ELLIPTIC COHOMOLOGY AND ELLIPTIC CURVES (FELIX KLEIN LECTURES, BONN 2015)

CHARLES REZK

Abstract. Lecture notes for a series of talks given in Bonn, June 2015. Most of the topics covered touched in one way or another on the role of power operations in elliptic cohomology.

In June of 2015 I gave a series of six lectures (the Felix Klein lectures) in Bonn. These are some of my lecture notes for those talks. I had hoped to polish them more carefully, but that hasn't happend yet, and at this point probably will not. I am making them available more or less as-is.

I include only the notes for the first five lectures. Note that some bits in these notes never made it into the spoken lectures. The notes for the final lecture are too disjointed to be very useful, so I have omitted them. I hope to soon have preprints on some aspects of what I spoke about in that lecture.

I'd like to thank the Hausdorff Institute for their hospitality, and for the opportunity to give these talks, which were a great challenge, but also great fun.

1. What is elliptic cohomology?

I'll start with a brief "pseudo-historical" account of elliptic cohomology. This is meant to be an imprecise overview. The idea is to introduce the basic questions and objects we're interested in, and to highlight the main themes of these lectures, which could be summarized as "power operations" and "isogenies".

1.1. **Genera.** A **genus** is a function which assigns to each closed manifold M of some type an element $\Phi(M) \in R$ of a commutative ring R, satisfying

- $\Phi(M_1 \amalg M_2) = \Phi(M_1) + \Phi(M_2)$.
- $\Phi(M_1 \times M_2) = \Phi(M_1)\Phi(M_2)$.
- $\Phi(\partial N) = 0$.

This is the same as giving a ring homomorphism from a suitable cobordism ring, e.g., $\Phi\colon MSO_* \to R$ or $\Phi\colon MU_* \to R$.

Genera with values in R with $\mathbb{Q} \subset R$ can be described entirely in terms of characteristic classes, by a formalism due to Hirzebruch.[1] For instance, associated to a genus $\Phi\colon MU_* \to R \otimes \mathbb{Q}$ is a characteristic class for complex vector bundles

$$K_\Phi(V \to X) \in H^*(X; R \otimes \mathbb{Q}),$$

which is completly determined by its **characteristic series**, i.e., its value on the universal line bundle

$$K_\Phi(x) = K_\Phi(\mathcal{O}(1) \to BU(1)) \in H^*(BU(1); R \otimes \mathbb{Q}) = R \otimes \mathbb{Q}[\![x]\!]$$

Date: May 8, 2018.
This work was partially supported by the National Science Foundation, DMS-1406121.

[1] See for instance, [HBJ92].

where $x = c_1(\mathcal{O}(1)) \in H^*(BU(1))$ is the usual first chern class, together with a sum formula $K_\phi(V \oplus W) = K_\phi(V)K_\phi(W)$. Then for a stably almost-complex M,

$$\phi(M) = \langle K_\phi(TM), [M] \rangle.$$

Conversely, any such characteristic series $K(x)$ determines a genus $MU_* \to R \otimes \mathbb{Q}$. A series with $K(x) = K(-x)$ determines a genus $MSO_* \to R \otimes \mathbb{Q}$.

Genera are not created equal. The most interesting ones (1) have a geometric (or analytic) interpretation and (2) lift to integral invariants.

1.2. *Example* (Todd genus). Characteristic series $K_{\mathrm{Td}}(x) = x/(1 - e^{-x})$. On stably almost complex manifolds, $\mathrm{Td}(M) \in \mathbb{Z}$, and for complex manifolds

$$\mathrm{Td}(M) = \sum (-1)^k \dim H^k_{\mathrm{Coh}}(M, \mathcal{O}_M).$$

1.3. *Example* ($\widehat{A}$-genus). Characteristic series $K_{\widehat{A}}(x) = (x/2)/\sinh(x/2)$. On *spin* manifolds $\widehat{A}(M) \in \mathbb{Z}$, and for such computes the index of the Dirac operator on the spinor bundle over M.

These genera come with "families" versions. In either case, the integer value of the genus can be constructed as an "index". If we have a family $M \to S$ of spin manifolds M_s over a space S, the family genus defines a class $\phi(M) \in E^*(X)$ in a generalized cohomology theory of S. Thus, for a family of n-dimensional spin manifolds, we have $\widehat{A}(M) \in KO^{-n}(S)$.

In homotopy theory, all of this can be encoded in a single map: a map of ring spectra

$$\phi \colon MG \to E$$

where MG is an appropriate bordism spectrum and E a cohomology theory.

1.4. **Modular forms.** Modular forms are sections of certain line bundles over the moduli of elliptic curves. Here is how this works in the complex analytic category.

Let

$$\mathcal{X} := \{\mathrm{Im}(\tau) \neq 0\} \subset \mathbb{C}.$$

Over this we have a $\Gamma := GL(2, \mathbb{Z})$-equivariant bundle of tori.

$$\Gamma \curvearrowright \quad \begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] & \mathcal{E}_\tau \arrow[l, hook'] \arrow[d] \arrow[r, equal] & \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z}) \\ \mathcal{X} & \{\tau\} \arrow[l, hook'] \end{tikzcd}$$

Here $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \curvearrowright \mathcal{X}$ by $A\tau := \frac{a\tau + b}{c\tau + d}$, and $\Gamma \curvearrowright \mathcal{E}$ by

$$A(\tau, z) := (A\tau,\ (\det A)(c\tau + d)^{-1}\, z).$$

Each fiber is an elliptic curve, i.e., a complex analytic curve of genus 1 with distinguished point at the equivalence class of $z = 0$. In fact, the object (orbifold) $(\mathcal{E} \to \mathcal{X})//\Gamma$ is the *universal family* of such curves.

Let $\omega \to \mathcal{X}$ be the Γ-equivariant line bundle with $\omega_\tau = T_0^* \mathcal{E}_\tau$. A (complex analytic) **modular form of weight** k is an equivariant holomorphic section

$$f \in H^0(\mathcal{X}//\Gamma, \omega^{\otimes k}) =: MF_k \otimes \mathbb{C}$$

which satisfies a certain growth condition; namely, that for $\mathrm{Im}(\tau) \gg 0$, we have

$$f(q) = \sum_{n \geq 0} a_n q^n, \qquad q = e^{2\pi i \tau},$$

for some $a_n \in \mathbb{C}$, converging near $q = 0$. Note: we can regard this as sections of line bundles over a certain compactification $\overline{\mathcal{X}//\Gamma}$ of this orbifold, obtained by "putting in the cusp".

The **Eisenstein series** for $k \geq 1$ are described by the q-expansions

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sum_{d|n} d^{2k-1}\, q^n.$$

With $q = e^{2\pi i \tau}$, they extend to functions of τ; in fact $G_{2k}(\tau) \sim \sum_{(a,b)\neq(0,0)} \frac{1}{(a\tau+b)^{2k}}$. For $k \geq 2$ they are modular forms. (The series $G_2(q)$ is merely a "quasimodular form".)

The ring of modular forms (of "level 1") over $\mathbb{C}$ is given by $MF_* \otimes \mathbb{C} \approx \mathbb{C}[G_4, G_6]$.

Variants involve sections which are invariant for the action of certain subgroups of Γ.

Every elliptic curve is, canonically, an *algebraic curve* over $\mathbb{C}$. More generally, one may consider the algebraic analogue of the above moduli stack, classifying maps $E \to S$ which are proper smooth curves of relative dimension 1 equipped with a section. The picture is this:

$$\overline{\mathcal{X}//\Gamma} \longrightarrow \overline{\mathcal{M}_{\mathrm{Ell}}}$$

The object $\overline{\mathcal{M}_{\mathrm{Ell}}}$ is a compactification of the moduli stack $\mathcal{M}_{\mathrm{Ell}}$ of (algebraic) elliptic curves. Integral modular forms are $MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$. We have

$$MF_* \approx \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$

where $c_4 = 240\, G_4$ and $c_6 = 504\, G_6$. This ring consists precisely of analytic modular forms with q-expansions in $\mathbb{Z}[\![q]\!]$.

1.5. **Elliptic genera: the Witten genus.** Elliptic genera were first constructed by Ochanine, via a particular characteristic series with coefficients in modular forms. Another genus was defined by Witten. The **Witten genus**, with characteristic series

$$K_W(x) = K_W(x,\tau) = \exp\big(2 \sum_{k\geq 2} G_{2k}(\tau)\, \frac{x^{2k}}{(2k)!}\big).$$

(Note that the non-modular form G_2 is excluded in this expression.) This is $K_W(x) = x/\sigma(x)$, where $\sigma(x,\tau)$ is the **Weierstrass σ-function**, and the associated genus is a homomorphism $W\colon MSO_* \to MF_* \otimes \mathbb{C}$.

The Witten genus has the following remarkable property: applied to spin-manifolds M such that $\frac{p_1}{2}(M) = 0$, the Witten genus gives a modular form with integral q-expansion: $K_W(x) \in MF_*$. The idea is that in terms of $q = e^{2\pi i \tau}$ and $u = e^x$, the characteristic series has the product expansion

$$K_W(u,q) = \frac{x/2}{\sinh(x/2)} \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}\, e^{-G_2(\tau) x^2}.$$

If we remove the last term with $G_2(\tau)x^2$, this can be calculated in terms of the characteristic series of a twisted version of the $\widehat{A}$-genus, which takes values in $\mathbb{Z}[\![q]\!]$ for spin manifolds by the index theorem. The last term contributes nothing exactly when $(p_1/2)(M) = 0$.

Thus, remarkably, the Witten genus takes values in *integral* modular forms, which are invariants of algebraic elliptic curves.

This is a remarkable fact, and is one reason for the interest in this: somehow, we are led directly to arithmetic algebraic geometry.

1.6. **Elliptic genera and quantum field theory.** Even more remarkable is Witten's explanation for this, in terms of field theory, which has been elaborated and clarified by many others, including Segal [Seg88], [Seg07], and Stolz-Teichner [ST11]. Very loosely, we look at a 2-dimensional (extended) field theory, which is a symmetric monoidal functor

$$F\colon \mathrm{Bord}_2 \to \mathcal{C}$$

where Bord_2 is the bordism 2-category, and $\mathcal{C}$ is a 2-category in the world of linear algebra.

The example of topological field theories has become very familiar, due to work of Lurie. In this case, the field theory should be a supersymmetric conformal field theory: 2-morphisms of Bord_2 should be some kind of conformal (or Euclidean) manifolds (Riemann surfaces), and in fact should be $2|1$-dimensional supermanifolds.

The 2-category $\mathcal{C}$ should have 1-morphisms be some kind of hilbert space, and 2-morphisms as operators. One might additionally allow some kind of twists, obtaining 2-categories $\mathcal{C}^k$ for $k \in \mathbb{Z}$; and we might actually be looking at sections of a 2-functor $\widetilde{C}^k \to \mathrm{Bord}_2$.

A 2-morphism between empty 1-morphisms between empty 0-morphisms is a closed 2-manifold Σ. If we apply F to a torus $\mathbb{R}^{2|1}/\Lambda_\tau$, we obtain the "partition function"

$$Z_F(\tau) := F(\mathbb{R}^{2|1}/\Lambda_\tau) \in \mathbb{C},$$

a function depending only on the modular parameter. Thus, Z_F should be a modular function (or form).

Given a suitable manifold M, there is a 2-category $\mathrm{Bord}_2(M)$, in which elements U of Bord_2 are equipped with maps $U \to M$. The idea is that there should be a map

$$\{F\colon \mathrm{Bord}_2(M) \to \mathcal{C}\} \to \mathrm{Ell}(M).$$

In fact, we might hope these are closely related, i.e., that $\mathrm{Ell}(M)$ is the homotopy invariant approximation to field theories.

The Witten genus of M is supposed[2] to be associated to a particular field theory $F_W\colon \mathrm{Bord}_2(M) \to \mathcal{C}$ defined naturally on manifolds with string-structure, evaluated on suitable maps $\Sigma^2 \to * \to M$.

This is an example of "dimensional reduction": because the maps factor through a map from a 0-manifold, we are supposed to reinterpret this as a 0-dimensional field theory (classified by $H^*(-,\mathbb{C})$) which is decorated by some additional data coming from the $\mathbb{R}^{2|1}/\Lambda_\tau$.

Similarly, we might restrict F_W to circle bundles over 1-dimensional manifolds $E \to \Sigma^1 \to M$, obtaining dimensional reduction to a 1-dimensional field theory (classified by K-theory). (Because F associates to these operators on Hilbert space.) The action of the circle on E means we live in $K[\![q]\!]$.

1.7. **Rigidity.** Another property of elliptic genera discovered by Ochanine and Witten is **rigidity**. Here is one formulation. Given a manifold M with a $U(1)$-action, we obtain a bundle

$$M \times_{U(1)} EU(1) \to BU(1).$$

[2]Much of my understanding of this comes from [BE13].

Using the characteristic series $K_W(z)$ one defines a families elliptic genus

$$W(M \circlearrowleft U(1))(x) \in H^*(BU(1), MF_* \otimes \mathbb{C}) \approx MF_* \otimes \mathbb{C}[\![x]\!].$$

The **rigidity theorem** (proved by Bott and Taubes) says that for a $U(1)$-equivariant spin manifold with $p_1/2(M) = 0$, we have $W(M \circlearrowleft U(1))(x) = W(M \circlearrowleft U(1))(0)$.

1.8. **How is this explained in topology?** The modularity and integrality of the Witten genus is explained by a map of ring spectra $M\mathrm{String} \to \mathrm{tmf}$ [AHR06]. Here $M\mathrm{String}$ is the bordism for associated to the fiber of $B\mathrm{String} \to B\mathrm{Spin} \to K(\mathbb{Z}, 4)$. The object tmf is the elliptic cohomology theory associated to the universal algebraic elliptic curve.

Rigidity is explained by the existence of an equivariant version of elliptic cohomology, taking values in sheaves on the universal elliptic curve itself.

1.9. **Multiplicative structure in generalized cohomology theories and power operations.** A generalized cohomology theory h consists of a collection of functors $\{h^n\colon \mathrm{Top} \to \mathrm{Ab}\}_{n\in\mathbb{Z}}$ together with some additional data, e.g., suspension isomorphisms $\widetilde{h}^n(X) \to \widetilde{h}^{n+1}(S^1 \wedge X)$. Such an theory is represented by a spectrum E, which in the original (and simplest) formulation is a sequence of spaces $\{E_n\}_{n\in\mathbb{Z}}$ and weak homotopy equivalences $\{E_n \xrightarrow{\sim} \Omega E_{n+1}\}$, so that

$$h^n(X) = h^n(X, E) \approx \mathrm{Hom}_{h\mathrm{Top}}(X, E_n).$$

There is an associated homotopy category of spectra $h\mathrm{Sp}$.

The cohomology theories we are usually interested in are multiplicative: $h^*(X) \otimes h^*(X) \to h^*(X)$. The product pairing in cohomology is encoded using the *smash product* of spectra: $E \wedge E \to E$. Thus, if E is an associative and commutative monoid in the homotopy category of spectra, then $h^*(-, E)$ takes values in graded commutative rings.

In particular, such a cohomology theory has a natural m-th power map $h^*(X, E) \to h^*(X^m, E)$ by $x \mapsto x^{\times m}$; restriction along the diagonal embedding $X \subset X^m$ gives the internal m-th power.

If E is a structured commutative ring spectrum (e.g., a *commutative S-algebra* or equivalent notion), then the associated cohomology theory admits a refinement to a *power operation P_m*.

$$\begin{tikzcd}
 & h^*(X^m \times_{\Sigma_m} E\Sigma_m) \arrow[r, "\mathrm{diag}^*"] \arrow[d, "\mathrm{fib}^*"] & h^*(X \times B\Sigma_m) \arrow[d, "\mathrm{incl}^*"] \\
h^*(X) \arrow[ur, "P_m"] \arrow[r, "x\mapsto x^{\times m}"'] & h^*(X^m) \arrow[r, "\mathrm{diag}^*"'] & h^*(X)
\end{tikzcd}$$

We are using $X \times B\Sigma_m \subset X^m \times_{\Sigma_m} E\Sigma_m \to B\Sigma_m$.

1.10. **Power operations and mulitplicative transfer.** A theory with power operations (such as one coming from a commutative S-algebra) is endowed with *multiplicative transfers.* This associates to any finite n-fold covering map $f\colon Y \to X$ a *multiplicative* function

$$N_f\colon h^*(Y) \to h^*(X).$$

Here's a definition. Let $\pi\colon P \to X$ be the associated principal Σ_n-bundle of f. Thus,

$$P \approx \big\{\phi\colon \underline{n} \to Y \text{ which identify } \underline{n} \text{ with a fiber of } f\big\} \subset Y^m.$$

Passage to homotopy quotient by the Σ_m-action gives

$$X = P/\Sigma_m \xleftarrow{\sim} P \times_{\Sigma_m} E\Sigma_m \to Y^m \times_{\Sigma_m} E\Sigma_m,$$

i.e., up to homotopy a map $t_f \colon X \to Y^m \times_{\Sigma_m} E\Sigma_m$. The mulitpliicative transfer is the composite

$$N_f \colon h^*(Y) \xrightarrow{P_m} h^*(Y^m \times_{\Sigma_m} E\Sigma_m) \xrightarrow{t_f^*} h^*(X).$$

Conversely, the power operations can be recovered from the mulitplicative transfers. They are equivalent formalisms.

A few properties: $N_{gf} = N_g \circ N_f$, and a "push-pull" identity $g^* \circ N_f = N_{f'} \circ g'^*$ associated to a pullback of covering maps.

1.11. *Example* (Steenrod operations). Let $h^* = H^*(-, \mathbb{Z}/2)$. Then Steenrod defined an operation

$$P_2 \colon h^*(X) \to h^*(X \times B\Sigma_2) \approx h^*(X) \otimes \mathbb{Z}/2[t],$$

which is in fact the power operation associated to the mod 2 Eilenberg MacLane spectrum. For $x \in h^n(X)$ this has value

$$P_2(x) = \mathrm{Sq}^n(x) + \mathrm{Sq}^{n-1}(x)\, t + \mathrm{Sq}^{n-2}(x)\, t^2 + \cdots + \mathrm{Sq}^0(x)\, t^n \in h^{2n}(X \times B\Sigma_2).$$

The **Steenrod operations** Sq^k are defined by reading off the coefficients of powers of t. Note that $P_2(x)|_{t=0} = x^2$, which is $\mathrm{Sq}^n(x) = x^2$.

You can also use the language of multiplicative transfer to describe these, though it is not so common: in group cohomology, the multiplicative transfer is called the **Evens norm**.

1.12. **Power operations in K-theory.** K-theory was invented by Grothendieck for the sake of its power operations, which is the theory of lambda rings. It's easiest to describe this for topological K-theory using equivariant K-theory, following Atiyah [Ati66]. Thus, for $G \curvearrowright X$, we write $K_G^0(X) = K^0(X/\!/G)$ for the Grothendieck group of G-equivariant complex vector bundles on X.

In this case, the power operation is defined by tensor power of vector bundles, with evident symmetry.

$$\overline{P}_m \colon K^0(X) \xrightarrow{P_m} K^0(X^m/\!/\Sigma_m) \xrightarrow{\mathrm{diag}^*} K^0(X/\!/\Sigma_m) = K^0(X) \otimes R\Sigma_m,$$
$$V \mapsto \big(\Sigma_m \curvearrowright (V^{\boxtimes m} \to X^m)\big) \mapsto \big((\Sigma_m \curvearrowright V^{\otimes m}) \to X\big).$$

It is not immediately obvious that this construction defined on $\mathrm{Vect}(X/\!/G)$ extends to its Grothendieck group, since P_m is not additive, but this can be shown (for instance, by extending to complexes of bundles).

We can write $\overline{P}_m(x) = \sum_\pi \phi_\pi(x) \otimes V_\pi$, sum over irreducible representations of Σ_m. In particular, $\phi_{\mathrm{sgn}}(x) = \lambda^m(x)$, the operation defined by the mth exterior power.

All ϕ_π can be written as polynomials in λ^ks. These operations give $K^0(X)$ the structure of a **λ-ring**: a ring equipped with functions $\lambda^m \colon K^0(X) \to K^0(X)$ satsifying some properties of the type satisfied by exterior powers.

1.13. θ^p**-rings and** λ**-rings.** Suppose p is prime, and restrict to the cyclic subgroup $C_p \subset \Sigma_p$. Write

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{\overline{P}_{C_p}} & K^0(X//C_p) \approx K^0(X) \otimes RC_p \\ x & \mapsto & \psi^p(x) \otimes 1 \;-\; \theta^p(x) \otimes (1 + T + \cdots + T^{p-1}). \end{array}$$

Here T is the standard 1-dimensional complex rep of C_p, so $RC_p = \mathbb{Z}[T]/(T^p - 1)$. The coefficients are natural functions $\psi^p, \theta^p \colon K^0(X) \to K^0(X)$.

Properties.

- Multiplicativity: $\overline{P}_{C_p}(xy) = \overline{P}_{C_p}(x)\overline{P}_{C_p}(y)$ and $\overline{P}_{C_p}(1) = 1$.
- Setting $T \mapsto 1$ (i.e., forgetting the C_p action) identifies $\overline{P}_{C_p}(x)|_{T=1} = x^p$, and thus
$$\psi^p(x) = x^p + p\,\theta^p(x).$$
- Setting $T \mapsto e^{2\pi i/p}$ (evaluation of character of C_p-representation at generator) sends $N = 1 + T + \cdots + T^{p-1} \mapsto 0$, so
$$\overline{P}_{C_p}(x)|_{T=e^{2\pi i/p}} = \psi^p(x).$$
- The "binomial formula" for tensor product gives
$$\overline{P}_{C_p}(V + W) = \overline{P}_{C_p}(V) + \cdots + V^{\otimes k} \otimes W^{\otimes p-k} \otimes \tfrac{1}{p}\binom{p}{k} N + \cdots + \overline{P}_{C_p}(W)$$
where $N = 1 + T + \cdots + T^{p-1}$. In particular, evaluation at $T \mapsto e^{2\pi i/p}$ shows that $\psi^p \colon K^0(X) \to K^0(X)$ is a ring homomorphism.

Investigating the behavior of θ^p in the same way gives identities

$$\begin{aligned} \theta^p(1) &= 0, \\ \theta^p(xy) &= \theta^p(x)y^p + x^p\theta^p(y) + p\,\theta^p(x)\theta^p(y) \\ \theta^p(x+y) &= \theta^p(x) + \theta^p(y) - \sum_{0<k<p} \tfrac{1}{p}\binom{p}{k} x^k y^{p-k}. \end{aligned}$$

Thus, the ring $K^0(X)$ admits the structure of a θ^p**-ring** (Bousfield [Bou96]) (also called a δ-ring (Joyal [Joy85a]) or a ring with p-derivation (Buium [Bui05])). Note that the identity $\psi^p(x) = x^p + p\theta^p(x)$ recovers the Adams operation from θ^p, and the above identities imply that ψ^p is a ring endomorphism which lifts Frobenius.

We think of θ^p as a "witness" to the fact that ψ^p is a lift of Frobenius.

If p and q are *distinct* primes, then it's not hard to show that $\overline{P}_{C_p}\overline{P}_{C_q} = \overline{P}_{C_{pq}} = \overline{P}_{C_q}\overline{P}_{C_p}$, from which it is not hard to see that $\psi^p\psi^q = \psi^q\psi^p$, and in fact $\psi^p\theta^q = \theta^q\psi^p$.

1.14. **Theorem** (Joyal [Joy85b]; see also [Rez14])**.** *There is an equivalence of categories*

$$\big\{\lambda\text{-}rings\big\} \leftrightarrows \big\{(R, \{\theta^p\}_{primes\ p})\ s.t.\ \psi^p\theta^q = \theta^q\psi^p\big\}.$$

So these are equivalent descriptions of this theory.

We note the following consequence (which was in fact proved first).

1.15. **Theorem** (Wilkerson [Wil82])**.** *Torsion free λ-rings are the same as rings equipped with a family $\{\psi^p\}$ of pairwise commuting lifts of Frobenius.*

In some sense, the category of λ-rings can be completly reconstructed from this observation.

1.16. **Application: Hopf invariant one.** Having set all this up, it seems too good not to give a famous application: a proof of the Hopf invariant 1 theorem due to Adams-Atiyah [AA66].

1.17. **Theorem** (Adams)**.** *There exists a two-cell CW-complex* $X = S^{2k} \cup D^{4k}$ *with* $H^*(X) \approx \mathbb{Z}[x]/(x^3)$ *iff* $k \in \{1, 2, 4\}$. *(E.g.,* $\mathbb{CP}^2$, $\mathbb{HP}^2$, $\mathbb{OP}^2$.*)*

Proof. Suppose we have such an X. A standard argument lets you replace the statement about $H^*(X)$ with the same one for $K^0(X)$. We know how Adams operations act on $K^0(S^{2k})$, so we know that for a prime p,

$$\psi^p(x) = p^k\, x + a_p\, x^2, \qquad \text{for some } a_p \in \mathbb{Z}.$$

For $p = 2$,

$$\psi^2(x) = 2^k\, x + a_2\, x^2, \qquad a_2 \equiv 1 \mod 2,$$

because ψ^2 is a lift of Frobenius. Apply the identity $\psi^2\psi^p = \psi^p\psi^2$ to x with p odd; equating the coefficients of x^2 gives

$$(2^k - 1)2^k\, a_p = (p^k - 1)p^k\, a_2.$$

Since p and a_2 are odd, we must have $2^k|(p^k - 1)$, which is only possible if $k \in \{1, 2, 4\}$ (as can be seen by checking for $p = 3$.) □

1.18. **Power operations and the multiplicative group.** The above constructions make sense for equivariant K-theory generally. Consider

$$K^0(*//U(1)) = RU(1) = \mathbb{Z}[T, T^{-1}].$$

Since $\mu\colon U(1) \times U(1) \to U(1)$ is a group homomorphism, this is an abelian Hopf algebra, with $\mu^*(T) = T \otimes T$. It is natural to regard $RU(1) = \mathcal{O}_{\mathbb{G}_m}$, the ring of functions on the multiplicative group scheme.

We have $\psi^p(T) = T^p$. By naturality, ψ^p is a map of Hopf algebras, corresponding to the pth power map $[p]\colon \mathbb{G}_m \to \mathbb{G}_m$.

1.19. **Refinements of power operations.** I pulled a switch: I said that power operations come from a commutative S-algebra, then I produced power operations for K-theory using equivariant K-theory, which doesn't come from having the commutative ring spectrum.

K theory is given by a commutative S-algebra. However, the operations I describe are a refinement:

$$\begin{tikzcd} & K^0(X//\Sigma_m) = K^0(X) \otimes R\Sigma_m \arrow[d, "\text{completion}"] \\ K^0(X) \arrow[ur] \arrow[r] & K^0(X \times B\Sigma_m) \end{tikzcd}$$

In this case, the difference is not huge, but it is there. In the "classical" case, you can only give a power operation construction of ψ^p, θ^p as functions $K^0(X) \to K^0(X; \mathbb{Z}_p)$. The equivariant setting lets you constuct these operations integrally.

The equivariant K-theory power operations come from the fact that equivariant K-theory is represented by a *globally equivariant, ultracommutative ring spectrum* (see [Sch16]).

We can go in the other direction, and replace K-theory by p-adic K-theory:

$$\begin{tikzcd} K^0(X) \arrow[r] \arrow[d] & K^0(X \times B\Sigma_m) \arrow[d] \\ K^0(X;\mathbb{Z}_p) \arrow[r] & K^0(X \times B\Sigma_m;\mathbb{Z}_p) \end{tikzcd}$$

Something strange happens: the operations ψ^q (for primes other than p) do not arise as power operations on $K^0(-;\mathbb{Z}_p)$. However, they still exist: they come from automorphisms $\psi^q\colon K_p \to K_p$ of the representing spectrum for p-adic K-theory. In fact, they extend to an action $\mathbb{Z}_p^\times \curvearrowright K_p$. The operations ψ^p and θ^p persist as power operations. In fact: the p-completed K theory of commutative S-algebras is naturally a θ^p-ring with a compatible action by $\mathbb{Z}_p^\times$. This is a key feature in the original construction of tmf, and has been used by Laures, Lawson-Naumann and others to carry out constructions in $K(1)$-local homotopy theory.

The moral here is that power operations for K-theory relate to isogenies of the multiplicative group. One therefore expect that power operations for elliptic cohomology relate to isogenies of elliptic curves. This is one of the main themes of these lectures.

1.20. **Elliptic spectra.** Let me jump ahead and describe the current state of the art about elliptic cohomology.

An **even periodic ring spectrum** is a homotopy commutative ring spectrum E such that the groups $\widetilde{E}^n(S^0) = \pi_{-n}E$ are 0 for n-odd, and such that the map $\pi_2 E \otimes_{\pi_0 E} \pi_{-2}E \to \pi_0 E$ is an isomorphism.

For such a spectrum, we have that $E^0 BU(1) \approx E^0(*)[\![x]\!]$ (non-canonically). The natural map induced by multiplication $\mu\colon U(1)\times U(1) \to U(1)$ gives rise to the structure of a **formal group** (commutative of dimension 1). We write

$$\mathbb{G}_E := \mathrm{Spf}(E^0 BU(1)).$$

The restriction along $S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^\infty = BU(1)$ gives a canonical identification of $\omega_E = \pi_2 E = \widetilde{E}^0(S^2)$ with the cotangent space of $\mathbb{G}_E$ at the identity.

Thus, every even periodic ring is *complex orientable*.

An **elliptic spectrum** is (E, C, α), consisting of an even periodic ring E, an elliptic curve $C/\mathrm{Spec}\pi_0 E$, and an isomorphism $\alpha\colon \mathbb{G}_E \to \widehat{C}$ to the formal completion of C at the identity section.

Many elliptic spectra can be constructed by the following observation.

1.21. **Theorem.** *If $C/\mathrm{Spec}A$ is an elliptic curve whose representing morphism $C\colon \mathrm{Spec}A \to \mathcal{M}_{\mathrm{Ell}}$ is* flat*, then there exists an elliptic spectrum $(\mathrm{Ell}_C, C, \alpha)$ with $\pi_0\mathrm{Ell}_C = A$.*

The flatness condition in this theorem turns out to depend only on the formal completion $\widehat{C}$ of the curve, and the theorem itself amounts to a special case of the Landweber exact functor theorem. In particular, if the line bundle ω_C admits a trivialization, we can describe the resulting homology theory by

$$(\mathrm{Ell}_C)_*(X) = MU_*(X) \otimes_{MU_*} A[u^{\pm}].$$

1.22. *Example* (Landweber-Ravenel-Stong). Let $A = \mathbb{Z}[\frac{1}{6}, c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$, and let C be the projective curve with affine eqaution $y^2 = x^3 - \frac{c_4}{48}x - \frac{1}{864}c_6$. There is a corresponding elliptic spectrum, which is a periodic version of the Landweber-Ravenel-Stong elliptic cohomology theory.

One would like to regard this construction as some kind of "sheaf of spectra" on the stack of elliptic curves with the flat topology (say). However, the assignment $C/\mathrm{Spec}A \mapsto \mathrm{Ell}_C$ is only a functor to the homotopy category of spectra.

1.23. **Theorem** (Goerss-Hopkins-Miller; see the "Talbot book" [DFHH14])**.** *There is a presheaf $\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}$ of commutative S-algebras over the étale site of $\mathcal{M}_{\mathrm{Ell}}$, so that for each étale map $C\colon \mathrm{Spec}A \to \mathcal{M}_\ell$ the ring spectrum $\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}(C)$ is an elliptic spectrum with curve $C/\mathrm{Spec}A$.*

We can think of $\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}$ as a homotopy theoretic lift of the "classical" structure sheaf $\mathcal{O}_{\mathrm{Ell}}$ over $\mathcal{M}_{\mathrm{Ell}}$, whose value $\mathcal{O}_{\mathrm{Ell}}(C/\mathrm{Spec}A) = A$. Furthermore, we have isomorphisms of sheaves

$$\pi_{2n}\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}} \approx \omega^{\otimes n}.$$

There is an extension of this result to the compactification $\overline{\mathcal{M}}_{\mathrm{Ell}}$.

1.24. **Theorem** (Goerss-Hopkins-Miller, Hill-Lawson [HL16])**.** *The above construction extends to a presheaf $\mathcal{O}^{\mathrm{Top}}_{\overline{\mathrm{Ell}}}$ of commutative S-algebras over the log-étale site of $\overline{\mathcal{M}}_{\mathrm{Ell}}$, with an analogous correspondence to the "classical" structure sheaf.*

We can evaluate this sheaf over any log-étale map $\mathcal{N} \to \mathcal{M}$, obtaining a commutative S-algebra $\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}(\mathcal{N})$. Furthermore, there is a spectral sequence

$$H^s(\mathcal{N}, \omega^{t/2}) \Longrightarrow \pi_{t-s}\mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}(\mathcal{N}).$$

1.25. *Example.* Define $\mathrm{TMF} := \mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}(\mathcal{M}_{\mathrm{Ell}})$. This is the ring of **periodic topological modular forms**. The edge map of the spectral sequence

$$\pi_*\mathrm{TMF} \to H^0(\mathcal{M}_{\mathrm{Ell}}, \omega^*) = MF_*[\Delta^{-1}]$$

is not an isomorphism, though it becomes one after 6 is inverted. The discriminant $\Delta \in MF_{12}$ is not in the image; however, the spectrum TMF is periodic on $\Delta^{24} \in \pi_{576}\mathrm{TMF}$.

1.26. *Example.* Define $\mathrm{Tmf} := \mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}}(\overline{\mathcal{M}}_{\mathrm{Ell}})$. This is the ring of **topological modular forms**. The edge map of the spectral sequence

$$\pi_*\mathrm{Tmf} \to H^0(\overline{\mathcal{M}}_{\mathrm{Ell}}, \omega^*) = MF_*$$

is not an isomorphism. Though MF_* is concentrated in non-negative degree, Tmf has much homotopy in negative degrees, much of it contributed by $H^1(\overline{\mathcal{M}}_{\mathrm{Ell}}, \omega^*)$.

1.27. *Example.* Define $\mathrm{tmf} := \mathrm{Tmf}_{\geq 0}$, the (-1)-connected cover of Tmf. This is the ring of **connective topological modular forms**.

1.28. *Example.* There is a map $\overline{\mathcal{M}(n)} \to \overline{\mathcal{M}}$ which over $\mathcal{M}$ carries the universal example of an elliptic curve C equipped with a (naive) full level n structure (i.e., an isomorphism $C[n] \approx \underline{(\mathbb{Z}/n)^2}$ of group schemes). The resulting spectrum of global sections is called $\mathrm{Tmf}(n)$.

The object $(\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{Top}}_{\mathrm{Ell}})$ is a *derived Deligne-Mumford stack*. Lurie has given an interpretation of this object using the notion of a *derived elliptic curve* [Lur09].

Let A be a commutative S-algebra. A **derived elliptic curve** over A is a derived abelian group scheme $C \to \mathrm{Spec}A$ which is an abelian group object in derived schemes over $\mathrm{Spec}A$, and such that the underlying map of schemes is an elliptic curve over $\pi_0 A$ (in the ordinary sense).

An **oriented derived elliptic curve** is data (A, C, α) consisting of a derived elliptic curve $C \to \mathrm{Spec} A$ together with an isomorphism

$$\alpha \colon \widehat{C} \xrightarrow{\sim} \mathrm{Spf}\, A^{BU(1)_+}$$

of formal derived group schemes.

1.29. **Theorem** (Lurie).

- *The object $(\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}_{\mathrm{Ell}}^{\mathrm{Top}})$ is the moduli stack of oriented elliptic curves. In particular, there is a universal oriented derived elliptic curve $\mathcal{C} \to \mathcal{M}_{\mathrm{Ell}}$.*
- *Given a map $\mathcal{C} \colon \mathrm{Spec} A \to \mathcal{M}_{\mathrm{Ell}}$ classifying an oriented derived elliptic curve, there exists a globally equivariant cohomology theory associated to it: i.e., for each compact Lie group, an equivariant cohomology theory $\mathrm{Ell}_{\mathcal{C}}(-//G) \colon G\mathrm{Top}^{\mathrm{op}} \to \mathrm{Sp}$, with change of group isomorphisms $\mathrm{Ell}^*_{\mathcal{C}}((X \times_H G)//G) \approx \mathrm{Ell}^*_{\mathcal{C}}(X//H)$ whenever $H \subseteq G$ and $H \curvearrowright X$.*
- *When G is an abelian compact Lie group and X is a finite G-CW-complex, the value $\mathrm{Ell}^*_{\mathcal{C}}(X)$ is naturally the global sections of a coherent sheaf $\mathcal{F}(X)$ on the derived group scheme $\mathcal{C} \otimes \widehat{G}$.*

In particular, for $G = U(1)$, we obtain for each $\mathcal{C} \to \mathrm{Spec} A$ a $U(1)$-equivariant cohomology theory taking values in coherent sheaves on $\mathcal{C}$.

Some remarks.

- This is a great theorem. However, it is not so easy to use, since in general it is hard to construct maps $\mathrm{Spec} A \to (\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}_{\mathrm{Ell}}^{\mathrm{Top}})$. Note that there is one case where such maps exist for free: those whose underlying map is étale.
- There are many examples of elliptic cohomology theories which don't fit into this set, or at least aren't known to. For instance, there is an elliptic cohomology theory associated to the general Weierstrass equation, defined over the ring $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$. The spectrum is
$$\mathrm{Ell}_{\mathrm{Weier}} = \mathrm{TMF} \wedge (\Omega SU(4))^{\gamma},$$
where the last term is the Thom spectrum on the Bott map $\Omega SU(4) \to BU$. This is an E_2-ring; I don't know if it can be realized as a commutative S-algebra.

  Another such example is the ring of *topological quasimodular forms.* There is a stack $\mathcal{Q} \to \mathcal{M}_{\mathrm{Ell}}$, which classifies the data (C, s) consisting of an elliptic curve $C/\mathrm{Spec} A$ together with a splitting s of the "Hodge extension"
$$0 \to H^0(C, \Omega^1_C) \to H^1_{\mathrm{dR}}(C/S) \to H^1(C, \mathcal{O}_C) \to 0$$
(an exact sequence of A-modules). You can extend this over the cusp, and the resulting ring of global sections is
$$\mathrm{QMF}_* = H^0(\overline{\mathcal{Q}}, \omega^*) \approx \mathbb{Z}[b_2, b_4, b_6, \tfrac{1}{4}(b_2 b_6 - b_4^2)].$$
The element $b_2 \in \mathrm{QMF}_2$ can be related to the Eisenstein series $12\, G_2(\tau)$. There is a candidate topological version:
$$\mathrm{tqmf} := \mathrm{tmf} \wedge (\Omega S^5)^{\gamma},$$

with a map $\pi_*\mathrm{tqmf} \to QMF_*$ (the class b_2 isn't in the image, but $2b_2 = 24\, G_2(\tau)$ is). This is a strictly associative ring spectrum, and is also homotopy commutative. Can it be made strictly commutative, or even just E_2?

- Lurie's technology is functorial with respect to isomorphisms of (oriented derived) elliptic curves. One might ask whether one obtains interesting maps between equivariant elliptic cohomology theories coming from *homomorphisms* between elliptic curves.

  There is an obvious constraint. Equivariant cohomology theories are naturally associated to the pair $(\mathcal{C}/S, \alpha)$, where α is an orientation. Any map $\mathcal{C} \to \mathcal{C}'$ of oriented derived elliptic curves over A is necessarily étale, since preserving the orientation implies that $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is an isomorphism. Thus, we can only expect to construct maps of equivariant cohomology theories associated to *separable* isogenies.

I'm not going to explain the proofs of the above theorems. Nowadays there are decent references for the Goerss-Hopkins-Miller construction, e.g., in the Talbot book.

1.30. **The σ-orientation.** There are several results relating to the Witten genus. Work of Ando-Hopkins-me, building on work of Ando-Hopkins-Strickland [AHS01], [AHS04], proves

1.31. **Theorem** (Ando-Hopkins-Rezk)**.** *The Witten genus refines to a map $M\mathrm{String} \to \mathrm{tmf}$ of commutative S-algebras.*

I will discuss some aspects of this in later talks. The proof involves commutative S-algebra models for elliptic cohomology, and depends in the end on power operations for various theories.

Rigidity can be explained using models of equivariant elliptic cohomology: in the complex analytic case by Rosu [Ros01] and Ando-Basterra [AB02]. Lurie has a more general formulation of this.

2. Descent for isogenies and deformations of formal groups

There is one piece of equivariant elliptic cohomology which is accessible from non-equivariant algebraic topology. This is the one associated to the *universal deformation* of a supersingular elliptic curve, for groups G which are finite p-groups. This cohomology theory can be identified with *Borel equivariant* Morava E-theory.

2.1. **Formal groups in finite characteristic.** The theory of power operations for Morava E-theory is due to Ando, Hopkins, and Strickland. It is a template for power operations in equivariant elliptic cohomology. Morava E-theory is the theory associated to the universal deformation of some formal group of finite height.

We consider formal groups over some ring R, and homomomorphisms between such. A particularly important example in characteristic p is the

2.2. *Example* (Frobenius homomorphism). If $\mathbb{F}_p \subseteq R$, write $\sigma\colon R \to R$ for the absolute Frobenius. The relative Frobenius is a homomorphism

$$F^r\colon G \to (\sigma^r)^* G.$$

defined on functions by

$$\mathrm{id} \otimes \sigma\colon R^{\sigma^r} \otimes_R \mathcal{O}_G = \mathcal{O}_{\sigma^{r*}G} \to \mathcal{O}_G.$$

In terms of any local coordinate on G, it is given by $x \mapsto x^{p^r}$.

If G/k where k is a field of characteristic p, then any non-zero homomorphism $f\colon G \to G'$ of formal groups can be factored uniquely as $f = g \circ F^r$, where $g\colon (\sigma^r)^*G \to G'$ is an isomorphism. We say f has degree p^r.

Suppose k has characteristic p, and consider the homomorphism $[p]\colon G \to G$. We say that G has **height** h if $\deg[p] = p^h$, or infinite height if $[p] = 0$. We have $h \in \{1, 2, \dots, \infty\}$.

The category of formal groups over a field k of char p is well understood.

2.3. **Theorem.** *If k is separably closed, then two formal groups over k are isomorphic if and only if they have the same height. For a given G of finite height h, the endomorphisms* $\mathrm{End}(G)$ *are a finite pro-etale ring scheme, non-canonically isomorphic to $D_h \approx \mathbb{W}\mathbb{F}_{p^h}\langle S\rangle/(S^h - p)$ (where $Sa = a^\sigma S$), a maximal order in a central division algebra of rank h^2 and invariant $1/h$ over $\mathbb{Q}_p$.*

The automorphism group $\mathbb{S}_h = \mathrm{Aut}(G_0)$ is called the Morava stabilizer group.

2.4. **Deformations of formal groups.** Let R be a complete Noetherian local ring, with $p \in \mathfrak{m}$. Say that a formal group G/R is a **deformation** of a height h formal group if $G_{R/\mathfrak{m}}$ is height h.

Morphisms between such are completely determined by their restriction to the special fiber.

2.5. **Proposition.** *Suppose $f, f'\colon G \to G'$ are homomorphisms between formal groups over R which are deformations of height h formal groups. Then $f = f'$ if and only if $f_{R/\mathfrak{m}} = f'_{R/\mathfrak{m}}$.*

Proof. Because we can add homomorphisms, it suffices to consider the case with $f' = 0$. Because R is complete and noetherian, the result follows by proving it inductively for R with $\mathfrak{m}^n = 0$. In fact, suppose $R \supset \mathfrak{m} \supset I$ with $\mathfrak{m}I = 0$ (e.g., $I = \mathfrak{m}^{n-1}$), and suppose $f_{R/I} = 0$. We want to show $f = 0$.

We use the identity $f \circ [p]_G = [p]_{G'} \circ f$. In choices of coordinates for G and G', write

$$f(x) = ax^k + \text{higher degree} \in I[\![x]\!],$$
$$[p]_G(x) \equiv cx^{p^h} + \text{higher degree} \mod \mathfrak{m},$$
$$[p]_{G'}(y) \equiv c'x^{p^h} + \text{higher degree} \mod \mathfrak{m},$$

where c, c' are units. Comparing leading coefficients (of x^{kp^h}) in the identity, we get $ac^k = c'a^{p^h}$. Since c and c' are units and $a \in I$ squares to 0, we must have $a = 0$. $\square$

Fix a formal group G_0 of height $h < \infty$, over a perfect field k.

A **deformation** of G_0 to G is data (G, i, α), where

- G is a formal group over R,
- $i\colon k \to R/\mathfrak{m}$ is an inclusion of fields,
- $\alpha\colon G_{R/\mathfrak{m}} \xrightarrow{\sim} i^*G_0$ is an isomorphism of formal groups over $R/\mathfrak{m}$.

(If $k = \mathbb{F}_p$, we can omit i.)

The data (i, α) will also be called a G_0**-deformation structure** on G. Write $\mathcal{D}_{G_0}(G/R)$ for the set of such. Any isomorphism $f\colon G \to G'$ of formal groups over R induces a map $f_!\colon \mathcal{D}_{G_0}(G/R) \to \mathcal{D}_{G_0}(G'/R)$, sending

$$f_!\colon (i, \alpha) \mapsto (i, f_{R/\mathfrak{m}}^{-1} \circ \alpha).$$

$$G \xrightarrow[\sim]{f} G'$$

$$\begin{tikzcd} G_{R/\mathfrak{m}} \arrow[r, "f_{R/\mathfrak{m}}", "\sim"'] \arrow[d, "\alpha"', "\sim"] & G'_{R/\mathfrak{m}} \arrow[d, "\sim"', "\alpha'"] \\ i^*G_0 \arrow[r, "\sim", "\mathrm{id}"'] & i^*G_0 \end{tikzcd}$$

By the above proposition, if (G, i, α) and (G', i', α') are G_0-deformations, there exists *at most one* isomorphism $f\colon G \to G'$ compatible with the deformation structure (in the sense that $i = i'$ and $\alpha' \circ f_{R/\mathfrak{m}} = \alpha$; these are sometimes called $\star$-isomorphisms). In particular, deformations have no non-trivial automorphisms.

2.6. **Theorem** (Lubin-Tate)**.** *The functor*

$$R \mapsto \textit{iso. classes of } G_0\textit{-deformations over } R$$

is representable by a complete local ring $A = A_{G_0}$. Furthermore, there is a non-canonical isomorphism

$$A \approx \mathbb{W}k[\![a_1, \ldots, a_{h-1}]\!].$$

The tautological example is the universal deformation G_{univ} of G_0 over A.

We can also think of A_{G_0} as classifying deformation structures on a given formal group. Given a formal group G/R, there is a bijection

$$\mathcal{D}_{G_0}(G/R) = \{G_0 \text{ def. str. on } G/R\} \leftrightarrows \{\phi\colon A_{G_0} \to R \text{ s.t. } \phi^* G_{\mathrm{univ}} \approx G \text{ as f.g. } /R\}$$

There's an obvious action of $\mathrm{Aut}(G_0) \curvearrowright \mathcal{D}_{G_0}(G/R)$, (by $\gamma \cdot (i, \alpha) = (i, i^*(\gamma) \circ \alpha)$) and thus an action $\mathrm{Aut}(G_0) \curvearrowright A_{G_0}$.

2.7. **Deformation structures and isogenies.** We say that a homomorphism $f\colon G \to G'$ of formal groups is an **isogeny** if the induced map on functions $f^*\colon \mathcal{O}_{G'} \to \mathcal{O}_G$ is finite and locally free. Any homomorphism f between deformations is an isogeny iff $f_{R/\mathfrak{m}}$ is a non-zero homomorphism.

Isogenies have well-defined kernels: $\mathrm{Ker}\, f = \mathrm{Spec}\mathcal{O}_G \otimes_{\mathcal{O}_{G'}} R$ is a finite abelian group scheme over R.

We can extend the construction $\mathcal{D}_{G_0}$ to a functor

$$\{\text{f.g. over } R \text{ and isogenies}\} \to \{\text{Sets}\}$$

as follows. Given an isogeny $f\colon G \to G'$ of degree p^r over R and $(i, \alpha) \in \mathcal{D}_{G_0}(G/R)$, define $f_!(i, \alpha) = (i', \alpha') \in \mathcal{D}_{G_0}(G'/R)$ using the diagram

$$G \xrightarrow{f} G'$$

$$\begin{tikzcd} G_{R/\mathfrak{m}} \arrow[r, "f_{R/\mathfrak{m}}"] \arrow[d, "\alpha"', "\sim"] & G'_{R/\mathfrak{m}} \arrow[d, "\sim"', "\alpha'"] \\ i^*G_0 \arrow[r, "F^r"'] & i^*(\sigma^r)^*G_0 \end{tikzcd}$$

so that $i' := i \circ \sigma^r$ and α' is the unique isomorphism making the diagram commute. This works because all isogenies of degree p^r between formal groups over k factor through F^r.

In other words, we can push-forward a deformation structure along isogenies, by factoring it through the appropriate power of Frobenius.

Conversely, given deformations (G, i, α) and (G', i', α') and an $r \geq 0$, there exists *at most one* isogeny $f \colon G \to G'$ which is compatible with F^r, in the sense that the above diagram commutes. In fact, the data "pair of deformations related by an isogeny compatible with F^r" (up to $\star$-isomorphism) is representable by a ring

$$A_r = A_{G_0, F^r} \approx (A_{G_0} \widehat{\otimes} A_{G_0})/J_r.$$

This ring carries the universal example of an isogeny $s^* G_{\mathrm{univ}} \to t^* G_{\mathrm{univ}}$ which deforms $F^r \colon G_0 \to (\sigma^r)^* G_0$.

Observation: if we fix the source deformation (G, i, α), then an isogeny $f \colon (G, i, \alpha) \to (G', i', \alpha')$ which deforms F^r is determined, up to $\star$-isomorphism, by the kernel $H := \operatorname{Ker} f$, which is a finite subgroup scheme of G. Thus,

$$\{A_r \to R\} \leftrightarrows \{(G_0\text{-def. } (G, i, \alpha),\ H \leq G \text{ subgp. of rank } p^r)\}$$

2.8. **Morava E-theory.** For any G_0/k (finite height), there is an even periodic ring spectrum $K = K_{G_0}$ with $\pi_0 K = k$ and $\mathbb{G}_K = G_0$. (With the caveat that if $p = 2$, it is not actually homotopy commutative.) This admits a strictly associative ring structure (Robinson, Baker), but it does not admit the structure of a commutative S-algebra. Such K are sometimes called **periodic Morava K-theories**.

(The actual Morava K-theory spectrum $K(h)$ is an "indecomposable summand" of K_{G_0}.)

For any G_0/k, there is an even periodic ring spectrum $E = E_{G_0}$ associated to the universal deformation of G_0, with $\pi_0 E = A_{G_0}$, and $\mathbb{G}_E = G_{\mathrm{univ}}$. It can be constructed as the spectrum representing a Landweber exact cohomology theory.

2.9. **Theorem** (Goerss-Hopkins-Miller). *Every E_{G_0} admits (essentially uniquely) the structure of a commutative S-algebra. Furthermore, the assignment $G_0/k \mapsto E_{G_0}$ refines to a functor*

$$\{\textit{finite ht. f.g. over perfect fields and isos.}\} \to \{\textit{comm } S\textit{-alg.}\}.$$

In particular, the automorphism group of G_0 acts on E_{G_0} though maps of commutative S-algebras.

2.10. **Power operations for Morava E-theory.** Consider the power operation associated to the commutative S-algebra E. This has the form

$$P_m \colon E^0 X \to E^0(X \times B\Sigma_m).$$

Explicitly, this is defined by the construction which takes a map $a \colon \Sigma^\infty_+ X \to E$ of spectra, and produces the map

$$\Sigma^\infty_+ X \to \Sigma^\infty_+ X^{\times m}_{h\Sigma_m} \approx (\Sigma^\infty_+ X)^{\wedge m}_{h\Sigma_m} \xrightarrow{f^{\wedge m}} E^{\wedge m}_{h\Sigma_m} \to E.$$

This is multiplicative $P_m(ab) = P_m(a) P_m(b)$ but not additive. However, the failure to be additive is encoded in a formula

$$P_m(a + b) = \sum_{i+j=m} \operatorname{Tr}^{\Sigma_m}_{\Sigma_i \times \Sigma_j} P_i(a) \times P_j(b).$$

Here $P_i(a) \times P_j(b) \in E^0(X \times B\Sigma_i \times B\Sigma_j)$ is a partial product, and $\mathrm{Tr}^{\Sigma_m}_{\Sigma_i \times \Sigma_j}$ is the transfer map. This is a consequence of the "binomial formula" for symmetric products

$$(X \amalg Y)^{\times m}_{h\Sigma_m} \approx \coprod_{i+j=m} (X^{\times i} \times Y^{\times j} \times \Sigma_m/(\Sigma_i \times \Sigma_j))_{h\Sigma_m}.$$

The **transfer ideal** $I \subset E^0 X \times B\Sigma_m$ is the sum of the images of all transfer maps

$$\mathrm{Tr}\colon E^0 X \times B\Sigma_i \times B\Sigma_j \to E^0 X \times B\Sigma_m$$

associated to *proper* subgroups of the form $\Sigma_i \times \Sigma_{m-i} \subset \Sigma_m$. The resulting map to the quotient

$$E^0 X \to E^0(X \times B\Sigma_m) \to E^0(X \times B\Sigma_m)/I$$

is a ring homomorphism.

It is a fact (HKR) that $E^0 B\Sigma_m$ is finite and free as a $\pi_0 E$-module. So we can rewrite this as

$$\overline{P}_m\colon E^0 X \to E^0(X) \otimes_{\pi_0 E} E^0 B\Sigma_m \to E^0 X \otimes_{\pi_0 E} E^0 B\Sigma_m/I.$$

Because E is a p-local theory, $I = E^0 B\Sigma_m$ unless $m = p^r$.

2.11. **Theorem** (Strickland [Str97], [Str98]). *There is an isomorphism of rings $A_r \approx E^0 B\Sigma_{p^r}/I$. Furthermore, the universal deformation of F^r is obtained by evaluating the map at $X = BU(1)$, in the sense that*

- *$s\colon A_0 \to A_r$, classifying the source, is $E^0(*) \to E^0 B\Sigma_{p^r}/I$ induced by $B\Sigma_m \to *$;*
- *$t\colon A_0 \to A_r$, classifying the target, is $\overline{P}_{p^r}\colon E^0(*) \to E^0 B\Sigma_{p^r}/I$;*
- *the universal lift $s^* G_{\mathrm{univ}} \to t^* G_{\mathrm{univ}}$ defined over A_r is represented by the ring map*

$$E^0 BU(1) \otimes_{A_0} {}^t A_r \to E^0 BU(1) \otimes_{A_0} {}^s A_r$$

produced by evaluating the commutative square

$$\begin{array}{ccc} E^0(*) & \xrightarrow{\overline{P}_{p^r}} & E^0(*) \otimes_{E^0(*)} E^0 B\Sigma_{p^r}/I \\ \downarrow & & \downarrow \\ E^0 X & \xrightarrow[\overline{P}_{p^r}]{} & E^0 X \otimes_{E_0(*)} E^0 B\Sigma_{p^r}/I \end{array}$$

Remarks.

- We have $E^0(*)/\mathfrak{m} \otimes_{E^0(*)} E^0 B\Sigma_{p^k}/I \approx E^0(*)/\mathfrak{m}$. Thus,

$$\begin{tikzcd} E^0 X \arrow[r] \arrow[dr, "x \mapsto x^{p^r}"'] & E^0 X \otimes_{E^0(*)} E^0 B\Sigma_{p^r} \arrow[r] \arrow[d] & E^0 X \otimes_{E^0} E^0 B\Sigma_{p^r} \arrow[d] \\ & E^0 X \otimes_{E^0} E^0(*) \arrow[r] & E^0 X \otimes_{E^0} E^0/\mathfrak{m} \end{tikzcd}$$

  showing that the isogeny defined by topology really is a deformation of F^r.
- An important part of proving this involves showing that $E^0 B\Sigma_{p^r}/I$ is itself a free E^0-module. This is a consequence of results of Kashiwabara [Kas98] that $K(n)_*(QS^0)$ is a polynomial algebra.

- The rest of Strickland's argument is quite complicated. (1) He shows that $s\colon A_0 \to A_r$ is finite and free of the expected degree (same as the rank of $E^0 B\Sigma_{p^r}/I$ as an E_0-module). (2) The map $A_r \to E^0 B\Sigma_{p^r}/I$ is shown to be injective after tensoring with $A_0/\mathfrak{m}$ by a delicate calculation using a description of certain elements in terms of euler classes of vector bundles.

  Schlank and Stapleton [SS15] have an alternate proof of the second part of the argument, using a "transchromatic character map" to reduce to the case of height 1.

2.12. **Operations on the homotopy of $K(n)$-local commutative S-algebras.** I've described power operations as operations on E^0X.

In fact, the proper context is $K(n)$-local commutative E-algebras. Given a commutative E-algebra R, the power construction defines a function

$$P_m\colon \pi_0 R \to \pi_0 R^{B\Sigma_m^+},$$

where $R^{Y_+} = \mathcal{F}(\Sigma_+^\infty Y, R)$. The case of $R = E^{X_+}$ is what is discussed above.

A spectrum F is **$K(n)$-local** if $[X, F] \approx 0$ for any spectrum X with $K(n)_*X = 0$. The map

$$R \wedge_E E^{B\Sigma_m^+} \to R^{B\Sigma_m^+}$$

is not a weak equivalence in general, but is if R is assumed to be $K(n)$-local as well. Thus we obtain a function

$$P_m\colon \pi_0 R \to \pi_0(R \wedge_E E^{B\Sigma_m^+}) \approx \pi_0 R \otimes_{\pi_0 E} E^0 B\Sigma_m,$$

and we proceed as above.

Let $B = \pi_0 R$. The data of power operations provides for each $r \geq 0$ a map of A_0-algebras

$$\psi_r\colon B \to B \otimes_{A_0} {}^s A_r{}^t.$$

Furthermore, for each r, r' the diagram

$$\begin{tikzcd} B \arrow[r, "\psi^r"] \arrow[d, "\psi^{r+r'}"'] & B \otimes_{A_0} {}^s A_r{}^t \arrow[d, "\psi^{r'} \otimes \mathrm{id}"] \\ B \otimes_{A_0} {}^s A_{r+r'}{}^t \arrow[r, "\mathrm{id} \otimes \mu^*"'] & B \otimes_{A_0} {}^s A_{r'}{}^t \otimes_{A_0} {}^s A_r \end{tikzcd}$$

commutes, where $\mu^*\colon A_{r'+r} \to A_{r'}{}^t \otimes {}^s A_r$ classifies the composition of a deformation of F^r with a deformation of $F^{r'}$.

We say that the A_0-algebra B is equipped with "descent for isogenies". That is, any G_0-deformation to R determines a ring

$$\underline{B}_R(G, (i, \alpha)) := B \otimes_{A_0} {}^{G,(i,\alpha)} R,$$

while any isogeny $f\colon G \to G'$ determines a map of rings

$$\underline{B}_R(G, (i, \alpha)) \to \underline{B}(G', f_!(i, \alpha)),$$

defined using

$$B \otimes_{A_0} {}^t A_r \xrightarrow{\psi_r \otimes 1} B \otimes_{A_0} {}^s A_r$$

and the map $A_r \to R$ classifying f. Recall that $f_!(i, \alpha)$ is the pushforward of the deformation structure along the isogeny described earlier.

The data $\underline{B}_R$ is a functor

$$\{G_0\text{-defs. to } R \text{ and isogenies}\} \to \{\text{comm. } R\text{-algebra}\}.$$

2.13. **Frobenius congruence.** There is a pullback square of rings

$$\begin{tikzcd} E^0 B\Sigma_p \arrow[r] \arrow[d, "(*\to B\Sigma_p)^*"'] & E^0 B\Sigma_p/I \arrow[d] \\ E^0(*) \arrow[r] & E^0(*)/p \end{tikzcd}$$

The right-hand column is a map $A_1 \to A_0/p$. The ring A_0/p supports the universal deformation of G_0 to any ring R of characteristic p. There is a distinguished (but not unique) example of an isogeny between deformations in characteristic p, namely the (relative) Frobenius isogeny

$$F\colon G \to \sigma^* G.$$

The fact that $\psi_1\colon \pi_0 R \to \pi_0 R \otimes_{A_0}{}^s A_1$ lifts to a map $\pi_0 R \to \pi_0 R \otimes_{E_0(*)} E^0 B\Sigma_p$ means that

$$\begin{tikzcd} \pi_0 R \arrow[r, "\psi_1"] \arrow[d] & \pi_0 R \otimes_{A_0} A_1 \arrow[d] \\ \pi_0 R/p \arrow[r, "x\mapsto x^p"'] & \pi_0 R/p = \pi_0 R \otimes_{A_0} A_0/p \end{tikzcd}$$

commutes. We call this the *Frobenius congruence* [Rez09]. The operation $P_p\colon \pi_0 R \to \pi_0 R \otimes E^0 B\Sigma_p$ is an explicit lift of ψ_1, which we think of as a *witness* for the Frobenius congruence.

Frobenius congruence amounts to saying that, for any $R \supset \mathbb{F}_p$ the functor $\overline{B}_R$ carries the Frobenius isogeny $F\colon G \to \sigma^* G$ to the relative pth power map between R-algebras.

2.14. **Example: multiplicative group.** Let $G_0/\mathbb{F}_p$ be the multiplicative group. Then $A_0 = \mathbb{Z}_p$, and G_{univ} can also be taken to be the multiplicative group. The Morava E-theory can be identified with $E = K_p$, p-complete K-theory.

We have that $A_r = \mathbb{Z}_p$ for all r.

Thus, the data of descent for isogenies on an A_0-algebra B amounts to giving a single ring homomorphism

$$\psi\colon B \to B,$$

which must satisfy $\psi(x) \equiv x^p \mod pB$. The lift to a function $B \to B \otimes E^0 B\Sigma_p$ gives an explicit witness to this congruence.

Thus, π_0 of a $K(1)$-local K_p-algebra is a θ^p-ring.

2.15. **Supersingular elliptic curves.** Let C_0/k be an elliptic curve over a field of characteristic p, and let $\widehat{C}_0$ be its formal completion. Then either

- $\widehat{C}_0$ has height 1, in which case we say C is **ordinary**, or
- $\widehat{C}_0$ has height 2, in which case we say C is **supersingular**.

There are only finitely many supersingular curves (up to isomorphism) at any prime p.

Suppose C_0/k is supersingular. Then C_0/k has a universal deformation, which is an elliptic curve C_{univ}/A_0 with $A_0 \approx \mathbb{W}k[\![a]\!]$. The underlying formal group $\widehat{C}_{\text{univ}}$ is in fact the universal deformation of the height 2-formal group $\widehat{C}_0$.

The associated Morava E-theory is an elliptic spectrum.

2.16. *Example* ([Rez08]). Let C_0/F_2 be the curve with (affine) Weierstrass equation $y^2+y=x^3$. Write $E=E_{\widehat{C}_0}$ for the associated Morava E-theory, with $\pi_0 E = A_0 = \mathbb{W}k[\![a]\!]$.

We have $A_1 \approx A_0[d]/(d^3-ad-2)$, with

$$A_0 \xrightarrow{s} A_1 \xleftarrow{t} A_0, \qquad a \mapsto a, \quad a' = a^2+3d-ad^2 \mapsfrom a.$$

The map $A_1 \to A_0/p$ classifying Frobenius is given by $a \mapsto a$, $d \mapsto 0$.

If $B=\pi_0 R$, then the above structure is entirely determined by the single map $P_p\colon B \to B \otimes E^0 B\Sigma_p$. The map P_p produces a ring a homomorphism

$$B \xrightarrow{\psi_1} B \otimes_{A_0} {}^s A_1,$$

together with a witness for the Frobenius congruence (which says $\psi_1(x) \equiv x^2 \mod d$). The homomorphism ψ_1 is subject to a single relation, namely that there *exists* a ψ_2-factoring

$$\begin{tikzcd} B \arrow[r, "\psi_1"] \arrow[d, dotted, "\psi_2"'] & B \otimes_{A_0} {}^s A_1{}^t \arrow[d, "\psi_1 \otimes \mathrm{id}"] \\ B \otimes_{A_0} {}^s A_2{}^t \arrow[r, tail, "\mathrm{id} \otimes \mu^*"'] & B \otimes_{A_0} {}^s A_1{}^t \otimes_{A_0} {}^s A_1 \end{tikzcd}$$

(The inclusion $A_2 \subset A_1 \otimes A_1$ is split as a map of left A_0-modules.) All higher ψ_r are uniquely determined once this is known. This is a generic feature for descent for isogenies associated to any formal group: the bialgebra $\prod A_r$ is *quadratic* in a precise sense [Rez12a].

For a formal group of height 2, the subring $A_2 \subset A_1 \otimes A_1$ is a pullback

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d] & A_1{}^t \otimes_{A_0} {}^s A_1 \arrow[d, "\mathrm{id} \otimes w"] \\ A_0 \arrow[r, "s"'] & A_1 \end{tikzcd}$$

where This map w classifies the "dual isogeny". Every degree p-isogeny f factors the multiplication by p map, so there exists a unique $\widehat{f}$ such that $\widehat{f} f = p$. When the formal group has height 2, $\widehat{f}$ is also a degree p-isogeny, and $f \mapsto \widehat{f}$ is represented by w. In our $p=2$ example, w acts by $a \mapsto a'$ and $d \mapsto d' = a - d^2$.

Analogous calculations are known for s.s. curves over $\mathbb{F}_3$ and $\mathbb{F}_5$, by work of Yifei Zhu [Zhu14], [Zhu15]. For more about calculation of power operations in the height 2 case, see [Rez13].

3. Modular isogeny complexes, and the Koszul property

Last time, I described how power operations for Morava E-theory are basically equivalent to giving "descent for isogenies" to deformations of formal groups.

I want to put this is a wider context, by thinking about "descent for isogenies" for elliptic curves.

3.1. **Descent for isogenies in the elliptic moduli stack.** Recall $\mathcal{M} = \mathcal{M}_{\mathrm{Ell}}$, the moduli stack of (smooth) elliptic curves.

$$\{S \to \mathcal{M}\} \leftrightarrow \{(\text{groupoid of}) \text{ ell. curves } C \to S\}$$

For each $N \geq 1$, there is an object $\mathcal{M}_{\mathrm{Isog}\,N}$, classifying N-isogenies of elliptic curves.

$$\{S \to \mathcal{M}_{\mathrm{Isog}\,N}\} \leftrightarrow \{\text{gpd. of } N\text{-isogenes } C \to C' \text{ def. over } S\}.$$

There are maps $\mathcal{M} \xleftarrow{s} \mathcal{M}_{\mathrm{Isog}\,N} \xrightarrow{t} \mathcal{M}$ encoding the source and target. We can think of the map s as carrying the "universal subgroup of rank N". That is, every N-isogeny $f\colon C \to C'$ over S deterimines a family $G := \operatorname{Ker} f \leq C$ of subgroup schemes (finite flat over S of rank N). Conversely, given such $G \leq C$ we can form the quotient map $g\colon C \to C/G$, and any N-isogeny $f\colon C \to C'$ with $\ker f = G$ factors uniquely through g.

The map s is **representable**. This means that for every $C\colon \mathrm{Spec} A \to \mathcal{M}$, there is a pullback square

$$\begin{tikzcd}
\mathrm{Spec} A_{\mathrm{Isog}\,N} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{Isog}\,N} \arrow[d, "s"'] \\
\mathrm{Spec} A \arrow[r, "C"'] & \mathcal{M}
\end{tikzcd}$$

of stacks.

3.2. **Theorem** (Katz-Mazur [KM85]). *Each $s\colon A \to A_{\mathrm{Isog}\,N}$ is flat and locally free, of rank $= \#\{$subgps. of $(\mathbb{R}/\mathbb{Z})^2$ of order $N\}$.*

Let $\mathcal{M}_{\mathrm{Isog}} := \coprod_{N\geq 1} \mathcal{M}_{\mathrm{Isog}\,N}$. There is a simplicial object $\mathcal{M}_\bullet$ of the form

$$\mathcal{M} \leftleftarrows \mathcal{M}_{\mathrm{Isog}} \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \mathcal{M}_{\mathrm{Isog}}{}^{t} \times_{\mathcal{M}} {}^{s}\mathcal{M}_{\mathrm{Isog}} \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \cdots$$

which encodes the fact that "elliptic curves and isogenies" forms a category. We can think of $\mathcal{M}_d$ as the thing that represents sequences $\{C_0 \to \cdots \to C_d\}$ of isogenies over S, or equivalently as representing chains $G_1 \leq G_2 \leq \cdots \leq G_d \leq C_0$ of finite subgroups, where $G_i = \operatorname{Ker}[C_0 \to C_i]$.

3.3. **Curves with descent for isogenies.** Fix a map of stacks $\mathcal{Y} \to \mathcal{M}$. We can think of this as characterizing an elliptic curve over $\mathcal{Y}$. Alternately, Given $C\colon S \to \mathcal{M}$, we can consider the collection (set or groupoid or ∞-groupoid) of lifts

$$Y(C/S) := \left\{ \begin{tikzcd} & \mathcal{Y} \arrow[d] \\ S \arrow[r] \arrow[ur, dotted] & \mathcal{M} \end{tikzcd} \right\}$$

The element $\alpha \in Y(C/S)$ is a "Y-structure on C/S", and we can regard $\mathcal{Y}$ as the moduil stack of elliptic curves equipped with a chosen Y-structure.

3.4. *Example.* The universal curve $\mathcal{Y} = \mathcal{E} \to \mathcal{M}$. Here $Y(C/S) = C(S) = \Gamma\big(C \to S\big)$ the set of sections.

3.5. *Example.* A "full level n-structure" on C/S is a choice C/S of isomorphism $\lambda\colon (\mathbb{Z}/n)^2_S \xrightarrow{\sim} C[n]$ of group schemes. There is a corresponding stack $\mathcal{Y} = \mathcal{M}(n) \to \mathcal{M}$. Note that $\mathcal{M}(n) = \mathrm{Spec} M(n)$ for $n \geq 4$, and the map is étale.

Say $\mathcal{Y} \to \mathcal{M}$ has **descent for isogenies** if it is equipped with a map $\mathcal{Y}_\bullet \to \mathcal{M}_\bullet$ of simplicial stacks such that

- $\mathcal{Y}_0 \to \mathcal{M}_0$ is the given map,
- for each k, the square
$$\begin{tikzcd} \mathcal{Y}_k \arrow[r] \arrow[d, "s"'] & \mathcal{M}_k \arrow[d, "s"] \\ \mathcal{Y}_0 \arrow[r] & \mathcal{M}_0 \end{tikzcd}$$
is a pullback of stacks. (Here s is the operator induced by $\langle 0 \rangle \colon [0] \to [k]$.)

3.6. *Remark.* Here is how to think about this. Note that $\mathcal{Y}_1 \approx \mathcal{Y} \times_{\mathcal{M}} {}^{s}\mathcal{M}_{\mathrm{Isog}}$. We have
$$\{S \to \mathcal{Y}_1\} \leftrightarrow \{(f \colon C \to C', \alpha \in Y(C/S))\}.$$
The simplicial structure in $\mathcal{Y}_\bullet$ induces
$$\begin{tikzcd} S \arrow[r, "{(C,\alpha,f)}"] \arrow[rr, bend left, "{(C',\alpha')}"] \arrow[dr, "{(C,\alpha)}"'] & \mathcal{Y}_1 \arrow[r, "t"'] \arrow[d, "s"] & \mathcal{Y} \\ & \mathcal{Y} & \end{tikzcd}$$
Thus we obtain $f_! \colon Y(C/S) \to Y(C'/S)$, by $f_!(\alpha) = \alpha'$. Thus, descent for isogenies provides a way to "push Y-structures forward along isogenies". The properties imposed on $\mathcal{Y}_\bullet$ imply that this pushforward is natural and functorial.

3.7. *Remark.* Suppose $\mathcal{Y} = \mathrm{Spec} A \to \mathcal{M}$ is representable by a ring A, and has descent for isogenies. Then $\mathcal{Y}_1 = \coprod \mathrm{Spec} A_{\mathrm{Isog}\, N}$. Thus the maps $\mathcal{Y} \xleftarrow{s} \mathcal{Y}_1 \xrightarrow{t} \mathcal{Y}$ are represented a collection of ring maps
$$A \xrightarrow{s} A_{\mathrm{Isog}\, N} \xleftarrow{t} A.$$
There are also maps $A_{\mathrm{Isog}\, NN'} \to A_{\mathrm{Isog}\, N}{}^{t} \otimes_A {}^{s}A_{\mathrm{Isog}\, N'}$ representing composition.

Associated to this is a category $\mathrm{Mod}(\mathcal{Y})$ of **isogeny modules on** $\mathcal{Y}$. The objects are A-modules M equipped with A-module maps
$$\psi_{\mathrm{Isog}\, N} \colon M \to M \otimes_A {}^{s}A_{\mathrm{Isog}\, N}{}^{t}$$
satisfying the identities of a comodule structure. (You can define a category $\mathrm{Mod}(\mathcal{Y})$ whenever $\mathcal{Y} \to \mathcal{M}$ is representable and has descent for isogenies.)

3.8. **Examples.**

3.9. *Example.* The universal curve $\mathcal{E} \to \mathcal{M}$ has descent for isogenies tautologically. For $f \colon C \to C'$ over S, the map $f_! \colon \mathcal{E}(C/S) \to \mathcal{E}(C'/S)$ is just $f \colon C(S) \to C'(S)$.

3.10. *Example.* The stack $\mathcal{Y} = \mathcal{M}(n) \to \mathcal{M}$ of level n-structures. For any isogeny $f \colon C \to C'$ of degree *prime to* n, we have an induced isomorphism $C[n] \to C'[n]$ of group schemes, and hence a map $f_! \colon \mathcal{Y}(C/S) \to \mathcal{Y}(C'/S)$.

Thus, $\mathcal{M}(n) \to \mathcal{M}$ has *descent for isogenies of degree prime to* n.

3.11. *Example.* Fix a supersingular curve C_0/k at some prime p. There is a formal stack $\mathcal{Y} = \mathcal{M}^\wedge_{C_0} \to \mathcal{M}$, which classifies deformations of C_0 to complete local rings with p topologically nilpotent.

This object has *descent for pth power isogenies*, by the same recipe I described last time for deformations of formal groups. Thus $A_r = A_{\mathrm{Isog}\, p^r}$.

3.12. *Example.* The **Tate curve** $\mathrm{Tate}\colon \mathrm{Spec}\mathbb{Z}((q)) \to \mathcal{M}$ is the "deleted formal neighborhood of infinity" in $\mathcal{M}$. This object has descent for isogenies. This data turns out to encode the power operations for equivariant $\mathrm{Ell}_{\mathrm{Tate}}$ constructed by Ganter [Gan07], [Gan13].

3.13. *Example.* Consider the analytic moduli space of elliptic curves. This is $\mathcal{M}^{\mathrm{an}} = \mathcal{X}/\!/\Gamma$, where $\mathcal{X} = \{\, \tau \in \mathbb{C} \mid \mathrm{Im}\tau \neq 0 \,\}$ and $\Gamma = GL(2, \mathbb{Z})$. The universal curve is $\mathcal{E}/\!/\Gamma$, where $\mathcal{E}_\tau = \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z})$.

Let $M = GL(2, \mathbb{Q}) \cap M_{2\times 2}(\mathbb{Z})$, the monoid of integer matrices with non-zero determinant. We can extend the action to $M \curvearrowright \mathcal{X}$, by the same formula. This is covered by an action $M \curvearrowright \mathcal{E}$ defined by $A(\tau, z) = (A\tau, (\det A)(c\tau + d)^{-1}z)$. The induced map $A\colon \mathcal{E}_\tau \to \mathcal{E}_{A\tau}$ is an isogeny of degree $\det A$.

Then
$$\mathcal{M}^{\mathrm{an}}_{\mathrm{Isog}} = (M \times \mathcal{X})/\!/(\Gamma \times \Gamma),$$
by the action $(B, C)(A, \tau) = (BAC^{-1}, C\tau)$.

I'll focus on $\mathrm{Spec}A = \mathcal{Y} \to \mathcal{M}$ which have descent for pth power isogenies.

3.14. **Koszul duality.** Koszul duality is a particularly nice case of bar-cobar duality.

Consider a coalgebra A over k. For a comodule M, the **cobar construction** is a cochain complex
$$\mathcal{C}(M, A, k) \approx M \otimes^h_A k.$$
Note this complex is naturally a module over $B := \mathcal{C}(k, A, k)$, which is a dga. We can soup this up to dg-comodules M, or even assume C is a codga. Thus we have a functor of derived categories
$$D(\mathrm{Comod}_A) \to D(\mathrm{Mod}_B),$$
Bar-cobar duality is the observation that this wants to be an equivalence, at least if you apply some adjectives and/or replace source and target by certain core full subcategories.

Koszul duality is a situation where the above correspondence is particularly computable. Suppose $A = \bigoplus A_r$ is a graded coaugmented coalgebra, with $A_0 = k$. Then the dga $B := \mathcal{C}(k, A, k)$ inherits a grading $B = \oplus B_{[r]}$.

$$\begin{array}{lllll}
B_{[0]}\colon & A_0 & & & \\[2em]
B_{[1]}\colon & & A_1 & & \\[2em]
B_{[2]}\colon & & A_2 \longrightarrow A_1 \otimes A_1 & & \\[2em]
B_{[3]}\colon & & A_3 \longrightarrow \begin{matrix} A_2 \otimes A_1 \\ + \\ A_1 \otimes A_2 \end{matrix} \longrightarrow A_1 \otimes A_1 \otimes A_1 & &
\end{array}$$

We say that A is **Koszul** if $H_*(B_{[r]}) \approx 0$ for $* \neq r$. This implies that the dga B is equivalent to a smaller, formal dga C with $C_r = H_r(B_{[r]})$.

3.15. *Example.* $A = (SV)^*$, the coalgebra dual to the symmetric algebra on a finite dimensional vector space V. Then $C = \Lambda(V^*[-1])$, an exterior algebra with $d = 0$.

An immediate payoff of the Koszul property is a the **Koszul complex**, a quasi-isomorphism

$$M \approx \mathcal{C}(M, C, A) \approx \left(M \otimes C_0 \otimes A \to M \otimes C_1 \otimes A \to M \otimes C_2 \otimes A \to \cdots \right)$$

of dg-comodules.

Another consequence is that A is **quadratic**: it is "cogenerated" by A_1, the map $A_2 \to A_1 \otimes A_1$ is injective, and all relations are determined by this one inclusion. Also, the dual algebra C is formally determined by A: thus $C \approx T(A)/\langle A_2 \rangle$.

3.16. **Descent coalgebroids are Koszul.** Fix $\mathcal{Y} = \mathrm{Spec} A_0 \to \mathcal{M}$ with descent for pth power isogenies. The coalgebra $\{A_r = A_{\mathrm{Isog}\, p^r}\}$ over A_0 has an associated cobar complex $B = \bigoplus B_{[r]}$, which we call the **modular isogeny complex**.

3.17. **Theorem** ([Rez12b])**.** *A is Koszul, i.e., $H_*(B_{[r]}) \approx 0$ for $* \neq r$. Furthermore,*

- *$C_r = H_r(B_{[r]}) = 0$ if $r \geq 3$.*
- *C_0, C_1, C_2 are locally free A_0-modules of ranks $1, p+1, p$ respectively.*

This immediately implies the claims I made last time about power operations for Morava E-theory, in the case of *height* 2.

I give an idea of the proof. Let $\mathcal{M}_{\mathrm{Isog}\, p^*} = \coprod \mathcal{M}_{\mathrm{Isog}\, p^r}$, for the associated simplicial object $\mathcal{M}_\bullet$, and consider $\mathcal{Y}_\bullet \to \mathcal{M}_\bullet$ with descent for pth power isogenies for $\mathcal{Y} = \mathrm{Spec} A_0$. The first observation is that forming the modular isogeny complex only involves part of the simplicial structure

$$\mathcal{Y}_0 \longleftarrow \mathcal{Y}_1 \begin{smallmatrix}\longleftarrow\\ \longleftarrow\end{smallmatrix} \mathcal{Y}_2 \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \cdots$$

namely that which always preserves the "source". In particular, $B_{[*]}$ is a complex of A_0-modules. This means we can basechange by any ring map $A_0 \to R$. In particular, we can form the modular isogeny complex for $\mathrm{Spec} R \to \mathcal{M}$, even if *it does not admit descent by pth power isogenies.*

Each A_r is a flat and locally free A_0-module. Thus, by standard commutative algebra, it suffices to show that

$$H_*(k \otimes_{A_0} B_{[r]}) \approx 0 \quad \text{for } * \neq r$$

for $A_0 \to k$ where k is an algebraically closed field.

- If $p^{-1} \in k$, then the statement is purely combinatorial. In this case, $C[p^r] \approx (\mathbb{Z}/p^r)^2$, so

$$(B_{[r]})_d \approx \prod_{\substack{G_1 \leq \cdots \leq G_d \leq C[p^\infty] \\ |G_d| = p^r}} k.$$

  In fact, $B_{[r]} \approx \bigoplus_{\substack{G \leq C[p^\infty] \\ |G| = p^r}} \widetilde{C}^{*-2}(P_G; k)$, the cochains on the "order complex" of subgroups of G. The complex P_G is contractible unless G is elementary abelian.
- When $p = 0$ in k, there are two cases: C is *ordinary* ($\widehat{C}$ is height 1), and C is *supersingular* ($\widehat{C}$ is height 2). I'll describe the height 2 case, which is what is important for Morava E-theory.

Fix a supersingular curve C_0/k, and consider

$$\mathrm{Spec} A_0 \to \mathcal{M}$$

where $A_0 = k[\![a]\!]$ is the ring which classifies deformations of C_0 to rings of characteristic p. (Thus, $A_0 = \pi_0 E_{\widehat{C}_0}/p$, the mod p-reduction of the object of the last talk.) This has descent for pth power isogenies, so we obtain an associated modular complex $B_{[r]}$. We prove the Koszul property for this, then specialize to $B_{[r]} \otimes_{A_0} k$ to get the result for C.

In fact, Katz-Mazur describe these rings explicitly. We have

$$A_1 \approx k[\![a, a']\!]/((a^p - a')(a - a'^p)), \qquad a' = \psi_1(a).$$

This is an avatar of the "classical" Kronecker congruence for the modular equation of degree p isogenies. More generally,

$$A_r \approx k[\![a, a']\!]/(\prod_{i+j=r} (a^{p^i} - a'^{p^j})).$$

Thus, we have complete explicit control of $B_{[r]}$; proving the vanishing of H^* is a calculation.

3.18. **The building picture.** A version of the above was conjectured by Ando-Hopkins-Strickland. Let me explain their picture.

Let $\mathcal{M}_\bullet$ be the simplicial object encoding pth power isogenies. The idea is that the category $\mathrm{Mod}(\mathcal{M}_\bullet)$ of pth power isogeny modules is entirely determined by a subobject of $\mathcal{M}_\bullet$ associated to isogenies whose kernel is killed by p.

That is, $\mathcal{M}_\bullet$ contains a sub-semisimplicial object

$$\mathcal{N}_\bullet := \left\{ \mathcal{M} \begin{array}{c}\xleftarrow{(s,t)}\\ \xleftarrow{(\mathrm{id},\Psi)}\end{array} \begin{array}{c}\mathcal{M}_{\mathrm{Isog}\,p}\\ \amalg \\ \mathcal{M}\end{array} \begin{array}{c}\xleftarrow{\mathrm{id}}\\ \xleftarrow{s}\\ \xleftarrow{w}\end{array} \mathcal{M}_{\mathrm{Isog}\,p} \right\}$$

Here $\mathcal{M}_{\mathrm{Isog}\,p} \amalg \mathcal{M} \subset \mathcal{M}_{\mathrm{Isog}\,p} \amalg \mathcal{M}_{\mathrm{Isog}\,p^2} \subset \mathcal{M}_1$, where the second term classifies isogenies $f\colon C \to C'$ with $\ker f = C[p]$. Note that any such f is *canonically* isomorphic to $[p]\colon C \to C$.

Likewise $\mathcal{M}_{\mathrm{Isog}\,p} \subset \mathcal{M}_{\mathrm{Isog}\,p} \times_{\mathcal{M}} \mathcal{M}_{\mathrm{Isog}\,p} \subset \mathcal{M}_2$ classifies composites $C \xrightarrow{f} C' \xrightarrow{g} C''$ with $\ker gf = C[p]$. Given f, the sequence $C \xrightarrow{f} C' \xrightarrow{g} C''$ is canonically isomorphic to $C \xrightarrow{f} C' \xrightarrow{g'} C$, by the same argument as above.

The content of the Koszul theorem is roughly that this semi-simplicial object "generates" $\mathcal{M}_\bullet$. If $\mathrm{Spec} A \to \mathcal{M}$ has descent for pth power isogenies, then an isogeny module M for this is exactly: an A-module, together with A-module maps

$$\psi_1\colon M \to M \otimes_A {}^{s}A_1{}^{t}, \qquad \phi\colon M \to M \otimes_A A^{\Psi}$$

such that

$$\begin{tikzcd} M \arrow[r, "\psi_1"] \arrow[d, "\phi"'] & M \otimes_A {}^{s}A_1{}^{t} \arrow[d, "((\mathrm{id}\times w)\otimes\mathrm{id})(\psi_1\otimes\mathrm{id})"] \\ M \otimes_A A^{\Psi} \arrow[r, "\mathrm{id}\otimes s"'] & M \otimes_A {}^{s}A_1{}^{s\Psi} \end{tikzcd}$$

Here $w\colon A_1 \to A_1$ is the map classifying dual isogenies, as in the previous lecture.

3.19. **Koszul property for power operations for Morava E-theory.** The general case requires a different argument, which relies on topology, not algebraic geometry.

Let $\Sigma_m \curvearrowright X$ be a finite Σ_m-set. We can consider $E^0(X_{h\Sigma_m})$, the E-cohomology of the homotopy orbit space.

Let $(2^{\underline{m}} - 2) \subset 2^{\underline{m}}$ be the subset of the power set of an m-element set with $\varnothing$ and $\underline{m}$ removed. Set

$$Q_m(X) := \operatorname{Cok}\left[E^0(X \times (2^{\underline{m}} - 2))_{h\Sigma_m} \xrightarrow{\text{Transfer}} E^0(X_{h\Sigma_m})\right]$$

In fact,

$$Q_{p^r}(*) \approx E^0(B\Sigma_{p^r})/I_{\text{transfer}} = A_r.$$

Observe that Q_m is actually a *Mackey functor* for Σ_m.

The Koszul result follows from the following two observations.

- The modular isogeny complex $B_{[r]}$ is *isomorphic* to
$$B_{[r]} = Q(\overline{\mathcal{P}}_{p^r}).$$
Here $\overline{\mathcal{P}}_{p^r}$ is the reduced partition complex. Thus
$$H^* B_{[r]} = H^*_{\text{Bre}}(\overline{\mathcal{P}}_{p^r}; Q).$$
- Arone-Dwyer-Lesh [ADL16] show that for suitable Mackey functors Q (including this one), $H^*_{\text{Bre}}(\overline{\mathcal{P}}_{p^r}; Q) \approx 0$ for $* \neq r$.

3.20. **Example: The Behrens $Q(\ell)$ spectrum.** Let ℓ be a prime, and consider $\mathcal{M}[\frac{1}{\ell}]$, the stack which classfies elliptic curves $C \to S$ over base schemes S which lie over $\operatorname{Spec}\mathbb{Z}[\frac{1}{\ell}]$.

In this case, each $\mathcal{M}_{\operatorname{Isog}\ell^r}[\frac{1}{\ell}] \to \mathcal{M}$ is étale, and in fact all maps between stacks in $\mathcal{M}_\bullet[\frac{1}{\ell}]$ are étale. Therefore, the general machinery of Goerss-Hopkins-Miller applies. We obtain a cosimplicial commutative S-algebra

$$[n] \mapsto \Gamma(\mathcal{M}_n[\tfrac{1}{\ell}], \mathcal{O}^{\text{Top}}),$$

whose inverse limit is the spectrum $Q(\ell)$ constructed by Mark Behrens [Beh06], [Beh07], [BL06].

The building picture says that you can construct $Q(\ell)$ as the inverse limit of a semi-cosimplicial ring

$$\mathrm{TMF}[\tfrac{1}{\ell}] \rightrightarrows \begin{matrix} \mathrm{TMF}_{\operatorname{Isog} p}[\frac{1}{\ell}] \\ \times \\ \mathrm{TMF}[\frac{1}{\ell}] \end{matrix} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \mathrm{TMF}_{\operatorname{Isog} p}[\tfrac{1}{\ell}]$$

(In fact, you can build this with TMF replaced by Tmf, etc., by Hill-Lawson.) We can $K(2)$-localize at a prime other than p. Behrens' conjecture is that for p odd and ℓ chosen suitably, there is a cofiber sequence

$$DQ(\ell)_{K(2),p} \to S_{K(2),p} \to Q(\ell)_{K(2),p},$$

and this is proved for $p = 3$ and $\ell = 2$.

3.21. **Example: $\Phi_h S^{2d-1}$.** There is no construction of $Q(p)$ *at the prime* p, i.e., without inverting p.

However, there turns out to be something that seems to play its role. This is $\Phi_h S^{2d+1}$, the Bousfield-Kuhn functor applied to an odd sphere.

Fix a Morava E-theory of height h associated to G_0/k. Let $\mathcal{C}$ denote the category of E_*-modules equipped with power operations. E.g., if G_0 is the formal completion of a s.s. curve, then $\mathcal{C}$ is the category of pth power isogeny modules (except that things are allowed to have odd grading).

Work of Behrens and me (in progress), shows that there is a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathcal{C}}(\omega^{d-1}, \omega^{(t-1)/2} \otimes \mathrm{nul}) \Rightarrow E^{\wedge}_{t-s}\Phi_h S^{2d-1}.$$

Here $\mathrm{nul} = E_0$ with coaction map $\psi_1 \equiv 0$, while $\omega = \widetilde{E}^0 S^2$.

4. Multiplicative orientation

Let's calculate power operations in an important example.

4.1. *$B\mathcal{V}$.* Let E be an even periodic ring theory. Thus $E^0 BU(1) \approx E^0[\![x]\!]$. The choice of a generator x is called a **coordinate**. A choice of class determines a notion of Euler class for line bundles:

$$(L \to X) \rightsquigarrow x(L) \in E^0 X$$

defined by

$$X \xrightarrow{L} BU(1) \quad \rightsquigarrow \quad \begin{array}{ccc} E^0 BU(1) & \xrightarrow{E^0(L)} & E^0 X \\ x & \mapsto & x(L) \end{array}$$

Note that $x = x(L_{\mathrm{univ}})$ where $L_{\mathrm{univ}} \to BU(1)$ is the universal line bundle.

The associated formal group law $x_1 +_F x_2 \in E^0[\![x_1, x_2]\!]$ is defined so that $x(L_1 \otimes L_2) = x(L_1) +_F x(L_2)$; the group law depends on x.

Let

$$B\mathcal{V} = \coprod BU(n),$$

the classifying space for complex vector bundles. This space has a multiplication $B\mathcal{V} \times B\mathcal{V} \to B\mathcal{V}$ encoding direct sum of bundles. We can identify the ring $E_* B\mathcal{V}$ by

$$\begin{tikzcd} E_* B\mathcal{V} \arrow[r, equal] & E_*[b_k,\ k \geq 0] \\ E_* BU(1) \arrow[u, "E_*(L_{\mathrm{univ}})"] \arrow[r, equal] & E_*\{b_k,\ k \geq 0\} \arrow[u] \end{tikzcd}$$

where $L_{\mathrm{univ}} \colon BU(1) \to B\mathcal{V}$ classifies the universal line bundle. The E_*-module basis $\{b_k\}$ of $E_* BU(1)$ is defined to be that which is dual to the "monomial basis" $\{x^k\}$ of $E^* BU(1) = E_*[\![x]\!]$. (More precisely, the basis $b_0, \ldots, b_n$ of $E_* \mathbb{CP}^n$ is dual to the monomial basis of $E^* \mathbb{CP}^n$ for each n.)

4.2. **The spectrum $F = E\wedge\Sigma^\infty_+ B\mathcal{V}$.** Let $F := E\wedge\Sigma^\infty_+ B\mathcal{V}$. This is a homotopy commutative ring spectrum.

Given a vector bundle $V \to X$, there is a characteristic class

$$\langle V\rangle \in F^0 X,$$

defined tautologously: if $V\colon X \to B\mathcal{V}$ is the map representing the bundle, then $\langle V\rangle$ is the composite

$$\Sigma^\infty_+ X \xrightarrow{\Sigma^\infty_+ V} \Sigma^\infty_+ B\mathcal{V} \xrightarrow{1\wedge\mathrm{id}} E\wedge\Sigma^\infty_+ B\mathcal{V} = F.$$

This class satisfies the whitney sum formula

$$\langle V\oplus W\rangle = \langle V\rangle\langle W\rangle.$$

4.3. *Example.* Consider $L_{\mathrm{univ}} \to BU(1)$. You can calculate

$$\langle L_{\mathrm{univ}}\rangle = \sum_{k\geq 0} b_k x^k \in F^0 BU(1)$$

Here x is the image of $x \in E^0BU(1) \to F^0BU(1)$. Note that the right-hand side appears to depend on the choice of x (which determines the b_k as well), but the left-hand side does not.

The element $\gamma_{\mathrm{univ}} := \langle L_{\mathrm{univ}}\rangle$ is the universal example of a function on the formal group $\mathbb{G}_E$ of E. That is, given $\phi\colon E_0 \to R$ and an element $\gamma \in \mathcal{O}_{\phi^*G_E} = E^0BU(1)\otimes_{E_0}{}^{\phi}R$, there exists a unique ring homomorphism $F_0 \to R$ exending ϕ so that

$$F_0BU(1) = E^0BU(1)\otimes_{E_0} F_0 \to E^0BU(1)\otimes_{E_0} R \quad \text{sends} \quad \gamma_{\mathrm{univ}} \mapsto \gamma.$$

In terms of a coordinate x, if $\gamma(x) = \sum c_k x^k$, then $F_0 \to R$ sends $b_k \mapsto c_k$.

4.4. *Remark.* The diagonal map $\Delta\colon B\mathcal{V} \to B\mathcal{V}\times B\mathcal{V}$ makes $E_0B\mathcal{V}$ into a Hopf algebra. What does the comultiplication represent? The calculation of Δ_* happens on $E_0BU(1)$, and we have $\Delta_*(b_k) = \sum_{i+j=k} b_i\otimes b_j$ (it is dual to the cup product on $E^0BU(1)$). This means that Δ_* represents *multiplication* of functions

$$(\gamma_1,\gamma_2) \mapsto \gamma_1\gamma_2\colon \mathcal{O}_{\phi^*G}\times\mathcal{O}_{\phi^*G} \to \mathcal{O}_{\phi^*G}.$$

Note that if $L \to X$ is any line bundle, then

$$\langle L\rangle = \gamma_{\mathrm{univ}}(x(L)),$$

and more generally

$$\langle L_1\oplus\cdots\oplus L_k\rangle = \gamma_{\mathrm{univ}}(x(L_1))\cdots\gamma_{\mathrm{univ}}(x(L_k))$$

for a sum of line bundles. You can use the "splitting principle" to compute $\langle V\rangle$ for any bundle. The class $\langle V\rangle$ is a kind of universal characteristic class taking sums to products.

4.5. **Power operations for F.** The space $B\mathcal{V}$ is actually an E_∞-space. This implies that $\Sigma^\infty_+ B\mathcal{V}$ is an E_∞-ring spectrum, and so can be realized as a commutative S-algebra.

Suppose E is also a commutative S-algebra. Then so is F. I would like to compute power operations for F, e.g., the map

$$P_m\colon F^0X \to F^0(X\times B\Sigma_m).$$

This is too difficult to make sense of for arbitrary elements of F^0X. However, there is a formula for classes of the form $\langle V\rangle$.

4.6. **Proposition.** *We have*

$$P_m(\langle V\rangle) = \langle V \boxtimes \rho_m\rangle.$$

Here we write $\rho_m = (\mathbb{C}^m \curvearrowleft \Sigma_m)$ for the permutation representation of Σ_m, and also for the induced bundle $\mathbb{C}^m \times_{\Sigma_m} E\Sigma_m \to B\Sigma_m$.

Proof. The point is that the class $\langle V\rangle$ is produced by a map of spaces. Since $B\mathcal{V}$ is an E_∞-space, we have a power construction

$$X \times B\Sigma_m \xrightarrow{\text{diag}} X^m \times_{\Sigma_m} E\Sigma_m \to B\mathcal{V}^m \times_{\Sigma_m} E\Sigma_m \to B\mathcal{V}$$

The induced bundle is $V^{\oplus m} \times_{\Sigma_m} E\Sigma_m \to B\Sigma_m$, which is what we have called $V \boxtimes \rho_m$. Now use commutativity of

$$\begin{tikzcd}
\Sigma^\infty_+(X \times B\Sigma_m) \arrow[r] \arrow[drr, "V\boxtimes\rho_m"'] & \Sigma^\infty_+(X^m_{h\Sigma_m}) \arrow[r] & \Sigma^\infty_+(B\mathcal{V}^m_{h\Sigma_m}) \arrow[r] \arrow[d] & F^{\wedge m}_{h\Sigma_m} \arrow[d] \\
 & & \Sigma^\infty_+ B\mathcal{V} \arrow[r] & F
\end{tikzcd}$$

□

To understand what this formula says, it is useful to restrict from the symmetric group to certain abelian subgroups. Consider $A \leq \Sigma_m$ which is abelian and *transitive*, i.e., the induced action $A \curvearrowright \underline{m}$ is transitive; this implies $|A| = m$. One such example is a cyclic group $C_m \leq \Sigma_m$.

The restriction of the permutation representation along $A \subseteq \Sigma_m$ is a regular representation:

$$\rho_m|_A \approx \bigoplus_{\lambda\in\widehat{A}} \lambda.$$

Here $\widehat{A} = \mathrm{Hom}(A, \mathbb{C}^\times)$. Thus, the restriction of $P_m(\langle V\rangle)$ to $F^0(X \times BA)$ is

$$\langle V \boxtimes \big(\sum_{\lambda\in\widehat{A}} \lambda\big)\rangle = \prod_{\lambda\in\widehat{A}} \langle V \boxtimes \lambda\rangle.$$

I'm using λ as notation for the associated line bundle over BA. If we let $V = L_{\text{univ}} \to BU(1)$, and choose a coordinate $x \in E^0BU(1)$, this becomes

$$\prod_{\lambda\in\widehat{A}} \langle L_{\text{univ}} \boxtimes \lambda\rangle = \prod_{\lambda\in\widehat{A}} \gamma_{\text{univ}}(x(L_{\text{univ}} \boxtimes \lambda)) = \prod_{\lambda\in\widehat{A}} \gamma_{\text{univ}}(x +_F x(\lambda)). \tag{4.7}$$

4.8. **The case of Morava E-theory.** Now assume that $E = E_{G_0/k}$ is a Morava E-theory for a height h formal group. To plug in the theory of power operations described in lecture 2, we need to consider

$$\widehat{F} := L_{K(h)}F, \qquad \widehat{F}_0 \approx (F_0)^\wedge_{\mathfrak{m}_E}.$$

This ring still represents functions on the formal group, as long as we use ring homomorphisms which are continuous wrt the $\mathfrak{m}_E$-adic topology.

We want to describe the induced descent-for-isogenies structure on $\mathrm{Spec}F_0$. This ring represents the functor

$$(G/R, i, \alpha) \mapsto \mathcal{O}_G.$$

Descent for isogenies means that for every isogeny $f\colon G \to G'$ between deformations of G_0 to R, there is an induced pushforward map

$$f_!\colon \mathcal{O}_G \to \mathcal{O}_{G'}.$$

One thing we can say about this is that $f_!$ is multiplicative: $f_!(\gamma_1\gamma_2) = f_!(\gamma_1)f_!(\gamma_2)$, because the product of functions is represented by the diagonal map, which is a map of E_∞-spaces. Also, the "Frobenius congruence" implies that for a pth power Frobenius $F\colon G \to \sigma^* G$, we must have $F_!(\gamma) = \gamma^p$.

An isogeny $f\colon G \to G'$ induces a map of function rings

$$f^*\colon \mathcal{O}_{G'} \to \mathcal{O}_G.$$

Because f is an isogeny of degree p^r, this map presents $\mathcal{O}_G$ as a free $\mathcal{O}_{G'}$-module of rank p^r.

4.9. **Theorem** ([AHS04]). *The function $f_!\colon \mathcal{O}_G \to \mathcal{O}_{G'}$ is the multiplicative norm. I.e., $f_!(\gamma)$ is determinant of $\gamma\cdot\colon \mathcal{O}_G \to \mathcal{O}_G$ as a map of $\mathcal{O}_{G'}$-modules.*

(1) It is enough to compute this for the universal example $f\colon s^*G_{\text{univ}} \to t^*G_{\text{univ}}$ of an isogeny of degree p^r. The universal example $s^*\gamma_{\text{univ}}$ of a function on s^*G_{univ} is represented by the identity map of $\widehat{F}_0 \otimes_{E_0} {}^s A_r$, and the desired function $f_!(s^*\gamma_{\text{univ}})$ on t^*G_{univ} is represented by the power operation

$$\widehat{F}_0 \xrightarrow{\psi_r} \widehat{F}_0 \otimes_{E_0} {}^s A_r.$$

(2) There is a tautological commutative diagram

$$\begin{tikzcd}
\widehat{F}_0 \arrow[r, "\text{represents } f_!(\gamma_{\text{univ}})"] \arrow[d] & \widehat{F}_0 \otimes_{E_0} {}^s A_1 \arrow[d] \\
\widehat{F}^0 BU(1) \arrow[r, "\gamma_{\text{univ}} \mapsto \psi_r(\gamma_{\text{univ}})"'] & \widehat{F}^0 BU(1) \otimes_{E_0} {}^s A_r
\end{tikzcd} \tag{4.10}$$

Note that $\widehat{F}^0 BU(1) \otimes_{E_0} {}^s A_r \approx \mathcal{O}_{s^*G_{\text{univ}}}$. We claim that

$$\psi_r(\gamma_{\text{univ}}) = f^* f_!(\gamma_{\text{univ}}).$$

This is sufficient to read off $f_!(\gamma_{\text{univ}})$, since $f^*\colon \mathcal{O}_{t^*G_{\text{univ}}} \to \mathcal{O}_{s^*G_{\text{univ}}}$ is injective.

Here's a proof. Given a fixed E_0-algebra R which is complete with respect to some ideal, classifying a deformation G, write

$$G(R) := \operatorname{Hom}(E^0 BU(1), R)$$

for the set of all continuous homomorphisms (using the $\mathfrak{m}_E + (x)$-topology on $E^0BU(1)$.) We think of these as R-valued points of G. If $\gamma \in E^0BU(1)$ is a function on $\mathbb{G}_E$, we write "$\gamma(p)$" for the image of γ under p (i.e., $\gamma(p) := p(\gamma)$ (!)). Note that if $x \in E^0BU(1)$ is a coordinate, then

$$x(p_1 + p_2) = x(p_1) +_F x(p_2).$$

More generally, $p \in G(R)$, then the map

$$E^0 BU(1) \widehat{\otimes}_{E_0} \widehat{F}_0 = \widehat{F}^0 BU(1) \xrightarrow{p} R$$

classifying data $(G, \gamma \in \mathcal{O}_G, p \in G(R))$ sends $\gamma_{\text{univ}} \mapsto \gamma(p)$.

Fix a homomorphism $\phi\colon \widehat{F}^0 BU(1) \otimes_{E_0} {}^s A_r \to R$, i.e., a map

$$E^0 BU(1) \widehat{\otimes} \widehat{F}_0 \otimes_{E_0} {}^s A_r \to R.$$

This classifies data

$$(f\colon G \to G', \gamma \in \mathcal{O}_G, p \in G(R)),$$

where f is an isogeny between deformations, γ a function on G, and p is a R-point in G.

Now consider the composite

$$\phi' \colon E^0BU(1)\widehat{\otimes}\widehat{F}_0 \xrightarrow{\psi_r} E^0BU(1)\widehat{\otimes}\widehat{F}_0 \otimes {}^s A_r \xrightarrow{\phi} R.$$

This classifies data

$$(\gamma' \in \mathcal{O}_{G'}, f(p) \in G'(R)),$$

and in fact $\gamma' = f_!(\gamma)$ by commutativity of (4.10).

Now let's follow the universal function γ_{univ} through this map:

$$\gamma_{\mathrm{univ}} \mapsto \psi_r(\gamma_{\mathrm{univ}}) \mapsto \phi(\psi_r(\gamma_{\mathrm{univ}})) = \gamma'(f(p)) = f_!(\gamma)(f(p)).$$

Taking ϕ to be the identity map, we see that $\psi_r(\gamma_{\mathrm{univ}}) = f^* f_!(\gamma_{\mathrm{univ}})$, as desired.

(3) Now we need to relate $\psi_r(\gamma_{\mathrm{univ}})$ to the multiplicative norm of γ along f. Let $A \subseteq \Sigma_{p^r}$ be a transitive abelian subgroup. There is a commutative diagram

$$\begin{tikzcd}
E^0B\Sigma_{p^r} \arrow[r] \arrow[d] & E^0BA \arrow[d] \\
E^0B\Sigma_{p^r}/I \arrow[r] & E^0BA/I'
\end{tikzcd}$$

where I' is the ideal generated by transfers from proper subgroups of A. It turns out (see [Str98]) that the restriction map

$$E^0B\Sigma_{p^r}/I \to \prod E^0BA/I'$$

is injective, where the product is over transitive abelian subgroups A. Thus, to compute $\psi_r(\gamma_{\mathrm{univ}})$, it it is enough to compute its projections to $\widehat{F}^0BU(1) \otimes_{E_0} {}^s E^0BA/I'$.

The element $\psi_r(\gamma_{\mathrm{univ}})$ is itself the image of $P_{p^r}(\langle L_{\mathrm{univ}}\rangle) \in \widehat{F}^0BU(1) \otimes_{E_0} {}^s E^0B\Sigma_{p^r}$. By (4.7), we know that

$$P_m(\langle L_{\mathrm{univ}}\rangle)|_{BA} = \prod_{\lambda \in \widehat{A}} \gamma_{\mathrm{univ}}(x +_F x(\lambda))$$

in terms of a coordinate $x \in E^0BU(1)$.

On the other hand, given an exact sequence $0 \to \widehat{A} \xrightarrow{\lambda} G \xrightarrow{f} G' \to 0$ of "physical" groups, where A is finite, the multiplicative norm can be computed by

$$f^* N_f(\gamma(g)) = \prod_{a \in A} (\mathrm{Trans}^*_{\lambda(a)}\gamma)(g) = \prod_{a \in A} \gamma(g +_G g(a)).$$

I.e., if you use the basis of $\mathcal{O}_G$ over $\mathcal{O}_{G'} = \prod_A \mathcal{O}_G$ obtained by picking a set theoretic section of $G \to G'$, diagonal matrix.

Such a "physical" exact sequence exists over a fraction field of the quotient ring E^0BA/I'. The tautological function

$$\ell \colon \widehat{A} \to G(E^0BA), \qquad \lambda \mapsto (E^0BU(1) \xrightarrow{E^0B\lambda} E^0BA),$$

is a homomorphism of groups, which is universal for homomorphisms from A:

$$\{\widehat{A} \to G(R)\} \leftrightarrow \{\text{defs. } (G, i, \alpha) \text{ to } R, \text{ hom. } \ell \colon \widehat{A} \to G(R)\ \}.$$

Over E^0BA/I', the function ℓ is an example of a **Drinfel'd level structure** on $G_{E^0BA/I'}$.

Pulled back to the ring $S^{-1}(E^0BA/I')$ obtained by inverting euler classes of all non-trivial line bundles, ℓ gives an *isomorphism* between A and the kernel of the universal isogeny (which now must be considered not as a map of formal groups, but as a map of p-divisible groups). It turns out that $E^0BA/I' \to S^{-1}(E^0BA/I')$ is injective, so you can read off the desired formula for the multiplicative norm in this case. (See [AHS04] for the details about this argument.)

4.11. **Some friends of $B\mathcal{V}$.** This norm interpretation of power operations on $B\mathcal{V}$ comes from Ando's thesis [And95]. It leads to similar interpretations for friends of $B\mathcal{V}$. The following diagram shows (i) some E_∞-spaces or spectra R, (ii) the corresponding functor represented by E_0R, and (iii) the "universal element" for the representable functor.

$$\begin{tikzcd} & BU \arrow[d] \\ B\mathcal{V} = \coprod BU(m) \arrow[r] & \mathbb{Z} \times BU \end{tikzcd} \qquad \begin{tikzcd} & MU \arrow[d] \\ M\mathcal{V} = \bigvee MU(m) \arrow[r] & MUP \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{O}_G^\times / R^\times \\ \mathcal{O}_G & \mathcal{O}_G^\times \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Triv}^{\mathrm{rigid}}(\mathcal{I}) \\ \mathcal{I} & \mathrm{Triv}(\mathcal{I}) \arrow[l] \arrow[u] \end{tikzcd}$$

$$(BU(1) \xrightarrow{1-L} BU) \qquad\qquad (BU(1)^{1-L} \to MU)$$

$$(BU(1) \xrightarrow{L} B\mathcal{V}) \qquad (BU(1) \xrightarrow{L} \mathbb{Z} \times BU) \qquad (BU(1)^L \to M\mathcal{V}) \qquad (BU(1)^L \to MUP)$$

Here $\mathcal{I} \subset \mathcal{O}_G(e)$ is the augmentation ideal, i.e., the functions on G which vanish at the identity element. A trivializaton of the ideal is a choice of generator; a rigid trivialization is a section of the projection $\mathcal{I} \xrightarrow{D} \omega_G(e)$.

Let $x \in E^0BU(1)$ be a coordinate for G_{univ}. This gives rise, for any deformation (G, i, α) of G_0 to R, a coordinate $x_{G,i,\alpha} \in \mathcal{O}_G$ (by $E^0BU(1) \to E^0BU(1) \otimes_{E_0} R$).

4.12. **Theorem** (Ando [And95])**.** *A* necessary *condition for the coordinate x to come from a map $MUP \to E$ of commutative S-algebras, is that*

$$N_f(x_{G,i,\alpha}) = x_{G',i',\alpha'}$$

for any isogeny $f\colon G \to G'$ compatible with deformation structures.

(To check this condition, it suffices to show it for the universal p-isogeny over A_1.)

Matt Ando proved that such coordinates exist, at least for E_{G_0} where G_0 satisfies $F^h = p$.

There is an extension to connected covers of BU [AHS04].

$$\begin{array}{ccc}
BU\langle 6\rangle & \Theta^3(G,\mathcal{O}_G) & (BU(1)^{\wedge 3} \xrightarrow{(1-L_1)(1-L_2)(1-L_3)} BU\langle 6\rangle) \\
\downarrow & \uparrow & \\
BSU & \Theta^2(G,\mathcal{O}_G) & (BU(1)^{\wedge 2} \xrightarrow{(1-L_1)(1-L_2)} BSU) \\
\downarrow & \uparrow & \\
0\times BU & \Theta^1(G,\mathcal{O}_G) & (BU(1) \xrightarrow{1-L} BU)
\end{array}$$

The set $\Theta^k(G,\mathcal{O}_G)$ is the set of functions f on G^k such that

- $f(0,\dots,0)=1$,
- f is symmetric,
- $f(a+b,c,\dots)f(a,b,\dots)=f(a,b+c,\dots)f(a,b,\dots)$.

The sequence runs out here, since these are the only covers of BU with even cohomology.

Similar results hold for MSU and $MU\langle 6\rangle$, with $\mathcal{O}_G$ replaced by $\mathcal{I}$. The action of power operations on these is described by a norm formula.

An element of $\Theta^3(G,\mathcal{I})$ is called a **cubical structure**. Any elliptic curve C has a *unique* cubical structure, which therefore prescribes a preferred cubical structure on its formal completion $\widehat{C}$. Therefore, the above correspondence picks out, for any elliptic spectrum, a unique map $MU\langle 6\rangle\to E$ of ring spectra. Furthermore, the norm of a cubical structure is another one, so if E is an elliptic spectrum, then the above map is H_∞.

Finally, the unique cubical structure can be described using the Weierstrass σ-function, which is a particular choice of section of $\Theta^1(C,\mathcal{I})$ when C is the Tate curve: the cubical structure is

$$s(a,b,c)=\frac{\sigma(0)\sigma(a+b)\sigma(a+c)\sigma(b+c)}{\sigma(a)\sigma(b)\sigma(c)\sigma(a+b+c)}.$$

This is partial progress towards constructing

$$\begin{array}{ccc}
MU\langle 6\rangle & \longrightarrow & E \\
\downarrow & & \uparrow \\
M\mathrm{String} & \longrightarrow & \mathrm{tmf}
\end{array}$$

as a map of commutative S-algebras.

4.13. **The string orientation.** The eventual construction of the S-algebra map $M\mathrm{String}\to\mathrm{tmf}$ is very different in detail.

Given a commutative S-algebra, there is an associated units spectrum $\mathrm{gl}_1(R)$, whose underlying space is $GL_1(R)$. Its delooping $BGL_1(R)$ classifies stable spherical fibrations.

There is an adjoint pair

$$\Sigma^\infty_+\Omega^\infty\colon ((-1)\text{-connected spectra}) \rightleftarrows (\text{comm } S\text{-algebras}) : \mathrm{gl}_1,$$

analogous to $\mathbb{Z}[-]\colon(\text{ab gps})\rightleftarrows(\text{comm rings}):(-)^\times$.

Let $o\to\mathrm{gl}_1(S)$ be the J-homomorphism. There is a correspondence between null-homotopies of the composite

$$g\to o\to \mathrm{gl}_1(S)\to\mathrm{gl}_1(R)$$

and the space of commutative S-algebra maps $MG \to R$, as torsors over $\mathrm{Map}(g, \mathrm{gl}_1(R)) \approx \mathrm{Map}(\Sigma^\infty_+ BG, R)$.

Thus, to construct orienations, we need to understand the homotopy type of $\mathrm{gl}_1(R)$.

For complex oriented E, we have

$$\{(\text{maps of ring spectra } MU \to E)\} \leftrightarrow \{\text{coordinates } x \in E^0 BU(1)\}.$$

To describe this set, we can use the following formalism. In the rationalization $E\mathbb{Q}$ of E, there is a standard coordinate $x_{H\mathbb{Q}}$ coming from $H\mathbb{Q} \to E\mathbb{Q}$. Thus, associated to $f\colon MU \to E$ is a coordinate $x_f \in E^0 BU(1)$, and we can write $K_f(x_{H\mathbb{Q}}) = x_{H\mathbb{Q}}/x_f$. This is precisely the Hirzebruch characteristic series, corresponding to a map $\kappa_f\colon BU \to GL_1(E\mathbb{Q})$. This map measures the "difference" between the two sides of the non-commuting square

$$\begin{tikzcd} MU \arrow[r, "f"] \arrow[d, "g"'] & E \arrow[d] \\ H\mathbb{Q} \arrow[r] & E\mathbb{Q} \end{tikzcd}$$

Write

$$K_f(x) = \exp\Big(\sum_{k \geq 1} t_k \frac{x^k}{k!}\Big).$$

4.14. **Proposition.** *In degree $2k$, $\kappa_f\colon \pi_{2k} BU \to \pi_{2k}(E\mathbb{Q}) = \pi_{2k} E \otimes \mathbb{Q}$ sends the Bott generator to $(-1)^k t_k$.*

This describes a function

$$\mathrm{Hom}(MU, E) \to \prod \pi_{2k} E \otimes \mathbb{Q}.$$

We ask the question: which elements in the image come from commutative S-algebra maps?

Likewise, there is a function

$$\mathrm{Hom}(\Sigma^\infty_+ BU, E) \to \prod \pi_{2k} E,$$

which associates $f\colon \Sigma^\infty_+ BU \to E$, corresponding to $f \in E^0 BU(1)$ with $f(0) = 1$, to the sequence t_k defined by

$$f(x_{H\mathbb{Q}}) = \exp\Big(\sum t_k \frac{x^k}{k!}\Big).$$

4.15. *Example.* The Todd genus is given by the characteristic series

$$K_{\mathrm{Td}}(x) = \frac{x}{1 - e^{-x}} = \exp\Big(-\sum_{k \geq 1} \frac{B_k}{k} \frac{x^k}{k!}\Big).$$

The only bernoulli number with k odd is $B_1 = 1/2$. Thus we can modify this to a formula

$$K_{\widehat{A}}(x) = \frac{x}{e^{x/2} - e^{x/2}} = \exp\Big(-\sum_{k \geq 2} \frac{B_k}{k} \frac{x^k}{k!}\Big).$$

4.16. *Example.* The Witten genus corresponds to

$$K_W(x) = \exp\Big(-2 \sum_{k \geq 4} G_k \frac{x^k}{k!}\Big),$$

where

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum\sum d^{2k-1}q^n.$$

The appearance of bernoulli numbers is not accidental. We can consider the universal case of the orientation $1\colon MU \to MU$.

A good toy example for this kind of calculation is maps from $\Sigma^\infty_+ BU$ to K.

4.17. **Theorem** ([Wal09]). *The image of*

$$\mathrm{Hom}_{S-\mathrm{alg}}(\Sigma^\infty_+ BU, K) \approx \mathrm{Hom}_{\Omega^\infty}(BU, \mathrm{gl}_1 K) \to \prod \pi_{2k} K$$

is the collection of sequences $(t_k)_{k\geq 1}$ of integers such that for each prime p, the sequence $t^*_{k,p} := t_k(1-p^{k-1})$ is a set of moments for a p-adic measure μ_p on $\mathbb{Z}_p^\times$. That is,

$$t^*_{k,p} = \int_{\mathbb{Z}_p^\times} x^k \, d\mu_p(x).$$

This turns out to be the same as the set of H_∞-maps $BU \to K$, which corresponds to elements $f \in (1 + x\mathbb{Z}[\![x]\!]) \subset \mathcal{O}^\times_{G_m}$ which are compatible with norms after completing at all primes p.

5. Logarithm

5.1. **Units spectrum.** Given a commutative S-algebra R, let $GL_1(R)$ be the pullback

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array}$$

It represents the functor

$$(R^0(X))^\times = [X, GL_1(R)]$$

The space $GL_1(R)$ is an infinite loop space. Its corresponding connective spectrum is denoted $\mathrm{gl}_1 R$.

Problem: say something about the homotopy type of the spectrum $\mathrm{gl}_1 R$.

5.2. **The idea.** Let E and F be spectra. Consider

$$[F, E]_{\mathrm{Sp}} \to [\Omega^\infty F, \Omega^\infty E]_H \subseteq [\Omega^\infty F, \Omega^\infty E]_{\mathrm{Top}_*},$$

induced by the functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{Top}_*$. The image lands in the set of H-space maps.

In general, there is no expectation that this map be either surjective or injective.

5.3. **Rational linearization.** Let us suppose

$$F \text{ is 0-conn.,} \qquad E = E_{\mathbb{Q}}, \text{ i.e., } \pi_* E \approx \pi_* E \otimes \mathbb{Q}.$$

5.4. **Proposition.** *In this case, the map $\Omega^\infty\colon [F, E]_{\mathrm{Sp}} \to [\Omega^\infty F, \Omega^\infty E]_{\mathrm{Top}_*}$ admits a retraction.*

Here is the construction of the retraction. Recall that since $E = E_{\mathbb{Q}}$,

$$[F, E]_{\mathrm{Sp}} \xrightarrow{\sim} \mathrm{Hom}(\pi_* F, \pi_* E),$$

because $E \approx \prod \Sigma^n H(\pi_n E)$.

Define $r\colon [\Omega^\infty F, \Omega^\infty E]_{\mathrm{Top}_*} \to [F,E]_{\mathrm{Sp}}$ by

$$[\Omega^\infty F, \Omega^\infty E]_{\mathrm{Top}_*} \to \mathrm{Hom}(\pi_*\Omega^\infty F, \pi_*\Omega^\infty E) \approx \mathrm{Hom}(\pi_* F, \pi_* E) \xleftarrow{\sim} [F,E]_{\mathrm{Sp}}.$$

This works because $\Omega^\infty F$ is connected, so the induced map $\pi_*\Omega^\infty F \to \pi_*\Omega^\infty E$ is a *homomorphism* of groups.

You can also show that all H-space maps are infinite loop maps:

$$[F,E]_{\mathrm{Sp}} \xrightarrow{\sim} [\Omega^\infty F, \Omega^\infty E]_H.$$

We can apply this to $GL_1(R)$ when R is rational.

5.5. **Rational linearization, again.** Let me do this again, but more complicated.

Write $\mathcal{L}_{\mathbb{Q}}\colon [\Omega^\infty F, \Omega^\infty E] \to [\Omega^\infty F, \Omega^\infty E]$ for the idempotent implicitly defined by the retraction above. Given $f \in [\Omega^\infty F, \Omega^\infty E]$, we can regard it as defining a cohomology operation

$$f\colon F^0(X) \to E^0(X).$$

The problem is to calculate the cohomology operation $\mathcal{L}f$ given f.

5.6. **Proposition.** *For X a finite dimensional connected CW-complex,*

$$(\mathcal{L}_{\mathbb{Q}} f)(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \mathrm{Cr}_n\, f(x,\dots,x). \tag{5.7}$$

Here Cr_n is the n**th cross-effect**. Given any function $f\colon A \to B$ between abelian groups, define functions $\mathrm{Cr}_n\, f\colon A^{\times n} \to B$ by

$$\begin{aligned}
\mathrm{Cr}_1\, f(x) &:= f(x) - f(0),\\
\mathrm{Cr}_2\, f(x_1,x_2) &:= f(x_1+x_2) - f(x_1) - f(x_2) + f(0),\\
&\cdots\\
\mathrm{Cr}_n\, f(x_1,\dots,x_n) &:= \sum_{I\subseteq \underline{n}} (-1)^{n-|I|} f(\sum_{i\in I} x_i).
\end{aligned}$$

Note that for any $f\colon \Omega^\infty F \to \Omega^\infty E$, the operation $\mathrm{Cr}_n\, f\colon (\Omega^\infty F)^{\times n} \to \Omega^\infty E$ factors through the quotient map $(\Omega^\infty F)^{\times n} \to (\Omega^\infty F)^{\wedge n}$. This means that for $n > \dim X$, we have $\mathrm{Cr}_n\, f(x,\dots,x) = 0$, so the sum is finite.

Proof. If $B = B\otimes\mathbb{Q}$, then

$$\mathrm{Map}(A,B) \approx \mathrm{Hom}(\mathbb{Q}[A], B)$$

by the correspondence $f \leftrightarrow \widetilde{f}$ given by

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\delta\colon a\mapsto [a]"'] & B \\
\mathbb{Q}[A] \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$

If $\mathrm{Cr}_n\, f = 0$, then we can factor $\widetilde{f}$ through $\mathbb{Q}[A]/I^n$, where $I = \mathrm{Ker}(\mathbb{Q}[A] \to \mathbb{Q})$.

If we formally define $\mathcal{L}$ by the right-hand side of (5.7), then $\mathcal{L}f = \widetilde{f}(\mathcal{L}\delta)$. We compute

$$(\mathcal{L}\delta)(a) = \sum \frac{(-1)^{n-1}}{n} \mathrm{Cr}_n\, \delta(a,\dots,a) = \sum \frac{(-1)^{n-1}}{n} ([a]-1)^n,$$

and it is clear that $\mathcal{L}\delta\colon A \to B$ is a group homomorphism. Applied to $A = F^0(-)$ and $B = E^0(-)$, we get an additive cohomology operation. If $X = S^k$, then the terms $n \geq 2$ vanish in (5.7), so we see that $\mathcal{L}f$ has the same effect on homotopy groups as f.

$\square$

5.8. *Exercise.* Consider $BU \xrightarrow{\mathrm{Sym}^m} \Omega^\infty K \to \Omega^\infty K_{\mathbb{Q}}$, where Sym^m is the operation corresponding to mth symmetric power of vector bundles. Then $\mathcal{L}_{\mathbb{Q}}\mathrm{Sym}^m = \frac{1}{m}\psi^m$. (Hint: consider $S_t = \sum \mathrm{Sym}^m t^m\colon BU \to K[\![t]\!]$ first.)

5.9. **Rational logarithm.** We can apply the above if $R = R\mathbb{Q}$, to the map

$$s\colon GL_1(R)_{\geq 1} \xrightarrow{x \mapsto x-1} \Omega^\infty R,$$

where $GL_1(R)_{\geq 1}$ is the basepoint component of $GL_1(R)$. For a connected X, we get a natural homomorphism

$$\mathcal{L}_{\mathbb{Q}}s\colon (R^0X)^\times \to R^0X,$$

and we compute

$$(\mathcal{L}_{\mathbb{Q}}s)(x) = \sum \frac{(-1)^{n-1}}{n}(x-1)^n = \log x.$$

5.10. K**-theory.** Let's consider the case of $E = F = K_p$, following Madsen-Snaith-Tornehave [MST77].

$$[K_p, K_p] \xrightarrow{\Omega^\infty} [\Omega^\infty K_p, \Omega^\infty K_p]_H \subset [\Omega^\infty K_p, \Omega^\infty K_p].$$

This map Ω^∞ is injective. In fact,

$$[\Omega^\infty K_p, \Omega^\infty K_p]_H \approx \mathbb{Z}_p[\![\mathbb{Z}_p]\!] \approx \lim \mathbb{Z}_p[\mathbb{Z}_p/p^n],$$

with generators $[\lambda]$ corresponding to the *Adams operation* ψ^λ. And

$$[K_p, K_p] \approx \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!],$$

the map induced by the evident inclusion $\mathbb{Z}_p^\times \to \mathbb{Z}_p$.

5.11. **The idempotent.** In fact, we can identify the image of Ω^∞ in terms of an idempotent $\mathcal{E}$ on the set $[\Omega^\infty K_p, \Omega^\infty K_p]_H$. Here are several recipes.

- Recall the Bott periodicity map $\beta\colon K_p \xrightarrow{\sim} \Omega^2 K_p$. Given $f\colon K_p \to K_p$, define ωf by

$$\begin{tikzcd} K_p \arrow[r, dotted, "\omega f"] \arrow[d, "\beta"', "\sim"] & K_p \arrow[d, "\beta", "\sim"'] \\ \Omega^2 K_p \arrow[r, "\Omega^2 f"'] & \Omega^2 K_p \end{tikzcd}$$

  Compute that

$$\omega(\psi^\lambda) = \lambda\,\psi^\lambda.$$

  Now consider

$$\mathcal{E}f := \lim_{k\to\infty} \omega^{(p-1)p^k}(f).$$

  That this converges and gives the right result amounts to the fact that, p-adically, $\lambda \mapsto \lim_{k\to\infty} \lambda^{(p-1)p^k}$ is the characteristic function of $\mathbb{Z}_p^\times$.

- Here's what MST do. Recall that associated to any finite covering map $g\colon Y \to X$ is an associated transfer map $\Sigma^\infty_+ X \to \Sigma^\infty_+ Y$, which induces an (additive) transfer $g_!\colon E^*X \to E^*Y$ in any cohomology theory.

  Any infinite loop map commutes with transfers. MST prove:

  5.12. **Proposition.** *$f \in [\Omega^\infty K_p, \Omega^\infty K_p]_H$ is infinite loop iff*

$$\begin{tikzcd} K_p^0(X \times EC_p) \arrow[r, "f"] \arrow[d, "\tau_p"'] & K_p^0(X \times EC_p) \arrow[d, "\tau_p"] \\ K_p^0(X \times BC_p) \arrow[r, "f"'] & K_p^0(X \times BC_p) \end{tikzcd}$$

  commutes, where τ_p is the transfer associated to the covering map $EC_p \to BC_p$.

  Proof. We have

$$K_p^0(X \times BC_p) \approx K_p^0(X)[T]/(T^p - 1),$$

  where $T =$ line bundle on BC_p associated to the representation $C_p \subset \mathbb{C}^\times$. Furthermore, the transfer in K-theory is given by $\tau_p(x) = x \boxtimes N$, where $N = 1 + T + \cdots + T^{p-1}$.

  The proof follows from the calculation (for integers λ)

$$\psi^\lambda(N) = \begin{cases} p & \text{if } p|\lambda, \\ N & \text{if } p \nmid \lambda. \end{cases}$$

  Since

$$\tau_p\psi^\lambda(x) = \psi^\lambda(x) \boxtimes N, \qquad \psi^\lambda\tau_p(x) = \psi^\lambda(x) \boxtimes \psi^\lambda(N),$$

  we see that the image of $\Omega^\infty\colon [K_p, K_p] \to [\Omega^\infty K_p, \Omega^\infty K_p]_H$ (i.e., the subset $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \subset \mathbb{Z}_p[\![Z_p]\!]$) is exactly as asserted. $\square$

We can turn this into a formula for a projection operator $\mathcal{E}$ on $[\Omega^\infty K_p, \Omega^\infty K_p]_H$, by

$$(\mathcal{E}f)(x) = f(x) - \frac{1}{p}\langle f(\tau_p(x)), c\rangle.$$

Here $\langle -, c\rangle\colon K_p^0(X \times BC_p) \to K_p^0(X) \otimes \mathbb{Z}(\zeta_p)$ is a ring homomorphism defined by $T \mapsto \zeta_p = e^{2\pi i/p}$. (You can think of it as pairing with a homology class $c \in K_0^\wedge(BC_p) \otimes \mathbb{C}$.) The point is that

$$\langle N, c\rangle = 0, \qquad \langle 1, c\rangle = 1.$$

5.13. **Application to $GL_1(K_p)$.** You can apply this to

$$[\mathrm{gl}_1(K_p), K_p] \xrightarrow{\Omega^\infty} [GL_1(K_p), \Omega^\infty K_p]_H \subset [GL_1(K_p), \Omega^\infty K_p].$$

This is because of the theorem of Adams-Priddy [AP76]: there exists an equivalence

$$\mathrm{gl}_1(K_p)_{\geq 4} \approx (K_p)_{\geq 4}$$

of spectra. The Adams-Priddy is non-constructive: they show that if $\Omega^\infty F$ has the same homotopy groups and k-invariants as BSU_p, then F has the same homotopy type as $(K_p)_{\geq 4}$.

In fact, there is a splitting $\mathrm{gl}_1(K_p) \approx Z \times (K_p)_{\geq 4}$. As a consequence, you can show that the image of Ω^∞ is exactly the set of H-maps $f\colon GL_1(K_p) \to \Omega^\infty K_p$ which commute with transfer.

Now, the infinite loop space structure on $GL_1(K)$ comes from the multiplicative structure. Thus, a map $X \to GL_1(K)$ has an $EC_p \to BC_p$ transfer map

$$X \times BC_p \to X^p \times_{C_p} EC_p \to GL_1(K)^p \times_{C_p} EC_p \to GL_1(K)$$

which coincides with the power operation P_p on $K_p^0(X)$.

5.14. **Proposition** ([MST77]). *An H-space map $f\colon GL_1(K_p) \to \Omega^\infty K_p$ is infinite loop if and only if*

$$\begin{tikzcd} K_p^0(X \times EC_p)^\times \arrow[r, "f"] \arrow[d, "P^p"'] & K_p^0(X \times EC_p) \arrow[d, "\tau_p"] \\ K_p^0(X \times BC_p)^\times \arrow[r, "f"'] & K_p^0(X \times BC_p) \end{tikzcd}$$

commutes.

5.15. **tom Dieck's logarithm.** Recall that the total power operation

$$P_p\colon K_p^0(X) \to K_p^0(X \times BC_p) \approx K_p^0(X)[T]/(T^p - 1)$$

has the form

$$P_p(x) = \psi^p(x) - \theta^p(x)N,$$

where ψ^p is the Adams operation, and $N = 1 + T + \cdots + T^{p-1}$ is the regular representation of C_p. Furthermore, forgetting about the C_p-action, which amounts to setting $T \to 1$, gives the identity $\psi^p(x) - p\theta^p(x) = x^p$.

5.16. **Theorem** (tom Dieck [tD89]). *There is a spectrum map $\mathrm{gl}_1(K_p) \to K_p$ inducing a cohomology operation $\ell\colon K_p^0(X)^\times \to K_p^0(X)$ described by*

$$\ell(x) = \frac{1}{p} \log \frac{x^p}{\psi^p(x)} = \sum_{m \geq 1} \frac{(-1)^m p^{m-1}}{m} (\theta^p(x)/x)^m.$$

Proof. Verify that $\ell P_p = \tau_p \ell$. We can compute

$$\begin{aligned} \ell P_p(x) &= \frac{1}{p} \log \frac{(P_p(x))^p}{\psi^p(P_p(x))} \\ &= \frac{1}{p} \log \frac{(\psi^p(x) - \theta^p(x)N)^p}{\psi^p\psi^p(x) - \psi^p\theta^p(x)p} \\ &= \begin{cases} \frac{1}{p} \log \frac{x^{p^2}}{\psi^p(x^p)} = p\ell(x) & \text{under } T \mapsto 0, \\ \frac{1}{p} \log \frac{(\psi^p(x))^p}{\psi^p(x^p)} = 0 & \text{under } T \mapsto \zeta_p. \end{cases} \end{aligned}$$

The same is true for $\tau_p \ell(x) = \ell(x)N$. □

5.17. **Corollary.** *The map $\ell\colon\ \mathrm{gl}_1(K_p) \to K_p$ giving the tom Dieck logarithm is an equivalence on 3-connected covers.*

Proof. Compute its effect on π_{2n} by computing it on $K^0(S^{2n}) = \mathbb{Z}[\epsilon]/(\epsilon^2)$. We get

$$\ell(1+\epsilon) = \frac{1}{p} \log \frac{(1+\epsilon)^p}{\psi^p(1+\epsilon)} = \frac{1}{p} \log \frac{1 + p\epsilon}{1 + p^n\epsilon} = (1 - p^{n-1})\epsilon.$$

□

So tom Dieck's logarithm gives a "canonical" example of the equivalence $\mathrm{gl}_1(K_p)_{\geq 4} \approx (K_p)_{\geq 4}$ proved by Adams-Priddy.

5.18. **Where does this come from?** Recall that we obtained a rational logarithm by "linearizing" the map $x \mapsto x - 1\colon GL_1(R) \to R$ for a rational R.

The analogue of this can be done much more generally.

5.19. **Proposition.** *Let F be any spectrum, then*

$$\Omega^\infty\colon [F, K_p] \to [\Omega^\infty F, \Omega^\infty K_p]$$

is injective, with image equal to the image of an idempotent $\mathcal{E}$ on $[\Omega^\infty F, \Omega^\infty K_p]$, computed (on finite dimensonal X) by the formula

$$(\mathcal{E}f)(x) = \sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\left(\mathrm{Cr}_m f(x,\dots,x) - \frac{1}{p}\langle \mathrm{Cr}_m f(\pi_1^*\tau_p(x),\dots,\pi_m^*\tau_p(x)), c^{\times m}\rangle\right).$$

Here $\pi_i\colon X \times BC_p^{\times m} \to X \times BC_p$ is projection to the ith factor, and

$$\langle -, c^{\times m}\rangle\colon K_p^0(X \times BC_p^{\times m}) \to K_p^0(X) \otimes \mathbb{Z}_p[\zeta_p]$$

is "character evaluation" at the element $(c,\dots,c) \in C_p^m$.

If f is an H-map, then $\mathrm{Cr}_m f = 0$ for $m \geq 2$, so this simplifies to the formula $f(x) - \frac{1}{p}\langle f(\tau_p(x)), c\rangle$.

Now apply this to the shift map $s\colon GL_1(K_p) \to \Omega^\infty K_p$ defined by $s(x) = x - 1$.

5.20. **Corollary.** *The map $\mathcal{E}s\colon GL_1(K_p) \to \Omega^\infty K_p$ is the one given by tom Dieck's formula.*

Proof. Explicitly, the formula reduces to

$$(Es)(x) = \sum \frac{(-1)^{m-1}}{m}\big((x-1)^m - \frac{1}{p}(\psi^p(x) - 1)^m\big) = \log x - \frac{1}{p}\log\psi^p(x).$$

□

It turns out we can replace K_p with many other commutative S-algebras, which are "$K(n)$-local". The idea is (to some extent) suggested by the "Bott peroidicity" construction of the idempotent for K-theory.

5.21. **Bousfield-Kuhn functor.** A finite CW-complex V is **type** n if $K(n)_*V \neq 0$ but $K(n-1)_*V = 0$.

The *periodicity theorem* of Hopkins-Smith says that, for any type n finite V, there exists a (stable) map $f\colon \Sigma^d V \to V$ with $d = 2(p^n - 1)p^k > 0$ so that $K(n)_*f$ is multiplication by $v_n^{p^k}$. Such a map is called a v_n**-self map**.

Given a finite CW-complex V and a map $f\colon \Sigma^d V \to V$ with $d > 0$, we can define

$$\Phi_{V,f}\colon \mathrm{Top}_* \to \mathrm{Sp}$$

by sending a space X to the spectrum $E = \{\underline{E}_k\}$ with

$$\underline{E}_{kd} := \mathrm{Map}_*(V, X), \qquad k \geq 0,$$

with structure map $\underline{E}_{kd} \to \Omega^d \underline{E}_{kd+d}$ given by

$$\mathrm{Map}_*(V, X) \xrightarrow{\circ f} \mathrm{Map}_*(\Sigma^d V, X) \approx \Omega^d \mathrm{Map}_*(V, X).$$

Since $E = \operatorname{hocolim} \Sigma^{-kd}\Sigma^\infty \underline{E}_{kd}$, we can write this as

$$\Phi_{V,f} X \approx \operatorname{hocolim}_k \Sigma^{-kd}\Sigma^\infty \operatorname{Map}_*(V, X).$$

Observe that if $X = \Omega^\infty F$, then we can write this as

$$\Phi_{V,f}\Omega^\infty F \approx \operatorname{hocolim}_k \Sigma^{-kd}\underline{\operatorname{Hom}}(\Sigma^\infty V, F) \approx (\Sigma^\infty f)^{-1}\underline{\operatorname{Hom}}(\Sigma^\infty V, F) =: \operatorname{Tel}_{V,f} F.$$

In other words, we have a factorization

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "\mathrm{Tel}_{V,f}"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp} \\ \mathrm{Top}_* \arrow[ur, "\Phi_{V,f}"'] & \end{tikzcd}$$

The localization functor $L_{K(n)}\colon \mathrm{Sp} \to \mathrm{Sp}$ factors through an inverse limit of telescopes: $L_{K(n)} = L_{K(n)} \operatorname{holim}_i \Sigma^{-q_i} \operatorname{Tel}_{V_i,f_i}$. Bousfield and Kuhn show ([Bou87], [Kuh89]) that with the above technology, you can factor this through spaces:

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "L_{K(n)}"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp} \\ \mathrm{Top}_* \arrow[ur, "\Phi_n"'] & \end{tikzcd}$$

As a consequence, if $E = L_{K(n)}E$, then you get a canonically defined retraction

$$[\Omega^\infty F, \Omega^\infty E] \to [\Phi\Omega^\infty F, \Phi\Omega^\infty E] \approx [LF, LE] \approx [F, E]$$

of the map $[F, E] \to [\Omega^\infty F, \Omega^\infty E]$.

5.22. **Computing the Bousfield-Kuhn idempotent.** In many cases you can compute the resulting idempontent $\mathcal{E}$ on $[\Omega^\infty F, \Omega^\infty E]$, whose image is infinite loop maps.

One way is suggested by the construction of Φ, and should lead to a formula generalizing the "Bott periodicity" definition for K-theory. Stacey and Whitehouse [SW08] carried this out in the case that E and F are $K(n)$ or similar.

5.23. **A sketch of the idea.** In the approach I describe here, the key case turns out to be $F = \mathbb{S}$. Given $a \in E^0(\Omega^\infty\mathbb{S}) = [\Omega^\infty\mathbb{S}, \Omega^\infty E]$, we have

$$\begin{tikzcd} \Omega^\infty\mathbb{S} \arrow[r, "a"] \arrow[d, "\eta_{\Omega^\infty\mathbb{S}}"'] & \Omega^\infty E \\ \Omega^\infty\Sigma^\infty(\Omega^\infty\mathbb{S}) \arrow[ur, "\Omega^\infty\widetilde{a}"'] & \end{tikzcd}$$

where $\widetilde{a}\colon \Sigma^\infty\Omega^\infty\mathbb{S} \to E$ is the adjoint to a. Applying Φ gives

$$\begin{tikzcd} L\mathbb{S} \arrow[r, "\Phi a"] \arrow[d, "\lambda:=\Phi\eta"'] & LE = E \\ L\Sigma^\infty\Omega^\infty\mathbb{S} \arrow[ur, "L\widetilde{a}"'] & \end{tikzcd}$$

Here, $\lambda \in \pi_0 L\Omega^\infty\mathbb{S}$.

In other words, $[\Omega^\infty\mathbb{S}, \Omega^\infty E] \to [\mathbb{S}, E]$ is computed by

$$\langle -, \lambda\rangle\colon E^0\Omega^\infty\mathbb{S} \to E^0\mathbb{S},$$

which can be thought of in terms of pairing with the Hurewicz image of of λ in $E_0^\wedge \Omega^\infty \mathbb{S}$.

Fix $L = L_{K(n)}$.

5.24. **Theorem** ([Rez06]). *Given a space X, spectra E, F with $E = LE$, and a map $f\colon \Omega^\infty F \to \Omega^\infty E$, and $x \in F^0 X$, we have*

$$(\mathcal{E}f)(x) = \langle f(\tau x), \lambda \rangle.$$

That is, the diagram

$$\begin{tikzcd} F^0 X \arrow[rr, "\mathcal{E}f"] \arrow[d, "\tau"'] & & E^0 X \\ F^0(X \times \Omega^\infty \mathbb{S}) \arrow[rr, "f"'] & & E^0(X \times \Omega^\infty \mathbb{S}) \arrow[u, "\langle -,\lambda \rangle"'] \end{tikzcd}$$

commutes, where τ is the "transfer".

5.25. **A formula for E_n.** Computing the Bousfield-Kuhn idempotent amounts to "naming" the element $\lambda \in \pi_0 L\Omega^\infty \mathbb{S}$. It is easier to name its image in $E_0^\wedge \Omega^\infty \mathbb{S}$, because we have better access to the E homology of symmetric groups, using $\Omega^\infty \mathbb{S} \approx \left(\coprod B\Sigma_m\right)^+$, and HKR character theory.

5.26. **Proposition.** *Let E be a height n Morava E-theory. The BK idempotent on $[\Omega^\infty F, \Omega^\infty E]_H$ is computed by the formula*

$$(\mathcal{E}f)(x) = \sum_{k=0}^{n} (-1)^r p^{\binom{r}{2}-r} \sum_{[\alpha]} \langle f(\tau_{p^r}(x)), \omega(\alpha) \rangle.$$

Here

- *$\tau_r(x)\colon X \times B(\mathbb{Z}/p)^r \to \Omega^\infty F$ is the transfer of x along $X \times E(\mathbb{Z}/p)^r \to X \times B(\mathbb{Z}/p)^r$;*
- *the inner sum is over conjugacy classes of surjective homomorphisms $\alpha\colon \mathbb{Z}_p^n \to (\mathbb{Z}/p)^r$;*
- *$\langle -, \alpha \rangle\colon E^0(X \times B(\mathbb{Z}/p)^r) \to E^0 X \otimes_{E_0} D$ is evaluation of the HKR character map at α.*

5.27. **Corollary.** *If $[\Omega^\infty F, \Omega^\infty E]$ is p-torsion free, then an H-map $f\colon \Omega^\infty F \to \Omega^\infty E$ is infinite loop iff $f\tau_r = \tau_r f$ for all $r = 1, \dots, n$.*

Proof. Amounts to $\langle \tau_k(1), \alpha \rangle = 0$ when $r \geq 1$, by the transfer formula from HKR. □

There is a more complicated formula for non-H-maps, using cross-effects. Using this, you get the formula for $\ell_n = \mathcal{E}s$, where $s\colon GL_1(E) \to \Omega^\infty E$ is the shift map.

5.28. **Theorem** ([Rez06]).

$$\ell_n(x) = \frac{1}{p} \log \left(\prod_{k=0}^{n} \Big(\prod_{[\alpha]} \psi_\alpha(x) \Big)^{(-1)^r p^{\binom{r}{2}-r+1}} \right).$$

The functions $\psi_\alpha\colon E^0 X \to E^0 X \otimes_{E_0} D$ are certain power operations of the form

$$E^0 X \to E^0 X \otimes E^0 B\Sigma_{p^r}/I \to E^0 X \otimes E^0 B(\mathbb{Z}/p)^r/I \to E^0 X \otimes D.$$

5.29. **The height 2 case.** Suppose E has height 2. Then we can write the above formula in the following form.

$$\ell_2(x) = \frac{1}{p} \log \frac{x^p\, N_{p^2}(x)}{N_p(x)}.$$

Here $N_p(x) = \psi_{\alpha_0}(x) \dots \psi_{\alpha_p}(x)$ corresponding to the $p+1$ conjugacy classes of $\mathbb{Z}_p^2 \to \mathbb{Z}/p$, while $N_2(x) = \psi_\alpha(x)$ corresponding to any projection $\mathbb{Z}_p^2 \to (\mathbb{Z}/p)^2$.

Recall that power operations for E-theory produce maps

$$\psi_r \colon E^0(X) \to E^0(X) \otimes_{E_0} {}^s A_r.$$

- The map N_p is the composite

$$E^0(X) \xrightarrow{\psi_1} E^0(X) \otimes_{E_0} {}^s A_1 \xrightarrow{\text{norm}} E^0 X,$$

  where the second map is the Galois norm associated to $s\colon A_0 \to A_1$, which is finite and free of rank $p+1$. The operation N_p is multiplicative but not additive.
- The map N_{p^2} is the composite

$$E^0(X) \xrightarrow{\psi_2} E^0(X) \otimes_{E_0} {}^s A_2 \to E^0(X),$$

  using $A_2 \to A_0$ which classifies the subgroup $G[p] \subset G$.

The fact that this is well defined relies on a congruence

$$x^p\, N_{p^2}(x) \equiv N_p(x) \mod pE^0(X),$$

which is a consequence of the Frobenius congruence for power operations.

To compute ℓ_2 on homotopy groups, compute it on $E^0(S^{2n}) = E_0[\epsilon]/(\epsilon^2)$. The formula becomes linearized, so on π_{2n} we have

$$\ell_2(f) = f - T_p(f) + pT_{p^2}(f).$$

Here $T_p = p^{-1} \sum \psi_{\alpha_i}$, or equivalently, p^{-1} times the trace version of N_p, and $T_{p^2} = p^{-2} N_{p^2}$. The maps are versions of Hecke operators.

5.30. **Application to** tmf**.** This can be applied to tmf, obtaining a map of spectra

$$\ell_2 \colon \operatorname{gl}_1 \mathrm{tmf} \to L_{K(2)}\mathrm{tmf}$$

whose effect on π_{2k} (up to torsion) is $f \mapsto f - T_p(f) + p^{k-1} f$.

Using the Hasse square and the $K(1)$-local version of this, we show [AHR06] that you can factor this through a map

$$\operatorname{gl}_1 \mathrm{tmf} \xrightarrow{\ell_{\mathrm{tmf}}} \mathrm{tmf}_p \to L_{K(2)}\mathrm{tmf}.$$

The existence of ℓ_{tmf} is ad hoc, and relies on some calculations. Such a map doesn't exist for general commuative S-algebras. We obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{gl}_1 \mathrm{tmf} \arrow[r, dotted] \arrow[rr, bend left, "\ell_2"] \arrow[d, "\ell_1"'] & \mathrm{tmf}_p \arrow[r, "\iota_2"'] \arrow[d, "\iota_1"'] & L_{K(2)}\mathrm{tmf} \arrow[d, "\iota_1"'] \\
L_{K(1)}\mathrm{tmf} \arrow[r, "1-U"'] & L_{K(1)}\mathrm{tmf} \arrow[r, "L_{K(1)}\iota_2"'] & L_{K(1)}L_{K(2)}\mathrm{tmf}
\end{tikzcd}$$

In the case of tmf, here is a clue for why this ought to work: the above formula given for ℓ_2 actually makes sense for any elliptic curve with descent for isogenies.

Getting control of gl_1 tmf in this way is one of the steps in the construction of the string orientation for tmf.

References

[AA66] J. F. Adams and M. F. Atiyah, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 31–38.

[AP76] J. F. Adams and S. B. Priddy, *Uniqueness of BSO*, Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 3, 475–509.

[And95] Matthew Ando, *Isogenies of formal group laws and power operations in the cohomology theories E_n*, Duke Math. J. **79** (1995), no. 2, 423–485.

[AB02] Matthew Ando and Maria Basterra, *The Witten genus and equivariant elliptic cohomology*, Math. Z. **240** (2002), no. 4, 787–822.

[AHR06] M. Ando, M. J. Hopkins, and C. Rezk, *Multiplicative Orientations of KO-theory and of the spectrum of topological modular forms* (2006), in preparation, available at `http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf`.

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146** (2001), no. 3, 595–687.

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland, *The sigma orientation is an H_∞ map*, Amer. J. Math. **126** (2004), no. 2, 247–334.

[ADL16] G. Z. Arone, W. G. Dwyer, and K. Lesh, *Bredon homology of partition complexes*, Doc. Math. **21** (2016), 1227–1268.

[Ati66] M. F. Atiyah, *Power operations in K-theory*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 165–193.

[Beh06] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402.

[Beh07] ———, *Buildings, elliptic curves, and the $K(2)$-local sphere*, Amer. J. Math. **129** (2007), no. 6, 1513–1563.

[BL06] Mark Behrens and Tyler Lawson, *Isogenies of elliptic curves and the Morava stabilizer group*, J. Pure Appl. Algebra **207** (2006), no. 1, 37–49, available at `arXiv:math/0508079`.

[BE13] Daniel Berwick-Evans, *Perturbative sigma models, elliptic cohomology and the Witten genus* (2013), available at `arXiv:1311.6836`.

[Bou87] A. K. Bousfield, *Uniqueness of infinite deloopings for K-theoretic spaces*, Pacific J. Math. **129** (1987), no. 1, 1–31.

[Bou96] ———, *On λ-rings and the K-theory of infinite loop spaces*, K-Theory **10** (1996), no. 1, 1–30.

[Bui05] Alexandru Buium, *Arithmetic differential equations*, Mathematical Surveys and Monographs, vol. 118, American Mathematical Society, Providence, RI, 2005.

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014.

[Gan07] Nora Ganter, *Stringy power operations in Tate K-theory* (2007), available at `arXiv:math/0701565`.

[Gan13] ———, *Power operations in orbifold Tate K-theory*, Homology Homotopy Appl. **15** (2013), no. 1, 313–342.

[HL16] Michael Hill and Tyler Lawson, *Topological modular forms with level structure*, Invent. Math. **203** (2016), no. 2, 359–416.

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung, *Manifolds and modular forms*, Aspects of Mathematics, E20, Friedr. Vieweg & Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum.

[Joy85a] André Joyal, *δ-anneaux et vecteurs de Witt*, C. R. Math. Rep. Acad. Sci. Canada **7** (1985), no. 3, 177–182 (French).

[Joy85b] ———, *δ-anneaux et λ-anneaux*, C. R. Math. Rep. Acad. Sci. Canada **7** (1985), no. 4, 227–232 (French).

[Kas98] Takuji Kashiwabara, *Brown-Peterson cohomology of $\Omega^\infty\Sigma^\infty S^{2n}$*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 195, 345–362.

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.

[Kuh89] Nicholas J. Kuhn, *Morava K-theories and infinite loop spaces*, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math., vol. 1370, Springer, Berlin, 1989, pp. 243–257.

[Lur09] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Abel Symp., vol. 4, Springer, Berlin, 2009, pp. 219–277.

[MST77] I. Madsen, V. Snaith, and J. Tornehave, *Infinite loop maps in geometric topology*, Math. Proc. Cambridge Philos. Soc. **81** (1977), no. 3, 399–430.

[Rez06] Charles Rezk, *The units of a ring spectrum and a logarithmic cohomology operation*, J. Amer. Math. Soc. **19** (2006), no. 4, 969–1014 (electronic).

[Rez08] ———, *Power operations for Morava E-theory of height* 2 *at the prime* 2 (2008), available at `arXiv:0812.1320(math.AT)`.

[Rez09] ———, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379, available at `arXiv:0902.2499`.

[Rez12a] ———, *Rings of power operations for Morava E-theories are Koszul* (2012), available at `arXiv:1204.4831`.

[Rez12b] ———, *Modular isogeny complexes*, Algebraic & Geometric Topology **12** (2012), no. 3, 1373–1403, available at `:arXiv:1102.5022`.

[Rez13] ———, *Power operations in Morava E-theory: structure and calculations* (2013), available at `http://www.math.uiuc.edu/~rezk/power-ops-ht-2.pdf`.

[Rez14] ———, *Etale extensions of λ-rings* (2014), available at `http://www.math.uiuc.edu/~rezk/etale-lambda.pdf`.

[Ros01] Ioanid Rosu, *Equivariant elliptic cohomology and rigidity*, Amer. J. Math. **123** (2001), no. 4, 647–677.

[SS15] Tomer M. Schlank and Nathaniel Stapleton, *A transchromatic proof of Strickland's theorem*, Adv. Math. **285** (2015), 1415–1447.

[Sch16] Stefan Schwede, *Global homotopy theory* (2016), available at `http://www.math.uni-bonn.de/~schwede/global.pdf`.

[Seg88] Graeme Segal, *Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others)*, Astérisque **161-162** (1988), Exp. No. 695, 4, 187–201 (1989). Séminaire Bourbaki, Vol. 1987/88.

[Seg07] ———, *What is an elliptic object?*, Elliptic cohomology, London Math. Soc. Lecture Note Ser., vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 306–317.

[SW08] Andrew Stacey and Sarah Whitehouse, *Stable and unstable operations in mod p cohomology theories*, Algebr. Geom. Topol. **8** (2008), no. 2, 1059–1091.

[ST11] Stephan Stolz and Peter Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 279–340.

[Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208.

[Str98] N. P. Strickland, *Morava E-theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779, available at `arXiv:math/9801125`.

[tD89] Tammo tom Dieck, *The Artin-Hasse logarithm for λ-rings*, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math., vol. 1370, Springer, Berlin, 1989, pp. 409–415.

[Wal09] Barry Walker, *Orientations and p-adic analysis* (2009), available at `arXiv:0905.0022`.

[Wil82] Clarence Wilkerson, *Lambda-rings, binomial domains, and vector bundles over* $\mathbf{C}P(\infty)$, Comm. Algebra **10** (1982), no. 3, 311–328.

[Zhu14] Yifei Zhu, *The power operation structure on Morava E-theory of height 2 at the prime 3*, Algebr. Geom. Topol. **14** (2014), no. 2, 953–977.

[Zhu15] ———, *The Hecke algebra action on Morava E-theory of height 2* (2015), available at `arXiv:1505.06377`.

Department of Mathematics, University of Illinois, Urbana, IL
E-mail address: `rezk@math.uiuc.edu`