Contemporary Mathematics
Volume 167, 1994

Quantization and C^*-Algebras

MARC A. RIEFFEL

ABSTRACT. We survey recent developments concerning the quantization of Poisson manifolds, within the setting of C^*-algebras, with emphasis on strict deformation quantization and Berezin-Toeplitz quantization.

One of von Neumann's primary motivations for initiating the theory of operator algebras around 1930 was to establish a firm foundation for quantum physics. Over a decade later, after Gelfand and Naimark [**GN**] launched the theory of what we now call C^*- algebras, Segal [**Se1**] gave general reasons why C^*-algebras might provide an appropriate foundation for quantum physics. Nevertheless, another decade passed before many specific operator algebras began to be employed in a significant way in quantum physics. However, during the past three decades the relationship between quantum physics and operator algebras has flourished, and by now operator algebras play an extensive role in several areas of quantum physics, such as quantum statistical mechanics [**BR1, BR2**] and quantum field theory [**Con, Dp, GF, Ha**].

It would be far too vast an undertaking to try to survey here the present state of the relationship between operator algebras and quantum physics. Instead I will confine my attention to one aspect of this relationship, which goes by the general name of "quantization", and which concerns the passage from classical systems to quantum systems. I will restrict my attention primarily to finite systems. There exist somewhat standard ways, which I will not discuss here, for then passing to systems with unbounded numbers of particles which are being annihilated and created (second quantization), or for passing to the infinite systems of quantum statistical mechanics. The survey given here should not be considered to be well-balanced; it is the product of the interaction between my comprehension of

1991 *Mathematics Subject Classification.* Primary 46L60; Secondary 46L87, 81S99.

Some of the research reported on here was supported in part by National Science Foundation grant DMS-8912907. This paper is in final form and no version of it will be submitted for publication elsewhere.

the moment and the deadline for submitting the manuscript for these conference proceedings.

The first two sections of this survey will discuss various quantizations of $\mathbb{R}^{2n}$, while the next two sections will discuss some of their generalizations to more general manifolds. I will be placing emphasis on the passage to the semi-classical limit, that is, to the relationship between the quantized space and the original Poisson manifold.

I will devote the rest of this introduction to going quickly through some of the familiar incantations which set the stage for the relationship between classical mechanics and quantum mechanics.

Our universe has the remarkable property that ordinarily the future evolution of a physical system is determined just by knowing at a given time the positions and velocities (or momenta) of its constituents (all of this of course needing suitable interpretation for any specific type of system). Typically, no higher derivatives need be known, for example. Thus one says that the *state* of a system at any given time is determined by its position and momentum at that time. For a finite classical system, and under the usual idealizations, the possible positions range over a certain manifold, the configuration space. For a given position, the possible momenta of the system are the cotangent vectors of the configuration manifold at that position. Thus, the collection of possible states of the system is identified with the cotangent bundle of the configuration space. Consequently this cotangent bundle is often referred to as the "state space" or "phase space" of the classical system. Now the cotangent bundle of any manifold carries a canonical symplectic form, with corresponding Poisson bracket. In the case of a phase space, this Poisson bracket plays a basic role in the Hamiltonian formulation of classical mechanics.

An observable quantity of a classical physical system will provide a number (usually real) for each state of the system. Thus it will be a function on the state space. Conversely, any function on the state space can be considered as giving an observable. It is easy to argue that, when convenient, one can restrict attention to functions which are continuous, or smooth, and also to ones which are bounded or vanish at infinity. In particular, one can restrict attention to the commutative C^*-algebra, $C_\infty(S)$, of continuous complex- valued functions on S which vanish at infinity, where S is the state space. When statistical considerations are involved, one can have "mixed states", which are probability measures on the state space, so that the expected value of an observable for a mixed state will be the integral of the corresponding function with respect to the probability measure. That is, the mixed states are just the usual "states" of operator algebra theory, on the C^*-algebra $C_\infty(S)$. The points of S, and their corresponding probability measures, are then the "pure states".

One of the hallmarks of the physical regimes where quantum theory holds sway is that observables are no longer always simultaneously observable. For example, the position and momentum coordinates in a given direction are usually not simultaneously observable. To model quantum mechanical systems, one traditionally associates self-adjoint operators on Hilbert space to observables in such a way that the non-simultaneous-observability of two observables corresponds exactly to the non-commutativity of the corresponding operators. The rays in the Hilbert space are the pure states, while the mixed states are determined by trace-class operators.

Usually the quantum mechanical system is considered to be a quantum version of a specific classical system. One then wants many of the important observables of the classical system to have quantum counterparts, and one wants this correspondence to satisfy favorable properties. A suitable process for making operators correspond to functions will be called a quantization. Later we will give precise statements of some of the properties a quantization should have. But to motivate this, we will first examine the fundamental example in which the configuration space is $\mathbb{R}^n$. The cotangent space (state space) is then $\mathbb{R}^{2n}$, and the Poisson bracket is the standard one given by

$$\{f, g\} = \sum \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial q_j}$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n$ are the coordinates for $\mathbb{R}^{2n}$. We will let p_j and q_j also denote the corresponding coordinate functions on $\mathbb{R}^{2n}$, which are the observables for momentum and position respectively.

A corresponding quantum system should have observables for momentum and position modeled by self-adjoint operators P_j and Q_j. These are required to satisfy the fundamental Heisenberg commutation relation

$$[P_j, Q_j] = i\hbar I$$

where $\hbar$ is Planck's constant (divided by 2π), reflecting the non-simultaneous- observability of the corresponding observables. (Here $[A, B] = AB - BA$.) All other pairs will commute, that is, $[P_j, P_k] = 0 = [Q_j, Q_k]$ for all j, k, and $[P_j, Q_k] = 0$ for $j \neq k$. Since the p_j's and q_k's more-or-less generate the algebra of all functions on $\mathbb{R}^{2n}$, we can hope to extend the correspondence $p_j \mapsto P_j$, $q_k \mapsto Q_k$ to more-or-less all functions on $\mathbb{R}^{2n}$, associating in this way quantum observables to other classical observables.

The first prescription for making such an extension was given by Weyl in 1931 (section 14 of chapter IV of [**Wey**]). Even today it is the most elegant of the known prescriptions. We will describe it in the next section. Then in the following section we will describe some other prescriptions which are also in current use. In the last two sections of this survey we will then consider various generalizations of these prescriptions to the setting of general Poisson manifolds.

1. Weyl Quantization

Suppose we have (unbounded) self-adjoint operators P_j and Q_j, $j = 1, \ldots, n$, on some Hilbert space, which satisfy the Heisenberg commutation relations. Let

P denote the n-tuple of the P_j's, and similarly for Q. For $x \in \mathbb{R}^n$ let $x \cdot P = \sum x_j P_j$, and similarly for $x \cdot Q$. Then for fixed $(x, y) \in \mathbb{R}^{2n}$, $x \cdot P + y \cdot Q$ will be a self-adjoint operator (we are speaking heuristically here, not worrying about domains), and so we can form the unitary operator $U_{(x,y)} = e^{i(x \cdot P + y \cdot Q)}$. Weyl [**Wey**] said that this is the operator which we should associate to the function $(p, q) \mapsto e^{i(x \cdot p + y \cdot q)}$. Now (heuristically speaking) any function f on the state space $\mathbb{R}^{2n}$ can be expressed in terms of its Fourier transform $\hat{f}$ as

$$f(p, q) = \int \hat{f}(x, y) e^{i(x \cdot p + y \cdot q)} dx\ dy\ .$$

Weyl [**Wey**] said that we should then associate to f the operator

$$L_f = \int \hat{f}(x, y) e^{i(x \cdot P + y \cdot Q)} dx\ dy = \int \hat{f}(w) U_w\ dw\ ,$$

where $w = (x, y)$. This is the general form of the process known as Weyl quantization.

This process was carried a step further by von Neumann in 1931 [**Ne**]. He pointed out, in effect, that Weyl quantization induces a new product on functions. More specifically, let f and g be functions on $\mathbb{R}^{2n}$. Then $L_f L_g = L_h$ where h is determined by $\hat{h} = \hat{f} *_\omega \hat{g}$, the twisted convolution of $\hat{f}$ and $\hat{g}$ for the cocycle ω on $\mathbb{R}^{2n}$ defined by

$$\omega(w, z) = \exp(iJw \cdot z)$$

for $w, z \in \mathbb{R}^{2n}$, where J is the standard symplectic matrix on $\mathbb{R}^{2n}$ multiplied by $\hbar$.

We would like to make all of the above more precise. In the process we would like to bring out the functorial nature of the construction, and also change to notation which will be more convenient when we consider general manifolds. Accordingly, let V denote a finite-dimensional real vector space (alias for $\mathbb{R}^{2n}$, though V need not be even-dimensional), and let V' denote its dual vector space. A translation-invariant Poisson bracket on V is determined by a bivector at 0, which can be viewed equivalently as a skew bilinear form on V', or as a linear map J from V' to V whose transpose J^t satisfies $J^t = -J$. We choose such a J, which we allow to be degenerate. As in the previous paragraph, we absorb $\hbar$ into J. For later convenience we now change the usage of the variables x, p, etc., from that employed at the beginning of this section. We denote the pairing of $x \in V$ with $p \in V'$ by $p \cdot x$. Let e denote the function $e(t) = e^{2\pi it}$, where t is a real variable. Then V' can be identified with the dual group of V by means of the pairing

$$(x, p) \mapsto e(p \cdot x)\ .$$

Choose any Haar (Lebesgue) measure on V. We define the Fourier transform for $f \in L^1(V)$ by

$$\hat{f}(p) = \int f(x) e(p \cdot x) dx\ .$$

Then on V' we will always take as measure the corresponding Plancherel (Lebesgue) measure for the given choice of Haar measure on V, so making the Fourier transform a unitary operator between L^2-spaces. The corresponding product measure on $V' \times V$ will then be independent of the choice of Haar measure on V.

Some fiddling with the twisted convolution of Fourier transforms mentioned earlier shows that von Neumann's product can be rewritten, at least formally, in a way not explicitly involving the Fourier transform, namely, as

$$(f \times_J g)(x) = \int_{V'} \int_V f(x + Jp) g(x + v) e(p \cdot v) dv \, dp, \tag{1.1}$$

where our conventions are slightly different from those suggested at the beginning of this section (by factors of π, -1, etc.). This way of writing the deformed product probably first appeared essentially in VIII.5a of **[Po]**. We will call the product $\times_J$ the "deformed product" (deformed by J). We remark that in defining this product in **[Rf6]** we placed an inner product on V and used it to identify V' with V. This was convenient for obtaining the various operator norm estimates which we needed, but it obscures some of the functoriality.

Let $\mathcal{S}$ denote the space of Schwartz functions on V. Then for $f, g \in \mathcal{S}$ the integral in (1.1) can be viewed as an iterated integral, and the integration over V can be viewed as a Fourier transform (after making the change of variables $v \mapsto v - x$), so that the deformed product can be rewritten as

$$(f \times_J g)(x) = \int_{V'} f(x + Jp) \hat{g}(p) \bar{e}(p \cdot x) \, . \tag{1.2}$$

(We will begin routinely omitting the dp etc. from integrals when this will not cause confusion.) Since $\hat{g} \in \mathcal{S}$ if $g \in \mathcal{S}$, the above integral is well-defined as an ordinary integral. Further examination **[GV, Rf6]** shows easily that $f \times_J g \in \mathcal{S}$ for $f, g \in \mathcal{S}$. Thus the deformed product is well-defined on $\mathcal{S}$.

But in fact, the deformed product is much more widely defined **[GV, Ho]**. We will only describe here the extension which is most pertinent to our needs and fits best into the framework of C^*-algebras. Let $\mathcal{B}$ denote the algebra of all smooth bounded functions on V all of whose derivatives of all degrees are bounded. Then for $f, g \in \mathcal{B}$ the integral in (1.1) for the product $f \times_J g$ can be shown to make sense as an oscillatory integral **[Ho, Rf6, SaR]**, and furthermore $f \times_J g \in \mathcal{B}$. This product on $\mathcal{B}$ is associative, although because of the involvement of oscillatory integrals the proof is a bit delicate (**[Rf6, SaR]**). Some further analysis shows that for $f \in \mathcal{B}$ and $g \in \mathcal{S}$ the product as defined by (1.2) is still well- defined, and $f \times_J g \in \mathcal{S}$. Carrying this a bit further, one finds that $\mathcal{S}$ is a two-sided ideal in $\mathcal{B}$ for $\times_J$.

Examination of the operator L_f defined near the beginning of this section shows that $(L_f)^* = L_{\bar{f}}$ for $f \in \mathcal{S}$, where $\bar{f}$ is the complex conjugate of f. (It is here that we need J to be skew-symmetric.) This suggests that we define an involution on $\mathcal{B}$ by $f^* = \bar{f}$; this is easily verified to be an involution for the product $\times_J$.

We now want to define an operator norm on $\mathcal{B}$ for its deformed product and the above involution. For this purpose we place on $\mathcal{S}$ the usual inner product, defined by

$$\langle f, g\rangle = \int f(x)\overline{g}(x)dx \ .$$

(It is not hard to see that this is just the GNS inner product for the positive (for $\times_J$) linear functional on $\mathcal{S}$ defined by $f \mapsto \int f(x)dx$.) Then for any $g \in \mathcal{B}$ the operator L_g on $\mathcal{S}$ defined by $L_g f = g \times_J f$ is a bounded operator for the above inner product. This fact is a substantial theorem [**Rf6**], which is essentially the Calderón-Vaillancourt theorem [**ClV, Ho**] from the theory of pseudo-differential operators. (Within the context of pseudo-differential operators the function g is usually called the "symbol" of L_g.) We place on $\mathcal{B}$ the corresponding operator norm, and denote its completion by $\overline{\mathcal{B}}_J$. The completion, $\overline{\mathcal{S}}_J$, of $\mathcal{S}$ for the operator norm is a two-sided ideal in $\overline{\mathcal{B}}_J$. This ideal can be seen to be essential, so that $\overline{\mathcal{B}}_J$ can be viewed as a subalgebra of the multiplier algebra of $\overline{\mathcal{S}}_J$. Of course $\overline{\mathcal{B}}_J$ will be realized as a C^*-algebra of operators on the Hilbert space completion of $\mathcal{S}$. From the definition of the deformed product it is clear that the action of V on $\mathcal{B}$ by translation gives an action by $*$-automorphisms for the deformed product. This action is easily seen to extend to a strongly continuous action on $\overline{\mathcal{B}}_J$.

Thus we have achieved much of our goal — we have obtained a representation of at least all of the functions in $\mathcal{B}$ as bounded operators on a Hilbert space in a way compatible with the deformed product. The deformed product can also be extended to polynomial functions, which will give unbounded operators on $\mathcal{S}$. When $V = \mathbb{R}^{2n}$ and J is the standard skew-form, it is easily seen that the coordinate functions will, as operators, satisfy the Heisenberg commutation relations for $\hbar = 1$ (at least up to factors of π etc.); and the representation of $\mathcal{B}$ can then be viewed as an extension of these relations to all of $\mathcal{B}$. More generally, when J is non-degenerate, the ideal $\overline{\mathcal{S}}_J$ can be shown to be isomorphic to the algebras of compact operators on a Hilbert space; while if J is degenerate, then $\overline{\mathcal{S}}_J$ can be shown [**Rf6**] to be a continuous field of compact operator algebras indexed by the quotient of V by the range of J. In particular, when $J = 0$ the deformed product is just the original pointwise product on $\mathcal{B}$. This can be seen easily from (1.2).

When J is non-degenerate so that $\overline{\mathcal{S}}_J$ is the algebra of compact operators, we can ask to represent $\overline{\mathcal{B}}_J$ *irreducibly* on a Hilbert space, rather than by its left regular representation, as we have in effect been doing above. For this purpose we can choose bases in such a way that we are exactly in the situation of $V = \mathbb{R}^{2n}$ with J giving the standard Poisson bracket. Denote the position and momentum variables by (s_j) and (ξ_k) respectively, and let $\mathbb{R}^n$ be the configuration space coordinatized by $s = (s_j)$. Then we can define a representation of the Heisenberg commutation relations (for $\hbar = 1$) on $L^2(\mathbb{R}^n)$ by

$$(Q_j\phi)(s) = s_j\phi(s) \ , \quad (P_k\phi)(s) = -i\frac{\partial\phi}{\partial s_k}(s) \ .$$

It is appropriate to call this representation the "Schrödinger representation". Then one can calculate [**Fl, GLS**] that

$$(e^{2\pi i(xP+yQ)}\phi)(t) = e^{2\pi i(t\cdot y+x\cdot y/2)}\phi(t+x) \ .$$

Let us denote this operator by $W_{(x,y)}$. This gives a projective irreducible representation of $\mathbb{R}^{2n}$ on $L^2(\mathbb{R}^n)$. On applying the definition of Weyl's quantization to this representation and rearranging, we find that

$$(L_f\phi)(s) = \iint f\left(\frac{s+t}{2}, \xi\right) e((s-t)\cdot\xi)\phi(t)dt\, d\xi \tag{1.3}$$

for $f \in \mathcal{B}$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. (See Proposition 3.1 of Chapter 1 of [**T2**], apart from some factors of π, or 1.29 of [**Fl**].) This gives an irreducible isometric *-representation of $\mathcal{B}$ as bounded operators on $L^2(\mathbb{R}^n)$, and these operators are the usual general pseudo-differential operators of the Weyl calculus, of order 0, with symbol f [**Ho**]. The reason that we do not emphasize this representation is that when J is degenerate this representation is not faithful. In particular, when $J = 0$ we obtain the representation of $\mathcal{B}(\mathbb{R}^{2n})$ on $L^2(\mathbb{R}^n)$ by pointwise multiplication, and this sees only the first n coordinates of functions in $\mathcal{B}$. This makes it awkward to use this representation to examine the limit as $\hbar$ (i.e. J) goes to 0.

Let us now consider this limit. For this purpose we will fix J, and we will write $\times_\hbar$ for $\times_{\hbar J}$. We will also denote the operator norm on $\mathcal{B}$ for $\hbar J$ by $\| \ \|_\hbar$, and we will denote the corresponding completion of $\mathcal{B}$ by $\overline{\mathcal{B}}_\hbar$. Then it can be shown [**Rf6**] that

(1) The algebras $\overline{\mathcal{B}}_\hbar$ form a continuous field of C^*-algebras, so that in particular, $\hbar \mapsto \|f\|_\hbar$ is a continuous function of $\hbar$ for any $f \in \mathcal{B}$.
(2) For any $f, g \in \mathcal{B}$ we have $\|(f \times_\hbar g - g \times_\hbar f)/\hbar - i\{f,g\}_J\|_\hbar \longrightarrow 0$ as $\hbar \to 0$, where $\{\ ,\ \}_J$ is the Poisson bracket defined by J (with a factor of π).

This second property, together with the continuity at $\hbar = 0$ of the first property, can be viewed as the version in the present context of the famous Correspondence Principle of quantum mechanics, which says that as $h \to 0$ the quantum mechanical model should converge to the classical model, with commutators converging (after division by $\hbar$) to the Poisson bracket (multiplied by i).

If in our definition of the deformed product one substitutes for the function e its Taylor expansion, and rearranges using Fourier transforms, then one obtains, for J the standard Poisson bracket, an expansion of the form

$$f \times_\hbar g = \sum_{|\alpha|=0}^{\infty} \left(\frac{i\hbar}{2}\right)^{|\alpha|} (\alpha!)^{-1}(-1)^{(\alpha)}\partial^\alpha f \partial^\alpha g$$

where α is a multi-index in the usual way and $\langle\alpha\rangle = a_{n+1} + \cdots + \alpha_{2n}$. See [**EGV**]. Of course there are serious questions about when and how such a series will converge. But as a formal power series in $\hbar$ this formula is the basic example underlying the extensive subject of *formal deformation quantization*, in which one looks for similar "products" defined just in terms of formal power series, for general manifolds with Poisson bracket. (See [**BaF**] as well as references in [**Rf6**].) Since this does not involve operator algebras (and, indeed, for a few authors it is explicitly an attempt to remove quantum mechanics from its reliance on Hilbert spaces), we will not discuss it further here. However, since it is in some sense easier to construct "products" in terms of formal power series than to construct products analytically, the results of formal deformation quantization can sometimes serve as a useful guide for what to hope for in the analytical case.

Let us also mention here that we will not be discussing the subject of geometric quantization, since it too has not had much interaction with C^*-algebras. But see [**Bt**] for a brief discussion of its relation to other quantizations.

We now examine the functoriality of Weyl quantization. Since our treatment here goes somewhat beyond that given in [**Rf6**], we will be somewhat less sketchy than above. Let (V, J) and (W, K) be finite-dimensional real vector spaces equipped with skew-symmetric bilinear forms on their duals. Call these "Poisson vector spaces". By a morphism from (V, J) to (W, K) we mean a linear map T from V to W which is compatible with the forms in the sense that

$$J(T'w_1', T'w_2') = K(w_1', w_2')$$

for all $w_1', w_2' \in W'$ (where T' denotes the dual, or transpose, of T). If we view J and K as linear operators as done earlier, this can be rewritten as

$$\langle J(T'w_1'), T'w_2'\rangle = \langle Kw_1', w_2'\rangle,$$

or

$$\langle (TJT')w_1', w_2'\rangle = \langle Kw_1', w_2'\rangle,$$

so that $TJT' = K$, much as in the definition of symplectic linear maps. With these morphisms, the Poisson vector spaces form a category.

We want to show that any morphism induces a homomorphism of the corresponding C^*-algebras. We work first at the level of functions. Let $f, g \in \mathcal{B}(W)$, so that $f \circ T$ and $g \circ T \in \mathcal{B}(V)$. Then

$$((f \circ T) \times_J (g \circ T))(x) = \iint f(Tx + TJp)g(Tx + Tv)e(p \cdot v) \ .$$

We need the following fact:

1.2 PROPOSITION. *Let $F \in \mathcal{B}(V' \times W)$. Then for any linear transformation $T : V \to W$ we have*

$$\int_{V'} \int_V F(p, Tv)e(p \cdot v) = \int_{W'} \int_W F(T'm, w)e(m \cdot w) \ .$$

PROOF. This is basically an extension of proposition 1.13 of **[Rf6]**, but will not follow directly from it since V and W may have different dimensions. But we can appeal to other results in **[Rf6]** as follows. Choose any basis for V such that the unit cube has unit Haar measure. Take as isomorphism from V to V' the map which sends each basis vector to its dual basis vector. This isomorphism determines an inner product on V for which the basis is orthonormal. Under this isomorphism the Plancherel Haar measure on V' will be the image of the Haar measure on V. Then it is easily seen that proposition 1.11 of **[Rf6]** yields:

1.3 PROPOSITION. *Let $F \in \mathcal{B}(V' \times V)$ and suppose that in its second variable F is constant on the cosets of a subspace V_0 of V. Then*

$$\int_{V'}\int_V F(p,v)e(p\cdot v) = \int_{V_0^\perp}\int_{V/V_0} F(p,v)e(p\cdot v) .$$

A similar statement holds if instead F is in its first variable constant on the cosets of some subspace of V'.

If we apply this to the situation of Proposition 1.2 with $V_0 = \text{kernel}(T)$, we see that

$$\int_{V'}\int_V F(p,Tv)e(p\cdot v) = \int_{V_0^\perp}\int_{V/V_0} F(p,Tv)e(p\cdot v) .$$

Let W_1 be the range of T, so that T is an isomorphism from V/V_0 onto W_1. Then T' is an isomorphism from W_1' onto $(V/V_0)' = V_0^\perp$. Notice that $(T'm)\cdot(T^{-1}w) = m\cdot w$ for $m \in W_1'$, $w \in W_1$. It follows that $T' \times T^{-1}$ will carry a Plancherel Haar measure on $W_1' \times W_1$ to one on $V_0^\perp \times (V/V_0)$. Consequently, for any $G \in \mathcal{B}(V_0^\perp \times (V/V_0))$ we have

$$\int_{V_0^\perp}\int_{V/V_0} G(p,v)e(p\cdot v) = \int_{W_1'}\int_{W_1} G(T'm, T^{-1}w)e(m\cdot w) .$$

When we apply this to the integral obtained earlier, we get

$$\int_{V'}\int_V F(p,Tv)e(p\cdot v) = \int_{W_1'}\int_{W_1} F(T'm,w)e(m\cdot w) .$$

Let Z denote the kernel of T' as a transformation from W'. Then by the second part of Proposition 1.3 we have

$$\int_{W'}\int_W F(T'm,w)e(m\cdot w) = \int_{W'/Z}\int_{Z^\perp} F(T'm,w)e(m\cdot w) .$$

But since Z is the kernel of T' we have $Z = W_1^\perp$, so that $Z^\perp = W_1$ and $W'/Z = W_1'$. This completes the proof of Proposition 1.2.

When we apply Proposition 1.2 to the integral appearing just before its statement, we obtain

$$\begin{aligned}((f\circ T)\times_J(g\circ T))(x) &= \int_{W'}\int_W f(Tx+TJT'm)g(Tx+w)e(m\cdot w)\\ &= \iint f(Tx+Km)g(Tx+w)e(m\cdot w)\\ &= (F\times_K g)(Tx)\ .\end{aligned}$$

That is,

$$(f\circ T)\times_J(g\circ T)=(f\times_K g)\circ T\ .$$

It is clear that $(f\circ T)^-=\bar f\circ T$. Thus composition with T gives a *- homomorphism from $\mathcal{B}(W)$ into $\mathcal{B}(V)$.

We want to show that this homomorphism is norm non-increasing. A direct proof can undoubtedly be given. But it will be briefer for us to appeal to one of the deeper results in [**Rf6**], namely theorem 7.1, which implies that $\mathcal{B}_K(W)$ is exactly the algebra of C^∞-vectors within the C^*-algebra $\overline{\mathcal{B}}_K(W)$ for the action of W by translation. It follows that $\mathcal{B}_K(W)$ is closed under the holomorphic functional calculus in $\overline{\mathcal{B}}_K(W)$. Hence $\mathcal{B}_K(W)$ is a local C^*- algebra as defined in 3.1.1 of [**Bl**]. But by corollary 3.1.5 of [**Bl**], any *-homomorphism from a local C^*-algebra into a C^*-algebra must be norm non- increasing. Thus our homomorphism from $\mathcal{B}_K(W)$ into $\overline{\mathcal{B}}_J(V)$ is norm non-increasing, as desired. We will denote the extension of this homomorphism to $\overline{\mathcal{B}}_K(W)$ by $\overline{\mathcal{B}}(T)$.

It is clear that if (X,L) is another Poisson vector space and if S is a morphism from (W,K) to (X,L), then

$$\overline{\mathcal{B}}(T)\circ\overline{\mathcal{B}}(S)=\overline{\mathcal{B}}(S\circ T)\ .$$

We summarize the above as:

1.4 Theorem. *The process of attaching to a Poisson vector space (V,J) the corresponding C^*-algebra $\overline{\mathcal{B}}_J(V)$ is a contravariant functor from the category of Poisson vector spaces to the category of unital C^*-algebras with unital homomorphisms.*

We also need the following refinement of the above ideas:

1.5 Proposition. *With V and J as above, let Q be a linear map of V onto a vector space U, and let $K=QJQ'$, so that Q can be viewed as a morphism from (V,J) to (U,K). Then the homomorphism $\overline{\mathcal{B}}(Q)$ from $\overline{\mathcal{B}}(U)_K$ to $\overline{\mathcal{B}}(V)_J$ is injective (and isometric).*

Proof. By choosing a basis for V as discussed earlier, we can throw this back to the setting of [**Rf6**] involving an inner product on V. Let α denote the action of V on $\overline{\mathcal{B}}(V)$ by translation. Then $\overline{\mathcal{B}}(U)$ can be viewed as a subalgebra of $\overline{\mathcal{B}}(V)$ which is α-invariant. (Here $\overline{\mathcal{B}}$ denotes closure in the supremum norm.) Let β denote α as action of V on $\overline{\mathcal{B}}(U)$. By proposition 5.8 of [**Rf6**], the map from $\overline{\mathcal{B}}(U)^\beta_J$ into $\overline{\mathcal{B}}(V)^\alpha_J$ is injective. Let P denote the orthogonal projection of V onto the orthogonal complement, V_1, of the kernel of Q. So there is a natural identification of V_1 with U. Then it is clear that $\beta = \beta \circ P$. By theorem 8.11 of **[Rf6]** we then have

$$\overline{\mathcal{B}}(U)^\beta_J = \overline{\mathcal{B}}(U)^\beta_{PJP} \ .$$

By theorem 8.7 of **[Rf6]** it follows that $\overline{\mathcal{B}}(U)^\beta_{PJP} = \overline{\mathcal{B}}(U)^\gamma_L$ where γ is the restriction of β to V, and L is the restriction of PJP to V_1. But under the natural identification of V_1 with U we have $L = K$ and γ is just the action by translation, so $\overline{\mathcal{B}}(U)^\gamma_L = \overline{\mathcal{B}}(U)_K$. $\square$

Let us consider the following interesting application of the above ideas, which is motivated by the discussion in **[KM]**. Let V be an infinite-dimensional real vector space, and let V' be some vector space which is in non-degenerate duality with V. Let J be a linear map from V' into V which is skew-symmetric in the sense that

$$\langle Jq, p\rangle = -\langle Jp, q\rangle$$

for all $p, q \in V'$. For example, let M be a manifold, let $V = \mathcal{E}'(M)$ be the vector space of all complex-valued distributions on M, and let $V' = C_c^\infty(M)$ be the vector space of complex-valued test functions (of compact support) on M, with the evident duality between V' and V. Choose a smooth positive measure on M of full support, and use it to define an inner product, $[\ ,\ ]$, on V'. Now view V and V' as real vector spaces, and define the linear map J from V' to V by

$$\langle \psi, J\phi\rangle = \mathrm{Im}([\phi, \psi]) \ .$$

As another, currently popular, example, let $V = W$ be the vector space of real-valued smooth functions on the circle, paired by the usual inner product. Define J by

$$J\phi = \phi' \in V$$

for $\phi \in W$, where ϕ' denotes the usual derivative of ϕ.

We return to the case of general V, V' and J. Let $\mathcal{F}$ denote the set of finite-dimensional subspaces of V', ordered by inclusion. For $F \in \mathcal{F}$ let $V_F = V/F^\perp$, so that $(V_F)' = F$ in the natural way. Let J_F be the "restriction" of J to $(V_F)'$, defined by $J_F(p) = Jp + F^\perp$ for $p \in F = (V_F)'$. Thus (V_F, J_F) is a Poisson vector space.

Suppose that $E \in \mathcal{F}$ with $E \subseteq F$. Then we have an evident surjection, P_{FE}, of V_F onto V_E. It is easily verified that $P_{FE}J_FP'_{FE} = J_E$. Thus P_{FE} is a morphism from (V_F, J_F) onto (V_E, J_E). From Theorem 1.4 and Proposition 1.5 it follows that $\overline{\mathcal{B}}(P_{FE})$ is an injection from $\overline{\mathcal{B}}(V_E)_{J_E}$ into $\overline{\mathcal{B}}(V_F)_{J_F}$ (and so is isometric). It is easily seen that if $D \in \mathcal{F}$ with $D \subseteq E$ then $\overline{\mathcal{B}}(P_{FD}) = \overline{\mathcal{B}}(P_{FE}) \circ \overline{\mathcal{B}}(P_{ED})$. Thus the collection $\{\overline{\mathcal{B}}(V_F), \overline{\mathcal{B}}(P_{FE}) : E, F \in \mathcal{F}\}$ is a directed system of C^*-algebras with isometric morphisms. Thus we can form its direct limit C^*-algebra.

1.6 DEFINITION. The direct limit of $\{\overline{\mathcal{B}}(V_F), \overline{\mathcal{B}}(P_{FE})\}$ will be denoted by $\overline{\mathcal{B}}(V)_J$, and will be called the Weyl C^*-algebra for (V, J) (where we think of J as remembering that its domain is the particular V' being employed).

This Weyl C^*-algebra for infinite dimensional (V, J) is larger, and so probably more convenient and flexible, than that given in theorem 2.6 of [**KM**]. Let us see that it contains the appropriate Weyl unitary operators. Suppose that $p \in W$. For any $F \in \mathcal{F}$ for which $p \in F$ let f_p^F be the function in $\mathcal{B}(V_F)$ defined by

$$f_p^F(v + F^\perp) = e(\langle v, p\rangle).$$

If $p, q \in F$, then a quick calculation shows that

$$f_p^F \times_{J_F} f_q^F = e(\langle Jp, q\rangle) f_{p+q}^F,$$

so that the f_p^F's satisfy the Weyl commutation relations for J. If $p \in E \subseteq F$, then $\bar{\mathcal{B}}(P_{FE})(f_p^E) = f_p^F$. Thus for fixed $p \in W$ the f_p^F's form a coherent family as F runs over elements of $\mathcal{F}$ containing p, and so determine an element, f_p, in $\bar{\mathcal{B}}(V)_J$. These f_p's will be unitary elements of $\bar{\mathcal{B}}(V)_J$ which satisfy the Weyl commutation relations for J.

We now turn briefly to some more traditional aspects of the functoriality of our construction. Let (V, J) be a finite-dimensional Poisson vector space. Let G_J denote the group of all invertible operators T on V such that $TJT' = J$, that is, the group of automorphisms of (V, J). This can be viewed as the "symplectic" group for (V, J). Then by the functoriality discussed above, it follows that G_J acts on $\overline{\mathcal{B}}_J$ as a group of automorphisms, though we must insert an inverse since our construction is contravariant. We saw above that V also acts by translation on $\overline{\mathcal{B}}_J$. It is easily seen that the semi-direct product $V \rtimes G_J$ of V by the evident action of G_J then acts on $\overline{\mathcal{B}}_J$. This action will carry the ideal $\overline{\mathcal{S}}_J$ into itself.

Suppose now that J is non-degenerate. Then, as discussed earlier, $\overline{\mathcal{S}}_J$ is isomorphic to the algebra of compact operators on Hilbert space. The action of $V \rtimes G_J$ on $\overline{\mathcal{S}}_J$ will then give a projective unitary representation of $V \rtimes G_J$ on the Hilbert space, H, of any irreducible representation of $\overline{\mathcal{S}}_J$. Restricted to G_J, this projective representation is the much- studied metaplectic representation [**Fl**] of G_J. Restricted to V it gives an ordinary representation of the Heisenberg group for (V, J) on H. When this representation is used to conjugate operators on H, it gives an action of V on the algebra $B(H)$ of all bounded operators on H. Since $\mathcal{B}_J$ is contained in the multiplier algebra of $\overline{\mathcal{S}}_J$, it also will act on H, and it is easily seen that the action of V on $B(H)$ essentially corresponds under this representation to the action of V on $\mathcal{B}_J$ by translation. (See 2.13 of [**Fl**], where the conventions are slightly different.) From this it is easily seen that the operators from $\mathcal{B}_J$ are smooth vectors for the action of V on $B(H)$. For the Kohn-Nirenberg quantization (see the next section) Cordes [**Cr,Pa**] has obtained the lovely results that the operators from $\mathcal{B}_J$ are exactly *all* the smooth vectors. Presumably the same is true in the present setting. Thus we find that $\mathcal{B}_J$ can be considered to be a smooth version of $B(H)$.

2. Other Quantizations of $\mathbb{R}^{2n}$

A. The Kohn–Nirenberg quantization.

The idea behind the Weyl quantization was to associate to the function $(p,q) \mapsto e^{i(x\cdot p+y\cdot q)}$ the unitary operator $e^{i(x\cdot P+y\cdot Q)}$ and then to extend it by using the Fourier decomposition of functions. But we could instead associate to $e^{i(x\cdot p+y\cdot q)}$ the unitary operator $e^{ix\cdot P}e^{iy\cdot Q}$. This can be viewed **[GLS]** as the idea behind the Kohn–Nirenberg quantization **[KN]**. When P and Q are the operators of the Schrödinger representation, this gives for suitable functions f on $\mathbb{R}^{2n}$ (i.e. "symbols") the operator K_f on $L^2(\mathbb{R}^n)$ defined by

$$(K_f\varphi)(s) = \iint f(s,\xi)e((s-t)\cdot\xi)\varphi(t)dtd\xi. \tag{2.1}$$

(See 2.31 of **[Fl]** or 3.25 of Chapter 1 of **[T2]**.) The symbol for the product of two such operators has an expression quite similar to that for the Weyl quantization given in 1.1. (See the appendix of **[GLS]**.) However an awkward feature of the Kohn–Nirenberg quantization is that because of the unsymmetric ordering of the P's and Q's, real-valued functions do not always go to self-adjoint operators. This unsymmetric treatment also makes the functorial and symmetry properties less clear. But a major advantage of the Kohn–Nirenberg quantization is that, exactly because of its unsymmetric treatment of the variables, it can be localized in the space variable. More specifically, let Ω be an open subset of $\mathbb{R}^n$, and let f be defined just on $\Omega \times \mathbb{R}^n$ (the cotangent bundle of Ω). Then the formula 2.1 still makes sense for $s \in \Omega$ (unlike the corresponding formula 1.3 for the Weyl quantization), and defines an operator on the spaces of functions on Ω. Such operators are called pseudo-differential operators on Ω. These operators have attractive properties with respect to smooth changes of coordinates. Making use of this, one can define what it means for an operator on the space of smooth functions on a manifold M to be locally a Kohn–Nirenberg pseudo-differential operator, independent of any specific choices of coordinate charts. (See §5 of Chapter II of **[T1]**.) Such an operator is called a pseudo-differential operator on M.

But for our purposes we must emphasize that this process does not give a specific way for associating operators to functions on the cotangent bundle of M. Thus it does not provide a quantization of the cotangent bundle. For this reason we will not discuss the Kohn–Nirenberg quantization further, even though pseudo-differential operators on manifolds can generate C^*-algebras of great interest **[Con, Km1, Km2]**. The general question as to which cotangent bundles admit a quantization remains a mysterious and fascinating one. (But see **[Wi1, Wi2]**.)

For relations between the Kohn–Nirenberg quantization and the Weyl quantization see **[Fl]**.

B. The Wick and anti-Wick quantization.

We give here a brief summary of the lucid exposition of these quantizations given in **[Fl]**. For clarity of ideas we will be sloppy about factors of π. Let P_j and Q_j be operators as before which satisfy the Heisenberg commutation relations. Define the corresponding "annihilation" operator by $Z_j = Q_j + iP_j$. Then $Z_j^* = Q_j - iP_j$, which is the corresponding "creation" operator. These operators satisfy the commutation relation $[Z_j, Z_k^*] = \delta_{jk} 2\hbar I$. A product of Z_j's and Z_j^*'s is said to be "Wick-ordered" if all the Z^*'s are to the left of all the Z's. By using the commutation relation, one sees that every element of the algebra generated by the Z's and Z^*'s can be written as a sum of Wick-ordered terms. This suggests the following quantization for polynomials on $\mathbb{R}^{2n}$. Define complex coordinates on $\mathbb{R}^{2n}$ by $z = x+i\xi$ and $\bar{z} = x - i\xi$ for $(x,\xi) \in \mathbb{R}^{2n}$. Any polynomial p on $\mathbb{R}^{2n}$ can be expressed in terms of these coordinates as

$$p(z,\bar{z}) = \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Then the Wick quantization associates to p the operator

$$p(Z,Z^*) = \sum a_{\alpha\beta} Z^{*\beta} Z^\alpha.$$

If instead we associate to p the operator

$$\sum a_{\alpha\beta} Z^\alpha Z^{*\beta},$$

we obtain the anti-Wick quantization (of polynomials). This suggested to Bargman **[Bar]** and Segal **[Se2, Se3]** that one seek a representation of the commutation relations for Z_j and Z_j^* on a Hilbert space of analytic functions. In describing this we will, for notational simplicity, set $\hbar = 1$. Accordingly, as in **[Fl]**, let $\mathcal{T}_n$ be the Hilbert space of entire functions F on $\mathbb{C}^n$ such that

$$\|F\|^2 = \int |F(z)|^2 e^{-\pi z\cdot\bar{z}} dz < \infty.$$

Let Z_j be the operator $(1/\sqrt{\pi})\partial/\partial z_j$ on $\mathcal{T}_n$, and let Z_j^* be the operator of pointwise multiplication by $\sqrt{\pi} z_j$. It is easily seen that these (unbounded) operators satisfy the given commutation relations. A bit more surprising is the fact that Z_j^* is the adjoint of Z_j. Some manipulation **[Fl]** then shows that if p is a polynomial, expressed as $p(z,\bar{z})$, and if T_p^W and T_p^{AW} are the Wick and anti-Wick quantizations, expressed now as operators on $\mathcal{T}_n$, then they can be written as

$$(T_p^W F)(z) = \int p(\bar{w}, z) e^{\pi z\cdot\bar{w}} F(w) e^{-\pi w\cdot\bar{w}} dw$$

$$(T_p^{AW} F)(z) = \int p(\bar{w}, w) e^{\pi z\cdot\bar{w}} F(w) e^{-\pi w\cdot\bar{w}} dw.$$

It is now feasible to try to use these formulas for functions p which are not polynomials. But in spite of their similarity, these two potential quantizations have quite different properties. Since the range of T_p^W must consist of entire functions, p will need to be an entire function on $\mathbb{C}^n \times \mathbb{C}^n$. There seems to be no way to define T_p^W for general (non-holomorphic) functions on $\mathbb{R}^{2n}$. Thus we do not obtain a quantization of $\mathbb{R}^{2n}$ in our sense. However, T_p^W does have the favorable property that all bounded operators on $\mathcal{T}_n$, and many unbounded operators, can be expressed as T_p^W for some (entire) p. In fact the p for an operator T is recovered by the formula

$$p(\bar{w}, z) = e^{-\pi z \cdot \bar{w}}(T(E_w))(z)$$

where $E_w(z) = e^{\pi z \cdot \bar{w}}$ (a "reproducing kernel" or "coherent state").

On the other hand, we see that the formula for T_p^{AW} can yield an entire function even when p is not holomorphic, so we can hope to make sense of T_p^{AW} for a fairly broad class of measurable functions on $\mathbb{C}^n$. In fact T_p^{AW} is a kind of Toeplitz operator. To see this, note that $\mathcal{T}_n$ is a subspace of $L^2(\mathbb{C}^n, e^{-\pi z \cdot \bar{z}} dz)$. Let P denote the projection onto this subspace. It can be seen **[Fl]** that P is given by

$$(PF)(z) = \int F(w) e^{\pi z \cdot \bar{w}} e^{-\pi w \cdot \bar{w}} dw.$$

Comparing this with the earlier expression for $T_p^{AW} F$, we see that T_p^{AW} consists of forming the pointwise product pF by the measurable function p, and then projecting back into $\mathcal{T}_n$. This is the traditional form for a Toeplitz operator. It follows in particular that T_p^{AW} is defined and is a bounded operator on $\mathcal{T}_n$ whenever $p \in L^\infty(\mathbb{C}^n) = L^\infty(\mathbb{R}^{2n})$. Thus one can hope that T^{AW} will give a quantization.

A convenient way to examine this question is to note first that the representation of the Heisenberg commutation relations on $\mathcal{T}_n$ is irreducible. By the Stone–von Neumann theorem all such irreducible representations are unitarily equivalent, and so the representation on $\mathcal{T}_n$ must be equivalent to the Schrödinger representation. The unitary operator intertwining these representations is the Bargmann transform **[Fl]**. By means of the Bargmann transform we can compare the anti-Wick quantization with the Weyl quantization. It turns out **[Fl, Gu]** that T_p^{AW} will correspond under the Bargmann transform to the Weyl quantization operator L_f whose symbol is given by $f = H_{\pi/2}p$, where H_t is the heat-diffusion semigroup on $\mathbb{R}^{2n}$. Since the operators H_t are smoothing operators, it follows that only Weyl operators with quite smooth symbols come from anti-Wick operators. In fact **[Fl]**, f must be the restriction to $\mathbb{R}^{2n}$ of an entire function on $\mathbb{C}^n$.

As this might suggest, if p and q are two anti-Wick symbols, the product $T_p^{AW} T_q^{AW}$ may fail to be of the form T_r^{AW} for some (bounded) symbol r. It follows that the anti-Wick quantization does not lead to a twisted product on symbols, in contrast to the Weyl quantization. Consequently we will not consider it a *deformation* quantization (although this term is occasionally applied to it in the literature).

However, with $\hbar$ restored in the formula, the crucial relationship with the

Poisson bracket, namely

$$\|[T_f^\hbar, T_g^\hbar]/\hbar - T_{i\{f,g\}}^\hbar\|_\hbar \to 0$$

when $\hbar \to 0$, is established in [Co2] for symbols f and g vanishing at infinity, and somewhat more widely. It remains an interesting question as to just how widely this holds.

For various classes of symbols one can investigate the structure of the C^*-algebras generated by all the resulting operators T_p^{AW}. Interesting results about this can be found in [Co3].

3. Deformation Quantization

In this section we will describe some generalizations of the Weyl quantization, within the setting of C^*-algebras, based on five different constructions. Given a manifold M with Poisson bracket, $\{\ ,\ \}$, we seek to deform the pointwise product on suitable smooth functions on M "in the direction of the Poisson bracket". We also want to have corresponding deformed involutions and C^*-norms. The completions will then be C^*-algebras which are in general non-commutative. It turns out to be very useful to treat the more general situation of a possibly non-commutative C^*-algebra A with a "Poisson bracket" and to seek to deform the product in A. The Poisson bracket will be defined on a dense *-subalgebra A^o of A. A definition of what is meant by a Poisson bracket in the non-commutative case has been given in [X], and in slightly more concrete form in [N2], though more experience is probably needed before it will be clear precisely what definition is optimal. The definition involves a bilinear map from A^o to itself, denoted by z in [N2], which is a Hochschild 2-cocycle and satisfies two additional conditions, which we will not repeat here as they are slightly complicated and we do not need them in what follows. Following [N2], we call A^o equipped with z a "strict Poisson *-algebra".

When A is non-commutative it is awkward to consider the commutator $a \times_\hbar b - b \times_\hbar a$ for a deformed product, because of the interchanged order of a and b. Rather we simply compare $a \times_\hbar b$ with the original product ab. The basic definition, motivated by the properties of the Weyl quantization, is:

3.1 Definition. **[Rf1, Rf2, Rf6, N2]** Let (A^o, z) be a strict Poisson *-algebra, where A^o is a dense *-subalgebra of a C^*-algebra A. By a *strict deformation quantization* of A in the direction of z we mean an interval I of the real line containing 0, and for each $\hbar \in I$ a product $\times_\hbar$, an involution $^{*_\hbar}$, and a C^*-norm $\|\ \|_\hbar$ on A^o, such that for $\hbar = 0$ they are the original product, involution and norm, and such that

(1) The completions of A^o for the various C^*-norms form a continuous field of C^*-algebras over I.
(2) $\|(a \times_\hbar b - ab)/\hbar - iz(a,b)\|_\hbar \to 0$ as $\hbar \to 0$, for all $a, b \in A^o$.

As in section 1, these two conditions express the famous "correspondence principle" of quantum mechanics, under which, when one lets $\hbar \to 0$ in the model for the quantization of a classical system, one should recover the model of the classical system, including the Poisson bracket. The inclusion of the Poisson bracket as data for this limit is often signified by speaking of the "semi-classical" limit.

Condition 2 is what we had in mind when we said "in the direction of the Poisson bracket" in the first paragraph. Note that it is just an infinitesimal condition at $\hbar = 0$, so in general there will be no uniqueness for deformation quantizations.

As mentioned earlier, there is a large literature dealing with deformation quantization of Poisson manifolds in terms of formal power-series. (See references in [**Rf1, Rf6**].) In that setting the deformed products are often called "star products".

We now describe some specific constructions of strict deformation quantizations.

A. Actions of $V = \mathbb{R}^d$.

This construction [**Rf6**] is closest to the original Weyl quantization described earlier. The data will consist of a C^*-algebra A, an action α of the vector group V on A, and, exactly as in section 1, a linear map J from V' to V satisfying $J^t = -J$. Let A^∞ denote the space of C^∞-vectors in A for α. It is well-known that A^∞ is a dense *-subalgebra of A. As usual, the tangent space at 0 for V is identified with V, but when elements of V are viewed as tangent vectors we will denote them by X, Y, etc. Then α_X will denote the corresponding derivation of A^∞ in the direction of X. Let $\{X_j\}$ be a basis for V, and let $\{P_j\}$ denote the dual basis for V'. Then in analogy with the Poisson brackets on $C^\infty(V)$ considered in section 1, it is natural to define a "Poisson bracket" on A^∞ by $z(a,b) = \{a,b\} = \sum \alpha_{JP_j}(a)\alpha_{X_j}(b)$. This will be independent of the choice of the basis, and will make A^∞ into a strict Poisson *-algebra along the lines sketched above.

We wish to construct a strict deformation quantization of A in the direction of $\{\ ,\ \}$. Once we notice that in formula 1.1 for the twisted product of the Weyl quantization the term $f(x + Jp)$ is just the translate of f by Jp, so that what is involved is the action of V on $C^\infty(V)$ by translation, it is natural to generalize that formula by setting

$$a \times_J b = \int_{V'} \int_V \alpha_{Jp}(a)\alpha_v(b)e(p \cdot v) ,$$

where this integral is taken as an oscillatory integral [**Rf6**]. As involution we keep the original involution from A. To define a C^*-norm, we let $\mathcal{S}^A$ denote the space of A-valued Schwartz functions on V, as right A-module, and we define an

A-valued inner-product on $\mathcal{S}^A$ by

$$\langle f, g \rangle_A = \int f(x)^* g(x) .$$

For $a \in A$ we define an operator, L_a, on $\mathcal{S}^A$ by

$$(L_a f)(x) = \iint \alpha_{x+Jp}(a) f(x+v) e(p \cdot v) .$$

Then one finds that L is a *-homomorphism from A^∞ with product $\times_J$ into the algebra of operators on $\mathcal{S}^A$. A suitable version of the Calderon-Vaillancourt theorem shows [**Rf6**] that L_a is a bounded operator on $\mathcal{S}^A$ for the A- valued inner product. We place the corresponding operator norm on A (for $\times_J$), and denote the completed C^*-algebra by A_J. If we hold J fixed but replace J by $\hbar J$ in the above formulas, then we obtain [**Rf6**] a strict deformation quantization of A in the direction of $\{\ ,\ \}$.

This construction is functorial with respect to equivariant homomorphisms between C^*- algebras on which V acts. Even more, if I is an α-invariant ideal in A, so that the quotient, B, also carries an action of V and we have an equivariant short exact sequence of C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 ,$$

then the corresponding sequence of deformed algebras

$$0 \longrightarrow I_J \longrightarrow A_J \longrightarrow B_J \longrightarrow 0 ,$$

will be exact [**Rf6**]. This property can be quite useful in determining the structure of A_J.

One way to get a feeling for the meaning of this process of strict deformation quantization is to look at the spectral subspaces for α. Suppose that the action α of V on A factors through a compact Abelian group G. For each character p of G one has the corresponding spectral subspace, A_p, of A, and the direct sum of the spectral subspaces is dense in A. Now characters of G give characters of V, and so elements of V'. Let this set of characters be denoted by K. Then one can show (as in Proposition 2.2 of [**Rf6**]) that if $p, q \in K$ and if $a \in A_p$ and $b \in A_q$, then the deformed product takes the attractive form

$$a \times_J b = \overline{e}(p \cdot Jq) ab .$$

Of course when the action does not factor through a compact group we no longer have spectral subspaces in general. But intuitively we can think of infinitesimal spectral subspaces for each point of V', with the deformed product given by the above formula. One situation in which this does make sense is when A is the cross-sectional algebra of one of Fell's C^*-algebraic bundles [**FD**], over V'. For then the bundle structure gives exactly infinitesimal spectral subspaces over the points of V' for the dual action of V on the cross-section algebra, and the corresponding deformed algebra is determined by the above formula. It does not seem to be known when an action of an Abelian group G on a C^*-algebra A comes about as the dual action for a C^*-algebraic bundle structure for A over the dual group $\hat{G}$, but it seems to me likely that this is closely related to the notion of proper actions on C^*-algebras discussed in [**Rf4**].

We now give some specific examples.

EXAMPLE 1. Let $A = C_u(V)$, the space of bounded uniformly continuous functions on V, and let α be the action of V on A by translation. Then A^∞ in this case is just the algebra $\mathcal{B}$ of section 1, and the deformed algebra is just the Weyl quantization of section 1. The space $I = C_\infty(V)$ of functions vanishing at infinity on V will be an α-invariant ideal, and if J is non-degenerate then I_J will be isomorphic to the algebra of compact operators on a Hilbert space. Then $B = A/I$ will be the algebra of functions on the "uniformly continuous fringe" of V, a compact space on which V acts quite non- trivially. And B^∞ with twisted product can be thought of as a smooth version of the Calkin algebra, with B_J a uniformly continuous version of the Calkin algebra, for an action of a Heisenberg group on an underlying Hilbert space. (Question: is B_J simple?)

We consider next several subalgebras of A.

EXAMPLE 2. Let M be the closed disk consisting of $V = \mathbb{R}^2$ with circle T adjoined at infinity, and let α be the action of V on $A = C(M)$ by translation. (So A is a subalgebra of $C_u(V)$.) With I as in example 1, we see that $A/I = C(T)$, on which α acts trivially. For non-degenerate J we then see that A_J is an extension of the compacts by $C(T)$. This extension can be seen to be the usual Toeplitz extension [**Rf6**].

EXAMPLE 3. Following [**BC2**] we let $BCESV$ denote the subalgebra of $C_u(V)$ consisting of the eventually slowly varying continuous functions, that is, the functions f such that

$$0 = \lim_{R \to \infty} \sup\{|f(x-t) - f(x)| : |t| \leq 1, |x| \geq R\} .$$

Of course $BCESV$ contains the ideal I of example 1, and it can be seen that $BCESV$ consists of exactly the functions in $C_u(V)$ whose image in $C_u(V)/I$ is fixed by α. If F denotes the maximal ideal space of $BCESV/I$, then the action α on F will be trivial, and we will obtain an exact sequence

$$0 \longrightarrow I_J \longrightarrow BLESV_J \longrightarrow C(F) \longrightarrow 0$$

of deformed algebras. (There exist even larger subalgebras B of $C_u(V)$ for which $(B/I)_J$ is commutative ($= B/I$).) If J is non-degenerate so that I_J is the algebra of compact operators, then a function in $BCESV^\infty$ will determine a Fredholm element in $BCESV_J$ exactly if its image in $C(F)$ never vanishes, or equivalently if outside some sufficiently large ball in V it is bounded away from 0. For such functions one presumably has an index theorem in terms of the values of the function on sufficiently large spheres, much as in Theorem 19 of [**BC2**]. See also section 6 of [**Pw**].

EXAMPLE 4. For any $p \in V'$ let u_p denote the character on V defined by $u_p(x) = e(p \cdot x)$. Each u_p is in $C_u(V)$, and is carried to multiples of itself by the action of Example 1. The closed subalgebra of $C_u(V)$ generated by the u_p's is the commutative C^*-algebra $AP = AP(V)$ of almost periodic functions on V. It is α-invariant. Thus we can construct the deformed algebra AP_J. It is easily checked that, much as in Example 10.2 of **[Rf6]** and in our discussion above of spectral subspaces,

$$u_p \times_J u_q = \bar{e}(p \cdot Jq) u_{p+q} \ .$$

(This is, of course, related to the fact that the action of V on AP factors through the Bohr compactification of V.) In particular, each u_p will be a unitary element of AP_J. If J is non-degenerate, it follows that the u_p's give a representation of the canonical commutation relations (CCR) for the symplectic space (V', J) as defined in **[BR2]**. Since the C^*-algebras generated by such representations are all simple, and isomorphic to each other (see Theorem 5.2.8 of **[BR2]**), and are called the CCR algebra for (V', J), it follows that AP_J is just the CCR-algebra for (V', J); and in particular it is simple. Thus we see that the CCR-algebra is just a deformation quantization of the algebra of almost periodic functions. This point of view appears in **[BC1, Co1, Co3]**. Presumably other results in these papers have corresponding versions in the present setting. For some related algebras which are fun to consider see **[Sa]**.

EXAMPLE 5. Let T^d denote the d-torus, let $A = C(T^d)$, and let α denote the action of $V = R^d$ on A by translation. (This is a subalgebra of the previous example.) Then A_J is a quantum d-torus — see example 10.2 of **[Rf6]**.

Further examples can be found in **[Rf2, Rf6]**. We will now describe here only the class of examples described in **[Rf8]**, which provides a construction of some quantum groups.

EXAMPLE 6. Let A and B be C^*-algebras, and let α and β be actions of $\mathbb{R}$ on A and B respectively. Then we have the evident product action $\alpha \otimes \beta$ of $\mathbb{R}^2$ on $A \otimes B$, where we let $\otimes$ be any C^*- algebra tensor product. Let J be the standard symplectic matrix on $\mathbb{R}^2$. Then for any $\hbar$ we can construct the deformed algebra $(A \otimes B)_{\hbar J}$, which we denote for brevity by $A \otimes_\hbar B$. This can be viewed as a "twisted" tensor product of A and B, in the sense that if B has an identity element 1, then $A \otimes 1$ will be a subalgebra of $A \otimes_\hbar B$ on which the product from $A \otimes_\hbar B$ gives the original product on A, and similarly for B, and these two subalgebras together generate $A \otimes_\hbar B$ and meet only in the scalar multiples of $1 \otimes 1$. However, these subalgebras do not, in general, commute with each other, hence the term "twisted". (In the absence of identity elements, one works with the multiplier algebras.) This construction is functorial on the category of C^*-algebras with $\mathbb{R}$-action and equivariant homomorphisms. (If one wants $A \otimes_\hbar B$ to be again in that category, one can restrict the action $\alpha \otimes \beta$ which one obtains on $A \otimes_\hbar B$ to the diagonal of $\mathbb{R}^2$.) There are evident generalizations of this construction to actions of $\mathbb{R}^n$. Quantum 2-tori, and certain higher-dimensional quantum tori, are examples of this construction. Other examples can be seen in examples 10.5 and 10.6 of **[Rf6]**, as well as in the "quantum quadrant" and "algebraist's real quantum plane" of sections 11 and 12 of **[Rf6]**.

EXAMPLE 7. Let G be a compact Lie group and let H be a closed connected Abelian subgroup, so that H is a torus. Let $\mathfrak{h}$ be the Lie algebra of H, and let η denote the exponential map from $\mathfrak{h}$ to H viewed as a map into G. Let $V = \mathfrak{h} \oplus \mathfrak{h}$ and let α denote the action of V on $C(G)$ defined by

$$(\alpha_{(s,t)} f)(x) = f(\eta(s) x \eta(t))$$

for $s, t \in \mathfrak{h}$ and $x \in G$. Let $K : \mathfrak{h}' \to \mathfrak{h}$ be any skew operator, and let $J = K \oplus (-K)$, so that $J : V' \to V$. Then we can construct the deformed C^*-algebra $C(G)_J$.

Now $C(G)$ is a Hopf algebra, with comultiplication Δ defined by

$$(\Delta f)(x, y) = f(xy) ,$$

and with corresponding coidentity and coinverse. What is surprising is that Δ, restricted to $C^\infty(G)$, determines a homomorphism Δ_J from $C(G)_J$ into $C(G)_J \otimes C(G)_J$. (The algebras $C(G)_J$ are nuclear by **[Rf7]**, so we don't need to specify the tensor product.) Of course Δ_J will still be associative. The coidentity and coinverse will persist on $C(G)_J$, and so $C(G)_J$ is a quantum group. Similar results hold for non-compact groups if suitable restrictions are placed on the position of H in G **[Rf9]**.

We now turn to a very different construction of strict deformation quantizations.

B. Linear Poisson brackets.

Let $\mathfrak{g}$ be a Lie algebra, with dual $\mathfrak{g}'$. Then on $\mathfrak{g}'$ there is a canonical Poisson bracket, called a linear Poisson bracket, coming from the Lie algebra structure of $\mathfrak{g}$. For $f, g \in C^\infty(\mathfrak{g}')$ and $\mu \in \mathfrak{g}'$ it is defined by

$$\{f, g\}(\mu) = \langle [df(\mu), dg(\mu)], \mu \rangle .$$

We seek a deformation quantization of $\mathfrak{g}'$ in the direction of this Poisson bracket. Suppose first that $\mathfrak{g}$ is nilpotent. Then the exponential map is a diffeomorphism from $\mathfrak{g}$ onto its simply-connected Lie group. So we can view the group structure as being on $\mathfrak{g}$ itself. For each $\hbar$ let $\mathfrak{g}_\hbar$ be $\mathfrak{g}$ but with bracket $\hbar[\ ,\ ]$, and let $G_\hbar$ denote $\mathfrak{g}$ but with group law coming from $\mathfrak{g}_\hbar$. In particular, on the Schwartz space $\mathcal{S}(\mathfrak{g})$ we have convolutions $*_\hbar$ coming from $G_\hbar$. Let $\hat{}$ denote the Fourier transform from $\mathcal{S}(\mathfrak{g}')$ to $\mathcal{S}(\mathfrak{g})$, and let $^\vee$ denote the inverse Fourier transform. Then for each $\hbar$ we can define a product, $\times_\hbar$, on $\mathcal{S}(\mathfrak{g}')$ by setting, for every $f, g \in \mathcal{S}(\mathfrak{g}')$,

$$f \times_\hbar g = (\hat{f} *_\hbar \hat{g})^\vee .$$

We can also define $\|f\|_\hbar$ to be the norm of $\hat{f}$ in the group C^*-algebra $C^*(G_\hbar)$. We obtain in this way a strict deformation quantization **[Rf3]**. Of course, the completed C^*-algebra will be isomorphic to $C^*(G_\hbar)$. Thus we can say that $C^*(G_\hbar)$ is the deformation quantization of $\mathfrak{g}'$ for its canonical Poisson bracket. Some related results, closely connected to star products, can be found in [**AC1, AC2, AG, M1, M2**]. One very interesting question is to what extent this construction can be modified to produce such deformation quantizations which have the additional property that they respect the symplectic leaf structure of $\mathfrak{g}'$, in the sense that the functions which vanish on a symplectic leaf will form an ideal for the deformed product [**BeA**]. These papers also contain references to papers treating the quantization of coadjoint orbits of Lie groups, again often in the setting of star products.

The procedure described above has been used to construct some non-compact solvable quantum groups in [**Rf5**]. Their dual quantum groups are constructed in [**VaD**]. For closely related quantum groups see [**SZ**].

For more general Lie groups, for which the exponential map is not as nicely behaved, one can obtain [**Rf3**] a considerably weakened form of the above construction, in which the deformed product $a \times_\hbar b$ of two elements is only defined when $\hbar$ is sufficiently small, depending on a and b. This idea has been used more recently by Landsman [**L1, L2, L3**] to treat the cotangent bundle of a homogeneous space G/H with H compact, and more generally the space T^*P/H where P is a principle bundle for a compact group H.

C. Quantum groups, and generators and relations.

Drinfeld [**Dr1, Dr2**] has singled out certain Poisson brackets on Lie groups as the ones giving the directions in which one can hope to deform the Lie groups into corresponding quantum groups. These are called the "compatible" Poisson brackets. The quantum groups mentioned near the end of the previous subsection correspond to certain compatible Poisson brackets, as do those described in example 6 of the first subsection. For semi-simple Lie groups much is known about the classification of compatible Poisson brackets [**LS**]; but for solvable Lie groups little seems to be known.

The construction of quantum groups corresponding to semisimple Lie groups has progressed rapidly at the algebraic level. At the C^*-algebra level most of the progress has been restricted to compact Lie groups. Quantizations of $SU(2)$ within the C^*- algebra framework were constructed by Woronowicz in [**Wr1**] as well as by Vaksman and Soibelman [**VaS**], and then extended to other simple compact Lie groups by various authors, see e.g. [**Wr2, Wr3, LS, Rs, An**]. Although intuitively these quantizations are deformations of the ordinary compact Lie groups, the sense in which this is true was not made precise until Sheu [**Sh2, Sh3, Sh4**] showed that for $SU(2)$ they are strict deformation quantizations (see also [**Bau1, Bau2**]), and then Nagy [**Ng**] showed that this is true for $SU(n)$ for all n. The other cases have not yet been treated, but it seems highly likely that they also are strict deformation quantizations.

Unlike the approach taken in the previous two subsections, in the present case it is not known up to now how to deal with the algebra of *all* smooth functions on

$SU(n)$. Instead, one restricts attention to the "representative" functions, that is, the span of the coordinate functions of the finite-dimensional representations of $SU(n)$. Sheu capitalized on recent work of Dubois-Violette [**DV**] in which he defined a "Weyl transform" from the commutative algebra of representative functions on $SU(2)$ into the algebra of the quantum $SU(2)$. Sheu [**Sh4**] showed that this Weyl transform gives a strict deformation quantization. (But he points out that this quantization does not respect the symplectic leaf structure of $SU(2)$, in the sense that the functions which vanish on one of the symplectic leaves of the Poison bracket do not form an ideal for the deformed product. It is a very interesting question as to whether a quantization with this additional property can be found, or whether a genuine obstacle to doing this exists.)

Nagy based his approach on the original presentation of quantum $SU(n)$ in terms of generators and relations given by Woronowicz [**Wr3**]. In fact Nagy provides a somewhat general framework for treating deformation quantizations of C^*-algebras which are defined by generators and relations. This permits him to treat a variety of other examples in addition to quantum groups, including examples 2 and 5 of subsection A above. (See [**NN, Ni**] for related examples.) The key idea in Nagy's approach is to look for a field of faithful states which one is able to prove is a *continuous* field. For the quantum $SU(n)$'s he is able to show that the Haar states are faithful [**N1**] and form a continuous field [**N2**]. (For quantum versions of the other classical simple Lie groups this faithfulness seems still not to be known, though presumably it is true.) The details are too lengthy for us to describe here.

For other discussions of continuous fields of quantum groups, see [**Bau1, Bau2, Bn, Mu1**].

In addition to the few quantizations of non-compact Lie groups mentioned earlier, there has been considerable investigation of quantum versions of the group of Euclidean motions of the plane [**Wr4, Wr5, Wr7, Wr8, Ba**], and of the Lorentz group and $SL(2,C)$ [**PoW, Wr6, WZ1, WZ2**]. But there has been essentially no precise discussion of the passage to the semi-classical limit for these quantum groups. In this connection however, see the very interesting comments in [**Wr6**].

D. Riemannian symmetric spaces.

The Weyl quantization for $\mathbb{R}^{2n}$ and its Schrödinger representation can be recast in a very interesting and suggestive alternate form. Let S_0 denote the symmetry of $L^2(\mathbb{R}^n)$ defined by $(S_0\phi)(t) = \phi(-t)$. Let W be the projective representation defined just before equation 1.3. Notice that $W_w S_0 = S_0 W_{-w}$. For each $w \in \mathbb{R}^{2n}$ define a "symmetry", S_w, of $L^2(R^n)$ by $S_w = S_0 W_{2w}$. Notice that $S_w^2 = I$. Then equation 1.3 can be rewritten as

$$L_f\phi = \int_{\mathbb{R}^{2n}} f(w) S_w \phi \, dw \ .$$

This suggests that we may be able to quantize Riemannian symmetric spaces

M in the following way. Let G_M denote the group of isometries of M, and let μ denote a G_M-invariant measure on M. For each $m \in M$ let S_m denote the geodesic symmetry about m, so that $S_m \in G_M$. Let π be any unitary representation of G_M on a Hilbert space H. Then for any function f on M we can attempt to define an operator, L_f, on H by

$$L_f = \int f(m) S_m \, d\mu(m) .$$

Of course if $f \in L^1(M, \mu)$ then L_f will be a well-defined bounded operator. But, much as with the original Weyl calculus, we can hope that this will also be true for a much wider class of functions f. And we can hope that this procedure will provide a quantization of functions on M.

Investigation of all of this plunges one into the very rich structure of Riemannian symmetric spaces. Some interesting specific situations have been explored by Gracia-Bondia and his collaborators — see [**Gr**] and its references. Other leaders in this investigation have been the Unterbergers [**UU1, UU2, UU3**] and Upmeier [**Up3, Up4, Up5**]. Most of the attention so far has been concentrated on obtaining suitable formulas and then showing that the operators L_f are bounded for wide classes of f's. In most cases it has not yet been shown that the operators L_f form an algebra, which would then give a deformed product on the functions on M. Very little seems to have been done about letting a Planck's constant vary and showing that in the semi-classical limit the situation is related to a Poisson bracket in the way required by our definition of a strict deformation quantization. But the situation looks promising enough that one can hope that in favorable cases one will eventually be able to show that one obtains a strict deformation quantization.

Attractive recent surveys of this approach have been written by Upmeier [**Up3, Up4**], so we will not try to give more details here. Let us only mention that the Unterbergers have also developed several closely related "calculi" generalizing the Weyl calculus, which one can also hope will eventually be shown to provide strict deformation quantizations. Specifically, they describe a "Bessel calculus" in [**UU3**], a "Fuchs calculus" in [**Un1, Un2**], and a "Klein-Gordon calculus" in [**Un3, Un4, Un5**]. Very recently use of the Bessel calculus for constructing deformation quantizations has been studied by Müller in [**Mu2**].

E. Symplectic groupoids.

We conclude this section by mentioning a highly interesting program initiated separately by Karasev [**Kr**], Zakrzewski [**Z**], and Weinstein [**We1**], though up to now it is less complete than the approaches described above. A detailed description of this program can be found in [**We2**]. Very briefly, the idea is that to a Poisson manifold P one can often associate a symplectic groupoid G whose unit space is P and whose groupoid structure induces the Poisson bracket on P. One can then try to apply Renault's theory [**Re**] for associating a C^*-algebra to a locally compact groupoid. Actually one needs to twist this construction by a suitable line bundle (or 1-cocycle) much as one does in geometric quantization. The C^*-algebra constructed in this way is too large, and one must then try to reduce its size by some kind of choice of a "polarization". This program is carried out for quantum tori in a very attractive way in [**We1**]. Further pieces of this program appear in [**WX1**]. It seems reasonable to hope that when this program is more fully developed it will provide interesting strict deformation quantizations for a variety of situations.

4. Berezin-Toeplitz Quantization

A generalization of the anti-Wick quantization was proposed by Berezin [**Be1, Be2, Be3, Be4**], employing Toeplitz-type operators. In recent years it has seen rapid development. We will be fairly vague about the details, for which we send the reader to the papers [**Pe, Up4, Up5, UUp**], as well as to other papers mentioned below. One starts with a complex manifold M and a measure μ. Let H denote the subspace of $L^2(M,\mu)$ spanned by the holomorphic functions, and let P denote the orthogonal projection on H. For any bounded measurable function f on M let B_f denote the operator on $L^2(M,\mu)$ of pointwise multiplication by f. The corresponding Toeplitz operator, T_f, is the compression of B_f to H, that is, PB_fP on H. One can then ask whether the correspondence $f \mapsto T_f$ is a quantization, especially when one has a natural way to vary the measure μ, and thus H, as a function of a Planck's constant.

We emphasize that in this approach one does not seek a deformed product on an algebra of functions on the manifold. For this reason we will not refer to these as *deformation* quantizations. One only seeks *linear* maps $T^\hbar$ from a Poisson algebra of functions into operators on Hilbert spaces such that in the limit as $\hbar \to 0$ one obtains as semi-classical limit the original Poisson algebra. As before, the "semi" in semi-classical refers exactly to the fact that one keeps track of the Poisson bracket. More specifically, one requires that

$$\|[T_f^\hbar, T_g^\hbar]/\hbar - T_{i\{f,g\}}^\hbar\|_\hbar \longrightarrow 0 \tag{4.1}$$

as $\hbar \to 0$.

Of course, for matters to work out well one needs more structure. The main technical tool is that of reproducing kernels, which already made an appearance in our discussion of Wick quantization in section 2. A reproducing kernel for H is a continuous function E on $M \times M$ such that for each $m \in M$ the function $E_m(n) = E(n,m)$ is in H, and for every $\phi \in H$ one has

$$\phi(m) = \langle \phi, E_m \rangle \ .$$

This plunges one into the detailed theory of complex manifolds, especially Kähler manifolds. We will not attempt to describe the details here. (See [**Up4, Up5, UUp, RCG**].) But the structure of some of the resulting C^*-algebras has been worked out in great detail, and can be quite fascinating [**SSU1, SSU2, Sh1, Up1, Up2, Up6**].

In this setting there has been greater progress in verifying that one often obtains a quantization, than has been seen for the related Weyl setting described in subsection D of the last section. To begin with, Klimek and Lesniewski [**KL1**] study the closed unit disk with its usual action of $SU(1,1)$. They construct a Berezin-Toeplitz quantization (depending on a parameter) which has the important property of respecting the action of $SU(1,1)$. They verify property (4.1) for their construction. (The sections of [**N2**] which deal with the quantum disk can be viewed as strengthening the results of [**KL1**] to show that if one takes as dense subalgebra the polynomials in z and $\bar{z}$ then one actually can obtain a *deformation* quantization.)

Once one has a quantization of the disk which respects the action of $SU(1,1)$, it is very tempting to try to construct quantum compact Riemann surfaces by choosing in $SU(1,1)$ a lattice which is the fundamental group of an ordinary Riemann surface for the action of $SU(1,1)$ on the ordinary disk, and then trying to "divide" the quantum disk by the action of this lattice. There appear to be severe technical difficulties in carrying this out, but Klimek and Lesniewski do manage to overcome them in a quite restricted setting in [**KL2**]. In a very interesting related paper [**KL3**] they construct a two-parameter Berezin-Toeplitz quantization of the unit disk, for a family of Poisson brackets, such that in the second parameter the quantum disks carry an action of the quantum universal enveloping algebra $U_q(sl(2))$, and so, heuristically speaking, an action of the quantum group $SU_q(1,1)$ if it were well-understood what this meant. (See [**Wr4**] for a discussion of why $SU_q(1,1)$ may not exist.)

More recently, Borthwick, Lesniewski and Upmeier [**BLU**] have greatly generalized the above results to construct Berezin-Toeplitz quantizations for bounded irreducible Hermitian symmetric spaces. An important ingredient in their construction is again the transitive Lie group of automorphisms which these spaces possess. Their approach also relies heavily on the application of Jordan algebra techniques in ways which have been extensively developed by Upmeier in earlier work [**Up5**]. The details are too complicated to describe here. However, they are so far only able to establish relation (4.1) for the case in which one of the two functions involved has compact support.

In a striking new direction, Borthwick, Klimek, Lesniewski and Rinaldi [**BK1, BK2**] have extended these ideas to supermanifold versions of Hermitian symmetric manifolds. Here one would not initially expect to obtain C^*-algebras since the (super)-commutative algebra in this case is not a C^*-algebra. But its component of degree 0 is a C^*-algebra, and this turns out to be sufficient.

References

[An] N. Andruskiewitsch, *Some exceptional compact matrix pseudogroups*, Bull. Soc. Math. France **120** (1992), 297–325.

[AC1] D. Arnal and J. C. Cortet, *Représentations * des groupes exponentiels*, J. Funct. Anal. **92** (1990), 103–135.

[AC2] ———, *Star representations of E(2)*, Letters Math. Phys. **20** (1990), 141–149.

[AG] D. Arnal and S. Gutt, *Décomposition de $L^2(G)$ et transformation de Fourier adaptée pour un groupe G nilpotent*, C. R. Acad. Sci. Paris **306** (1988), 25–28.

[Ba] S. Baaj, *Representation reguliere du groupe quantique $E_\mu(2)$ de Woronowicz*, C. R. Acad. Sci. Paris **314** (1992), 1021–1026.

[Bar] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[Bau1] A. Bauval, *Deformation du groupe de Lie-Poisson SU(2)*, preprint.

[Bau2] ———, *Quantum group- and Poisson- deformation of SU(2)*, preprint.

[BaF] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization, I, II*, Ann. Phys. **110** (1978), 61–110, 111–151.

[BeA] N. Ben Amar, *Tangential deformations on the dual of nilpotent special Lie algebras*, preprint.

[Be1] F. A. Berezin, *Quantization in complex bounded domains*, Soviet Math. Dokl. **14** (1973), 1209–1213.

[Be2] ———, *Quantization*, Math. USSR Izv. **8** (1974), 1109–1163.

[Be3] ———, *Quantization in complex symmetric spaces*, Math. USSR Izv. **9** (1975), 341–379.

[Be4] ———, *General concept of quantization*, Comm. Math. Phys. **40** (1975), 153–174.

[BC1] C. A. Berger and L. A. Coburn, *Toeplitz operators and quantum mechanics*, J. Funct. Anal. **68** (1986), 273–299.

[BC2] ———, *Toeplitz operators on the Segal-Bargmann space*, Trans. Amer. Math. Soc. **301** (1987), 813–829.

[Bl] B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, New York Berlin Heidelberg, 1986.

[Bn] E. Blanchard, *Continuous fields of multiplicative unitaries*, preprint.

[Bt] R. J. Blattner, *Some remarks on quantization*, Symplectic Geometry and Mathematical Physics, Progr. in Math. 99, Birkhäuser, Boston, 1991, pp. 37–47.

[BK1] D. Borthwick, S. Klimek, A. Lesniewski, and M. Rinaldi, *Matrix Cartan superdomains, super Toeplitz operators, and quantization*, preprint.

[BK2] ———, *Super Toeplitz operators and non-perturbative deformation quantization of supermanifolds*, Comm. Math. Phys. **153** (1993), 49–76.

[BLU] D. Borthwick, A. Lesniewski and Harald Upmeier, *Non-perturbative deformation quantization of Cartan domains*, J. Funct. Anal. **113** (1993), 153–176.

[BR1] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, New York, Heidelberg, Berlin, 1979.

[BR2] ———, *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[ClV] A. P. Calderón and R. Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan **23** (1971), 374–378.

[Co1] L. A. Coburn, *Toeplitz operators, quantum mechanics, and mean oscillation in the Bergman metric*, Proc. Symp. Pure Math. vol. 51, Amer. Math. Soc., 1990, pp. 97–104.

[Co2] ———, *Deformation estimates for the Berezin-Toeplitz quantization*, Comm. Math. Phys. **149** (1992), 415–424.

[Co3] ———, *Berezin-Toeplitz quantization*, preprint.

[Con] A. Connes, *Géométrie Non Commutative*, InterEditions, Paris, 1990.

[Cr] H. O. Cordes, *On pseudo-differential operators and smoothness of special Lie-group representations*, Manuscripta Math. **28** (1979), 51–69.

[D] S. Doplicher, *Abstract Compact Group Duals, Operator Algebras and Quantum Field Theory*, Proceedings of the International Congress of Mathematicians, Kyoto 1990, Springer-Verlag, Tokyo, Berlin, Heidelberg, 1991, pp. 1319–1333.

[Dr1] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, Dokl. Akac. Nauk SSSR **268** (1983), 285–287 (Russian); English transl. in Soviet Math. Dokl. **27** (1983), 68–71.

[Dr2] ———, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Berkeley, Amer. Math. Soc., Providence, R. I., 1987, pp. 798–820.

[DV] M. Dubois-Violette, *On the theory of quantum groups*, Letters Math. Phys. **19** (1990), 121–126.

[EGV] R. Estrada, J. M. Gracia-Bondia and J. C. Varilly, *On asymptotic expansions of twisted products*, J. Math. Phys. **30** (1989), 2789–2796.

[FD] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles, I, II*, Academic Press, Boston, 1988.

[Fl] G. B. Folland, *Harmonic analysis in phase space*, Ann. Math. Studies no. 122, Princeton U. Press, Princeton, 1989.

[GF] F. Gabbiani and J. Frölich, *Operator algebras and conformal field theory*, preprint.

[GN] I. M. Gelfand and M. A. Naimark, *On the embedding of normed rings into the ring of operators in Hilbert space (in Russian)*, Mat. Sb. **12** (1943), 197–213.

[GB] J. M. Gracia-Bondia, *Generalized Moyal quantization on homogeneous symplectic spaces*, Deformation Theory and Quantum Groups with Applications to Mathematical Physics (M. Gerstenhaber and J. Stasheff, eds.), Contemporary Math. 134, Amer. Math. Soc., Providence, R. I., 1992, pp. 93–114.

[GV] J. M. Gracia-Bondia and J. C. Varilly, *Algebras of distributions suitable for phase-space quantum mechanics. I*, J. Math. Phys. **29** (1988), 869–878.

[GLS] A. Grossman, G. Loupias and E. M. Stein, *An algebra of pseudodifferential operators and quantum mechanics in phase space*, Ann. Inst. Fourier (Grenoble) **18** (1968), 343–368.

[Gu] Guillemin, V., *Toeplitz operators in n-dimensions*, Integ. Eq. Operator Theory **7** (1984), 145–205.

[Ha] R. Haag, *Local Quantum Physics*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[Ho] L. Hormander, *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[Km1] J. Kaminker, *Pseudo-differential operators and differential structures I*, Contemp. Math. **10** (1982), 99–128.

[Km2] ———, *Pseudo-differential operators and differential structures II*, Amer. Math. J. **108** (1986), 703–718.

[Kr] M. V. Karasev, *Analogues of objects of Lie group theory for nonlinear Poisson brackets*, Math. USSR Isv. **28** (1987), 497–527.

[KM] D. Kastler and M. Mebkhout, *Revisiting the Mackey-Stone-VonNeumann theorem; The C*-algebra of a presymplectic space*, Nuclear Physics B, Proceedings Suppl. (1991), 200–211.

[KL1] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces I. The unit disk*, Comm. Math. Phys. **146** (1992), 103–122.

[KL2] ———, *Quantum Riemann Surfaces II. The discrete series*, Letters in Math. Phys. **24** (1992), 125–139.

[KL3] ———, *A two-parameter deformation of the unit disk*, J. Funct. Anal. **115** (1993), 1–23.

[KN] J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

[L1] N. P. Landsman, *Deformations of algebras of observables and the classical limit of quantum mechanics*, Rev. Math. Phys. (to appear).

[L2] ———, *Strict deformation quantization of a particle in external gravitational and Yang-Mills fields*, J. Geom. Phys. **12** (1993), 1–40.

[L3] ———, *Classical and quantum representation theory*, preprint.

[LS] Levendorskii, S. and Soibelman, Y., *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Commun. Math. Phys. **139** (1991), 141–170.

[M1] D. Manchon, *Calcul symbolique sur les groupes de Lie nilpotents et applications*, J. Funct. Anal. **102** (1991), 206–251.

[M2] ———, *Weyl symbolic calculus on any Lie group*, Acta Appl. Math. (to appear).

[Mr1] C. Moreno, **-Products on some Kähler manifolds*, Letters Math. Phys. **11** (1986), 361–372.

[Mr2] ———, *Geodesic symmetries and invariant star products on Kähler symmetric spaces*, Lett. Math. Phys. **13** (1987), 245–257.

[Mr3] ———, *Produits star sur certains G/K Kähleriens. Equation de Yang-Baxter et produits star sur G.*, Geometrie Symplectique et Mecanique (C. Albert, ed.), Springer Lecture Notes Math. no. 1416, Springer-Verlag, Berlin Heidelberg, 1990, pp. 210–234.

[MO] C. Moreno and P. Ortega-Navarro, *Deformation of the algebra of functions on hermitian symmetric spaces resulting from quantization*, Ann. Inst. Henri Poincare **38** (1983), 215–241.

[My] J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc Camb. Phil. Soc. **45** (1949), 99–124.

[Mu1] A. Müller, *Classifying spaces for quantum principal bundles*, Commun. Math. Phys. **149** (1992), 495–512.

[Mu2] ———, *Quantization of $SU(1,1)$-actions*, preliminary draft.

[N1] G. Nagy, *On the Haar measure of the quantum SU(N) group*, Comm. Math. Phys. **153** (1993), 217–228.

[N2] ———, *A framework for deformation quantization*, Doctoral dissertation, University of California, Berkeley, 1992.

[NN] Nagy, G. and Nica, A., *On the"quantum disk" and a "non-commutative circle"*, preprint.

[Ne] J. von Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Ann. **104** (1931), 570–578.

[Ni] A. Nica, *On a quantization related to the Toeplitz algebra*, preprint.

[P] K. R. Payne, *Smooth tame Frechet algebras and Lie groups of pseudo-differential operators*, Comm. Pure Appl. Math. **44** (1991), 309–337.

[Pe] J. Peetre, *The Berezin transform and Ha-plitz operators*, J. Operator Theory **24** (1990), 165–186.

[PoW] P. Podles and S. I. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.

[Po] J. C. T. Pool, *Mathematical aspects of the Weyl correspondence*, J. Math. Phys. **7** (1966), 66–76.

[Pw] S. C. Power, *Commutator ideals and pseudo-differential C^*-algebras*, Quart. J. Math. Oxford **31** (1980), 467–489.

[RCG] J. Rawnsley, M. Cahen and S. Gutt, *Quantization of Kähler manifolds. Geometric interpretation of Berezin's quantization*, J. Geometry and Phys. **7** (1990), 45–62.

[Re] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes Math., vol. 793, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[Rf1] M. A. Rieffel, *Deformation quantization of Heisenberg manifolds*, Comm. Math Phys. **122** (1989), 531–562.

[Rf2] ———, *Deformation quantization and operator algebras*, Proc. Symp. Pure Math., vol. 51, Amer. Math. Soc., Providence, 1990, pp. 411–423.

[Rf3] ———, *Lie group convolution algebras as deformation quantizations of linear Poisson structures*, Amer. J. Math. **112** (1990), 657–686.

[Rf4] ———, *Proper actions of groups on C^*-algebras*, Mappings of Operator Algebras (H. Araki and R. V. Kadison, eds.), Birkhauser, Boston Basel Berlin, 1990, pp. 141–182.

[Rf5] ———, *Some solvable quantum groups*, Operator Algebras and Topology, Pitman Research Notes Math. 270 (W. B. Arveson, A. S. Mischenko, M. Putinar, M. A. Rieffel, and S. Stratila, eds.), Proc. OATE2 Conf: Romania 1989, Longman, Burnt Mill, England, 1992, pp. 146–159.

[Rf6] ———, *Deformation quantization for actions of R^d*, Memoirs A. M. S., vol. 506, Amer. Math. Soc., Providence, 1993.

[Rf7] ———, *K-theory for C^*-algebras deformed by actions of R^d*, J. Funct. Anal. **116** (1993), 199–214.

[Rf8] ———, *Compact quantum groups associated with toral subgroups*, Representation Theory of Groups and Algebras (J. Adams, R. Herb, S. Kudla, J.-S. Li, R. L. Lipsman, J. Rosenberg, eds.), Contemp. Math. 145, Amer. Math. Soc., Providence, R. I., 1993, pp. 465–491.

[Rf9] ———, *Non-compact quantum groups associated with Abelian subgroups*, in preparation.

[Rs] M. Rosso, *Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul diffŕentiel non commutatif*, Duke Math. J. **61** (1990), 11–40.

[SaR] X. Saint Raymond, *Elementary Introduction to the Theory of Pseudodifferential Operators*, CRC Press, Boca Raton, 1991.

[SSU1] N. Salinas, A. J.-L. Sheu and H. Upmeier, *Toeplitz C^*-algebras over pseudoconvex Reinhardt domains*, Bull. Amer. Math. Soc. **20** (1989), 173–176.

[SSU2] ———, *Toeplitz operators on pseudo-convex domains and foliation C^*-algebras*, Annals Math. **130** (1989), 531–565.

[Sa] D. Sarason, *Remotely almost periodic functions*, Contemp. Math. **32** (1984), 237–242.

[Se1] I. E. Segal, *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc. **53** (1947), 73–88.

[Se2] ———, *Foundations of the theory of dynamical systems of infinitely many degrees of freedom I*, Mat.-Fys. Medd. Dansk. Vid. Selsk. **31** (1959), no. 12.

[Se3] ———, *Mathematical Problems of Relativistic Physics*, Amer. Math. Soc., Providence, R. I., 1963.

[Sh1] A. J.-L. Sheu, *Reinhardt domains, boundary geometry and Toeplitz C^*-algebras*, J. Funct. Anal. **92** (1990), 264–311.

[Sh2] ———, *Quantization of the Poisson $SU(2)$ and its Poisson homogeneous space — the 2-sphere*, Comm. Math. Phys. **135** (1990), 217–232.

[Sh3] ———, *Quantum Poisson $SU(2)$ and quantum Poisson spheres*, Deformation Theory and Quantum Groups with Applications to Mathematical Physics (M. Gerstenhaber and J. Stasheff, eds.), Contemporary Math. 134, Amer. Math. Soc., Providence, R. I., 1992, pp. 247–258.

[Sh4] ———, *The Weyl quantization of Poisson $SU(2)$*, preprint.

[SZ] I. Szymczak and S. Zakrzewski, *Quantum deformations of the Heisenberg group obtained by geometric quantization*, J. Geom. Physics **7** (1990), 553–569.

[T1] M. E. Taylor, *Pseudodifferential operators*, Princeton U. Press, Princeton, 1981.

[T2] ———, *Noncommutative harmonic analysis*, Math. Surveys & Monographs, vol. 22, Amer. Math. Soc., Providence, 1986.

[Un1] A. Unterberger, *The calculus of pseudodifferential operators of Fuchs type*, Comm. Part Diff. Equ. **9** (1984), 1179–1236.

[Un2] ———, *L'analyse harmonique et l'analyse pseudodifferentielle du cône de lunière*, Astérisque **156** (1987), 1–199.

[Un3] ———, *Analyse relativiste*, C. R. Acad. Sci. Paris **305** (1987), 415–418.

[Un4] ———, *Pseudodifferential analysis, quantum mechanics and relativity*, Comm. Partial Diff. Eq. **13** (1988), 847–894.

[Un5] ———, *L'analyse de Klein-Gordon*, C. R. Acad. Sci. Paris **308** (1989), 397–399.

[Un6] ———, *A relativistic pseudodifferential analysis*, Texte zur Mathematik **117** (1990), 292–308.

[UU1] A. Unterberger and J. Unterberger, *La serie discrete de SL(2,R) et les operateurs pseudo-differentiels sur une demi-droite*, Ann. Scient. Ec. Norm. Sup. **17** (1984), 83–116.

[UU2] ———, *A quantization of the Cartan domain BD I (q=2) and operators on the light cone*, J. Funct. Anal. **72** (1987), 279–319.

[UU3] ———, *Quantification et analyse pseudo-differentielle*, Ann. Scient. Ec. Norm. Sup. **21** (1988), 133–158.

[UUp] A. Unterberger and H. Upmeier, *The Berezin transform and invariant differential operators*, preprint.

[Up1] H. Upmeier, *Toeplitz C^*-algebras on bounded symmetric domains*, Ann. of Math. **119** (1984), 549–576.

[Up2] ———, *Fredholm indices for Toeplitz operators on bounded symmetric domains*, Amer. J. Math. **110** (1988), 811–832.

[Up3] ———, *Weyl quantization of symmetric spaces. I. Hyperbolic matrix domains*, J. Funct. Anal. **96** (1991), 297–330.

[Up4] ———, *Weyl quantization of complex domains*, Operator Algebras and Topology, Pitman Research Notes Math. 270 (W. B. Arveson, A. S. Mischenko, M. Putinar, M. A. Rieffel, and S. Stratila, ed.), Proc. OATE2 Conf: Romania 1989, Longman, Burnt Mill, England, 1992, pp. 160–178.

[Up5] ———, *Jordan algebras, complex analysis and quantization*, preprint.

[Up6] ———, *Toeplitz C^*-algebras and noncommutative duality*, J. Operator Theory **26** (1991), 407–423.

[VaS] L. Vaksman and Ya. Soibelman, *The algebra of functions on the quantum group* $SU(2)$, English transl. in Funct. Anal. Appl., Funkts. Anal. Pril. **22** (1988), 1–14.

[VaD] A. Van Daele, *Quantum deformation of the Heisenberg group*, Proceedings of the Satellite Conference of ICM-90, World Scientific, Singapore, 1991, pp. 314–325.

[We1] A. Weinstein, *Symplectic groupoids, geometric quantization, and irrational rotation algebras*, Symplectic geometry, groupoids, and integrable systems, P. Dazord, A. Weinstein ed., Springer-Verlag, New York Berlin Heidelberg, 1991, pp. 281–290.

[We2] ———, *Noncommutative geometry and geometric quantization*, Symplectic Geometry and Mathematical Physics, Progr. in Math. 99, Birkhäuser, Boston, 1991, pp. 446–461.

[WX1] A. Weinstein and P. Xu, *Extensions of symplectic groupoids and quantization*, J. Reine Ang. Math. **417** (1991), 159–189.

[Wey] H. Weyl, *The Theory of Groups and Quantum Mechanics*, reprinted by Dover Publ., New York, 1950, Methuen, London, 1931.

[Wi1] H. Widom, *Families of pseudodifferential operators*, Topics in Functional Analysis, Advances in Mathematics Supplementary Studies, Vol. 3, Academic Press, Boston, 1978, pp. 345–395.

[Wi2] ———, *A complete symbolic calculus for pseudodifferential operators*, Bull. Sc. Math. **104** (1980), 19–63.

[Wr1] S. L. Woronowicz, *Twisted* $SU(2)$*. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. Kyoto **23** (1987), 117–181.

[Wr2] ———, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

[Wr3] ———, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted* $SU(N)$ *groups*, Invent Math. **93** (1988), 35–76.

[Wr4] ———, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups*, Comm Math. Phys. **136** (1991), 399–432.

[Wr5] ———, *Quantum E(2) group and its Pontryagin dual*, Letters Math. Phys. **23** (1991), 251–263.

[Wr6] ———, *New quantum deformation of SL(2,C). Hopf algebra level*, Reports Math. Phys. **30** (1991), 259–269.

[Wr7] ———, *Operator equalities related to the quantum E(2) group*, Comm. Math. Phys. **144** (1992), 417–428.

[Wr8] ———, *Quantum SU(2) and E(2) groups. Contraction procedure*, Comm. Math. Phys. **149** (1992), 637–652.

[WZ1] S. L. Woronowicz and S. Zakrzewski, *Quantum deformation of Lorentz group. Hopf *-algebra level*, preprint.

[WZ2] ———, *Quantum Lorentz group having Gauss decomposition property*, Publ. RIMS Kyoto U. **28** (1992), 809–824.

[X] P. Xu, *Non-commutative Poisson algebras*, Amer. J. Math. (to appear).

[Z] S. Zakrzewski, *Quantum and classical pseudogroups, I and II*, Comm. Math. Phys. **134** (1990), 347–395.

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail address: rieffel@math.berkeley.edu