# Advanced Course
# on
# Quasideterminants
# and
# Universal Localization

## Notes of the Course

January 30 to February 10, 2007

Centre de Recerca Matemàtica

Bellaterra (Spain)

# Contents

# Foreword

These notes correspond to the Advanced Course on Quasideterminants and Universal localization that will take place from January 30 to February 10, 2007 at the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona). It is one of the activities of the *Research Programme on Continuous and Discrete Methods on Ring Theory*.

The Advanced Course is organized in three, largely independent, series of lectures delivered by Professors Robert Lee Wilson and Vladimir Retakh both from Rutgers University. The first series will concentrate on the topic of *Quasideterminants*, a non commutative approach to the determinants introduced by I. S. Gelfand and V. Retakh. The aim of the theory of quasideterminants is to be an organizing tool in noncommutative algebra. The second series of lectures *Factorization of noncommutative polynomials and noncommutative symmetric functions* as well as the third one on *Universal localization* will show this point of view.

We thank the lecturers for their effort in the preparation of these notes and for having them on time to assure that the volume will be ready at the beginning of the course. We believe it will be of great help to the participants.

We want to express our gratitude to the director and the staff of the CRM who helped us in the organization of this course. We hope that this course will be profitable to all the participants and that all of us will remember these days with great pleasure.

Silvana Bazzoni
Università di Padova

Dolors Herbera
Universitat Autònoma de Barcelona

Co-ordinators

The notes contained in this booklet were printed directly from files supplied by the authors before the course.

# Lectures on Quasideterminants

# Contents

# Introduction

The ubiquitous notion of a determinant has a long history, both visible and invisible. The determinant has been a main organizing tool in commutative linear algebra and we cannot accept the point of view of a modern textbook [FIS] that "determinants ... are of much less importance than they once were".

The history of commutative determinants can be described by the following quotation from Wikipedia.

"Historically, determinants were considered before matrices. Originally, a determinant was defined as a property of a system of linear equations. The determinant "determines" whether the system has a unique solution (which occurs precisely if the determinant is non-zero). In this sense, two-by-two determinants were considered by Cardano at the end of the 16th century and larger ones by Leibniz about 100 years later. Following him Cramer (1750) added to the theory, treating the subject in relation to sets of equations. The recurrent law was first announced by Bezout (1764).

"It was Vandermonde (1771) who first recognized determinants as independent functions. Laplace (1772) gave the general method of expanding a determinant in terms of its complementary minors: Vandermonde had already given a special case. Immediately following, Lagrange (1773) treated determinants of the second and third order. Lagrange was the first to apply determinants to questions outside elimination theory; he proved many special cases of general identities.

"Gauss (1801) made the next advance. Like Lagrange, he made much use of determinants in the theory of numbers. He introduced the word determinants (Laplace had used resultant), though not in the present signification, but rather as applied to the discriminant of a quantic. Gauss also arrived at the notion of reciprocal (inverse) determinants, and came very near the multiplication theorem.

"The next contributor of importance is Binet (1811, 1812), who formally stated the theorem relating to the product of two matrices of $m$ columns and $n$ rows, which for the special case of $m = n$ reduces to the multiplication theorem. On the same day (Nov. 30, 1812) that Binet presented his paper to the Academy, Cauchy also presented one on the subject. (See Cauchy-Binet formula.) In this he used the word determinant in its present sense, summarized and simplified what was then known on the subject, improved the notation, and gave the multiplication theorem with a proof more satisfactory than Binet's. With him begins the theory in its generality.

"The next important figure was Jacobi (from 1827). He early used the functional determinant which Sylvester later called the Jacobian, and in his memoirs in Crelle for 1841 he specially treats this subject, as well as the class of alternating functions which Sylvester has called alternants. About the time of Jacobi's last memoirs, Sylvester (1839) and Cayley began their work.

Attempts to define a determinant for matrices with noncommutative entries started more than 150 years ago and also include several great names. For many years the most famous examples of matrices of noncommutative objects were quaternionic matrices and block matrices. It is not suprising that the first noncommutative determinants or similar notions were defined for such structures.

A. Cayley [C] was the first to define, in 1845, the determinant of a matrix with noncommutative entries. He mentioned that for a quaternionic matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the expressions $a_{11}a_{22} - a_{12}a_{21}$ and $a_{11}a_{22} - a_{21}a_{12}$ are different and suggested choosing one of them as the determinant of the matrix $A$. The analog of this construction for $3\times 3$-matrices was also proposed in [C] and later developed in [J]. This "naive" approach is now known to work for quantum determinants and some other cases. Different forms of quaternionic determinants were considered later by E. Study [St], E.H. Moore [Mo] and F. Dyson [Dy].

There were no direct "determinantal" attacks on block matrices (excluding evident cases) but important insights were given by G. Frobenius [Fr] and I. Schur [Schur] who introduced Schur compliments for such matrices.

A theory of determinants of matrices with general noncommutative entries was in fact originated by J.H.M. Wedderburn in 1913. In [W] he constructed a theory of noncommutative continued fractions or, in modern terms, "determinants" of noncommutative Jacobi matrices.

In 1926-1928 A. Heyting [H] and A. Richardson [Ri, Ri1] suggested analogs of a determinant for matrices over division rings. Heyting is known as a founder of intuitionist logic and Richardson as a creator of the Littlewood-Richardson rule. Heyting tried to construct a noncommutative projective geometry. As a computational tool, he introduced the "designant" of a noncommutative matrix. The designant of a $2 \times 2$-matrix $A = (a_{ij})$ is defined as $a_{11} - a_{12}a_{22}^{-1}a_{21}$. The designant of an $n \times n$-matrix is defined then by a complicated inductive procedure. The inductive procedures used by Richardson were even more complicated. It is important to mention that determinants of Heyting and Richardson in general are rational functions (and not polynomials!) in matrix entries.

The idea to have non-polynomial determinants was strongly criticized by O. Ore [O]. In [O] he defined a polynomial determinant for matrices over an imporatnt class of noncommutative rings (now known as Ore rings).

The most famous and widely used noncommutaive determinant is the Dieudonne determinant. It was defined for matrices over a division ring $R$ by J. Dieudonne in 1943 [D]. His idea was to consider determinants with values in $R^*/[R^*, R^*]$ where $R^*$ is the monoid of invertible elements in $R$. The properties of Dieudonne determinants are close to those of commutative ones, but, evidently, Dieudonne determinants cannot be used for solving systems of linear equations.

An interesting generalization of commutative determinants belongs to F. Berezin [B, Le]. He defined determinants for matrices over so called super-commutative algebras. In particular, Berezin also understood that it is impossible to avoid rational functions in matrix entries in his definition.

Other famous examples of noncommutative determinants developed for different special cases are: quantum determinants [KS, Ma], Capelli determinants [We], determinants introduced by Cartier-Foata [CF, F] and Birman-Williams [BW], Yangians [MNO], etc. A relation of noncommutative determinants with noncommutative generalization of the MacMahon Master Theorem can be found in [KP]. As we explain later (using another universal notion, that of quasideterminants) these determinants and the determinants of Dieudonne, Study, Moore, etc., are related to each other much more than one would expect.

In particular, one can formulate the following experimental principle:

• All well-known noncommutative determinants are products of a commuting family of quasiminors (may be with a shift) or their inverses.

The notion of quasideterminants for matrices over a noncommutative division ring was introduced in [GR, GR1, GR2]. Quasideterminants are defined in the "most noncommutative case", namely, for matrices over free division rings. We believe that quasideterminants should be one of main organizing tools in noncommutative algebra giving them the same role determinants play in commutative algebra. The quasideterminant is not an analog of the commutative determinant but rather of a ratio of the determinant of an $n \times n$-matricx to the determinant of an $(n-1) \times (n-1)$-submatrix.

Another experimental principle says

• All commutative formulas containing a ratio of determinants have their noncommutative analogues.

The main property of quasideterminants is a "heredity principle": let $A$ be a square matrix over a division ring and $(A_{ij})$ a block decomposition of $A$ (into submatrices of $A$). Consider the $A_{ij}$'s as elements of a matrix $X$. Then the quasideterminant of the matrix X will be a matrix $B$, and (under natural assumptions) the quasideterminant of $B$ is equal to a suitable quasideterminant of $A$. Since determinants of block matrices are not defined, there is no analog of this principle for ordinary (commutative) determinants.

Quasideterminants have been effective in many areas including noncommutative symmetric functions [GKLLRT, GR3, GR4], noncommutative integrable systems [RS, EGR, EGR1, Ha], quantum algebras and Yangians [GR, GR1, GR2, KL, Mol, Mol1, MolR ], and so on [P, Sch, RSh, RRV]. Quasideterminants and related quasi-Plücker coordinates are also important in various approches to noncommutative algebraic geometry (e.g., [K, KR, SvB, BR])

Many areas of noncommutative mathematics (Ore rings, rings of differential operators, theory of factors, "quantum mathematics", Clifford algebras, etc) were developed separately from each other. Our approach shows an advantage of working with totally noncommutative variables (over free rings and division rings). It leads us to a large variety of results, and their specialization to different noncommutative areas implies known theorems with additional information.

The price one pays for this is a huge number of inversions in rational noncommutative expressions. The minimal number of successive inversions required to express an element is called the height of this element. This invariant (inversion height) reflects the "degree of noncommutativity" and it is of a great interest by itself.

Our experience shows that in dealing with noncommutative objects one should not imitate the classical commutative mathematics, but follow "the way it is" starting with basics. In these series of lectures we concentrate on two problems: noncommutative Plücker coordinates (as a background of a noncommutative geometry) and the noncommutative Bezout and Viète theorems (as a background of noncommutative algebra). We apply the obtained results to the theory of noncommutative symmetric functions started in [GKLLRT].

We have already said that the universal notion of a determinant has a long history, both visible and invisible. The visible history of determinants comes from the fact that they are constructed from another class of universal objects: matrices.

The invisible history of determinants is related with the Heredity principle for matrices: matrices can be viewed as matrices with matrix entries (block matrices) and some matrix properties come from the corresponding properties of block matrices. In some cases, when the matrix entries of the block matrix commute, the determinant of a matrix can be computed in terms of the determinants of its blocks, but in general it is not possible: the determinant of a matrix with matrix entries is not defined because the entries do not commute. In other words, the determinant does not satisfy the Heredity principle.

Quasideterminants are defined for matrices over division rings and satisfy the Heredity Principle. Their definition can be specialized for matrices over a ring (including noncommutative rings) and can be connected with different "famous" determinants. This reflects another general principle:

• In many cases noncommutative algebra can be made simpler and more natural than commutative algebra.

These lectures describe the first 15 years of development of this very active area, and we hope that future work will bring many new interesting results.

The first ten years of the theory are described in [GGRW].

The paper is organized as follows. In Section 1 a definition of quasideterminants is given and the main properties of quasideterminants (including the Heredity principle) are described.

In Section 2 we discuss an important example: quasideterminants of quaternionic matrices. These quasideterminants can be written as polynomials with real coefficients in the matrix entries and their quaternionic conjugates.

As we already mentioned, mathematics knows a lot of different versions of noncommutative determinants. In Section 3 we give a general definition of determinants of noncommutative matrices (in general, there are many determinants of a fixed matrix) and show how to obtain some well-known noncommutative determinants as specializations of our definition.

In Section 4 we introduce noncommutative versions of Plücker and flag coordinates for rectangular matrices over division rings and discuss some results related to the theory of noncommutative double Bruhat cells for $GL_n(R)$ developed in [BR] where $R$ is a noncommutative associative division ring.

In Section 5 we present another approach to the theory of noncommutative determinants, traces, etc., and relate it to the results presented in Section 3.

Some applications to noncommutative continued fractions, characteristic functions of graphs, noncommutative orthogonal polynomials and integrable systems are given in Section 6.

We are very grateful to I. Gelfand who suggested to one of us (V.R.) in 1989 to study the very non-fashionable subject of noncommutative determinants and proposed to start the theory by looking onto systems of linear equations over noncommutative rings.

# Chapter 1

# General theory and main identities

All rings considered in this paper are associative unital rings.

## 1.1 Definition of quasideterminants

Quasideterminants were defined by I. Gelfand and V. Retakh in 1991 by returning to the "roots" of linear algebra, i.e. as a tool for solving system of linear equations over noncommutative rings (cf. the approach by Cardano). We will present here several definitions of quasideterminants starting with an approach suggested by B. Osofsky twelve years later.

An $n$ by $n$ matrix $A$ over a not necessarily commutative ring $R$ has, in general, $n^2$ quasideterminants, denoted $|A|_{ij}, 1 \leq i, j \leq n$. If $|A|_{ij}$ exists it is a rational function in the entries of $A$. Here are several essentially equivalent definitions of $|A|_{ij}$.

**I**. Definition of quasideterminant via Gaussian elimination (following B. Osofsky):

Let where $I_k$ is the identity $k$ by $k$ matrix. Suppose the matrix $A$ can be transformed into a matrix of the form

$$\begin{pmatrix} I_{n-1} & b \\ 0 & c \end{pmatrix}$$

by a sequence of elementary row operations which do not interchange rows, which do not multiply the $n$-th row by a scalar and which do not add add multiples of the $n$-th row to any other row, i.e., by operations of the following forms:

• multiply each entry of the $i$-th row (where $1 \leq i < n$) on the left by some $r \in R$ and leave all other rows unchanged;

• replace the $i$-th row (where $1 \leq i \leq n$) by the sum of the $i$-th row and the $j$-th row (where $1 \leq j < n$) and leave all other rows unchanged.

(These operations allow us to replace $A$ by $MA$ where $M = [m_{ij}]$ is an $n$ by $n$ matrix with $m_{in} = \delta_{in}$ for $1 \leq i \leq n$.)

Then the quasideterminant $|A|_{nn}$ exists and

$$|A|_{nn} = c.$$

If $P$ and $Q$ are the permutation matrices corresponding to the transpositions $(in)$ and $(jn)$ respectively and $|PAQ|_{nn}$ exists, then $|A|_{ij}$ exists and

$$|A|_{ij} = |PAQ|_{nn}.$$

**II**. Definition of quasideterminants of $A$ via the inverse of $A$: Suppose the matrix $A$ is invertible with inverse $B = (b_{ij})$ and that $b_{ji}$ is invertible in $R$. Then the quasideterminant $|A|_{ij}$ exists and

$$|A|_{ij} = b_{ji}^{-1}.$$

**III**. Definition of quasideterminants of $A$ via inverses of minors of $A$:

For $1 \leq i, j \leq n$ let $A^{ij}$ denote the matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column. Let $r_k^j$ denote the (row) vector obtained from the $k$-th row of $A$ by deleting the $j$-th entry and let $s_l^i$ denote the (column) vector obtained from the $l$-th column of $A$ by deleting the $i$-th entry. Assume that $A^{ij}$ is invertible. Then $|A|_{ij}$ exists and

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} s_j^i.$$

We will refer to this definition as the *basic definition*.

**IV**. Inductive definition of quasideterminant: If $A = (a)$ is a 1 by 1 matrix define $|A|_{11} = a$. Assume quasideterminants have been defined for $n-1$ by $n-1$ matrices and that $A = (a_{ij})$ is an $n$ by $n$ matrix. Using the notation of (3), assume that each quasideterminant of $A^{ij}$ exists and is invertible. Let $C$ denote the $n-1$ by $n-1$ matrix with rows indexed by $\{k|1 \leq k \leq n, k \neq i\}$ and columns indexed by $\{l|1 \leq l \leq n, l \neq j\}$ and whose entry in the $(k,l)$ position is $[A^{ij}]_{lk}^{-1}$. Then the quasideterminant $|A|_{ij}$ exists and

$$|A_{ij}| = a_{ij} - r_i^j C s_j^i.$$

Equivalence of these definitions:

Suppose $MA = \begin{pmatrix} I_{n-1} & b \\ 0 & c \end{pmatrix}$ where $M$ is a product of appropriate elementary matrices. Then if $c$ is invertible

$$\begin{pmatrix} I_{n-1} & -bc^{-1} \\ 0 & 1 \end{pmatrix} MA = \begin{pmatrix} I_{n-1} & 0 \\ 0 & c \end{pmatrix}$$

and so, $c^{-1}$ is the entry $(n,n)$ entry of $A^{-1}$. This shows the equivalence of (1) and (2).

If we write the $n$ by $n$ matrix $A$ as the block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11}$ is an $n-1$ by $n-1$ matrix and assume that $A_{11}$ is invertible, then

$$\begin{pmatrix} I_{n-1} & 0 \\ -A_{21} & I_1 \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & I_1 \end{pmatrix} A = \begin{pmatrix} I_{n-1} & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

Since $A_{11} = A^{nn}, A_{21} = r_n^n, A_{12} = s_n^n$, this shows the equivalence of (1) and (3). The equivalence of (3) and (4) now follows from (2).

If $A$ is the 2 by 2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then we have

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21},$$

$$|A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22},$$

$$|A|_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11},$$

$$|A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}.$$

For the $3 \times 3$-matrix $A = (a_{ij})$, $i, j = 1, 2, 3$, there are 9 quasideterminants. One of them is

$$|A|_{11} = a_{11} - a_{12}(a_{22} - a_{23}a_{33}^{-1}a_{32})^{-1}a_{21} - a_{12}(a_{32} - a_{33} \cdot a_{23}^{-1}a_{22})^{-1}a_{31} \tag{1.1}$$
$$- a_{13}(a_{23} - a_{22}a_{32}^{-1}a_{33})^{-1}a_{21} - a_{13}(a_{33} - a_{32} \cdot a_{22}^{-1}a_{23})^{-1}a_{31}. \tag{1.2}$$

One can see a pattern of "noncommutative continued fraction" in this formula. We will explore this line later.

## 1.2 Comparison with the commutative determinants

Suppose $A$ is a matrix over a commutative ring $R$. How is the quasideterminant $|A|_{ij}$ related to $det\ A$?

It is well known that, if $A$ is invertible, the $(j,i)$ entry of $A^{-1}$ is $(-1)^{i+j}\frac{det\ A^{i,j}}{det\ A}$. Thus, in view of characterization (2) of $|A|_{i,j}$ we have

$$|A|_{ij} = (-1)^{i+j}\frac{det\ A}{det\ A^{ij}}.$$

## 1.3 Effect of row and column operations on quasideterminants

We now describe the effect of certain row and column operations on the values of quasideterminants. This is useful for comparison with properties of determinants over commutative rings and will also be used for some computations in later lectures.

(i) The quasideterminant $|A|_{pq}$ does not depend on permutations of rows and columns in the matrix $A$ that do not involve the $p$-th row and the $q$-th column.

(ii) *The multiplication of rows and columns.* Let the matrix $B = (b_{ij})$ be obtained from the matrix $A$ by multiplying the $i$-th row by $\lambda \in R$ from the left, i.e., $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for
$k \neq i$. Then

$$|B|_{kj} = \lambda |A|_{ij} \text{ if } k = i,$$

and

$$|B|_{kj} = |A|_{kj} \text{ if } k \neq i \text{ and } \lambda \text{ is invertible.}$$

Let the matrix $C = (c_{ij})$ be obtained from the matrix A by multiplying the $j$-th column by $\mu \in R$ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all $i$ and $l \neq j$. Then

$$|C|_{i\ell} = [A]_{ij}\mu \text{ if } l = j,$$

and

$$|C|_{i\ell} = |A|_{i\ell} \text{ if } l \neq j \text{ and } \mu \text{ is invertible.}$$

(iii) *The addition of rows and columns.* Let the matrix $B$ be obtained from $A$ by replacing the $k$-th row of $A$ with the sum of the $k$-th and $l$-th rows, i.e., $b_{kj} = a_{kj} + a_{lj}$, $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$|A|_{ij} = |B|_{ij}, \qquad i = 1, \dots k-1, k+1, \dots n, \quad j = 1, \dots, n.$$

Let the matrix $C$ be obtained from $A$ by replacing the $k$-th column of $A$ with the sum of the $k$-th and $l$-th columns, i.e., $c_{ik} = a_{ik} + a_{il}$, $c_{ij} = a_{ij}$ for $j \neq k$. Then

$$|A|_{ij} = |C|_{ij}, \qquad i = 1, \dots, n, \quad , \dots, \ell - 1, \ell + 1, \dots n.$$

## 1.4 Applications to linear systems

Solutions of systems of linear systems over an arbitraty ring can be expressed in terms of quasideterminants.

Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring $R$.

**Theorem 1.4.1.** *Assume that all the quasideterminants $|A|_{ij}$ are defined and invertible. Then the system of equations*

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \ 1 \leq i \leq n$$

*has the unique solution*

$$x_i = \sum_{j=1}^{n} |A|_{ji}^{-1} b_j. \qquad i = 1, \dots, n.$$

Proof: In view of characterization (2) of quasideterminants, the assumption that every $|A|_{i,j}$ is defined and invertible implies that A is invertible. The result now follows by using $A^{-1}$ to write the solution of the system and replacing the elements of $A^{-1}$ by quasideterminants.
**Cramer's rule**. Let $A_\ell(b)$ be the $n \times n$-matrix obtained by replacing the $\ell$-th column of the matrix $A$ with the column $(b_1, \dots, b_n)$.

**Theorem 1.4.2.** *In notation of Theorem 1.1.1, if the quasideterminants $|A|_{ij}$ and $|A_j(b)|_{ij}$ are defined, then*

$$|A|_{ij} x_j = |A_j(b)|_{ij}.$$

## 1.5 Further properties of quasideterminants

In many cases it is more convenient to define quasideterminants for matrices whose entries are indexed by two finite sets of the same cardinality $n$.

Denote those sets by $I$ and $J$. Let $A = (a_{ij})$, $i \in I$, $j \in J$. Suppose that $n \geq 2$ and let $A^{ij}$ be the $(n-1) \times (n-1)$-matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column. Then the basic definition for the quasidetrminant $|A|_{ij}$ is

$$|A|_{ij} = a_{ij} - \sum a_{ii'} (|A^{ij}|_{j'i'})^{-1} a_{j'j}.$$

Here the sum is taken over $i' \in I \setminus \{i\}$, $j' \in J \setminus \{j\}$.

It sometimes convenient to use another notation for quasideterminants by boxing the leading element, i.e.

$$|A|_{ij} = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \boxed{a_{ij}} & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

Note that according to our definitions, the quasideterminant $|A|_{ij}$ of a matrix $A$ over a ring $R$ is an element of $R$. The action of the product of symmetric groups $S_n \times S_n$ on $I \times J$, $|I| = |J| = n$, induces the action of $S_n \times S_n$ on the the set of variables $\{a_{ij}\}$, $i \in I$, $j \in J$. The following proposition shows that the definition of the quasideterminant is compatible with this action.

**Proposition 1.5.1.** *For $(\sigma, \tau) \in S_n \times S_n$ we have $(\sigma, \tau)\big(|A|_{ij}\big) = |A|_{\sigma(i)\tau(j)}$.*

In particular, the stabilizer subgroup of $|A|_{ij}$ under the action of $S_n \times S_n$ is isomorphic to $S_{n-1} \times S_{n-1}$.

Proposition 1.2.4 shows that in the definition of the quasideterminant, we do not need to require $I$ and $J$ to be ordered or a bijective correspondence between $I$ and $J$ to be given.

If $A$ is a generic $n \times n$-matrix (in the sense that all square submatrices of $A$ are invertible), then there exist $n^2$ quasideterminants of $A$. However, a non-generic matrix may have $k$ quaisdeterminants, where $0 \leq k \leq n^2$. Example 1.2.3(a) shows that each of the quasideterminants $|A|_{11}$, $|A|_{12}$, $|A|_{21}$, $|A|_{22}$ of a $2 \times 2$-matrix $A$ is defined whenever the corresponding element $a_{22}$, $a_{21}$, $a_{12}$, $a_{11}$ is invertible.

*Remark.* The definition of the quasidereminant can be generalized to define $|A|_{ij}$ for a matrix $A = (a_{ij})$ in which each $a_{ij}$ is an invertible morphism $V_j \to V_i$ in an additive category $C$ and the matrix $A^{pq}$ of morphisms is invertible. In this case the quasideterminant $|A|_{pq}$ is a morphism from the object $V_q$ to the object $V_p$.

Recall that if the elements $a_{ij}$ of the matrix $A$ commute, then

$$|A|_{pq} = (-1)^{p+q} \frac{\det A}{\det A^{pq}}.$$

This example shows that the notion of a quasideterminant is not a generalization of a determinant over a commutative ring, but rather a generalization of a ratio of two determinants.

We will show in Section 3 that similar expressions for quasideterminants can be given for quantum matrices, quaternionic matrices, Capelli matrices and other cases listed in the Introduction.

In general quasideterminants are not polynomials in their entries, but (noncommutative) rational functions. Very interesting properties of these rational functions will be discussed in other series of these lectures.

In the commutative case determinants are finite sums of monomials with appropriate coefficients. As is shown in [GR1, GR2], in the noncommutative case quasideterminants of a matrix $A = (a_{ij})$ with formal entries $a_{ij}$ can be identified with formal power series in the matrix entries or their inverse. A simple example of this type is described below.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix with formal entries. Denote by $I_n$ the identity matrix of order $n$ and by $\Gamma_n$ the complete oriented graph with vertices $\{1, 2, \dots, n\}$, with the arrow from $i$ to $j$ labeled by $a_{ij}$. A path $p : i \to k_1 \to k_2 \to \dots \to k_t \to j$ is labeled by the word $w = a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \dots a_{k_t j}$.

Denote by $P_{ij}$ the set of words labelling paths going from $i$ to $j$, i.e. the set of words of the form $w = a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \dots a_{k_t j}$. A simple path is a path $p$ such that $k_s \neq i, j$ for every $s$. Denote by $P'_{ij}$ the set of words labelling simple paths from $i$ to $j$.

Let $R$ be the ring of formal power series in $x_{ij}$ over a field. From [Co], Section 4, it follows that there is a canonical embedding of $R$ in a division ring $D$ such that the image of $R$ generates $D$. We identify $R$ with its image in $D$.

**Proposition 1.5.2.** *Let $i, j$ be two distinct integers between 1 and $n$. The rational functions $|I_n - A|_{ii}$, $|I_n - A|_{ij}^{-1}$ are defined in $D$ and*

$$|I_n - A|_{ii} = 1 - \sum_{w \in P'_{ii}} w, \qquad |I_n - A|_{ij}^{-1} = \sum_{w \in P_{ij}} w.$$

**Example.** For $n = 2$,

$$|I_2 - A|_{11} = 1 - a_{11} - \sum_{p \geq 0} a_{12} a_{22}^p a_{21}.$$

For some matrices of special form over a ring, quasideterminants can be expressed as polynomials in the entries of the matrix. The next proposition shows that this holds, in particular, for the so-called almost triangular matrices. Such matrices play am important role in many papers, including [DS, Ko, Gi].

**Proposition 1.5.3.** *The following quasideterminant is a polynomial in its entries:*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & -1 & a_{33} & \dots & a_{3n} \\ & & \dots & & \\ 0 & & \dots & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1 \leq j_1 < j_2 < \dots < j_k < n} a_{1j_1} a_{j_1+1, j_2} a_{j_2+1, j_3} \dots a_{j_k+1, n}.$$

*Remark.* Denote the expression on right-hand side by $P(A)$. Note that $(-1)^{n-1} P(A)$ equals to the determinant of the almost upper-triangular matrix over a commutative ring. For non-commutative almost upper triangular matrices, Givental [Gi] (and others) defined the determinant as $(-1)^{n-1} P(A)$.

**Example.** For $n = 3$ we have

$$P(A) = a_{13} + a_{11} a_{23} + a_{12} a_{33} + a_{11} a_{22} a_{33}.$$

## 1.6 General properties of quasideterminants

### 1.6.1 Two involutions (see [GR4]).

For a square matrix $A = (a_{ij})$ over a ring $R$, denote by $IA = A^{-1}$ the inverse matrix (if it exists), and by $HA = (a_{ji}^{-1})$ the Hadamard inverse matrix (a;so if it exists). It is evident that if $IA$ exists, then $I^2 A = A$, and if $HA$ exists, then $H^2 A = A$.

Let $A^{-1} = (b_{ij})$. According to Theorem 1.2.1, $b_{ij} = |A|_{ji}^{-1}$. This formula can be rewritten in the following form.

**Theorem 1.6.1.** *For a square matrix $A$ over a ring $R$,*

$$HI(A) = (|A|_{ij})$$

provided that all quasideterminants $|A|_{ij}$ exist.

### 1.6.2 Homological relations (see [GR])

Let $X = (x_{ij})$ be a square matrix of order $n$ with formal entries. For $1 \le k, l \le n$ let $X^{kl}$ be the submatrix of order $n-1$ of the matrix $X$ obtained by deleting the $k$-th row and the $l$-th column. Quasideterminants of the matrix $X$ and the submatrices are connected by the following *homological relations.*

**Theorem 1.6.2.** (i) *Row homological relations:*

$$-|A|_{ij} \cdot |A^{i\ell}|_{sj}^{-1} = |A|_{i\ell} \cdot |A^{ij}|_{s\ell}^{-1}, \qquad s \neq i$$

(ii) *Column homological relations:*

$$-|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj}, \qquad t \neq j$$

The same relations hold for matrices over a ring $R$ provided the corresponding quasideterminants exist and are invertible.

A consequence of homological relations is that the ratio of two quasideterminants of an $n \times n$ matrix (each being a rational function of inversion height $n-1$) actually equals a ration of two rational functions each having inversion height $< n-1$.

### 1.6.3 Heredity

Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring $R$, and let

$$A = \begin{pmatrix} A_{11} & \dots & A_{1s} \\ & & \\ A_{s1} & \dots & A_{ss} \end{pmatrix} \tag{1.6.2}$$

be a block decomposition of $A$, where each $A_{pq}$ is a $k_p \times l_q$ matrix, $k_1 + \dots + k_s = l_1 + \dots + l_s = n$. Let us choose $p'$ and $q'$ such that $k_{p'} = l_{q'}$, so that $A_{p'q'}$ is a square matrix.

Let also $X = (x_{pq})$ be a matrix with formal variables and $|X|_{p'q'}$ be the $p'q'$-quasideterminant of $X$. In the formula for $|X|_{p'q'}$ as a rational function in variable $x_{pq}$ we can substitute each variable $x_{pq}$ with the corresponding matrix $A_{pq}$, obtaining a rational expression $F(A_{pq})$. Let us note that all matrix operations in this rational expression formally make sense, i.e., in each addition, the orders of summands coincide, in each multiplication, the number of columns of the first multiplier equals the number of rows of the second multiplier, and each matrix that has to be inverted is a square matrix. Let us assume that all matrices in this rational expression for that need to be inverted, are indeed invertible over $R$. Computing $F(A_{pq})$, we obtain an $k_{p'} \times l_{q'}$ matrix over $R$, whose rows are naturally numbered by indices

$$i = k_1 + \dots + k_{p'-1} + 1, \dots, k_1 + \dots + k_{p'} \tag{1.6.3}$$

and columns are numbered by indices

$$j = l_1 + \dots + l_{q'-1} + 1, \dots, l_1 + \dots + l_{q'}. \tag{1.6.4}$$

We denote this matrix by $|X|_{p'q'}(A)$.

Let us note that under our assumptions, $k_{p'} = l_{q'}$, so that $|X|_{p'q'}(A)$ is a square matrix over $R$.

**Theorem 1.6.3.** *Let the index $i$ lies in the range (1.4.3) and the index $j$ lies in the range (1.6.4). Let as assume that the matrix $|X|_{p'q'}(A)$ is defined. Then each of the quasideterminants $|A|_{ij}$ and $||X|_{p'q'}(A)|_{ij}$ exist if and only of the other exists, and in this case*

$$|A|_{ij} = ||X|_{p'q'}(A)|_{ij}. \tag{1.6.5}$$

**Example 1.** Let in (1.4.2) $s = 2$, $p' = q' = 1$ and $k_1 = l + 1 = 1$. Then formula (S0) becomes the inductive definition of the quasideterminant $|A|_{ij}$ (see Definition 1.2.5).

**Example 2.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Take the decomposition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ of $A$ into four $2 \times 2$ matrices, so that $A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}$, $A_{21} = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$, $A_{22} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$.
Let us use formula (1.4.5) to find the quasideterminant $|A|_{13}$. We have

$$\begin{aligned} |X|_{12}(A) &= A_{12} - A_{11}A_{21}^{-1}A_{22} \\ &= \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}^{-1} \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \\ &= \begin{pmatrix} a_{13} - \dots & a_{14} - \dots \\ a_{23} - \dots & a_{24} - \dots \end{pmatrix}. \end{aligned}$$

Denote the matrix in the right-hand side of this formula by $\begin{pmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{pmatrix}$. Then

$$|A|_{13} = \left| \begin{pmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{pmatrix} \right|_{13},$$

or, in other notation,

$$|A|_{13} = \left| \begin{pmatrix} \boxed{c_{13}} & c_{14} \\ c_{23} & c_{24} \end{pmatrix} \right|.$$

### 1.6.4 A generalization of the homological relations

Homological relations admit the following generalization. For a matrix $A = (a_{ij})$, $i \in I$, $j \in J$, and two subsets $L \subset I$, $M \subset J$ denote by $A^{L,M}$ the submatrix of the matrix $A$ obtained by deleting the rows with the indexes $\ell \in L$ and the columns with the indexes $m \in M$. Let $A$ be a square matrix, $L = (\ell_1, \dots, \ell_k)$, $M = (m_0, \dots, m_k)$. Set $M_i = M \smallsetminus \{m_i\}$, $i = 0, \dots, k$.

**Theorem 1.6.4.** [GR1, GR2] *For $p \notin L$ we have*

$$\sum_{i=0}^{k} |A^{L,M_i}|_{pm_i} \cdot |A|^{-1}_{\ell m_i} = \delta_{p\ell},$$
$$\sum_{i=0}^{k} |A|^{-1}_{m_i \ell} \cdot |A^{M_i,L}|_{m_i p} = \delta_{\ell p},$$

*provided the corresponding quasideterminants are defined and the matrices $|A|^{-1}_{m_i \ell}$, $|A|^{-1}_{\ell m_i}$ are invertible over $R$.*

### 1.6.5 Quasideterminants and Kronecker tensor products

Let $A = (a_{ij})$, $B = (b_{\alpha\beta})$ be matrices over a ring $R$. Denote by $C = A \otimes B$ the Kronecker tensor product, i.e., the matrix with entries numbered by indices $(i\alpha, j\beta)$, and with the $(i\alpha, j\beta)$-th entry equal to $c_{i\alpha, j\beta} = a_{ij} b_{\alpha\beta}$.

**Proposition 1.6.5.** *If quasideterminants $|A|_{ij}$ and $|B|_{\alpha\beta}$ are defined, then the quasideterminant $|A \otimes B|_{i\alpha, j\beta}$ is defined and*

$$|A \otimes B|_{i\alpha, j\beta} = |A|_{ij} |B|_{\alpha\beta}.$$

Note that in the commutative case the corresponding identity determinants is different. Namely, if $A$ is a $m \times m$-matrix and $B$ is a $n \times n$-matrix over a commutative ring, then $\det(A \otimes B) = (\det A)^n (\det B)^m$.

### 1.6.6 Quasideterminants and matrix rank

Let $A = (a_{ij})$ be a matrix over a division ring.

**Proposition 1.6.6.** *If the quasideterminant $|A|_{ij}$ is defined, then the following statements are equivalent.*

*(i)* $|A|_{ij} = 0$,

*(ii) the $i$-th row of the matrix $A$ is a left linear combination of the other rows of $A$;*

*(iii) the $j$-th column of the matrix $A$ is a right linear combination of the other columns of $A$.*

**Example.** Let $i, j = 1, 2$ and $|A|_{11} = 0$, i.e., $a_{11} - a_{12}a_{22}^{-1}a_{21} = 0$. Therefore, $a_{11} = \lambda a_{21}$, where $\lambda = a_{12}a_{22}^{-1}$. Since $a_{12} = (a_{12}a_{22}^{-1})a_{22}$, the first row of $A$ is proportional to the second row.

There exists the notion of linear dependence for elements of a (right or left) vector space over a division ring. So there exists the notion of the row rank (the dimension of the left vector space spanned by the rows) and the notion of the column rank (the dimension of the right vector space spanned by the columns) and these ranks are equal [Ja, Co]. This also follows from Proposition 1.6.6.

By definition, an $r$-quasiminor of a square matrix $A$ is a quasideterminant of an $r \times r$-submatrix of $A$.

**Proposition 1.6.7.** *The rank of the matrix $A$ over a division algebra is $\geq r$ if and only if at least one $r$-quasiminor of the matrix $A$ is defined and is not equal to zero.*

## 1.7 Basic identities

### 1.7.1 Row and column decomposition

The following result is an analogue of the classical expansion of a determinant by a row or a column.

**Proposition 1.7.1.** *Let $A$ be a matrix over a ring $R$. For each $k \neq p$ and each $\ell \neq q$ we have*

$$|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj}(|A^{pq}|_{kj})^{-1}|A^{pj}|_{kq},$$
$$|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pi}(|A^{pq}|_{i\ell})^{-1}a_{iq},$$

*provided all terms in right-hand sides of these expressions are defined.*

As it was pointed out in [KL], Propostiion 1.7.1 immediately follows from the homological relations.

### 1.7.2 Sylvester's identity

Let $A = (A_{ij}), i, j = 1, \dots, n$, be a matrix over a ring $R$ and $A_0 = (a_{ij}), i, j = 1, \dots, k$, a submatrix of $A$ that is invertible over $R$. For $p, q = k+1, \dots, n$ set

$$c_{pq} = \begin{vmatrix} & & & a_{1q} \\ & A_0 & & \vdots \\ & & & a_{kq} \\ a_{p1} & \dots & a_{pk} & a_{pq} \end{vmatrix}_{pq} .$$

These quasidetrminants are defined because matrix $A_0$ is invertible.

Consider the $(n-k) \times (n-k)$ matrix

$$C = (c_{pq}), \quad p, q = k+1, \dots, n.$$

The submatrix $A_0$ is called the *pivot* for the matrix $C$.

**Theorem 1.7.2.** (see [GR]) *For $i, j = k+1, \dots, n$,*

$$|A|_{ij} = |C|_{ij}$$

The commutative version of Theorem 1.7.2 is the following Sylvester's theorem.

**Theorem 1.7.3.** *Let $A = (a_{ij}), i, j = 1, \dots, n$, be a matrix over a commutative ring. Suppose that the submatrix $A_0 = (a_{ij}), i, j = 1, \dots, k$, of $A$ is invertible. For $p, q = k+1, \dots, n$ set*

$$\tilde{b}_{pq} = \det \begin{pmatrix} & & & a_{1q} \\ A_0 & & & \vdots \\ & & & a_{kq} \\ a_{p1} & \dots & a_{pk} & a_{pq} \end{pmatrix},$$

$$\tilde{B} = (\tilde{b}_{pq}), \quad p, q = k+1, \dots, n.$$

*Then*

$$\det A = \frac{\det \tilde{B}}{(\det A_0)^{n-k-1}}.$$

*Remark 1.* A quasideterminant of an $n \times n$-matrix $A$ is equal to the corresponding quasideterminant of a $2 \times 2$-matrix consisting of $(n-1) \times (n-1)$-quasiminors of the matrix $A$, or to the quasideterminant of an $(n-1)\times(n-1)$-matrix consisting of $2\times 2$-quasiminors of the matrix $A$. One can use any of these procedures for an inductive definition of quasideterminants. In fact, Heyting [H] essentially defined the quasideterminants $|A|_{nn}$ for matrices $A=(a_{ij})$, $i,j=1,\dots,n$, in this way.

*Remark 2.* Theorem 1.7.2 can be generalized to the case where $A_0$ is a square submatrix of $A$ formed by some (not necessarily consecutive and not necessarily the same) rows and columns of $A$. In particular, in the case where $A_0=(a_{ij}), i,j=2,\dots,n-1$, Theorem 1.7.2 is an analogue of a well-known commutative identity which is called the “Lewis Carroll identity” (see, for example, [Ho]).

### 1.7.3 Inversion for quasiminors

The following theorem was formulated in [GR]. For a matrix $A=(a_{ij})$, $i\in I$, $j\in J$, over a ring $A$ and subsets $P\subset I$, $Q\subset J$ denote by $A_{PQ}$ the submatrix

$$A_{PQ}=(a_{\alpha\beta}),\qquad \alpha\in P,\quad \beta\in Q.$$

Let $|I|=|J|$ and $B=A^{-1}=(b_{rs})$. Suppose that $|P|=|Q|$.

**Theorem 1.7.4.** *Let $k\notin P, \ell\notin Q$. Then*

$$|A_{P\cup\{k\},Q\cup\{\ell\}}|_{k\ell}\cdot|B_{I\setminus P,J\setminus Q}|_{\ell k}=1.$$

Set $P=I\setminus\{k\}, Q=J\setminus\{\ell\}$. Then this theorem leads to the already mentioned identity

$$|A|_{k\ell}\cdot b_{\ell k}=1.$$

**Example.** Theorem 1.7.4 implies the following identity for principal quasiminors. Let $A=(a_{ij})$, $i,j=1,\dots,n$ be an invertible matrix over $R$ and $B=(b_{ij})=A^{-1}$. For a fixed $k$, $1\le k\le n$, set $A_{(k)}=(a_{ij})$, $i,j=1,\dots,k$ and $B^{(k)}=(b_{ij})$, $i,j=k,\dots,n$. Then

$$|A_{(k)}|_{kk}\cdot|B^{(k)}|_{kk}=1.$$

### 1.7.4 Multiplicative properties of quasideterminants

Let $X=(x_{pq}), Y=(y_{rs})$ be $n\times n$-matrices. The following statement follows directly from from Definition 1.2.2.

**Theorem 1.7.5.** *We have*

$$|XY|_{ij}^{-1}=\sum_{p=1}^{n}|Y|_{pj}^{-1}|X|_{ip}^{-1}.$$

### 1.7.5 Cayley–Hamilton theorem

Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a matrix over a ring $R$. Denote by $E_n$ the identity matrix of order $n$.

Let $t$ be a formal variable. Set $f_{ij} = |tE_n - A|_{ij}$ for $1 \leq i, j \leq n$. Then $f_{ij}(t)$ is a rational function in $t$. Define the matrix function $\tilde{f}_{ij}(t)$ by replacing in $f_{ij}(t)$ each element $a_{ij}$ with the matrix $\tilde{a}_{ij} = a_{ij}E_n$ of order $n$ and the variable $t$ by the matrix $A$. The functions $f_{ij}(t)$ are called the characteristic functions of the matrix $A$.

The following theorem was stated in [GR1, GR2].

**Theorem 1.7.5.** $\tilde{f}_{ij}(A) = 0$ *for all* $i, j = 1, \ldots, n$.

# Chapter 2

# Important example: quaternionic quasideterminants

As an example, we compute here quasideterminants of quaternionic matrices.

## 2.1 Norms of quaternionic matrices

Let $\mathbb{H}$ be the algebra of quaternions. Algebra $\mathbb{H}$ is an algebra over the field of real numbers $\mathbb{R}$ with generators $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$. It follows from the definition that $\mathbf{ij} + \mathbf{ji} = 0$, $\mathbf{ik} + \mathbf{ki} = 0$, $\mathbf{jk} + \mathbf{kj} = 0$.

Algebra $\mathbb{H}$ posseses a standard anti-involution $a \mapsto \bar{a}$: if $a = x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k}$, $x, y, z, t \in \mathbb{R}$, then $\bar{a} = x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$. It follows that $a\bar{a} = x^2 + y^2 + z^2 + t^2$. The multiplicative functional $\nu : \mathbb{H} \to \mathbb{R}_{\geq 0}$ where $\nu(a) = a\bar{a}$ is called the norm of $a$. One can see that $a^{-1} = \frac{\bar{a}}{\nu(a)}$ for $a \neq 0$.

We will need the following generalization of the norm $\nu$ to quaternionic matrices. Let $M(n, \mathbb{H})$ be the $\mathbb{R}$-algebra of quaternionic matrices of order $n$. There exists a unique multiplicative functional $\nu : M(n, \mathbb{H}) \to \mathbb{R}_{\geq 0}$ such that

(i) $\nu(A) = 0$ if and only if the matrix $A$ is non-invertible,

(ii) If $A'$ is obtained from $A$ by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $\nu(A') = \nu(A)$.

(iii) $\nu(E_n) = 1$ where $E_n$ is the identity matrix of order $n$.

The number $\nu(A)$ is called the *norm* of the quaternionic matrix $A$.

For a quaternionic matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$, denote by $A^* = (\bar{a}_{ji})$ the conjugate matrix. It is known that $\nu(A)$ coincides with the Dieudonne determinant of $A$ and with the Moore determinant of $AA^*$ (see [As] and Subsections 3.2–3.4 below). The norm $\nu(A)$ is a real number and it is equal to an alternating sum of monomials of order $2n$ in the $a_{ij}$ and $\bar{a}_{ij}$. An expression for $\nu(A)$ is given by Theorem 2.1.2 below.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a quaternionic matrix. Let $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$ be two ordered sets of natural numbers such that all $i_p$ and all $j_p$ are distinct. Set

$$z_{I,J} = a_{i_1 j_1} \bar{a}_{i_2 j_1} a_{i_2 j_2} \dots a_{i_k j_k} \bar{a}_{i_1 j_k}.$$

Denote by $\mu_i(A)$ the sum of all $z_{I,J}(A)$ such that $i_1 = i$. One can easily see that $\mu_i(A)$ is a real number since with each monomial $z_{I,J}$ it contains the conjugate monomial $\bar{z}_{I,J} = z_{I',J'}$, where $I' = \{i_1, i_k, i_{k-1} \dots, i_2\}$, $J = \{j_k, j_{k-1}, \dots, j_1\}$.

**Proposition 2.1.1.** *The sum $\mu_i(A)$ does not depend on $i$.*

**Example.** For $n = 1$ the statement is obvious. For $n = 2$ we have

$$\mu_1(A) = a_{11}\bar{a}_{21}a_{22}\bar{a}_{12} + a_{12}\bar{a}_{22}a_{21}\bar{a}_{11},$$
$$\mu_2(A) = a_{22}\bar{a}_{12}a_{11}\bar{a}_{21} + a_{21}\bar{a}_{11}a_{12}\bar{a}_{22}.$$

Note that for two arbitrary quaternions $x, y$ we have $xy + \bar{y}\bar{x} = 2\Re(xy) = 2\Re(yx) = yx + \bar{x}\bar{y}$, where $\Re(a)$ is the real part of the quaternion $a$. By setting $x = a_{11}\bar{a}_{21}$, $y = a_{22}\bar{a}_{12}$ one see that $\mu_1(A) = \mu_2(A)$.

Proposition 2.1.1 shows that we may omit the index $i$ in $\mu_i(A)$ and denote it by $\mu(A)$.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$ be a matrix. We call an (unordered) set of square submatrices $\{A_1, \dots, A_s\}$ where $A_p = (a_{ij})$, $i \in I_p$, $j \in J_p$ a *complete set* if $I_p \cap I_q = J_p \cap J_q = \emptyset$ for all $p \neq q$ and $\cup_p I_p = \cup_p J_p = \{1, \dots, n\}$.

**Theorem 2.1.2.** *Let $A = (a_{ij})$, $i, j = 1, \dots, n$ be a quaternionic matrix. Then*

$$\nu(A) = \sum (-1)^{k_1 + \dots + k_p - p} \mu(A_1) \dots \mu(A_p),$$

*where the sum is taken over all complete sets $(A_1, \dots, A_p)$ of submatrices of $A$, $k_i$ is the order of the matrix $A_i$.*

**Example.** For $n = 2$ we have

$$\nu(A) = \nu(a_{11})\nu(a_{22}) + \nu(a_{12})\nu(a_{21}) - (a_{11}\bar{a}_{21}a_{22}\bar{a}_{12} + a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}).$$

**Corollary 2.1.3.** *Let $A$ be a square quaternionic matrix. Fix an arbitrary $i \in \{1, \dots, n\}$. Then*

$$\nu(A) = \sum (-1)^{k(B_1)-1} \nu(B_1)\mu(B_2)$$

*where the sum is taken over all complete sets of submatrices $(B_1, B_2)$ such that $B_2$ contains an element from the $i$-th row, $k(B_1)$ the order of $B_1$, and $\nu(B_1) = 1$ if $B_2 = A$.*

## 2.2 Quasideterminants of quaternionic matrices

This section contains results from [GRW1].

Let $A = (a_{ij})$ , $i, j = 1, \dots, n$, be a quaternionic matrix. Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ be two ordered sets of natural numbers $1 \le i_1, i_2, \dots, i_k \le n$ and $1 \le j_1, j_2, \dots, j_k \le n$ such that all $i_p$ are distinct and all $j_p$ are distinct. For $k = 1$ set $m_{I,J}(A) = a_{i_1 j_1}$. For $k \ge 2$ set

$$m_{I,J}(A) = a_{i_1 j_2} \bar{a}_{i_2 j_2} a_{i_2 j_3} \bar{a}_{i_3 j_3} a_{i_3 j_4} \dots \bar{a}_{i_k j_k} a_{i_k j_1}.$$

If the matrix $A$ is Hermitian, i.e., $a_{ji} = \bar{a}_{ij}$ for all $i, j$, then

$$m_{I,J}(A) = a_{i_1 j_2} a_{j_2 i_2} a_{i_2 j_3} a_{j_3 i_3} a_{i_3 j_4} \dots a_{j_k i_k} a_{i_k j_1}.$$

To a quaternionic matrix $A = (a_{pq})$, $p, q = 1, \dots, n$, and to a fixed row index $i$ and a column index $j$ we associate a polynomial in $a_{pq}$, $\bar{a}_{pq}$, which we call the $(i, j)$-th double permanent of $A$.

**Definition 2.2.1.** The $(i, j)$-th double permanent of $A$ is the sum

$$\pi_{ij}(A) = \sum m_{I,J}(A),$$

taken over all orderings $I = \{i_1, \dots, i_n\}$, $J = \{j_1, \dots, j_n\}$ of $\{1, \dots, n\}$ such that $i_1 = i$ and $j_1 = j$ .

**Example.** For $n = 2$

$$\pi_{11}(A) = a_{12} \bar{a}_{22} a_{21}.$$

For $n = 3$

$$\pi_{11}(A) = a_{12} \bar{a}_{32} a_{33} \bar{a}_{23} a_{21} + a_{12} \bar{a}_{22} a_{23} \bar{a}_{33} a_{31} + a_{13} \bar{a}_{33} a_{32} \bar{a}_{22} a_{21} + a_{13} \bar{a}_{23} a_{22} \bar{a}_{32} a_{31}.$$

For a submatrix $B$ of $A$ denote by $B^c$ the matrix obtained from $A$ by deleting all rows and columns containing elements from $B$. If $B$ is a $k \times k$-matrix, then $B^c$ is a $(n-k) \times (n-k)$-matrix. $B^c$ is called the complementary submatrix of $B$.

Quasideterminants of a matrix $A = (a_{ij})$ are rational functions of elements $a_{ij}$. Therefore, for a quaternionic matrix $A$, its quasideterminants are polynomials in $a_{ij}$ and their conjugates, with coefficients that are rational functions of $a_{ij}$ always taking rational values. The following theorem gives expressions for these polynomials.

**Theorem 2.2.2** If the quasideterminant $|A|_{ij}$ of a quaternionic matrix is defined, then

$$\nu(A^{ij})|A|_{ij} = \sum (-1)^{k(B)-1} \nu(B^c) \pi_{ij}(B) \tag{2.2.1}$$

where the sum is taken over all square submatrices $B$ of $A$ containing $a_{ij}$, $k(B)$ is the order of $B$, and we set $\nu(B^c) = 1$ for $B = A$.

Recall that the quasideterminant $|A|_{ij}$ is defined if the matrix $A^{ij}$ is invertible. In this case $\nu(A^{ij})$ is invertible, so that formula (2.2.1) indeed gives an expression for $|A|_{ij}$.

The right-hand side in (2.2.1) is a linear combination with real coefficients of monomials of lengths $1, 3, \dots, 2n-1$ in $a_{ij}$ and $\bar{a}_{ij}$. The number $\mu(n)$ of such monomials for a matrix of order $n$ is $\mu(n) = 1 + (n-1)^2\mu(n-1)$.

**Example.** For $n = 2$

$$\nu(a_{22})|A|_{11} = \nu(a_{22})a_{11} - a_{12}\bar{a}_{22}a_{21}.$$

For $n = 3$

$$\begin{aligned}\nu(A^{11})|A|_{11} = \nu(A^{11})a_{11} &- \nu(a_{33})a_{12}\bar{a}_{22}a_{21} - \nu(a_{23})a_{12}\bar{a}_{32}a_{31} - \\ &- \nu(a_{32})a_{13}\bar{a}_{23}a_{21} - \nu(a_{22})a_{13}\bar{a}_{33}a_{31} + a_{12}\bar{a}_{32}a_{33}\bar{a}_{23}a_{21} + \\ &+ a_{12}\bar{a}_{22}a_{23}\bar{a}_{33}a_{31} + a_{13}\bar{a}_{33}a_{32}\bar{a}_{22}a_{21} + a_{13}\bar{a}_{23}a_{22}\bar{a}_{32}a_{31}.\end{aligned}$$

The example shows how to simplify the general formula for quasideterminants of matrix of order 3 for quaternionic matrices.

The following theorem, which is similar to Corollary 2.1.3, shows that the coefficients in formula (2.2.1) are uniquely defined.

**Theorem 2.2.3.** Let quasideterminants $|A|_{ij}$ of quaternionic matrices are given by the formula

$$\xi(A^{ij})|A|_{ij} = \sum(-1)^{k(B)-1}\xi(B^c)\pi_{ij}(B)$$

and all coefficients $\xi(C)$ depend of submatrix $C$ only, then $\xi(C) = \nu(C)$ for all square matrix $C$.

**Example.** For $n = 2$ set $a_{11} = 0$. Then $\xi(a_{22})a_{12}a_{22}^{-1}a_{21} = a_{12}\bar{a}_{22}a_{21}$. This implies that $\xi(a_{22}) = \bar{a}_{22}a_{22} = \nu(a_{22})$.

# Chapter 3

# Noncommutative determinants

Noncommutative determinants were defined in different and, sometimes, not related situations. In this section we present some results from [GR, GR1, GR2, GRW1] describing a universal approach to noncommutative determinants and norms of noncommutative matrices based on the notion of quasideterminants.

## 3.1 Noncommutative determinants as products of quasiminors

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix over a division ring $R$ such that all square submatrices of $A$ are invertible. For $\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ define $A^{i_1 \dots i_k, j_1 \dots j_k}$ to be the submatrix of $A$ obtained by deleting rows with indices $i_1, \dots, i_k$ and columns with indices $j_1, \dots, j_k$. Next, for any orderings $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ set

$$D_{I,J}(A) = |A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2 j_2} |A^{i_1 i_2, j_1 j_2}|_{i_3 j_3} \dots a_{i_n j_n}.$$

In the commutative case $D_{I,J}(A)$ is, up the the sign, the determinant of $A$. When $A$ is a quantum matrix $D_{I,J}(A)$ differs from the quantum determinant of $A$ by a factor depending on $q$ [GR, GR1, KL]. The same is true for some other noncommutative algebras. This suggests to call $D_{I,J}(A)$ the $(I, J)$-predeterminants of $A$. From the "categorical point of view" the expressions $D_{I,\tilde{I}}(A)$ where $I = (i_1, i_2, \dots, i_n)$, $\tilde{I} = (i_2, i_3, \dots, i_n, i_1)$ are particularly important. We denote $D_I(A) = D_{I,\tilde{I}}(A)$. It is also convenient to have the basic predeterminant

$$\Delta(A) = D_{\{12 \dots n\},\{23 \dots n1\}}. \tag{3.1.1}$$

We use the homological relations for quasideterminants to compare different $D_{I,J}$. Here we restrict ourselves to elementary transformations of $I$ and $J$.

Let $I = (i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$ and $J = (j_1, \ldots, j_p, j_{p+1}, \ldots, j_n)$. Set $I' = (i_1, \ldots, i_{p+1}, i_p, \ldots, i_n)$, $J' = (j_1, \ldots, j_{p+1}, j_p, \ldots, j_n)$. Set also

$$
\begin{aligned}
X &= |A|_{i_1,j_1}|A^{i_1,j_1}|_{i_2,j_2} \ldots |A^{i_1\ldots i_{p-2},j_1,\ldots,j_{p-2}}|_{i_{p-1},j_{p-1}},\\
Y &= |A^{i_1\ldots i_{p+1},j_1,\ldots,j_{p+1}}|_{i_{p+2},j_{p+2}} \ldots a_{i_n,j_n},\\
u &= |A^{i_1\ldots i_p,j_1,\ldots,j_p}|_{i_{p+1},j_{p+1}},\\
w_1 &= |A^{i_1\ldots i_{p-1}i_{p+1},j_1,\ldots,j_p}|_{i_p,j_{p+1}},\\
w_2 &= |A^{i_1,\ldots i_p,j_1,\ldots,j_{p-1}}|_{i_{p+1},j_{p+1}}.
\end{aligned}
$$

**Proposition 3.1.1.** We have

$$
\begin{aligned}
&D_{I,J'} = -D_{I,J}Y^{-1}u^{-1}w_2^{-1}uw_2Y,\\
&D_{I',J} = -Xuw_1^{-1}X^{-1}D_{I,J}Y^{-1}u^{-1}w_1Y.
\end{aligned}
$$

Let $C$ be a commutative ring with a unit and $f : R \to C$ be a multiplicative map, i.e. $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_n)$ be any orderings of $(1, \ldots, n)$. For an element $\sigma$ from the symmetric group of $n$-th order set $\sigma(I) = (\sigma(i_1), \ldots, \sigma(i_n))$. Let $p(\sigma)$ be the parity of $\sigma$.

Proposition 3.1.1 immediately implies the following theorem.

**Theorem 3.1.2.** In notations of Section 3.1 we have

$$f(D_{I,J}(A)) = f(-1)^{p(\sigma_1)+p(\sigma_2)}f(D_{\sigma(I),\sigma(J)}(A)).$$

It follows that $f(D_{I,J}(A)$ is uniquely defined up to a power of $f(-1)$. We call $f(D_{1\ldots n,1\ldots n})(A))$ the $f$-determinant $A$ and denote it by $fD(A)$. Note that if $f$ is a homomorphism then $f$-determinant $fD(A)$ equals to the usual determinant of the commutative matrix $f(A)$.

**Corollary 3.1.3.** We have

$$fD(AB) = fD(A) \cdot fD(B).$$

When $R$ is the algebra of quaternions and $f(a) = \nu(a) = a\bar{a}$, or, in other words, $f$ is the quaternionic norm, then one can see that $fD(a)$ is the matrix quaternionic norm $\nu(A)$ (see Section 2.1).

In Theorems 3.1.4–3.1.6 we present formulas for determinants of triangular and almost triangular matrices. A matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$, is called an upper-triangular matrix if $a_{ij} = 0$ for $i > j$. An upper-triangular matrix $A$ is called a generic upper-triangular matrix if every square submatrix $A$ consisting of the rows $i_1 \le i_2 \le \cdots \le i_k$ and the columns $j_1 \le j_2 \le \cdots \le j_k$ such that $i_1 \le j_1$, $i_2 \le j_2$, ..., $i_k \le j_k$, is invertible.

**Theorem 3.1.4.** *Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a generic upper-triangular matrix. The determinants $D_{i_1 i_2 \ldots i_n}(A)$ are defined if and only if $i_1 = n$. In this case*

$$D_{ni_2\ldots i_{n-1}}(A) = \\ = a_{nn} \cdot |A^{n,i_2}|^{-1}_{i_2 n} \cdot a_{i_2 i_2} \cdot |A^{n,i_2}|_{i_2 n} \cdot |A^{ni_2,i_2 i_3}|^{-1}_{i_3 n} \cdot a_{i_3 i_3} |A^{ni_2,i_2 i_3}|_{i_3 n} \cdot \ldots \\ \cdot |A^{ni_2 i_3\ldots i_{n-1},i_2 i_3\ldots i_n}|^{-1}_{i_n n} \cdot a_{i_n i_n} \cdot |A^{ni_2 i_3 \ldots i_{n-1}, i_2 i_3 \ldots i_n}|_{i_n n}.$$

In particular,

$$D_{n,n-1\ldots 2,1}(A) = a_{nn} a^{-1}_{n-1,n} a_{n-1,n-1} a_{n-1,n} \ldots a^{-1}_{1n} a_{11} a_{1n}.$$

A matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$, is called an almost upper-triangular matrix if $a_{ij} = 0$ for $i > j + 1$. An almost upper-triangular matrix $A$ is called a Frobenius matrix if $a_{ij} = 0$ for all $j \ne n$ and $i \ne j + 1$, and $a_{j+1j} = 1$ for $j = 1, \ldots, n - 1$.

**Theorem 3.1.5.** *If $A$ is invertible upper-triangular matrix, then*

$$D_{1,n,n-1\ldots 2}(A) = |A|_{1n} a_{n,n-1} a_{n-1,n-2} \ldots a_{21}.$$

By Proposition 1.2.7, the determinant $D_{1,n,n-1\ldots 2}(A)$ of an uppen-triangular matrix $A$ is polynomial in $a_{ij}$.

Let $p(I)$ be the parity of the ordering $I = (i_1, \ldots, i_n)$.

**Theorem 3.1.6.** *If $A$ is a Frobenius matrix and the determinant $D_I(A)$ is defined, then $D_I(A) = (-1)^{p(I)+1} a_{1n}$.*

Now let $R$ be a division ring, $R^* = R \setminus \{0\}$ the monoid of invertible elements in $R$ and $\pi : R^* \to R^*/[R^*, R^*]$ the canonical homomorphism. To the abelian group $R^*/[R^*, R^*]$ we adjoin the zero element 0 with obvious multiplication, and denote the obtained semi-group by $\tilde{R}$. Extend $\pi$ to a map $R \to \tilde{R}$ by setting $\pi(0) = 0$.

We recall here the classical notion of the Dieudonne determinant (see [D, A]). There exists a unique homomorphism

$$\det : M_n(R) \to \tilde{R}$$

such that

(i) $\det A' = \tilde{\mu} \det A$ for any matrix $A'$ obtained from $A \in M_n(R)$ by multiplying one row of $A$ from the left by $\mu$;

(ii) $\det A'' = \det A$ for any matrix $A''$ obtained from $A$ by adding one row to another;

(iii) $\det(E_n) = 1$ for the identity matrix $E_n$.

The homomorphism det is called the Dieudonne determinant.

It is known that $\det A = 0$ if $\mathrm{rank}(A) < n$ (see [A], Chapter 4). The next proposition gives a construction of the Dieudonne determinant in the case where $\mathrm{rank}(A) = n$.

**Proposition 3.1.7.** *Let $A$ be an $n \times n$-matrix over a division ring $R$. If rank$(A) = n$, then*

(i) *There exist orderings $I$ and $J$ of $\{1, \dots, n\}$ such that $D_{I,J}(A)$ is defined.*

(ii) *If $D_{I,J}(A)$ is defined, then the Diedonne determinant is given by the formula* $\det A = p(I)p(J)\pi(D_{I,J}(A))$*, where $p(I)$ is the parity of the ordering $I$.*

Note that in [Dr] Draxl introduced the Dieudonne predeterminant, denoted $\delta\epsilon\tau$. For a generic matrix $A$ over a division ring there exists the Gauss decomposition $A = UDL$ where $U, D, L$ are upper-unipotent, diagonal, and lower-unipotent matrices. Then Draxl $\delta\epsilon\tau(A)$ is defined as the product of diagonal elements in $D$ from top to the bottom. For nongeneric matrices Draxl used the Bruhat decomposition instead of the Gauss decomposition.

**Proposition 3.1.8.** $\delta\epsilon\tau(A) = \Delta(A)$*, where $\Delta(A)$ is given by* (3.1.1).

Proof (for a generic $A$)Let $y_1, \dots, y_n$ be the diagonal elements in $D$ from top to the bottom. As shown in [GR1, GR2] (see also 4.9), $y_k = |A^{12\dots k-1, 12\dots k-1}|_{kk}$. Then $\delta\epsilon\tau(A) = y_1 y_2 \dots y_n = \Delta(A)$. □

Below we consider below special examples of noncommutative determinants.

## 3.2 Dieudonne determinant for quaternions

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a quaternionic matrix. If $A$ is not inverstible, then the Dieudonne determinant of $A$ equals zero. By Proposition 3.1.7, if $A$ is invertible, there exist orderings $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ such that the following expressions are defined:

$$D_{I,J}(A) = |A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2, j_2} |A^{i_1 i_2, j_1 j_2}|_{i_3 j_3} \dots a_{i_n j_n}.$$

By Theorem 2.2.2, $D_{I,J}(A)$ can be expressed as a polynomial in $a_{ij}$ and $\overline{a_{ij}}$ with real coefficients.

In the quaternionic case the Dieudonne determinant $D$ coincides with the map

$$\det : M_n(\mathbb{H}) \to \mathbb{R}_{\geq 0}$$

(see [As]).

The following proposition generalizes a result in [VP].

**Proposition 3.2.1** *In the quaternionic case for each $I, J$ we have*

$$\det A = \nu(D_{I,J}(A))^{1/2}$$

(*the positive square root*).

The proof of Proposition 3.2.1 follows from the homological relations for quasideterminants.

## 3.3 Moore determinants of Herimitian quaternionic matrices

A quaternionic matrix $A = (a_{ij})$, $i, j = 1, \dots, n$, is called Hermitian if $a_{ji} = \bar{a}_{ji}$ for all $i, j$. It follows that all diagonal elements of $A$ are real numbers and that the submatrices $A^{11}$, $A^{12,12}$, $\dots$ are Hermitian.

The notion of determinant for Hermitian quaternionic matrices was introduced by E. M. Moore in 1922 [M, MB]. Here is the original definition.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix over a ring. Let $\sigma$ be a permutation of $\{1, \dots, n\}$. Write $\sigma$ as a product of disjoint cycles. Since disjoint cycles commute, we may write

$$\sigma = (k_{11} \dots k_{1j_1})(k_{21} \dots k_{2j_2}) \dots (k_{m1} \dots k_{mj_m})$$

where for each $i$, we have $k_{i1} < k_{ij}$ for all $j > 1$, and $k_{11} > k_{21} > \dots > k_{m1}$. This expression is unique. Let $p(\sigma)$ be the parity of $\sigma$. The Moore determinant $M(A)$ is defined as follows:

$$M(A) = \sum_{\sigma \in S_n} p(\sigma) a_{k_{11},k_{12}} \dots a_{k_{1j_1},k_{11}} a_{k_{21},k_{22}} \dots a_{k_{mj_m},k_{m1}}. \tag{3.3.1}$$

(There are equaivalent formulations of this definition; e.g., one can require $k_{i1} > k_{ij}$ for all $j > 1$.) If $A$ is Hermitian quaternionic matrix then $M(A)$ is a real number. Moore determinants have nice features and are widely used (see, for example, [Al, As, Dy1]).

We will show (Theorem 3.3.2) that determinants of Hermitian quaternionic matrices can be obtained using our general approach. First we prove that for a quaternionic Hermitian matrix $A$, the determinants $D_{I,I'}(A)$ coincide up to a sign.

Recall that $\Delta(A) = D_{I,I'}(A)$ for $I = \{1, \dots, n\}$ and that $\Delta(A)$ is a pre-Dieudonne determinant in the sense of [Dr]. If $A$ is Hermitian, then $\Delta(A)$ is a product of real numbers and, therefore, $\Delta(A)$ is real.

**Proposition 3.3.1.** *Let $p(I)$ be the parity of the ordering $I$. Then $\Delta(A) = p(I)p(J)D_{I,J}(A)$.* The proof follows from homological relations for quasideterminants.

**Theorem 3.3.2.** Let $A$ be a Hermitian quaternionic matrix. Then $\Delta(A) = M(A)$ (see (3.3.1)).

*Proof.* We use the noncommutative Sylvester formula for quasideterminants (Theorem 1.5.2).

For $i, j = 2, \dots, n$ define a Hermitian matrix $B_{ij}$ by the formula

$$B_{ij} = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}.$$

Let $b_{ij} = M(B_{ij})$ and $c_{ij} = |B_{ij}|_{11}$.

Note that $B = (b_{ij})$ and $C = (c_{ij})$ also are Hermitian matrices. It follows from (3.3.1) that $M(A) = a_{nn}^{2-n} M(B)$. Note, that $M(B) = a_{nn}^{n-1} M(C)$, therefore, $M(A) = a_{nn} M(C)$.

By noncommutative Sylvester identity, $|A|_{11} = |C|_{11}$, $|A^{11}|_{22} = |C^{11}|_{22}$, .... So,

$$|A^{11}|_{22}|A^{11}|_{22} \dots |A^{12\dots n-1,12\dots n-1}|_{n-1,n-1} = \\ = |C^{11}|_{22}|C^{11}|_{22} \dots |C^{12\dots n-1,12\dots n-1}|_{n-1,n-1}.$$

The product on the left-hand side equals $\Delta(A) a_{nn}^{-1}$ and the product on right-hand side equals $\Delta(C)$, so $\Delta(A) = \Delta(C) a_{nn} = M(A)$. □ □

## 3.4 Moore determinants and norms of quaternionic matrices

**Proposition 3.4.1.** *For generic matrices $A, B$ we have*

$$\nu(A) = \Delta(A)\Delta(A^*) = \Delta(AA^*).$$

Since $AA^*$ is a Hermitian matrix, one has the following

**Corollary 3.4.2.** $\nu(A) = M(AA^*)$.

## 3.5 Study determinants

An embedding of the field of complex numbers $\mathbb{C}$ into $\mathbb{H}$ is defined by an image of $\mathbf{i} \in \mathbb{C}$. Chose the embedding given by $x + y\mathbf{i} \mapsto x + y\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, where $x, y \in \mathbb{R}$ and identify $\mathbb{C}$ with its image in $\mathbb{H}$. Then any quaternion $a$ can be uniquely written as $a = \alpha + \mathbf{j}\beta$ where $\alpha, \beta \in \mathbb{C}$.

Let $M(n, F)$ be the algebra of matrices of order $n$ over a field $F$. Define a homomorphism $\theta : \mathbb{H} \to M(2, \mathbb{C})$ by setting

$$\theta(a) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$

For $A = (a_{ij}) \in M(n, \mathbb{H})$, set $\theta_n(A) = (\theta(a_{ij}))$. This extends $\theta$ to homomorhism of matrix algebras

$$\theta_n : M(n, \mathbb{H}) \to M(2n, \mathbb{C}).$$

In 1920, Study [S] defined a determinant $S(A)$ of a quaternionic matrix $A$ of order $n$ by setting $S(A) = \det \theta_n(A)$. Here det is the standard determinant of a complex matrix. The following proposition is well known (see [As]).

**Proposition 3.5.1.** *For any quaternionic matrix $A$*

$$S(A) = M(AA^*).$$

The proof in [As] was based on properties of eigenvalues of quaternionic matrices. Our proof based on Sylvester's identity and homological relations actually shows that $S(A) = \nu(A)$ for a generic matrix $A$.

## 3.6 Quantum determinants

Note, first of all, that quantum determinants and the Capelli determinants (to be discussed in Section 3.7) are not defined for all matrices over the corresponding algebras. For this reason, they are not actual determinants, but, rather, "determinant-like" expressions. However, using the traditional terminology, we will talk about quantum and Capelli determinants.

We say that $A = (a_{ij}), i, j = 1, \dots, n$, is a *quantum matrix* if, for some central invertible element $q \in F$, the elements $a_{ij}$ satisfy the following commutation relations:

$$\begin{aligned}
a_{ik}a_{il} &= q^{-1}a_{il}a_{ik} \quad \text{for } k < l,\\
a_{ik}a_{jk} &= q^{-1}a_{jk}a_{ik} \quad \text{for } i < j,\\
a_{il}a_{jk} &= a_{jk}a_{il} \quad \text{for } i < j, k < l,\\
a_{ik}a_{jl} - a_{jl}a_{ik} &= (q^{-1} - q)a_{il}a_{jk} \quad \text{for } i < j, k < l.
\end{aligned} \tag{3.6.1}$$

Denote by $\mathcal{A}(n, q)$ the algebra with generators $(a_{ij})$, $i, j = 1, \dots, n$, satisfying relations (3.6.1). The center of this algebra is the one-dimensional subspace generated by the so called *quantum determinant* of $A$.

The quantum determinant $\det_q A$ is defined as follows:

$$\det{}_q A = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where $l(\sigma)$ is the number of inversions in $\sigma$.

If $A$ is a quantum matrix, then any square submatrix of $A$ also is a quantum matrix with the same $q$.

Note that the algebra $\mathcal{A}(n, q)$ admits the ring of fractions.

**Theorem 3.6.1.** ([GR, KL]) *In the ring of fractions of the algebra $\mathcal{A}(n, q)$ we have we have*

$$\det{}_q A = (-q)^{i-j} |A|_{ij} \cdot \det{}_q A^{ij} = (-q)^{i-j} \det{}_q A^{ij} \cdot |A|_{ij}.$$

**Corollary 3.6.2.** ([GR, KL]) *In the ring of fractions of the algebra $\mathcal{A}(n,q)$ we have*

$$\det{}_q A = |A|_{11}|A^{11}|_{22}\dots a_{nn}$$

*and all factors on the right-hand side commute.*

An important generalization of this result for matrices satisfying Faddeev–Reshetikhin–Takhtadjan relations is given in [ER].

## 3.7 Capelli determinants

Let $X = (x_{ij})$, $i,j = 1,\dots,n$ be a matrix of formal commuting variables and $X^T$ the transposed matrix. Let $D = (\partial_{ij})$, $\partial_{ij} = \partial/\partial x_{ij}$, be the matrix of the corresponding differential operators. Since each of the matrices $X$, $D$ consists of commuting entries, $\det X$ and $\det D$ make sense. Let us set $X^T D = (f_{ij})$, so that $f_{ij} = \sum_k x_{ki}\partial/\partial x_{kj}$.

Let $W$ be a diagonal matrix, $W = \mathrm{diag}(0,1,2,\dots,n)$.

By definition, the Capelli determinant $\det_{Cap}$ of $X^T D - W$ equals to the sum

$$\sum_{\sigma\in S_n}(-1)^{l(\sigma)}f_{\sigma(1)1}(f_{\sigma(2)2}-\delta_{\sigma(2)2})\dots(f_{\sigma(n)n}-(n-1)\delta_{\sigma(n)n}).$$

The classical Capelli identity says that the sum is equal to $\det X\det D$.

Set $Z = X^T D - I_n$. It was shown in [GR1, GR2] that the Capelli determinant can be expressed as a product of quasideterminants. More precisely, let $\mathcal{D}$ be the algebra of polynomial differential operators with variables $x_{ij}$.

**Theorem 3.7.1.** *In the ring of fractions of the algebra $\mathcal{D}$ we have*

$$|Z|_{11}|Z^{11}|_{22}\dots z_{nn} = \det X\det D$$

*and all factors on the left-hand side commute.*

It is known [We] that the right-hand side in the theorem is equal to the Capelli determinant.

This theorem can also be interpreted in a different way.

Let $A = (e_{ij})$, $i,j = 1,\dots n$ be the matrix of the standard generators of the universal enveloping algebra $U(gl_n)$. Recall that these generators satisfy the relations

$$[e_{ij},e_{kl}] = \delta_{jk}e_{il}-\delta_{li}e_{kj}.$$

Let $E_n$ be the identity matrix of order $n$. It is well known (see, for example, [Ho]) that coefficients of the polynomial in a central variable $t$

$$\det(I_n+tA) := \sum_{\sigma\in S_n}(-1)^{l(\sigma)}(\delta_{\sigma(1)1}+te_{\sigma(1)1})\dots(\delta_{\sigma(n)n}+t(e_{\sigma(n)n}-(n-1)\delta_{\sigma(n)n}))$$

generate the center of $U(gl_n)$.

Theorem 3.7.1 can be reformulated in the following way [GKLLRT].

**Theorem 3.7.2** $\det(I_n + tA)$ can be factored in the algebra of formal power series in $t$ with coefficients in $U(gl_n)$:

$$\det(I_n+tA) \quad = (1+te_{11})\begin{vmatrix} 1+t(e_{11}-1) & te_{12} \\ te_{21} & \boxed{1+t(e_{22}-1)} \end{vmatrix} \cdot \ldots$$

$$\cdot \begin{vmatrix} 1+t(e_{11}-n+1) & \ldots & te_{1n} \\ \ldots & \ldots & \ldots \\ te_{n1} & \ldots & \boxed{1+t(e_{nn}-n+1)} \end{vmatrix}$$

and the factors on the right-hand side commute with each other.

The above version is obtained by using the classical embedding of $U(gl_n)$ into the Weyl algebra generated by $(x_{ij}, \partial/\partial x_{ij})$, $i,j = 1,\ldots,n$, where $e_{ij}$ corresponds to

$$f_{ij} = \sum_{k=1}^{n} x_{ki}\partial/\partial x_{kj}$$

.

## 3.8 Berezinians

Let $p(k)$ be the parity of an integer $k$, i.e. $p(k) = 0$ if $k$ is even and $p(k) = 1$ if $k$ is odd. A (commutative) super-ring over $R^0$ is a ring $R = R^0 \oplus R^1$ such that

(i) $a_i a_j \in R^{p(i+j)}$ for any $a_m \in R^m$, $m = 0, 1$,

(ii) $ab = ba$ for any $a \in R^0$, $b \in R$, and $cd = -dc$ for any $c, d \in R^1$.

Let $A = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ be an $(m+n)\times(m+n)$-block-matrix over a super-ring $R = R^0 \oplus R^1$, where $X$ is an $m\times m$-matrix over $R^0$, $T$ is an $n\times n$-matrix over $R^0$, and $Y, Z$ are matrices over $R^1$. If $T$ is an invertible matrix, then $X - YT^{-1}Z$ is an invertible matrix over commutative ring $R^0$. Super-determinant, or Berezinian, of $A$ is defined by the following formula:

$$\operatorname{Ber} A = \det(X - YT^{-1}Z)\det T^{-1}.$$

Note that $\operatorname{Ber} A \in R^0$.

**Theorem 3.8.1** Let $R^0$ be a field. Set $J_k = \{1, 2, \ldots, k\}$, $k \le m+n$ and $A^{(k)} = A^{J_k, J_k}$. Then $\operatorname{Ber} A$ is a product of elements of $R^0$:

$$\operatorname{Ber} A = |A|_{11}|A^{(1)}|_{22}\ldots|A^{(m-1)}|_{mm}|A^{(m)}|^{-1}_{m+1,m+1}\cdots|A^{(m+n-1)}|^{-1}_{m+n,m+n}.$$

## 3.9 Cartier-Foata determinants

Let $A = (a_{ij})$, $i, j = 1, \ldots, n$ be a matrix such that the entries $a_{ij}$ and $a_{kl}$ commute when $i \neq k$. In this case Cartier and Foata [CF, F] defined a determinant of $A$ as

$$\det{}_{CF} A = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}.$$

The order of factors in monomials $a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$ is insignificant.

Let $C_n$ be the algebra over a field $F$ generated by $(a_{ij})$, $i, j = 1, \ldots, n$, with relations $a_{ij} a_{kl} = a_{kl} a_{ij}$ if $i \neq k$. Algebra $C_n$ admits the ring of fractions.

**Theorem 3.9.1.** In the ring of fractions of algebra $C_n$, let $A = (a_{ij})$, $i, j = 1, \ldots, n$ be a matrix such that the entries $a_{ij}$ and $a_{kl}$ commute when $i \neq k$.

$$|A|_{pq} = (-1)^{p+q} \det{}_{CF} \ (A^{pq})^{-1} \det{}_{CF} A 3.9.1$$

and all factors in (3.9.1) commute.

**Corollary 3.9.2.** In the ring of fractions of algebra $C_n$ we have

$$det_{CF} = |A|_{11} |A^{11}|_{22} \ldots a_{nn}$$

and all factors commute.

# Chapter 4

# Noncommutative Plücker and Flag Coordinates

Most of the results described in this section were obtained in [GR4].

## 4.1 Commutative Plücker coordinates

Let $k \leq n$ and $A$ be a $k \times n$-matrix over a commutative ring $R$. Denote by $A(i_1, \dots, i_k)$ the $k \times k$-submatrix of $A$ consisting of columns labeled by the indices $i_1, \dots, i_k$. Define $p_{i_1 \dots i_k}(A) := \det A(i_1, \dots, i_k)$. The elements $p_{i_1 \dots i_k}(A) \in R$ are called Plücker coordinates of the matrix $A$. The Plücker coordinates $p_{i_1 \dots i_k}(A)$ satisfy the following properties:

(i) (invariance) $p_{i_1 \dots i_k}(XA) = \det X \cdot p_{i_1 \dots i_k}(A)$ for any $k \times k$-matrix $X$ over $R$;

(ii) (skew-symmetry) $p_{i_1 \dots i_k}(A)$ are skew-symmetric in indices $i_1, \dots, i_k$; in particular, $p_{i_1 \dots i_k}(A) = 0$ if a pair of indices coincides;

(iii) (Plücker relations) Let $i_1, \dots, i_{k-1}$ be $k-1$ distinct numbers which are chosen from the set $1, \dots, n$, and $j_1, \dots, j_{k+1}$ be $k+1$ distinct numbers chosen from the same set. Then

$$\sum_{t=1}^{k} (-1)^t p_{i_1 \dots i_{k-1} j_t}(A) p_{j_1 \dots j_{t-1} j_{t+1} \dots j_{k+1}}(A) = 0.$$

**Example.** For $k = 2$ and $n = 4$ the Plücker relations in (iii) imply the famous identity

$$p_{12}(A)p_{34}(A) - p_{13}(A)p_{24}(A) + p_{23}(A)p_{14}(A) = 0. \tag{4.1.1}$$

Historically, Plücker coordinates were introduced as coordinates on Grassmann manifolds. Namely, let $R = F$ be a field and $G_{k,n}$ the Grassmannian of

$k$-dimensional subspaces in the $n$-dimensional vector space $F^n$. To each $k \times n$-matrix $A$ of rank $k$ we associate the subspace of $F^n$ generated by the rows of $A$. By the invariance property (i), we can view each Plücker coordinate $p_{i_1 \dots i_k}$ as a section of a certain ample line bundle on $G_{k,n}$, and all these sections together define an embedding of $G_{k,n}$ into the projective space $\mathbb{P}^N$ of dimension $N = \binom{k}{n} - 1$. In this sense, Plücker coordinates are projective coordinates on $G_{k,n}$.

## 4.2 Quasi-Plücker coordinates for $n \times (n+1)$- and $(n+1) \times n$-matrices

Let $A = (a_{ij})$, $i = 1, \dots, n$, $j = 0, 1, \dots, n$, be a matrix over a division ring $R$. Denote by $A^{(k)}$ the $n \times n$-submatrix of $A$ obtained from $A$ by removing the $k$-th column and suppose that all $A^{(k)}$ are invertible. Choose an arbitrary $s \in \{1, \dots, n\}$, and denote

$$q_{ij}^{(s)}(A) = |A^{(j)}|_{si}^{-1} |A^{(i)}|_{sj}.$$

**Proposition 4.2.1.** *The element $q_{ij}^{(s)}(A) \in R$ does not depend on $s$.* We denote the common value of $q_{ij}^{(s)}(A)$ by $q_{ij}(A)$ and call $q_{ij}(A)$ the *left quasi-Plücker coordinates* of the matrix $A$.

*Proof.* of Proposition 4.2.1 Considering the columns of the matrix $A$ as $n+1$ vectors in the right $n$-dimensional space $R^n$ over $R$, we see that there exists a nonzero $(n+1)$-vector $(x_1, \dots, x_{n+1}) \in R^{n+1}$ such that

$$A \begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix} = 0.$$

This means that

$$A^{(j)} \begin{pmatrix} x_0 \\ \dots \\ \widehat{x}_j \\ \dots \\ x_n \end{pmatrix} = - \begin{pmatrix} a_{1j} \\ \\ \dots \\ \\ a_{nj} \end{pmatrix} x_j.$$

Since all submatrices $A^{(k)}$ are invertible, each $x_I$ is a nonzero element of $R$. Cramer's rule and transformations properties for quasideterminanats imply that $|A^{(j)}|_{si} x_i = -|A^{(i)}|_{sj} x_j$. Therefore,

$$q_{ij}^{(s)}(A) = |A^{(j)}|_{si}^{-1} |A^{(i)}|_{sj} = -x_i x_j^{-1} \tag{4.1}$$

does not depend on $s$. $\square$

**Proposition 4.2.2.** *If $g$ is an invertible $n \times n$-matrix over $R$, then $q_{ij}(gA) = q_{ij}(A)$.*

*Proof.* We have $gA\begin{pmatrix}x_0\\ \dots\\ x_n\end{pmatrix} = 0$. Therefore, $q_{ij}(gA) = -x_i x_j^{-1} = q_{ij}(A)$. □ □

In the commutative case, $q_{ij}(A)$ is a ratio of two Plücker coordinates: $q_{ij}(A) = p_{1,\dots,\widehat{j},\dots,n}/p_{1,\dots,\widehat{i},\dots,n} = \det A^{(j)}/\det A^{(i)}$.

Similarly, we define the right quasi-Plücker coordinates $r_{ij}(B)$ for $(n+1)\times n$-matrix $B = (b_{ji})$. Denote by $B^{(k)}$ the submatrix of $B$ obtained from $B$ by removing the $k$-th row. Suppose that all $B^{(k)}$ are invertible, choose $s \in \{1,\dots,n\}$, and set $r_{ij}^{(s)}(B) = |B^{(j)}|_{is}|B^{(i)}|_{js}^{-1}$.

**Proposition 4.2.3.** (i) *The element $r_{ij}^{(s)}(B)$ does not depend of $s$.*

*Denote the common value of elements $r_{ij}^{(s)}(B)$ by $r_{ij}(B)$.*

(ii) *If $g$ is an invertible $n\times n$-matrix over $R$, then $r_{ij}(Bg) = r_{ij}(B)$.*

In the commutative case, $r_{ij}(A) = \det B^{(j)}/\det B^{(i)}$.

## 4.3 Definition of left quasi-Plücker coordinates. General case

Let $A = (a_{pq})$, $p = 1,\dots,k$, $q = 1,\dots,n$, $k < n$, be a matrix over a division ring $R$. Choose $1 \le i, j, i_1, \dots, i_{k-1} \le n$ such that $i \notin I = \{i_1,\dots,i_{k-1}\}$. Let $A(i,j,i_1,\dots,i_{k-1})$ be the $k\times(k+1)$-submatrix of $A$ with columns labeled by $i, j, i_1, \dots, i_{k-1}$.

**Definition 4.3.1** Define left quasi-Plücker coordinates $q_{ij}^I(A)$ of the matrix $A$ by the formula

$$q_{ij}^I(A) = q_{ij}(A(i,j,i_1,\dots,i_{k-1})).$$

By Proposition 4.2.1, left quasi-Plücker coordinates are given by the formula

$$q_{ij}^I(A) = \begin{vmatrix} a_{1i}a_{1i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \\ a_{ki}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{si}^{-1} \cdot \begin{vmatrix} a_{1j}a_{1,i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \\ a_{kj}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{sj}$$

for an arbitrary $s$, $1 \le s \le k$.

**Proposition 4.3.2.** *If $g$ is an invertible $k\times k$-matrix over $R$, then $q_{ij}^I(gA) = q_{ij}^I(A)$.*

*Proof.* Use Proposition 4.2.2. □

In the commutative case $q_{ij}^I = p_{jI}/p_{iI}$, where $p_{\alpha_1\dots\alpha_k}$ are the standard Plücker coordinates.

## 4.4 Identities for the left quasi-Plücker coordinates

The following properties of $q^I_{ij}$ immediately follow from the definition.

(i) $q^I_{ij}$ does not depend on the ordering on elements in $I$;

(ii) $q^I_{ij} = 0$ for $j \in I$;

(iii) $q^I_{ii} = 1$ and $q^I_{ij} \cdot q^I_{jk} = q^I_{ik}$.

**Theorem 4.4.1.** (Skew-symmetry) *Let $N$, $|N| = k+1$, be a set of indices, $i, j, m \in N$. Then*

$$q_{ij}^{N \smallsetminus \{i,j\}} \cdot q_{jm}^{N \smallsetminus \{j,m\}} \cdot q_{mi}^{N \setminus \{m,i\}} = -1.$$

**Theorem 4.4.2.**(Plücker relations) *Fix $M = (m_1, \dots, m_{k-1})$, $L = (\ell_1, \dots, \ell_k)$. Let $i \notin M$. Then*

$$\sum_{j \in L} q^M_{ij} \cdot q^{L \setminus \{j\}}_{ji} = 1.$$

**Examples.** Suppose that $k = 2$.

1) From Theorem 4.4.1 it follows that

$$q^{\{\ell\}}_{ij} \cdot q^{\{i\}}_{j\ell} \cdot q^{\{j\}}_{\ell i} = -1.$$

In the commutative case, $q^{\{\ell\}}_{ij} = \frac{p_{j\ell}}{p_{i\ell}}$ so this identity follows from the skew-symmetry $p_{ij} = -p_{ji}$.

2) From Theorem 4.4.2 it follows that for any $i, j, \ell, m$

$$q^{\{\ell\}}_{ij} \cdot q^{\{m\}}_{ji} + q^{\{\ell\}}_{im} \cdot q^{\{j\}}_{mi} = 1.$$

In the commutative case this identity implies the standard identity (cf. (4.1.1))

$$p_{ij} \cdot p_{\ell m} - p_{i\ell} \cdot p_{jm} + p_{im} \cdot p_{\ell j} = 0.$$

*Remark.* The products $p^{\{\ell\}}_{ij} p^{\{m\}}_{ji}$ (which in the commutative case are equal to $\frac{p_{j\ell}}{p_{i\ell}} \cdot \frac{p_{im}}{p_{jm}}$) can be viewed as noncommutative cross-ratios.

To prove Theorems 4.4.1 and 4.4.2 we need the following lemma. Let $A = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, n$, $k < n$, be a matrix over a division ring. Denote by $A_{j_1,\dots,j_\ell}$, $\ell \le n$, the $k \times \ell$-submatrix $(a_{ij})$, $i = 1, \dots, k$, $j = j_1, \dots, j_\ell$. Consider the $n \times n$-matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & E_{n-k} \end{pmatrix},$$

where $E_m$ is the identity matrix of order $m$.

**Lemma 4.4.3.** *Let $j < k < i$. If $q_{ij}^{1\ldots\hat{j}\ldots k}(A)$ is defined, then $|X|_{ij}$ is defined and*

$$|X|_{ij} = -q_{ij}^{1\ldots\hat{j}\ldots k}(A). \tag{4.4.1}$$

*Proof.* We must prove that

$$|X|_{ij} = -|A_{1\ldots\hat{j}\ldots ki}|_{si}^{-1} \cdot |A_{1\ldots k}|_{sj} \tag{4.4.2}$$

provided the right-hand side is defined. We will prove this by induction on $\ell = n-k$. Let us assume that formula (2.2) holds for $l = m$ and prove it for $\ell = m+1$. Without loss of generality we can take $j = 1, i = k+1$. By homological relations (Theorem 1.4.3)

$$|X|_{k+1,1} = -|X^{k+1,1}|_{s,k+1}^{-1} \cdot |X^{k+1,k+1}|_{s1}$$

for an appropriate $1 \leq s \leq k$. Here

$$X^{k+1,1} = \begin{pmatrix} A_{2\ldots k+1} & A_{k+2\ldots n} \\ 0 & E_{n-k-1} \end{pmatrix},$$
$$X^{k+1,k+1} = \begin{pmatrix} A_{1\ldots k} & A_{k+2\ldots n} \\ 0 & E_{n-k-1} \end{pmatrix}.$$

By the induction assumption

$$|X^{k+1,1}|_{s,k+1} = -|A_{23\ldots kk+2}|_{s,k+2}^{-1} \cdot |A_{23\ldots k+1}|_{s,k+1},$$
$$|X^{k+1,k+1}|_{s1} = -|A_{23\ldots kk+2}|_{s,k+2}^{-1} \cdot |A_{1\ldots k}|_{s1}$$

and $|X|_{k+1,1} = -p_{k+1,1}^{23\ldots k}$. □ □

To prove Theorem 4.4.2 we apply the second formula in Theorem 1.6.4 to the matrix

$$X = \begin{pmatrix} A_{1\ldots k} & A_{k+1\ldots n} \\ 0 & E_{n-k} \end{pmatrix}$$

for $M = (k+1,\ldots,n)$ and any $L$ such that $|L| = n-k-1$. By Lemma 4.4.3, $|X|_{m_i\ell} = -q^{1\ldots\hat{\ell}\ldots k}(A)$, $|X^{M_i,L}|_{m_iq} = -p_{m_iq}^{1\ldots n\setminus L}(A)$, and Theorem 4.4.2 follows from Theorem 1.6.4. □

To prove Theorem 4.4.1 it is sufficient to take the matrix $X$ for $n = k+1$ and use homological relations. □

**Theorem 4.4.4.** *Let $A = (a_{ij})$, $i = 1,\ldots,k$, $j = 1,\ldots,n$, be a matrix with formal entries and $f(a_{ij})$ an element of a free skew-field $F$ generated by $a_{ij}$. Let $f$ be invariant under the transformations*

$$A \to gA$$

*for all invertible $k \times k$-matrices $g$ over $F$. Then $f$ is a rational function of the quasi-Plücker coordinates.*

*Proof.* Let $b_{ij} = a_{ij}$ for $i, j = 1, \ldots, k$. Consider the matrix $B = (b_{ij})$. Then $B^{-1} = (|B|_{ji}^{-1})$. Set $C = (c_{ij}) = B^{-1}A$. Then

$$c_{ij} = \begin{cases} \delta_{ij} & \text{for } j \leq k, \\ q_{ij}^{1\ldots\hat{\imath}\ldots k}(A) & \text{for } j > k. \end{cases}$$

By invariance, $f$ is a rational expression of $c_{ij}$ with $j > k$. □

## 4.5 Right quasi-Plücker coordinates

Consider a matrix $B = (b_{pq})$, $p = 1, \ldots, n$; $q = 1, \ldots, k$, $k < n$ over a division ring $F$. Choose $1 \leq i, j, i_1, \ldots, i_{k-1} \leq n$ such that $j \notin I = (i_1, \ldots, i_{k-1})$. Let $B(i, j, i_1, \ldots, i_{k-1})$ be the $(k+1) \times k$-submatrix of $B$ with rows labeled by $i, j, i_1, \ldots, i_{k-1}$.

**Definition 4.5.1** Define right quasi-Plücker coordinates $r_{ij}^I(B)$ of the matrix $B$ by the formula

$$r_{ij}^I(B) = r_{ij}(B(i, j, i_1, \ldots, i_{k-1})).$$

By Proposition 4.2.3, right quasi-Plücker coordinates are given by the formula

$$r_{ij}^I(B) = \begin{vmatrix} b_{i1} & \ldots & b_{ik} \\ b_{i_1 1} & \ldots & b_{i_1 k} \\ & \ldots & \\ b_{i_{k-1}1} & \ldots & b_{i_{k-1}k} \end{vmatrix}_{it} \cdot \begin{vmatrix} b_{j1} & \ldots & b_{jk} \\ b_{i_1 1} & \ldots & b_{i_1 k} \\ & \ldots & \\ b_{i_{k-1}1} & \ldots & b_{i_{k-1}k} \end{vmatrix}_{jt}^{-1}$$

for an arbitrary $t$, $1 \leq t \leq k$.

**Proposition 4.5.2.** *$r_{ij}^I(Bg) = r_{ij}^I(B)$ for each invertible $k \times k$-matrix $g$ over $F$.*

## 4.6 Identities for the right quasi-Plücker coordinates

Identities for $r_{ij}^I$ are dual to correspoding identities for the left quasi-Plücker coordinates $q_{ij}^I$. Namely,

(i) $r_{ij}^I$ does not depend on the ordering on elements of $I$;

(ii) $r_{ij}^I = 0$ for $i \in I$;

(iii) $r_{ii}^I = 1$ and $r_{ij}^I \cdot r_{jk}^I = r_{ik}^I$.

**Theorem 4.6.1.** (Skew-symmetry) *Let $N$, $|N| = k+1$, be a set of indices, $i, j, m \in N$. Then*

$$r_{ij}^{N\setminus\{i,j\}} \cdot r_{jm}^{N\setminus\{j,m\}} \cdot r_{mi}^{N\setminus\{m,i\}} = -1.$$

**Theorem 4.6.2.** (Plücker relations) *Fix $M = (m_1, \dots, m_{k-1})$, $L = (\ell_1, \dots, \ell_k)$. Let $i \notin M$. Then*

$$\sum_{j \in L} r_{ij}^{L\setminus\{j\}} r_{ij}^{M} = 1.$$

## 4.7 Duality between quasi-Plücker coordinates

Let $A = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, n$; and $B = (b_{rs})$, $r = 1, \dots, n$, $s = 1, \dots, n-k$. Suppose that $AB = 0$. (This is equivalent to the statement that the subspace generated by the rows of $A$ in the left linear space $F^n$ is dual to the subspace generated by the columns of $B$ in the dual right linear space.) Choose indices $1 \le i, j \le n$ and a subset $I \subset [1, n]$, $|I| = k-1$, such that $i \notin I$. Set $J = ([1, n] \setminus I) \setminus \{i, j\}$.

**Theorem 4.7.1.** *We have*

$$q_{ij}^{I}(A) + r_{ij}^{J}(B) = 0.$$

## 4.8 Quasi-Plücker coordinates for $k \times n$-matrices for different $k$

Let $A = (a_{\alpha\beta})$, $\alpha = 1, \dots, k$, $\beta = 1, \dots, n$, be a $k \times n$-matrix over a noncommutative division ring $R$ and $A'$ a $(k-1) \times n$-submatrix of $A$. Choose $1 \le i, j, m, j_1, \dots, j_{k-2} \le n$ such that $i \ne m$ and $i, m \notin J = \{j_1, \dots, j_{k-2}\}$.

**Proposition 4.8.1.** *We have*

$$q_{ij}^{J}(A') = q_{ij}^{J\cup\{m\}}(A) + q_{im}^{J}(A') \cdot q_{mj}^{J\cup\{i\}}(A).$$

## 4.9 Applications of quasi-Plücker coordinates

## Row and column expansion of a quasideterminant

Some of the results obtained in [GR], [GR1], [GR2] and partially presented in Section I can be rewritten in terms of quasi-Plücker coordinates.

Let $A = (a_{ij}), i, j = 1, \dots, n$, be a matrix over a division ring $R$. Choose $1 \le \alpha, \beta \le n$. Using the notation of section I let $B = A^{\{\alpha\},\emptyset}, C = A^{\emptyset,\{\beta\}}$ be the $(n-1) \times n$ and $n \times (n-1)$ submatrices of $A$ obtained by deleting the $\alpha$-th row and the $\beta$-th column respectively. For $j \ne \beta$ and $i \ne \alpha$ set

$$q_{j\beta} = q_{j\beta}^{1\dots\hat{j}\dots\hat{\beta}\dots n}(B),$$
$$r_{\alpha i} = r_{\alpha i}^{1\dots\hat{\alpha}\dots\hat{i}\dots n}(C).$$

**Proposition 4.9.1.** (i) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{j\ne\beta} a_{\alpha j} q_{j\beta}$,
(ii) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{i\ne\alpha} r_{\alpha i} a_{i\beta}$
*provided the terms in the right-hand side of these formulas are defined.*

## Homological relations

**Proposition 4.9.2.** *In the previous notation,*
(i) $|A|_{ij}^{-1} \cdot |A|_{i\ell} = -q_{j\ell}$ *(row relations)*
(ii) $|A|_{ij} \cdot |A|_{kj}^{-1} = -r_{ik}$ *(column relations).*

**Corollary 4.9.3.** *In the previous notation, let $(i_1, \dots, i_s), (j_1, \dots, j_t)$ be sequences of indices such that $i \ne i_1$, $i_1 \ne i_2, \dots, i_{s-1} \ne i_s$; $j \ne j_1$, $j_1 \ne j_2, \dots, j_{t-1} \ne j_t$. Then*

$$|A|_{i_s j_t} = q_{i_s i_{s-1}} \dots q_{i_2 i_1} q_{i_1 i} \cdot |A|_{ij} \cdot r_{j j_1} r_{j_1 j_2} \dots r_{j_{t-1} j_t}.$$

**Example.** For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have

$$|A|_{22} = a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{22}^{-1} \cdot a_{22},$$
$$|A|_{11} = a_{12} \cdot a_{22}^{-1} \cdot a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{21}^{-1} \cdot a_{22} \cdot a_{12}^{-1} \cdot a_{11}.$$

## Matrix multiplication

The following formula was already used in the proof of Theorem 4.4.4. Let $A = (a_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$, $n < m$, $B = (a_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, n$, $C = (a_{ik})$, $i = 1, \dots, n$, $k = n+1, \dots, m$.

**Proposition 4.9.4.** *Let the matrix $B$ be invertible. Then $q_{ik}^{1\dots\hat{i}\dots n}(A)$ are defined for $i = 1, \dots n, k = n+1, \dots m$, and*

$$B^{-1}C = (q_{ik}^{1\dots\hat{i}\dots n}(A)), \qquad i = 1, \dots, n, \quad k = n+1, \dots, m.$$

## Quasideterminant of the product

Let $A = (a_{ij})$, $B = (b_{ij})$, $i, j = 1, \dots n$ be matrices over a division ring $R$. Choose $1 \le k \le n$. Consider the $(n-1) \times n$-matrix $A' = (a_{ij})$, $i \ne k$, and the $n \times (n-1)$-matrix $B'' = (b_{ij})$, $j \ne k$.

**Proposition 4.9.5.** *We have*

$$|B|_{kk} \cdot |AB|_{kk}^{-1} \cdot |A|_{kk} = 1 + \sum_{\alpha \ne k} r_{k\alpha} \cdot q_{\alpha k},$$

*where* $r_{k\alpha} = r_{k\alpha}^{1 \dots \hat{\alpha} \dots n}(B'')$ *are right quasi-Plücker coordinates and* $q_{\alpha k} = q_{\alpha k}^{1 \dots \hat{\alpha} \dots n}(A')$ *are left quasi-Plücker coordinates, provided all expressions are defined.*

The proof follows from the multiplicative property of quasideterminants and Proposition 4.9.2.

## Gauss decomposition

Consider a matrix $A = (a_{ij})$, $i, j = 1, \dots, n$, over a division ring $R$. Let $A_k = (a_{ij})$, $i, j = k, \dots n$, $B_k = (a_{ij})$, $i = 1, \dots n$, $j = k, \dots n$, and $C_k = (a_{ij})$, $i = k, \dots n$, $j = 1, \dots n$. These are submatrices of sizes $(n-k+1) \times (n-k+1)$, $n \times (n-k+1)$, and $(n-k+1) \times n$ respectively. Suppose that the quasideterminants

$$y_k = |A_k|_{kk}, \qquad k = 1, \dots, n,$$

are defined and invertible in $R$.

**Theorem 4.9.6.** (see [GR1, GR2])

$$A = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix},$$

*where*

$$x_{\alpha\beta} = r_{\alpha\beta}^{\beta+1 \dots n}(B_\beta), \quad 1 \le \alpha < \beta \le n,$$
$$z_{\beta\alpha} = q_{\beta\alpha}^{\beta+1 \dots n}(C_\beta), \quad 1 \le \alpha < \beta \le n.$$

Similarly, let $A^{(k)} = (a_{ij})$, $i, j = 1, \dots, k$, $B^{(k)} = (a_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, k$, $C^{(k)} = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, n$. Suppose that the quasideterminants

$$y'_k = |A^{(k)}|_{kk}, \qquad k = 1, \dots, n,$$

are defined and invertible in $R$.

**Theorem 4.9.7** We have

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x'_{\beta\alpha} & & 1 \end{pmatrix} \begin{pmatrix} y'_1 & & 0 \\ & \ddots & \\ 0 & & y'_n \end{pmatrix} \begin{pmatrix} 1 & & z'_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

where

$$x'_{\beta\alpha} = r^{1\dots\alpha-1}_{\beta\alpha}(B^{(\alpha)}), \quad 1 \le \alpha < \beta \le n,$$
$$z'_{\alpha\beta} = q^{1\dots\alpha-1}_{\alpha\beta}(C^{(\alpha)}), \quad 1 \le \alpha < \beta \le n.$$

## Bruhat decompositions

A generalization of Theorem 4.9.6 is given by the following noncommutative analog of the Bruhat decomposition.

**Definition.** A square matrix $P$ with entries 0 and 1 is called a permutation matrix if in each row of $P$ and in each column of $P$ there is exactly one entry 1.

**Theorem 4.9.8** (Bruhat decomposition) For an invertible matrix $A$ over a division ring there exist an upper-unipotent matrix $X$, a low-unipotent matrix $Y$, a diagonal matrix $D$ and a permutation matrix $P$ such that

$$A = XPDY.$$

Under the additional condition that $P^{-1}XP$ is an upper-unipotent matrix, the matrices $X, P, D, Y$ are uniquely determined by $A$.

Note that one can always find a decomposition $A = XPDY$ that satisfies the additional condition.

The entries of matrices $X$ and $Y$ can be written in terms of quasi-Plücker coordinates of submatrices of $A$. The entries of $D$ can be expressed as quasiminors of $A$.

**Examples.** Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If $a_{22} \neq 0$, then

$$A = \begin{pmatrix} 1 & a_{12}a_{22}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |A|_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{22}^{-1}a_{21} & 1 \end{pmatrix}.$$

If $a_{22} = 0$ and the matrix $A$ is invertible, then $a_{12} \neq 0$. In this case,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{21} & 0 \\ 0 & a_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{12}^{-1}a_{11} & 1 \end{pmatrix}.$$

An important example of quasi-Plücker coordinates for the Vandermonde matrix will be considered later.

## 4.10 Flag coordinates

Noncommutative flag coordinates were introduced in [GR1, GR2].

Let $A = (a_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, be a matrix over a division ring $R$. Let $F_p$ be the subspace of the left vector space $R^n$ generated by the first $p$ rows of $A$. Then $mathcalF = (F_1 \subset F_2 \subset \cdots \subset F_k)$ is a flag in $R^n$. Put

$$f_{j_1 \ldots j_k}(\mathcal{F}) = \begin{vmatrix} a_{1j_1} & \ldots & a_{1j_k} \\ & \ldots & \\ a_{kj_1} & \ldots & a_{kj_k} \end{vmatrix}_{kj_1} .$$

In [GR1, GR2] the functions $f_{j_1 \ldots j_k}(\mathcal{F})$ were called the *flag coordinates* of $\mathcal{F}$. Transformations properties of quasideterminants imply that $f_{j_1 \ldots j_k}(\mathcal{F})$ does not depend on the order of the indices $j_2, \ldots, j_k$.

**Proposition 4.10.1.** (see [GR1, GR2]) *The functions $f_{j_1 \ldots j_m}(\mathcal{F})$ do not change under left multiplication of $A$ by an upper unipotent matrix.*

**Theorem 4.10.2.** (see [GR1, GR2]) *The functions $f_{j_1 \ldots j_k}(\mathcal{F})$ possess the following relations:*

$$\begin{aligned} &f_{j_1 j_2 j_3 \ldots j_k}(\mathcal{F}) f_{j_1 j_3 \ldots j_k}(\mathcal{F})^{-1} = -f_{j_2 j_1 \ldots j_k}(\mathcal{F}) f_{j_2 j_3 \ldots j_k}(\mathcal{F})^{-1}, \\ &f_{j_1 \ldots j_k}(\mathcal{F}) f_{j_1 \ldots j_{k-1}}(\mathcal{F})^{-1} + f_{j_2 \ldots j_k j_1}(\mathcal{F}) f_{j_2 \ldots j_k}(\mathcal{F})^{-1} \\ &\qquad + \cdots + f_{j_k j_1 \ldots j_{k-1}}(\mathcal{F}) f_{j_k j_1 \ldots j_{k-2}}(\mathcal{F})^{-1} = 0 \end{aligned}$$

**Example.** Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. Then $f_{12}(\mathcal{F}) a_{11}^{-1} = -f_{21}(\mathcal{F}) a_{12}^{-1}$ and $f_{12}(\mathcal{F}) a_{11}^{-1} + f_{23}(\mathcal{F}) a_{12}^{-1} + f_{31}(\mathcal{F}) a_{13}^{-1} = 0$.

It is easy to see that

$$q_{ij}^{i_1 \ldots i_{k-1}}(A) = \left(f_{i i_1 \ldots i_{k-1}}(\mathcal{F})\right)^{-1} \cdot f_{j i_1 \ldots i_{k-1}}(\mathcal{F}).$$

Theorems 4.4.1 and 4.4.2 can be deduced from Theorem 4.10.2.

## 4.11 Positive quasiminors

The results in this subsection appeared in [BR].

Recall, that for a given matrix $A \in Mat_n(R)$ and $I, J \subset [1, n] = \{1, 2, \ldots, n\}$ we denote by $A_{I,J}$ the submatrix with the $I$ rows and columns $J$. And, if $|I| = |J|$, i.e., when $A_{I,J}$ is a square matrix, for any $i \in I$, $j \in J$ denote by $|A_{I,J}|_{i,j}$ the quasideterminant of the submatrix $A_{I,J}$ with the marked position $(i, j)$.

Let us denote by $\Delta^i(A)$ the principal $i \times i$-quasiminor of $A \in Mat_n(R)$, i.e.,

$$\Delta^i(A) = |A_{\{1,2,\ldots,i\},\{1,2,\ldots,i\}}|_{i,i} .$$

The following fact is obvious.

**Lemma 4.11.1** For any $I, J \subset \{1, 2, \ldots, n\}$ such that $|I| = |J| = k$ and any $i \in I$, $j \in J$ there exist permutations $u, v$ of $\{1, 2, \ldots, n\}$ such that $I = u\{1, \ldots, k\}$, $J = v\{1, \ldots, k\}$, $i = u(k)$, $j = v(k)$, and for any $A \in Mat_n(R)$ we have:

$$\Delta^k(u^{-1} \cdot A \cdot v) = |A_{I,J}|_{i,j} \ .$$

(where we identified the permutations $u$ and $v$ with their corresponding $n \times n$ matrices).

**Definition.** For $I, J \subset [1, n]$, $|I| = |J|$, $i \in I$, $j \in J$ define the *positive quasiminor* $\Delta^{i,j}_{I,J}$ as follows.

$$\Delta^{i,j}_{I,J}(A) = (-1)^{d_i(I)+d_j(J)}|A_{I,J}|_{i,j}$$

where $d_i(I)$ (resp. $d_j(J)$) is the number of those elements of $I$ (resp. of $J$) which are greater than $i$ (resp. than $j$).

The definition is motivated by the fact that for a commutative ring $R$ one has

$$\Delta^{i,j}_{I,J}(A) = \frac{\det(A_{I,J})}{\det(A_{I',J'})} \ ,$$

where $I' = I \setminus \{i\}$, $J' = J \setminus \{j\}$. That is, a positive quasiminor is a positive ratio of minors.

Let $S_n$ be the group of permutations on $\{1, 2, \ldots, n\}$ and $k \in [1, n]$. For any permutations $u, v \in S_n$ set

$$\Delta^k_{u,v}(A) := \Delta^{i,j}_{I,J}(A) = (-1)^{d_i(I)+d_j(J)}\Delta^k(u^{-1} \cdot A \cdot v) \ ,$$

where $I = u\{1, \ldots, k\}$, $J = v\{1, \ldots, k\}$, $i = u(k)$, $j = v(k)$.

Denote by $D_n = D_n(R)$ the set of all diagonal $n \times n$ matrices over $R$.

Clearly, positive quasiminors satisfy the relation:

$$\Delta^k_{u,v}(hAh') = h_{u(k)}\Delta^k_{u,v}(A)h'_{v(k)}$$

for $h = diag(h_1, \ldots, h_n), h' = diag(h'_1, \ldots, h'_n) \in D_n$, and $k = 1, 2, \ldots, n$.

Let $\sigma$ be an involutive automorphism of $Mat_n(R)$ defined by

$$\sigma(A)_{ij} = a_{n+1-i,n+1-j} \ ,$$

The following fact follows from the elementary properties of quasideterminants Let $w_0 = (n, n-1, \ldots, 1)$ be the longest permutation in $S_n$.

**Lemma 4.11.2** For any $u, v \in S_n$, and $A \in Mat_n(R)$ we have

$$\Delta^i_{u,v}(\sigma(A)) = \Delta^i_{w_0u,w_0v}(A)$$

Now we present some less obvious identities for positive quasiminors. For each permutation $v \in S_n$ denote by $\ell(v)$ the number of inversions of $v$. Also for $i = 1, 2, \ldots, n-1$ denote by $s_i$ the simple transposition $(i, i+1) \in S_n$.

**Proposition 4.11.3** Let $u, v \in S_n$ and $i \in [1, n-1]$ be such that $l(us_i) = l(u) + 1$ and $l(vs_i) = l(v) + 1$. Then

$$\Delta^i_{us_i,vs_i} = \Delta^i_{us_i,v}(\Delta^i_{u,v})^{-1}\Delta^i_{u,vs_i} + \Delta^{i+1}_{u,v} \ ,$$

$$(\Delta^i_{us_i,v})^{-1}\Delta^{i+1}_{u,v} = (\Delta^i_{u,v})^{-1}\Delta^{i+1}_{us_i,v} \ , \Delta^{i+1}_{u,v}(\Delta^i_{u,vs_i})^{-1} = \Delta^{i+1}_{u,vs_i}(\Delta^i_{u,v})^{-1},$$
$$\Delta^{i+1}_{u,v}(\Delta^{i+1}_{us_i,v})^{-1} = \Delta^i_{us_i,v}(\Delta^i_{u,v})^{-1} \ , (\Delta^{i+1}_{u,vs_i})^{-1}\Delta^{i+1}_{u,v} = (\Delta^i_{u,v})^{-1}\Delta^i_{u,vs_i}.$$

*Proof.* Clearly, the fourth and the fifth identities follow from the second and the third. Using Lemma 4.11.1 and the Gauss factorization it suffices to take $u = v = 1$, $i = 1$ in the first three identities, i.e., work with $2 \times 2$ matrices. Then the first three identities will take respectively the following obvious forms:

$$a_{22} = a_{21}a_{11}^{-1}a_{12} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} \ ,$$

$$a_{21}^{-1}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = -a_{11}^{-1}\begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} \ , \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} a_{12}^{-1} = -\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} a_{11}^{-1} \ .$$

One can prove the next proposition presenting some generalized Plücker relations.

**Proposition 4.11.4** Let $u, v \in S_n$ and $i \in [1, n-2]$. If $l(us_is_{i+1}s_i) = l(u) + 3$, then

$$\Delta^{i+1}_{us_{i+1},v} = \Delta^{i+1}_{us_is_{i+1},v} + \Delta^i_{us_{i+1}s_i,v}(\Delta^i_{us_i,v})^{-1}\Delta^{i+1}_{u,v} \ .$$

If $l(vs_is_{i+1}s_i) = l(v) + 3$, then

$$\Delta^{i+1}_{u,vs_{i+1}} = \Delta^{i+1}_{u,vs_is_{i+1}} + \Delta^{i+1}_{u,v}(\Delta^i_{u,vs_i})^{-1}\Delta^i_{u,vs_{i+1}s_i} \ .$$

# Chapter 5

# Noncommutative traces, determinants and eigenvalues

In this section we discuss noncommutative traces, determinants and eigenvalues. Our approach to noncommutative determinants in this Section is different from our approach described in Section 3.

Classical (commutative) determinants play a key role in representation theory. Frobenius developed his theory of group characters by studying factorizations of group determinants (see [L]). Therefore, one cannot start a noncommutative representation theory without looking at possible definition of noncommutative determinants and traces. The definition of a noncommutative determinant given in this Section is different from the definition given in Section 3. However, for matrices over commutative algebras, quantum and Capelli matrices both approach give the same results.

## 5.1 Determinants and cyclic vectors

Let $R$ be an algebra with unit and $A : R^m \to R^m$ a linear map of right vector spaces, A vector $v \in R^m$ is an $A$-*cyclic vector* if $v, Av, \ldots, A^{m-1}v$ is a basis in $R^m$ regarded as a right $R$-module. In this case there exist $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$, such that

$$(-1)^m v\Lambda_m(v, A) + (-1)^{m-1}(Av)\Lambda_{m-1}(v, A) + \cdots - (A^{m-1}v)\Lambda_1(v, A) + A^m v = 0.$$

**Definition 5.1.1.** We call $\Lambda_m(v, A)$ the *determinant* of $(v, A)$ and $\Lambda_1(v, A)$ the *trace* of $(v, A)$.

We may express $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$, as quasi-Plücker coordinates of the $m \times (m + 1)$ matrix with columns $v, Av, \ldots, A^n v$ (following [GR4]).

In the basis $v, Av, \dots, A^{m-1}$ the map $A$ is represented by the Frobenius matrix $A_v$ with the last column equal to $((-1)^m\Lambda_m(v,A), \dots, -\Lambda_1(v,A))^T$. Theorem 3.1.3 implies that if determinants of $A_v$ are defined, then they coincide up to a sign with $\Lambda_m(V,A)$. This justifies our definition.

Also, when $R$ is a commutative algebra, $\Lambda_m(v,A)$ is the determinant of $A$ and $\Lambda_1(v,A)$ is the trace of $A$.

When $R$ is noncommutative, the expressions $\Lambda_i(v,A) \in R$, $i = 1, \dots, m$, depend on vector $v$. However, they provide some information about $A$. For example, the following statement is true.

**Proposition 5.1.2.** *If the determinant $\Lambda_m(v,A)$ equals zero, then the map $A$ is not invertible.*

Definition 5.1.1 of noncommutative determinants and traces was essentially used in [GKLLRT] for linear maps given by matrices $A = (a_{ij})$, $i, j = 1, \dots, m$ and unit vectors $e_s$, $s = 1, \dots, m$. In this case $\Lambda_i(e_s, A)$ are quasi-Plücker coordinates of the corresponding Krylov matrix $K_s(A)$. Here (see [G]) $K_s(A)$ is the matrix $(b_{ij})$, $i = m, m-1, \dots, 1, 0$, $j = 1, \dots, m$, where $b_{ij}$ is the $(sj)$-entry of $A^i$.

**Example.** Let $A = (a_{ij})$ be an $m \times m$-matrix and $v = e_1 = (1, 0, \dots, 0)^T$. Denote by $a_{ij}^{(k)}$ the corresponding entries of $A^k$. Then

$$\Lambda_m(v,A) = (-1)^{m-1} \begin{vmatrix} \boxed{a_{11}^{(m)}} & a_{12}^{(m)} & \dots & a_{1m}^{(m)} \\ a_{11}^{(m-1)} & a_{12}^{(m-1)} & \dots & a_{1m}^{(m-1)} \\ & \dots & \dots & \\ a_{11} & a_{12} & \dots & a_{1m} \end{vmatrix}.$$

For $m = 2$ the "noncommutative trace" $\Lambda_1$ equals $a_{11} + a_{12}a_{22}a_{12}^{-1}$ and the "noncommutative determinant" $\Lambda_2$ equals $a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21}$.

It was shown in [GKLLRT] that if $A$ is a quantum matrix, then $\Lambda_m$ equals $\det_q A$ and $A$ is a Capelli matrix, then $\Lambda_m$ equals the Capelli determinant.

A construction of a noncommutative determinant and a noncommutative trace in terms of cyclic vectors in a special case was used in [Ki].

One can view the elements $\Lambda_i(v,A)$ as elementary symmetric functions of "eigenvalues" of $A$.

One can introduce complete symmetric functions $S_i(v,A)$, $i = 1, 2, \dots$, of "eigenvalues" of $A$ as follows. Let $t$ be a formal commutative variable. Set $\lambda(t) = 1 + \Lambda_1(v,A)t + \dots + \Lambda_m(v,A)t^m$ and define the elements $S_i(v,A)$ by the formulas

$$\sigma(t) := 1 + \sum_{k>0} S_k t^k = \lambda(-t)^{-1}.$$

Recall that $R_{1^k l}$ is the ribbon Schur function corresponding to the hook with $k$ vertical and $l$ horizontal boxes. In particular, $\Lambda_k = R_{1^k}$, $S_l = R_l$.

Let $A : R^m \to R^m$ be a linear map of right linear spaces.

**Proposition 5.1.3.** For $k \geq 0$

$$A^{m+k}v = (-1)^{m-1}vR_{1^{m-1}(k+1)} + (-1)^{m-2}(Av(R_{1^{m-2}(k+1)} + \cdots + (A^{m-1}v)R_{k+1}.$$

Let $A = \operatorname{diag}(x_1, \ldots, x_m)$. In the general case for a cyclic vector one can take $v = (1, \ldots, 1)^T$. In this case, the following two results hold.

**Proposition 5.1.4.** *For $k = 1, \ldots, m$*

$$\Lambda_k(v, A) =$$

$$= \begin{vmatrix} 1 & \ldots & \boxed{x_m^{m-k}} & \ldots & x_m^{m-1} \\ & & \ldots & & \\ 1 & \ldots & x_1 & \ldots & x_1^{m-1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1 & \ldots & x_m^{m-k-1} & x_m^{m-k+1} & \ldots & \boxed{x_m^m} \\ & & \ldots & & & \\ 1 & \ldots & x_1^{m-k-1} & x_1^{m-k+1} & \ldots & x_1^m \end{vmatrix}.$$

**Proposition 5.1.5.** *For any $k > 0$*

$$S_k(v, A) = \begin{vmatrix} 1 & \ldots & \boxed{x_m^{m-1}} \\ & \ldots & \\ 1 & \ldots & x_1^{m-1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1 & \ldots & x_m^{m-2} & \boxed{x_m^{m+k-1}} \\ & & \ldots & \\ 1 & \ldots & x_1^{m-2} & x_1^{m+k-1} \end{vmatrix}.$$

Note that formulas for $S_k$ look somewhat simpler than formulas for $\Lambda_k$.

## 5.2 Noncommutative determinants and noncommutative eigenvalues

One can also express $\Lambda_i(v, A) \in R$ in terms of left eigenvalues of $A$.

Let a linear map $A : R^m \to R^m$ of the right vector spaces is represented by the matrix $(a_{ij})$.

**Definition 5.2.1.** A nonzero row-vector $u = (u_1, \ldots, u_m)$ is a left eigenvector of $A$ if there exists $\lambda \in R$ such that $uA = \lambda u$.

We call $\lambda$ a *left eigenvalue* of $A$ corresponding to vector $u$. Note, that $\lambda$ is the eigenvalue of $A$ corresponding to a left eigenvector $u$ then, for each $\alpha \in R$, $\alpha\lambda\alpha^{-1}$ is the eigenvalue corresponding to the left eigenvector $\alpha u$. Indeed, $(\alpha u)A = \alpha\lambda\alpha^{-1}(\alpha u)$.

For a row vector $u = (u_1, \ldots, u_m)$ and a column vector $v = (v_1, \ldots, v_m)^T$ denote by $\langle u, v\rangle$ the inner product $\langle u, v\rangle = u_1v_1 + \ldots u_mv_m$.

**Proposition 5.2.2.** *Suppose that $u = (u_1, \ldots, u_m)$ is a left eigenvector of $A$ with the eigenvalue $\lambda$, $v = (v_1, \ldots, v_m)^T$ is a cyclic vector of $A$, and $\langle u, v\rangle = 1$. Then The eigenvalue $\lambda$ satisfies the equation*

$$(5.1)\quad (-1)^m\Lambda_m(v, A) + (-1)^{m-1}\lambda\Lambda_{m-1}(v, A) + \cdots - \lambda^{m-1}\Lambda_1(v, A) + \lambda^m = 0.$$

Equation (5.2.1) and the corresponding Viète theorem (see Section 3) show that if the map $A : R^m \to R^m$ has left eigenvectors $u^1, \dots, u^m$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$ such that $\langle u^i, v\rangle = 1$ for $i = 1, \dots, m$ and any submatrix of the Vandermonde matrix $(\lambda_i^j)$ is invertible, then all $\Lambda_i(v, A)$ can be expressed in terms of $\lambda_1, \dots, \lambda_m$ as "noncommutative elementary symmetric functions".

## 5.3 Multiplicativity of determinants

In the commutative case the multiplicativity of determinants and the additivity of traces are related to computations of determinants and traces with diagonal block-matrices. In the noncommutative case we suggest to consider the following construction.

Let $R$ be an algebra with a unit. Let $A : R^m \to R^m$ and $D : R^n \to R^n$ be linear maps of right vector spaces, $v \in R^m$ an $A$-cyclic vector and $w \in R^n$ a $D$-cyclic vector.

There exist $\Lambda_i(w, D) \in R$, $i = 1, \dots, n$, such that

$$(-1)^n v\Lambda_n(w, D) + (-1)^{n-1}(Dw)\Lambda_{n-1}(w, D) + \dots - (D^{m-1}v)\Lambda_1(w, D) + D^n w = 0.$$

Denote also by $S_i(w, D)$, $i = 1, 2, \dots$, the corresponding complete symmetric functions.

The matrix $C = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ acts on $R^{m+n}$. Suppose that the vector $u = \begin{pmatrix} v \\ w \end{pmatrix}$ is a cyclic vector for matrix $C$. We want to express $\Lambda_i(u, C)$, $i = 1, \dots, m + n$ in terms of $\Lambda_j(v, A)$, $S_k(v, A)$, $\Lambda_p(w, D)$, and $S_q(w, D)$.

Denote, for brevity, $\Lambda_j(v, A) = \Lambda_j$, $S_k(v, A) = S_k$, $\Lambda_p(w, D) = \Lambda'_p$, $S_q(w, D) = S'_q$.

For two sets of variables $\alpha = \{a_1, a_2, \dots, \}$ and $\beta = \{b_1, b_2, \dots, \}$ introduce the following $(m + n) \times (m + n)$-matrix $M(m, n; \alpha, \beta)$:

$$\begin{pmatrix}
1 & a_1 & a_2 & \dots & \dots & \dots & a_{m-1} & \dots & a_{m+n-1} \\
0 & 1 & a_1 & a_2 & \dots & \dots & a_{m-1} & \dots & a_{m+n-2} \\
 & & & & & \dots & & & \\
0 & 0 & 0 & \dots & \dots & \dots & 1 & \dots & a_m \\
1 & b_1 & b_2 & \dots & b_{n-1} & \dots & \dots & \dots & b_{m+n-1} \\
0 & 1 & b_1 & b_2 & \dots & \dots & \dots & \dots & b_{m+n-2} \\
 & & & & & \dots & & & \\
0 & 0 & 0 & \dots & 1 & b_1 & \dots & \dots & b_n
\end{pmatrix}.$$

**Proposition 5.3.1.** *For any $j = 2, \dots, m + n$ we have*

$$|M(m, n; \alpha, \beta)|_{1j} = -|M(m, n; \alpha, \beta)|_{m+1,j}.$$

The elements $S_i(u, C)$, $i = 1, 2, \dots,$ can be computed as follows. Denote by $N_k(m, n; \alpha, \beta)$ the matrix obtained from $M$ by replacing its last column by the following column:

$$(a_{m+n+k-1}, a_{m+n+k-2}, \dots, a_{n+k-1}, b_{m+n+k-1}, b_{m+n+k-2}, \dots, b_{m+k-1})^T.$$

Set $\alpha = \{-S_1, S_2, \dots, (-1)^k S_k, \dots\}$, $\alpha' = \{-S_1', S_2', \dots, (-1)^k S_k', \dots\}$.

**Theorem 5.3.2.** *For $k = 1, 2, \dots$ we have*

$$S_k(u, C) = |M(m, n; \alpha, \alpha')|_{1m+n}^{-1} \cdot |N_k(m, n; \alpha, \alpha')|_{1m+n}.$$

**Example.** For $m = 3$, $n = 2$ and $k = 1, 2, \dots$. Then

$$S_k(u, C) =$$

$$= (-1)^{k-1} \begin{vmatrix} 1 & -S_1 & S_2 & -S_3 & S_4 \\ 0 & 1 & -S_1 & S_2 & -S_3 \\ 0 & 0 & 1 & -S_1 & S_2 \\ 1 & -S_1' & S_2' & -S_3' & S_4' \\ 0 & 1 & -S_1' & S_2' & -S_3' \end{vmatrix}_{15}^{-1} \times \begin{vmatrix} 1 & -S_1 & S_2 & -S_3 & S_{4+k} \\ 0 & 1 & -S_1 & S_2 & -S_{3+k} \\ 0 & 0 & 1 & -S_1 & S_{2+k} \\ 1 & -S_1' & S_2' & -S_3' & S_{4+k}' \\ 0 & 1 & -S_1' & S_2' & -S_{3+k}' \end{vmatrix}_{15}.$$

For $n = 1$ denote $\Lambda_1(D) = S_1(D)$ by $\lambda'$.

**Corollary 5.3.3.** *If $n = 1$, then for $k = 1, 2, \dots$ we have*

$$S_k(u, C) = S_k(v, A) + S_{k-1}(v, A)|M(m, n; \alpha, \alpha')|_{1m+n}^{-1} \lambda' |M(m, n; \alpha, \alpha')|_{1m+n+1}.$$

Note that

$$\Lambda_{m+1}(u, C) = |M(m, n; \alpha, \alpha')|_{1m+n}^{-1} \lambda' |M(m, n; \alpha, \alpha')|_{1m+n+1} \Lambda_m(v, A),$$

i.e. the “determinant” of the diagonal matrix equals the product of two “determinants”.

# Chapter 6

# Some applications

In this section we mainly present some results from [GR1, GR2, GR4].

## 6.1 Continued fractions and almost triangular matrices

Consider an infinite matrix $A$ over a skew-field:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \dots \\ -1 & a_{22} & a_{23} & \dots & a_{2n} \dots \\ 0 & -1 & a_{33} & \dots & a_{3n} \dots \\ 0 & 0 & -1 & \dots & \dots \end{pmatrix}$$

It was pointed out in [GR1], [GR2] that the quasideterminant $|A|_{11}$ can be written as a generalized continued fraction

$$|A|_{11} = a_{11} + \sum_{j_1 \neq 1} a_{1j_1} \frac{1}{a_{2j_1} + \sum_{j_2 \neq 1, j_1} a_{2j_2} \frac{1}{a_{3j_2} + \dots}}.$$

Let

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ -1 & a_{22} & \dots & a_{2n} \\ 0 & -1 & \dots & a_{3n} \\ & & \dots & \\ \dots & 0 & -1 & a_{nn} \end{pmatrix}.$$

The following proposition was formulated in [GR1], [GR2].

**Proposition 6.1.1.** $|A_n|_{11} = P_n Q_n^{-1}$, *where*

$$P_n = \sum_{1\le j_1<\dots<j_k<n} a_{1j_1}a_{j_1+1,j_2}a_{j_2+1,j_3}\dots a_{j_k+1,n}, \tag{6.1.1}$$

$$Q_n = \sum_{2\le j_1<\dots<j_k<n} a_{2j_1}a_{j_1+1,j_2}a_{j_2+1,j_3}\dots a_{j_k+1,n}. \tag{6.1.2}$$

*Proof.* From the homological relations one has

$$|A_n|_{11}|A_n^{1n}|_{21}^{-1} = -|A_n|_{1n}|A_n^{11}|_{2n}^{-1}.$$

We will apply formula (1.2.2) to compute $|A_n|_{1n}$, $|A_n^{11}|_{2n}$, and $|A_n^{1n}|_{21}$. It is easy to see that $|A_n^{1n}|_{21} = -1$. To compute the two other quasideterminants, we have to invert triangular matrices. Setting $P_n = |A_n|_{1n}$ and $Q_n = |A_n^{11}|_{2n}$ we arrive at formulas (6.1.1), (6.1.2). □ □

*Remark*In the commutative case Proposition 6.1.1 is well known. In this case $P_n = |A_n|_{1n} = (-1)^n \det A_n$ and $Q_n = (-1)^{n-1}\det A_n^{11}$.

Formulas (6.1.1), (6.1.2) imply the following result (see [GR1, GR2]).

**Corollary 6.1.2.** *The polynomials $P_k$ for $k \ge 0$ and $Q_k$ for $k \ge 1$ are related by the formulas*

$$P_k = \sum_{s=0}^{k-1} P_s a_{s+1,k}, \qquad P_0 = 1, \tag{6.1.3}$$

$$Q_k = \sum_{s=1}^{k-1} Q_s a_{s+1,k}, \qquad Q_1 = 1. \tag{6.1.4}$$

**Corollary 6.1.3.** *Suppose that for any $i \ne j$ and any $p, q$ the elements of the matrix $A$ satisfy the conditions*

$$a_{ij}a_{pq} = a_{pq}a_{ij}$$
$$a_{jj}a_{ii} - a_{ii}a_{jj} = a_{ij}, \quad 1 \le i < j \le n.$$

*Then*

$$P_n = |A_n|_{1n} = a_{nn}a_{n-1n-1}\dots a_{11}. \tag{6.1.5}$$

The proof follows from (6.1.3).

**Corollary 6.1.4.** ([GR1, GR2]) *For the Jacoby matrix*

$$A = \begin{pmatrix} a_1 & 1 & 0 & \dots \\ -1 & a_2 & 1 & \\ 0 & -1 & a_3 & \dots \end{pmatrix}$$

*we have*

$$|A|_{11} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}},$$

*and*

$$P_0 = 1, \quad P_1 = a_1, \quad P_k = P_{k-1}a_k + P_{k-2}, \ \textit{for } k \geq 2;$$
$$Q_1 = 1, \quad Q_2 = a_2, \quad Q_k = Q_{k+1}a_k + Q_{k-2}, \ \textit{for } k \geq 3;.$$

In this case $P_k$ is a polynomial in $a_1, \dots, a_k$ and $Q_k$ is a polynomial in $a_2, \dots, a_k$.

## 6.2 Continued fractions and formal series

In the notation of the previous subsection the infinite continued fraction $|A|_{11}$ may be written as a ratio of formal series in the letters $a_{ij}$ and $a_{ii}^{-1}$. Namely, set

$$P_\infty = \sum_{\substack{1 \leq j_1 < j_2 \dots < j_k < r-1 \\ r = 1,2,3,\dots}} a_{1j_1} a_{j_1+1j_2} \dots a_{j_k+1r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1}$$
$$= 1 + a_{12}a_{22}^{-1}a_{11}^{-1} + a_{13}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{11}a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \dots,$$

and

$$Q_\infty = a_{11}^{-1} + \sum_{\substack{2 \leq j_1 < j_2 \dots < j_k < r-1 \\ r = 2,3\dots}} a_{2j_1} a_{j_1+1j_2} \dots a_{j_k+1r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1}$$
$$= a_{11}^{-1} + a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{24}a_{44}^{-1}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \dots.$$

Since each monomial appears in these sums at most once, these are well-defined formal series.

The following theorem was proved in [PPR]. Another proof was given in [GR4].

**Theorem 6.2.1.** *We have*

$$|A|_{11} = P_\infty \cdot Q_\infty^{-1}.$$

*Proof.* Set $b_{ij} = a_{ij}a_{jj}^{-1}$ and consider matrix $B = (b_{ij})$, $i, j = 1, 2, 3, \dots$. According to a property of quasideterminants $|A|_{11} = |B|_{11}a_{11}$. Applying the noncommutative Sylvester theorem to $B$ with matrix $(b_{ij}), i, j \geq 3$, as the pivot, we have

$$|B|_{11} = 1 + |B^{21}|_{12}|B^{11}|_{22}^{-1}a_{11}^{-1}.$$

Therefore

$$|A|_{11} = (a_{11}|B^{11}|_{22}a_{11}^{-1} + |B^{21}|_{12}a_{11}^{-1})(|B^{11}|_{22}a_{11}^{-1})^{-1}. \tag{6.2.1}$$

By [GKLLRT], Proposition 2.4, the first factor in (6.2.1) equals $P_\infty$, and the second equals $Q_\infty^{-1}$. □
□

## 6.3 Noncommutative Rogers-Ramanujan continued fraction

The following application of Theorem 6.2.1 to Rogers-Ramanujan continued fraction was given in [PPR]. Consider a continued fraction with two formal variables $x$ and $y$:

$$A(x,y) = \frac{1}{1 + x\frac{1}{1+x\frac{1}{1+\dots}y}y}.$$

It is easy to see that

$$A(x,y) = \begin{vmatrix} 1 & x & & \cdot & & & \\ -y & 1 & x & & \cdot & & 0 \\ & -y & 1 & x & & \cdot & \\ & & & 1 & & & \cdot \\ & 0 & & \ddots & \ddots & & \cdot \end{vmatrix}_{11}^{-1} = \begin{vmatrix} 1 & x & 0 & & \\ -1 & y^{-1} & xy^{-1} & & \\ 0 & -1 & y^{-1} & xy^{-1} & \\ & & -1 & y^{-1} & \ddots \end{vmatrix}_{11}$$

Theorem 9.2.1 implies the following result.

**Corollary 6.3.1** $A(x,y) = P \cdot Q^{-1}$, where $Q = yPy^{-1}$ and

$$P = 1 + \sum_{k \geq 1 n_1, \dots, n_k \geq 1} y^{-n_1} x y^{-n_2} x \dots y^{-n_k} x y^{k+n_1+n_2+\dots+n_k}.$$

Following [PPR], let us assume that $xy = qyx$, where $q$ commutes with $x$ and $y$. Set $z = yx$. Then Corollary 6.3.1 implies Rogers-Ramanujan continued fraction identity

$$A(x,y) = \frac{1}{1 + \frac{qz}{1+\frac{q^2 z}{1+\dots}}} = \frac{1 + \sum_{k\geq 1} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)} z^k}{1 + \sum_{k \geq 1} \frac{q^{k^2}}{(1-q)\dots(1-q^k)} z^k}.$$

## 6.4 Quasideterminants and characterisric functions of graphs

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, where $a_{ij}$ are formal noncommuting variables. Fix $p, q \in \{1, \dots, n\}$ and a set $J \subset \{1, \dots, \hat{p}, \dots, n\} \times \{1, \dots, \hat{q}, \dots, n\}$ such that $|J| = n - 1$ and both projections of $J$ onto $\{1, \dots, \hat{p}, \dots, n\}$ and $\{1, \dots, \hat{q}, \dots, n\}$ are surjective. Introduce new variables $b_{kl}$, $k, l = 1, \dots, n$, by the formulas $b_{kl} = a_{kl}$

for $(l,k) \notin J$, $b_{kl} = a_{lk}^{-1}$ for $(l,k) \in J$. Let $F_J$ be a ring of formal series in variables $b_{kl}$.

**Proposition 6.4.1.** *The quasideterminant $|A|_{ij}$ is defined in the ring $F_J$ and is given by the formula*

(9.4.1)
$$|A|_{ij} = b_{ij} - \sum (-1)^s b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}.$$

*The sum is taken over all sequences $i_1, \dots, i_s$ such that $i_k \neq i, j$ for $k = 1, \dots, s$.*

**Proposition 6.4.2.** *The inverse to $|A|_{ij}$ is also defined in the ring $F_J$ and is given by the following formula*

(6.4.2)
$$|A|_{ij} = b_{ij} - \sum (-1)^s b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}.$$

*The sum is taken over all sequences $i_1, \dots, i_s$.*

All relations between quasideterminants, including the Sylvester identity, can be deduced from formulas (6.4.1) and (6.4.2).

Formulas (6.4.1) and (6.4.2) can be interpreted in terms of graph theory. Let $\Gamma_n$ be a complete oriented graph with vertices $1, \dots, n$ and edges $e_{kl}$, where $k, l = 1, \dots, n$. Introduce a bijective correspondence between edges of the graph and elements $b_{kl}$ such that $e_{kl} \mapsto b_{kl}$.

Then there exist a bijective correspondence between the monomials $b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}$ and the paths from the vertex $i$ to the vertex $j$.

## 6.5 Factorizations of differential operators and noncommutative variation of constants

Let $R$ be an algebra with a derivation $D : R \to R$. Denote $Dg$ by $g'$ and $D^k g$ by $g^{(k)}$. Let $P(D) = D^n + a_1 D^{n-1} + \dots + a_n$ be a differential operator acting on $R$ and $\phi_i$, $i = 1, \dots, n$, be solutions of the homogeneous equation $P(D)\phi = 0$, i.e., $P(D)\phi_i = 0$ for all $i$.

For $k = 1, \dots, n$ consider the Wronski matrix

$$W_k = \begin{pmatrix} \phi_1^{(k-1)} & \dots & \phi_k^{(k-1)} \\ & \dots & \\ \phi_1 & \dots & \phi_k \end{pmatrix}$$

and suppose that any square submatrix of $W_n$ is invertible.

Set $w_k = |W|_{1k}$ and $b_k = w_k' w_k^{-1}$, $k = 1, \dots, n$.

**Theorem 6.5.1.** [EGR]

$$P(D) = (D - b_n)(D - b_{n-1}) \dots (D - b_1).$$

**Corollary 6.5.2.** *Operator $P(D)$ can be factorized as*

$$P(D) = (w_n \cdot D \cdot w_n^{-1})(w_{n-1} \cdot D \cdot w_{n-1}^{-1}) \dots (w_1 \cdot D \cdot w_1^{-1}).$$

One can also construct solutions of the nonhomogeneous equation $P(D)\psi = f$, $f \in R$, starting with solutions $\phi_1, \dots, \phi_n$ of the homogeneous equation. Suppose that any square submatrix of $W_n$ is invertible and that there exist elements $u_j \in R$, $j = 1, \dots, n$, such that

$$u'_j = |W|_{1j}^{-1} f. \tag{6.5.1}$$

**Theorem 6.5.3** The element $\psi = \sum_{j=1}^{j=n} \phi_j u_j$ satisfies the equation

$$(D^n + a_1 D^{n-1} + \dots + a_n)\psi = f.$$

In the case where $R$ is the algebra of complex valued functions $g(x)$, $x \in \mathbb{R}$ the solution $\psi$ of the nonhomogeneous equation is given by the classical formula

$$\psi(x) = \sum_{j=1}^{j=n} \phi_j \int \frac{\det W_j}{\det W} dx \tag{6.5.2}$$

where matrix $W_j$ is obtained fron the Wronski matrix $W$ by replacing the entries in the $j$-th column of $W$ by $f, 0, \dots, 0$. It is easy to see that formula (6.5.1) and Theorem 6.5.3 imply formula (6.5.2).

## 6.6 Iterated Darboux transformations

Let $R$ be a differential algebra with a derivation $D : R \to R$ and $\phi \in R$ be an invertible element. Recall that we denote $D(g) = g'$ and $D^k(g) = g^{(k)}$. In particular $D^{(0)}(g) = g$.

For $f \in R$ define $\mathcal{D}(\phi; f) = f' - \phi'\phi^{-1} f$. Following [Mat] we call $\mathcal{D}(\phi; f)$ the *Darboux transformation* of $f$ defined by $\phi$. This definition was known for matrix functions $f(x)$ and $D = \partial_x$. Note that

$$\mathcal{D}(\phi; f) = \begin{vmatrix} \boxed{f'} & \phi' \\ f & \phi \end{vmatrix}.$$

Let $\phi_1, \dots, \phi_k$. Define the *iterated* Darboux transformation $\mathcal{D}(\phi_k, \dots \phi_1; f)$ by induction as follows. For $k = 1$, it coincides with the Darboux transformation defined above. Assume that $k > 1$. The expression $\mathcal{D}(\phi_k, \dots, \phi_1; f)$ is defined if $\mathcal{D}(\phi_k, \dots, \phi_2; f)$ is defined and invertible and $\mathcal{D}(\phi_k; f)$ is defined. In this case,

$$\mathcal{D}(\phi_k, \dots \phi_1; f) = \mathcal{D}(\mathcal{D}(\phi_k, \dots \phi_2; f); \mathcal{D}(\phi_1; f)).$$

**Theorem 6.6.1.** *If all square submatrices of matrix* $(\phi_i^{(j)})$, $i = 1, \dots, k$; $j = k-1, \dots, 0$ *are invertible, then*

$$\mathcal{D}(\phi_k, \dots, \phi_1; f) = \begin{vmatrix} \boxed{f^{(k)}} & \phi_1^{(k)} & \dots & \phi_k^{(k)} \\ \dots & \dots & \dots & \dots \\ f & \phi_1 & \dots & \phi_k \end{vmatrix}.$$

The proof follows from the noncommutative Sylvester theorem.

**Corollary 6.6.2.** *The iterated Darboux transformation* $\mathcal{D}(\phi_k, \dots \phi_1; f)$ *is symmetric in* $\phi_1, \dots, \phi_k$.

The proof follows from the symmetricity of quasideterminants.

**Corollary 6.6.3.** ([Mat]) *In commutative case, the iterated Darboux transformation is a ratio of two Wronskians,*

$$\mathcal{D}(\phi_k, \dots, \phi_1; f) = \frac{W(\phi_1, \dots \phi_k, f)}{W(\phi_1, \dots, \phi_k)}.$$

## 6.7 Noncommutative Sylvester–Toda lattices

Let $R$ be a division ring with a derivation $D : R \to R$. Let $\phi \in R$ and the quasideterminants

$$T_n(\phi) = \begin{vmatrix} \phi & D\phi & \dots & D^{n-1}\phi \\ D\phi & D^2\phi & \dots & D^n\phi \\ \dots & \dots & \dots & \dots \\ D^{n-1}\phi & D^n\phi & \dots & \boxed{D^{2n-2}\phi} \end{vmatrix} \quad 6.7.1$$

are defined and invertible. Set $\phi_1 = \phi$ and $\phi_n = T_n(\phi)$, $n = 2, 3, \dots$.

**Theorem 6.7.1.** *Elements* $\phi_n$, $n = 1, 2, \dots,$ *satisfy the following system of equations:*

$$D((D\phi_1)\phi_1^{-1}) = \phi_2\phi_1^{-1},$$
$$D((D\phi_n)\phi_n^{-1}) = \phi_{n+1}\phi_n^{-1} - \phi_n\phi_{n-1}^{-1}, \quad n \geq 2.$$

If $R$ is commutative, the determinants of matrices used in formulas (6.7.1) satisfy a nonlinear system of differential equations. In the modern literature this system is called the Toda lattice (see, for example, [Ok] but in fact it was discovered by Sylvester in 1862 [Syl] and, probably, should be called the Sylvester–Toda lattice. Our system can be viewed as a noncommutative generalization of the

Sylvester–Toda lattice. Theorem 6.7.1 appeared in [GR1, GR2] and was generalized in [RS] and [EGR].

The following theorem is a noncommutative analog of the famous Hirota identities.

**Theorem 6.7.2.** *For $n \geq 2$*

$$T_{n+1}(\phi) = T_n(D^2\phi) - T_n(D\phi) \cdot ((T_{n-1}(D^2\phi)^{-1} - T_n(\phi)^{-1})^{-1} \cdot T_n(D\phi).$$

## 6.8 Noncommutative orthogonal polynomials

The results described in this subsection were obtained in [GKLLRT]. Let $S_0, S_1, S_2, \dots$ be elements of a skew-field $R$ and $x$ be a commutative variable. Define a sequence of elements $P_i(x) \in R[x]$, $i = 0, 1, \dots$, by setting $P_0 = S_0$ and

$$P_n(x) = \begin{vmatrix} S_n & \dots & S_{2n-1} & \boxed{x^n} \\ S_{n-1} & \dots & S_{2n-2} & x^{n-1} \\ \dots & \dots & \dots & \dots \\ S_0 & \dots & S_{n-1} & 1 \end{vmatrix} \tag{6.8.1}$$

for $n \geq 1$. We suppose here that quasideterminants in (9.8.1) are defined. One can see that $P_n(x)$ is a polynomial of degree $n$. If $R$ is commutative, then $P_n, n \geq 0$, are orthogonal polynomials defined by the moments $S_n$, $n \geq 0$. We are going to show that if $R$ is a free division ring generated by $S_n$, $n \geq 0$, then polynomials $P_n$ are indeed orthogonal with regard a natural noncommutative $R$-valued product on $R[x]$.

Let $R$ be a free skew-field generated by $c_n$, $n \geq 0$. Define on $R$ a natural anti-involution $a \mapsto a^*$ by setting $c_n^* = c_n$ for all $n$. Extend the involution to $R[x]$ by setting $(\sum a_i x^i)^* = \sum a_i x^i$. Define the $R$-valued inner product on $R[x]$ by setting

$$\left\langle \sum a_i x^i, \sum b_j x^j \right\rangle = \sum a_i c_{i+j} b_j^*.$$

**Theorem 6.8.1.** *For $n \neq m$ we have*

$$\langle P_n(x), P_m(x) \rangle = 0.$$

The three term relation for noncommutative orhogonal polynomials $P_n(x)$ can be expressed in terms of noncommutative quasi-Schur functions (see [GKLLRT]).

# Bibliography


[A] E. Artin, *Geometric algebra*, Wiley, New York, 1988.

[Al] S. Alesker, *Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables*, arXiv: math.CV/010429 (2002).

[As] H. Aslaksen, *Quaternionic determinants*, Math. Intelligencer, **18** (1996), no. 3, 57–65.

[B] F. A. Berezin, *Introduction to algebra and analysis with anticommuting variables*, Izd. Moskov. Univ., Moscow, 1983. (Russian).

[BR] A. Berenstein and V. Retakh, *Noncommutative double Bruhat celles and their factorizations*, IMRN, (2005), no.8, 477–516.

[BW] J. Birman and R. Williams, *Knotted periodic orbits I: Lorenz knots*, Topology, **22** (1983) 47–62.

[C] A. Cayley, *On certain results relating to quaternions*, Phil. Mag., **26** (1845) 141–145.

[CF] P. Cartier and D. Foata, *Problemes combinatoires de commutation et rearrangements*, Lecture Notes Math., vol. 85, Springer-Verlag, Berlin–Heidelberg, 1969.

[CS] A. Connes and A. Schwarz, *Matrix Viète theorem revisited*, Lett. Math. Phys., **39** (1997), 349–353.

[Co] P. M. Cohn, *Skew-field constructions* , London Math. Soc. Lecture Notes, vol. 27, Cambridge Univ. Press , Cambridge, 1977.

[D] J. Dieudonne, *Les determinantes sur un corps non-commutatif*, Bull. Soc. Math. France, **71** (1943), 27–45.

[Dr] P. K. Draxl, *Skew fields* , London Math. Soc. Lecture Note Series, **81** (1983), Cambridge Univ. Press, Cambridge.

[DS] V. G. Drinfeld and V.V. Sokolov , *Lie algebras and equations of Korteweg-de Vries type*, jour Current problems in mathematics. Itogi Nauki i Tekniki, Akad. Nauk. SSSR, vol 24, VINITI, Moscow, 1984, pp. 81–180, 1984 (Russian).

[Dy] F. J. Dyson, *Quaternion determinants* , Helv. Phys. Acta, **45** (1972), 289–302.

[Dy1] ——— , *Correlations between eigenvalues of random matrices*, Comm. Math. Phys., **19** (1970), 235–250.

[EGR] P. Etingof, I. Gelfand, and V. Retakh, *Factorization of differential operators, quasideterminants, and nonabelian Toda field equation*s, Math. Res. Letters, **4** (1997), no. 2-3, 413–425.

[EGR1] ——— , *Nonabelian integrable systems, quasideterminants, and Marchenko lemma*, Math. Res. Letters, **5** (1998), no. 1-2, 1–12.

[ER] P. Etingof and V. Retakh, *Quantum determinants and quasideterminants*, Asian J. Math., **3** (1999), no. 2, 345–352.

[F] D. Foata, *A noncommutative version of the matrix inversion formula*, Advances in Math. vol 31 (1979), 330–349.

[Fr] G. Frobenius, *Über lineare Substituenen und bilinear Forme*n, J. Reine Angew. Math., **84** (1877), 1–63.

[FIS] S. H. Friedberg, A. J. Insel, and L. E. Spence , *Linear algebra*, Third Edition, Prentice Hall, Upper Saddle River, NJ, 1997.

[G] F.R. Gantmacher, *The theory of matrices* , Chelsea, New York, 1959.

[GGR] I. Gelfand, S. Gelfand, and V. Retakh, *Noncommutative algebras associated to complexes and graph*s, Selecta Math (N.S.) **7** (2001) 525–531.

[GGRSW] I. Gelfand, S. Gelfand, V. Retakh, S. Serconek, and R. Wilson, *Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomial*s, J. Algebra, **254** (2002), 279–299.

[GGRW] I. Gelfand, S. Gelfand, V. Retakh, and R. Wilson, *Quasideterminants*, Advances in Math., **193** (2005), 56–141.

[GGRW1] I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, *Factorizations of polynomials over noncommutative algebras and sufficient sets of edges in directed graphs*, Lett. Math. Physics, **74** (2005), 153–167.

[GKLLRT] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, *Noncommutative symmetric functions*, Advances in Math. **112** (1995), no. 2, 218–348.

[GR] I. Gelfand and V. Retakh, *Determinants of matrices over moncommutative rings*, Funct. Anal. Appl., **25** (1991), no. 2, 91–102.

[GR1] ——— , A *theory of noncommutative determinants and characteristic functions of graphs*, Funct. Anal. Appl. **26** (1992), no. 4, 1–20.

[GR2] ——— , *A theory of noncommutative determinants and characteristic functions of graphs.* I, Publ. LACIM, UQAM, Montreal **14** (1993), 1–26.

[GR3] ——— , *Noncommutative Vieta theorem and symmetric functions*, Gelfand Mathematical Seminars 1993–95, Birkhäuser, Boston, 1996, pp. 93–100.

[GR4] ——— , *Quasideterminants*, I , Selecta Math. (N.S.) **3** (1997), no. 4, 517–546.

[GRSW] I. Gelfand, V. Retakh, S. Serconek, and R. L. Wilson , *On a class of algebras associated to directed graphs*, Selecta Math. **11** (2005), 281–295.

[GRW] I. Gelfand, V. Retakh, and R. Wilson, *Quadratic-linear algebras associated with decompositions of noncommutative polynomials and Differential polynomials*, Selecta Math. (N.S.) **7** (2001), 493–523.

[GRW1] ——— , *Quaternionic quasideterminants and determinants, Lie groups and symmetric spaces*, Amer. Math. Soc. Transl. Series 2,vol. 210, Amer. Math. Soc., Providence, RI, 2003, pp. 111–123.

[Gi] A. Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in Singularity Theory, Amer. Math. Soc. Transl. Series 2, vol. 180, Amer. Math. Soc., Providence, RI, 1997, pp. 103–115.

[GV] V. Goncharenko and A. Veselov, *Darboux transformations for multidemensional Schrödinger operato*rs, J. Phys. A: Math. Gen., **31** (1998), 5315–5326.

[H] A. Heyting, *Die Theorie der linearen Gleichungen in einer Zahlenspezies mit nichtkommutativer Multiplication*, Math. Ann., **98** (1927), 465–490.

[Ha] M. Hamanaka, *Notes on exact multi-soliton solutions of noncommutative integrable hierarchies*, info hep-th/061006 (2006), 1–17.

[Ho] R. Howe, *Perspectives in invariant theory: Schur duality, multiplicity-free actions and beyond*, Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ, Ramat Gan, 1995, pp.1–182.

[Ho1] ———, *Remarks on classical invariant theory* , Trans. Amer. Math. Soc., **313**, 1 89, 539–570.

[J] C. Joly, *Supplement to oevres of Hamilton*, W. R. Hamilton, Elements of Quaternions, Chelsea , New York, 1969.

[Ja] N. Jacobson, L*ectures in abstract algebra.* Vol. II , Springer-Verlag, New York–Berlin, 1975.

[Ja2] ———, *Structure and representations of Jordan algebras*, Colloquium Publications, vol. 39, Amer. Math. Soc., Providence, RI, 1968.

[K] M. Kapranov, *Noncommutative geometry based on commutator expansions*, J. Reine Angew. Math., **505** (1998), 73–118.

[Ki] A. Kirillov, *Introduction to family algebras*, Moscow Math. J., **1** (2001), no. 1, 49–63.

[KP] M. Konvalinka, I. Pak , *Non-commutative extensions of the MacMahon Master Theorem*, math.CO/0607737 (2006).

[KR] M. Kontsevich and A. Rosenberg, *Noncommutative smooth spaces*, The Gelfand Mathematical Seminars, 1996–1999, Birkhäuser, Boston, 2000, pp. 85–108.

[Ko] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Advances in Math. **34** (1979), 195–338.

[KL] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), no. 1, 1–23.

[KLT] D. Krob, B. Leclerc, and J.-Y. Thibon, *Noncommutative symmetric functions.* II. Transformations of alphabets, Internat. J. Algebra Comput. **7** (1997), 181–264.

[KS] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method.* Recent developments, Lecture Notes Phys., vol. 151, Springer-Verlag, Belin–Heidelberg–New York, 1982, pp. 61-119.

[L] T. Y. Lam, *Representations of finite groups: A hundred years*, Part I, Notices Amer. Math. Soc., **45** (1998), no. 4, 361–372.

[Le] D. A. Leites, *A certain analog of the determinant* , Russian Math. Surv. **35** (1975), no. 3, 156 (Russian).

[LST] B. Leclerc, T. Scharf, and J.-Y. Thibon, *Noncommutative cyclic characters of symmetric groups, J. Combinatorial Theory* A **75** (1996), 55–69.

[L] Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Lecture Notes in Math. **1183** (1986), 291-338.,

[IM] I.Macdonald, *Symmetric functions and Hall polynomials*, Second edition , The Clarendon Press, Oxford University Press, New York, 1995

[M] P. A. MacMacon, *Combinatorial analysis*, Cambridge Univ. Press, Cambridge, 1915.

[Ma] Y. Manin, *Multiparameter quantum deformations of the linear supergroup*, Comm. Math. Phys. **113** (1989), 163–175.

[Mat] V. B. Matveev, *Darboux transformations, covariance theorems and integrable systems, L. D. Faddeev's Seminar on Mathematical Physics*, Amer. Math. Soc. Transl. Series, vol 201, Amer. Math. Soc., Providence, RI, 2000, pp. 179–209.

[MNO] A. Molev, M. Nazarov, and G. Olshansky, *Yangians and classical Lie algebras*, Russian Math. Surv. **51** (1996), no. 2, 205–282.

[Mol] A. Molev, *Noncommutative symmetric functions and Laplace operators for classical Lie algebras*, Lett. Math. Phys. **35** (1995),no. 2, 135–143.

[Mol1] ——— , *Gelfand-Tsetlin bases for representations of Yangians*, Lett. Math. Phys. **30** (1994), no. 2, 53–60.

[MolR ] A. Molev and V. Retakh, *Quasideterminants and Casimir elements for the general Lie superalgebras*, Intern. Math. Res. Notes (2004), no. 13, 611-619.

[MB] E. H. Moore and R. W. Barnard, *General analysis pt. I* , The American phylosophical society, Philadelphia, 1935.

[Mo] E. H. Moore, *On the determinant of an Hermitian matrix of quaternionic elements*, Bull. Amer. Math. Soc. **28** (1922), 161–162.

[O] O. Ore, *Liner equations in noncommutative ring*s , Ann. Math. **32** (1931), 463–477.

[Ok] K. Okamoto, *Bäcklund transformations of classical orthogonal polynomials*, Algebraic Analysis. II, (M. Kashiwara, T. Kawai eds), Academic Press,r Boston, MA, 1998, pp. 647–657.

[Os] B. L. Osofsky, *Quasideterminants and roots of polynomials over division rings*, Proc. Intern. Lisboa Conf. on Algebras, Modules and Rings, 2003, World Sci. Publ. 2006, pp. 241–263.

[P] A. Polischuk, *Triple Massey products on curves, Fay's trisecant identity and tangents to the canonical embedding*, math.AG/0109007 (2001).

[PPR] I. Pak, A. Postnikov and V. Retakh, *Noncommutative Lagrange inversion*, info preprint (1995).

[RS] A. Razumov and M. Saveliev, *Maximally nonabelian Toda systems*, Nuclear Phys. B, **494** (1997) no. 3, 657–686.

[RRV] V. Retakh, C. Reutenauer, and A. Vaintrob, *Noncommutative rational functions and Farber's invariants of boundary links, Differential Topology, Infinite-dimensional Lie Algebras and Applications*, Amer. Math. Soc. Transl. Series 2, vol 194, Amer. Math. Soc., Providence, RI, 1999, pp. 237–246.

[RSW] V. Retakh, S. Serconek, and R.L. Wilson , *On a class of Koszul algebras associated to directed graphs , J. of Algebra* **304** (2006), 1114–1129.

[RSW1] V. Retakh, S. Serconek, and R.L. Wilson, *Hilbert series of algebras associated to directed graphs*, math.CO/0601224. J. of Algebra (to appear).

[RSW2] V. Retakh, S. Serconek, and R.L. Wilson , *Construction of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials*, math.RA/0603327, Proceedings of the Conference "Lie Algebras, Vertex Operator Algebras and Their Applications".

[RSh] V. Retakh and V. Shander, Schwarz derivative for noncommutative differential algebras, Unconvential Lie Algebras , Adv. Soviet Math. vol. 17, Amer. Math. Soc., Providence, RI, 1993, pp. 130–154.

[Re] C. Reutenauer, *Inversion height in free fields* , Selecta Math. (N.S.) **2** (1996), no. 1, 93–109.

[Ri] A. R. Richardson, *Hypercomplex determinants*, Messenger of Math. **55** (1926), 145–152.

[Ri1] ——— , *Simultaneous linear equations over a division algebra*, Proc. London Math. Soc. **28** (1928) 395–420.

[SvB] J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. **38** (2001) no. 2, 171–216.

[SW] S. Serconek and R. L. Wilson, *Quadratic algebras associated with decompositions of noncommutative polynomials are Koszul algebras*, in preparation.

[Sch] M. Schork, *The bc-system of higher rank revisited* , J. Phys. A: Math. Gen. **35** (2001), 2627–2637.

[Schur] I. Schur, *Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen* , Math. Ann., **66** (1909), 488–510.

[St] E. Study, *Zur Theorie der lineare Gleichungen* , Acta Math. **42** (1920), 1–61.

[Syl] J. J. Sylvester, *Sur une classe nouvelle d'equations differéntielles et d'equations aux differences finies d'une forme intégrable* , Compt. Rend. Acad. Sci., **54** (1862), 129-132, 170.

[Thi] J.-Y. Thibon, *Lectures on moncommutative symmetric functions*, MSJ Mem. **11** (2001), 39–94.

[VP] P. Van Praag, *Sur le norme reduite du determinant de Dieudonne des matrices quaterniennes*, J. Algebra **136** (1991), 265–274.

[W] J. H. M. Wedderburn, *On continued fractions in noncommutative quantities*, Ann. Math. **15** (1913), 101–105.

[We] H. Weyl, *Classical groups, their invariants and representations* , Princeton Univ. Press, Princeton, NJ, 1946.

[Wi] R. L. Wilson, *Invariant polynomials in the free skew field*, Selecta Math. **7** (2001), 565–586.

# Factorization of Noncommutative Polynomials and Noncommutative Symmetric Functions

# Contents

# Chapter 1

# Basic results on the commutative case

Consider the (commutative) polynomial algebra $F[x_1, ..., x_n]$ on $n$ indeterminates. The symmetric group $\mathfrak{S}_n$ acts on $F[x_1, ..., x_n]$ with $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in \mathfrak{S}_n$ and $1 \leq i \leq n$. The elements of

$$F[x_1, ..., x_n]^{\mathfrak{S}_n} = \{f \in F[x_1, ..., x_n] | f = \sigma(f)\ \forall \sigma \in \mathfrak{S}_n\}$$

are called symmetric polynomials.

As usual, we extend the action of $\mathfrak{S}_n$ on $F[x_1, ..., x_n]$ to an action on $F[x_1, ..., x_n][t]$ by setting $\sigma(t) = t$ for all $\sigma \in \mathfrak{S}_n$.

Let

$$P(t) = (t - x_n)...(t - x_1) = \sum_{i=0}^{n} (-1)^{n-i} \Lambda_{n-i}(x_1, ..., x_n) t^i.$$

Thus, writing $\Lambda_i$ for $\Lambda_i(x_1, ..., x_n)$ we have

$$\Lambda_0 = 1,$$

$$\Lambda_1 = x_1 + x_2 + ... + x_n,$$

$$\Lambda_i = \sum_{n \geq j_1 > j_2 > ... > ge j_i \geq 1} x_{j_1} x_{j_2} ... x_{j_i},$$

$$\Lambda_n = x_n x_{n-1} ... x_1.$$

Since any $\sigma \in \mathfrak{S}_n$ permutes the factors in the expression for $P(t)$, it is clear that each $\Lambda_i$ is a symmetric polynomial. (Of course, this is also clear from the above expressions for the $\Lambda_i$.)

The Fundamental Theorem of Symmetric Functions states that any symmetric polynomial in $F[x_1, ..., x_n]$ may be expressed as a polynomial in

$$\{\Lambda_1(x_1, ..., x_n), ..., \Lambda_n(x_1, ..., x_n)\}$$

and that $\Lambda_1, ..., \Lambda_n$ are algebraically independent. Thus

$$F[x_1, ..., x_n]^{\mathfrak{S}_n} \cong F[\Lambda_1, ..., \Lambda_n].$$

There are other natural symmetric polynomials in $F[x_1, ..., x_n]$. In particular, the *complete s ymmetric functions*

$$S_i = \sum_{n \geq j_1 \geq j_2 \geq ... \geq j_i \geq 1} x_{j_1} x_{j_2} ... x_{j_i}, 0 \leq i,$$

and the *power sum symmetric functions*

$$\Psi_i = x_1^i + x_2^i + ... + x_n^i, 1 \leq i$$

are in $F[x_1, ..., x_n]^{\mathfrak{S}_n}$. The Fundamental Theorem then implies that these functions can be expressed as polynomials in the elementary symmetric functions. In fact, the relations among these functions may be elegantly described in terms of generating functions. Thus, letting

$$\lambda(t) = \sum_{i=0}^{n} \Lambda_i t^i,$$

$$\sigma(t) = \sum_{i \geq 0} S_i t^i,$$

and

$$\psi(t) = \sum_{i \geq 1} \Psi_i t^{i-1}$$

we have

$$\lambda(t)\sigma(t) = 1,$$

and

$$\sigma(t)\psi(t) = \frac{d}{dt}\sigma(t).$$

Note that the last equation may be rewritten as

$$\psi(t)\sigma(t) = \frac{d}{dt}\sigma(t)$$

or as

$$\sum_{i \geq 1} \frac{\Psi_i}{i} t^i = \log(\sigma(t)).$$

# Chapter 2

# Generalizations to the noncommutiative case

## 2.1 $F < x_1, ..., x_n >^{\mathfrak{S}_n}$ - the Fundamental Theorem does not generalize

The most natural generalization of symmetric functions to the noncommutative case is to replace the polynomial algebra $F[x_1, ..., x_n]$ by the free algebra $F < x_1, ..., x_n >$. The symmetric group $\mathfrak{S}_n$ acts on $F < x_1, ..., x_n >$ (again by setting $\sigma(x_i) = x_{\sigma(i)}$). Then $F < x_1, ..., x_n >^{\mathfrak{S}_n}$ contains polynomials analogous to the elementary symmetric functions, namely,

$$\tilde{\lambda}_i(x_1, ..., x_n) = \Sigma x_{j_1} ... x_{j_i}$$

where the sum is over all sequences $(j_1, ..., j_i)$ of $i$ distinct elements of $\{1, ..., n\}$.

It was shown by M. Wolf (1936) that the $\tilde{\Lambda}_i$ do not generate $F{<}x_1, ..., x_n{>}^{\mathfrak{S}_n}$ and by Bergman and Cohn (1969) that $F < x_1, ..., x_n >^{\mathfrak{S}_n}$ is not finitely generated. We will not discuss this algebra further here, but refer the reader to the work of Rosas and Sagan (http://www.newton.cam.ac.uk/preprints/NI03033.pdf).

## 2.2 Families of formal noncommutative symmetric functions

A second way to generalize results about symmetric functions of commuting variables to the noncommutative case (begun by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon in [3]) is to consider the elementary symmetric functions as the primitive objects and to define other families of symmtric functions in terms of the elementary symmetric functions by generating function identities analogous to those used in the commutative case.

Thus one works in the free algebra, $\mathbf{Sym} = F < \Lambda_1, \Lambda_2, ... >$ and defines $\lambda(t) \in \mathbf{Sym}[[t]]$

$$\lambda(t) = 1 + \sum_{i \geq 1} \Lambda_i t^i.$$

Then we may define the *(noncommutative) complete symmetric functions*

$$S_1, S_2, ...$$

by setting

$$\sigma(t) = 1 + \sum_{i \geq 1} S_i t^i = \lambda(-t)^{-1}.$$

In the absence of commutativity, the three generating function identities used to define the power sum symmetric functions in the commutative case are no longer equivalent. We are thus led to define three families of noncommutative power sum symmetric functions. These are:

*(noncommutative) power sum symmetric functions of the first kind*

$$\{\Psi_i | i \geq 1\},$$

defined by

$$\psi(t) = \sum_{i \geq 1} \Psi_i t^{i-1}$$

and

$$\psi(t)\sigma(t) = \frac{d}{dt}\sigma(t);$$

*(noncommutative) power sum symmetric functions of the second kind*

$$\{\Phi_i | i \geq 1\},$$

defined by

$$\sum_{i \geq 1} \frac{\Phi_i}{i} = \log(\sigma(t));$$

and

*(noncommutative) power sum symmetric functions of the third kind*

$$\{\Xi_i | i \geq 1\},$$

defined by

$$\xi(t) = \sum_{i \geq 1} \Xi_i t^{i-1}$$

and

$$\sigma(t)\xi(t) = \frac{d}{dt}\sigma(t).$$

Relations among these families of noncommutative symmetric functions may be given in terms of quasideterminants.

## 2.3 The Viète theorem and an analogue of the Fundamental Theorem

One of the earliest theorems in algebra, due to Francois Viète (who also discovered the law of cosines), expresses the coefficients of a polynomial equation in terms of its roots: If the polynomial equation $f(x) = 0$, where $f(x)$ is monic of degree $n$ over a (commutative) field, has $n$ roots $x_1, ..., x_n$, then $f(x) = (x - x_1)...(x - x_n)$.

Gelfand and Retakh ([**?**, **?**] have used the theory of quasideterminants to give a generalization to equations over noncommutative rings. Their result, stated precisely in Chapter 4, shows that if

$$P(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$$

is a polynomial over a noncommutative ring $R$ and if $\{x_1, ..., x_n\}$ is a set of roots of the equation $P(x) = 0$ that is $i$ndependent (in the sense that certain quasideterminants of certain Vandermonde matrices are nonzero), then there exist rational expressions $y_1, ..., y_n$ in $x_1, ..., x_n$ such that

$$a_i = (-1)^i \sum_{n \geq j_1 > ... > j_i \geq 1} y_{j_1} ... y_{j_i}.$$

Since the $a_i$ are independent of the ordering of the roots $x_1, ..., x_n$ we see that they are symmetric functions of $x_1, ..., x_n$ (with respect to the action of $\mathfrak{S}_n$ defined by $\sigma(x_i) = s_{\sigma(i)}$. They are also, of course, polynomials in $y_1, ..., y_n$. The following analogue of the Fundamental Theorem for (commutative) symmetric polynomials was conjectured by Gelfand and Retakh [**?**] and proved by Wilson [6]: Any polynomial in $y_1, ..., y_n$ invariant under the action of $\mathfrak{S}_n$ is a polynomial in $a_1, ..., a_n$.

# Chapter 3

# Relations among families of noncommutative symmetric functions

## 3.1 Quasideterminental formulas

Proposition 3.1: For every $k \geq 1$, one has

$$S_k = (-1)^{k-1} \begin{vmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_{k-1} & \boxed{\Lambda_k} \\ 1 & \Lambda_1 & \dots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \dots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Lambda_k = (-1)^{k-1} \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 \\ S_2 & S_1 & 1 & \dots & 0 \\ S_3 & S_2 & S_1 & \dots & 0 \\ & & \dots & & \\ \boxed{S_k} & S_{k-1} & S_{k-2} & \dots & S_1 \end{vmatrix},$$

$$kS_k = \begin{vmatrix} \Psi_1 & \Psi_2 & \dots & \Psi_{k-1} & \boxed{\Psi_k} \\ -1 & \Psi_1 & \dots & \Psi_{k-2} & \Psi_{k-1} \\ 0 & -2 & \dots & \Psi_{k-3} & \Psi_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & -n+1 & \Psi_1 \end{vmatrix},$$

$$k\Lambda_k = \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ \Psi_2 & \Psi_1 & 2 & \dots & 0 \\ & & \dots & & \\ \boxed{\Psi_k} & \Psi_{k-1} & \Psi_{k-2} & \dots & \Psi_1 \end{vmatrix},$$

$$\Psi_k = (-1)^{k-1} \begin{vmatrix} \Lambda_1 & 2\Lambda_2 & \dots & (k-1)\Lambda_{k-1} & \boxed{k\Lambda_k} \\ 1 & \Lambda_1 & \dots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \dots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Psi_k = \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 \\ 2S_2 & S_1 & 1 & \dots & 0 \\ & & \dots & & \\ \boxed{kS_k} & S_{k-1} & S_{k-2} & \dots & S_1 \end{vmatrix}.$$

Each of the four sequences $(\Lambda_k)$, $(S_k)$, $(\Psi_k)$, and $(\Phi_k)$ is a set of generators in **Sym**. Therefore, each of the four sets of products $F_{i_1} \dots F_{i_N}$, $i_1, \dots, i_N \geq 1$, where $F_{i_k}$ equals to $\Lambda_{i_k}$, $S_{i_k}$, $\Psi_{i_k}$, or $\Phi i_k$, is a linear basis in $\mathbf{Sym}_+$. Linear relations between these bases were given in [3].

## 3.2 Ribbon Schur functions

Another important example of a linear basis in $\mathbf{Sym}_+$ is given by *ribbon Schur functions*. Commutative ribbon Schur functions were defined by MacMahon [5]. Here we follow his ideas.

Let $I = (i_1, \dots, i_k)$, $i_1, \dots, i_k \geq 1$, be an ordered set.

Definition 3.2 [3]:

The ribbon Schur function $R_I$ is defined by the formula

$$R_I = (-1)^{k-1} \begin{vmatrix} S_{i_1} & S_{i_1+i_2} & S_{i_1+i_2+i_3} & \dots & \boxed{S_{i_1+\dots+i_k}} \\ 1 & S_{i_2} & S_{i_2+i_3} & \dots & S_{i_2+\dots+i_k} \\ 0 & 1 & S_{i_3} & \dots & S_{i_3+\dots+i_k} \\ & & & \dots & \\ 0 & 0 & 0 & \dots & S_{i_k} \end{vmatrix}.$$

This definition allows us to express $R_I$'s as polynomials in $S_k$'s. To do this we need the following ordering of sets of integers.

Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$. We say that $I \leq J$ if $i_1 = j_1 + j_2 + \dots + j_{t_1}$, $i_2 = j_{t_1+1} + \dots + j_{t_2}$, $\dots$, $i_s = j_{t_{s-1}+1} + \dots + j_s$. For example, if $I \leq (12)$, then $I = (12)$ or $I = (3)$. If $I \leq (321)$, then $I$ is equal to one of the sets $(321)$, $(51)$, $(33)$, or $(6)$.

For $I = (i_1, \dots, i_r)$ set $l(I) = r$ and $S^I = S_{i_1} S_{i_2} \dots S_{i_r}$.

Proposition 3.3 [3] (p. 254):

$$R_J = \sum_{I \leq J} (-1)^{l(J)-l(I)} S^I.$$

Example $R_{123} = S_6 - S_3^2 - S_1S_5 + S_1S_2S_3$.

Definition 3.2 implies that $R_I = S_m$ for $I = \{m\}$ and $R_I = \Lambda_k$ for $i_1 = \cdots = i_k = 1$.

In [3] similar formulas expressing $R_I$ as quasideterminants of matrices with entries $\Lambda_k$, as well as linear relations with different bases in $\mathbf{Sym}_+$ are given.

Natural bases in algebra $Sym$ of commutative symmetric functions are indexed by weakly decreasing (or, weakly increasing) finite sequences of integers. Examples are products of elementary symmetric functions $e_{i_1} \dots e_{i_k}$ where $i_1 \geq i_2 \cdots \geq i_k$ and Schur functions $s_\lambda$ where $\lambda = (i_1, \dots, i_k)$. The following theorem gives a natural basis in the algebra of noncommutative symmetric functions. Elements of this basis are indexed by all finite sequences of integers.

Theorem 3.4 [3] The ribbon Schur functions $R_I$ form a linear basis in **Sym**.

Let $\pi : \mathbf{Sym} \to Sym$ be the canonical morphism. Then it is known (see [5]) that the commutative ribbon Schur functions $\pi(R_I)$ are not linearly independent. For example, commutative ribbon Schur functions defined by sets $(ij)$ and $(ji)$ coincide. This means that the kernel $Ker\ \pi$ is nontrivial.

In the commutative case, ribbon Schur functions $\pi(R_I)$ with with weakly decreasing $I$ constitute a basis in the space of symmetric functions. However, this basis is not frequently used.

The description of the kernel $Ker\pi$ in terms of ribbon Schur functions is given by the following theorem.

For an ordered set $I$ is denote by $u(I)$ the corresponding unordered set.

Theorem 3.5 The kernel of $\pi$ is linearly generated by the elements

$$\Delta_{J,J'} = \sum_{I \leq J} R_I - \sum_{I' \leq J'} R_{I'}$$

for all $J$, $J'$ such that $u(J) = u(J')$.

Example 1. Let $J = (12)$, $J' = (21)$. Then $\Delta_{J,J'} = (R_{12}+R_3)-(R_{21}+R_3) = R_{12} - R_{21}$ and $\pi(R_{12}) = \pi(R_{21})$.

2. Let $J = (123)$, $J' = (213)$. Then $\Delta_{J,J'} = (R_{123} + R_{33} + R_{15} + R_6) - (R_{213} + R_{33} + R_{24} + R_6) = R_{123} + R_{15} - R_{213} - R_{24}$. This shows, in particular, that $\pi(R_{123}) - \pi(R_{213}) = \pi(R_{24}) - \pi(R_{15}) \neq 0$.

The homological relations for quasideterminants imply the multiplication rule for the ribbon Schur functions. Let $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$, $i_p \geq 1$, $j_q \geq 1$ for all $p, q$. Set $I + J = (i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s)$ and $I \cdot J = (i_1, \dots, i_r, j_1, \dots, j_s)$.

These operations may be described geometrically in a fashion that explains the origin of the name "ribbon Schur functions". To each ordered set $I = \{i_1, i_2, \dots, i_k\}$ we can associate a ribbon (i.e., a sequence of square cells with horizontal and vertical sices of length one and centers on $\mathbf{Z} \times \mathbf{Z}$ starting at the square $(0,0)$ and with each successive square being either directly below or directly to the right of the preious square) with $i_1$ squares in the first column, $i_2$ squares in the second column, and so on. Then the construction of ribbons $I+J$ and $I \cdot J$ has a simple geometric meaning: $I+J$ is obtained by placing the first square of $J$ below the last square of $I$; $I.J$ is obtained by placing the first square of $J$ to the right of the last square of $I$.

Theorem 3.6 [3] We have

$$R_I R_J = R_{I+J} + R_{I \cdot J}.$$

The commutative version of this multiplication rule is due to MacMahon.

6.3. Algebras with two multiplications

The relations between the functions $\Lambda_k$ and the functions $S_k$ can be illuminated by noting that the ideal $\mathbf{Sym}_+$ has two natural associative multiplications $*_1$ and $*_2$. In terms of ribbon Schur functions it can be given as $R_I *_1 R_J = R_{I.J}$ and $R_I *_2 R_J = R_{I+J}$. We formalize this notion as follows.

Definition 6.3.1 A linear space $A$ with two bilinear products $\circ_1$ and $\circ_2$ is called a biassociative algebra with products $\circ_1$ and $\circ_2$ if

$$(a \ \circ_i \ b) \ \circ_j \ c = a \ \circ_i \ (b \ \circ_j \ c)$$

for all $a, b, c \in A$ and all $i, j \in \{1, 2\}$.

Note that if the products $\circ_1$ and $\circ_2$ in a biassociative algebra $A$ have a common identity element 1 (i.e., if $1 \ \circ_i \ a = a \ \circ_i 1 = a$ for all $a \in A$ and $i = 1, 2$, then

$$a \ \circ_1 \ b = (a \ \circ_2 1) \ \circ_1 b = a \ \circ_2 (1 \ \circ_1 b) = a \ \circ_2 b$$

for all $a, b \in A$ and so $\circ_1 = \circ_2$.

Note also that if $A$ is a biassociative algebra with two products $\circ_1$ and $\circ_2$, then for $r, s \in F$ one can define the linear combination $\circ_{r,s} = r \circ_1 + s \circ_2$ by the formula

$$a \ \circ_{r,s} \ b = r(a \ \circ_1 \ b) + s(a \ \circ_2 \ b), \qquad a, b \in A.$$

Then $A$ is a biassociative algebra with the products $\circ_{r,s}$ and $\circ_{t,u}$ for each $r, s, t, u \in F$.

Jacobson's discussion of isotopy and homotopy of Jordan algebras (see [4], p. 56,ff) shows that if $A$ is an associative algebra with the product $\circ$ and $\circ_a$ for $a \in A$ is defined by the formula

$$b \ \circ_a c = b \ \circ a \ \circ c,$$

then $A$ is a biassociative algebra with the products $\circ$ and $\circ_a$.

We now endow the ideal $\mathbf{Sym}_+ \subset \mathbf{Sym}$ with the structure of a biassociative algebra in two different ways. Recall that the nontrivial monomials $(\Lambda_{i_1} \ldots \Lambda_{i_r})$ as well as the nontrivial monomials $(S_{i_1} \ldots S_{i_r})$ form linear bases in $\mathbf{Sym}+$.

Definition 3.8 Define the linear map $*_1 : \mathbf{Sym}_+ \otimes \mathbf{Sym}_+ \to \mathbf{Sym}_+$ by

$$(\Lambda_{i_1} \dots \Lambda_{i_r}) *_1 (\Lambda_{j_1} \dots \Lambda_{j_s}) = \Lambda_{i_1} \dots \Lambda_{i_{r-1}} \Lambda_{i_r+j_1} \Lambda_{j_2} \dots \Lambda_{j_s}$$

and the linear map $*_2 : \mathbf{Sym}_+ \otimes \mathbf{Sym}_+ \to \mathbf{Sym}_+$ by

$$(S_{i_1} \dots S_{i_r}) *_2 (S_{j_1} \dots S_{j_s}) = S_{i_1} \dots S_{i_{r-1}} S_{i_r+j_1} S_{j_2} \dots S_{j_s}.$$

Write $ab = a *_0 b$ for $a, b \in \mathbf{Sym}_+$. Then it is clear that $a *_i (b *_j c) = (a *_i b) *_j c$ for all $a, b, c \in \mathbf{Sym}_+$ and $i, j = 0, 1$ or $i, j = 0, 2$. Thus we have the following result.

Lemma 3.9 $\mathbf{Sym}_+$ is a biassociative algebra with products $*_0$ and $*_1$ and also a biassociative algebra with products $*_0$ and $*_2$.

In fact, $*_0$, $*_1$ and $*_2$ are closely related.

The following Lemma is just a restatement of Theorem 6.2.5.

Lemma 3.10 $*_0 = *_1 + *_2$.

Proof We have

$$\lambda(-t)^{-1} = \Big(1 + \sum_{i>0} (-1)^i \Lambda_i t^i\Big)^{-1} = 1 + \sum_{j>0} \sum_{i_1+\dots+i_l=j} (-1)^{l+j} \Lambda_{i_1} \dots \Lambda_{i_l} t^j.$$

Since $\Lambda_i = \Lambda_1 *_1 \Lambda_1 *_1 \dots *_1 \Lambda_1$, where there are $i-1$ occurences of $*_1$, the coefficient at $t^j$ in $\lambda(-t)^{-1}$ is

$$\sum_{u_1,\dots,u_{j-1} \in \{0,1\}} (-1)^k \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \dots *_{u_{j-1}} \Lambda_1,$$

where $k$ is the number of $u_t$ equal to 1.

Since $1 + \sum S_i t^i = \lambda(-t)^{-1}$ we have

$$S_j = \sum_{u_1,\dots u_{j-1} \in \{0,1\}} (-1)^k \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \dots *_{u_{j-1}} \Lambda_1.$$

Therefore $S_i *_0 S_j - S_i *_1 S_j = S_i(*_0 - *_1)S_j = S_{i+j} = S_i *_2 S_j$ and $*_2 = *_0 - *_1$, as required.

Now let $U$ be the two-dimensional vector space with basis $\{u_0, u_1\}$ and

$$F\langle U \rangle = \sum_{k \geq 0} F\langle U \rangle_k$$

the (graded) free associative algebra on $U$, with the homogeneous components

$$F\langle U \rangle_k = U^{\otimes k}.$$

We use the products $*_1$ and $*_2$ to define two isomorphisms, $\phi_1$ and $\phi_2$, of $F\langle U \rangle$ to $\mathbf{Sym}_+$. Namely, for a basis element $u_{i_1} \dots u_{i_l} \in F\langle U \rangle_k$ set

$$\phi_1 : u_{i_1} \dots u_{i_l} \mapsto \Lambda_1 *_{i_1} \Lambda_1 *_{i_2} \dots *_{i_l} \Lambda_1 \in (\mathbf{Sym}_+)_{l+1}$$

and

$$\phi_2 : u_{i_1} \dots u_{i_l} \mapsto \Lambda_1 *_{j_1} \Lambda_1 *_{j_2} \cdots *_{j_l} \Lambda_1 \in (\mathbf{Sym}_+)_{l+1}$$

where $j_t = 0$ if $i_t = 0$ and $j_t = 2$ if $i_t = 1$. Note that $\phi_1$ and $\phi_2$ shift degree.

Define the involution $\theta$ of $U$ by $\theta(u_0) = u_0$ and $\theta(u_1) = u_0 - u_1$. Then $\theta$ extends to an automorphism $\Theta$ of $F\langle U\rangle$ and the restriction $\Theta_k$ of this automorphism to $F\langle U\rangle_k$ is the $k$-th tensor power of $\theta$. Clearly $\phi_1\Theta = \phi_2$ and so we recover Proposition 4.13 in [3], which describes, in terms of tensor powers, the relation between the bases of $\mathbf{Sym}_+$ consisting of nontrivial monomials in $\Lambda_i$ and of nontrivial monomials in $S_i$.

Similarly, taking the identity

$$a^{n-1} + (-1)^n b^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} a^{k-1}(a+b)b^{n-k-1} = 0,$$

valid in any associative algebra, setting $a = u_0 - u_1$, $b = u_1$, and applying $\phi_1$, we obtain the identity

$$0 = \sum_{k=0}^{n} (-1)^{n-k} \Lambda_k S_{n-k}$$

between the elementary and complete symmetric functions (Proposition 3.3 in [3]). Using Proposition 3.1, one can express these identities in terms of quasideterminants.

# Chapter 4

# The noncommutative Viète Theorem

Let

$$P(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

be a polynomial over a noncommutative ring $R$ and let $\{x_1, ..., x_n\}$ be a set of roots. We will determine the $a_i$ in terms of $x_1, ..., x_n$ in case the set of roots $\{x_1, ..., x_n\}$ satisfies certain conditions.

For quadratic equations this is not hard. If $x_1$ and $x_2$ are roots of $x^2 + a_1x + a_2 = 0$, then we have

$$x_1^2 + a_1x_1 + a_2 = 0$$

and

$$x_2^2 + a_1x_2 + a_2 = 0.$$

Taking the difference gives

$$x_1^2 - x_2^2 + a_1(x_1 - x_2) = 0.$$

This may be rewritten as

$$x_1(x_1 - x_2) + (x_1 - x_2)x_2 + a_1(x_1 - x_2) = 0$$

which gives

$$a_1 + x_1 = -(x_1 - x_2)x_2(x_1 - x_2)^{-1}$$

so that

$$a_1 = -x_1 - (x_1 - x_2)x_2(x_1 - x_2)^{-1}.$$

It is then easy to see that

$$a_2 = (x_1 - x_2)x_2(x_1 - x_2)^{-1}x_1.$$

In general, let $x_1, x_2, \ldots, x_n$ be a set of elements of a ring $R$. For $1 \leq k \leq n$ set

$$\mathfrak{V}(x_1, ..., x_k) = \begin{bmatrix} x_1^{k-1} & \ldots & x_k^{k-1} \\ & \ldots & \\ x_1 & \ldots & x_k \\ 1 & \ldots & 1 \end{bmatrix}$$

. Then the quasideterminant

$$V(x_1, \ldots, x_k) = \begin{bmatrix} x_1^{k-1} & \ldots & x_k^{k-1} \\ & \ldots & \\ x_1 & \ldots & x_k \\ 1 & \ldots & 1 \end{bmatrix}_{1k}$$

is called the $V$andermonde quasideterminant

We say that a sequence of elements $x_1, \ldots, x_n \in R$ is $i$ndependent if all quasideterminants

$$V(x_1, \ldots, x_k), \; k = 2, \ldots, n,$$

are defined and invertible.

We need the following easy result.

Lemma 4.1: If the polynomial equation $a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$ has a set of $n$ independent roots, then $a_1 = ... = a_n = 0$.

Proof: Consider the system of linear equations in the variables $a_1, ..., a_n$:

$$a_1 x_i^{n-1} + ... + a_{n-1}x_i + a_n = 0, 1 \leq i \leq n.$$

This may be written in the form

$$\begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1^{n-1} & \ldots & x_n^{n-1} \\ & \ldots & \\ x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{bmatrix}.$$

Since $\{x_1, ..., x_n\}$ is independent, the quasideterminant

$$V(x_1, \ldots, x_k) = \begin{bmatrix} x_1^{n-1} & \ldots & x_n^{n-1} \\ & \ldots & \\ x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{bmatrix}_{1n}$$

exists and so the matrix

$$\begin{bmatrix} x_1^{n-1} & \ldots & x_n^{n-1} \\ & \ldots & \\ x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{bmatrix}$$

is invertible, giving the result.

For independent sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}, z$ set

$$y_1 = x_1, \quad z_1 = z$$

$$y_k = V(x_1, \dots, x_k) x_k V(x_1, \dots, x_k)^{-1}, \quad 1 < k \le n$$

$$z_k = V(x_1, \dots, x_{k-1}, z) z V(x_1, \dots, x_{k-1}, z)^{-1}, \quad 1 < k \le n.$$

In the commutative case $y_k = x_k$ and $z_k = z$ for $k = 1, \dots, n$.

Define

$$\Lambda_k(x_1, ..., x_n) =$$

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} y_{i_k} y_{i_{k-1}} \dots y_1.$$

Theorem 4.2: (Noncommutative Viète Theorem) Let $\{x_1, ..., x_n\}$ be an independent sequence of roots of the polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

over $R$. Then

$$a_i = (-1)^i \Lambda_i(x_1, ..., x_n)$$

for $1 \le i \le n$.

To prove this, we need the following preliminary result which is of independent interest.

Theorem 4.3 (Bezout decomposition of a Vandermonde quasideterminant). Suppose that sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}, z$ are independent. Then

$$V(x_1, \dots, x_n, z) = (z_n - y_n)(z_{n-1} - y_{n-1}) \cdots (z_1 - y_1).$$

Proof (A. Lauve): By induction (base case being trivial) we need only check that

$$V(x_1, \dots, x_n, z) = (z_n - y_n) V(x_1, \dots, x_n, z)$$

$$= \; V(x_1, \dots, x_{n-1}, z) z - V(x_1, \dots, x_{n-1}) x_n V(x_1, \dots, x_n)^{-1} V(x_1, \dots, x_n, z).$$

Use Sylvester's Identity on the left-hand side.

$$
\begin{aligned}
lhs &= \begin{vmatrix} x_1^n & \cdots & x_{n-1}^n & x_n^n & \boxed{z^n} \\ x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & z^{n-1} \\ \vdots & & \vdots & \vdots & \vdots \\ x_1 & \cdots & x_{n-1} & x_n & z \\ 1 & \cdots & 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} x_1^n \cdots x_{n-1}^n & \boxed{z^n} \\ & z^{n-1} \\ X_0 & \vdots \\ & z \end{vmatrix} - \begin{vmatrix} x_1^n \cdots x_{n-1}^n & \boxed{x_n^n} \\ & x_n^{n-1} \\ X_0 & \vdots \\ & x_n \end{vmatrix} \times \\
&\qquad \begin{vmatrix} & x_n^{n-1} \\ X_0 & \vdots \\ & x_n \\ 1 \cdots 1 & \boxed{1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} & z^{n-1} \\ X_0 & \vdots \\ & z \\ 1 \cdots 1 & \boxed{1} \end{vmatrix} \\
&= (\star).
\end{aligned}
$$

Now, by column transformation properties and homological relations, we have

$$
\begin{aligned}
(\star) &= V(x_1, \ldots, x_{n-1}, z) \cdot z - V(x_1, \ldots, x_{n-1}, x_n) \cdot x_n \times \\
&\qquad \begin{vmatrix} & \boxed{x_n^{n-1}} \\ X_0 & \vdots \\ & x_n \\ 1 \cdots 1 & 1 \end{vmatrix}^{-1} \cdot \begin{vmatrix} & \boxed{z^{n-1}} \\ X_0 & \vdots \\ & z \\ 1 \cdots 1 & 1 \end{vmatrix} \\
&= V(x_1,\ldots,x_{n-1},z)z - V(x_1,\ldots,x_{n-1},x_n)x_n \cdot V(x_1,\ldots,x_n)^{-1} V(x_1,\ldots,x_n,z)
\end{aligned}
$$

as needed.

Theorem 4.4 (Viète decomposition of the Vandermonde quasideterminant) Suppose that the sequences $x_1, ..., x_n$ and $x_1, ..., x_{n-1}, z$ are independent. Then

$$V(x_1, \ldots, x_n, z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

where

$$a_k = (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} y_{i_k} y_{i_{k-1}} \ldots y_1.$$

In particular

$$a_1 = -(y_1 + \cdots + y_n),$$

$$a_2 = \sum_{1 \le i < j \le n} y_j y_i,$$
$$\dots$$
$$a_n = (-1)^n y_n \dots y_1.$$

Proof: By induction on $n$ we show that Theorem 5.2.2 follows from Theorem 5.2.1. For $n = 1$ one has $V(x_1, z) = z - x_1$ and so the result holds. Suppose that the result holds for $m = n - 1$. By Theorem 4.3

$$V(x_1, \dots, x_n, z) = (z_n - y_n)V(x_1, \dots, x_{n-1}, z)$$
$$= (V(x_1, \dots, x_{n-1}, z) \cdot z) - (y_n \cdot V(x_1, \dots, x_{n-1}, z)).$$

By induction,

$$V(x_1, \dots, x_{n-1}, z) = z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1},$$

where

$$b_1 = -(y_1 + \dots + y_{n-1}),$$
$$\dots$$
$$b_{n-1} = (-1)^n y_{n-1} \cdot \dots \cdot y_1.$$

Therefore,

$$V(x_1, \dots, x_n, z) = z^n + (b_1 - y_n)z^{n-1} + (b_2 - y_n b_1)z^{n-2} + \dots - y_n b_n$$
$$= z^n + a_1 z^{n-1} + \dots + a_n,$$

where $a_1, \dots, a_n$ are given as desired.

Proof of Theorem 4.1: Since a quasideterminant vanishes if two columns are equal, we have $V(x_1, ..., x_n, x_i) = 0$ for all $i$. Thus, if $\{x_1, ..., x_n\}$ is an independent set of roots of a monic polynomial $P(x)$ of degree $n$, it is also an independent set of roots of $P(x) - V(x_1, ..., x_n, z)$. Then by the lemma we have $P(x) = V(x_1, ..., x_n, z)$ and the theorem follows from the Viète decomposition of the Vandermonde quasideterminant.

# Chapter 5

# An analogue of the Fundamental Theorem

## 5.1 Statement of the theorem

Let $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ denote the free skew field on generators $x_1, ..., x_n$. Note that the symmetric group $\mathfrak{S}_n$ acts on $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ by $\sigma(x_i) = x_{\Sigma(i)}$.

As in Chapter 4, define $y_1 = 1$ and $y_k = V(x_1, ..., x_k)x_k V(x_1, ..., x_k)^{-1}$ for $2 \leq k \leq n$. Set

$$\Lambda_i = \sum_{n \geq j_1 > ... > j_i \geq 1} y_{j_1} ... y_{j_i}$$

for $1 \leq i \leq n$. By the Viète Theorem, each $\Lambda_i \in F\langle\!\langle x_1, ..., x_n \rangle\!\rangle^{\mathfrak{S}_n}$.

Theorem 5.1

Let $f \in F[y_1, ..., y_n]$ and suppose $\sigma f = f$ for all $\sigma \in \mathfrak{S}_n$. Then $f \in F[\Lambda_1, ..., \Lambda_n]$.

## 5.2 Algebraic independence results

As usual, we say a set of elements $\{v_1, ..., v_k\} \subseteq F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ is *a*lgebraically independent if $g \in F < u_1, ..., u_k >$ and $g(v_1, ..., v_k) = 0$ imply $g = 0$. Our proof of the fundamental theorem will depend on showing that certain subsets of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ are algebraically independent.

Fix $i, 1 \leq i \leq n-1$ and let $\sigma \in \mathfrak{S}_n$ denote the transposition which interchanges $i$ and $i+1$. Set

$$z_i = y_i - \frac{\sigma(y_i)}{2}$$

and

$$z_i = (y_i + \sigma(y_i))s_i^{-1}.$$

Proposition 5.2: (a) $\{y_1, ..., y_n\}$ is algebraically independent in $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$.

(b) $\{y_1, ..., y_{i-1}, z_i, s_i, y_{i+2}, ..., y_n\}$ is algebraically independent in $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$.

In view of the universal property of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ it is sufficient to find a skew field K and an injection $\mu : F < x_1, ..., x_n > \to K$ such that $V(x_1, ..., x_k)$ is defined and invertible in $K$ for all $k, 1 \le k \le n$, that $s_i$ is invertible in $K$ and that the asserted algebraic independence results hold in $K$.

Our construction of such a $K$ is based on results of Jategaonkar and of Fisher. Consider the commutative field $L = F(t_{i,j}|1 \le i \le n, j \ge 1)$ generated by the algebraically independent elements $\{t_{i,j}|1 \le i \le n, j \ge 1\}$. Embed this in the skew polynomial algebra $S = F(t_{i,j}|1 \le i \le n, j \ge 1)[D]$ where $Dt_{i,j} = t_{i,j+1}D$. Then $S$ is embedded in the skew field

$$K = \{\sum_{n\ge 0} D^{-n} \sum_{i=0}^{\infty} f_{n,i}D^i | f_{n,i} \in L\}.$$

The map $\mu : x_i \mapsto t_{i,1}D, 1 \le i \le n$ extends to a monomorphism of $F < x_1, ..., x_n >$ into $K$.

One can show that

$$\mu(V(x_1, ..., x_k) =$$

$$(det \begin{bmatrix} t_{1,1}t_{1,2}...t_{1,k-1} & ... & t_{k,1}t_{k,2}...t_{k,k-1} \\ t_{1,2}...t_{1,k-1} & ... & t_{k,2}...t_{k,k-1} \\ . & ... & . \\ t_{1,k-1}7... & t_{k,k-1} & \\ 1 & ... & 1 \end{bmatrix})\times$$

$$(det \begin{bmatrix} t_{1,2}t_{1,3}...t_{1,k-1} & ... & t_{k-1,2}t_{k-1,3}...t_{k-1,k-1} \\ t1,3...t1,k-1 & ... & t_{k,3}...t_{k-1,k-1} \\ . & ... & . \\ t_{1,k-1}7... & t_{k-1,k-1} & \\ 1 & ... & 1 \end{bmatrix})^{-1}D.$$

One uses this to prove Proposition 5.2.

## 5.3 Completion of the proof

Let $A$ be an associative algebra over $F$ and $\sigma \in Aut\ A$. For any subset $X \subset A$, let $F[X]$ denote the $F$-subalgebra of $A$ generated by $X$. Let $\{a_1, ..., a_k, s, z\}$ be an algebraically independent subset of $A$. Assume that $\sigma z = -z, \sigma s = -s$ and $\sigma a_i = a_i$ for all $i$.

Let $b_1 = (sz + z)/2$ and $b_2 = (zs - z)/2$.

Proposition 5.3

Let $0 \neq f \in B = F[b_1, b_2, a_1, ..., a_k]$ satisfy $\sigma f = cf$ for some $c \in F$. Then $c = 1$ and $f \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$.

**Proof:** We may assume, without loss of generality, that $f$ is homogeneous of degree $l$ (as a polynomial in $b_1, b_2, a_1, ..., a_k$). The result clearly holds if $l = 0$. We will proceed by induction on $l$. Thus we assume $l \geq 1$ and write

$$f = b_1 f_1 + b_2 f_2 + \sum_{j=1}^{k} a_j g_j$$

where $f_1, f_2, g_1, ..., g_k \in B$ are homogeneous of degree $l-1$. Then $2f = (s+1)zf_1 + z(s-1)f_2 + 2\sum_{j=1}^{k} a_j g_j = s(zf_1) + z(f_1 + sf_2 - f_2) + 2\sum_{j=1}^{k} a_j g_j$ and so

$$0 = 2(\sigma f - cf) \in sz(\sigma f_1 - cf_1) + 2\sum_{j=1}^{k} a_j(\sigma g_j - cg_j) + zB.$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, we have $0 = \sigma f_1 - cf_1 = \sigma g_1 - cg_1 = ... = \sigma g_k - cg_k$. Then the induction assumption implies that $f_1, g_1, ..., g_k \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$. Replacing $f$ by $f - (b_1 + b_2)f_1 - \sum_{j=1}^{k} a_j g_j$, we may assume that $0 = f_1 = g_1 = ... = g_k$. Note that (as $\sigma b_2 \notin F b_2$) this proves the proposition in the case $l = 1$.

Now assume $l \geq 2$ and write

$$f = b_2(b_1 h_1 + b_2 h_2 + \sum_{j=1}^{k} a_j p_j)$$

where $h_1, h_2, p_1, ..., p_k \in B$. Then $4f = (z(s^2-1)z - z^2)h_1 + (z(s-1)z(s-1))h_2 + 2(zs - z)\sum_{j=1}^{k} a_j p_j)$ and so

$$0 = 4(\sigma f - cf) \in zs^2 z(\sigma h_1 - ch_1)$$

$$+2zs\sum_{j=1}^{k} a_j(\sigma p_j - cp_j) + 2s\sum_{j=1}^{k} a_j(\sigma p_j + cp_j) + z^2 B + zszB.$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, we have $0 = \sigma h_1 - ch_1 = \sigma p_1 - cp_1 = ... = \sigma p_k - cp_k = \sigma p_1 + cp_1 = ... = \sigma p_k + cp_k$. Consequently, $p_1 = ... = p_k = 0$. Furthermore, the induction assumption implies that $h_1 \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$ and so, replacing $f$ by $f - b_2 b_1 h_1$, we see that we may assume $h_1 = 0$ and so $4f = 4b_2^2 h_2 = z(s-1)z(s-1)h_2 = zsz(s-1)h_2 - z^2(s-1)h_2 = zszq - z^2 q$ where $q = (s-1)h_2 \in B$. Then

$$0 = 4(\sigma f - cf) = -zsz(\sigma q + cq) - z^2(\sigma q - cq).$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, it follows that $\sigma q + cq = 0$ and $\sigma q - cq = 0$. Thus $q = 0$ and so $h_2 = 0$, proving the proposition.

Let $y_1, ..., y_n$ be algebraically independent elements of an associative algebra $A$ over a field $F$. Let

$$Y = F[y_1, ..., y_n]$$

be the subalgebra of $A$ generated by $y_1, ..., y_n$. Note that, by the algebraic independence of the $y_i$, we have

$$Y = \oplus_{i=1}^{n} y_i Y.$$

Write

$$Y_i = \sum_{l=i}^{n} y_l Y$$

for $1 \le i \le n$ and set $Y_{n+1} = (0)$. For $1 \le i < n$, set

$$Y^{[i]} = F[y_1, ..., y_{i-1}, y_i + y_{i+1}, y_{i+1}y_i, y_{i+2}, ..., y_n].$$

For $1 \le j < i \le n$, define

$$\Lambda_{i,j} = 0$$

and for $1 \le j \le n$, define

$$\Lambda_{0,j} = 1.$$

For $1 \le i \le j \le n$, define

$$\Lambda_{i,j} = \sum_{j \ge l_1 > ... > l_i \ge 1} y_{l_1} y_{l_2} ... y_{l_i}.$$

For $1 \le j \le n$, set

$$M_j = F[\Lambda_{1,j}, ..., \Lambda_{j,j}, y_{j+1}, ..., y_n].$$

Lemma 5.4:

If $1 \le j \le n-1$, then $\Lambda_j \cap Y^{[j]} = M_{j+1}$.

Proof: Note that for $1 \le j \le n-1$ and $1 \le i \le n$, we have

$$\Lambda_{i,j+1} = y_{j+1}\Lambda_{i-1,j} + \Lambda_{i,j}$$

and consequently

$$M_{j+1} \subseteq M_j.$$

Furthermore, we also see that

$$\Lambda_{i,j+1} = y_{j+1}y_j\Lambda_{i-2,j-1} + (y_{j+1} + y_j)\Lambda_{i-1,j-1} + \Lambda_{i,j-1}$$

for $2 \le i \le n, 1 \le j \le n-1$, and so

$$M_{j+1} \subseteq Y^{[j]}.$$

Thus

$$M_{j+1} \subseteq M_j \cap Y^{[j]}.$$

Hence we need to show that if $f \in M_j \cap Y^{[j]}$, then $f \in M_{j+1}$. Without loss of generality, we may assume that $f$ is homogeneous of degree $t \geq 0$ in $\{y_1, ..., y_n\}$. The assertion is clearly true if $t = 0$. We now proceed by induction on $t$, assuming that the assertion is true for homogeneous polynomials of degree $< t$.

Suppose that for $1 \leq i \leq n$, we have $f \in Y_i \cap M_j \cap Y^{[j]}$. Then we may write

$$f = y_i f_i + ... + y_n f_n$$

where $f_i, ..., f_n \in Y$,

$$f = \Lambda_{1,j} g_1 + ... + \Lambda_{j,j} g_j + y_{j+1} g_{j+1} + ... + y_n g_n$$

where $g_1, ..., g_n \in M_j$, and $f =$

$$y_1 h_1 + ... + y_{j-1} h_{j-1} + (y_j + y_{j+1}) h_j + y_{j+1} y_j h_{j+1} + y_{j+2} h_{j+2} + ... + y_n h_n,$$

where $h_1, ..., h_n \in Y^{[j]}$. We will show that

$$f \in Y_i \cap M_{j+1} + Y_{i+1} \cap M_j \cap Y^{[j]}.$$

Note that, since $Y_1 = Y$ and $Y_{n+1} = 0$, iterating this result proves the lemma.

To prove our assertion, first suppose that $i \leq j$. Then $g_1 = ... = g_{i-1} = h_1 = ... = h_{i-1} = 0$ and $y_i f_i = y_i y_{i-1} ... y_1 g_i = y_i h_i$. Thus $y_{i-1} ... y_1 g_i \in Y^{[j]}$, and so $g_i \in Y^{[j]}$. But then $g_i \in M_j \cap Y^{[j]}$ and so, by the induction assumption, $g_i \in M_{j+1}$. But then $\Lambda_{i,j+1} g_i \in Y_i \cap M_{j+1}$ and so, since $f - \Lambda_{i,j+1} g_i = f - (y_{j+1} \Lambda_{i-1,j} + \Lambda_{i,j}) g_i \in Y_{i+1}$, we have $f - \Lambda_{i,j+1} g_i \in Y_i \cap M_{j+1}$, proving our assertion.

Next suppose that $i = j + 1$. Then we have $y_{j+1} g_{j+1} = y_{j+1} y_j h_{j+1}$, and so $g_{j+1} = y_j h_{j+1} \in M_j$. Then $g_{j+1} = \Lambda_{j,j} h'_{j+1}$ with $h'_{j+1} \in M_j$ and so $h_{j+1} = y_{j-1} ... y_1 h'_{j+1} \in Y^{[j]}$. It follows that $h'_{j+1} \in M_{j+1}$, so by the induction assumption $h'_{j+1} \in M_{j+1}$. Then $y_{j+1} g_{j+1} = y_{j+1} \Lambda_{j,j} h'_{j+1} = \Lambda_{i+1,j+1} h'_{j+1} \in M_{j+1}$. Since $f - y_{j+1} g_{j+1} \in Y_{i+1}$, our assertion is proved in this case.

Finally, suppose $i > j+1$. Then $y_i g_i = y_i h_i$ so $g_i = h_i \in M_j \cap Y^{[j]}$ and, by the induction assumption $g_i \in M_{j+1}$. Therefore $y_i g_i \in M_{j+1}$. Since $f - y_i g_i \in Y_{i+1}$, our assertion is proved in this case as well, completing the proof of the lemma.

Noting that $Y_1 = M_2$ we obtain the following immediate consequence of Lemma 3.2:

Proposition 3.3 For $2 \leq j \leq n$, we have $\cap_{i=1}^{j-1} Y^{[i]} = \Lambda_j$.

Proof of Theorem 5.1: Let $\sigma_i \in \mathfrak{S}_n$ be the transposition that interchanges $i$ and $i + 1$. It is clear from the definition that

$$\sigma_i(y_j) = y_j$$

whenever $j < i$. By Theorem 4.2 we have $y_{i+1}y_iy_{i-1}...y_1 = \Lambda_{i+1,i+1} = \sigma_i(\Lambda_{i+1,i+1}) = \sigma_i(y_{i+1})\sigma_i(y_i)(y_{i-1}...y_1)$ and so

$$\sigma_i(y_{i+1})\sigma_i(y_i) = y_{i+1}y_i.$$

Similarly, we have $\Lambda_{i,i+1} = (y_{i+1} + y_i)\Lambda_{i-1,i-1} + y_{i+1}y_i\Lambda_{i-2,i-1} = \sigma_i(\Lambda_{i,i+1}) = (y_{i+1}+y_i)\Lambda_{i-1,i-1}+y_{i+1}y_i\Lambda_{i-2,i-1} = \sigma_i(y_{i+1}+y_i)\Lambda_{i-1,i-1}+y_{i+1}y_i\Lambda_{i-2,i-1}$. Therefore

$$\sigma_i(y_{i+1} + y_i) = y_{i+1} + y_i.$$

Also, for $j > i + 1$ we have $\Lambda_{j,j} = y_j\Lambda_{j-1,j-1} = \sigma_i y_j(\Lambda_{j-1,j-1}) = \sigma_i(y_j)\Lambda_{j-1,j-1}$. Thus

$$\sigma_i(y_j) = y_j$$

whenever $j > i + 1$.

Let $u_i = (y_i + \sigma_i(y_i))/2, v_i = (y_{i+1} + \sigma_i(y_{i+1}))/2$, and $z_i = (y_i - \sigma_i(y_i))/2$. Since $\sigma_i(y_{i+1} + y_i) = y_{i+1} + y_i$, we also have $z_i = -(y_{i+1} - \sigma_i(y_{i+1}))/2$. Set $s_i = u_iz_i^{-1}$. By Proposition 5.2 2.4 we have that the set $\{y_1, ..., y_{i-1}, s_i, z_i, y_{i+2}, ..., y_n\}$ is algebraically independent. Since $\sigma_i(y_{i+1})\sigma_i(y_i) = y_{i+1}y_i$, we have that $z_iu_i = (\sigma_i(y_{i+1})y_i - y_{i+1}\sigma(y_i))/4 = v_iz_i$. Then we have

$$y_i = u_i + z_i = s_iz_i + z_i,$$

$$y_{i+1} = v_i - z_i = z_is_i - z_i.$$

Note that $\sigma_i(s_i) = -s_i$ and $\sigma_i(z_i) = -z_i$. Therefore the hypotheses of Proposition 5.3 are satisfied (with $k = n-2, a_j = y_j$ for $1 \leq j \leq i-1, s = s_i, z = z_i, a_j = y_{j+2}$ for $i \leq j \leq n-2$ and $\sigma = \sigma_i$). Therefore $f \in F[y_1, ..., y_{i-1}, y_i + y_{i+1}, y_{i+1}y_i, y_{i+2}, ..., y_n]$ or, in the notation of Proposition 5.5, $f \in Y^{[i]}$. Since this holds for all $i, 1 \leq i \leq n-1$, Proposition 5.5 shows that $f \in M_n$, proving the theorem.

# Bibliography


[1] Bergman, G. and Cohn, P. M., *Symmetric elements in free powers of rings*, J. London Math. Soc., **1** (1969), 525-534.

[2] Gelfand, I., Gelfand, S., Retakh, V., Wilson, R.L., *Quasideterminants*, Adv. in Math. **193** (2005), 56–141.

[3] Gelfand, I., Krob, D., Lascoux, A., Leclerc, R., Retakh, V., Thibon, J.-Y., *Noncommutative symmetric functions*, Adv. Math., **112** (1995), 218–348.

[4] Jacobson, N., *Structure and representations of Jordan algebras*, Colloquium Publications **39** (1968), Amer. Math. Soc., RI.

[5] MacMacon, P. A., *Combinatorial analysis*, (1915), Cambridge Univ. Press, Cambridge.

[6] Wilson, R. L., *Invariant polynomials in the free skew field*, Selecta Math. (N.S.) **7** (2001), 565–586.

[7] Wolf, M. C., *Symmetric functions of noncommuting elements*, Duke Math. J. **2** (1936), 626–637.

# Noncommutative Localization

# Contents

# Chapter 1

## 1.1 Why the study of quasideterminants leads to the study of localization

The determinant of an $n$ by $n$ matrix $A = [a_{i,j}]$ over a commutatuve ring $R$ may be expressed as a polynomial in the matrix entries:

$$det\ A = \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) a_{1,\sigma(1)} \ldots a_{n,\sigma(n)}.$$

As we will see (in other lectures) the analogous definition of the quasideterminant $[A]_{i,j}$ of an $n$ by $n$ matrix $A = [a_{i,j}]$ over a (possibly) noncommutative ring $R$ inductively gives $[A]_{i,j}$ as a rational expression in the matrix entries. Thus for $n = 1$

$$[A]_{1,1} = a_{1,1}$$

and for $n > 1$

$$[A]_{i,j} = a_{i,j} - \sum_{k \neq i, l \neq j} a_{i,l} [A^{i,j}]^{-1}_{k,l} a_{k,j}$$

where $A^{i,j}$ denotes the minor obtained from $A$ by deleting the $i$-th row and the $j$-th column.

For example:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}_{1,1} = a_{1,1} - a_{1,2} a_{2,2}^{-1} a_{2,1},$$

and

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}_{1,1} =$$

$$
\begin{aligned}
&= a_{1,1} - a_{1,2}\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}^{-1}_{2,2} a_{2,1} \\
&\quad - a_{1,3}\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}^{-1}_{2,3} a_{2,1} \\
&\quad - a_{1,2}\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}^{-1}_{3,2} a_{3,1} \\
&\quad - a_{1,3}\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}^{-1}_{3,3} a_{3,1} \\
&= a_{1,1} - a_{1,2}(a_{2,2} - a_{2,3}a_{3,3}^{-1}a_{3,2})^{-1}a_{2,1} \\
&\quad - a_{1,3}(a_{2,3} - a_{2,2}a_{3,2}^{-1}a_{3,3})^{-1}a_{2,1} \\
&\quad - a_{1,2}(a_{3,2} - a_{3,3}a_{2,3}^{-1}a_{2,2})^{-1}a_{3,1} \\
&\quad - a_{1,3}(a_{3,3} - a_{3,2}a_{2,2}^{-1}a_{2,3})^{-1}a_{3,1}.
\end{aligned}
$$

Of course, if the $a_{i,j}$ are elements of an arbitrary ring $R$, these expressions may not exist, since all the required inverses may not exist. We may attempt to overcome this difficulty by requiring that $R$ be a skew field so that all we need to check is that each of the expressions to be inverted is nonzero. We may further regard the matrix elements $a_{i,j}$ as independent noncommuting indeterminates over some field $F$ (satisfying no relations except those forced by the axioms of a skew field), i.e., assume that $R$ is the "free skew field over $F$ generated by the $a_{i,j}$" (which we denote by $F\langle\!\langle a_{i,j}|1 \leq i,j \leq n \rangle\!\rangle$). If we are to do this, we need to be precise about the properties of the "free skew field" and show that such an object actually exists. We will do this, following the development of P. M. Cohn [2].

Now our independence assumption on the $a_{i,j}$ implies that the subalgebra of $F\langle\!\langle a_{i,j}|1 \leq i,j \leq n \rangle\!\rangle$ generated by $\{a_{i,j}|1 \leq i,j \leq n\}$ is just the free algebra $F < a_{i,j}|1 \leq i,j \leq n >$, a reasonably well understood object. Thus the study of $F\langle\!\langle a_{i,j}|1 \leq i,j \leq n \rangle\!\rangle$ may be thought of as a special case of the more general problem: find all skew fields containing the free algebra $F < a_{i,j}|1 \leq i,j \leq n >$. This, in turn, is a special case of the more general problem: Let $R$ be a ring, find all injective homomorphisms $\phi : R \to K$ where $K$ is a skew field. This is the general problem of *localization*, which P. M. Cohn describes as "the process of introducing fractions in a ring".

Once we have established the existence and properties of $F\langle\!\langle a_{i,j}|1 \leq i,j \leq n \rangle\!\rangle$, important questions about quasideterminants remain. In particular, note that the rational expressions giving the quasideterminant of an $n$ by $n$ matrix get successively more complicated in the sense that for an $n$ by $n$ matrix the expression involves $n-1$ successive (or nested) inversions. However, it could be that there are simpler expressions allowing the quasideterminants of an $n$ by $n$

matrix to be expressed with fewer invesions. (Consider, for example, *Hua's identity* (cf. [6]): $(a^{-1}+(b^{-1}-a)^{-1})^{-1} = a - aba$.) Reutenauer has proved that this is not the case, i.e., any expression for a quasideterminant of the $n$ by $n$ matrix $[a_{i,j}]$ over $F \langle\!\langle a_{i,j} | 1 \leq i,j \leq n \rangle\!\rangle$ must involve $n-1$ successive inversions.

We will also be interested in studying whether certain sets of elements in $F \langle\!\langle a_{i,j} | 1 \leq i,j \leq n \rangle\!\rangle$ are algebraically independent. To do this, we must have a way of determining whether or not certain expressions in $F \langle\!\langle a_{i,j} | 1 \leq i,j \leq n \rangle\!\rangle$ represent 0. We will see that the universal property of $F \langle\!\langle a_{i,j} | 1 \leq i,j \leq n \rangle\!\rangle$ allows to answer this question by considering expressions in any skew field $K$ containing $F < a_{i,j} | 1 \leq i,j \leq n >$. We will see that a constuction due to Fisher [4] (using techniques of Jateogaonkar [7]) produces skew fields that are useful in this regard.

We will begin these lectures (Section 2) by reviewing the well-known case of commutative rings. Here if $R$ is embedded in a field $S$, then the set of elements $\{rs^{-1} | r,s \in R, s \neq 0\}$ is a subfield of $S$. We will then follow Ore in characterizing rings $R$ (not hecessarily commutative) that may be embedded in a skew field $S$ is such a way that $\{rs^{-1} | r,s \in R, s \neq 0\}$ is a skew field. We will then consider examples of embeddings $R \subseteq S$ where $S$ such that $\{rs^{-1} | r,s \in R, s \neq 0\}$ is not a skew field. These examples will suggest that it is necessary to consider the subring of $S$ generated by all entries of the inverses of matrices over $R$ that are invertible in $S$. This provides motivation for the later introduction of the matrix ideals and prime matrix ideals belonging to a ring $R$. Constructions due to Jategaonkar and Fisher (Section 3) provide particularly important examples $R \subseteq S$ in which $R$ is a free algebra and $\{rs^{-1} | r,s \in R, s \neq 0\}$ is not a skew field. In Section 4 we will define the category of $R$-fields and specializations and formulate the appropriate universal condition to define the free skew field. Section 5 places "the subring of $S$ generated by all entries of the inverses of matrices over $R$ that are invertible in $S$" in a more general context. Sections 6 - 8 present Cohn's development of the correspondence between prime matrix ideals and epic $R$-fields. Section 9 proves the existence of the free skew field and Section 10 gives a normal form (from [3]) for elements of the free skew field that is necessary for the proof of Reutenauer's Theorem. Section 11 presents the proof (from [10]) of that theorem.

## 1.2 Classical cases - commutative rings and Ore rings

Let $K$ be a skew field and $S$ be a subring of $K$. Then $S$ is an *integral domain.* That is, if $a, b \in S, a, b \neq 0$ then $ab \neq 0$. (Note that we do not include commutativity in the definition of an integral domain.)

Let $S$ be a commutative integral domain and write $S^{\times} = \{s \in S | s \neq 0\}$. Define an equivalence relation $\sim$ on $S \times S^{\times}$ by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Then letting $Q(S)$ denote the set of equivalence classes of $S \times S^{\times}$ under $\sim$, denoting the equivalence class of $(a, b)$ by $[a, b]$, and setting

$$[a, b] + [c, d] = [ad + bc, bd]$$

and

$$[a, b][c, d] = [ac, bd]$$

we see that $Q(S)$ is a field, the map

$$\phi : S \to Q(S)$$

$$\phi : s \mapsto [s, 1]$$

is an injective homomorphism of rings and if $K$ is any field and

$$\psi : S \to K$$

is an injective homomorphism of rings, then there is an injective homomorphism

$$\tau : Q(S) \to K$$

such that

$$\phi\tau = \psi.$$

Now let $S$ be any commutative ring and $\phi : S \to K$ be a homomorphism into a field $K$. Then $S$ is a commutative integral domain and so ker $\phi$ is a prime ideal in $S$. If $\phi(S)$ generates $K$ (as a field), then we have $Q(S) = K$. Hence the homomorphisms $\phi : S \to K$ such that $\phi(S)$ generates $K$ are in one-to-one correspondence with the prime ideals of $S$.

Ore [9] considered embeddings $R \subseteq K$ where $R$ is a ring (not necessarily commutative), $K$ is a skew field and $K = \{rs^{-1} | r \in R, 0 \neq s \in R\}$. Then $R$ must be an integral domain and, furthermore, for any $t, u \in R, u \neq 0$ we have $u^{-1}t \in K$ so

$$u^{-1}t = rs^{-1}$$

for some $r, s \in R, s \neq 0$. Consequently, for any nonzero $u, t \in R$ there exist $r, s \in R$ such that

$$ur = ts \neq 0$$

. A ring satisfying these conditions is called a $r$ight Ore domain. (Left Ore domains are defined similarly.) If $R$ is a right Ore domain, we may construct a skew field containing $R$ by a construction analogous to that for commutative integral domains. Thus we define an equivalence relation $\sim$ on $R \times R^{\times}$ by

$$(a, b) \sim (c, d) \Leftrightarrow \exists u, v \in R^{times} \ such\ that\ au = cv\ and\ bu = dv.$$

Then letting $Q(R)$ denote the set of equivalence classes of $R \times R^{\times}$ under $\sim$, denoting the equivalence class of $(a, b)$ by $[a, b]$ and setting

$$[a, b] + [c, d] = [au + bv, t] \ where\ bu = dv = t \neq 0$$

and

$$[a, b][c, d] = [ar, ds] \ where\ br = cs \neq 0$$

we see that $Q(R)$ is a skew field, the map

$$\phi : R \to Q(R)$$

$$\phi r \mapsto [r, 1]$$

is an injective homomorphism of rings and if $K$ is any skew field and

$$\psi : R \to K$$

is an injective homomorphism of rings, then there is an injective homomorphism

$$\tau : Q(R) \to K$$

such that

$$\phi\tau = \psi.$$

It is known ([5], p. 166) that if $L$ is a finite-dimensional Lie algebra over a field, then $U(L)$, the universal enveloping algebra of $L$ is an Ore domain. Therefore, there is a homomorphism of $U(L)$ into the skew field $Q(U(L))$. If $L$ is any Lie algebra, we let $L^i$ denote the span of all products $[a_1[a_2[...[a_{i-1}, a_i]...]]$. Then $L^i$ is an ideal in $L$. Let $\mathfrak{F}_n$ denote the free Lie algebra on $n$ generators $x_1, ..., x_n$ and $\mathfrak{F}_n(k)$ denote the quotient of $\mathfrak{F}_n$ by the ideal $\mathfrak{F}_n^k$. Now $U(\mathfrak{F}_n) \cong F < x_1, ..., x_n >$, the free algebra generated by $x_1, ..., x_n$. Thus we have

$$F < x_1, ..., x_n > \to U(\mathfrak{F}_n) \to U(\mathfrak{F}_n(k)) \to Q(U(\mathfrak{F}_n(k))),$$

a homomorphism of the free algebra $F < x_1, ..., x_n$ onto a skew field.

## 1.3 The constructions of Jategaonkar and Fisher

The constructions of this section depend on the following lemma of Jategaonkar.

Lemma: Let $R$ be an integral domain with center $F$ and $\{x_i|i \in I\}$ be a subset of $R$ containing at least two elements. Assume that $\sum_{i\in I} x_iR$ is a direct sum of non-zero right ideals of $R$. Then the subring $A$ of $R$ generated by $\{x_i|i \in I\}$ over $F$ is isomorphic to the free $F$-algebra $F < y_i|i \in I >$ .

Proof: For a sequence $\mathbf{j} = (j_1, ..., j_k)$ let $y^{\mathbf{j}}$ denote the monomial

$$y_{j_1}...y_{j_k}.$$

We also set

$$|\mathbf{j}| = j_1 + ... + j_k.$$

For $f = \sum_{\mathbf{j}} f_{\mathbf{j}} y^{\mathbf{j}} \in F < y_i|i \in I >$ let

$$|f| = min\{|\mathbf{j}||f_{\mathbf{j}} \neq 0\}$$

and

$$||f|| = |\{\mathbf{j}|f_{\mathbf{j}} \neq 0\}.$$

The homomorphism $\phi : F < y_i|i \in I >\longrightarrow R$ defined by $\phi(y_i) = x_i$ for all $i$ is surjective. Suppose $\phi$ is not injective. Order the set

$$S = \{(||f||, |f|)|0 \neq f \in ker(\phi)\}$$

lexicographically and suppose that $0 \neq g \in ker(\phi)$ and $(||g||, |g|)$ is minimal in $S$. We may write

$$g = g_\emptyset + \sum_{j\in I} y_j g_j$$

where $g_\emptyset \in F$ and $g_j \in F < y_i|i \in I >$ . If $g_j = 0$ for all $j \in I$, then $g = g_\emptyset$ and so $0 = \phi(g) = g_\emptyset$, so $g = 0$, contradicting the choice of $g$. Thus there is some $l \in I$ such that $g_l \neq 0$.

Now suppose $g_\emptyset \neq 0$. Then if $j \in I, i \neq l$ we have

$$0 = \phi(gy_j) = g_\emptyset x_j + \sum_{i\in I} x_i\phi(g_i)x_j.$$

As $\sum_{i\in I} x_iR$ is direct we have $0 = g_\emptyset x_j + x_j\phi(g_j)x_j$ and $0 = x_i\phi(g_i)x_j$ for all $i \in I, i \neq j$. Thus $0 \neq g_\emptyset y_j + y_jg_jy_j$ and $0 \neq y_lg_ly_j$ are elements of $ker(\phi)$. But $||y_lg_ly_j|| \leq ||g|| - 1$, contradicting the choice of $g$. Thus $g_\emptyset = 0$ and so $|g| > 0$.

Now $\phi(g) = \phi(\sum_{j\in I} y_jg_j) = \sum_{j\in I} x_j\phi(g_j)$. Since $\sum_{i\in I} x_iR$ is direct, we have each $x_j\phi(g_j) = 0$. In particular, $0 \neq y_lg_l \in ker(\phi)$. Then by the choice of $g$ we have $g = y_lg_l$. But then $0 = \phi(g) = x_l\phi(g_l)$ so $g_l \in ker(\phi)$. Since $||g_l|| = ||g||$ and $|g_l| = |g| - 1$, this contradicts the choice of $g$, completing the proof.

Jategaonkar uses this lemma to give an embedding of the free algebra $F < x_i|i \in I >$ over a field $F$ on an arbitary set of generators $\{x_i|i \in I\}$ in a skew field. Let $L = F(t_{i,j}|i \in I, j \in \mathbf{Z}, j \geq 1)$, the (commutative) transcendental field extension of $F$ generated by $\{t_{i,j}|i \in I, j \in \mathbf{Z}, j \geq 1\}$. Let $L[D]$ denote the ring of skew Laurent polynomials

$$\{\sum_{i=0}^{m} f_i D^i | f_i \in L, m \geq 0\}$$

where $Dt_{i,j} = t_{i,j+1}D$. Now if $f = \sum_{i=0}^{m} f_i D^i$ and $g = \sum_{i=0}^{n} g_i D^i \in L[D]$ with $f_m \neq 0$ and $m \leq n$, then

$$g - (g_n(D^{n-m} f_m)^{-1} D_{n-m})f = \sum_{i=0}^{n-1} (g_i - (g_n(D^{n-m} f_m)^{-1} D_{n-m}) f_i D^i.$$

Thus $L[D]$ has a left division algorithm and so is a left principal ideal domain. Therefore $L[D]$ is a left Ore domain and hence has a left ring of quotients $K$ which is a skew field containing $L[D]$.

Theorem (Jateogaonkar [7]): The homomorphism

$$\phi : F < x_i | i \in I > \rightarrow K$$

defined by

$$\phi : x_i \mapsto t_{i,1} D$$

is an embedding of the free algebra $F < x_1, ..., x_n >$ into $K$.

Proof: Since $\sum_{i \in I} (t_{i,1} D) L[D]$ is direct, the lemma gives the result.

Fisher ([4]) gives a related embedding of $F < x_1, x_2 >$ into a skew field. In the notation of Jategaonkar's Theorem, let $I = \{1, 2\}$, set $t_{1,j} = 1$ for all $j \geq 1$ and write $t_j$ for $t_{2,j}$. Write $L = F(t_j | j \in \mathbf{Z}, j \geq 1)$, set $Dt_j = t_{j+1} D$ for all $j \geq 1$, and define $L[D]$ and $K$ as above.

Theorem (Fisher): The homomorphism

$$\phi : F < x_1, x_2 > \rightarrow K$$

defined by

$$\phi : x_1 \mapsto D$$

and

$$\phi : x_2 \mapsto t_1 D$$

is an embedding of the free algebra $F < x_1, ..., x_n >$ into $K$.

Fisher proves an important property of this embedding. Suppose $\phi : R \rightarrow K$ is an isomorphism of a ring $R$ into a skew field $K$. Define

$$Q_0(R, \phi) = \phi(R)$$

and, for $i > 0$, define

$$Q_i(R, \phi) = < Q_{i-1}(R, \phi), \{r^{-1} | 0 \neq r \in Q_{i-1}(R, \phi)\} >,$$

the subring of $K$ generated by $Q_{i-1}(R,\phi)$ and $\{r^{-1}|0 \neq r \in Q_{i-1}(R,\phi)\}$. Then

$$\cup_{i\geq 0} Q_i(R,\phi)$$

is a skew field. We say that the embedding $\phi : R \to K$ has *height* $m$ if $m = min\{n|Q_n(R,\phi) = K\}$. We say that an element $x \in K$ has height $m$ (with respect to $R$ and $\phi$) if $m = min\{n|x \in Q_n(R,\phi)\}$. Thus if $R$ is a skew field the embedding $R \to R$ has height 0 and if $R$ is an Ore domain but not a skew field the embedding $R \to Q(R)$ has height 1.

Theorem ([4]): Let $F$ be a field. Let the embedding $\phi : F < x_1, x_2 > \to K$ be as described in the previous theorem. Set $K_1 = \cup_{i\geq 0} Q_i(F < x_1, x_2 >, \phi)$. Then the embedding $\phi : F < x_1, x_2 > \to K_1$ is of height 2.

Proof: We first show that the embedding is of height $\leq 2$. Write $Q_i$ for $Q_i(F < x_1, x_2 >), \phi)$. Then $Q_1$ contains $D$ and $D^{-1}$. Hence $Q_1$ also contains $D^{i-1}(t_1 D)D^i = t_i$ for all $i \geq 1$. Thus $Q_1 \supseteq L[D]$. Since the ring of quotients $K$ is generated by $L[D]$ and $Q_1 \supseteq L[D]$ we have $Q_2 \supseteq K$. Furthermore, $K \supseteq Q_1$ and so $K \supseteq Q_2$. Thus $K = Q_2$.

Now $F < x_1, x_2 >$ is the span of all monomials

$$x_1^{i_0} x_2 x_1^{i_1 - 1} x_2 ... x_2 x_1^{i_{k-1}-1} x_2 x_1^{i_k}$$

where $i_0, i_k \geq 0$ and $i_1, ..., i_{k-1} \geq 1$. Now

$$\phi(x_1^{i_0} x_2 x_1^{i_1 - 1} x_2 ... x_2 x_1^{i_{k-1}-1} x_2 x_1^{i_k}) =$$

$$t_{i_0} t_{i_0+i_1} ... t_{i_0+...+i_{k-1}} D^{i_0+...+i_k}.$$

Set $M = span\{t_{j_1} t_{j_2} ... t_{j_m} | m \geq 0, j_1 < j_2 < ... j_m\}$. Thus if $f = \sum_{i=0}^m f_i D^i \in \phi(F < x_1, x_2 >)$ we have $f_i \in M$ for all $i \geq 0$. Let $N$ denote the set of all products $l_1 ... l_k$, where $k \geq 1$ and $0 \neq l_i \in M$ for $1 \leq i \leq k$. Let $V = \{rs^{-1}|r \in N, 0 \neq s \in N\}$. Let $U = \{D^{-n} \sum_{i=0}^\infty v_{n,i} D_i | n \geq 0, v_{n,i} \in V\}$. It is easy to check that $U$ is a subring containing $\phi(F < x_1, x_2 >) \cup \{s^{-1}|0 \neq s \in \phi(F < x_1, x_2 >\}$. Thus $U$ contains $Q_1$. Now suppose $u \in F[t_i|i \geq 1]$ is an irreducible element such that $u^{-1} \in Q_1$. Then $u^{-1} \in U$ and so $u^{-1} = rs^{-1}$ where $r, s \in N$ for some $m$. Then $ur = s$. Since $u$ is irreducible, $u$ divides some element of $M$. Since every monomial summand of every element of $M$ is square free, the same must be true of every monomial summand of $u$. But there are irreducible elements of $F[t_i|i \geq 1]$ (such as $t_i - t_{i+1}^n$ for $n \geq 2, i \geq 1$) for which this is not true. Thus $K \neq Q_1$ and so the embedding $\phi : F < x_1, x_2 > \to K_1$ is of height 2.

Fisher [4] also gives a related embedding of height 1 of $F < x_1, x_2 >$ in a skew field

# Chapter 2

## 2.1 $R$-rings, $R$-fields and specializations

Let $R$ be a ring. An $R$-ring is pair $(\phi, S)$ where $\phi : R \to S$ is a homomorphism of rings. For a fixed $R$, the $R$-rings form a category in which the morphisms from $(\phi, S)$ to $(\phi', S')$ are ring homomorphisms $\psi : S \to S'$ such that $\psi\phi = \phi'$.

If $(\phi, S)$ is an $R$-ring and $S$ is a skew field we call $(\phi, S)$ and $R$-field. If $S$ is generated (as a skew field) by $\phi(R)$ we say that $(\phi, S)$ is an *epic* $R$-field. If $(\phi, S)$ is an epic $R$-field and $\phi$ is injective, we say that $S$ is a *field of fractions* of $R$. Thus if $R$ is a commutative integral domain or an Ore domain, $Q(R)$ is a field of fractions of $R$.

Any morphism between epic $R$-fields must be an isomorphism. For if $\psi$ is a morphism from the epic $R$-field $(\phi, S)$ to the epic $R$-field $(\phi', S')$, then $\psi$, being a (ring) homomorphism of between skew fields, must be injective. Furthermore, $\psi(S) \subseteq S'$ is a skew field containing $\phi(R)$ and so must equal $S'$. Thus, if the category of $R$-fields is to have an interesting structure, we must define a more general notion of morphism.

Define a *local homomorphism* $\psi$ from the $R$-field $(\phi, S)$ to the $R$-field $(\phi', S')$ to be a morphism from an $R$-subring $(\phi, S_0)$ of $(\phi, S)$ to the $R$-ring $(\phi', S')$ such that any element $S_0$ not in the kernel of $\psi$ has an inverse in $S_0$. This implies that *ker* $\psi$ is a maximal ideal of $S_0$ and is the set of all non-units of $S_0$. Thus $\psi(S)$ is a skew field containing $\phi'(R) = \psi\phi(R)$. Hence any local homomorphism to an epic $R$-field is surjective.

Recall the definition of the skew field $Q(U(\mathfrak{F}_n(k)))$ from Section 2. Denote this skew field by $Q_n(k)$ and let $\phi_{n,k} : F < x_1, ..., x_n > \to Q_n(k)$ be the homomorphism defined in Section 2. Then, for each $n$ and $k$, $(\phi_{n,k}, Q_n(k))$ is an epic $F < x_1, ..., x_n >$-field. If $m > k$ define $V(n, m, k) \subseteq Q_n(m)$ to be

$$\{rs^{-1} | r, s \in U(\mathfrak{F}_n(m)), s \notin U(\mathfrak{F}_n(m))(\mathfrak{F}_n(m))^k\}.$$

Then $\phi_{n,k}$ is a local homomorphism from $(\phi_{n,m}, Q_n(m))$ to $(\phi_{n,k}, Q_n(k))$.

Two local homomorphisms between $R$- fields $(\phi, S)$ and $(\phi', S')$ are said to be *e*quivalent if they agree on a subring $S_0$ of $S$ and if their (common) restriction to $(\phi, S_0)$ is a local homomorphism.

An equivalence class of local homomorphisms between $R$-fields $(\phi, S)$ and $(\phi', S')$ is called a *specialization*. Then $R$-fields and specializations form a category. An initial object $U$ in the full subcategory of epic $R$-fields is called a *universal* $R$-field.

We will see that there is a universal $F < x_1, ..., x_n >$-field which we will call the free skew field.

## 2.2 Rational closures

Let $R$ and $S$ be rings and $f : R \to S$ be a homomorphism. As usual, we may extend $f$ to a map from the set of $m$ by $n$ matrices over $R$ to the set of $m$ by $n$ matrices over $S$. Let $\Sigma$ be a set of matrices over $R$. We say that $f$ is $\Sigma$-inverting if every element of $f(\Sigma)$ is invertible. In this case we let $R_\Sigma(S)$ denote the subset of $S$ consisting of all entries of inverses of matrices in $f(R)$.

We will find sufficient conditions on $\Sigma$ for $R_\Sigma$ to be a subring of $S$.

We say that a set $\Sigma$ of matrices is *upper multiplicative* (respectively, *lower multiplicative*) if $1 \in \Sigma$ and if whenever $A, B \in \Sigma$ and $C$ is a matrix of appropriate size then $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \Sigma$ (respectively, $\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \in \Sigma$). An upper multiplicative set that is invariant under right and left multiplication by permutation matrices is called a *multiplicative set*.

Lemma ([2], Proposition 7.1.1): Let $f : R \to S$ be a homomorphism and let $\Sigma$ be the set of all matrices over $R$ whose images under $f$ are invertible. Then $\Sigma$ is multiplicative.

Proof: $f(1)$ is invertible and if $f(A)$ and $f(B)$ are invertible, then

$$\begin{bmatrix} f(A) & f(C) \\ 0 & f(B) \end{bmatrix}^{-1} = \begin{bmatrix} f(A)^{-1} & -f(A)^{-1}f(C)f(B)^{-1} \\ 0 & f(B)^{-1} \end{bmatrix}.$$

Theorem ([2], Theorem 7.1.2): Let $\Sigma$ be an upper multiplicative set of matrices over $R$ and $f : R \to S$ be a $\Sigma$-inverting homomorphism. Then $R_\Sigma(S)$ is a subring of $S$ containing $f(R)$ and the following conditions are equivalent:

(a) $x \in R_\Sigma(S)$

(b) $x$ is a component of a solution of a matrix equation

$$Au - a = 0, A \in f(\Sigma)$$

(c) $x = bA^{-1}c, A \in f(\Sigma), b = \begin{bmatrix} b_1 & ... & b_m \end{bmatrix}, c = \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_m \end{bmatrix}, b_1, ..., b_m, c_1, ..., c_m \in$ $f(R)$.

Now the equation in (b) can be rewritten in terms of the augmented matrix $(a, A)$ as $(a, A)u' = 0$ where $u' = \begin{bmatrix} 1 \\ u \end{bmatrix}$. With this as motivation we adopt the following notation for an $m$ by $m+1$ matrix $A$:

$$A = (A_0, A_1, ..., A_m) = (A_0, A_*, A_\infty)$$

where $A_0, ..., A_m$ denote the columns of $A$ and $A_*$ denotes the $m$ by $m-1$ matrix $(A_1, ..., A_{m-1})$. We call $(A_0, A_*)$ the *numerator* of $A$, $A_*$ the *core* of $A$, and $(A_*, A_\infty)$ the *denominator* of $A$.

## 2.3 Matrix ideals

Suppose $(\phi, K)$ is an $R$-field. It is natural to ask what matrices over $R$ map to non-invertible matrices over $K$. We call the set of such matrices the *singular kernel* of $\phi$. Here are some such matrices.

We say that an $n$ by $n$ matrix $A$ over $R$ is *full* if it cannot be written as a product $A = BC$ where $B$ is an $n$ by $r$ matrix over $R$, $C$ is an $r$ by $n$ matrix over $R$, and $r < n$. Then if $A$ is a non-full matrix we have $\phi(A) = \phi(B)\phi(C)$ where $\phi(B)$ and $\phi(C)$ are both $< n$. Hence $\phi(A)$ is not invertible and so $A$ belongs to the singular kernel.

If $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are matrices of the same size and if for some $j$ we have $A_i = B_i$ for all $i \neq j$, we define the *determinental sum* of $A$ and $B$ (with respect to the $j$-th column) to be the matrix $C = (C_1, ..., C_n)$ with $C_i = A_i$ for $i \neq j$ and $C_j = A_j + B_j$. We denote the determinental sum by $A \nabla B$. Also, if the determinental sum of the transposed matrices $A^t$ and $B^t$ with respect to the $j$-th column is defined, we define the determinental sum of $A$ and $B$ with respect to the $j$-th row to be $A \nabla B = (A^t \nabla B^t)^t$. Note that the operation $\nabla$ is not always defined, may depend on specification of a row or column, and is not associative. Now if $A$ and $B$ are non-square, then $A \nabla B$ is non-square and so belongs to the singular kernel of $\phi$. If $A$ and $B$ are square matrices belonging to the singular kernel of $\phi$, then $rank(\phi(A) \nabla \phi(b)) \leq \ max\{rank\ \phi(A),\ rank\ \phi(B)\}$. Thus $A \nabla B$ belongs to the singular kernel.

For matrices $A$ and $B$ define $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Clearly if either $A$ or $B$ belongs to the singular kernel, then $A \oplus B$ does as well. Furthermore, if $A \oplus I$ belongs to the singular kernel, so does $A$.

We say a set $\mathfrak{P}$ of matrices over $R$ is a *matrix pre-ideal* if the following three conditions are satisfied.

(1)$\mathfrak{P}$ contains all non-full matrices.

(2) If $A, B \in \mathfrak{P}$ and $A \nabla B$ is defined, then $A \nabla B \in \mathfrak{P}$

(3) If $A \in \mathfrak{P}$ then $A \oplus B \in \mathfrak{P}$ for all matrices $B$ over $R$.

If, in addition,

(4) $A \oplus I \in \mathfrak{P} \Rightarrow A \in \mathfrak{P}$

we say that $\mathfrak{P}$ is a *matrix ideal*.

## 2.4 Prime matrix ideals

The singular kernel of an $R$-field $(\phi, K)$ has an additional property. If $A \oplus B$ belongs to the singular kernel, then either $A$ or $B$ belongs to the singular kernel. This is the motivation for the following definition.

A matrix ideal $\mathfrak{P}$ is called a *prime matrix ideal* if $A \oplus B \in \mathfrak{P}$ implies $A \in \mathfrak{P}$ or $B \in \mathfrak{P}$.

The theory of prime matrix ideals resembles the theory of prime ideals. For example, we may use the following theorem to construct prime matrix ideals.

Theorem ([2], Theorem 7.4.3): Let $R$ be a ring, $\Sigma$ be a non-empty set of matrices over $R$ closed under diagonal sums, and let $\mathfrak{A}$ be a matrix ideal such that $\mathfrak{A} \cap \Sigma = \emptyset$. Then there is a matrix ideal $\mathfrak{P}$ maximal with respect to $\mathfrak{P} \subseteq \mathfrak{A}$ and $\mathfrak{P} \cap \Sigma = \emptyset$. The matrix ideal $\mathfrak{P}$ is a prime.

## 2.5 The correspondence between prime matrix ideals and epic $R$-fields

Theorem (([2]Theorem 7.4.8): Let $R$ be a ring and $\mathfrak{P}$ be a prime matrix ideal. Then there is an epic $R$-field $(\phi, K)$ with singular kernel $\mathfrak{P}$.

Outline of proof: We say that an $m$ by $n$ matrix has index $n - m$. Let $M$ denote the set of matrices of index 1

$$A = (A_0, A_*, A_\infty)$$

such that the denominator $(A_*, A_\infty) \notin \mathfrak{P}$. Let $M_0$ denote the set of all $A \in M$ such that the numerator $(A_0, A_*) \in \mathfrak{P}$. For $A \in M$ set $A^0 = (A_0, A_*, -A_\infty)$

We say that two matrices $A, B \in M$ are *trivially related* if $A = PBQ^*$ where $P$ is invertible and

$$Q^* = \begin{bmatrix} 1 & 0 & 0 \\ Q_0 & Q_* & Q_\infty \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible. We abuse notation by writing $A$ for the class of $A$ under this equivalence relation.

Now, for $A, B \in M$ define

$$A \sharp B = \begin{bmatrix} B_0 & B_* & B_\infty & 0 & 0 \\ A_0 & 0 & -A_\infty & A_* & A_\infty \end{bmatrix}$$

and

$$A.B = \begin{bmatrix} B_0 & B_* & B_\infty & 0 & 0 \\ 0 & 0 & A_0 & A_* & A_\infty \end{bmatrix}.$$

Define an equivalence relation $\sim$ on $M$ by $A \sim B$ if and only if $A \sharp C \sharp B^0 \in M_0$ for some $C \in M_0$. Let $K$ denote the set of equivalence classes. Then $\sharp$ and . induce operations on $K$ that give $K$ the structure of a skew field. Define $\phi : R \to K$ by $f(a) = \begin{bmatrix} a & -1 \end{bmatrix}$.

Observe that if $y \in K$ and $y$ is represented by the matrix $A \in M$, then

$$Au = 0$$

for some

$$u = \begin{bmatrix} u_0 \\ u_* \\ u_\infty \end{bmatrix}$$

with $u_0 = 1, u_\infty = p$.

# Chapter 3

## 3.1 Existence of $F \langle\!\langle a_{i,j} | 1 \leq i, j \leq n \rangle\!\rangle$

For any set $X$ of indeterminates, $F < X >$ is a free ideal ring and therefore the set of all non-full matrices is a prime matrix ideal (and so, the minimal prime matrix ideal). Therefore, $F < X >$ has a universal field of fractions $F \langle\!\langle X \rangle\!\rangle$.

## 3.2 A normal form for elements of $F \langle\!\langle a_{i,j} | 1 \leq i, j \leq n \rangle\!\rangle$

Let $F$ be a field and $D$ be a skew field with center $F$. Define $D_F < X >$ to be the free product ofer $F$ of $F < X >$ and $D$. Then $D_F < X >$ has a universal field of fractions $D_F \langle\!\langle X \rangle\!\rangle$. Since this field of fractions corresponds to the prime matrix ideal consisting of all non-full matrices, the characterization of elements of the $R$-field corresponding to a prime matrix ideal given at the end of Section 8 may be restated by saying that each element $f \in D_F \langle\!\langle X \rangle\!\rangle$ admits a respresentation $(\lambda, M, \gamma)$ where $\lambda$ in an $n$-dimensional row vector over $D$, $\gamma$ is an $n$-dimensional column vector over $D$ and $M$ is an $n$ by $n$ full matrix over $D_F \langle\!\langle X \rangle\!\rangle$. Furthermore, each element of $M$ may be taken to be of degree $\leq 1$ in $X$, and

$$f = \lambda M^{-1} \gamma.$$

This representation is said to be of dimension $n$.

Such a representation is said to be *minimal* if $n$ is minimal among all representations of $f$.

Two representations $(\lambda, M, \gamma)$ and $(\lambda', M', \gamma')$ of dimension $n$ are *equivalent* if for some invertible matrices $P, Q \in GL_n(D)$ we have

$$\lambda' = \lambda P, M' = QMP, \gamma' = Q\gamma.$$

Theorem ([3], Theorem 4.3): Any two minimal representations are equivalent.

## 3.3 Reutenauer's Theorem

Let $M = [m_{i,j}]$ be an $n$ by $n$ matrix. We say that $M$ is *canonically invertible* if $M = UL$ where $U$ is upper triangular and invertible, and $L$ is lower triangular and invertible.We say that $M$ has a *height structure of height* $\leq l$ if there us a function $h : \{1, ..., n\} \to \{0, ..., l\}$ such that:

If $h(i) = 0$, then $m_{i,i} = 1$ and $m_{i,j} = 0$ for $j > i$.

If $h(i) \geq 1$, $S_i = \{j > i | h(j) \geq h(i)\} \neq \emptyset$ and $n(i) = min\ S_i$, then $m_{i,j} = 0$ for $j \geq n(i)$.

Theorem ([10], Theorem 2.2): Let $D$ be a skew field with infinite center $F$ and of infinite dimension over $F$. An element of $D_F\langle\!\langle X \rangle\!\rangle$ has height $\leq l$ if and only if it admits a minimal representation $(\lambda, M, \gamma)$ where $M$ is canonically invertible and has a height structure of height $\leq l$.

Using this result, Reutenauer proves that each entry of $[a_{i,j}]^{-1}$ has height $n$ in $F\langle\!\langle a_{i,j} | 1 \leq i, j \leq n \rangle\!\rangle$. Since each entry of this inverse matrix is the inverse of a quasideterminant of height $\leq n{-}1$ we see that the height of every quasideterminant of the $n$ by $n$ matrix $[a_{i,j}]$ has height $n - 1$.

# Bibliography


[1] Cohn, P. M. *Rings of fractions*, Amer. Math. Monthly, **78** (1971), 596–615.

[2] Cohn, P. M.*Free rings and their relations*, (1985), Academic Press, London.

[3] Cohn, P. M. and Reutenauer, C. *A normal form in free fields*, Can. J. Math. **46** (1994), 517-531 .

[4] Fisher, J. L. *Embedding freee algebras in skew fields*, Proc. Amer. Math. Soc., **30** (1971), 453–458.

[5] Jacobson, Nathan, *Lie algebras*, (1962),

[6] Jacobson, Nathan, *Structure and representations of Jordan algebras*, Colloquium Publications, Volume 39 (1968), Amer. Math. Soc., Providence, RI.

[7] Jategaonkar, Arun, *Ore donains and free algebras*, Bull. London Math. Soc. **1** (1969), 45–46.

[8] Malcev, A. I., *On the immersion of an algebraic ring into a field*, Math. Ann. **113** (1937), 686-691.

[9] Ore, O., *Linear equations in noncommutative fields*, Ann. of Math. **32** (1931), 463–477.

[10] Reutenauer, C. *Inversion height in free fields*, Selecta Math. (N.S.) **2** (1996), 93–109.