Rational Valued Characters

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I. INTRODUCTION

In this note, G always denotes a finite group. A representation of G is said to be *rational* if it is afforded by a vector space V over $\mathbb{Q}$. The purpose of this article is to investigate several relationships between the characters of rational representations of G and some induced characters of G. Let H be a subgroup of G and π be a linear representation of H. Recall that the we defined the induced representaion $i_H^G\pi$ on the vector space

$$\mathcal{V} = \{f : G \to V | f(hg) = \pi(h)f(g), h \in H, g \in G\}.$$

If $\pi = 1$ is trivial, then $f(hg) = f(g)$ for all $f \in \mathcal{V}$. Therefore, $\mathcal{V}$ is the space of left H-invariant functions on G, i.e., functions on the set of right cosets $H\backslash G$. The induced representation $i_H^G 1$ is said to be a *permutation representation* of G. In particular, it is the *right regular representation* when $H = \{1\}$. By the Frobenius character formula,

$$\chi_{i_H^G 1}(g) = [G : H] \cdot \frac{\text{Number of conjugates of } g \text{ lying in } H}{\text{Total number of conjugates of } g \text{ in } G}$$

for $g \in G$. These permutation characters $\chi_{i_H^G 1}$ induced by subgroups H of G actually span all permutation characters of G. In other words, if χ is a permutation character of G, then there exist subgroups $H_1, H_2, \cdots, H_r$ and integers $c_1, c_2, \cdots, c_r$ such that $\chi = \sum_{k=1}^r c_k \chi_{i_{H_k}^G 1}$. Moreover, the subgroups H_k that occur in the decomposition are precisely the stablizers of the orbits of the permutation representation ([3]). On the other hand, characters of rational representations of G can also be expressed as $\mathbb{Q}$-linear combinations of induced characters from subgroups of G. This famous result is the essence of the next section.

II. ARTIN INDUCTION THEOREM

In this section, we will state the Artin induction theorem and present its proof, partly due to Brauer. Roughly speaking, the theorem tells us that the character of any rational representation of G is a $\mathbb{Q}$-linear combination of induced characters from cyclic subgroups of G. Its proof utilizes a result of algebraic number theory, namely that the cyclotomic polynomials are irreducible over $\mathbb{Q}$. This is equivalent to saying that for each $n \in \mathbb{Z}$, all of the primitive nth roots of unity are conjugate over $\mathbb{Q}$.

Lemma 1. *Let χ be a rational valued character of G and let $x, y \in G$ with $\langle x \rangle = \langle y \rangle$. Then, $\chi(x) = \chi(y)$.*

Proof. Let $n = |\langle x \rangle| = |\langle y \rangle|$ and let ζ be a primitive nth root of unity. Then, $y = x^m$ where $(m, n) = 1$ and ζ^m is also a primitive nth root of unity. Consider the Galois extension $\mathbb{Q}(\zeta)/\mathbb{Q}$. By our remark preceding the lemma, there exists $\sigma \in \mathbf{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q})$ such that $\zeta^m = \sigma(\zeta)$. Now $\chi(x) = \sum_{i=1}^{\chi(1)} \zeta_i$, where each ζ_i is an nth root of unity and hence a power of ζ. So,

$$\chi(y) = \chi(x^m) = \sum_{i=1}^{\chi(1)} \zeta_i^m = \sum_{i=1}^{\chi(1)} \sigma(\zeta_i) = \sigma(\sum_{i=1}^{\chi(1)} \zeta_i) = \sigma(\chi(x)).$$

Since $\chi(x)$ is rational, it is fixed by σ and we have $\chi(y) = \sigma(\chi(x)) = \chi(x)$. □

Lemma 2. *Let χ be a rational valued character of G. For all $g \in G$, $\chi(g) \in \mathbb{Z}$.*

Proof. Every algebraic integer in $\mathbb{Q}$ is an integer. □

Before we proceed further, we shall set up the following notations and definitions for later work. Let ϕ denote the Euler function. Define an equivalence relation $\equiv$ on G by $x \equiv y$ iff $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k$ be the distinct equivalence classes of G and let Φ_i be the characteristic function for $\mathcal{C}_i$ such that $\Phi_i(x) = 1$ if $x \in \mathcal{C}_i$ and $\Phi_i(x) = 0$ otherwise. Let x_i be a representative for $\mathcal{C}_i$ and let $H_i = \langle x_i \rangle$ and $n_i = |H_i|$. We write $x_i \leq_G x_j$ if there exists $g \in G$ such that $g^{-1} H_i g \leq H_j$. Clearly the relation $\leq_G$ is a partial ordering on $G/\equiv$. At last, let 1_G and 1_{H_i} be the characters of the trivial linear representations of G and H_i, respectively.

Theorem 3. *(E. Artin) Let χ be a rational valued character of G. Then,*

$$\chi = \sum_{i=1}^{k} \frac{a_i}{[\mathbf{N}(H_i) : H_i]} \chi_{i^G_{H_i} 1}, \tag{1}$$

where $a_i \in \mathbb{Z}$.

Proof. For $x, y \in G$, if $x \equiv y$, then there exists $g \in G$ such that $\langle x \rangle = g^{-1}\langle y \rangle g = \langle g^{-1}yg \rangle$. It follows from Lemma 1 that $\chi(y) = \chi(g^{-1}yg) = \chi(x)$. Thus, by Lemma 2, χ is a $\mathbb{Z}$-linear combination of the Φ_i's, i.e.,

$$\chi = \sum_{i=1}^{k} c_i \Phi_i, c_i \in \mathbb{Z}. \tag{2}$$

Since H_i has $\phi(n_i)$ generators and there are $[G : \mathbf{N}(H_i)]$ distinct conjugates of H_i, we have

$$|\mathcal{C}_i| = [G : \mathbf{N}(H_i)] \cdot \phi(n_i). \tag{3}$$

Next, we prove by induction on n_i that

$$|\mathbf{N}(H_i)| \cdot \Phi_i = \sum_j a_j n_j \chi_{i^G_{H_j} 1} \tag{4}$$

for $a_j \in \mathbb{Z}$, where j runs over the set of subscripts for which $x_j \leq_G x_i$.

If $n_i = 1$, then $\mathcal{C}_i = H_i = \{1\}$ and $|G|\Phi_i = \chi_{i^G_{H_i} 1}$. So, (4) holds. Suppose $n_i > 1$. Since $\chi_{i^G_{H_i} 1}$ is a rational valued character of G, we can write $\chi_{i^G_{H_i} 1} = \sum_{j=1}^{k} b_j \Phi_j$ and compute the coefficients b_j as follows:

$$|G|^{-1} b_j |\mathcal{C}_j| = \langle b_j \Phi_j, \Phi_j \rangle = \langle \sum_{j=1}^{k} b_j \Phi_j, \Phi_j \rangle = \langle \chi_{i^G_{H_i} 1}, \Phi_j \rangle = \langle 1_{H_i}, r^{H_i}_G \Phi_j \rangle,$$

where the last equality follows from the Frobenius reciprocity theorem. Now $r^{H_i}_G \Phi_j = 0$ unless H_i contains a cyclic subgroup K conjugate to H_j, i.e., $x_j \leq_G x_i$. In that case, $r^{H_i}_G \Phi_j$ takes on the value 1 on each generator of K and vanishes elsewhere. Therefore, $\langle 1_{H_i}, r^{H_i}_G \Phi_j \rangle = \phi(n_j)/n_i$ and we have

$$b_j = |\mathcal{C}_j|^{-1} |G| \phi(n_j)/n_i = |\mathbf{N}(H_j)|/n_i$$

using equation (3).

So,

$$n_i \chi_{i^G_{H_i} 1} = \sum_j |\mathbf{N}(H_j)| \Phi_j, \tag{5}$$

where j runs over the set of subscripts for which $x_j \leq_G x_i$. Notice that $|n_j| < |n_i|$ for all $j \neq i$. Rewrite (5) as

$$|\mathbf{N}(H_i)| \Phi_i = n_i \chi_{i^G_{H_i} 1} - \sum_{j \neq i} |\mathbf{N}(H_j)| \Phi_j$$

and applying the induction hypothesis on each j yields (4).

Suppose $x_j \leq_G x_i$ and let $g \in G$ such that $g^{-1}H_j g \leq H_i$. If $g_0 \in \mathbf{N}(H_i)$, then $gg_0 \in \mathbf{N}(H_j)$. Thus, we have $g\mathbf{N}(H_i) \leq \mathbf{N}(H_j) \implies |\mathbf{N}(H_i)|$ divides $|\mathbf{N}(H_j)|$. The result then follows from (2) and (4). □

Corollary 4. *Let χ be any rational valued character of G. Then $|G|\chi = \sum_{i=1}^{k} c_i \chi_{i_{H_i}^G 1}$, where $c_i \in \mathbb{Z}$.*

Let $\Lambda(G) = \{\sum_{i=1}^{k} c_i \chi_{H_i}^G 1 : c_i \in \mathbb{Z}\}$. Corollary 4 tells us that if χ is any rational valued character of G, then $|G|\chi \in \Lambda(G)$. We define the *Artin exponent of* G to be the smallest positive integer $A(G)$ such that $A(G)\chi \in \Lambda(G)$ for **all** rational valued characters χ of G. Clearly, $A(G)$ is at most $|G|$ and in particular, divides $|G|$. We will come back to this topic in section IV and determine the Artin exponents of various types of groups.

III. SOME BASIC THEOREMS

In this section, we present two results that are related to rational valued characters. Let ρ_G denote the character afforded by a right regular representation of G.

Lemma 5. *Let G be a cyclic group. For $g \in G$ define*

$$\chi_G(g) = \begin{cases} |G| & \text{if } G = \langle g \rangle, \\ 0 & \text{if } G \neq \langle g \rangle. \end{cases} \tag{6}$$

Then $\chi_G \in \Lambda(G)$.

Proof. We proceed by induction on $|G|$. If $|G| = 1$ then $\chi_G = 1_G$ and (6) holds. Suppose that $|G| > 1$. Let $H \leq G$ such that $H \neq G$. For $h \in H$,

$$i_H^G \chi_H(h) = \begin{cases} [G : H]\chi_H(h) = |G| & \text{if } H = \langle h \rangle, \\ 0 & \text{if } H \neq \langle h \rangle. \end{cases}$$

So if $\tau = \sum i_H^G \chi_H$ where H ranges over all cyclic subgroups of G with $H \neq G$, then $\tau(g) = |G|$ if $\langle g \rangle = H$ for some $H \leq G$ and $\tau(g) = 0$ if $\langle g \rangle = G$. Hence $\chi_G = |G|1_G - \tau$ and the result follows by induction. □

Theorem 6. *There exist cyclic subgroups H_j of G and nontrivial linear characters χ_j of H_j such that*

$$\rho_G = 1_G + \sum a_j i_{H_j}^G \chi_j,$$

where $a_j \in \mathbb{Q}$.

Proof. For a cyclic subgroup H_j of G, let χ_{H_j} be defined as in Lemma 5 and let $\chi_j = \phi(|H_j|)\rho_{H_j} - \chi_{H_j}$. Clearly χ_j is a nontrivial character of H_j. We now show that $\rho_G - 1_G = |G|^{-1} \sum i^G_{H_j}\chi_j$, where H_j ranges over all cyclic subgroups of G.

Let χ be an irreducible character of G. Then $\langle \chi, \rho_G - 1_G\rangle = \chi(1) - \langle \chi, 1_G\rangle$ by the linearity of the inner product. On the other hand,

$$\begin{aligned} \langle \chi, |G|^{-1} \textstyle\sum i^G_{H_j}\chi_j\rangle &= |G|^{-1} \textstyle\sum \langle \chi, i^G_{H_j}\chi_j\rangle_G \text{ (By the linearity of the inner product)} \\ &= |G|^{-1} \textstyle\sum \langle r^{H_j}_G \chi, \chi_j\rangle_{H_j} \text{ (By the Frobenius reciprocity theorem)} \\ &= |G|^{-1} \textstyle\sum \phi(|H_j|)\chi(1) - |G|^{-1} \sum |H_j|^{-1} \sum_{\langle g_j\rangle = H_j} |H_j|\chi(g_j) \\ &= |G|^{-1}|G|\chi(1) - |G|^{-1} \textstyle\sum_{g\in G} \chi(g) \\ &= \chi(1) - \langle \chi, 1_G\rangle. \end{aligned}$$

Since χ was chosen arbitrarily, $\rho_G - 1_G = |G|^{-1} \sum i^G_{H_j}\chi_j$ as required. □

IV. CYCLIC GROUPS

We now return to the topic of Artin exponents. To get our hands wet, we start by considering one of the most basic types of groups: the cyclic groups. Recall that the Artin exponent of a finite group G is denoted by $A(G)$. To prove the major theorem in this section, we need several lemmas.

Lemma 7. *Let L be the least common multiple of $[\mathbf{N}(H_j) : H_j]$, where j runs over the set of indices for which H_j is maximal. Then, L divides $A(G)$.*

Proof. Let $A(G)1_G = \sum_{i=1}^k c_i \chi_{i^G_{H_i}1}$ where $c_i \in \mathbb{Z}$. Furthermore, let $C(g)$ denote the conjugacy class of g in G. Notice that for each maximal cyclic subgroup H_j, $\chi_{i^G_{H_i}1} x_j \neq 0$ iff $i = j$. Then,

$$\begin{aligned} A(G) = A(G)1_G(x_j) &= \textstyle\sum_{i=1}^k c_i \chi_{i^G_{H_i}1}(x_j) \\ &= c_j \chi_{i^G_{H_j}1}(x_j) \\ &= c_j[G : H_j]/|C(g)| \\ &= c_j[\mathbf{N}(H_j) : H_j]. \end{aligned}$$

Therefore, $[\mathbf{N}(H_j) : H_j]$ divides $A(G)$ for each j and hence L divides $A(G)$. □

The next lemma is parallel to the last one and has a very similar proof.

Lemma 8. *Let M be the greatest common divisor of $[G : H_i]$, where $i = 1, 2, \cdots k$ runs over the whole set of indices. Then, M divides $A(G)$.*

Proof. As before, we have

$$A(G)1_G = \sum_{i=1}^{k} c_i \chi_{i_{H_i}^G 1}.$$

Evaluating at the identity we obtain

$$\begin{aligned} A(G) &= \sum_{i=1}^{k} c_i \chi_{i_{H_i}^G 1}(x_1) \\ &= \sum_{i=1}^{k} c_i [G : H_i]. \end{aligned}$$

Therefore, $[G : H_i]$ divides $A(G)$ for each i and hence M divides $A(G)$. □

Here, we want to remark that L and M do not have to equal $A(G)$ in general. For $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $A(G) = 8$ but $L = 4$. For $G = S_3$, $A(G) = 2$ but $M = 1$.

Recall that the ordinary exponent of a group G is equal to the order of any element of G of maximal order, i.e., the cardinality of any maximal order cyclic subgroup of G. We define the p-exponent of G, denoted by $exp(G, p)$, to be the exponent of a Sylow p-subgroup P of G. This definition leads to the following lemma.

Lemma 9. *Let P be a Sylow p-subgroup of G of cardinality p^n. If $exp(G, p) \leq p^{n-m}$, then p^m divides $A(G)$.*

Proof. Assume that $exp(G, p) \leq p^{n-m}$. We shall prove that $p^m | [G : G_i]$ for all $i = 1, 2, \cdots, k$, and the result will then follow from Lemma 8. Suppose that p^m does not divide $[G : G_i]$ for some i. Then, p^{n-m+1} divides $|G_i|$, and so the cyclic subgroup G_i has a subgroup H of order p^{n-m+1}. By Sylow's theorem, there exists a Sylow p-subgroup P of G which contains H as a subgroup. But then, we have $p^{n-m+1} \leq |P| = exp(G, p) \leq p^{n-m}$, which is a contradiction. □

Corollary 10. *$p|A(G)$ unless every Sylow p-subgroup of G is cyclic.*

Recall that a group G is *metacyclic* if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic. It follows that all metacyclic groups have a normal series of length two. The following proposition gives a sufficient condition for a group G being metacyclic. Its proof can be found in many group theory texts, such as ([6]). Only within the scope of the proposition we let G' denote the derived group of G.

Proposition 1. *Suppose all Sylow subgroups of G are cyclic. Then G is solvable. Moreover, G/G' and G' are both cyclic, so that G is metacyclic.*

Theorem 11. $A(G) = 1$ *if and only if G is cyclic.*

Proof. (*Necessity*) Assume that $A(G) = 1$. By Corollary 10, we see that the Sylow p-subgroups of G are all cyclic. Therefore, G is metacyclic and hence has a normal series $\{1\} \trianglelefteq H \trianglelefteq G$ of length two. If K is any subgroup of G containing H, then $K \trianglelefteq G$ by the correspondence theorem. So, without loss of generality we could assume that H is a maximal cyclic subgroup of G. By Lemma 7 and the hypothesis that $A(G) = 1$, $\mathbf{N}(H)$ must be equal to H. But H is normal in G, so $G = \mathbf{N}(H) = H$ and hence G is cyclic.

(*Sufficiency*) Suppose that G is cyclic. We enumerate all divisors of $|G|$ in an increasing sequence of numbers, say $s_1 = 1, s_2, \cdots, s_q = |G|$. Then, there are exactly q irreducible rational representations of G, whose characters $\chi_{s_1}, \cdots, \chi_{s_q}$ are afforded by the represention modules $\mathbb{Q}(\zeta_{s_1}), \cdots, \mathbb{Q}(\zeta_{s_q})$, where $\zeta_{s_j} = e^{2\pi i/s_j}$ is a primitive s_jth root of unity. Here we let G act on $\mathbb{Q}(\zeta_{s_j})$ by agreeing that a fixed generator g acts as multiplication by ζ_{s_j}. For each s_j dividing $|G|$, there is a unique subgroup $H_j \leq G$ such that $[G : H_j] = s_j$. We now claim that for all j between 1 and q,

$$\chi_{i_{H_j}^G 1} = \sum_{s|s_j} \chi_s. \tag{7}$$

By the Frobenius character formula, if $h \in H_j = \langle g^{s_j} \rangle$, then $\chi_{i_{H_j}^G 1}(h) = [G : H_j] = s_j$ and $\sum_{s|s_j} \chi_s(h) = \sum_{s|s_j} \chi_s(1) = \sum_{s|s_j} \phi(s) = s_j$. On the other hand, if $g' = g^k \notin H_j$ and $(k, s_j) = d$, then it is easy to calculate that $\chi_{i_{H_j}^G 1}(g') = 0$ and $\sum_{s|s_j} \chi_s(g') = d \cdot \sum(\frac{s_j}{d}\text{th root of unity}) = d \cdot 0 = 0$. Now, by arguing inductively, it is clear that any χ_{s_j} is an integral combination of the induced characters $\chi_{i_{H_k}^G 1}$, $1 \leq k \leq q$, and therefore so is any rational character χ. We conclude that $A(G) = 1$. □

V. p-GROUPS ($p \geq 3$)

Throughout the remaining sections, let μ denote the Möbius function. In 1950, Brauer proved the following deep theorem in his paper ([1]) by applying techniques from algebraic number theory.

Theorem 12. *Let χ be a rational valued character of G. Then,*

$$\chi = \sum a_H \chi_{i_H^G 1} \text{ with } a_H = \frac{1}{[G:H]} \sum \mu([G':H])\chi(g'),$$

where the first summation is taken over ***all*** *cyclic subgroups H of G and the second summation over all cyclic subgroups $G' = \langle g' \rangle \geq H$.*

In the special case where $\chi = 1_G$, we obtain

$$1_G = \sum a_H \chi_{i_H^G 1} \text{ with } a_H = \frac{1}{[G:H]} \sum \mu([G':H]). \tag{8}$$

We are now going to calculate the Artin exponents of a large collection of p-groups ($p \geq 3$). Following Lam's notations, let $z(G) = z$ be the number of subgroups of order p in G and let $sol(G)$ be the number of solutions of the equation $x^p = 1$ for $x \in G$. In 1895, Frobenius proved that $sol(G)$ is divisible by p.

Theorem 13. *Let $p \geq 3$ be a prime and let G be a p-group of order p^n, $n \geq 2$. If $sol(G)$ is not congruent to p modulo p^2, then $A(G) = p^{n-1}$.*

Proof. In equation 8, if we group together terms in the first summation with respect to conjugacy classes of H, then we end up with the following relation which should look somewhat familiar to the readers:

$$1_G = \sum_{i=1}^{k} a_i \chi_{i_{H_i}^G 1} \text{ with } a_i = \frac{1}{[\mathbf{N}(H_i):H_i]} \sum \mu([G':H_i]),$$

where the second summation is taken over all cyclic subgroups G' of G containing H_i as a subgroup. Let us evaluate the coefficient a_1. In the expression for a_1, $\mu([G':H_1]) = \mu(|G'|) \neq 0$ iff $|G'|$ is squarefree, i.e., $G' = \{1\}$ or G' is cyclic of order p. Therefore,

$$a_1 = |G|^{-1}(1 + \sum_{|G'|=p} \mu(|G'|)) = (1-z)/p^n.$$

Notice that if $x^p = 1$ for $x \in G$ and $x \neq 1$, then x is contained in an unique cyclic subgroup of order p. By our assumption that $sol(G)$ is not congruent to p modulo p^2, we can easily deduce that z is not congruent to 1 modulo p^2. If $A(G) = p^m$, then by definition $p^m a_1$ must be an integer. This implies that m is at least $n-1$. On the other hand, the Frobenius theorem states that $sol(G) \equiv 0 \pmod p \Longrightarrow z \equiv 1 \pmod p \Longrightarrow p^{n-1}a_1 \in \mathbb{Z}$. For $j \geq 2$, $[\mathbf{N}(H_j):H_j]$ divides p^{n-1}, so $p^{n-1}a_j \in \mathbb{Z}$. Altogether we have that $A(G) = p^{n-1}$, as desired. □

VI. 2-GROUPS

In this section we shall complete the discussion of p-groups by calculating the Artin exponents of 2-groups. First let us give a definition of three types of *exceptional* 2*-groups*.

Definition 1. *Let G be a* 2*-group of order 2^n generated by a and b. We say that*

1. G is quaternion*, if a and b satisfy the relations*

$$a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ bab^{-1} = a^{-1};$$

2. G is dihedral*, if a and b satisfy the relations*

$$a^{2^{n-1}} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1};$$

3. G is semi-dihedral*, if a and b satisfy the relations*

$$a^{2^{n-1}} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1+2^{n-2}}.$$

Thompson proved the following theorem which will help us greatly to compute $A(G)$ for 2-groups.

Theorem 14. *Let G be a noncyclic* 2*-group. Then, $sol(G) \equiv 0 \pmod 4$ unless G is quaternion, dihedral or semi-dihedral, for which cases $sol(G)$ are* 2*, $2^{n-1}+2$, and $2^{n-2}+2$ respectively.*

In addition to Thompson's theorem, we also need the following lemma when proving the main result which will follow after.

Lemma 15. *Let G be a finite group and $d \in \mathbb{Z}$. Suppose*

$$d \cdot 1_G = \sum_{i=1}^{n} a_i \chi_{i_{H_i}^G 1}$$

where $a_i \in \mathbb{Z}$. If $a_1, a_2, \cdots, a_n$ have no common factor, then $d = A(G)$.

Proof. This follows easily from the independence of the $\chi_{i_{H_i}^G 1}$'s and the definition of $A(G)$. □

Theorem 16. *Let G be a noncyclic* 2*-group of order 2^n. Then $A(G) = 2^{n-1}$ unless G is quaternion, dihedral or semi-dihedral, for which cases $A(G) = 2$.*

Proof. If $sol(G) \equiv 0 \pmod 4$, then a repetition of the argument in the proof of Theorem 13 shows that $A(G) = 2^{n-1}$. So, suppose that $sol(G)$ is not divisible by 4. Then, by Theorem 14, G is either quaternion, dihedral or semi-dihedral. We consider each individual case to verify that $A(G) = 2$.

Case 1. Suppose G is a quaternion group. We shall compute $A(G)$ directly from Theorem 12. Take any cyclic subgroup $H = \langle h \rangle$ of G. If $h \notin \langle a \rangle$, then without loss of generality we may assume that $h = b$. For any $m \in \mathbb{N}$, $a^{-m}ba^m = a^{-m}bab^{-1}ba^{m-1} = a^{-m-1}ba^{m-1} = a^{-2m}b$. Since $a^{-2m}b \in \langle b \rangle$ iff $a^{-2m} = 1$, it follows that $[\mathbf{N}(H) : H] = 2$. Notice that there is no proper subgroup of G strictly containing H. If $h \in \langle a \rangle$, but $H \neq 1$ and $\neq \langle a \rangle$, then the corresponding Brauer coefficient has numerator equal to $\sum \mu([G' : H])$, where G' ranges over all cyclic subgroups of G containing H. Now, $\sum \mu([G' : H]) = \mu(1) + \mu(2) = 0$. If $h = a$, then again $[\mathbf{N}(H) : H] = [G : H] = 2$. At last, if $h = 1$, then by our previous calculation the corresponding coefficient is $(1 - z)/2^n$. By Theorem 15, we have $z = sol(G) - 1 = 2 - 1 = 1$, hence $(1 - z)/2^n = 0$. Therefore, $2 \cdot 1_G \in \Lambda(G)$, so $A(G) = 2$ by Lemma 15.

Case 2. Suppose G is a dihedral group and $H = \langle h \rangle$ is a cyclic subgroup of G. Following similar arguments, $h \notin \langle a \rangle \Longrightarrow [\mathbf{N}(H) : H] = 2$, $h \in \langle a \rangle \Longrightarrow \sum \mu([G' : H]) = 0$, and $h = a \Longrightarrow [\mathbf{N}(H) : H] = 2$. If $h = 1$, then $z = 2^{n-1} + 2 - 1 = 2^{n-1} - 1$, hence $(1 - z)/2^n = -1/2$. Therefore, $2 \cdot 1_G \in \Lambda(G)$, so again $A(G) = 2$ by Lemma 15.

Case 3. This case is again similar to the first, so we omit the details. □

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