GLOBAL HOMOTOPY THEORY AND COHESION

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Abstract. We investigate the role of cohesion in unstable global equivariant homotopy theory.

Contents

1. Motivations

1.1. **Introduction.** **Global equivariant homotopy theory** is a way to amalgamate equivariant homotopy theory for "all" groups G into a single homotopy theory. Here "all groups" means something like "all compact Lie groups", though we could take it to mean a smaller class like "all compact abelian Lie groups" or "all finite groups".

The purpose of this document is to give a brief introduction to one definition, and to highlight the role of "cohesion" in relating ordinary equivariant homotopy theory with global equivariant homotopy theory.

Generally, I will work at the level of $(\infty,1)$**-categories**, here also called **homotopy theories**[1]. I will sometimes present a homotopy theory by giving a "relative category"; i.e., a pair (C,W) consisting of a category with a class of weak equivalences. Usually, I have a model category structure in mind as well, but I may not bother to mention it. At other times I'll present a homotopy theory by giving a category enriched over spaces. Phrases such as "colimit" usually really mean "homotopy colimit" (i.e., "$(\infty,1)$-categorical colimit"), unless otherwise indicated. I do not attempt to be especially precise about which $(\infty,1)$-categorical foundations are involved.

Sometimes I need to dig in and think about an honest category. (In particular, I'll mess around a bit with explicit topological groupoids.) Hopefully, it is clear when this is what

Date: October 11, 2014.

The author was supported under NSF grant DMS-1006054.

[1] I use the expressions "$(\infty,1)$-category" and "homotopy theory" synonymously, and generally prefer the latter.

I'm doing. However, the goal is always to give homotopy-meaningful statements, so that is the level at which I generally work, and unless otherwise stated, statements should be interpreted in this way.

I also assume the reader is fairly familiar with the theory of ∞-toposes. In particular, I will make much use of the property of "descent" in the homotopy theory of spaces, and in the homotopy theory of presheaves of spaces.

This document is evolving; in fact, it has changed substantially since the first version I posted. Some parts are unstable (both structurally and mathematically).

1.2. **Classical equivariant homotopy theory.** Recall that for any (compact Lie) group G,[2] we have a **G-equivariant homotopy theory**. We can take this to be the category GTop of spaces equipped with a G-action, in which a G-equivariant map $f\colon X \to Y$ is said to be a **G-equivariant equivalence** if for each closed subgroup H of G, the induced map

$$f^H\colon X^H \to Y^H$$

on H-fixed points is a weak equivalence of spaces. I'll write GTop for this homotopy theory. It is also modelled by the Top-enriched category whose objects are G-CW complexes.

The various equivariant homotopy theories can be related to each other via group homomorphisms. Thus, if $\phi\colon H \to G$ is a homomorphism of compact Lie groups, we obtain an "restriction" functor of homotopy theories

$$\phi^*\colon G\mathrm{Top} \to H\mathrm{Top},$$

where $\phi^* X$ is the H-space obtained by restriction the action of G on X along the homomorphism ϕ. Thus, we can think of $G \mapsto G$Top as a contravariant functor from the category of Lie groups to homotopy theories.

Note that if $\phi, \phi'\colon H \to G$ are homomorphisms which are conjugate via some $g \in G$, then there is an induced natural isomorphism of functors $c_g\colon \phi^* \xrightarrow{\sim} \phi'^*$. Thus we should properly think of $G \mapsto G$Top as a functor from the 2-category of Lie groups, homomorphisms, and intertwiners. We will realize this 2-category as a topologically enriched category, called Glo, the "global indexing category".

1.3. **Equivariant bundles.** Fix a compact Lie group Π. For any compact Lie group G, there is a notion of a **G-equivariant Π-principal bundle**. A Π-principal bundle over a G-space X is a $G \times \Pi$-space P together with a map $f\colon P \to X$ which is $G \times \Pi$-equivariant (using the trivial Π-action on X), and such that Π acts freely on P and $P/\Pi \to X$ is a homeomorphism.

For nice G-spaces (e.g., G-CW complexes) this notion is homotopy invariant, and thus we obtain a functor

$$P_\Pi^G\colon h(G\mathrm{Top})^{\mathrm{op}} \to \mathrm{Set},$$

which associates to a G-space X the set $P_\Pi^G(X)$ of equivalence classes of G-equivariant Π-principal bundles over X. The functor P_Π^G is represented by a G-space, namely the classifying space $B_G\Pi$ for G-equivariant Π-principal bundles.

The functors P_Π^G are related if we let G vary. For instance, if $\phi\colon H \to G$ is a homorphism, there is an evident map $P_\Pi^G(X) \to P_\Pi^H(\phi^* X)$ defined by restriction of the group action, and conjugate ϕ, ϕ' given rise to isomorphic maps. We may call such a collection $P_\Pi = \{P_\Pi^G\}$ of functors, together with the relevant restriction maps, a "global functor" (as in [Sch13a]).

[2] All groups will be compact Lie groups, except when they aren't.

We would like to be able to say that the global functor P_Π is represented by a "global space", which we will call $\mathbb{B}\Pi$.

Furthermore, any group homomorphism $\psi\colon \Pi \to \Pi'$ gives rise to maps $\psi_*\colon P_{G,\Pi} \to P_{G,\Pi'}$, which send a bundle $P \to X$ to the induced bundle $P \times_\Pi \Pi' \to X$, and hence to a map $\psi_*\colon P_\Pi \to P_{\Pi'}$ of global functors. We should expect that such maps are represented by maps of global spaces, which we would denote $\mathbb{B}\psi\colon \mathbb{B}\Pi \to \mathbb{B}\Pi'$.

1.4. **Global spaces.** The above discussion suggests that there should be a homotopy theory $\mathrm{Top}_{\mathrm{Glo}}$ of "global spaces", together with functors

$$\delta_G = \text{“}X \mapsto X/\!\!/G\text{”}\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}.$$

Then the global functor $P_\Pi = \{P_\Pi^G\}$ should extend to a functor $P_\Pi\colon h(\mathrm{Top}_{\mathrm{Glo}})^{\mathrm{op}} \to \mathrm{Set}$, so that

$$P_\Pi(X/\!\!/G) = P_\Pi^G(X).$$

Furthermore, the functor P_Π should be represented by a global space $\mathbb{B}\Pi$, and natural transformations $P_\Pi \to P_{\Pi'}$ between such functors should correspond to maps $\mathbb{B}\Pi \to \mathbb{B}\Pi'$ in $\mathrm{Top}_{\mathrm{Glo}}$.

In fact, it will turn out that $\mathbb{B}G \approx \delta_G(*) = */\!\!/G$, so that the global space corresponding to the terminal G-space is simultaneously the classifying space for P_Π. In particular, the functor $\delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}$ lifts to a functor $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$, and it turns out that the functor Δ_G will be fully-faithful, and will preserve homotopy limits and homotopy colimits.

The main goal of this note is to show that such a model of global homotopy theory exists, and that it comes with "cohesion" for each group G [Sch13b]. This amounts to an adjoint sequence $\Pi_G \dashv \Delta_G \dashv \Gamma_G \dashv \nabla_G$ of functors of the form

$$\begin{tikzcd} \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \arrow[d, shift right=4.5, "\Pi_G"'] \arrow[d, shift right=1.5, leftarrow, "\Delta_G"'] \arrow[d, shift left=1.5, "\Gamma_G"'] \arrow[d, shift left=4.5, leftarrow, "\nabla_G"'] \\ G\mathrm{Top} \end{tikzcd}$$

so that Δ_G and ∇_G are fully faithful, and Π_G preserves finite products. The functor Δ_G is the $X \mapsto X/\!\!/G$ functor (landing in the slice category over $*/\!\!/G \approx \mathbb{B}G$) indicated above. We note that in particular, $G\mathrm{Top}$ is thus equivalent (in two different ways!) to a full subcategory of the slice theory $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$.

1.5. **Orbispaces.** It turns out that the functors $\delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}$ have their values contained in a certain sub-homotopy theory $\mathrm{Top}_{\mathrm{Orb}}$, which we call the theory of "orbispaces". It is a non-full subtheory. For instance, the theory of orbispaces contains all the objects $\mathbb{B}G = \delta_G(*)$ described above, for all compact Lie groups G, but only contains morphisms $\mathbb{B}\phi\colon \mathbb{B}H \to \mathbb{B}G$ associated to *injective* homomorphisms $\phi\colon H \to G$ (while $\mathrm{Top}_{\mathrm{Glo}}$ contains $\mathbb{B}\phi$ for all homomorphisms ϕ.)

It turns out that the slice theory $\mathrm{Top}_{\mathrm{Orb}}/\mathbb{B}G$ is in fact equivalent to the theory of G-spaces. On the other hand, while the functors $P_\Pi^G\colon h(\mathrm{Top}_{\mathrm{Orb}})^{\mathrm{op}} \to \mathrm{Set}$ remain representable in $\mathrm{Top}_{\mathrm{Orb}}$, the representing objects are no longer the global spaces $\mathbb{B}G$.

We will show that for any orbispace X, there is an associated cohesion, which amounts to an adjoint sequence $\Pi_X \dashv \Delta_X \dashv \Gamma_X \dashv \nabla_X$ of functors with $\Delta_X\colon \mathrm{Top}_{\mathrm{Orb}}/X \to \mathrm{Top}_{\mathrm{Glo}}/X$ so that Δ_X and ∇_X are fully faithful and Π_G preserves finite products. When $X = \mathbb{B}G$, then

$\mathrm{Top}_{\mathrm{Orb}}/\mathbb{B}G \approx G\mathrm{Top}$, and this result recovers as a special case the cohesion $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ described above.

We note that there is a terminal orbispace $\mathcal{N}$ (which is *not* the same as the terminal global space), and thus in particular there is a cohesion $\mathrm{Top}_{\mathrm{Orb}} \to \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ relating orbispaces and global spaces sliced over $\mathcal{N}$. In particular, the theory of orbispaces is a full subtheory of global spaces sliced over $\mathcal{N}$.

Remark 1.5.1. The term "orbispace" comes from [GH07], where it is in fact used more generally; in particular, in their usage it might apply to either what we have called $\mathrm{Top}_{\mathrm{Glo}}$ and $\mathrm{Top}_{\mathrm{Orb}}$. It seems good to have separate terms for these concepts. Thus, I have stolen the term "orbispaces" to refer to objects of $\mathrm{Top}_{\mathrm{Orb}}$, while I use the term "global spaces" to refer to objects of $\mathrm{Top}_{\mathrm{Glo}}$, as this latter category seems to be the correct place to do "global homotopy theory". I realize this may cause some confusion; I am not strongly attached to this choice of terminology, and would be happy to consider an alternative.

1.6. **Models.** There are two kinds of models for the unstable global equivariant homotopy theory $\mathrm{Top}_{\mathrm{Glo}}$ that I know of: a presheaf model (as in Gepner-Henriques [GH07]), and the "orthogonal space model" of Schwede [Sch13a]. I'll describe the presheaf model $\mathrm{Top}_{\mathrm{Glo}}$ first, though in detail it won't be exactly the same as [GH07]. Then I'll briefly talk about the orthogonal space model $\mathbb{L}\mathrm{Top}$.

2. The category of compact Lie groups

Global spaces may be defined as presheaves of spaces on Glo, a topologically enriched category built from compact Lie groups. The idea is reminiscent of Elmendorff's theorem, which gives an equivalence of homotopy theories

$$G\mathrm{Top} \approx \mathrm{Top}_{\mathcal{O}_G}$$

between G-spaces and the homotopy theory of functors $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Top}$, where $\mathcal{O}$ is the orbit category of the group G.

Below I'll describe a model for Glo based on groupoids of representations. The category $\mathrm{Top}_{\mathrm{Glo}}$ of presheaves of spaces on the Top-enriched category Glo will be our basic model for global spaces.

2.1. **Topological groupoids.** Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids internal to Top. We write $\mathrm{Fun}(\mathcal{H},\mathcal{G})$ for the groupoid whose objects are (continuous) functors $\mathcal{H} \to \mathcal{G}$, and whose morphisms are natural transformations between such. There is an evident topology on the object and morphism sets of $\mathrm{Fun}(\mathcal{H},\mathcal{G})$ (given by the compact-open topology).

Given a topological groupoid $\mathcal{G}$, we write $B\mathcal{G}$ for the geometric realization of the nerve of $\mathcal{G}$ (which is itself a simplicial space).

For the most part, we will consider topological groupoids of a very special form, namely Lie groups G, and more generally action groupoids $G \curvearrowright X$ obtained from the action of a Lie group G on a space X.

2.2. **Representation groupoids and** Glo**.** Given Lie groups G and H, the functor groupoid $\mathrm{Fun}(H,G)$ has the form

$$\mathrm{Fun}(H,G) = \begin{cases} \mathbf{obj} & \mathrm{Hom}(H,G) \\ \mathbf{mor} & \mathrm{Hom}(H,G)\times G \end{cases} .$$

The objects are homomorphisms $H \to G$. An element $(\phi, g) \in \mathrm{Hom}(H,G) \times G$ is a morphism with source ϕ and target ψ, where $\psi(x) = g\phi(x)g^{-1}$; I'll write $g\colon \phi \to g\phi g^{-1}$ as a shorthand for ψ. The space $\mathrm{Hom}(H,G)$ is topologized with the compact-open topology.

Now let Glo be a category *enriched* over Top, whose objects are *compact* Lie groups, and whose morphism spaces are

$$\mathrm{Glo}(H,G) = B\,\mathrm{Fun}(H,G).$$

We can think of Glo as a "global indexing category".[3]

Remark 2.2.1. As a variant, we can consider a *topologically enriched* groupoid $\mathrm{Fun}'(H,G)$, whose objects are the *set* of homomorphisms $\phi\colon H \to G$, and whose morphisms $\phi \to \psi$ are the space $\{\, g \in G \mid \psi = g\phi g^{-1} \,\} \subseteq G$. Then we can define a topologically enriched category Glo' with $\mathrm{Glo}'(H,G) = B\,\mathrm{Fun}'(H,G)$.

As long as the Lie groups are *compact*, this doesn't make much difference. In fact, it is "well-known" that if H is compact, then sufficiently close homomorphisms $H \to G$ are conjugate ([CF64, Ch. VIII, Lemma 38.1]; see discussion at [MO13].) This means that under the conjugation G-action, $\mathrm{Hom}(H,G)$ decomposes (topologically) as a disjoint union of orbits; in fact, as a G-space,

$$\mathrm{Hom}(H,G) \approx \coprod_{[\phi]} G/C_G(\phi),$$

the coproduct taken over the set of conjugacy classes of homomorphisms. This implies that $B\,\mathrm{Fun}'(H,G)$ and $B\,\mathrm{Fun}(H,G)$ are weakly equivalent, and both are weakly equivalent to

$$\coprod_{[\phi]} BC_G(\phi).$$

Thus, $\mathrm{Top}_{\mathrm{Glo}} \approx \mathrm{Top}_{\mathrm{Glo}'}$.

If H is not compact, this kind of equivalence can fail badly. For instance, consider $B\,\mathrm{Fun}(\mathbb{R}, U(1))$ vs. $B\,\mathrm{Fun}'(\mathbb{R}, U(1))$; one is connected, while the other has uncountably many components.

2.3. **Relation to equivariant classifying spaces.** Recall that for an H-space X, we may consider the notion an **H-equivariant G-principal bundle** over X, which is a principal G-bundle $P \to X$ over X equipped with an continuous action of H on P covering the given action on X, so that the induced squares

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d] & P \arrow[d] \\ X \arrow[r, "h"'] & X \end{tikzcd}$$

[3] This is nearly the same as what in [GH07] is called Orb. Note that they don't restrict attention to compact Lie groups, and allow fairly general classes of isotropy groups. They also (sometimes) restrict attention to *injective* homomomorphisms, instead of all homomorphisms. A variant of Glo based on injective homomorphisms will also play a role and our story; we will call it Orb.

One more difference is that they use the "fat realization" of a simplicial space to construct the classifying space of a groupoid in Top. (The fat realization does not quotient out degeneracy relations, and thus has slightly better homotopy theoretic properties that the "thin realization", which does identify degeneracies; however, the thin realization is product-preserving on the nose, while the fat realization is not. As long as we stick to compact Lie groups, we should be able to use the thin realization.)

are G-bundle maps.

Write $\mathcal{P}_G(H \curvearrowright X)$ for the ∞-groupoid of H-equivariant principal G-bundles over X. (This can be modelled (up to some coherence issues) as a simplicial set, so that n-simplices are H-equivariant principal G-bundles over $X \times \Delta^n$.)

For compact Lie groups H and G, there is a universal equivariant classifying space $B_H G$, with the property that for nice X (e.g., G-CW complexes), we have

$$\mathcal{P}_G(H \curvearrowright X) \approx \mathrm{Map}_{H\mathrm{Top}}(X, B_H G).$$

We note that if $X = *$, then an H-equivariant principal G-bundle over X is nothing more than a G-representation. That is,

$$\mathcal{P}_G(H \curvearrowright *) \approx \mathrm{Map}_{H\mathrm{Top}}(*, B_H G) \approx B\,\mathrm{Fun}(H, G).$$

In other words, $\mathrm{Glo}(H, G) = B\,\mathrm{Fun}(H, G)$ defined above is equivalent to the space $(B_H G)^H$, the H-fixed points of the classifying space of H-equivariant G-principal bundles.

Remark 2.3.1. If H is compact Lie, and G is either a (i) finite or (ii) abelian compact Lie group (see [May90], or [LMS83] for the abelian case), then for a G-CW complex X we have[4]

$$\mathrm{Map}_{H\mathrm{Top}}(X, B_H G) \approx \mathrm{Map}_{\mathrm{Top}}(X_{hH}, BG),$$

where $X_{hH} = (X \times EH)_H$ is the usual homotopy orbit construction of the H-action on X. In particular, if H is compact Lie and G satisfies (i) or (ii) above, then the map

$$B\,\mathrm{Fun}(H, G) \to \mathrm{Map}(BH, BG),$$

which is defined so that a homomorphism $\phi\colon H \to G$ is sent to $B\phi\colon BH \to BG$, is a weak equivalence.

2.4. **Universal bundle model of** Glo**.** There is another model for Glo, obtained immediately from the construction of equivariant classifying spaces.

For each compact Lie group G, pick a universal principal G bundle $EG \to EG/G = BG$. Let O_{gl} be the topologically enriched category whose objects are compact Lie groups, and whose morphism spaces are

$$O_{\mathrm{gl}}(G, H) = [\mathrm{Top}(EG, EH)/H]^G.$$

That is, a point of $O_{\mathrm{gl}}(G, H)$ is an equivalence class of maps $f\colon EG \to EH$ such that for each $g \in G$, there exists $h \in H$ such that $f \cdot g = h \cdot f$. Because H acts freely on EH, the stabilizer group $\Lambda(f) \leq G \times H$ of such an f must be the graph of a continuous homomorphism $\phi\colon G \to H$. Composition is defined evidently. It can be shown [May90] that $O_{\mathrm{gl}}(G, H)$ is a model for the equivariant classifying space $B_G H$.

2.5. **Linear isometry model of** Orb**.** A variant of the universal bundle model for Glo built from linear isometries is the category Schwede calls $\mathbf{O}_{\mathrm{gl}}$ [Sch13a, I.6.13].

Schwede's model for $\mathbf{O}_{\mathrm{gl}}$ is obtained as follows. To describe this, let $\mathcal{L} = \mathbb{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ be the topological monoid of endomorphisms of the real inner product space $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$. A **universal subgroup** [Sch13a, I.6.1] of $\mathcal{L}$ is a closed subgroup $G \subset \mathcal{L}$ which (i) admits the structure of a Lie group, and (ii) is such that the induced G action on $\mathbb{R}^\infty$ makes it into a complete G-universe [Sch13a, I.1.5] (i.e., is isomorphic to a direct sum of countably many copies of each irreducible orthogonal G-representation).

[4] It seems likely to me that the result holds when G is any extension of a finite group by a compact torus, but I don't know a reference or a proof.

The objects of $\mathbf{O}_{\mathrm{gl}}$ are the universal subgroups of $\mathcal{L}$. The morphism space $\mathbf{O}_{\mathrm{gl}}(G, H)$ is

$$\mathbf{O}_{\mathrm{gl}}(G, H) = (\mathcal{L}/H)^G \approx \mathrm{Map}_{\mathcal{L}}(\mathcal{L}/G, \mathcal{L}/H),$$

a space of maps of spaces with left $\mathcal{L}$-action. Every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$ [Sch13a, I.6.2]. The space $(\mathcal{L}/H)^G$ is a model for the equivariant classifying space $B_G H$ (see [Sch13a, A.2.5]).

3. Global spaces

The model for global homotopy that we consider is simply that of presheaves of spaces on Glo.

3.1. **Global spaces as presheaves on** Glo**.** Let $\mathrm{Top}_{\mathrm{Glo}}$ denote the category of Top-enriched functors $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Top}$. We call such objects **global spaces**. We say that a map $f\colon X \to Y$ of global spaces is an **weak equivalence** if $f(G)\colon X(G) \to Y(G)$ is a weak equivalence for every group G.

Each G gives a representable functor,

$$\mathbb{B}G\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Top}, \qquad \mathbb{B}G(H) = \mathrm{Glo}(H, G),$$

and thus we obtain a fully faithful Yoneda embedding $\mathbb{B}\colon \mathrm{Glo} \to \mathrm{Top}_{\mathrm{Glo}}$.

3.2. **Constructing global spaces from G-spaces: functors δ_G and Δ_G.** Next we build a functor $\delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Orb}}$ for each compact Lie group G, which associates to each G-space a global space. The functor δ_G will not in any sense be an inclusion (i.e., fully faithful). However, the value of δ_G on the terminal object 1 of $G\mathrm{Top}$ will be $\mathbb{B}G$. Therefore, our functor δ_G will immediately give rise to a functor

$$\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Orb}}/\mathbb{B}G$$

to the slice category of global space over $\mathbb{B}G$, whose value at a G-space X will be the map $\delta_G(X) \to \delta_G(1) = \mathbb{B}G$. As we will see below, the functor Δ_G will induce a full embedding of the homotopy of G-spaces inside the slice homotopy theory $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$.

For a general G-space X, the object $\delta_G X$ should be some kind of "fiber bundle" over $\mathbb{B}G$, whose "fibers" (over points of each space $(\mathbb{B}G)(H)$) are fixed point spaces of the action of some closed subgroup of G on X. For instance, suppose $\phi \in (\mathbb{B}G)(H) = B\,\mathrm{Fun}(H, G)$ is a point corresponding to an actual homomorphism $\phi\colon H \to G$. We would like the fiber of

$$(\delta_G X)(H) \to B\,\mathrm{Fun}(H, G) \qquad \text{over } \phi$$

to be the space $X^{\phi(H)}$ of points in X fixed by the subgroup $\phi(H)$ of G. (Observe that $\phi(H) \leq G$ is a closed subgroup, since H is compact.)

We define

$$(\delta_G X)(H) := B\,\mathrm{Fun}(H, G \curvearrowright X),$$

the classifying space of the topological groupoid of functors to the action groupoid of G acting on X. Set theoretically this has the form

$$\mathrm{Fun}(H, G \curvearrowright X) = \begin{cases} \mathbf{obj} & \coprod_{\phi \in \mathrm{Hom}(H,G)} X^{\phi(H)} \\ \mathbf{mor} & \coprod_{\phi \in \mathrm{Hom}(H,G)} X^{\phi(H)} \times G. \end{cases}$$

The objects and morphisms are topologized as subspaces of $\mathrm{Hom}(H, G) \times X$ and $\mathrm{Hom}(H, G) \times X \times G$ respectively.

A morphism $(\phi, x) \to (\phi', x')$ in $\mathrm{Fun}(H, G \curvearrowright X)$, where $\phi, \phi' \colon H \to G$ and $x \in X^{\phi(H)}$ and $x' \in X^{\phi'(H)}$, is thus a $g \in G$ such that $\phi' = g\phi g^{-1}$ and $x' = gx$. The map $\coprod_{\phi \in \mathrm{Hom}(H,G)} X^{\phi(H)} \to \mathrm{Hom}(H, G)$ is actually a fiber bundle: under the isomorphism $\mathrm{Hom}(H, G) \approx \coprod_{[\phi]} G/C_G(\phi)$, the restriction of this bundle over $G/C_G(\phi)$ is $G \times_{C_G(\phi)} X^{\phi(H)} \to G/C_G(\phi)$.

Observe that after taking classifying spaces, the map

$$\pi \colon B\,\mathrm{Fun}(H, G \curvearrowright X) \to B\,\mathrm{Fun}(H, G)$$

remains a fiber bundle.

We thus have a topologically enriched functor

$$\delta_G \colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}, \qquad \delta_G(X)(H) \overset{\mathrm{def}}{=} B\,\mathrm{Fun}(H, G \curvearrowright X),$$

where the action on morphisms is defined via the evident composition functor $\mathrm{Fun}(H, G \curvearrowright X) \times \mathrm{Fun}(H', H) \to \mathrm{Fun}(H', G \curvearrowright X)$, which on objects sends $((\phi, x), \psi)$ to $(\phi\psi, x)$.

We also obtain a topologically enriched functor

$$\Delta_G \colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G, \qquad \Delta_G(X)(H) \overset{\mathrm{def}}{=} \big(\pi \colon B\,\mathrm{Fun}(H, G \curvearrowright X) \to B\,\mathrm{Fun}(H, G)\big).$$

Both δ_G and Δ_G take G-equivariant weak equivalences to weak equivalences of orbispaces.

3.3. **Compatibility of δ functors with induction.** Next, we note that the δ functors are compatible with "inducing up" from subgroups, in the following sense.

Proposition 3.3.1. *If $H \subseteq G$ is a closed subgroup*[5] *and X is a H-space, there is a weak equivalence $\delta_H(X) \to \delta_G(G \times_H X)$, natural in X. In particular, taking $X = *$ we find $\Delta_G(G/H) \approx (\mathbb{B}\iota_H \colon \mathbb{B}H \to \mathbb{B}G)$.*

Proof. This amounts to showing that the evident map $B\,\mathrm{Fun}(K, H \curvearrowright X) \to B\,\mathrm{Fun}(K, G \curvearrowright G \times_H X)$ is a weak equivalence for any Lie group K. The key calculation is the case of $X = *$, and to prove this case, note that for a homomorphism $\phi \colon K \to G$, it is straightforward to calculate that the homotopy fiber of $B\,\mathrm{Fun}(K, H) \to B\,\mathrm{Fun}(K, G)$ over ϕ is equivalent to $(G/H)^{\phi(K)}$. □

3.4. **Faithful morphisms of global spaces.** Say that a map $f \colon X \to Y$ in $\mathrm{Top}_{\mathrm{Glo}}$ is **faithful**[6] if for every compact Lie group G, and every closed normal subgroup $N \subseteq G$, the resulting square

$$\begin{array}{ccc} X(G/N) & \longrightarrow & X(G) \\ \downarrow & & \downarrow \\ Y(G/N) & \longrightarrow & Y(G) \end{array}$$

is a homotopy pullback of spaces.

We can characterize the faithful maps in the following way.

Proposition 3.4.1. *The faithful maps in $\mathrm{Top}_{\mathrm{Glo}}$ are precisely those which are right-orthogonal (in the $(\infty,1)$-categorical sense) to the set of maps $\{\mathbb{B}\phi\}$ where ϕ ranges over all surjective homomorphisms of compact Lie groups.*

Proof. This is a straightforward translation of the definition. □

[5] All subgroup we consider in this note will be closed.

[6] Alternately, we could say that f is **representable**. Both terms are used in [GH07] for notions like this.

We write $\mathrm{Top}_{\mathrm{Glo}}^{\mathrm{faith}}$ for the (non-full) subtheory of $\mathrm{Top}_{\mathrm{Glo}}$ consisting of all objects and all faithful maps. The derived mapping space $\mathrm{Map}_{\mathrm{Top}_{\mathrm{Glo}}^{\mathrm{faith}}}(X,Y)$ is equivalent to the union of path components of $\mathrm{Map}_{\mathrm{Top}_{\mathrm{Glo}}}(X,Y)$ which contain faithful maps.

Example 3.4.2. If $\phi\colon G \to H$ is a homomorphism of compact Lie groups, then $\mathbb{B}\phi\colon \mathbb{B}G \to \mathbb{B}H$ is faithful if and only if ϕ is injective.

Example 3.4.3. If X is a G-space, then the projection $\pi\colon \delta_G X \to \mathbb{B}G$ is a faithful map. To see this, observe that for a group H, the homotopy fiber of $(\delta_G X)(H) \to \mathbb{B}G(H)$ over a point corresponding to a homomorphism $\phi\colon H \to G$ is $X^{\phi(H)}$, which only depends on the image of ϕ inside G.

Exercise 3.4.4. Let X be a G-CW complex. Show that $\delta_G X \to *$ is a faithful map in $\mathrm{Top}_{\mathrm{Glo}}$ if and only if G acts freely on X.

We record the following properties of faithful maps, whose proof is straightforward.

Proposition 3.4.5.

(1) *The homotopy pullback of a faithful map is faithful.*
(2) *If $f\colon Y \to X$ is faithful, then $g\colon Z \to Y$ is faithful if and only if gf is faithful.*
(3) *Let $f\colon X \to Y$ be a natural transformation of functors $X, Y\colon \mathcal{C} \to \mathrm{Top}_{\mathrm{Glo}}$, such that each $f(c)\colon X(c) \to Y(c)$ is faithful. Then* $\mathrm{holim}_{\mathcal{C}} f$ *is faithful.*

3.5. $G\mathrm{Top}$ **corresponds to faithful objects of** $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$**.** Now we will show that the functor $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ is homotopically fully faithful, and thus embeds the homotopy theory of G-spaces inside the homotopy theory of global spaces over $\mathbb{B}G$. Furthermore, we identify the essential image of Δ_G as consisting of faithful morphisms to $\mathbb{B}G$.

Proposition 3.5.1. *The functor $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ between homotopy theories commutes with homotopy limits* and *homotopy colimits, and gives an equivalence between G-spaces and the full sub-homotopy-theory* $(\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G)_{\mathrm{faith}}$ *of* $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ *whose objects are faithful morphisms to $\mathbb{B}G$.*

Proof. By construction, the functor Δ_G factors as a composite

$$G\mathrm{Top} \xrightarrow{\sim} \mathrm{Top}_{\mathcal{O}_G} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G.$$

The first functor in the composite (obtained by taking fixed points) is an equivalence by Elmendorff's theorem, while the second functor preserves both homotopy limits and colimits, since it does so fiberwise.

We can describe a right adjoint $\Gamma_G\colon \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \to G\mathrm{Top}$ via a composite

$$\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \xrightarrow{\Gamma'_G} \mathrm{Top}_{\mathcal{O}_G} \xrightarrow{\sim} G\mathrm{Top},$$

where the second functor is Elmendorff's equivalence. The first functor Γ'_G may be defined by

$$(\Gamma'_G(X))(G/H) = (\mathrm{Top}_{\mathrm{Orb}}/\mathbb{B}G)(\Delta_G(G/H), X) \approx (\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G)(\mathbb{B}\iota_H, X),$$

where the last equivalence is by (3.3.1). Thus for a subgroup $H \leq G$, the space $\Gamma_G(X)^H$ is naturally equivalent to the homotopy fiber of $X(H) \to B\,\mathrm{Fun}(H,G)$ over ι_H.

It follows immediately that the (derived) counit map $\Gamma_G\Delta_G \to \mathrm{Id}_{\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G}$ is a weak equivalence, since the homotopy fiber of $\Delta_G T\colon \delta_G T \to B\,\mathrm{Fun}(H,G)$ over ι_H is the fiber, which by construction is T^H.

On the other hand, we see that $\mathrm{Id}_{\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G} \to \Delta_G\Gamma_G(X)$ is an equivalence if and only if X is faithful over $\mathbb{B}G$. This gives the desired equivalence of homotopy theories. □

We obtain as a corollary the following alternative characterization of faithful maps.

Proposition 3.5.2. *A map $f\colon X \to Y$ in $\mathrm{Top}_{\mathrm{Glo}}$ is faithful if and only if for every compact Lie group G and every map $y\colon \mathbb{B}G \to Y$, the homotopy pullback $X_y \to \mathbb{B}G$ of f along y is weakly equivalent in $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ to an object of the form $\Delta_G(T)$ for some G-space T.*

Proof. Straightforward. □

3.6. **The functor γ_G.** Note that the functor $\delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}$ is not itself homotopically fully faithful. However, it does have a homotopical right adjoint $\gamma_G\colon \mathrm{Top}_{\mathrm{Glo}} \to G\mathrm{Top}$, with the property that

$$(\gamma_G X)^K \approx X(K)$$

for a closed subgroup $K \subseteq G$. This is an immediate consequence of the identification $\delta_G(G/H) \approx \mathbb{B}H$.

In particular, note that for a group H and a subgroup $K \leq G$,

$$(\gamma_G \mathbb{B}H)^K \approx B\,\mathrm{Fun}(K, H).$$

This means that $\gamma_G\mathbb{B}H \approx B_G H$, the G-equivariant classifying space of H. As a consequence, we see that for a G-space X,

$$\mathrm{Map}_{\mathrm{Top}_{\mathrm{Glo}}}(\delta_G(X), \mathbb{B}H)$$

is weakly equivalent to the classifying space $\mathcal{P}_H(G \curvearrowright X)$ of G-equivariant H-principal bundles over X.

The unit map $X \to \gamma_G\delta_G(X) \approx B_G G$ of the adjunction, applied to a G-space X, classifies the G-equivariant G-principal bundle $\pi_1\colon X \times G \to X$, in which the $G \times G$-action on the total space is $(u, v) \cdot (x, g) = (ux, ugv^{-1})$.

Exercise 3.6.1. Let X be a G-space, and let $P \to G$ be a G-equivariant H-principal bundle over X, which is classified by a map $f\colon \delta_G X \to \mathbb{B}H$. Show that the map f is faithful if and only if G acts freely on P.

3.7. **The functor ∂_G.** Finally, we note that the functor $\gamma_G\colon \mathrm{Top}_{\mathrm{Orb}} \to G\mathrm{Top}$ itself admits a homotopical right adjoint $\partial_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Orb}}$, with the property that

$$(\partial_G X)(H) \approx \mathrm{Map}_{G\mathrm{Top}}(B_G H, X).$$

Altogether, we have functors

$$\begin{tikzcd} G\mathrm{Top} \arrow[d, shift right=6, "\delta_G"'] \arrow[d, shift left=6, "\partial_G"] \\ \mathrm{Top}_{\mathrm{Orb}} \arrow[u, "\gamma_G"'] \end{tikzcd}$$

4. Orbispaces

In this section we identify an interesting class of global spaces, called "orbispaces", which includes all global spaces obtained from spaces equipped with actions by a compact Lie group. There is a corresponding homotopy theory consisting of the orbispaces and the faithful maps between them, which turns out itself to be modelled as presheaves on a subcategory Orb of Glo.

4.1. **The normal subgroup classifier.** Define $\mathcal{N}$ in $\mathrm{Top}_{\mathrm{Glo}}$ by

$$\mathcal{N}(G) = \{\text{normal subgroups of } G\}.$$

The right-hand side is a set viewed as a discrete space. A homomorphism $\phi\colon G \to H$ induces the map $\mathcal{N}(\phi)$ which sends $N \trianglelefteq H$ to $\phi^{-1}N \trianglelefteq G$. The global space $\mathcal{N}$ is called the **normal subgroup classifier**.

Exercise 4.1.1. Show that $\mathrm{Map}_{\mathrm{Top}_{\mathrm{Glo}}}(\mathcal{N}, \mathbb{B}G) \approx BG$.

4.2. **Orbispaces.** We say that a global space X is an **orbispace** if there *exists* a *faithful* map $X \to \mathcal{N}$ to the normal subgroup classifier.

Proposition 4.2.1. *If Y is a global space such that there exists a faithful map $Y \to X$ to an orbispace X, then Y is an orbispace.*

Proof. Immediate from (3.4.5)(2). □

Example 4.2.2. The normal subgroup classifier is tautologically an orbispace.

Example 4.2.3. For any compact Lie group G, the global space $\mathbb{B}G$ is an orbispace. To see this, consider the map $I\colon \mathbb{B}G \to \mathcal{N}$ which classifies the trivial subgroup of G. Explicitly, I sends a point in $\mathbb{B}G(H)$ corresponding to a homomorphism $\phi\colon H \to G$ to the subgroup $\operatorname{Ker}\phi \trianglelefteq H$. The map I is a faithful map, a fact which amounts to the observation that if $\phi\colon H \to G$ is a homomorphism and $N \trianglelefteq H$ is such that $N \subseteq \operatorname{Ker}\phi$, then ϕ factors through the quotient H/N. This observation also shows that I is the unique faithful map $\mathbb{B}G \to \mathcal{N}$.

Example 4.2.4. By (4.2.1), it follows that $\delta_G(T)$ is an orbispace for any compact Lie group G and G-space T.

4.3. **The inertia map.** When X is an orbispace, any "G-point" of X factors uniquely through a "G/N-point" which is itself represented by a faithful map.

Proposition 4.3.1. *Suppose $f\colon X \to \mathcal{N}$ is a faithful map of global spaces. Let $x\colon \mathbb{B}G \to X$ be map and let $N \trianglelefteq G$ be a normal subgroup, with $\pi\colon G \to G/N$ the quotient homomorphism. There exists a factorization of x of the form*

$$\mathbb{B}G \xrightarrow{\mathbb{B}\pi} \mathbb{B}(G/N) \xrightarrow{\overline{x}} X$$

in which $\overline{x}$ is a faithful map, if and only if $N = f(x)$.

Proof. First, suppose such a factorization of x through some faithful $\overline{x}\colon \mathbb{B}(G/N) \to X$ exists. Then the composite $f \circ \overline{x}\colon \mathbb{B}(G/N) \to \mathcal{N}$ is faithful by (3.4.5)(2), and thus classifies the trivial subgroup of G/N, whence $f(x) = N$.

Conversely, let $N = f(x)$. Then there is a commutative square

$$\begin{tikzcd} \mathbb{B}G \arrow[r, "x"] \arrow[d, "\mathbb{B}\pi"'] & X \arrow[d, "f"] \\ \mathbb{B}(G/N) \arrow[r, "\{1\}"'] \arrow[ur, dotted, "\overline{x}"] & \mathcal{N} \end{tikzcd}$$

As faithful maps are right orthogonal to $\mathbb{B}\pi$ when π is surjective, a lift $\overline{x}$ exists making the diagram commute. As f and $\{1\}$ are faithful, it follows that $\overline{x}$ is faithful by (3.4.5)(2). □

Given an orbispace X and a map $x\colon \mathbb{B}G \to X$, the **inertia group** of x is the unique normal subgroup $I_X(x) \trianglelefteq G$ such that x factors through a faithful map $\mathbb{B}(G/I_X(x)) \to X$. It is straightforward to check that if $\phi\colon H \to G$ is a homomorphism, then $I_X(x \circ \mathbb{B}\phi) = \phi^{-1}(I_X(x))$. Thus for any orbispace X we may define its **inertia map** $I_X\colon X \to \mathcal{N}$ by $x \mapsto I_X(x)$ for $x \in X(G)$. The previous proposition implies that for an orbispace X, any faithful map $X \to \mathcal{N}$ coincides with the inertia map I_X, and thus the inertia map is itself faithful. We can summarize this discussion as follows.

Proposition 4.3.2. *Let X be an arbitrary global space. Then the space $\mathrm{Map}^{\mathrm{faith}}_{\mathrm{Top}_{\mathrm{Glo}}}(X, \mathcal{N})$ of faithful maps to $\mathcal{N}$ is a (-1)-type (i.e., is either empty or contractible). It is non-empty (and hence contractible) exactly when X is an orbispace, in which case the inertia map $I_X\colon X \to \mathcal{N}$ exhibits a point of $\mathrm{Map}^{\mathrm{faith}}_{\mathrm{Top}_{\mathrm{Glo}}}(X, \mathcal{N})$.*

We have the following criterion for a global space X to be an orbispace.

Proposition 4.3.3. *Let X be an arbitrary global space. The following are equivalent.*

1. *X is an orbispace.*
2. *For every $x\colon \mathbb{B}G \to X$, there exists a factorization up to homotopy of x of the form $\mathbb{B}G \to \mathbb{B}(G/N) \xrightarrow{\overline{x}} X$ where $\overline{x}$ is faithful.*
3. *For every $x\colon \mathbb{B}G \to X$, there exists a factorization up to homotopy of x of the form $\mathbb{B}G \xrightarrow{y} Y \xrightarrow{f} X$ where Y is an orbispace and f is faithful.*

Proof. Direction (1) $\Longrightarrow$ (2) follows from (4.3.1), and direction (2) $\Longrightarrow$ (3) is immediate since $\mathbb{B}(G/N)$ is an orbispace.

To show that (2) $\Longrightarrow$ (1), note that the factorization whose existence is proclaimed by (2) is essentially unique, by (3.4.1). We may use this to associate to each $x\colon \mathbb{B}G \to X$ the unique normal subgroup $I(x) \trianglelefteq G$ such that x factors through a faithful $\overline{x}\colon \mathbb{B}(G/I(x)) \to X$, which thus defines a map $I\colon X \to \mathcal{N}$, which is easily seen to be faithful using the uniqueness of the factorization. It follows that X is an orbispace.

To show that (3) $\Longrightarrow$ (2), note that given $x\colon \mathbb{B}G \to X$, we can factor it through some $\mathbb{B}G \xrightarrow{y} Y \xrightarrow{f} X$ with f faithful by (3), and we can then factor y through $\mathbb{B}G \to \mathbb{B}(G/I_Y(y)) \xrightarrow{\overline{y}} Y$ with $\overline{y}$ faithful by (4.3.1). Therefore $f\overline{y}\colon \mathbb{B}(G/I_Y(y)) \to X$ is a faithful (3.4.5) factorization of x. $\square$

4.4. **The homotopy theory of orbispaces.** We can now define a homotopy theory $\mathrm{Top}^{\mathrm{orbi}}_{\mathrm{Glo}}$ of orbispaces, as the topologically enriched category, whose objects are the orbispaces, and whose morphisms spaces are the spaces $\mathrm{Map}^{\mathrm{faith}}_{\mathrm{Top}_{\mathrm{Glo}}}(X, Y)$ of faithful morphisms (viewed as a subspace of the derived mapping space in $\mathrm{Top}_{\mathrm{Glo}}$). Thus, $\mathrm{Top}^{\mathrm{orbi}}_{\mathrm{Glo}} \subset \mathrm{Top}_{\mathrm{Glo}}$ can be regarded as a non-full sub-theory.

In view of (3.4.5)(2) and (4.3.2), there is an equivalence

$$(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}} \xrightarrow{\sim} \mathrm{Top}^{\mathrm{orbi}}_{\mathrm{Glo}}$$

between the theory of orbispaces and the full subtheory $(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}}$ of the slice category $\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ whose objects are precisely the faithful morphisms $X \to \mathcal{N}$. On objects, the equivalence sends a faithful map $X \to \mathcal{N}$ of global spaces to the orbispace X.

Thus, the theory of orbispaces can be seen (in view of (3.5.1)) as an analogue of the homotopy theory of G-spaces, with the role of $\mathbb{B}G$ replaced by $\mathcal{N}$.

4.5. **The global orbit category** Orb**.** Let $\mathrm{Orb} \subset \mathrm{Glo}$ denote the subtheory consisting of all objects, and injective homomorphisms. That is, $\mathrm{Orb}(G,H) = B\,\mathrm{Fun}_{\mathrm{inj}}(G,H)$, where $\mathrm{Fun}_{\mathrm{inj}}(G,H) \subseteq \mathrm{Fun}(G,H)$ is the full subgroupoid whose objects are injective homomorphisms. Write $\mathrm{Top}_{\mathrm{Orb}}$ for the category of presheaves on Orb, and write $\mathbb{B}_{\mathrm{Orb}}\colon \mathrm{Orb} \to \mathrm{Top}_{\mathrm{Orb}}$ for the Yoneda embedding. We refer to Orb as the **global orbit category**.[7]

Let $i\colon \mathrm{Orb} \to \mathrm{Glo}$ denote the evident inclusion functor. It gives rise to a series of homotopically adjoint functors

$$\begin{tikzcd} \mathrm{Top}_{\mathrm{Glo}} \arrow[d, "\gamma_{\mathrm{Orb}}"] \\ \mathrm{Top}_{\mathrm{Orb}} \arrow[u, bend left=60, "\delta_{\mathrm{Orb}}"] \arrow[u, bend right=60, "\partial_{\mathrm{Orb}}"'] \end{tikzcd}$$

The functor γ_{Orb} is defined by restriction along i, so that

$$(\gamma_{\mathrm{Orb}}X)(G) = X(iG).$$

The functor δ_{Orb} is defined by left Kan extension along i. It may be described explicitly on objects by

$$(\delta_{\mathrm{Orb}}X)(G) = \coprod_{N \trianglelefteq G} X(G/N).$$

A homomorphism $\phi\colon H \to G$ sends the summand $X(G/N) \subset (\delta_{\mathrm{Orb}}X)(G)$ to $X(H/\phi^{-1}N) \subset (\delta_{\mathrm{Orb}}X)(H)$ via the induced homomorphism $H/\phi^{-1}N \to G/N$. Note that $\delta_{\mathrm{Orb}}\mathbb{B}_{\mathrm{Orb}}G \approx \mathbb{B}G$.

The functor ∂_{Orb} is defined by right Kan extension along i. It satisfies $(\partial_{\mathrm{Orb}}X)(G) \approx \mathrm{Map}_{\mathrm{Top}_{\mathrm{Orb}}}(\gamma_{\mathrm{Orb}}\mathbb{B}G, X)$.

4.6. **Orbispaces are presheaves on** Orb**.** Note that $\delta_{\mathrm{Orb}}(1) \approx \mathcal{N}$. It is straightforward to check that for any X in $\mathrm{Top}_{\mathrm{Orb}}$, the map $\delta_{\mathrm{Orb}}X \to \delta_{\mathrm{Orb}}1 \approx \mathcal{N}$ is faithful. Thus, δ_{Orb} lifts to a functor

$$\Delta_{\mathrm{Orb}}\colon \mathrm{Top}_{\mathrm{Orb}} \to \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N},$$

whose image lies in the full subcategory $(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}}$. (Likewise, the functor $\delta_{\mathrm{Orb}}\colon \mathrm{Top}_{\mathrm{Orb}} \to \mathrm{Top}_{\mathrm{Glo}}$ takes values in the subcategory $\mathrm{Top}^{\mathrm{orbi}}_{\mathrm{Glo}} \subset \mathrm{Top}_{\mathrm{Glo}}$ of orbispaces.)

The functor Δ_{Orb} admits a right adjoint $\Gamma_{\mathrm{Orb}}\colon \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N} \to \mathrm{Top}_{\mathrm{Orb}}$, which can be computed by

$$(\Gamma_{\mathrm{Orb}}X)(G) \approx X(G;1),$$

where the space $X(G;N) \subseteq X(G)$ is the preimage of $I_X(G)\colon X(G) \to \mathcal{N}(G)$ over $N \in \mathcal{N}(G)$.

Proposition 4.6.1. *The functor* $\Delta_{\mathrm{Orb}}\colon \mathrm{Top}_{\mathrm{Orb}} \to \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ *is fully faithful, and its essential image is precisely* $(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}} \approx \mathrm{Top}^{\mathrm{orbi}}_{\mathrm{Glo}}$.

Proof. It is immediate from the above descriptions of the adjoint pair $\Delta_{\mathrm{Orb}}\colon \mathrm{Top}_{\mathrm{Orb}} \rightleftarrows \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N} : \Gamma_{\mathrm{Orb}}$ that the unit map $X \to \Gamma_{\mathrm{Orb}}\Delta_{\mathrm{Orb}}X$ is an equivalence for all X in $\mathrm{Top}_{\mathrm{Orb}}$. Conversely, for Y in $\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$, the counit map $\Delta_{\mathrm{Orb}}\Gamma_{\mathrm{Orb}}Y \to Y$ is given by

$$\Delta_{\mathrm{Orb}}\Gamma_{\mathrm{Orb}}Y(G) \approx \coprod_{N \trianglelefteq G} Y(G/N;1) \to \coprod_{N \trianglelefteq G} Y(G;N) \approx Y(G),$$

induced by the quotient maps $G \to G/N$. It is straightforward to check that $Y \to \mathcal{N}$ is faithful if and only if $Y(\phi)\colon Y(G;N) \to Y(H;\phi^{-1}N)$ is an equivalence for every surjective homomorphism $\phi\colon H \to G$ and normal $N \trianglelefteq G$, which holds if and only if $Y(G;N) \to$

[7] Recall that we call Glo that "global indexing category".

$Y(G/N;1)$ is an equivalence for every pair $N \trianglelefteq G$. Thus $\Delta_{\mathrm{Orb}}\Gamma_{\mathrm{Orb}}Y \to Y$ is an equivalence if and only if $Y \to \mathcal{N}$ is faithful, as desired. □

5. Cohesion

Cohesion is a category theoretic notion due to Lawvere, which has been generalized to the ∞-categorical setting by Schreiber [Sch13b]. By a **cohesion**, we mean a collection of functors

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\Pi"', shift right=6] \arrow[d, "\Gamma", shift left=2] \\ \mathcal{T} \arrow[u, "\Delta"', shift right=2] \arrow[u, "\nabla"', shift left=6] \end{tikzcd}$$

between homotopy theories, equipped with homotopical adjunctions $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla$ (so that Π is left adjoint to Δ, etc.), such that

(1) the natural unit and counit maps
$$\Pi \circ \Delta \to \mathrm{Id}_{\mathcal{T}} \to \Gamma \circ \Delta, \qquad \mathrm{Id}_{\mathcal{T}} \to \Gamma \circ \nabla$$
are equivalences, and

(2) the functor Π preserves finite products (up to weak equivalence).

Note that (1) is equivalent to saying that both Δ and ∇ are homotopically fully-faithful. As a consequence, $\mathcal{T}$ is equivalent to a full sub-homotopy-theory of $\mathcal{C}$ in two different ways. A brief notation for a cohesion is to just give the fully-faithful functor $\Delta\colon \mathcal{T} \to \mathcal{C}$, which in fact determines the rest of the structure.

Typically in a cohesion, both $\mathcal{T}$ and $\mathcal{C}$ are ∞-topoi. When this is the case, and $\mathcal{T} \approx \mathrm{Top}$, then we say that $\mathcal{C}$ is a **cohesive ∞-topos**. The motivation for cohesion comes from attempting to formalize notions of geometry based on collections of "points" which "cohere" via some kind of geometric structure[8].

Example 5.0.2. A familiar (to homotopy theorists) example of a cohesive ∞-topos is that of simplicial spaces. Write Top_Δ for the homotopy theory of functors $\Delta^{\mathrm{op}} \to \mathrm{Top}$. In this case, the functors

$$\begin{tikzcd} \mathrm{Top}_\Delta \arrow[d, "\Pi"', shift right=6] \arrow[d, "\Gamma", shift left=2] \\ \mathrm{Top} \arrow[u, "\Delta"', shift right=2] \arrow[u, "\nabla"', shift left=6] \end{tikzcd}$$

are as follows.

- Π is the homotopy colimit functor, which can be computed by geometric realization, which is finite product preserving.
- Δ is the constant simplicial object functor, sending T to $(\Delta T)([n]) = T$.
- Γ is the evaluation at $[0]$ functor, sending X to $\Gamma X = X([0])$; it is equivalent to the homotopy limit functor.
- ∇ is the functor given by $(\nabla T)([n]) = \mathrm{Map}([n], T)$, where the finite set $[n]$ is regarded as a discrete space.

[8]See [Sch14]. I don't actually understand this motivation, which doesn't seem to relate easily to the sorts of cohesion which appear in this note.

We are going to show that for any orbispace X we obtain a cohesion of the form

$$\begin{tikzcd} \mathrm{Top}_{\mathrm{Glo}}/X \arrow[d, shift right=6, "\Pi_X"'] \arrow[d, phantom] \arrow[d, shift left=6, "\Gamma_X"] \\ (\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}} \arrow[u, shift right=2, "\Delta_X"'] \arrow[u, shift left=10, "\nabla_X"'] \end{tikzcd}$$

associated to the full subtheory of the slice theory over X whose objects are precisely the faithful maps $Y \to X$. We will describe how this works in the particular examples of $X = *$, $X = \mathbb{B}G$, and $X = \mathcal{N}$ before giving a general proof.

5.1. **Cohesion for global spaces.** Global spaces are a cohesive $(\infty, 1)$-topos. That is there is a sequence of adjoint functors

$$\begin{tikzcd} \mathrm{Top}_{\mathrm{Glo}} \arrow[d, shift right=6, "\Pi"'] \arrow[d, shift left=6, "\Gamma"] \\ \mathrm{Top} \arrow[u, shift right=2, "\Delta"'] \arrow[u, shift left=10, "\nabla"'] \end{tikzcd}$$

such that $\Pi\Delta$, $\Gamma\Delta$, and $\Gamma\nabla$ are equivalent to the identity, and Π preserves finite products. We describe each of these functors in turn. (In fact, we have already described three of them in a previous section, as $\Delta = \delta_1$, $\Gamma = \gamma_1$, and $\nabla = \partial_1$.)

- The functor $\Delta\colon \mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}$ sends a space T to the constant functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Top}$ with value T. We will call Δ the **inclusion** of spaces into global spaces; it is the same as the functor δ_1 of §3.2.
- The functor Π is the **strict quotient** functor. It can be identified with the homotopy colimit functor: $\Pi(X) \approx \mathrm{hocolim}_{\mathrm{Glo}^{\mathrm{op}}}(X)$. This implies that $\Pi(\mathbb{B}G) \approx *$ for every compact Lie group G. Since Orb has finite products, $\Pi(\mathbb{B}G \times \mathbb{B}H) \approx \Pi(\mathbb{B}(G \times H)) \approx *$ and $\Pi(*) \approx \Pi B1 \approx *$. It follows from this that Π preserves finite products in general, since every global space is a homotopy colimit of a diagram of $\mathbb{B}G$s.

  Note that Π does not preserve pullbacks. For instance, the loop object of $\mathbb{B}G$ at a map $* \approx \mathbb{B}1 \to \mathbb{B}G$ is equivalent to $\Delta(G)$, but $\Pi\Delta(G) \approx G$ is not equivalent to $\Omega(\Pi(\mathbb{B}G)) \approx *$.

  For a group G and a G-CW complex X, we have that

  $$\Pi(\delta_G(X)) \approx X_G,$$

  the quotient space obtained by identifying G-translates in X. (The G-space X is built by attaching cells $G/K \times S^{n-1} \to G/K \times D^n$, and Π takes such maps to $S^{n-1} \to D^n$.) In particular, $\Pi\Delta \approx \mathrm{Id}$.
- The functor Γ is the **homotopy quotient** functor. It can be identified with the homotopy limit functor: $\Gamma(X) \approx \mathrm{holim}_{\mathrm{Glo}^{\mathrm{op}}}(X)$. Since Glo has a terminal object (the trivial group 1), we see that in fact $\Gamma(X) \approx X(1)$. It is immediate that $\Gamma\Delta \approx \mathrm{Id}$.

  For a group G and a G-space X, we have that

  $$\Gamma(\delta_G(X)) \approx X_{hG},$$

  the homotopy orbit space of X.

- The functor ∇ is the **co-inclusion** of spaces into global spaces[9]. It is a right Kan extension along $\{1\} \to \mathrm{Glo}^{\mathrm{op}}$. On a space T and an object $G \in \mathrm{Glo}$ it is given by
$$\nabla(T)(G) \approx \mathrm{Map}_{\mathrm{Top}}(BG, T).$$
It is clear that $\Gamma\nabla \approx \mathrm{Id}$. There is an evident map $\Delta(T) \to \nabla(T)$ from inclusion to co-inclusion, which is not generally an equivalence, though it is closer to an equivalence than one might initially think; Miller's theorem (the Sullivan conjecture) says that $\Delta(T)(G) \to \nabla(T)(G)$ is a weak equivalence when G is a finite group and T is a finite CW-complex.
By (2.3.1), we have that $\mathbb{B}G \approx \nabla(BG)$ for any G which is either a finite group or a compact abelian group (compare [Sch13a, 2.18]).

5.2. **Another model for the slice $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$.** In this section, we describe a presheaf model for $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$, which will be useful when we discuss the cohesion associated to $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$.

Let $\mathrm{Glo}_{\mathbb{B}G}$ be the full homotopy theory of $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$, whose objects are maps $\mathbb{B}\phi\colon \mathbb{B}K \to \mathbb{B}G$ corresponding to homomorphisms $\phi\colon K \to G$ of compact Lie groups. Then it is standard that as homotopy theories,
$$\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \approx \mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}}.$$
It is possible to give an explicit model for $\mathrm{Glo}_{\mathbb{B}G}$ as a category enriched over spaces, so that objects are homomorphisms $\phi\colon K \to G$, and $\mathrm{Glo}_{\mathbb{B}G}(\phi_1, \phi_2) = B\,\mathrm{Fun}_G(\phi_1, \phi_2)$, where $\mathrm{Fun}_G(\phi_1, \phi_2)$ is a topological groupoid having the form
$$\mathrm{Fun}_G(\phi_1,\phi_2) = \begin{cases} \textbf{obj} & (\rho\colon K_1 \to K_2, \gamma \in G) \mid \phi_2\rho = \gamma\phi_1\gamma^{-1} \\ \textbf{mor} & (\rho,\gamma) \xrightarrow{\delta} (\rho',\gamma')\colon\ \delta \in K_2 \mid \rho' = \delta\rho\delta^{-1},\ \gamma' = \phi_2(\delta)\gamma. \end{cases}$$
We can represent an object of this groupoid by the picture
$$\begin{tikzcd} K_1 \arrow[rr, "\rho"] \arrow[dr, "\phi_1"'] & \arrow[d, phantom, "\gamma"] & K_2 \arrow[dl, "\phi_2"] \\ & G & \end{tikzcd}$$
If $\phi_2\colon K_2 \to G$ is an inclusion of a subgroup $K_2 \subseteq G$, then
$$B\,\mathrm{Fun}_G(\phi_1,\phi_2) = \{\, \gamma \in G \mid \gamma\phi_1\gamma^{-1}(K_1) \subseteq K_2 \,\}_{hK_2} \approx (G/K_2)^{\phi_1(K_1)}.$$

5.3. **Cohesion for global spaces over $\mathbb{B}G$, relative to G-spaces.** Earlier we described cohesion for global spaces. There is an similar cohesion phenomenon for the slice categories $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$, which is *relative* to the homotopy of G-spaces $G\mathrm{Top}$. That is, there are homotopical adjoint functors
$$\begin{tikzcd} \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \arrow[d, shift right=9, "\Pi_G"] \arrow[d, shift right=3, phantom] \arrow[d, shift left=3, "\Gamma_G"] \\ G\mathrm{Top} \arrow[u, shift right=3, "\Delta_G"'] \arrow[u, shift left=9, "\nabla_G"'] \end{tikzcd}$$
so that the natural maps
$$\Pi_G \circ \Delta_G \to \mathrm{Id}_{G\mathrm{Top}} \to \Gamma_G \circ \Delta_G, \quad \mathrm{Id}_{G\mathrm{Top}} \to \Gamma_G \circ \nabla_G$$

[9]The essential image of ∇ corresponds to what Schwede calls *cofree* orthogonal spaces [Sch13a, 2.14].

are weak equivalences, and Π_G preserves finite products. I'll describe this structure below, making heavy use of the equivalence $\mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G \approx \mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}}$.

The most difficult part in showing cohesion is verifying that Π_G preserves finite products; the proof of this property will be deferred to the next section, where it is proved in a more general setting.

- The functor $\Delta_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}}/\mathbb{B}G$ was defined in §3.2. As we observed above, we may factor it naturally through a functor $G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}}$, which we also denote Δ_G, and which satisfies
$$(\Delta_G T)(\phi) = T^{\phi(K)} \qquad \text{for } \phi\colon K \to G.$$
- The (homotopical) left adjoint $\Pi_G\colon \mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}} \to G\mathrm{Top}$ to Δ_G is thus given on generators $\mathbb{B}\phi\colon \mathbb{B}K \to \mathbb{B}G$ by
$$\Pi_G(\mathbb{B}\phi) \approx G/\phi(K).$$
It is immediate that $\Pi_G \circ \Delta_G \to \mathrm{Id}_{G\mathrm{Top}}$ is an equivalence at orbits G/H, and thus is an equivalence on all objects, since both Π_G and Δ_G preserve homotopy colimits.

  The functor Π_G preserves finite products; we will prove this below.
- The functor $\Gamma_G\colon \mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}} \to G\mathrm{Top}$ is described (via the Elmendorff equivalence $G\mathrm{Top} \approx \mathrm{Top}_{\mathcal{O}_G}$) by
$$(\Gamma_G(X))^H \approx X(\iota_H),$$
where $\iota_H\colon H \to G$ is the inclusion of a subgroup. We have already noted (in the proof of (3.5.1)) that $\mathrm{Id}_{G\mathrm{Top}} \to \Gamma_G \circ \Delta_G$ is an equivalence. In particular, $\Gamma_G(\mathbb{B}\iota_H) \approx \Gamma_G\Delta_G(G/H) \approx G/H$.

  It is useful to have a description of $\Gamma_G(\mathbb{B}\phi)^H$, where $\phi\colon K \to G$ is a homomorphism and $H \subseteq G$ is a subgroup. To do this, note that
$$\mathrm{Fun}_G(\iota_H, \phi) = \begin{cases} \textbf{obj} & U \\ \textbf{mor} & U \times K \end{cases},$$
where
$$U = \{\, (\alpha, \gamma) \in \mathrm{Hom}(H, K) \times G \mid \gamma(\phi \circ \alpha)\gamma^{-1} = \iota_H \,\},$$
and where an element $((\alpha, \gamma), \delta) \in U \times K$ is regarded as a morphism $(\alpha, \gamma) \to (\delta\alpha\delta^{-1}, \phi(\delta)\gamma)$.

  Given a homomorphism $\alpha\colon H \to K$, let
$$U(\alpha) = \{\, \gamma \in G \mid \gamma^{-1}(\phi \circ \alpha)\gamma = \iota_H \,\} \subseteq G.$$
Then
$$\Gamma_G(\mathbb{B}\phi)^H \approx B\,\mathrm{Fun}_G(\iota_H, \phi) \approx \coprod_{[\alpha]} U(\alpha)_{hC_K(\alpha)},$$
where the coproduct is over K-conjugacy classes of $\alpha\colon H \to K$, and $C_K(\alpha) \curvearrowright U(\alpha)$ by $\delta \cdot \gamma = \phi(\delta)\gamma$.

  In the special case that $\pi\colon K \times G \to G$ is a projection homomorphism, we find that
$$\Gamma_G(\mathbb{B}\pi)^H \approx B\,\mathrm{Fun}(H, K),$$
and thus $\Gamma_G(\mathbb{B}\pi) \approx B_G K$. More generally, if Γ is a compact Lie group with normal subgroup $\Pi \trianglelefteq \Gamma$ and quotient homomorphism $\phi\colon \Gamma \to \Gamma/\Pi = G$, we have that
$$\Gamma_G(\mathbb{B}\phi) \approx B_G(\Pi; \Gamma),$$

where the right-hand side denotes the classifying space of G-equivariant principal $(\Pi;\Gamma)$-bundles, as in [May90].

- The functor $\nabla_G\colon G\mathrm{Top}\to\mathrm{Top}_{\mathrm{Glo}_{\mathbb{B}G}}$ is described by
$$\nabla_G(T)(\phi)\approx\mathrm{Map}_{G\mathrm{Top}}(\Gamma_G(\mathbb{B}\phi),T)$$
where $\phi\colon K\to G$ is a homomorphism. To show that $\Gamma_G\circ\nabla_G\to\mathrm{Id}_{G\mathrm{Top}}$ is an equivalence, note that
$$(\Gamma_G\nabla_G(T))^H\approx\nabla_G(T)(\iota_H)\approx\mathrm{Map}_{G\mathrm{Top}}(\Gamma_G(\mathbb{B}\iota_H),T)\approx\mathrm{Map}_{G\mathrm{Top}}(G/H,T)\approx T^H.$$

5.4. **Cohesion for global spaces over $\mathcal{N}$, relative to orbispaces.** There are homotopical adjoint functors
$$\begin{array}{c}\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}\\ \big\downarrow\Pi_{\mathrm{Orb}}\quad\big\uparrow\Delta_{\mathrm{Orb}}\quad\big\downarrow\Gamma_{\mathrm{Orb}}\quad\big\uparrow\nabla_{\mathrm{Orb}}\\ (\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}}\approx\mathrm{Top}_{\mathrm{Orb}}\end{array}$$
so that the natural maps
$$\Pi_{\mathrm{Orb}}\circ\Delta_{\mathrm{Orb}}\to\mathrm{Id}\to\Gamma_{\mathrm{Orb}}\circ\Delta_{\mathrm{Orb}},\quad\mathrm{Id}\to\Gamma_{\mathrm{Orb}}\circ\nabla_{\mathrm{Orb}}$$
are equivalences, and Π_{Orb} preserves finite products.

- The functor $\Delta_{\mathrm{Orb}}\colon(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}}\to\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ is the evident inclusion functor. In terms of the equivalence $(\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N})_{\mathrm{faith}}\approx\mathrm{Top}_{\mathrm{Orb}}$, it is the functor $\Delta_{\mathrm{Orb}}\colon\mathrm{Top}_{\mathrm{Orb}}\to\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ which sends X in $\mathrm{Top}_{\mathrm{Orb}}$ to the map $\delta_{\mathrm{Orb}}X\to\delta_{\mathrm{Orb}}1=\mathcal{N}$, where the functor $\delta_{\mathrm{Orb}}\colon\mathrm{Top}_{\mathrm{Orb}}\to\mathrm{Top}_{\mathrm{Glo}}$ is as described in §4.5. That is, for X in $\mathrm{Top}_{\mathrm{Orb}}$, the map $\Delta_{\mathrm{Orb}}X(G)\to\mathcal{N}(G)$ is given by
$$\coprod_{N\trianglelefteq G}X(G/N)\to\coprod_{N\trianglelefteq G}*.$$
We have noted that $\Delta_G(\mathbb{B}_{\mathrm{Orb}}G)\approx(\mathbb{B}G\xrightarrow{1}\mathcal{N})$ where the map labelled "1" classifies the trivial subgroup of G.
- The functor $\Pi_{\mathrm{Orb}}\colon\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}\to\mathrm{Top}_{\mathrm{Orb}}$ is homotopically left adjoint to Δ_{Orb}. It is given on generators $\mathbb{B}G\xrightarrow{N}\mathcal{N}$ of $\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ by
$$\Gamma_{\mathrm{Orb}}(\mathbb{B}G\xrightarrow{N}\mathcal{N})\approx\mathbb{B}_{\mathrm{Orb}}(G/N).$$
The functor Π_{Orb} preserves finite products; we will prove this below.
- The functor $\Gamma_{\mathrm{Orb}}\colon\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}\to\mathrm{Top}_{\mathrm{Orb}}$ is homotopically right adjoint to Δ_{Orb}. On an object $f_Y\colon Y\to\mathcal{N}$ of $\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ and an object G in Orb, it takes the value
$$(\Gamma_{\mathrm{Orb}}Y)(G)\approx Y(G;1),$$
where we write $Y(G;N)$ for the fiber of $f_Y(G)\colon Y(G)\to\mathcal{N}(G)$ over the point corresponding to $N\trianglelefteq G$.

  Consider an object $\mathbb{B}H\xrightarrow{N}\mathcal{N}$ of $\mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$. For G in Orb, we have
$$\Gamma_{\mathrm{Orb}}(\mathbb{B}H\xrightarrow{N}\mathcal{N})\approx B\,\mathrm{Fun}(G,H)_{1,N},$$
where $\mathrm{Fun}(G,H)_{1,N}$ denotes the full subgroupoid of $\mathrm{Fun}(G,H)$ whose objects are homomorphisms $\phi\colon G\to H$ such that $\phi^{-1}N=1$; i.e., such that the composite $G\xrightarrow{\phi}H\to H/N$ is injective.

In particular, $\Gamma_{\mathrm{Orb}}(\mathbb{B}H \xrightarrow{1} \mathcal{N}) \approx \mathbb{B}_{\mathrm{Orb}}H$, while $\Gamma_{\mathrm{Orb}}(\mathbb{B}H \xrightarrow{H} \mathcal{N}) \approx BH \times \mathbb{B}_{\mathrm{Orb}}1$.

- The functor $\nabla_{\mathrm{Orb}} \colon \mathrm{Top}_{\mathrm{Orb}} \to \mathrm{Top}_{\mathrm{Glo}}/\mathcal{N}$ is homotopically right adjoint to Γ_{Orb}. Formally, we have

$$(\nabla_{\mathrm{Orb}}X)(G) \approx \coprod_{N \trianglelefteq G} \mathrm{Map}_{\mathrm{Top}_{\mathrm{Orb}}}\big(\Gamma_{\mathrm{Orb}}(\mathbb{B}G \xrightarrow{N} \mathcal{N}), X\big).$$

6. Fiber products of Lie groups and the fiber product property

In our proofs of cohesion, the only difficult part will be the proof that the functors labelled Π_X are product preserving. This fact turns out to depend on something we call the *fiber product property* of Glo, which asserts that Glo contains homotopy pullbacks along surjective group homomorphisms.

6.1. **The fiber product property.** A finite product of compact Lie groups is a compact Lie group, and it is straightforward to see that Glo thus has (homotopical) finite products (including a terminal object), and thus that the Yoneda embedding $\mathbb{B} \colon \mathrm{Glo} \to \mathrm{Top}$ preserves such finite products.

Note that Glo does not have general homotopy pullbacks, even though fiber product of compact Lie groups always exist. However, some homotopy pullbacks do exist in Glo, namely those in which one of the homomorphisms is surjective.

Proposition 6.1.1 (Fiber product property)**.** *Let $\phi \colon G \to H$ and $\psi \colon H' \to H$ be homomorphisms of compact Lie groups, and let $G' = G \times_H H'$. If ϕ is surjective, then the evident map $\mathbb{B}G' \to \mathbb{B}G \times_{\mathbb{B}H} \mathbb{B}H'$ to the homotopy pullback of $\mathbb{B}\psi$ along $\mathbb{B}\phi$ is a weak equivalence of global spaces.*

6.2. **Proof of the fiber product property.** We prove this after a couple of lemmas.

Lemma 6.2.1. *Let $f \colon C \to D$ and $g \colon D' \to D$ be functors of groupoids enriched over* Top*, and let $C' = D' \times_D C$, the pullback of f along g in the category of groupoids enriched over* Top*. Suppose that f has the following properties.*

1. *For each object c of C, object d' of D, and morphism $\delta \in D(f(c), d)$, there exists an object c' of C and a morphism $\gamma \in C(c, c')$ such that $f(c') = d'$ and $f(\gamma) = \delta$.*
2. *For each pair of objects c, c' in C, the map $f \colon C(c, c') \to D(f(c), f(c'))$ is a fibration of spaces.*

Then $BC' \to BD' \times^h_{BD} BC$ is a weak equivalence; i.e., the resulting pullback square of classifying spaces is in fact a homotopy pullback square.

Proof. Given a topologically enriched groupoid C, let $C_\bullet$ denote the corresponding simplicial space whose realization is BC. Condition (2) implies that $C_\bullet \to D_\bullet$ is a levelwise fibration of simplicial spaces. We write $[m]$ for the representable simplicial space $\Delta(-, [m])$, and $i \colon \{0\} = [0] \to [m]$ for map induced by inclusion of the 0th vertex.a To prove the result, it suffices to show that the induced map

$$\{0\} \times_{D_\bullet} C_\bullet \xrightarrow{i \times_{D_\bullet} C_\bullet} [m] \times_{D_\bullet} C_\bullet$$

becomes a weak equivalence after geometric realization, or equivalently that $B(i) \colon B(\{0\} \times_D C) \to B([m] \times_D C)$ is a weak equivalence. It suffices to construct a functor $j \colon [m] \times_D C \to \{0\} \times_D C$ and a natural map $ji \to \mathrm{Id}$, which can be done using (1); this relies on the fact that C_0 and D_0 are discrete. $\square$

Lemma 6.2.2. *Let $\phi\colon G \to H$ be a surjective homomorphism of Lie groups, and $\alpha\colon K \to G$ a homomorphism from a compact Lie group K. Then the induced homomorphism $C_G(\alpha) \to C_H(\phi\alpha)$ of centralizers restricts to a surjective map on identity components (i.e., it is an open map).*

Proof. Without loss of generality, we may replace K with its image $\alpha(K)$, and replace α with the inclusion map. Write $Z = C_G(K)$ and $N = \mathrm{Ker}(\phi)$. Then

$$X = \{\, x \in G \mid kxk^{-1}x^{-1} \in N \;\; \forall k \in K \,\}$$

is a closed subgroup of G containing Z and N, and K acts on X by conjugation. To show that the homomorphism $Z \to X/N = C_H(\phi(K))$ is surjective on identity components, it suffices to show that $T_eX = T_eZ + T_eN$.

Given a representation $\rho\colon K \to GL(V)$, write $V_1 = \{\, v \in V \mid \rho(k)(v) = v \;\; \forall k \in K \,\}$ for the K-fixed subspace of V, and let V_2 be the span of $\{\, \rho(k)(v) - v \mid v \in V, k \in K \,\}$; both V_1 and V_2 are K-invariant subspaces of V. Because K is compact, $V = V_1 + V_2$. (Proof: V is a direct sum of irreducible K-representations.)

Now consider the adjoint action $\mathrm{ad}\colon K \curvearrowright V = T_eX$. We have that $V_1 \subseteq T_eZ$ by a standard argument using $k\exp(tv)k^{-1} = \exp(t\,\mathrm{ad}(k)(v))$, and that $V_2 \subseteq T_eN$ using the fact that $\mathrm{ad}(k)(v) - v = \gamma'(0)$, where $\gamma(t) = k\exp(tv)k^{-1}\exp(-tv) \in N$. Thus $T_eZ + T_eN = T_eX$ as desired. ∎

Proof of (6.1.1). Recall that we can model $(\mathbb{B}G)(K) \approx B\,\mathrm{Fun}'(K, G)$, where $\mathrm{Fun}'(K, G)$ is a groupoid enriched over Top. We note that the hypotheses (1) and (2) of (6.2.1) apply to the functor $\mathrm{Fun}'(K, \phi)\colon \mathrm{Fun}'(K, G) \to \mathrm{Fun}'(K, H)$. To see this, note that property (1) holds because ϕ is a surjective homomorphism, while property (2) is a consequence of (6.2.2). The proposition follows. ∎

6.3. **A model for $\mathbb{B}\pi\colon \mathbb{B}G \to \mathbb{B}H$ when π is surjective.** We can adapt the proof of the previous proposition to describe the homotopy fiber of $\mathbb{B}\pi\colon \mathbb{B}G \to \mathbb{B}H$ when π is a surjective homomorphism. Given homomorphisms $\pi\colon G \to H$ and $\phi\colon K \to H$, define a topological groupoid by

$$\mathrm{Lift}_H(\phi, \pi) = \begin{cases} \textbf{obj} & \alpha\colon K \to G \mid \pi\alpha = \phi \\ \textbf{mor} & \alpha \to \alpha'\colon \quad g \in G \mid \alpha' = g\alpha g^{-1},\ \pi(g) = 1, \end{cases}$$

where objects are topologized as a subspace of $\mathrm{Hom}(K, G)$.

Proposition 6.3.1. *If $\pi\colon G \to H$ is a surjective homomorphism, then $B\,\mathrm{Lift}_H(\phi, \pi)$ is weakly equivalent to the homotopy fiber of $B\,\mathrm{Fun}(K, \pi)\colon B\,\mathrm{Fun}(K, G) \to \mathbb{B}\,\mathrm{Fun}(K, H)$ over the point corresponding to ϕ.*

Proof. First, let's consider a category $\mathrm{Lift}'_H(\phi, \pi)$ enriched over Top, which set theoretically is the same as $\mathrm{Lift}_H(\phi, \pi)$, but has discrete object set. Then $\mathrm{Lift}'_H(\phi, \pi)$ is precisely the fiber of $\mathrm{Fun}'(K, G) \to \mathrm{Fun}'(K, H)$ over the object ϕ. The argument of the proof of (6.1.1) shows that $B\,\mathrm{Lift}'_H(\phi, \pi)$ is equivalent to the homotopy fiber of $B\,\mathrm{Fun}'(K, G) \to B\,\mathrm{Fun}'(K, H)$ over ϕ, using that π is a surjective homomorphism.

Next, I claim that $B\,\mathrm{Lift}'_H(\phi, \pi) \approx B\,\mathrm{Lift}_H(\phi, \pi)$. It will suffice to show that any two sufficiently nearby objects in $\mathrm{Lift}_H(\phi, \pi)$ are isomorphic in the groupoid. Fix $\alpha \in \mathrm{obj}\,\mathrm{Lift}_H(\phi, \pi)$, and set $L = \pi^{-1}(C_H(\phi))$ and $L_0 = \pi^{-1}(\pi(C_G(\alpha)))$, so that $L_0 \subseteq L \subseteq G$ are closed subgroups. Let $\rho\colon L \to \mathrm{obj}\,\mathrm{Lift}_H(\phi, \pi)$ be $\rho(g) = g\alpha g^{-1}$. The image $\rho(L) \approx L/C_G(\alpha)$ is an open neighborhood of α, as it is precisely the intersection in $\mathrm{Hom}(K,G)$ of $\operatorname{obj}\mathrm{Lift}_H(\phi,\pi)$ with $\{\, g\alpha g^{-1} \mid g\in G \,\}$ (2.2.1). The set $\rho(L_0)\approx L_0/C_G(\alpha)$ is precisely the set of elements of $\rho(L)$ which are isomorphic to α in the groupoid $\mathrm{Lift}_H(\phi,\pi)$.

By (6.2.2), $L/L_0\approx C_H(\phi)/\pi(C_G(\alpha))$ is discrete, so $\rho(L_0)$ is open in $\rho(L)$, and thus $\rho(L_0)$ is an open neighborhood of $\alpha\in\operatorname{obj}\mathrm{Lift}_H(\phi,\pi)$ consisting of points isomorphic to α. □

7. Cohesion relative to an orbispace

In this section, we prove our general cohesion result, which exhibits for each orbispace X in $\mathrm{Top}_{\mathrm{Glo}}$ a cohesion $\Delta_X\colon (\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}}\to \mathrm{Top}_{\mathrm{Glo}}/X$. In the case when $X=\mathbb{B}G$ or $X=\mathcal{N}$, this recovers the instances of cohesion discussed in §5.

7.1. **The indexing category of a global space.** Given a global space X in $\mathrm{Top}_{\mathrm{Glo}}$, let $\mathrm{Glo}_X\subset \mathrm{Top}_{\mathrm{Glo}}/X$, the full sub-homotopy theory of the slice category whose objects are of the form $\mathbb{B}G\to X$. It is standard that the inclusion functor $\mathrm{Glo}_X\to\mathrm{Top}_{\mathrm{Glo}}/X$ is dense, i.e., that it extends to an equivalence

$$\mathrm{Top}_{\mathrm{Glo}_X}\approx \mathrm{Top}_{\mathrm{Glo}}/X.$$

Thus, we can regard an object $Y\to X$ of $\mathrm{Top}_{\mathrm{Glo}}/X$ as a presheaf $\widetilde{Y}$ in Glo_X, whose value at $x\colon\mathbb{B}G\to X$ is the homotopy fiber of $Y(G)\to X(G)$ over the point corresponding to x.

From this point of view, we see that a map $f\colon Y\to X$ of global spaces is faithful if and only if $\widetilde{Y}(x)\to\widetilde{Y}(x\circ\mathbb{B}\phi)$ is a weak equivalence for every object $x\colon\mathbb{B}G\to X$ in Glo_X and every surjective homomorphism $\phi\colon H\to G$.

We call Glo_X the **indexing category** of the global space X.

7.2. **The orbit category of an orbispace.** Suppose now that the global space X is actually an orbispace. Let $\mathrm{Orb}_X=\mathrm{Glo}_X\cap(\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}}$, the full subtheory of Glo_X consisting of $x\colon\mathbb{B}G\to X$ which are faithful. We write $i_X\colon\mathrm{Orb}_X\to\mathrm{Glo}_X$ for the evident inclusion functor. It is evident that i_X admits a left adjoint

$$j_X\colon \mathrm{Glo}_X\rightleftarrows \mathrm{Orb}_X : i_X;$$

the value of j_X on $x\colon\mathbb{B}G\to X$ is the canonical factorization $j_X(x)\colon\mathbb{B}(G/I(x))\to X$ through a faithful map.

We call Orb_X the **orbit category** of the orbispace X.

Example 7.2.1. If $X=\mathcal{N}$, then $\mathrm{Orb}_{\mathcal{N}}\approx\mathrm{Orb}$, the theory formed from compact Lie groups and injective homomorphisms described in §4.5.

Example 7.2.2. If $X=\mathbb{B}G$, then $\mathrm{Orb}_{\mathbb{B}G}$ is equivalent to $\mathcal{O}_G$, the classical orbit category of the Lie group G.

Example 7.2.3. Suppose $X=\delta_G(T)$ for some G-space T. Then every object of Orb_X is equivalent to one of the form $\delta_G(t)\colon\delta_G(G/H)\to X$ where $t\colon G/H\to T$ is a map in $G\mathrm{Top}$ and $H\le G$ is a closed subgroup. In other words, objects of Orb_X correspond (up to equivalence) to pairs $(H\le G, t\in T^H)$.

The adjoint pair $j_X\colon \mathrm{Glo}_X \rightleftarrows \mathrm{Orb}_X : i_X$ gives rise to a chain of homotopical adjoint functors.

$$\begin{array}{c} \mathrm{Top}_{\mathrm{Glo}_X} \approx \mathrm{Top}_{\mathrm{Glo}}/X \\ (j_X)_\# \big\downarrow \qquad \big\uparrow (j_X)^*=(i_X)_\# \qquad (j_X)_*=(i_X)^* \big\downarrow \qquad \big\uparrow (i_X)_* \\ \mathrm{Top}_{\mathrm{Orb}_X} \end{array}$$

where $(j_X)_\#(i_X)_\# \approx \mathrm{Id}_{\mathrm{Top}_{\mathcal{O}_X}} \approx (i_X)^*(j_X)^*$ and $(j_X)_*(i_X)_* \approx \mathrm{Id}_{\mathrm{Top}_{\mathcal{O}_X}}$, which we will take as the definitions of the functors $\Pi_X \dashv \Delta_X \dashv \Gamma_X \dashv \nabla_X$. Note that Δ_X and ∇_X are thus fully faithful.

Theorem 7.2.4. *The functor* $\Delta_X = (i_X)_\#\colon \mathrm{Top}_{\mathrm{Orb}_X} \to \mathrm{Top}_{\mathrm{Glo}_X} \approx \mathrm{Top}_{\mathrm{Glo}}/X$ *gives an equivalence* $\mathrm{Top}_{\mathrm{Orb}_X} \xrightarrow{\sim} (\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}}$ *between presheaves on* Orb_X *and the full subtheory of faithful morphisms of global spaces to* X*.*

Theorem 7.2.5. *For any orbispace* X*, the above structure defines a cohesion*

$$\Delta_X\colon (\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}} \to \mathrm{Top}_{\mathrm{Glo}}/X.$$

We will prove both of these below.

7.3. **Recognizing cohesion.** Fix an adjoint pair $j\colon \mathcal{G} \rightleftarrows \mathcal{O} : i$ between small homotopy theories, with the property that $ji \to \mathrm{Id}_{\mathcal{O}}$ is an equivalence. We immediately obtain a sequence of homotopical adjoint functors

$$\begin{array}{c} \mathrm{Top}_{\mathcal{G}} \\ \Pi=j_\# \big\downarrow \qquad \big\uparrow \Delta=j^*=i_\# \qquad \Gamma=j_*=i^* \big\downarrow \qquad \big\uparrow \nabla=i_* \\ \mathrm{Top}_{\mathcal{O}} \end{array}$$

so that $\Pi\Delta \to \mathrm{Id} \to \Gamma\Delta$ and $\Gamma\nabla \to \mathrm{Id}$ are equivalences, so that both Δ and ∇ are fully faithful. In addition, Π preserves the terminal object, since $\Pi(1) \approx \Pi\Delta(1) \approx 1$.

We have two goals in this section: *first*, to identify the essential image of Δ, and *second*, to give a criterion which ensures that Δ is a cohesion.

Let $\mathcal{C} \subset \mathrm{Top}_{\mathcal{G}}$ denote the (non-full) subtheory consisting of morphisms f in $\mathrm{Top}_{\mathcal{G}}$ such that $\Pi(f) = j_\#(f)$ is an equivalence in $\mathrm{Top}_{\mathcal{O}}$. Because Δ is fully faithful, a morphism f is in $\mathcal{C}$ if and only if $\Delta\Pi(f)$ is an equivalence in $\mathrm{Top}_{\mathcal{G}}$.

In the following, we will identify $\mathcal{G}$ with its Yoneda image in $\mathrm{Top}_{\mathcal{G}}$. Then for objects A in $\mathcal{G}$, we have that $\Pi(A) = j_\#(A) \approx j(A)$. Thus $\mathcal{C} \cap \mathcal{G}$ is the subtheory consisting precisely of morphisms $f \in \mathcal{G}$ such that $j(f)$ is an equivalence in $\mathcal{O}$. Say that an object X in $\mathrm{Top}_{\mathcal{G}}$ is **$\mathcal{C} \cap \mathcal{G}$-local** if for every $U \to V$ in $\mathcal{C} \cap \mathcal{G}$, we have that $\mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(V, X) \to \mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(U, X)$ is an equivalence. Write $(\mathrm{Top}_{\mathcal{G}})_{\mathrm{local}} \subset \mathrm{Top}_{\mathcal{G}}$ for the full subtheory of $\mathcal{C} \cap \mathcal{G}$-local objects.

Proposition 7.3.1. *The functor* $\Delta = i_\#$ *restricts to an equivalence* $\mathrm{Top}_{\mathcal{O}} \to (\mathrm{Top}_{\mathcal{G}})_{\mathrm{local}}$*.*

Proof. Since $\Delta = i_\#$ is fully faithful, it suffices to show that the essential image of Δ consists precisely of $\mathcal{C} \cap \mathcal{G}$-local objects. It is immediate that all objects in the essential image of Δ are $\mathcal{C} \cap \mathcal{G}$-local; in fact, they are even $\mathcal{C}$-local, since $\mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(U, \Delta X) \approx \mathrm{Map}_{\mathrm{Top}_{\mathcal{O}}}(\Pi U, X)$, and Π takes elements of $\mathcal{C}$ to equivalences by definition.

Conversely, let X be an $\mathcal{C} \cap \mathcal{G}$-local object of $\mathrm{Top}_{\mathcal{G}}$. To show that X is in the essential image of Δ, it suffices to show that the counit map $\Delta\Gamma X \to X$ of the $\Delta \dashv \Gamma$ adjunction is an equivalence. Since objects of $\mathcal{G}$ generate $\mathrm{Top}_{\mathcal{G}}$, it suffices to show that this counit map induces equivalences $\mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(U, \Delta\Gamma X) \to \mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(U, X)$ for all U in $\mathcal{G}$. By the adjunction $\Delta\Pi \dashv \Delta\Gamma$ obtained by composing the adjunctions $\Pi \dashv \Delta$ and $\Delta \dashv \Gamma$, this is equivalent to the map $\mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(\Delta\Pi U, X) \to \mathrm{Map}_{\mathrm{Top}_{\mathcal{G}}}(U, X)$ induced by the unit $\eta(U)\colon U \to \Delta\Pi U$ of the $\Pi \dashv \Delta$ adjunction. The claim follows, because $\eta(U) \in \mathcal{C}$ since $\Pi\Delta \to \mathrm{Id}$ is an equivalence. □

We have the following criteria for constructing morphisms in $\mathcal{C}$.

Proposition 7.3.2. *Let $\mathcal{D}$ be a small topologically enriched category, and $f\colon F \to G$ a natural transformation of functors $\mathcal{D} \to \mathrm{Top}_{\mathcal{G}}$. If $f(d)\colon F(d) \to G(d)$ is contained in $\mathcal{C}$ for each object $d \in \mathcal{D}$, then* $\mathrm{hocolim}_{\mathcal{D}} f\colon \mathrm{hocolim}_{\mathcal{D}} F \to \mathrm{hocolim}_{\mathcal{D}} G$ *is contained in $\mathcal{C}$.*

Proof. A straightforward consequence of the fact that Π commutes with homotopy colimits. □

Proposition 7.3.3. *Let $f\colon V \to U$ be a morphism of $\mathrm{Top}_{\mathcal{G}}$. Suppose that for every morphism $g\colon B \to U$ in $\mathrm{Top}_{\mathcal{G}}$ such that B is an object of $\mathcal{G}$, the homotopy pullback $f'\colon C \to B$ of f along g is contained in $\mathcal{C}$. Then f is contained in $\mathcal{C}$.*

Proof. Because $\mathcal{G} \to \mathrm{Top}_{\mathcal{G}}$ is dense, there exists a small category $\mathcal{D}$, a functor $G\colon \mathcal{D} \to \mathcal{G} \subset \mathrm{Top}_{\mathcal{G}}$, and an equivalence $\mathrm{hocolim}_{\mathcal{D}} G \to U$. Define $F\colon \mathcal{D} \to \mathrm{Top}_{\mathcal{G}}$ by $F(d) := G(d) \times_U V$ (homotopy pullback), and let $f'\colon F \to G$ be the evident natural transformation. By hypothesis, each map $f(d)\colon F(d) \to G(d)$ is in $\mathcal{C}$, and thus $\mathrm{hocolim}_{\mathcal{D}} f'$ is in $\mathcal{C}$ by (7.3.2). Since $\mathrm{hocolim}_{\mathcal{D}} f'$ is equivalent to f, the result follows. □

With the same setup as above, we now give a condition for $\Delta\colon \mathrm{Top}_{\mathcal{O}} \to \mathrm{Top}_{\mathcal{G}}$ to be a cohesion.

Proposition 7.3.4. *Suppose the following holds:*

(*) *Homotopy pullbacks in $\mathrm{Top}_{\mathcal{G}}$ of morphisms in $\mathcal{C} \cap \mathcal{G}$ along arbitrary morphisms of $\mathcal{G}$ are contained in $\mathcal{C} \cap \mathcal{G}$.*

Then $\Delta\colon \mathrm{Top}_{\mathcal{O}} \to \mathrm{Top}_{\mathcal{G}}$ is a cohesion.

Proof. The existence of the adjoint functors, as well as the natural isomorphisms $\Pi\Delta \to \mathrm{Id} \to \Gamma\Delta$ and $\Gamma\nabla \to \mathrm{Id}$, is immediate, as is the fact that Π preserves the terminal object. Thus, it remains only to show that $\Pi = j_{\#}$ preserves binary products. Because $\Delta = i_{\#}$ preserves products and $i_{\#}j_{\#} = \Pi\Delta \xrightarrow{\sim} \mathrm{Id}$, it suffices to show that for any pair of objects X_1, X_2 in $\mathrm{Top}_{\mathcal{G}}$, the map $\eta(X_1) \times \eta(X_2)\colon X_1 \times X_2 \to \Delta\Pi X_1 \times \Delta\Pi X_2$ is contained in $\mathcal{C}$.

Choose functors $F_i\colon \mathcal{D}_i \to \mathcal{G} \subset \mathrm{Top}_{\mathcal{G}}$ and equivalences $\mathrm{hocolim}_{\mathcal{D}_i} F_i \to X_i$ for $i = 1, 2$, where $\mathcal{D}_i$ are small categories. Let $\overline{F}_i = \Delta\Pi F_i$, whence $\mathrm{hocolim}_{\mathcal{D}_i} \overline{F}_i \approx \Delta\Pi X_i$, since $\Delta\Pi$ preserves homotopy colimits. Note that $\Delta\Pi F_i(d) = i_{\#}j_{\#}(F_i(d)) \approx ij(F_i(d))$. Since binary products in $\mathrm{Top}_{\mathcal{G}}$ preserve colimits in each variable, the map $\eta(X_1) \times \eta(X_2)$ is equivalent to the homotopy colimit over $\mathcal{D}_1 \times \mathcal{D}_2$ of the maps

$$\eta(F_1(d)) \times \eta(F_2(d))\colon F_1(d) \times F_2(d) \to \overline{F}_1(d) \times \overline{F}_2(d),$$

where the maps $\eta(F_i(d))\colon F_i(d) \to \overline{F}_i(d)$ are contained in $\mathcal{C} \cap \mathcal{G}$. In view of of (7.3.2), it suffices to prove the following.

(**) For any pair of maps $f_i\colon A_i \to \overline{A}_i$ in $\mathcal{C} \cap \mathcal{G}$, the product map $f_1 \times f_2$ is contained in $\mathcal{C}$.

In view of (7.3.3), it thus suffices to prove the following.

(***) For any pair of maps $f_i\colon A_i \to \overline{A}_i$ in $\mathcal{C} \cap \mathcal{G}$ and any pair of maps $u_i\colon B \to \overline{A}_i$ in $\mathcal{G}$, the map g obtained as the homotopy pullback of $f = f_1 \times f_2\colon A_1 \times A_2 \to \overline{A}_1 \times \overline{A}_2$ along $u = (u_1, u_2)\colon B \to \overline{A}_1 \times \overline{A}_2$ in $\mathrm{Top}_{\mathcal{G}}$ is contained in $\mathcal{C}$.

Let $g_i\colon C_i \to B$ denote the homotopy pullback of f_i along u_i, and consider the homotopy pullback diagrams

$$\begin{tikzcd}
C_1 \times_B C_2 \arrow[r] \arrow[d, "h_i"'] & C_i \arrow[r] \arrow[d, "g_i"'] & A_i \arrow[d, "f_i"'] \\
C_j \arrow[r, "g_j"'] & B \arrow[r, "u_i"'] & \overline{A}_i
\end{tikzcd}$$

in $\mathrm{Top}_{\mathcal{G}}$, with $\{i, j\} = \{1, 2\}$. Hypothesis (*) implies that $g_i \in \mathcal{C} \cap \mathcal{G}$, and a second application of hypothesis (*) gives that $h_i \in \mathcal{C} \cap \mathcal{G}$. Thus $g = g_j h_i \in \mathcal{C}$. The map g is in fact the desired pullback of f along u, as demonstrated by the homotopy pullback squares

$$\begin{tikzcd}
C_1 \times_B C_2 \arrow[r] \arrow[d, "g"'] & C_1 \times C_2 \arrow[r] \arrow[d, "g_1 \times g_2"'] & A_1 \times A_2 \arrow[d, "f_1 \times f_2"] \\
B \arrow[r, "\mathrm{diag}"'] & B \times B \arrow[r, "u_1 \times u_2"'] & \overline{A}_1 \times \overline{A}_2
\end{tikzcd}$$

Thus, we have proved (***), and the result follows. $\square$

7.4. **Proofs of the theorems.** We now give the proofs which show that $\Delta_X\colon (\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}} \to \mathrm{Top}_{\mathrm{Glo}}/X$ is a cohesion when X is an orbispace. We now set the adjunction $j\colon \mathcal{G} \rightleftarrows \mathcal{O} : i$ to be $j_X\colon \mathrm{Glo}_X \rightleftarrows \mathrm{Orb}_X : i_X$ as defined earlier. First we identify the class $\mathcal{C} \cap \mathrm{Glo}_X \subset \mathrm{Top}_{\mathrm{Glo}_X}$.

Lemma 7.4.1. *A morphism $\mathbb{B}\phi\colon x \to x'$ in Glo_X (presented as $\mathbb{B}G \xrightarrow{\mathbb{B}\phi} \mathbb{B}G' \xrightarrow{x'} X$ with $x = x' \circ \mathbb{B}\phi$) is in $\mathcal{C}$ if and only if ϕ is surjective.*

Proof. Since X is an orbispace, there are canonical factorizations of x and x' of the form

$$\begin{tikzcd}
\mathbb{B}G \arrow[r] \arrow[d, "\mathbb{B}\phi"'] & \mathbb{B}(G/I(x)) \arrow[d, "\mathbb{B}\overline{\phi}"'] \arrow[dr, "\overline{x}"] & \\
\mathbb{B}G' \arrow[r] & \mathbb{B}(G'/I(x')) \arrow[r, "\overline{x}'"'] & X
\end{tikzcd}$$

with $\overline{x}$ and $\overline{x}'$ are faithful. This exhibits the result of j applied to $\mathbb{B}\phi\colon x \to x'$, namely $\mathbb{B}\overline{\phi}\colon \overline{x} \to \overline{x}'$. Since $\mathbb{B}\overline{\phi}$ is faithful (by (3.4.5)) we see that $\overline{\phi}$ is an injective homomorphism. Thus $\mathbb{B}\phi\colon x \to x'$ is in $\mathcal{C}$ if and only if $\overline{\phi}$ is surjectve, if and only if ϕ is surjective. $\square$

Proof of (7.2.4). By (7.3.1), the essential image of $\Delta_X\colon \mathrm{Top}_{\mathrm{Orb}_X} \to \mathrm{Top}_{\mathrm{Glo}_X} \approx \mathrm{Top}_{\mathrm{Glo}}/X$ thus consists precisely of $\mathcal{C} \cap \mathrm{Glo}_X$-local objects, which by (7.4.1) are exactly the faithful morphisms to X in $\mathrm{Top}_{\mathrm{Glo}}/X$; i.e.., the essential image of Δ_X is $(\mathrm{Top}_{\mathrm{Glo}}/X)_{\mathrm{faith}}$, as desired. $\square$

Proof of (7.2.5). By (7.3.4), we must show that pullbacks of morphisms in $\mathcal{C} \cap \mathrm{Glo}_X$ along morphisms of Glo_X are contained in $\mathcal{C} \cap \mathrm{Glo}_X$. By (7.4.1), this precisely reduces to the fiber product property (6.1.1) $\square$

8. Orthogonal spaces

I have nothing much to say not already in [Sch13a]. Mainly I try to link Schwede's constuctions to the ones I used in previous sections.

8.1. **The Stiefel category.** The **Stiefel category** $\mathbb{L}$ has as objects finite dimensional real inner product spaces, and as morphisms $f\colon V \to W$ all isometric embeddings. There is an evident topology on $\mathbb{L}(V, W)$, which is in fact a Stiefel manifold, so $\mathbb{L}$ is a topologically enriched category. Furthermore, $\mathbb{L}$ admits a symmetric monoidal structure $\oplus$, so that $V \oplus W$ is the orthogonal direct sum of V and W.

Given a Lie group G, write $\mathbb{L}^G$ for the category of functors $G \to \mathbb{L}$. Thus, an object of $\mathbb{L}^G$ is a pair (V, ρ) consisting of a real inner product space V and a homomorphism $\rho\colon G \to \mathbb{L}(V, V)$, while a morphism $(V, \rho) \to (V', \rho')$ is a map $f\colon V \to V'$ of $\mathbb{L}$ which is a map of G-representations.

Remark 8.1.1. I'm going to supress the homomorphism ρ when talking about objects of $\mathbb{L}^G$. Thus, an object of $\mathbb{L}^G$ is a G-representation V, which is a vector space with a left G-action: $g, v \mapsto g \cdot v$. I'll write $\rho_V\colon G \to \mathbb{L}(V, V)$ for the corresponding homomorphism, though sometimes I'll actually write "V" for this homomorphism.

For instance, if $\phi\colon H \to G$ is a group homomorphism and V is an object of $\mathbb{L}^G$, then I write $V\phi$ for the evident object of $\mathbb{L}^H$ obtained by restriction along ϕ.

Note that $\mathbb{L}^G$ is also a topologically enriched category, with $\mathbb{L}^G(V, V') \subseteq \mathbb{L}(V, V')$, and that $\mathbb{L}^G$ is also symmetric monoidal under orthogonal direct sum. There is an evident forgetful functor $\mathbb{L}^G \to \mathbb{L}$, which is compatible with direct sum.

More generally, given any homomorphism $\phi\colon H \to G$, we obtain a functor $\phi^*\colon \mathbb{L}^G \to \mathbb{L}^H$, given on objects by $V \mapsto V\phi$. If $g\colon \phi \Rightarrow \phi'$ is such that $\phi' = g\phi g^{-1}$, then we obtain a natural isomorphism of functors $c_g\colon \phi^* \to \phi'^*$, which to each object V of $\mathbb{L}^G$ associates the map $V\phi \to V\phi'$ defined by acting by g.

Remark 8.1.2. It is also possible to consider a category $\widetilde{\mathbb{L}}^G$ internal to Top, which set theoretically is the same as $\mathbb{L}^G$; the object space of $\widetilde{\mathbb{L}}^G$ is $\coprod_V \operatorname{Hom}(G, \mathbb{L}(V, V))$, where $\operatorname{Hom}(G, \mathbb{L}(V, V)) \approx \operatorname{Hom}(G, O(V))$ has the compact-open topology.

Proposition 8.1.3. *The* Top*-enriched categories* $\mathbb{L}^G$ *are homotopically filtered. That is,*

(a) *for every pair V_1, V_2 of objects in $\mathbb{L}^G$, there exists an object W and maps $V_i \to W$ in $\mathbb{L}^G$;*
(b) *for every $k \geq 0$, every pair of objects V, V' in $\mathbb{L}^G$, and every continuous map $f\colon S^{k-1} \to \mathbb{L}^G(V, V')$, there exists a map $\alpha\colon V' \to W$ of $\mathbb{L}^G$ and a continuous map $D^k \to \mathbb{L}^G(V, W)$ making*

$$\begin{tikzcd}
S^{k-1} \arrow[r, "f"] \arrow[d] & \mathbb{L}^G(V, V') \arrow[d, "\mathbb{L}^G(V,\alpha)"] \\
D^k \arrow[r, "g"'] & \mathbb{L}^G(V, W)
\end{tikzcd}$$

commute

Proof. For (a), simply use the direct sum for W.

For (b), recall that any orthogonal V representation admits an orthogonal direct sum

$$V \approx \bigoplus_{\lambda} U_\lambda \otimes \mathbb{R}^{n_\lambda(V)},$$

where U_λ is irreducible, the sum is over isomorphism classes of irreducibles, and $\sum n_\lambda(V) < \infty$. Observe that

$$\mathbb{L}^G\big(\bigoplus_{\lambda} U_\lambda \otimes \mathbb{R}^{n_\lambda}, \bigoplus_{\lambda} U_\lambda \otimes \mathbb{R}^{n'_\lambda}\big) \approx \prod_{\lambda} \mathbb{L}(\mathbb{R}^{n_\lambda}, \mathbb{R}^{n'_\lambda}),$$

and that $\mathbb{L}(\mathbb{R}^n, \mathbb{R}^{n+k})$ is $(k-2)$-connected, and is a one-point space if $n = 0$. Thus, we may set $W = V' \oplus (V \otimes \mathbb{R}^{k+1})$. ☐

8.2. **Faithful representations.** An object V of $\mathbb{L}^G$ is called a **G-representation**. Of especial importance are the **faithful** G-representations; we write $\mathbb{L}^G_{\text{faith}} \subset \mathbb{L}^G$ for the full Top-enriched subcategory of faithful G-representations. We observe that $\mathbb{L}^G_{\text{faith}}$ is also homotopically filtered, and that the inclusion $\mathbb{L}^G_{\text{faith}} \to L^G$ is cofinal.

8.3. **Orthogonal spaces.** An **orthogonal space** is a topologically enriched functor $X : \mathbb{L} \to \text{Top}$. We write $\mathbb{L}\text{Top}$ for the category of such functors.

Given a compact Lie group H, we obtain a functor $X^H : \mathbb{L}^H \to \text{Top}$ by

$$X^H(V) := X(V)^{\rho_V(H)}.$$

We write $X[H] = \operatorname{hocolim}_{\mathbb{L}^H} X^H$.

We say that a map $f : X \to Y$ of orthogonal spaces is a **global equivalence** if $X[H] \to Y[H]$ is a weak equivalence of spaces for every compact Lie group H. (This is essentially the same as Schwede's definition [Sch13a, 1.1.3]. His form is more elementary and is probably preferable.)

As we will note below, orthogonal spaces under global equivalence is a model for Top_{Orb}.

8.4. **Free orthogonal spaces.** For each V in $\mathbb{L}^G$, we have a functor

$$L_{G,V} : G\text{Top} \to \mathbb{L}\text{Top}$$

defined by

$$L_{G,V}(X)(W) = (\mathbb{L}(V, W) \times X)_G,$$

using the diagonal action of G (acting on $\mathbb{L}(V, W)$ through ρ). This is the free orthogonal space generated by X at (G, V) [Sch13a, §I.2].

Let $L_{G,V} = L_{G,V}(*)$. Then one shows that $L_{G,V}[H]$ is a model for the equivariant classifying space $B_H(\rho_V(G))$, where $\rho_V(G) \subseteq \mathbb{L}(V, V)$. In particular, if V is a *faithful* G-representation, then $L_{G,V}[H] \approx B_H G$ [Sch13a, I.2.6]. Furthermore, since any two faithful G-representations V_1, V_2 can be embedded in their orthogonal direct sum $V_1 \oplus V_2$, we see that there are global weak equivalences $L_{G,V_1}(X) \leftarrow L_{G,V_1 \oplus V_2}(X) \to L_{G,V_2}(X)$ [Sch13a, I.2.11].

The functor $L_{G,V} : G\text{Top} \to \mathbb{L}\text{Top}$ for any faithful V in $\mathbb{L}^G$ is a model for $\delta_G : G\text{Top} \to \text{Top}_{\text{Orb}}$.

8.5. **Cofree orthogonal spaces.** Let $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$ be a countable dimensional real inner product space, and write $\mathbb{L}(V, \mathbb{R}^\infty)$ for the space of linear isometric embeddings. We define $R\colon \mathrm{Top} \to \mathbb{L}\mathrm{Top}$ by

$$(RT)(V) = \mathrm{Map}_{\mathrm{Top}}(\mathbb{L}(V, \mathbb{R}^\infty), T).$$

Note that if (V, ρ) is a faithful H-representation, then H acts freely on $\mathbb{L}(V, \mathbb{R}^\infty)$, and thus $\mathbb{L}(V, \mathbb{R}^\infty)_H \approx BH$. Therefore, for faithful (V, ρ) we have

$$(RT)^H(V, \rho) = \mathrm{Map}_{\mathrm{Top}}(\mathbb{L}(V, \mathbb{R}^\infty)_H, T) \approx \mathrm{Map}_{\mathrm{Top}}(BH, T),$$

and thus, $(RT)[H] \approx \mathrm{Map}_{\mathrm{Top}}(BH, T)$. We say that RT is the **cofree** orthognal space on T [Sch13a, I.2.12].

The functor $R\colon \mathrm{Top} \to L\mathrm{Top}$ is a model for $\nabla\colon \mathrm{Top} \to \mathrm{Top}_{\mathrm{Orb}}$.

More generally, we can define $R_G\colon G\mathrm{Top} \to \mathbb{L}\mathrm{Top}$ by

$$(R_G T)(V) = \mathrm{Map}_{G\mathrm{Top}}(\mathbb{L}(V, \mathcal{U}_G), T),$$

where $\mathcal{U}_G$ is a complete G-universe [Sch13a, I.1.5][10]. If (V, ρ) is a faithful H-representation, then H acts freely on $\mathbb{L}(V, \mathcal{U}_G)$, and $\mathbb{L}(V, \mathcal{U}_G)_H \approx B_G H$ as a G-space. Thus, for faithful (V, ρ) we have

$$(R_G T)^H(V, \rho) = \mathrm{Map}_{G\mathrm{Top}}(\mathbb{L}(V, \mathcal{U}_G)_H, T) \approx \mathrm{Map}_{G\mathrm{Top}}(B_G H, T),$$

and thus $(R_G T)[H] \approx \mathrm{Map}_{G\mathrm{Top}}(B_G H, T)$.

The functor $R_G\colon G\mathrm{Top} \to \mathbb{L}\mathrm{Top}$ is a model for $\partial_G\colon G\mathrm{Top} \to \mathrm{Top}_{\mathrm{Orb}}$.

8.6. **Orthogonal spaces with G-action.** Just as functors $\mathbb{L} \to \mathrm{Top}$ model global spaces, it is also the case that functors $\mathbb{L} \to G\mathrm{Top}$ model global spaces sliced over $\mathbb{B}G$.

Fix a compact Lie group G. A **G-orthogonal space** is a topologically enriched functor $X\colon \mathbb{L} \to G\mathrm{Top}$. We write $\mathbb{L}G\mathrm{Top}$ for the category of such functors.

Given a compact Lie group H together with a homomorphism $\phi\colon H \to G$, we obtain a functor $X^\phi\colon \mathbb{L}^H \to \mathrm{Top}$ by

$$X^\phi(V) := X(V)^{(\phi, \rho_V)}.$$

That is, points in this space are $x \in X(V)$ such that for all $h \in H$, we have $\phi(h)\cdot x = x\cdot\rho_V(h)$, using the action of G on X and the action of H on the representation V. Set

$$X[\phi] := \mathrm{hocolim}_{\mathbb{L}^H} X^\phi.$$

Say that a map $X \to Y$ of G-orthogonal spaces is a **G-global equivalence** if $X[\phi] \to Y[\phi]$ is a weak equivalence of spaces for every homomorphism $\phi\colon H \to G$.

For each homomorphism $\phi\colon H \to G$ and object V in $\mathbb{L}^H$, we have an object $L_{\phi,V}$ in $\mathbb{L}G\mathrm{Top}$, defined by

$$L_{\phi,V}(W) := G \times_{\phi,H} \mathbb{L}(V, W).$$

That is, this is collection of pairs $(g, f) \in G \times \mathbb{L}(V, W)$, subject to the relation $(g, f) \sim (g\phi(h), f\rho_V(h))$ for all $h \in H$. It is clear that

$$\mathrm{Map}_{\mathbb{L}G\mathrm{Top}}(L_{\phi,V}, X) \approx X^\phi(V),$$

and therefore

$$X[\phi] \approx \mathrm{hocolim}_{V \in \mathbb{L}^H} \mathrm{Map}_{\mathbb{L}G\mathrm{Top}}(L_{\phi,V}, X).$$

[10] I do not see that Schwede discusses the functor R_G anywhere in [Sch13a].

Proposition 8.6.1. *Let $\phi\colon H \to G$ be a homomorphism, and let V be a faithful representation of H. Then for any homomorphism $\psi\colon K \to G$, there is a weak equivalence*

$$L_{\phi,V}[\psi] \approx \mathrm{Glo}_{\mathbb{B}G}(\psi, \phi).$$

Proof. First we compute $(L_{\phi,V})^{\psi}(W)$ for an object W of $\mathbb{L}^K$. Let $X(W)$ denote the space of pairs $(g, f) \in G \times \mathbb{L}(V, W)$ such that for every $k \in K$, there exists an $h \in H$ such that $(\psi(k)g, kf) = (g\phi(h), fh)$. The group H acts on $X(W)$ by $(g, f) \cdot h := (g\phi(h), fh)$, and we have that

$$(L_{\phi,V})^{\psi}(W) = X(W)_H,$$

and furthermore H acts freely on $X(W)$.

Because V is a faithful representation, for every $(g, f) \in X(W)$ there is a (unique) homomorphism $\sigma = \sigma_{g,f}\colon K \to H$ (depending on (g, f)), such that $h = \sigma(k)$ is the unique element such that $(\psi(k)g, kf) = (g\phi(h), fh)$. Thus, there is a map

$$(g, f) \mapsto \sigma_{g,f}\colon X(W) \to \mathrm{Hom}(K, H),$$

whose fiber $X_\sigma(W)$ over $\sigma \in \mathrm{Hom}(K, H)$ is the space

$$X_\sigma(W) = \{\, (g, f) \in G \times \mathbb{L}(V, W) \mid \phi\sigma = g^{-1}\psi g,\ kf = f\sigma(k)\, \forall k \in K \,\}$$

If we write $Y_\sigma = \{\, g \in G \mid \phi\sigma = g^{-1}\psi g \,\}$, then

$$X_\sigma(W) = Y_\sigma \times \mathbb{L}(V\sigma, W)^K.$$

□

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