) are the same as in the model category $\mathcal{M}$. The main difficulty is to prove the existence of factorizations into (trivial cofibration, fibration). The natural strategy is to apply the small object argument with respect to an adequate set of trivial S-cofibrations. The problem is exactly finding such a set which detects S-fibrations (or generates trivial S-cofibrations). At this point, some set-theoretical arguments are usually required – in the case of combinatorial model categories, these can be formulated in terms of accessibility properties.

Remark 4.5. Suppose, for simplicity, that $\mathcal{M}$ is a simplicial model category and S is a set of cofibrations between cofibrant objects (which are also small, in an appropriate sense). Then there is a natural choice of a set of trivial S-cofibrations:

$$S' = \{A \otimes \Delta^n \bigcup_{A \otimes \partial\Delta^n} B \otimes \partial\Delta^n \to B \otimes \Delta^n \mid (A \to B) \in S, n \geqslant 0\}.$$

This set of morphisms detects S-local objects. Moreover, it is sufficient for the construction of the homotopy idempotent functor of S-localization on $\mathcal{M}$ by applying the small object argument to the canonical morphisms $X \to 1$ (where 1 denotes a terminal object of $\mathcal{M}$). However, this does not guarantee the required factorizations, since the set of morphisms S' does not characterize S-fibrations in $L_S\mathcal{M}$ in general.

Hirschhorn [10] introduced the notion of a *cellular model category* as a special type of cofibrantly generated model category with additional properties inspired by certain formal properties of the model category $\mathcal{T}op$ of topological spaces. The model categories $\mathcal{S}\mathcal{S}et$, $\mathcal{T}op$ and (projective or injective) $\mathcal{S}\mathcal{S}et^C$ are cellular. Note that $\mathcal{T}op$ is not a combinatorial model category.

Theorem 4.6 (Hirschhorn [10])**.** *Let $\mathcal{M}$ be a left proper, cellular model category and S a set of maps. Then the left Bousfield localization $L_S\mathcal{M}$ exists and is left proper and cellular. The fibrant objects of $L_S\mathcal{M}$ are the S-local objects. If $\mathcal{M}$ is a simplicial model category, then so is $L_S\mathcal{M}$ with the same simplicial structure.*

4.2. **Dugger's theorem.** Theorem 2.5 characterized locally presentable categories as accessibly embedded reflective subcategories (or orthogonality classes) of presheaf categories. There is an analogous characterization for (model categories which are Quillen equivalent to) combinatorial model categories due to Dugger [6].

Theorem 4.7 (Dugger [6])**.** *Let $\mathcal{M}$ be a combinatorial model category. Then there is a small category $\mathcal{A}$, a set S of morphisms in the projective model category $\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}}$, and a Quillen equivalence*

$$L_S\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}} \rightleftarrows \mathcal{M}.$$

The idea of the proof is the same as for the case of ordinary locally presentable categories but carrying it out is considerably more complicated. We give below an outline of the proof and refer to [6] for more details.

The first part of the proof shows the existence of a *homotopically surjective map*

$$F : \mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}} \leftrightarrows \mathcal{M} : G.$$

This means a left Quillen functor F whose left derived functor is a reflection onto the homotopy category of $\mathcal{M}$. A left adjoint F as above is uniquely determined by its restriction to the representable presheaves:

$$f : \mathcal{A} \times \Delta \to \mathcal{M}.$$

Moreover, the cocontinuous extension of f is a left Quillen functor if and only if f is pointwise in $\mathcal{A}$ a Reedy cofibrant replacement of a cosimplicially constant object in the Reedy model category of cosimplicial objects in $\mathcal{M}$. In other words, for every $X \in \mathcal{A}$, the cosimplicial object $f(X, [\bullet]) \in c\mathcal{M}$ is a *cosimplicial resolution of* $f(X, [0])$ (= Reedy cofibrant and homotopically constant). Moreover, we have a weak equivalence

$$F^{\mathrm{cof}}(G(X)) \simeq \mathrm{hocolim}(\mathcal{A} \times \Delta \downarrow X)$$

for every fibrant object X and the derived counit transformation of the Quillen adjunction (F, G) is identified with the canonical map

$$\mathrm{hocolim}(\mathcal{A} \times \Delta \downarrow X) \to X.$$

(F, G) is homotopically surjective if and only if the derived counit transformation is a weak equivalence for every fibrant object X. (The superscript "cof" denotes cofibrant replacement.)

The proof of Theorem 4.7 shows that this can be achieved if we choose $\mathcal{A}$ to be the (essentially) small full subcategory $\mathcal{M}_\lambda^{\mathrm{cof}}$ of $\mathcal{M}$ consisting of the cofibrant λ-presentable objects for a sufficiently large regular cardinal λ. Let us assume for simplicity that $\mathcal{M}$ is a simplicial model category (so that, among other things, cosimplicial resolutions are easier to construct). Then we consider the left Quillen functor F specified by

$$f \colon \mathcal{M}_\lambda^{\mathrm{cof}} \times \Delta \to \mathcal{M},\ (X, [n]) \mapsto X \otimes \Delta^n.$$

Choose λ to be large enough so that the following are satisfied:

(a) $\mathcal{M}$ is λ-combinatorial,
(b) a cofibrant replacement functor in $\mathcal{M}$ preserves λ-presentable objects.

Then, using (a) and (b), it is shown that there are canonical weak equivalences for every fibrant object $X \in \mathcal{M}$:

$$\mathrm{hocolim}(\mathcal{M}_\lambda^{\mathrm{cof}} \times \Delta \downarrow X) \overset{\simeq}{\leftarrow} \mathrm{hocolim}(\mathcal{M}_\lambda^{\mathrm{cof}} \downarrow X) \overset{\simeq}{\to} \mathrm{hocolim}(\mathcal{M}_\lambda \downarrow X)$$

and

$$\mathrm{hocolim}(\mathcal{M}_\lambda \downarrow X) \overset{\simeq}{\to} \mathrm{colim}(\mathcal{M}_\lambda \downarrow X) \cong X.$$

Combining these weak equivalences, we may conclude that the derived counit transformation $F^{\mathrm{cof}}G(X) \to X$ is a natural weak equivalence for every fibrant X.

The second part of the proof shows that a homotopically surjective map (F, G) can be turned into a Quillen equivalence after Bousfield localization. To achieve this, we would like to localize $\mathcal{SS}et^{\mathcal{A}^{op}}$ at the components of the derived unit transformation of (F, G), which are canonical morphisms of the form

$$Y \to G^{\mathrm{fib}}F^{\mathrm{cof}}(Y), \tag{1}$$

but these defines a class of morphisms, rather than a set, and therefore we cannot directly conclude that this Bousfield localization exists. However, the derived unit transformation (1) is a natural transformation between accessible functors. In addition, note that filtered colimits in $\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}}$ are also homotopy colimits. Then the idea is to restrict to the set S of derived unit morphisms (1) only for λ-presentable objects Y, for some appropriate choice of λ, and consider the associated Bousfield localization $L_S\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}}$ (which exists by Theorem 4.6). The regular cardinal λ has the property that every derived unit morphism (1) is a (homotopy) filtered colimit of morphisms in S. As a consequence, every morphism in (1) becomes a weak equivalence in $L_S\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}}$. We can then conclude that (F, G) induces a Quillen equivalence $L_S\mathcal{S}\mathcal{S}et^{\mathcal{A}^{op}} \simeq \mathcal{M}$ as requred. This completes our outline of Dugger's proof.

There are also versions of Theorem 4.7 for pointed and for stable combinatorial model categories (see [7]).

5. Applications

5.1. **Brown representability for model categories.** Let $\mathcal{M}$ be a model category. We write $[X, Y]$ to denote the set of morphisms from X to Y in the homotopy category $\mathrm{Ho}(\mathcal{M})$. A functor $F : \mathcal{M}^{op} \to \mathcal{S}et$ is called *representable* if it is isomorphic to a functor of the form $[-, X] : \mathcal{M}^{op} \to \mathcal{S}et$ for some object X of $\mathcal{M}$. Every representable functor satisfies the conditions **B1** – **B3** below.

A model category $\mathcal{M}$ is said to *satisfy Brown representability* if for any given functor $F \colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}et$, F is representable if (and only if) the following conditions are satisfied.

B1. F sends weak equivalences to bijections.

B2. (wedge property) For any coproduct $\coprod_{i\in I} X_i$ of cofibrant objects in $\mathcal{M}$, the canonical morphism

$$F\left(\coprod_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} F(X_i)$$

is an isomorphism.

B3. (Mayer-Vietoris property) For every pushout diagram of cofibrant objects

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & B\coprod_A X \end{array}$$

where i is a cofibration, the canonical morphism

$$F(B\coprod_A X) \longrightarrow F(B) \times_{F(A)} F(X)$$

is an epimorphism.

This property is clearly invariant under Quillen equivalences. Brown's classical representability theorem shows that the model category of spectra $\mathcal{S}p$ satisfies Brown representability. Using similar arguments, it can be shown that the projective model category $\mathcal{S}p^C$ also satisfies Brown representability. More generally, every *compactly generated*[2] model category satisfies Brown representability (see [9, 12, 15]). Locally finitely presentable categories (with the trivial/discrete model structure) are also examples of compactly generated model categories.

If a model category $\mathcal{M}$ satisfies Brown representability, then the same holds for its Bousfield localizations. According to the stable version of Theorem 4.7 (see [7]), every stable combinatorial model category is Quillen equivalent to a Bousfield localization $L_S\mathcal{S}p^C$ of $\mathcal{S}p^C$ for some small category C and set of morphisms S. Thus, we conclude

Theorem 5.1. *Every stable combinatorial model category satisfies Brown representability.*

Corollary 5.2. *Let $\mathcal{T}$ be a triangulated category which is equivalent to the homotopy category of a stable combinatorial model category, as triangulated categories. Then a homological functor $F : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ is representable if and only if it sends small coproducts to products.*

Representability theorems are closely related to adjoint functor theorems. A consequence of Theorem 5.1 is the following "homotopical" adjoint functor theorem.

Theorem 5.3. *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories that preserves the weak equivalences. Suppose that $\mathcal{M}$ is stable and combinatorial. Then $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is a left adjoint if and only if the following conditions are satisfied.*

- **B2′.** *(wedge property) For any coproduct $\coprod_{i\in I} X_i$ of cofibrant objects in $\mathcal{M}$, the canonical morphism*

$$\coprod_{i\in I} F(X_i)^{\mathrm{cof}} \longrightarrow F(\coprod_{i\in I} X_i)$$

is an isomorphism in $\mathrm{Ho}(\mathcal{N})$.

- **B3′.** *(Mayer-Vietoris property) F sends homotopy pushout squares in $\mathcal{M}$ to weak pushout squares in $\mathrm{Ho}(\mathcal{N})$, i.e., for every pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & B\coprod_A X \end{array}$$

[2]We say that a model category $\mathcal{M}$ is *compactly generated* if there is a set of objects $\mathcal{G}$ with the following properties: (a) the objects in $\mathcal{G}$ jointly detect the isomorphisms in $\mathrm{Ho}(\mathcal{M})$, and (b) for every $G \in \mathcal{G}$, the functor $[G, -]\colon \mathcal{M} \to \mathcal{S}et$ sends homotopy sequential colimits to colimits. (We remark that this terminology is not standard in the literature.)

where i is a cofibration and all objects are cofibrant, the induced diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(B) & \longrightarrow & F(B \coprod_A X) \end{array}$$

defines a weak pushout square in $\mathrm{Ho}(\mathcal{N})$*.*

Proof. Suppose that $\mathrm{Ho}(F)$ is a left adjoint. Then it preserves coproducts and weak pushouts. Coproduct of cofibrant objects in $\mathcal{M}$ define coproducts in $\mathrm{Ho}(\mathcal{M})$, so it follows that **B2′** is satisfied. Moreover, a homotopy pushout in $\mathcal{M}$ defines a weak pushout in $\mathrm{Ho}(\mathcal{M})$, so **B3′** is also satisfied.

For the converse, suppose that **B2′**-**B3′** are satisfied. The functor $\mathrm{Ho}(F)$ is a left adjoint if (and only if) for every object N of $\mathcal{N}$ the functor

$$y_N : \mathcal{M}^{op} \to \mathcal{S}et, \quad M \mapsto [F(M), N]$$

is representable. We claim that y_N satisfies the conditions **B1**–**B3**. **B1** is obvious since F preserves the weak equivalences. **B2** and **B3** follow easily from **B2′** and **B3′** respectively. □

Example 5.4. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor from a locally presentable category to a category which admits small colimits and limits. We may regard $\mathcal{M}$ and $\mathcal{N}$ as model categories with the trivial/discrete model structure (where the weak equivalences are the isomorphisms and every morphism is a (co)fibration). The model category $\mathcal{M}$ satisfies Brown representability. (This follows, for example, from the fact that it is a reflective subcategory of a locally finitely presentable category. Locally finitely presentable categories are compactly generated, so they satisfy Brown representability.) Then, applying Theorem 5.3, F is a left adjoint if and only if F preserves coproducts and sends pushouts to weak pushouts. In this case F actually preserves all colimits.

5.2. **Bousfield localizations of combinatorial model categories.** Combinatorial model categories, similarly to cellular model categories (Theorem 4.6), are also useful because they admit Bousfield localizations at any *set* of morphisms.

Theorem 5.5. *Let $\mathcal{M}$ be a left proper, combinatorial model category and let S be a set of maps. Then the left Bousfield localization $L_S\mathcal{M}$ exists and is again left proper and combinatorial.*

The theorem can be shown by applying the recognition theorem for combinatorial model structures (Theorem 3.3). The main difficulty is to prove the accessibility of the class of S-local equivalences. This can be done by constructing an accessible S-localization functor $L : \mathcal{M} \to \mathcal{M}$ using the small object argument (cf. Remark 4.5). This functor comes together with a natural transformation $\eta_X : X \to L(X)$ which is an S-local equivalence to an S-local object. Then the class of S-local equivalences is the preimage under L of the class of weak equivalences in $\mathcal{M}$, so it is accessible and accessibly embedded in $\mathcal{M}^{\to}$ as a consequence of Proposition 2.7.

A different way of proving Theorem 5.5 is to use Dugger's theorem (Theorem 4.7) in order to pass to the projective model category $\mathcal{SS}et^C$, for which Bousfield localizations exist by Theorem 4.6, and then lift the corresponding Bousfield localization of $\mathcal{SS}et^C$ back to a Bousfield localization of $\mathcal{M}$.

See also [5] for an extension of Theorem 5.5 to the case where S is a (possibly proper) class of morphisms under the assumption of Vopěnka's principle.

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