Zeta Function Regularization

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September 25th 2009

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Submitted in partial fulfillment
of the requirements for the degree of Master of Science in
Quantum Fields and Fundamental Forces of Imperial College London

Abstract

The technique of zeta regularization is reviewed in quantum mechanics and field theory.
After introducing the zeta function rigorously we compute the partition function of bosonic and fermionic harmonic oscillators in quantum mechanics and study the generating functional of a quantum field in the presence of a source J by considering the determinant of its corresponding differential quadratic operator at 1-loop order.
The invariance of the potential in the bare Lagrangian of the φ^4 theory at the solution of the classical equations with source J is proved and explained and the partition function of the harmonic oscillator in field theory is computed and explained in the limit of high temperature.
The link between the zeta function, the heat kernel and the Mellin transform is explained and the equivalence between zeta and dimensional regularizations is shown and explicitly derived for the case φ^4 theory.
Furthemore, the transformation on the 1-loop effective Lagrangian is also illustrated.
Finally, the Casimir effect is introduced.

The formula

$$1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}$$

has got to mean *some*thing.

Anonymous

(...) *the use of the procedure of analytic continuation through the zeta function requires a good deal of mathematical work. It is no surprise that [it] has been often associated with mistakes and errors.*

E. Elizalde and A. Romeo

We may - paraphrasing the famous sentence of George Orwell - say that 'all mathematics is beautiful, yet some is more beautiful than the other'. But the most beautiful in all mathematics is the zeta function. There is no doubt about it.

Krzysztof Maslanka

Contents

1 Introduction

Special functions have arisen constantly and systematically in mathematical and theoretical physics throughout the XIX and XX centuries.
Indeed, the study of theoretical physics is plagued with special functions: Dirac, Legendre, Bessel, Hankel, Hermite functions - to name a few - are abundant in the indices of most modern treatises on physics. The most prominent ones have been the ever-present gamma function and those which are solutions to differential equations that model physical systems.

The Riemann zeta function is defined for a complex variable s as [8]

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

The above definition is valid for $\mathrm{Re}(s) > 1$ and it can be analytically continued to the whole complex plane except at $s = 1$ where it has a simple pole with residue 1. It will be shown that the product runs over all primes p.
It is not a solution to any physically motivated differential equation [1] which sets it apart from other special functions which have a more transparent physical meaning.
Traditionally, the Riemann ζ function has had its applications in analytic number theory and especially in the distribution of prime numbers. As such it has been regarded mostly as a function that fell completely within the realm of pure mathematics and it was temporarily excluded.

We will begin by studying the Γ and ζ functions. The lack of a course on special functions at Imperial College gives us a welcome opportunity to discuss these entities thoroughly. The exposition will be rigorous definition-theorem-proof style and the only essential pre-requisites are those of complex analysis: convergence, analytic continuation, residue calculus and Fourier and Mellin transforms.
We have limited the discussion, however, to the essential aspects of the ζ functions that we will need for the rest of the dissertation and thus commented the beautiful connection between number theory and the ζ function only briefly.
The functional equation is of particular importance

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{s\pi}{2}\right)\zeta(1-s)$$

as well as the formulas

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi)$$

which will be used time and again. Of course, they need a note of clarification. The ζ function can be written as

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

These two sums agree in the region where they converge. However, when $\mathrm{Re}(s) < 1$ the RHS is the step to analytic continuation and it goes to $-\frac{1}{2}$ as $s \to 0$ hence $\zeta(0) = -\frac{1}{2}$ in this sense.

During the last quarter of the XX century papers from S. Hawking [4], S. Elizalde, S. Odintsov and A. Romeo (EOR) [3] explained how the ζ regularization assigns finite values

to otherwise superficially divergent sums.
This is precisely what the two above formulas for $\zeta(0)$ and $\zeta'(0)$ accomplish.
One of the early uses of the ζ technique was made by Hendrik Casimir in 1948 to compute the vacuum energy of two uncharged metallic plates a few micrometers apart [2].
Another of the first instances of the Riemann's ζ function as a summation device comes from Hawking's paper [4]. Others before had used this device in connection with the renormalization of effective Lagrangians and vacuum energy-momentum tensors $T^{\mu\nu}$ on curved spaces applied to a scalar field in a de Sitter space background. What Hawking accomplished was to show that the ζ function could be used as a technique for yielding finite values to path integrals whose fields are curved. This, in turn, amounts to saying that the ζ function can be used to compute determinants of quadratic differential operators.

It is interesting to note that at a more academic level however ζ regularization is hardly ever mentioned in undergraduate quantum mechanics books, nor is it mentioned either in Peskin and Schroeder [6] or in Weinberg [9] which are some of the standard books on quantum field theory. It is precisely in QFT where the ζ function becomes apparent as a serious competitor to dimensional renormalization.

Let us briefly explain how the technique works in broad strokes. The determinant of an operator A can be written as the infinite product of its eigenvalues λ_n as [3], [4], [7]

$$\log\det A = \log\prod_n \lambda_n = \mathrm{Tr}\log A = \sum_n \log\lambda_n$$

The ζ function arises naturally by using

$$\zeta_A(s) := \mathrm{Tr}A^{-s} = \sum_n \frac{1}{\lambda_n^s}, \qquad \frac{d}{ds}\zeta_A(s)\Big|_{s=0} = -\sum_n \log\lambda_n,$$

and the functional determinant of the operator can then be written as

$$\det A = \exp(-\zeta_A'(0)).$$

When we use this definition we can find finite values for products which are otherwise divergent because the spectrum of their eigenvalues is unbounded.

Now, we have mentioned operators must be differential and quadratic.
In the bosonic quantum mechanical case, we take this operator to be the harmonic oscillator

$$A_{\mathrm{QM}} = -\frac{d^2}{d\tau^2} + \omega^2.$$

Using ζ regularization we will show that the partition function of the oscillator defined as

$$Z(\beta) = \mathrm{Tr}\exp(-\beta\hat{H})$$

is given by [5]

$$Z(\beta) = \int dx\, \langle x|\exp(-\beta H)|x\rangle = \frac{1}{2\sinh(\beta\omega/2)},$$

where β is a constant appearing in the partition function which accounts for the discretization of time, i.e. $\beta = N\varepsilon$ as $N\to\infty$ and $\varepsilon\to 0$ with N being number of time slices and ε the size of each time slice. Also, its inverse can be thought of as the temperature by setting

imaginary time $\beta = iT$.
A slightly more exotic ζ function is needed for fermions called the Hurwitz ζ function defined as [8]

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

with $a \neq 0, -1, -2, \cdots$. Since fermionic theories demand anti-commutation relations we will need Grassmann numbers and therefore we will explain these in detail but only in so much as is needed to compute partition functions for fermions.
The same technique applied with this ζ function to the partition function

$$\mathrm{Tr} e^{-\beta H} = \int d\theta^* d\theta \left\langle -\theta | e^{-\beta H} | \theta \right\rangle e^{-\theta^* \theta}.$$

where θ and θ^* are conjugate Grassmann numbers yields [5]

$$Z(\beta) = 2 \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{\pi(2n-1)} \right)^2 \right] = 2 \cosh \frac{\beta\omega}{2}.$$

In the quantum mechanical case the eigenvalues λ_n are explicitly known thus the computation of ζ_A is relatively easy compared to the field theoretical case. The key step in these cases is to relate the ζ_A function of the operator to the standard ζ function, for instance in the bosonic case

$$\zeta_{\text{boson}}(s) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{\beta} \right)^{-2s} = \left(\frac{\beta}{\pi} \right)^{2s} \zeta(2s).$$

whereas in the fermionic case

$$\zeta_{\text{fermion}}(s) = \sum_{k=1}^{\infty} \left[\frac{2\pi(k-1/2)}{\beta} \right]^{-s} = \left(\frac{\beta}{2\pi} \right)^{s} \zeta(s, 1/2),$$

We will make use of formulas for $\zeta(0)$ and $\zeta'(0)$ showing the necessity of having discussed the ζ function at length but more importantly also showing that in a certain sense regularization in quantum theory can be thought of as a technique of complex analysis, namely analytic continuation.

Because determinants of differential quadratic operators arise in field theory through path integrals in the presence of a source J we have devoted a whole chapter to explain this construction. This approach is complementary to the QFT/AQFT courses from the MSc. The only field theory we shall consider is φ^4 however it is important to note that ζ regularization can be applied to more complex field theories and even to string theory [3]. We shall keep this section brief and avoid topics such as Feynman diagrams whenever possible.

Equipped with all the tools we have developed in the preceding section we will see how the ζ function can be used in field theory. This will be the culmination of the formulas we have proved in Section 2, the techniques developed in Section 3 and the theory explained in Section 4. Furthermore, this will encapsulate the spirit of the technique.

Quantum field theories of a scalar field in the presence of an external source J will be studied. It will be shown how the generating functional or probability amplitude [6]

$$Z_E[J] = e^{-iE[J]} = \left\langle \Omega | e^{-iHT} | \Omega \right\rangle_J = \int D\phi \exp\left[i \int d^4x (L[\phi] + J\phi) \right]$$

can be written in Euclidean spacetime as [3], [7]

$$Z_E[J] = \frac{N'_E \exp\{-S_{\text{Euc}}[\phi(x_0), J]\}}{\sqrt{\det\left[(-\bar{\partial}_\mu \bar{\partial}^\mu + m^2 + V''[\phi(x_0)])\delta(x_1 - x_2)\right]}}$$

It is plainly visible that the determinant in the denominator already contains the seed of the technique, namely a differential quadratic operator. By taking the potential to be that of the φ^4 theory, the operator becomes

$$A_{\text{FT}} = -\partial^2 + m^2 + \frac{\lambda}{2}\phi_0^2(x),$$

In contrast to the quantum mechanical case, the link to the ζ_A function is much more complex and makes use of the heat kernel. With appropriate boundary conditions, the solution to the heat equation

$$A_x G_A(x, y, t) = -\frac{\partial}{\partial t} G_A(x, y, t)$$

is given by scalar product

$$G_A(x, y, t) = \langle x | e^{-tA} | y \rangle = \sum_n e^{-t\lambda_n} \psi_n(x) \psi_n^*(y).$$

where λ_n are the eigenvalues of the operator and ψ_n the orthogonal eigenvectors. The trace of the solution is

$$\int d^4x G_A(x, x, t) = \sum_n e^{-t\lambda_n} = \text{Tr} G_A(t)$$

and the link with the ζ_A function comes from the Mellin transform [4]

$$\zeta_A(s) = \sum_n \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} G_A(t).$$

We will show that by taking into account first order quantum corrections, the potential in the bare Lagrangian of the field theory is renormalized as

$$V(\phi_{\text{cl}}) = \frac{\lambda}{4!}\phi_{\text{cl}}^4 + \frac{\lambda^2 \phi_{\text{cl}}^4}{256\pi^2}\left(\log\frac{\phi_{\text{cl}}^2}{M^2} - \frac{25}{6}\right)$$

where ϕ_{cl} is the classical field defined in terms of the source J and ground state Ω as

$$\phi_{\text{cl}}(x) = \langle \Omega | \phi(x) | \Omega \rangle_J = -\frac{\delta}{\delta J(x)} E[J] = -i \frac{\delta}{\delta J(x)} \log Z.$$

We will then discuss how coupling constants evolve in terms of scale dependence and by exploring the analogy between field theory and statistical mechanics. Furthermore, we will show that [7]

$$\log Z = \frac{1}{2}\zeta'_A(0) = -\frac{1}{2}\omega\beta - \log(1 - e^{-\omega\beta}).$$

is the partition function of QFT harmonic oscillator. We can push the method more by showing

$$\log Z = \frac{1}{2}\zeta'_B(0) = V\left[\frac{M^3}{12\pi} + \frac{\pi^2}{90\beta^3} - \frac{M^2}{24\beta} + \frac{\beta M^4}{32\pi^2}\left(\gamma + \log\frac{\mu\beta}{4\pi}\right) + \cdots\right]$$

where

$$B = -\frac{\partial^2}{\partial t^2} + \beta^2 \left(-\nabla^2 + m^2 + \frac{\lambda}{2}\bar{\phi}_0^2 \right)^2$$

in the limit of high temperatue $\beta \to 0$. The ζ technique will also show how the effective Lagrangian is affected.

As a means of a check to make sure the technique yields the same results as other regularization techniques we will derive the same results by using more standard methods such as the one-loop expansion approach. However, we will do better than this by showing that the two techniques are equivalent and we will show this explictily for the case of φ^4 theory.

EOR devote a substantial amount of their book on the Casimir effect. Consequently there is a brief introduction to the calculation done by Casimir in the late 1940s. This will constitute the more 'applied' aspect of the thesis.

The conclusion contains a clarification on some of the prejudices regarding the use and ill definiteness of the ζ function as well as a repertoire of analogous equations concerning the distribution of the zeros of the ζ function, ζ regularization and dimensional regularization.

The appendix contains some formulas that did not fit in the presentation and makes the whole dissertation almost self-contained. References have been provided in each individual chapter, including pages where the main ideas have been explored. Overall, the literature on ζ regularization is sparse. The main sources for this dissertation have been Grosche and Steiner, Kleinert, Ramond and the superb treatise by Elizalde, Odintsov, and Romeo.

In terms of the knowledge of physics, the only pre-requisite comes from field theory up to the notes of QFT/AQFT from the MSc in QFFF.

It has been an objective to try to put results from different sources in a new light, by clarifying proofs and creating a coherent set of examples and applications which are related to each other and which are of increasing complexity.

Finally, a few remarks which I have not been able to find in the literature are now made concerning a potential relationship between ζ functions and families of elementary particles. Different ζ functions come into play in quantum mechanics by computing partition functions. As we have said above, the Riemann $\zeta(s)$ function is used in bosonic partition functions, whereas the Hurwitz $\zeta(s, a)$ function is needed for the fermionic partition function.

$$\text{Riemann } \zeta \text{ function} \Leftrightarrow \text{bosons} \qquad \text{Hurwitz } \zeta \text{ function} \Leftrightarrow \text{fermions}$$

It would be an interesting subject of research to investigate how the ζ function behaves with respect to its corresponding particle, and what kind of knowledge we can extract about the particle given its special ζ function. For instance, a photon has an associated ζ function, and its partition function can be computed by using known facts of this ζ function, on the other hand, known facts about the boson might bring clarifications to properties of this ζ function.

The celebrated Riemann hypothesis claims that all the nontrivial zeros of the Riemann ζ function are of the form $\zeta(\frac{1}{2} + it)$ with t real. Another observation I have not been able to find would be to consider an operator which brings the ζ function to the form $\zeta(\frac{1}{2}+it)$ and investigate the behaviour of ζ function with respect to this operator. This would mean a 'translation' of the Riemann hypothesis and a study of its physical interpretations.

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2 Introduction to the Riemann Zeta Function

2.1 The Gamma Function $\Gamma(s)$

Often in mathematics, it is more natural to define a function in terms of an integral depending on a parameter rather than through power series. The Γ function is one such case. Traditionally, it can be approached as a Weierstrass product or as a parameter-dependent integral. The approach chosen to introduce the Γ function follows from the courses in complex analysis such delivered by Freitag and usam [2], as well as Titchmarsch [6] and Whitaker and Watson [7].
We adopt the notation $s = \sigma + it$ which was introduced by Riemann in 1859 and which has become the standard in the literature of the ζ function. Let us first define the Euler Γ function.

Definition 1 The integral

$$\Gamma(s) := \int_0^\infty dt t^{s-1} e^{-t} \tag{2.1}$$

is well defined and defines a holomorphic function in the right half complex plane, where $\mathrm{Re}(s) = \sigma > 0$.

The first lemma generalises the factorial function as follows

Lemma 1 For any $n \in \mathbb{N}$ we have $\Gamma(n) = (n-1)!$.
Proof. Note that $\Gamma(1) := \int_0^\infty dt e^{-t}$ and by integration by parts we have

$$\Gamma(s+1) = \int_0^\infty dt t^s e^{-t} = -t^s e^{-t}|_0^\infty + s\int_0^\infty dt t^{s-1} e^{-t} = s\Gamma(s)$$

for any s in the right half-plane. Now, for any positive integer n, we have

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!\Gamma(1) = (n-1)! \tag{2.2}$$

so the result is proved.

In order to have a complete view of the Γ function we need to extend it to a meromorphic function in the whole complex plane.

Lemma 2 Let c_n where $n \in \mathbb{Z}^+$ be a sequence of complex numbers such that the sum $\sum_{n=0}^\infty |c_n|$ converges. Furthermore, let $S = \{-n | n \in \mathbb{Z}^+ \text{ and } c_n \neq 0\}$. Then

$$f(s) = \sum_{n=0}^\infty \frac{c_n}{s+n}$$

converges absolutely for $s \in \mathbb{C} - S$ and uniformly on bounded subsets of $\mathbb{C} - S$. The function f is a meromorphic function on $\mathbb{C}$ with simple poles at the points in S and the residues are by given $\mathrm{res}_{s=-n} f(s) = c_n$ by for any $-n \in S$.
Proof. Let us start by finding upper bounds. If $|s| < R$, then $|s+n| \geq |n-R|$ for all $n \geq R$. Therefore, we have $|s+n|^{-1} \leq (n-R)^{-1}$ for $|s| < R$ and $n \geq R$. From this we can deduce that for $n_0 > R$ we have

$$\left|\sum_{n=n_0}^\infty \frac{c_n}{s+n}\right| \leq \sum_{n=n_0}^\infty \frac{|c_n|}{|s+n|} \leq \sum_{n=n_0}^\infty \frac{|c_n|}{n-R} \leq \frac{1}{n-R}\sum_{n=n_0}^\infty |c_n|.$$

As such the series $\sum_{n>R} c_n/(s+n)$ converges absolutely and uniformly on the disk $|s|<R$ and defines there a holomorphic function. It follows that $\sum_{n=0}^{\infty} c_n/(s+n)$ is a meromorphic function on that disk with simple poles at the points of S on the disk $|s|<R$. Thus, $\sum_{n=0}^{\infty} c_n/(s+n)$ is a meromorphic function with simple poles at the points in S and for any $-n \in S$ we can write

$$f(s) = \frac{c_n}{s+n} + \sum_{-k\in S-\{n\}} \frac{c_k}{s+k} = \frac{c_n}{s+n} + g(s),$$

where g is holomorphic at $-n$. From this we see that residues are indeed $\mathrm{res}_{s=-n} f(s) = c_n$. This concludes the proof.

Equipped with this lemma we are in a position to extend the Γ function as we wanted.

Theorem 1 The Γ function extends to a meromorphic function on the complex plane. It has simple poles at $0, -1, -2, -3, \cdots$. The residues of Γ at are given by

$$\underset{s=-k}{\mathrm{res}}\, \Gamma(s) = \frac{(-1)^k}{k!}, \tag{2.3}$$

for any $k \in \mathbb{Z}^+$.
Proof. Let us split the Γ function as

$$\Gamma(s) = \int_0^1 dt t^{s-1} e^{-t} + \int_1^{\infty} dt t^{s-1} e^{-t},$$

the second integral converges for any complex s and it is an entire function. Let us expand the exponential function in the first integral

$$\int_0^1 dt t^{s-1} e^{-t} = \int_0^1 dt t^{s-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 dt t^{k+s-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k},$$

these operations are valid for $s \in \mathbb{C}$ as the exponential function is entire and converges uniformly on compact sets of the complex plane. The Γ function can now be written in a form where Lemma 2 can be used, i.e.

$$\Gamma(s) = \int_1^{\infty} dt t^{s-1} e^{-t} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}, \tag{2.4}$$

for any s in the right half-plane. By Lemma 2, the RHS defines a meromorphic function on the complex plane with simple poles at $0, -1, -2, -3, \cdots$. The residues are given as a direct application of the lemma.

Theorem 2 For $s \in \mathbb{C}$ any we have

$$\Gamma(s+1) = s\Gamma(s). \tag{2.5}$$

Proof. This follows directly from Lemma 1 and Theorem 1.

We three-dimensional representation of the Γ function looks like
Another important function related to the Γ function, and also discovered by Euler, is the

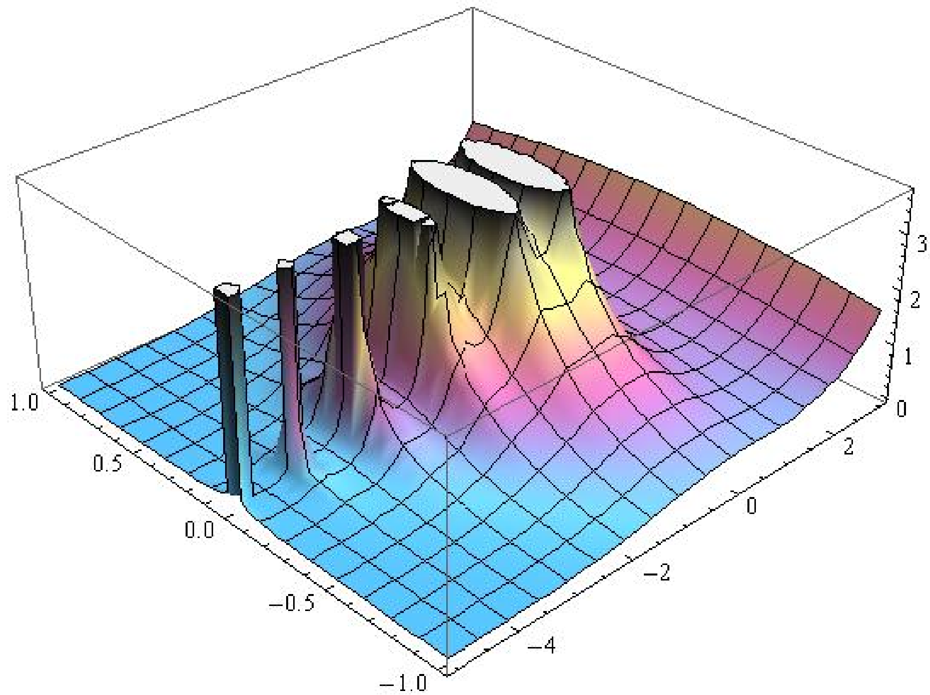

Figure 2.1: $|\Gamma(x+iy)|$ for $-5 \leq x \leq 3$ and $-1 \leq y \leq 1$

Beta function which we proceed to develop as follows. Let $\mathrm{Re}(p),\ \mathrm{Re}(q) > 0$ and in the integral that defines the Γ function, make the change of variable $t = u^2$ to obtain

$$\Gamma(p) = \int_0^\infty dt t^{p-1} e^{-t} = 2\int_0^\infty du u^{2p-1} e^{-u^2}.$$

In an analogous form we have

$$\Gamma(q) = 2\int_0^\infty dv v^{2q-1} e^{-v^2}$$

Multiplying these two together we have

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty du dv e^{-(u^2+v^2)} u^{2p-1} v^{2q-1},$$

and switching to polar coordinates $u = r\cos\theta,\ v = r\sin\theta,\ dudv = rdrd\theta$

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^\infty \int_0^{\pi/2} drd\theta\, e^{-r^2} r^{2(p+q)-1} \cos^{2p-1}\theta \sin^{2q-1}\theta \\ &= \left(2\int_0^\infty dr\, e^{-r^2} r^{2(p+q)-1}\right)\left(2\int_0^{\pi/2} d\theta\, \cos^{2p-1}\theta \sin^{2q-1}\theta\right)\end{aligned}$$

$$= 2\Gamma(p+q) \int_0^{\pi/2} d\theta \, \cos^{2p-1}\theta \sin^{2q-1}\theta$$

The integral can be simplified by setting $z = \sin^2\theta$

$$2 \int_0^{\pi/2} d\theta \, \cos^{2p-1}\theta \sin^{2q-1}\theta = \int_0^1 dz \, z^{q-1}(1-z)^{p-1}.$$

Next we define

$$B(p,q) := \int_0^1 dz \, z^{p-1}(1-z)^{q-1}$$

for $\mathrm{Re}(p), \mathrm{Re}(q) > 0$, and this gives the identity

$$B(p,q) = B(q,p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

We denote by B the Beta function. Moreover, if $0 < x < 1$ we have

$$\Gamma(x)\Gamma(1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = B(x, 1-x) = \int_0^1 dz \, z^{x-1}(1-z)^{-x}.$$

We will evaluate this integral with an appropriate contour, but first we need to make one last change $z = u/(u+1)$ which yields

$$\int_0^1 dz \, z^{x-1}(1-z)^{-x} = \int_0^\infty \frac{du}{(u+1)^2} \frac{u^{x-1}}{(u+1)^{x-1}} \left(1 - \frac{u}{u+1}\right)^{-x} = \int_0^\infty du \, \frac{u^{x-1}}{1+u}.$$

Lemma 3 For $0 < y < 1$ we have (1.10)

$$\int_0^\infty du \, \frac{u^{-y}}{1+u} = \frac{\pi}{\sin \pi y}. \tag{2.6}$$

Proof. Let us use a keyhole contour, which is accomplished by cutting the complex plane along the positive real axis. On this region we define the function

$$f(s) = \frac{s^{-y}}{1+s}$$

with argument of s^{-y} equal to 0 on the upper side of the cut. Furthermore, the function f has a first order pole at $s = -1$ with residue $e^{-iy\pi}$. See Figure 2.2 below.

We are now to integrate this function along the path described in Figure 2.2: the path goes along the upper side of the cut from $\epsilon > 0$ to R, then along the circle C_R of radius R centred at the origin, then along the side of the cut from R to ϵ and at the end around the origin via the circle C_ϵ of radius ϵ also centred at the origin. An application of the Cauchy residue theorem gives

$$2\pi i e^{-\pi i y} = \int_\varepsilon^R du \, \frac{u^{-y}}{1+u} + \oint_{C_R} dz \, \frac{z^{-y}}{1+z} - e^{-2\pi i y} \int_\varepsilon^R du \, \frac{u^{-y}}{1+u} - \oint_{C_\varepsilon} dz \, \frac{z^{-y}}{1+z}.$$

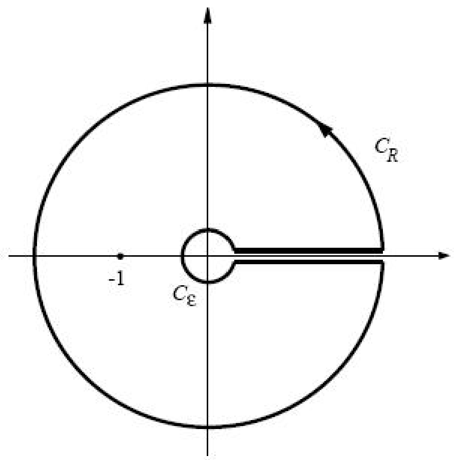

Figure 2.2: $C = C_R \cup C_\varepsilon \cup [\varepsilon, R] \cup [R, \varepsilon]$

We can get rid of the integrals around the arcs by appropriate estimates. Note that for we have the following

$$|z^{-y}| = |e^{-y\log z}| = e^{-y\mathrm{Re}(\log z)} = e^{-y\log|z|} = |z|^{-y},$$

$$\left|\frac{z^{-y}}{1+z}\right| \leq \frac{|z|^{-y}}{|1+z|} \leq \frac{|z|^{-y}}{|1-|z||},$$

and the integrals can be estimated

$$\left|\oint\limits_{C_R} dz\, \frac{z^{-y}}{1+z}\right| \leq 2\pi \frac{R^{1-y}}{R-1} \underset{R\to\infty}{\to} 0, \quad \left|\oint\limits_{C_\varepsilon} dz\, \frac{z^{-y}}{1+z}\right| \leq 2\pi \frac{\varepsilon^{1-y}}{1-\varepsilon} \underset{\varepsilon\to 0}{\to} 0;$$

so that we are left with

$$(1 - e^{-2\pi i y}) \int\limits_0^\infty du\, \frac{u^{-y}}{1+u} = 2\pi i e^{-\pi i y}.$$

We may re-write this to obtain the final result

$$(e^{\pi i y} - e^{-\pi i y}) \int\limits_0^\infty du\, \frac{u^{-y}}{1+u} = 2\pi i \quad \Rightarrow \quad \int\limits_0^\infty du\, \frac{u^{-y}}{1+u} = \frac{\pi}{\sin \pi y}.$$

This proves the claim of the lemma.

Let us now make the concluding remarks.

Theorem 3 (Euler Reflection Formula) For all $s \in \mathbb{C}$ one has

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{2.7}$$

Proof. The lemma we have just proved can be written as

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi(1-x)} = \frac{\pi}{\sin\pi x},$$

for $0 < x < 1$. However, both sides of the equation above are meromorphic, hence we have proved the theorem.

Corollary 1 One has

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty dt\, t^{-1/2} e^{-t} = \int_{-\infty}^{\infty} dt\, e^{-t^2} = \sqrt{\pi}. \tag{2.8}$$

Proof. This follows by substituting $s = 1/2$ in the Euler reflection formula (2.7).

Theorem 4 The Γ function has no zeroes.
Proof. Since $s \to \sin(\pi s)$ is an entire function, the RHS of Theorem 3 has no zeroes, therefore $\Gamma(s) = 0$ only happens where $s \to \Gamma(1-s)$ has poles. However, as we have argued before, the poles of Γ are at $0, -1, -2, -3, \cdots$ so it follows that $\Gamma(1-s)$ must have poles at $1, 2, 3, \cdots$. By the factorial formula, $\Gamma(n+1) = n! \neq 0$ and so Γ has no zeroes.

Intrinsically connected to the Γ function is the Euler γ constant. Let us first define it and prove its existence.

Lemma 4 If $s_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$, then $\lim_{n\to\infty} s_n$ exists. This limit is called the Euler γ constant.
Proof. Consider $t_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n$ geometrically, it represents the area of the $n-1$ regions between the upper Riemann sum and the exact value of $\int_1^n dx\, x^{-1}$. Therefore t_n increases with n. We can write

$$t_n = \sum_{k=1}^{n-1}\left[\frac{1}{k} - \log\frac{k+1}{k}\right], \qquad \lim_{n\to\infty} t_n = \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right].$$

The series on the right converges to a positive constant since

$$0 < \frac{1}{k} - \log\left(1+\frac{1}{k}\right) = \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - \cdots \leq \frac{1}{2k^2}.$$

Next, the following holds

$$s_{n+1} - s_n = \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right), \quad t_{n+1} - t_n = \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$$

which means that

$$\frac{1}{n+1} < \log\left(1+\frac{1}{n}\right) < \frac{1}{n} \Rightarrow s_{n+1} - s_n < 0 < t_{n+1} - t_n.$$

Convergence now follows because s_n decreases monotonically whereas t_n increase monotonically and the differences are negative. Hence s_n is monotonically decreasing and bounded below thus convergent.

The value of γ was computed by Mascheroni to be

$$\gamma = 0.5772156649015328606065120900824024310421593359399 2\cdots$$

and of course subsequent improvements have been made.

Lemma 5 Weierstrass product of the Γ function

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} s e^{s\gamma} \prod_{k=1}^{n} \left(1+\frac{s}{k}\right) e^{-s/k}. \tag{2.9}$$

Proof. Using the fact that $(1-t/n)^n \to e^{-t}$ as $n\to\infty$ it can be shown that

$$\Gamma(s) = \lim_{n\to\infty} \int_0^n dt\, t^{s-1} \left(1-\frac{t}{n}\right)^n = \lim_{n\to\infty} \frac{1}{n^n} \int_0^n dt\, t^{s-1}(n-t)^n$$

and integrating by parts yields

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \frac{1}{n^n}\frac{n}{s} \int_0^n dt\, t^s (n-t)^{n-1} = \lim_{n\to\infty} \frac{1}{n^n} \frac{n(n-1)\cdots 1}{s(s+1)\cdots(s+n-1)} \int_0^n dt\, t^{s+n-1} \\ &= \lim_{n\to\infty} \frac{n^s}{s} \left(\frac{1}{s+1}\right)\left(\frac{2}{s+2}\right)\cdots\left(\frac{n}{s+n}\right).\end{aligned}$$

Inverting both sides

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} s n^{-s}(s+1)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right) = \lim_{n\to\infty} s n^{-s} \prod_{k=1}^{n}\left(1+\frac{s}{k}\right)$$

In order to be limit we need to insert the convergence factor $e^{-s/k}$ to obtain

$$\begin{aligned}\frac{1}{\Gamma(s)} &= \lim_{n\to\infty} s n^{-s} e^{s(1+1/2+\cdots+1/n)} \prod_{k=1}^{n}\left(1+\frac{s}{k}\right) e^{-s/k} \\ &= \lim_{n\to\infty} e^{s(1+1/2+\cdots+1/n-\log n)} \left[s\prod_{k=1}^{n}\left(1+\frac{s}{k}\right)e^{-s/k}\right].\end{aligned}$$

However by the use of Lemma 4, we know that the sum converges γ to so that we have shown the result (2.9).

The derivative of the Γ function at -1 is $\Gamma'(1) = -\gamma$, as it can be seen by taking the logarithmic derivative of Weierstrass product (2.9)

$$-\log\Gamma(s) = \log s + \gamma s + \sum_{k=1}^{\infty}\left[\log\left(1+\frac{s}{k}\right) - \frac{s}{k}\right] \Rightarrow -\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{k=1}^{\infty}\left[\frac{1}{k+s} - \frac{1}{k}\right]$$

and hence

$$\Gamma'(1) = -1 - \gamma - \sum_{k=1}^{\infty}\left[\frac{1}{k+1} - \frac{1}{k}\right] = -\gamma. \tag{2.10}$$

2.2 The Hurwitz Function $\zeta(s,a)$

The Hurwitz $\zeta(s,a)$ function is initially defined for $\sigma > 1$ by the series

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}. \tag{2.11}$$

This is provided that $n+a \neq 0$ and $a \neq 0, -1, -2, \cdots$. The reason why we work with a generalized $\zeta(s,a)$ function, rather than with the $\zeta(s)$ function itself, is because fermions require this special kind of ζ function for their regularization. Note however that bosons require the Riemann $\zeta(s,1) = \zeta(s)$ function. For the special case of the Riemann ζ function, the 3-d plot looks like. The discussion presented here of the properties of the Riemann ζ function has its foundations in Titchmarsh [5]. Although the roots of the functional equation go back to Riemann, the development of the Hurwitz ζ can be traced back to Apostol [1] which in turn is taken from Ingham [3].

Let us now examine the properties of the Hurwitz ζ function.

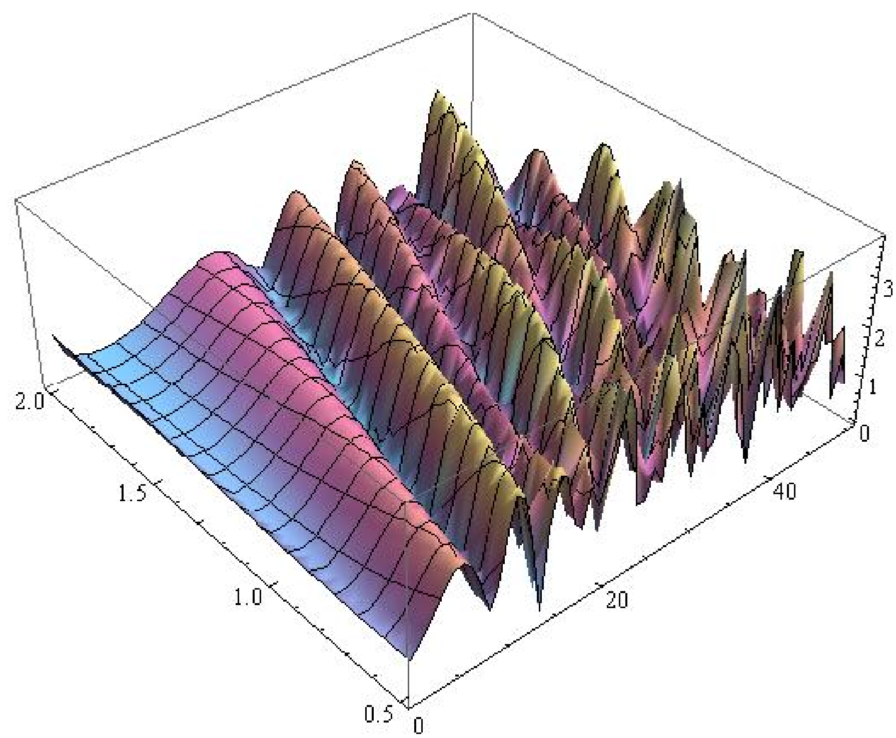

Figure 2.3: $|\zeta(\frac{1}{2}+iy,s)|$ for $1 \leq y \leq 50$ and $\frac{1}{2} \leq a \leq 2$

Proposition 1 The series $\zeta(s,a)$ for converges absolutely for $\sigma > 1$. The convergence is uniform in every half-plane $\sigma \geq 1+\delta$ with $\delta > 0$ so $\zeta(s,a)$ is analytic function of s in the half-plane $\sigma > 1$.

Proof. From the inequalities

$$\sum_{n=1}^{\infty} \left|(n+a)^{-s}\right| = \sum_{n=1}^{\infty} (n+a)^{-\sigma} \leq \sum_{n=1}^{\infty} (n+a)^{-(1+\delta)}.$$

all the statements follow and this proves the claim.

The analytic continuation of the ζ function to a meromorphic function in the complex plane is more complicated than in the case of the Γ function.

Proposition 2 For $\sigma > 1$ we have the integral representation

$$\Gamma(s)\zeta(s,a) = \int_0^{\infty} dx \frac{x^{s-1}e^{-ax}}{1-e^{-x}}. \tag{2.12}$$

In the case of the Riemann ζ function, that is when $a = 1$, we have

$$\Gamma(s)\zeta(s) = \int_0^{\infty} dx \frac{x^{s-1}e^{-x}}{1-e^{-x}}. \tag{2.13}$$

Proof. First we consider the case when s is real and $s > 1$, then extend the result to complex s by analytic continuation. In the integral for the Γ function we make the change of variable $x = (n+a)t$ where $n \geq 0$ and this yields

$$(n+a)^{-s}\Gamma(s) = \int_0^{\infty} dt e^{-nt}e^{-at}t^{s-1}.$$

Next, we sum over all $n \geq 0$ and this gives

$$\zeta(s,a)\Gamma(s) = \sum_{n=0}^{\infty} \int_0^{\infty} dt e^{-nt}e^{-at}t^{s-1},$$

where the series on the right is convergent if $\mathrm{Re}(s) > 1$. To finish the proof we need to interchange the sum and the integral signs. This interchange is valid by the theory of Lebesgue integration; however we do not proceed to prove this more rigorously because it would take us too far from the subject at matter. Therefore, we may write

$$\zeta(s,a)\Gamma(s) = \sum_{n=0}^{\infty} \int_0^{\infty} dt e^{-nt}e^{-at}t^{s-1} = \int_0^{\infty} dt \sum_{n=0}^{\infty} e^{-nt}e^{-at}t^{s-1}.$$

However, if $\mathrm{Im}(s) = t > 0$ we have $0 < e^{-t} < 1$ and hence we may sum

$$\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{1-e^{-t}},$$

by geometric summation. Thus the integrand becomes

$$\zeta(s,a)\Gamma(s) = \int_0^{\infty} dt \sum_{n=0}^{\infty} e^{-nt}e^{-at}t^{s-1} = \int_0^{\infty} dt \frac{e^{-at}t^{s-1}}{1-e^{-t}}. \tag{2.14}$$

Now we have the first part of the argument and we need to extend this to all complex s with $\text{Re}(s) > 1$. To this end, note that both members are analytic for $\text{Re}(s) > 1$. In order to show that the right member is analytic we assume $1 + \delta \leq \sigma \leq c$ where $c > 1$ and $\delta > 0$. We then have

$$\int_0^\infty dt \left| \frac{e^{-at} t^{s-1}}{1 - e^{-t}} \right| \leq \int_0^\infty dt \frac{e^{-at} t^{\sigma-1}}{1 - e^{-t}} = \left(\int_0^1 dt + \int_1^\infty dt \right) \frac{e^{-at} t^{\sigma-1}}{1 - e^{-t}}.$$

Notice the analogy of splitting the integral as in the proof of Theorem 1.
If $0 \leq t \leq 1$ we have $t^{\sigma-1} \leq t^\delta$ and if $t \geq 1$ we have $t^{\sigma-1} \leq t^{c-1}$. Also since $e^t - 1 \geq t$ for $t \geq 0$ we then have

$$\int_0^1 dt \frac{e^{-at} t^{\sigma-1}}{1 - e^{-t}} \leq \int_0^1 dt \frac{e^{(1-a)t} t^\delta}{e^t - 1} \leq e^{(1-a)} \int_0^1 dt\, t^{\delta-1} = \frac{e^{1-a}}{\delta},$$

and

$$\int_1^\infty dt \frac{e^{-at} t^{\sigma-1}}{1 - e^{-t}} \leq \int_1^\infty dt \frac{e^{-at} t^{c-1}}{1 - e^{-t}} \leq \int_0^\infty dt \frac{e^{-at} t^{c-1}}{1 - e^{-t}} = \zeta(c, a)\Gamma(c).$$

This proves that the integral in the statement of the theorem converges uniformly in every strip $1 + \delta \leq \sigma \leq c$, where $\delta > 0$, and therefore represents an analytic function in every such strip, hence also in the half-plane $\sigma = \text{Re}(s) > 1$. Therefore, by analytic continuation, (1.14) holds for all s with $\text{Re}(s) > 1$.

Consider the keyhole contour C: a loop around the negative real axis as shown in Figure 2.4. The loop is made of three parts C_1, C_2, and C_3. The C_2 part is a positively oriented circle of radius $\varepsilon < 2\pi$ above the origin, and C_1 and C_3 are the lower and upper edges of a cut in the z plane along the negative real axis.

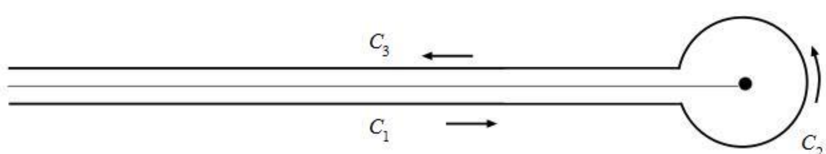

Figure 2.4: $C = C_1 \cup C_2 \cup C_3$

This can be translated into the following parametrizations: $z = re^{-\pi i}$ on C_1 and $z = re^{\pi i}$ on C_3 where r varies from ε to ∞.

Proposition 3 If $0 < a \leq 1$ the function defined by the contour integral

$$\Upsilon(s, a) = \frac{1}{2\pi i} \int_C dz \frac{z^{s-1} e^{az}}{1 - e^z} \tag{2.15}$$

is an entire function of s. Moreover, we have

$$\zeta(s, a) = \Gamma(1 - s)\Upsilon(s, a) \tag{2.16}$$

if $\mathrm{Re}(s) = \sigma > 1$.

Proof. Write $z^s = r^s e^{-\pi i s}$ on C_1 and $z^s = r^s e^{\pi i s}$ on C_3. Let us consider an arbitrary compact disk $|s| \leq M$ and we proceed to prove that the integrals along C_1 and C_3 converge uniformly on every such disk. Since the integrand is an entire function of s, this will prove that the integral $\Upsilon(s,a)$ is entire. Along C_1 we have for $r \geq 1$,

$$|z^{s-1}| = r^{\sigma-1}|e^{-\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{\pi t} \leq r^{M-1}e^{\pi M}$$

since $|s| \leq M$. The same on C_3 gives

$$|z^{s-1}| = r^{\sigma-1}|e^{\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{-\pi t} \leq r^{M-1}e^{\pi M}$$

also for $r \geq 1$. Therefore, independently on which side of the cut we place ourselves, we have that for $r \geq 1$,

$$\left|\frac{z^{s-1}e^{az}}{1-e^z}\right| \leq \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^{-r}} = \frac{r^{M-1}e^{\pi M}e^{(1-a)r}}{e^r-1}.$$

However, $e^r - 1 > e^r/2$ when $r > \log 2$ so the integrand is bounded by $\Omega_1 r^{M-1}e^{-ar}$ where Ω_1 is a constant depending on M but not on r. The integral $\int_\varepsilon^\infty dr\, r^{M-1}e^{-ar}$ converges if $\varepsilon > 0$ so this proves that the integrals along C_1 and C_3 converge uniformly on every compact disk $|s| \leq M$ and hence $\Upsilon(s,a)$ is indeed an entire function of s.

To prove the equation of the theorem, we have to split up the integral as

$$2\pi i\Upsilon(s,a) = \left(\int\limits_{C_1} dz + \int\limits_{C_2} dz + \int\limits_{C_3} dz\right) z^{s-1}g(z)$$

where $g(z) = e^{az}/(1-e^z)$. According to the parametrizations we have on C_1 and C_3 that $g(z) = g(-r)$ but on the circle C_2 we write $z = \varepsilon e^{i\theta}$, where $-\pi \leq \theta \leq \pi$. This gives us

$$2\pi i\Upsilon(s,a) = \int\limits_\infty^\varepsilon dr\, r^{s-1}e^{-i\pi s}g(-r) + i\int\limits_{-\pi}^{\pi} d\theta\, \varepsilon^{s-1}e^{(s-1)i\theta}\varepsilon e^{i\theta}g(\varepsilon e^{i\theta}) + \int\limits_\varepsilon^\infty dr\, r^{s-1}e^{i\pi s}g(-r).$$

Divide by $2i$ and name the integrals Υ_1 and Υ_2

$$\pi\Upsilon(s,a) = \sin(\pi s)\Upsilon_1(s,\varepsilon) + \Upsilon_2(s,\varepsilon).$$

If we let $\varepsilon \to 0$ we see that

$$\lim_{\varepsilon\to 0}\Upsilon_1(s,\varepsilon) = \int\limits_0^\infty dr\, \frac{r^{s-1}e^{-ar}}{1-e^{-r}} = \Gamma(s)\zeta(s,a),$$

as long as $\sigma > 1$. In $|z| < 2\pi$ the function g is analytic except for a first order pole at $z = 0$. Therefore $zg(z)$ is analytic everywhere inside $|z| < 2\pi$ and hence is bounded there, say $|g(z)| \leq \Omega_2/|z|$, where $|z| = \varepsilon < 2\pi$ and Ω_2 is a constant. We can then write

$$|\Upsilon_2(s,\varepsilon)| \leq \frac{\varepsilon^\sigma}{2}\int\limits_{-\pi}^{\pi} d\theta\left(e^{-t\theta}\frac{\Omega_2}{\varepsilon}\right) \leq \Omega_2 e^{\pi|t|}\varepsilon^{\sigma-1}.$$

When we let $\varepsilon \to 0$ and provided that $\sigma > 1$ we find that $\Upsilon_2(s,\varepsilon) \to 0$ hence we have $\pi\Upsilon(s,a) = \sin(\pi s)\Gamma(s)\zeta(s,a)$. Finally, by the use of the Euler relfection formula (2.7) we have a proof of (2.16).

2.3 Analytic continuation and the functional equation of $\zeta(s,a)$

Now we have to extend the previous result for complex numbers such that $\sigma \leq 1$. In the statement that we have just proved the functions $\Upsilon(s,a)$ and $\Gamma(1-s)$ make sense for every complex s, and thus we can use this equation to *define* $\zeta(s,a)$ for $\sigma \leq 1$.

Definition 2 If $\sigma \leq 1$ we define $\zeta(s,a)$ by the equation

$$\zeta(s,a) = \Upsilon(s,a)\Gamma(1-s). \tag{2.17}$$

This provides the analytic continuation of $\zeta(s,a)$ in the entire s plane.

Theorem 5 The function $\zeta(s,a)$ defined above is analytic for all s except for a simple pole at $s=1$ with residue 1.
Proof. The function $\Upsilon(s,a)$ is entire so the only possible singularities of $\zeta(s,a)$ must be the poles of $\Gamma(1-s)$, and we have shown those to be the points $s = 1, 2, 3, \cdots$. However Theorem 1 shows that $\zeta(s,a)$ is analytic at $s = 2, 3, \cdots$ so $s = 1$ is the only possible pole of $\zeta(s,a)$.
If s in an integer $s = n$ the integrand in the contour integral for $\Upsilon(s,a)$ takes the same value on both C_1 and C_3 and hence the integrals along C_1 and C_3 cancel, yielding

$$\Upsilon(n,a) = \frac{1}{2\pi i}\int_{C_2} dz \frac{z^{n-1}e^{az}}{1-e^z} = \operatorname*{res}_{z=0} \frac{z^{n-1}e^{az}}{1-e^z}.$$

In this case we have $s = 1$ and so

$$\Upsilon(1,a) = \operatorname*{res}_{z=0} \frac{e^{az}}{1-e^z} = \lim_{z\to 0} \frac{ze^{az}}{1-e^z} = \lim_{z\to 0} \frac{z}{1-e^z} = \lim_{z\to 0} \frac{-1}{e^z} = -1.$$

Finally, the residue of $\zeta(s,a)$ at $s=1$ is computed as

$$\lim_{s\to 1}(s-1)\zeta(s,a) = -\lim_{s\to 1}(1-s)\Gamma(1-s)\Upsilon(s,a) = -\Upsilon(1,a)\lim_{s\to 1}\Gamma(2-s) = \Gamma(1) = 1,$$

now the claim is complete: $\zeta(s,a)$ has a simple pole at $s=1$ with residue 1.

Let us remark that since $\zeta(s,a)$ is analytic at $s = 2, 3, \cdots$ and $\Gamma(1-s)$ has poles at these points, then (2.17) implies that $\Upsilon(s,a)$ vanishes at these points. Also we have proved that the Riemann $\zeta(s)$ function is analytic everywhere except for a simple pole at $s=1$ with residue 1.

Lemma 6 Let $S(r)$ designate the region that remains when we remove from the s plane all open circular disks of radius r, $0 < r < \pi$ with centres at $z = 2n\pi i$, $n = 0, \pm 1, \pm 2, \cdots$. Then if $0 < a \leq 1$ the function

$$g(s) := \frac{e^{as}}{1-e^s}$$

is bounded in $S(r)$.
Proof. With our usual notation: $s = \sigma + it$ we consider the rectangle $H(r)$ with the circle at $n = 0$ this rectangle has an indentation as follows

$$H(r) = \{s : |\sigma| \leq 1, |t| \leq \pi, |s| \geq r\},$$

as shown in Figure 2.5 below.

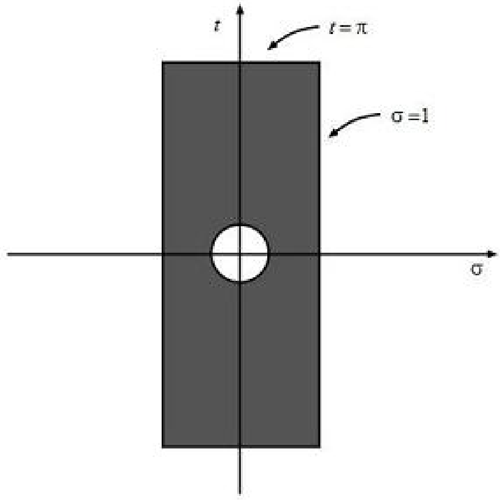

Figure 2.5: The region $H(r)$

The set H so defined is compact so g is bounded on H. Also, because of the periodicity $|g(s+2\pi i)| = |g(s)|$, g is bounded in the perforated infinite strip

$$\{s : |\sigma| \leq 1, |s - 2n\pi i| \geq r, n = 0, \pm 1, \pm 2, \cdots\}.$$

Let us suppose that $|\sigma| \geq 1$ and consider

$$|g(s)| = \left|\frac{e^{as}}{1-e^s}\right| = \frac{e^{a\sigma}}{|1-e^s|} \leq \frac{e^{a\sigma}}{|1-e^\sigma|}.$$

We can examine the numerator and denominator for $\sigma \geq 1$ giving $|1-e^\sigma| = e^\sigma - 1$ and $e^{a\sigma} \leq e^\sigma$ so

$$|g(s)| \leq \frac{e^\sigma}{e^\sigma - 1} = \frac{1}{1-e^{-\sigma}} \leq \frac{1}{1-e^{-1}} = \frac{e}{e-1}.$$

A similar argument when $\sigma \leq -1$ gives $|1-e^\sigma| = 1 - e^\sigma$ and so

$$|g(s)| \leq \frac{e^\sigma}{1-e^\sigma} \leq \frac{1}{1-e^\sigma} \leq \frac{1}{1-e^{-1}} = \frac{e}{e-1}.$$

And therefore, as we claimed $|g(s)| \leq e/(e-1)$ for $|\sigma| \leq 1$.

Definition 3 The periodic ζ function is defined as

$$\zeta_{\text{periodic}}(x,s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \tag{2.18}$$

where x is real and $\sigma > 1$.

Let us remark the following properties of the periodic zeta function. It is indeed a periodic function of x with period 1 and $\zeta_{\text{periodic}}(1,s) = \zeta(s)$.

Theorem 6 The series converges absolutely if $\sigma > 1$. If x is not an integer the series also converges (conditionally) for $\sigma > 0$.
Proof. This is because for each fixed non-integral x the coefficients have bounded partial sums.

Theorem 7 (Hurwitz's formula) If $0 < a \leq 1$ and $\sigma > 1$ we have

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s}(e^{-\pi is/2}\zeta_{\text{periodic}}(a,s) + e^{\pi is/2}\zeta_{\text{periodic}}(-a,s)). \tag{2.19}$$

If $a \neq 1$ this is also valid for $\sigma > 0$.
Proof. Consider the function defined by the contour integral

$$\Upsilon_N(s,a) := \frac{1}{2\pi i}\oint_{C(N)} dz \frac{z^{s-1}e^{az}}{1-e^z}, \tag{2.20}$$

where is the contour show in Figure 2.6 and N is an integer. It is the same keyhole contour as that of the gamma function, only that it has been rotated for convenience.

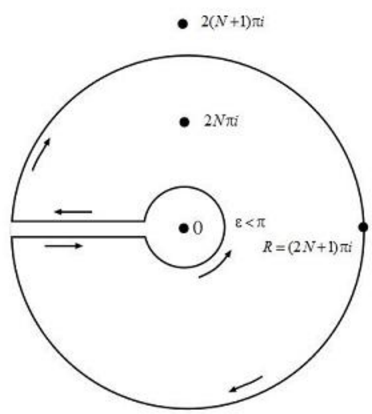

Figure 2.6: The keyhole contour $C(N)$

The poles are located on the y-axis, symmetric to the origin, at multiples of $2\pi i$. Let us first prove that if $\sigma < 0$ then $\lim_{N\to\infty}\Upsilon_N(s,a) = \Upsilon(s,a)$. The method to prove this is to show that the integral along the outer circle tends to 0 as $N \to \infty$. On the outer circle we have $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, hence

$$|z^{s-1}| = |R^{s-1}e^{i\theta(s-1)}| = R^{\sigma-1}e^{-t\theta} \leq R^{\sigma-1}e^{\pi|t|}.$$

The outter circle is inside the domain $S(r)$ described in Lemma 6, the integrand is bounded by $\Omega_3 e^{\pi|t|}R^{\sigma-1}$ where Ω_3 is the bound for $|g(s)|$ implied by Lemma 6, hence the whole integral is bounded by $2\pi e^{\pi|t|}R^{\sigma}$ which tends to 0 as R tends to infinity as long as $\sigma < 0$. Now when we replace s by $1-s$ we obtain

$$\lim_{N\to\infty}\Upsilon_N(1-s,a) = \lim_{N\to\infty}\frac{1}{2\pi i}\oint_{C(N)} dz \frac{z^{-s}e^{az}}{1-e^z} = \Upsilon(1-s,a)$$

for $\sigma > 1$. We are left with the problem of computing $\Upsilon_N(1-s,a)$ which we proceed to do by the use of the Cauchy residue theorem. Formally,

$$\Upsilon_N(1-s,a) = -\sum_{n=-N, n\neq 0}^{N} \operatorname*{res}_{z=n} f(z) = -\sum_{n=1}^{N} \left\{ \operatorname*{res}_{z=n} f(z) + \operatorname*{res}_{z=-n} f(z) \right\}$$

where $f(z)$ is the integrand of $\Upsilon_N(1-s,a)$ and residues are calculated as follows

$$\begin{aligned} \operatorname*{res}_{z=n} f(z) &= \operatorname*{res}_{z=2n\pi i} \left(\frac{z^{-s} e^{az}}{1-e^z} \right) = \lim_{z\to 2n\pi i} (z - 2n\pi i) \frac{z^{-s} e^{az}}{1-e^z} \\ &= \frac{e^{2n\pi i a}}{(2n\pi i)^s} \lim_{z\to 2n\pi i} \frac{z-2n\pi i}{1-e^z} = -\frac{e^{2n\pi i a}}{(2n\pi i)^s}, \end{aligned}$$

which in turn gives

$$\Upsilon_N(1-s,a) = \sum_{n=1}^{N} \frac{e^{2n\pi i a}}{(2n\pi i)^s} + \sum_{n=1}^{N} \frac{e^{-2n\pi i a}}{(2n\pi i)^s}.$$

Now we make the following replacements $i^{-s} = e^{-\pi i s/2}$ and $(-i)^{-s} = e^{\pi i s/2}$ which allow us to write

$$\Upsilon_N(1-s,a) = \frac{e^{-\pi i s/2}}{(2\pi)^s} \sum_{n=1}^{N} \frac{e^{2n\pi i a}}{n^s} + \frac{e^{\pi i s/2}}{(2\pi)^s} \sum_{n=1}^{N} \frac{e^{-2n\pi i a}}{(2n\pi i)^s}$$

and we let $N \to \infty$

$$\Upsilon(1-s,a) = \frac{e^{-\pi i s/2}}{(2\pi)^s} \zeta_{\text{periodic}}(a,s) + \frac{e^{\pi i s/2}}{(2\pi)^s} \zeta_{\text{periodic}}(-a,s).$$

We have thus arrived at the following result

$$\zeta(1-s,a) = \Gamma(s)\Upsilon(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{-\pi i s/2} \zeta_{\text{periodic}}(a,s) + e^{\pi i s/2} \zeta_{\text{periodic}}(-a,s) \right\}.$$

This proves the claim.

The simplest particular case (and the most important one) is when we take $a = 1$ this gives us the functional equation of the Riemann ζ function

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{-\pi i s/2} \zeta(s) + e^{\pi i s/2} \zeta(s) \right\} = \frac{\Gamma(s)}{(2\pi)^s} 2\cos\frac{\pi s}{2} \zeta(s). \tag{2.21}$$

This is valid for $\sigma > 1$ but it also holds for all s by analytic continuation. Another useful formulation can be obtained by switching s with $1-s$

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\zeta(1-s). \tag{2.22}$$

Let us now see the consequences of this equation. Taking $s = 2n+1$ in (2.21) when n is an integer the cosine factor vanishes and we find the trivial zeroes of $\zeta(s)$

$$\zeta(-2n) = 0 \qquad n = 1, 2, 3, \cdots. \tag{2.23}$$

Later we will need to use certain other values of the Riemann ζ function which we can now compute. In particular the value of $\zeta(-n,a)$ can be calculated if n is a non-negative integer. Taking $s = -n$ in the formula $\zeta(-s,a) = \Gamma(1-s)\Upsilon(s,a)$ we find that

$$\zeta(-n,a) = \Gamma(1+n)\Upsilon(-n,a) = n!\Upsilon(-n,a), \tag{2.24}$$

where

$$\Upsilon(-n,a) = \operatorname*{res}_{z=0}\left(\frac{z^{-n-1}e^{az}}{1-e^{z}}\right), \tag{2.25}$$

the evaluation of this residue requires special functions of its own (special type of polynomials, rather) which are known as the Bernoulli polynomials.

2.4 Bernoulli numbers and the value of $\zeta(0)$

Definition 4 For any complex s we define the functions $B_n(s)$ as

$$\frac{ze^{sz}}{e^{z}-1} = \sum_{n=0}^{\infty}\frac{B_n(s)}{n!}z^{n} \tag{2.26}$$

provided that $|z| < 2\pi$.

A particular case of the polynomials are the Bernoulli numbers $B_n = B_n(0)$ i.e.

$$\frac{z}{e^{z}-1} = \sum_{n=0}^{\infty}\frac{B_n(0)}{n!}z^{n}. \tag{2.27}$$

Lemma 7 One has the following equations $B_n(s) = \sum_{k=0}^{n}\binom{n}{k}B_k s^{n-k}$. In particular when $s=1$ this yields $B_n = \sum_{k=0}^{n}\binom{n}{k}B_k$.
Proof. Using a Taylor expansion and comparing coefficients on both sides we have

$$\sum_{n=0}^{\infty}\frac{B_n(s)}{n!}z^{n} = \frac{z}{e^{z}-1}e^{sz} = \left(\sum_{n=0}^{\infty}\frac{B_n}{n!}z^{n}\right)\left(\sum_{n=0}^{\infty}\frac{s^n}{n!}z^{n}\right),$$

$$\frac{B_n(s)}{n!} = \sum_{k=0}^{n}\frac{B_k}{k!}\frac{s^{n-k}}{(n-k)!}$$

and by passing $n!$ to the RHS we obtain the lemma.

Now we can write the values of the ζ function in terms of Bernoulli numbers.

Lemma 8 For every integer $n \geq 0$ we have

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}. \tag{2.28}$$

Proof. This follows from the previous observation (2.24) so we just have to evaluate the integral Υ by the Cauchy residue theorem.

$$\begin{aligned}\Upsilon(-n,a) &= \operatorname*{res}_{z=0}\left(\frac{z^{-n-1}e^{az}}{1-e^{z}}\right) = -\operatorname*{res}_{z=0}\left(\frac{z^{-n-2}ze^{az}}{e^{z}-1}\right)\\ &= \operatorname*{res}_{z=0}\left(z^{-n-2}\sum_{k=0}^{\infty}\frac{B_k(a)}{k!}z^{k}\right) = -\frac{B_{n+1}(a)}{(n+1)!},\end{aligned}$$

and multiplying by $n!$ we have the end of the proof.

Lemma 9 The recursion $B_n(s+1) - B_n(s) = ns^{n-1}$ is valid for Bernoulli polynomials if $n \geq 1$ and in particular for Bernoulli numbers when $n > 1$: $B_n(0) = B_n(1)$.
Proof. From the identity

$$z\frac{e^{(s+1)z}}{e^z - 1} - z\frac{e^{sz}}{e^z - 1} = ze^{sz}$$

it follows that

$$\sum_{n=0}^{\infty} \frac{B_n(s+1) - B_n(s)}{n!} z^n = \sum_{n=0}^{\infty} \frac{s^n}{n!} z^{n+1},$$

and as we did before, we equate coefficients of z^n to obtain the first statement and then set $s = 0$ to obtain the second statement.

Using the definition, the first Bernoulli number is $B_0 = 1$ and the rest can be computed by recursion. We obtain the values listed on the table below. Building on from the Bernoulli numbers we can construct the polynomials by the use of the lemmas, the first ones as

n	$B_n(0)$	$B_n(s)$
0	1	1
1	$-\frac{1}{2}$	$s - \frac{1}{2}$
2	$\frac{1}{6}$	$s^2 - s + \frac{1}{6}$
3	0	$s^3 - \frac{3}{2}s^2 + \frac{1}{s}s$
4	$-\frac{1}{30}$	$s^4 - 2s^3 + s^2 - \frac{1}{30}$
5	0	$\cdots$

Note that for $n \geq 0$ we have by setting $a = 1$ in (2.28)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{2.29}$$

and because of the trivial zeroes of the Riemann ζ function $\zeta(-2n) = 0$ we confirm our observation that the odd Bernoulli numbers are zero, i.e. $B_{2n+1} = 0$. Also note

$$\zeta(0) = -\frac{1}{2}. \tag{2.30}$$

Finally we can write a compact formula for the even values of the ζ function in terms of Bernoulli numbers.

Theorem 8 Suppose n is a positive integer, then

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n} B_{2n}}{2(2n)!}. \tag{2.31}$$

Proof. This follows from the functional equation by setting $s = 2n$

$$\zeta(1-2n) = 2(2\pi)^{-2n}\Gamma(2n)\cos(\pi n)\zeta(2n)$$

re-arranging we obtain

$$-\frac{B_{2n}}{2n} = 2(2\pi)^{-2n}(2n-1)!(-1)^n\zeta(2n),$$

from which the result follows.

Using the tabulated values given above we have the well-known Euler formulas

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}. \tag{2.32}$$

For instance the value of $\zeta(2)$ was used in the Quantum Information course in connection to the with the result that the probability of two random intergers being coprime is

$$p = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

a very uselful result in cryptography, as well as in the Cosmology course in connection to the entropy density of a species of particles in the ultra-relativistic limit. The second value occurs in the computation of the total energy u radiated by a blackbody in quantum mechanics

$$u = \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{8\pi k^4 T^4}{c^3 h^3} 3! \zeta(4),$$

as well as in the neutrino density (Fermi distribution) in the early history of the universe

$$\rho_\nu = \frac{4\pi}{h^3} \int_0^\infty dx \frac{x^3}{e^{x/(kT)} + 1} = \frac{7\pi^5}{30h^3} (kT)^4.$$

Let us remark that no such formula for odd values is known and in fact the value of $\zeta(3)$ remains one of the most elusive mysteries of modern mathematics. A significant advance was achieved by Roger Apery who was able to show in 1979 that it is an irrational number.

2.5 The value of $\zeta'(0)$

Theorem 9 One has

$$\zeta'(0) = -\frac{1}{2} \log(2\pi). \tag{2.33}$$

Proof. We may write (2.15) and (2.16) as

$$\frac{\zeta(s)}{\Gamma(1-s)} = \frac{1}{2\pi i} \int_C dz \frac{(-z)^{s-1}}{e^z - 1} = \frac{1}{2\pi i} \int_C \frac{dz}{z} \frac{(-z)^s}{e^z - 1}$$

where C is the same contour as that of Figure 2.4, except shifted to the *positive infinity* instead of negative infinity and to account for this we have $(-z)^s = \exp[s \log(-z)]$. Let us differentiate with respect to s and then set $s = 1$

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} \frac{(-z) \log(-z)}{e^z - 1}.$$

The integral on the RHS can be split as

$$\frac{1}{2\pi i} \int_{+\infty}^{\varepsilon} \frac{dz}{z} \frac{(-z)(\log z - i\pi)}{e^z - 1} + \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{dz}{z} \frac{(-z)(\log \varepsilon + i\theta - i\pi)}{e^z - 1} + \frac{1}{2\pi i} \int_{\varepsilon}^{\infty} \frac{dz}{z} \frac{(-z)(\log z + i\pi)}{e^z - 1}$$

and writing $z = \varepsilon e^{i(\phi+\pi)}$ in the middle integral we have

$$-\int_{\varepsilon}^{\infty} \frac{dz}{e^z - 1} - \frac{\log \varepsilon}{2\pi i} \oint_{|z|=\varepsilon} \frac{dz}{z} \frac{z}{e^z - 1} - \frac{1}{2\pi i} \int_{-\pi}^{\pi} d\phi \cdot \phi \frac{z}{e^z - 1},$$

at this point we need to evaluate all three integrals. The first one can be expanded as

$$-\int_{\varepsilon}^{\infty} \frac{dz}{e^z - 1} = \int_{\varepsilon}^{\infty} dz \sum_{n=1}^{\infty} e^{-nz} = -\sum_{n=1}^{\infty} \frac{e^{-nz}}{-n}\Bigg|_{z=\varepsilon}^{z=\infty} = -\sum_{n=1}^{\infty} \frac{(e^{-\varepsilon})^n}{n}$$

$$= \log(1 - e^{-\varepsilon}) = \log\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{6} - \cdots\right) = \log \varepsilon + \log\left(1 - \frac{\varepsilon}{2} + \cdots\right).$$

The second integral is solved by Cauchy's integral theorem by noting that at $z = 0$ we have $z(e^z - 1)^{-1} \to 1$, and therefore we have

$$-\frac{\log \varepsilon}{2\pi i} \oint_{|z|=\varepsilon} \frac{dz}{z} \frac{z}{e^z - 1} = -\log \varepsilon.$$

Finally, the third integral goes to zero as $\varepsilon \to 0$. Putting all these facts together we have shown that

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} \frac{(-z)\log(-z)}{e^z - 1} = \log \varepsilon - \log \varepsilon + O(\varepsilon^{\alpha}) = 0.$$

Re-arranging the functional equation (2.22)

$$\frac{\zeta(s)}{\Gamma(1-s)} = 2(2\pi)^{s-1}\zeta(1-s)\sin\frac{\pi s}{2}$$

and because the derivative at $s = 1$ is zero, as we have just shown, its logarithmic derivative

$$\log(2\pi) - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\pi}{2}\frac{\cos(\pi s/2)}{\sin(\pi s/2)}$$

must also be 0 at $s = 1$ and consequently

$$\log(2\pi) = \frac{\zeta'(0)}{\zeta(0)},$$

finally yielding the result of Theorem 9 by use of (2.30).

Theorem 10 One has

$$\zeta\left(0, \frac{1}{2}\right) = 0 \quad \text{and} \quad \zeta'\left(0, \frac{1}{2}\right) = -\frac{1}{2}\log 2. \tag{2.34}$$

Proof. We have the following identity

$$\zeta(s, 1/2) + \zeta(s) = 2^s \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s} + \frac{1}{(2n)^s}\right] = 2^s \zeta(s),$$

from which it follows that

$$\zeta(s, 1/2) = (2^s - 1)\zeta(s)$$

and hence the first formula is shown. By differentiating the above equation with respect to s we have the second formula of the theorem.

2.6 The polygamma function $\psi^{(m)}(s)$

A complete discussion of the digamma and polygamma functions would take us too far into the presentation of special functions therefore we will only list the definitions and properties of these functions.
The Weierstrass product can be re-written in such a form as to leave the Γ function as

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!}{s(s+1)(s+2)\cdots(s+n)} n^s$$

for $s \neq 0, -1, -2, \cdots$. Thus

$$\begin{aligned}\log\Gamma(s+1) = \log s\Gamma(s) &= \log \lim_{n\to\infty} \frac{n!}{(s+1)(s+2)\cdots(s+n)} n^s \\ &= \lim_{n\to\infty} \left[\log(n!) + s\log n - \log(s+1) - \log(s+2) - \cdots - \log(s+n)\right],\end{aligned}$$

bearing in mind that the logarithm of the limit is the limit of the logarithm. Differentiating with respect to s we can defined the digamma function ψ as

$$\psi(s+1) := \frac{d}{ds}\log\Gamma(s+1) = \lim_{n\to\infty}\left(\log n - \frac{1}{s+1} - \frac{1}{s+2} - \cdots - \frac{1}{s+n}\right). \tag{2.35}$$

We can bring in the definition of the Euler constant

$$\psi(s+1) = -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right) = -\gamma + \sum_{n=1}^{\infty}\frac{s}{n(s+n)} \tag{2.36}$$

where of course

$$\psi(1) = \Gamma'(1) = -\gamma. \tag{2.37}$$

The polygamma function is a generalization of the ψ defined as

$$\psi^{(m)}(s+1) := \frac{d^{m+1}}{ds^{m+1}}\log(s!) = (-1)^{m+1}m!\sum_{n=1}^{\infty}\frac{1}{(s+n)^{m+1}}. \tag{2.38}$$

We note that $\psi^{(0)}(s) = \psi(s)$. Some properties include

$$\psi^{(m)}(s) = \frac{d^{m+1}}{ds^{m+1}}\log\Gamma(s) = \frac{d^{m+1}}{ds^{m+1}}\psi(s), \tag{2.39}$$

$$\psi^{(m)}(1) = (-1)^{m+1}m!\zeta(m+1), \tag{2.40}$$

and the very useful MacLaurin expansion

$$\log\Gamma(s+1) = \sum_{n=1}^{\infty}\frac{s^n}{n!}\psi^{(n-1)}(1) = -\gamma s + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}s^n\zeta(n) \tag{2.41}$$

convergent for $|s| < 1$.

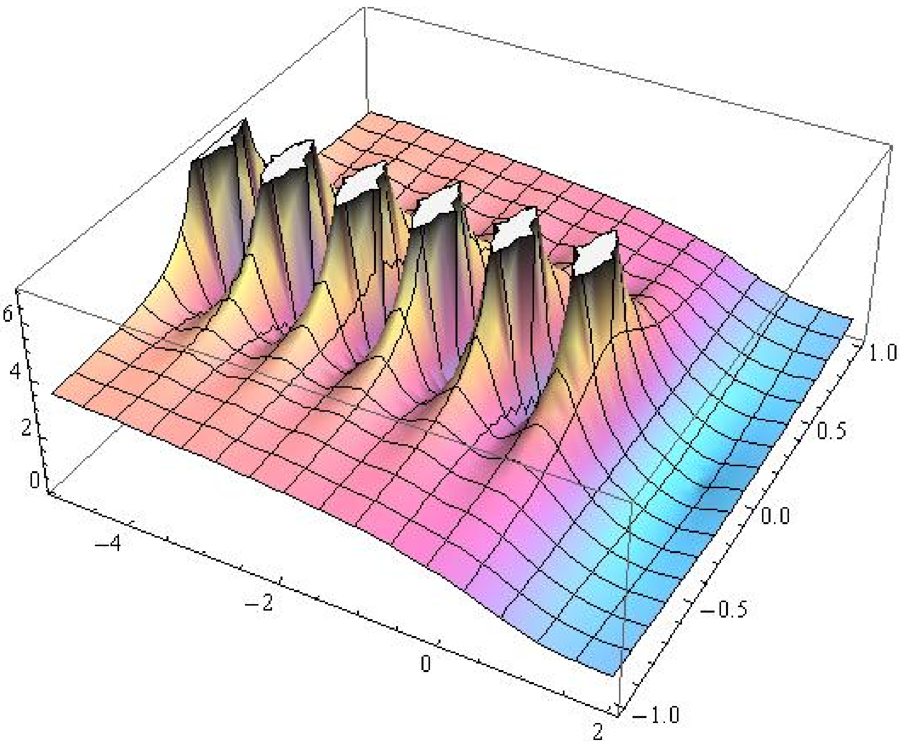

Figure 2.7: $|\psi(x+iy)|$ for $-5 \leq y \leq 2$ and $-1 \leq y \leq 1$

2.7 Laurent Series of the $\zeta(s)$ function

We will lastly need to write the Laurent series of the ζ function. In order to do so, we will use the following theorem from Titchmarsh [5], which we assume.

Theorem 11 Let $f(x)$ be any function with a continuous derivative in the interval $[a, b]$. Then if $[x]$ denotes the greatest integer not exceeding x,

$$\sum_{a<n\leq b} f(x) = \int_a^b dx f(x) + \int_a^b dx (x - [x] - \tfrac{1}{2}) f'(x) + (a - [a] - \tfrac{1}{2}) f(a) - (b - [b] - \tfrac{1}{2}) f(b).$$

If we take the case $f(n) = n^{-s}$ where a, b and n are integers and $s \neq 1$ then

$$\sum_{n=a+1}^{b} n^{-s} = \frac{b^{1-s} - a^{1-s}}{1-s} - s \int_a^b dx \frac{x - [x] - \frac{1}{2}}{x^{s+1}} + \frac{1}{2}(b^{-s} - a^{-s}).$$

By setting $a = 1$ and letting $b \to \infty$ with $\sigma > 1$ and adding 1 to each side we have

$$\zeta(s) = s \int_1^\infty dx \frac{x - [x] - \frac{1}{2}}{x^{s+1}} + \frac{1}{s-1} + \frac{1}{2}. \tag{2.42}$$

This equation contains a remarkable amount of information. First, there is clearly a simple pole at $s = 1$ with residue 1 (the numerator of the fraction). The RHS provides analytic continuation up to $\sigma = 0$. Following Ivic [4] we have

Theorem 12 (Stieltjes' representation) The Laurent series of the ζ function is

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n. \tag{2.43}$$

where γ_n are constants independent of s.
Proof. The first term is explained easily as the pole is at $s = 1$, it is simple and the residue is 1 as we know from Theorem 5. Furthermore

$$\begin{aligned}
\lim_{s\to 1}\left[\zeta(s) - \frac{1}{s-1}\right] &= \int_1^{\infty} dx \frac{x - [x] - \frac{1}{2}}{x^2} + \frac{1}{2} \\
&= \lim_{n\to\infty}\left(\int_1^{n} dx \frac{x-[x]}{x^2} + 1\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1} k \int_k^{k+1} \frac{dx}{x^2} - \log n + 1\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} \frac{1}{k+1} - \log n\right) = \gamma,
\end{aligned}$$

which shows the second term and the remaining terms are regular.

We will use this formula frequently. The constants γ_n are known as the Stieltjes constants and it can be shown that

$$\gamma_n = \frac{(-1)^n}{n!} \lim_{m\to\infty}\left[\sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1}\right], \tag{2.44}$$

however, we will not be needing these. Finally note that the following equation

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^s} = 2^{1-s} \sum_{k=1}^{\infty} \frac{1}{k^s} \Leftrightarrow \zeta(s) = \frac{1}{1-2^{1-2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \tag{2.45}$$

also provides analytic continuation for $\sigma > 0$.

2.8 Concluding remarks on the disbriution of prime numbers and the zeros of $\zeta(s)$

It would not be fair to end this chapter without commenting, briefly, on the link between the ζ function and prime numbers. The reason for doing this is twofold.
Firstly, it is precisely the connection between the continuous (ζ) and the discrete (primes) that makes this theory so powerful and elegant.
Secondly, the ζ function is - in a certain way - deeply connected to quantum theory, and hence this indicates that quantum theory is also connected with the primes numbers. This

connection is be worth investigating on a research level and many papers have been published [J. Brian Conrey *The Riemann Hypothesis*] on the link between primes, quantum theory and random matrices.
We shall proceed informally.
Euler discovered that for $|x| < 1$ we have $(1-x)^{-1} = 1 + x + x^2 + \cdots$ and with $x = p^{-s}$ for $p \neq 1$ we have

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + p^{-2s} + \cdots .$$

Now, the Fundamental Theorem of Arithmetic states that every integer n can be expressed uniquely as a product of only prime factors, hence

$$\prod_p \left(1 + p^{-s} + p^{-2s} + \cdots\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

or equivalently,

$$\prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.46}$$

for $s > 1$. In fact, the above equation is usually called the Euler product or the analytic version of the Fundamental Theorem of Arithmetic as it is its analytic equivalent. Once we have established analytic continuation of the ζ function as we have done in this Chapter and let $s \to 1$ we have the divergence of the harmonic series on the RHS which implies that the LHS must also diverge. This can only happen if there are an infinite number of primes. Interesting as this may be, we can go further by manipulating the equation for $s = 1$ as follows

$$\log \sum_{n=1}^{\infty} \frac{1}{n} = \log \prod_p \frac{1}{1-p^{-1}} = \sum_p \log \frac{1}{1-p^{-1}} = \sum_p \log\left(1 + \frac{1}{p-1}\right)$$

and $e^x > 1 + x \Rightarrow x > \log(1+x)$ so that

$$\sum_p \log\left(1 + \frac{1}{p-1}\right) < \sum_p \frac{1}{p-1},$$

effectively showing that $\sum_p (p-1)^{-1}$ diverges. Finally we note that $(p_k - 1)^{-1} < p_{k-1}^{-1}$ where p_k denotes the kth prime. Hence

$$\sum_p \frac{1}{p} \tag{2.47}$$

diverges. This result was the first attempt to quantify the distribution of prime numbers as it suggests that

$$\sum_{p<x} \frac{1}{p} \sim \log \log x, \tag{2.48}$$

where $\sim$ indicates that the ratio of the two functions tends to 1 as $x \to \infty$. This observation of Euler's was the point of no return in analytic number theory and the distribution of prime numbers.

During the XIX century, several mathematicians, including Gauss and Legendre, formulated the prime number theorem,

$$\pi(x) := \sum_{p \leq x} 1 \sim \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}, \tag{2.49}$$

where $\pi(x)$ denotes the number of primes less than or equal to x and the integral is called the logarithmic integral Li. As we have seen, the zeros $\zeta(s)$ are located at $s = -2, -4, -6, \cdots$ but these are only the trivial zeros. Riemann defined the function ξ as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{2.50}$$

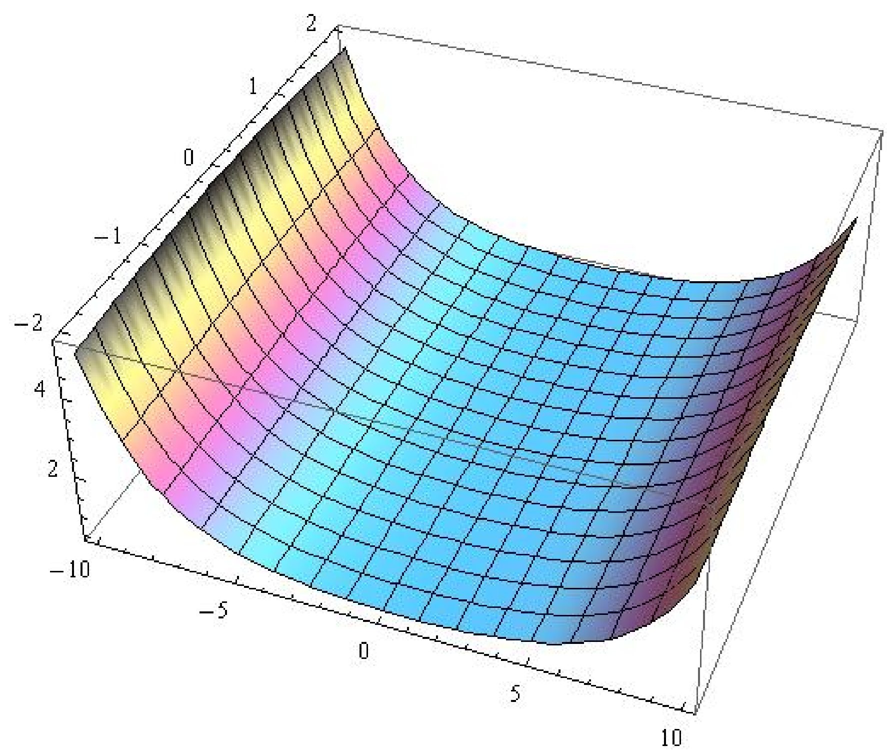

Figure 2.8: $|\xi(x+iy)|$ for $-10 \leq y \leq 10$ and $-2 \leq y \leq 2$

in which case the functional equations takes the elegant form

$$\xi(s) = \xi(1-s).$$

The Euler product shows that $\zeta(s)$ has no zeros in the halfplane $\mathrm{Re}(s) > 1$ because a convergent infinite product can be zero only if one of its factors in zero. Let $\rho_1, \rho_2, \cdots$ be the zeros of $\xi(s)$. It follows from the functional equation that $\zeta(s)$ has no zeros for $\rho < 0$ except for the trivial ones. This is because in the functional equation (2.22) $\zeta(1-s)$ has

no zeros for $\rho < 0$, and $\sin(\pi s/2)$ has simple zeros at $s = -2, -4, \cdots$ and $\Gamma(1-s)$ has no zeros.
Precisely, the trivial zeros of $\zeta(s)$ do not correspond to zeros of $\xi(s)$ since they are cancelled by the poles of $\Gamma(s/2)$. Therefore, it follows that $\xi(s)$ has no zeros for $\rho < 0$ and for $\rho > 1$. The zeros $\rho_1, \rho_2, \cdots$ lie in the strip $0 \le \sigma \le 1$. However, these are the zeros of $\zeta(s)$ also since $s(s-1)\Gamma(s/2)$ has no zeros in the strip except the one at $s = 1$ which is cancelled by the simple pole of $\zeta(s)$.
This proves that $\zeta(s)$ has an infinite number of zeros $\rho_1, \rho_2, \cdots$ in the strip $0 \le \sigma \le 1$ and since

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} > 0$$

for $0 < s < 1$ and $\zeta(0) = -1/2 \neq 0$ then $\zeta(s)$ has no zeros on the real axis between 0 and 1, i.e. the zeros $\rho_1, \rho_2, \cdots$ are all complex.
Here comes a critical observation. The zeros come in conjugate pairs: since $\zeta(s)$ is real on the real axis and if ρ is a zero so is $1-\rho$ by the functional equation then so is $1-\bar{\rho}$. If $\rho = \beta + i\gamma$ then $1 - \bar{\rho} = 1 - \beta + i\gamma$. Consequently the zeros either lie on $\rho = 1/2$ or occur in pairs symmetrical about this line.

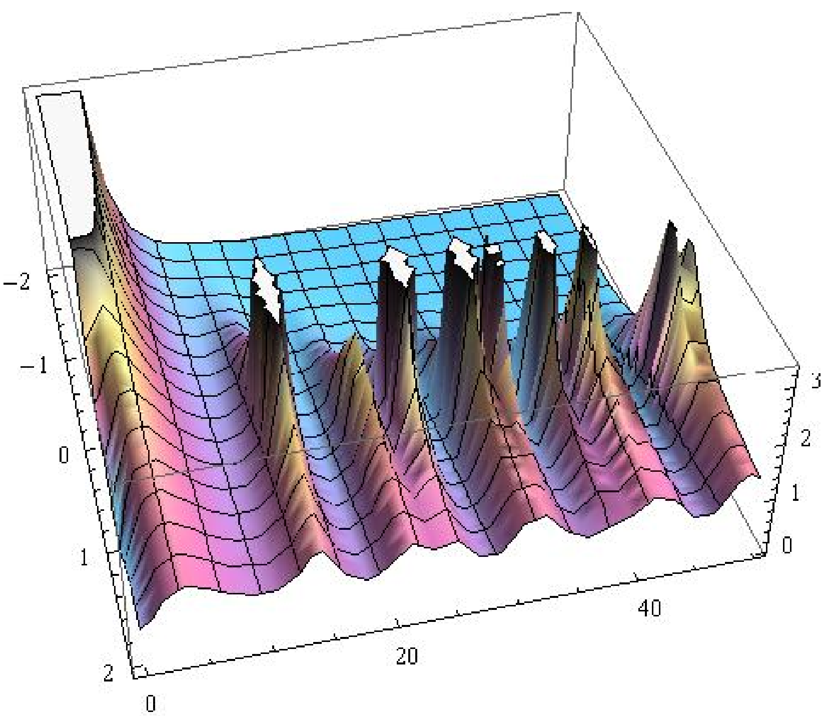

Figure 2.9: $|\zeta(x+iy)|^{-1}$ for $-2 \le x \le 2$ and $0 \le y \le 50$: the zeros become poles

Riemann gave a sketch of a proof of the approximation of the number $N(T)$ of these zeroes for $0 < t < T$ as

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{2.51}$$

This was subsequently showed by von Mangoldt. This equation has the same form as that of the behaviour of the effective Lagrangian at 1-loop order (see Section 5.6 Equation (5.189)). Riemann went even futher and conjectured that all the zeros are on the line $\mathrm{Re}(s) = \frac{1}{2}$.
In 1914 Hardy showed that there is an infinite number of zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ but this does not mean that *all* the zeros are located there.
Riemann's main accomplishement of his 1859 paper was the analytic continuation of the ζ function and his proofs (he gave two) of the functional equation. Furthemore, by the use of an innovative Fourier transform he showed that

$$\log \zeta(s) = s \int_2^\infty du \frac{\pi(u)}{u(u^s - 1)}$$

which enabled him to prove the following remarkable closed analytic formula for $\pi(x)$

$$\pi(x) = R(x) - \frac{1}{2}R(x^{1/2}) - \frac{1}{3}R(x^{1/3}) - \frac{1}{5}R(x^{1/5}) + \frac{1}{6}R(x^{1/6}) + \cdots = \sum_{n=1}^\infty \frac{\mu(n)}{n} R(x^{1/n})$$

where $\mu(n)$ is the Moebius function defined as 0 if n is divisible by a prime square, 1 if n is a product of an even number of distinct primes and -1 if n is a product of an odd number of distinct primes and

$$R(x) = \mathrm{Li}(x) - \sum_{\rho:\zeta(\rho)=0} \mathrm{Li}(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t},$$

where each term is paired with its 'twin', i.e. $\rho \leftrightarrow 1-\rho$ so that

$$\sum_\rho \mathrm{Li}(x^\rho) = \sum_{\mathrm{Im}\rho>0} \mathrm{Li}(x^\rho) + \mathrm{Li}(x^{1-\rho}).$$

In 1896 Hadamard and de la Vallee Poussin proved, independently and almost simultaneously, the prime number theorem by showing that it is equivalent to $\zeta(1+it) \neq 0$, i.e. no zeros on $\mathrm{Re}(s) = 1$.
Riemann's Hypothesis is equivalent to

$$\pi(x) - \int_2^x \frac{dt}{\log t} = O(x^{1/2}\log x)$$

as $x \to \infty$.
Finally, Hadamard also proved the product representation stated (without a valid proof) by Riemann

$$\zeta(s) = \frac{e^H s}{2(s-1)\Gamma\left(\frac{s}{2}+1\right)} \prod_{\rho:\zeta(\rho)=0} \left[\left(1 - \frac{s}{\rho}\right) e^{s/\rho}\right] \tag{2.52}$$

where $H = \log 2\pi - 1 - \gamma/2$.
The $\xi(s)$ function also admits a product representation

$$\xi(s) = \frac{1}{2}\prod_\rho \left(1 - \frac{s}{\rho}\right).$$

Let us re-write the definition of $\xi(s)$ as

$$\xi(s) = \Pi\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s)$$

where Π is Riemann's notation for the shifted Γ function

$$\Pi(s-1) := \Gamma(s).$$

The logarithmic derivative of $\xi(s)$ is on the one hand

$$\sum_\rho \frac{d}{ds}\log\left(1-\frac{s}{\rho}\right) = \sum_\rho \frac{1}{s-\rho}$$

and on the other hand

$$\frac{d}{ds}\log\Pi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi + \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}.$$

Evaluating both expressions at $s=0$ yields

$$\sum_\rho \frac{1}{\rho} = \frac{1}{2}\gamma + \frac{1}{2}\log\pi + 1 - \log 2\pi$$

or

$$\sum_{\mathrm{Im}\rho>0}\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right) = \frac{1}{2}[2+\gamma-\log 4\pi] \tag{2.53}$$

since $\Pi'(0) = \Gamma'(1) = -\gamma$. Note that $\gamma - \log 4\pi$ is a mathematical constant that will show up frequently in renormalization, indicating yet another link between quantum theory and the zeros of the ζ function. This formula can be used to compute the zeros ρ. Some of the first zeros $\rho = \frac{1}{2} + it_i$ are

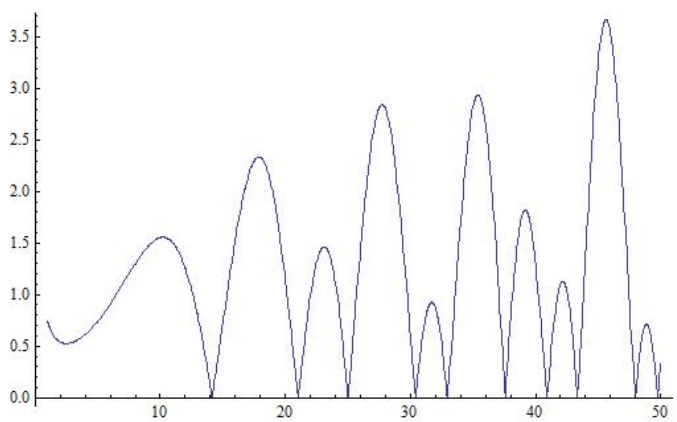

Figure 2.10: $|\zeta(\frac{1}{2}+it)|$ for $0 \le t \le 50$

that is,

$$\begin{aligned}
t_1 &= 14.134725\\
t_2 &= 21.022040\\
t_3 &= 25.010858\\
t_4 &= 30.424878\\
t_5 &= 32.935057
\end{aligned}$$

REFERENCES
Here we have followed the course from
- [1] Tom Apostol's *Introduction to Analytic Number Theory* for the presentation of the ζ function and its analytic continuation. Apostol in turn follows
- [3] *Distribution of Prime Numbers* by Ingham.
The section of the Γ function follows from the course (and the exercises) on
- [2] *Complex Analysis* by Freitag and Busam, Titchsmarch's *Theory of Functions* [6] and Whitaker and Watson's *Modern Analysis* [7].
There exist many sources for the development of the Riemann ζ function. The most comprehensive ones are by
- [4] Ivic's *Riemann's Zeta Function*, and - [5] E.C. Titchmarsh (revised by D.R. Heath-Brown) *The Theory of the Riemann Zeta Function*.

3 Zeta Regularization in Quantum Mechanics

3.1 Path integral of the harmonic oscillator

One of the first instances where the Riemann ζ function occurs in quantum physics is in the path integral development of the harmonic oscillator; specifically it takes place when computing the partition function of the spectrum of the harmonic oscillator.
We will follow Exercise 9.2 p 312 from Peskin and Schroeder [2] and the path integral development from Chapter 2 of Kleinert [1]. The action of the one-dimensional harmonic oscillator is given by

$$S = \int_{t_i}^{t_f} dt L, \tag{3.1}$$

where the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{3.2}$$

As we know from the functional approach to quantum mechanics, the transition amplitude is the functional integral

$$\langle x_f, t_f | x_i, t_i \rangle = \int Dx e^{iS[x(t)]}. \tag{3.3}$$

The extremum of S, $x_c(t)$, satisfies

$$\left.\frac{\delta S[x]}{\delta x}\right|_{x=x_c(t)} = 0. \tag{3.4}$$

We now proceed to expand the action around $x_c(t)$. This indicates that $x_c(t)$ is the classical trajectory connecting both space-time points of the amplitude and therefore it satisfies the Euler-Lagrange equation

$$\ddot{x}_c + \omega^2 x_c = 0. \tag{3.5}$$

The solution to the equation above with conditions $x_c(t_i) = x_i$ and $x_c(t_f) = x_f$ is

$$x_c(t) = (\sin\omega T)^{-1}[x_f \sin\omega(t - t_i) + x_i \sin\omega(t_f - t)], \tag{3.6}$$

where $T = t_f - t_i$. We next plug this solution into the action S

$$S_c := S[x_c] = \frac{m\omega}{2\sin\omega T}[(x_f^2 + x_i^2)\cos\omega T - 2x_f x_i]. \tag{3.7}$$

As we intended originally, we now expand $S[x]$ around $x = x_c$ to obtain

$$S[x_c + z] = S[x_c] + \int dt z(t) \left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x_c} + \frac{1}{2!}\int dt_1 dt_2 z(t_1) z(t_2) \left.\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \tag{3.8}$$

where $z(t)$ satisfies the boundary condition

$$z(t_i) = z(t_f) = 0. \tag{3.9}$$

The expansion ends at second order because the action is in second order in x. Noting that $\delta S[x]/\delta x = 0$ at $x = x_c$ we are left with the first and last terms only. Because the expansion is finite

$$S[x_c + z] = S[x_c] + \frac{1}{2!}\int dt_1 dt_2 z(t_1)z(t_2)\left.\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \tag{3.10}$$

the problem can be solved analytically.
Let us now compute the second order functional derivative in the integrand; this can be accomplished as follows

$$\frac{\delta}{\delta x(t_1)}\int_{t_i}^{t_f} dt\left[\frac{1}{2}m\dot{x}(t)^2 - \frac{1}{2}m\omega^2 x(t)^2\right] = -m\left(\frac{d^2}{dt_1^2} + \omega^2\right)x(t_1). \tag{3.11}$$

Next, using the rule

$$\frac{\delta x(t_1)}{\delta x(t_2)} = \delta(t_1 - t_2) \tag{3.12}$$

we obtain the following expression for the second order functional derivative

$$\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)} = -m\left(\frac{d^2}{dt_1^2} + \omega^2\right)\delta(t_1 - t_2). \tag{3.13}$$

Plugging this back into the equation for the expansion, we can use the delta function to get rid of the t variables

$$\begin{aligned} S[x_c + z] &= S[x_c] - \frac{m}{2!}\int dt_1 dt_2 z(t_1)z(t_2)\delta(t_1 - t_2)\left(\frac{d^2}{dt_1^2} + \omega^2\right) \\ &= S[x_c] + \frac{m}{2!}\int dt(\dot{z}^2 - \omega^2 z^2), \end{aligned} \tag{3.14}$$

where we have simplified the expression by using (3.9).
A crucial point to be made is that because Dx is invariant, we may replace it by Dz and this will give us

$$\langle x_f, t_f \mid x_i, t_i\rangle = e^{iS[x_c]}\int\limits_{z(t_i)=z(t_f)=0} Dz\,\exp\left[i\frac{m}{2}\int_{t_i}^{t_f} dt(\dot{z}^2 - \omega^2 z^2)\right]. \tag{3.15}$$

The fluctuation part (integral at $z(0) = z(T) = 0$)

$$I_f := \int\limits_{z(0)=z(T)=0} Dz\,\exp\left[i\frac{m}{2}\int_0^T dt(\dot{z}^2 - \omega^2 z^2)\right] \tag{3.16}$$

is computed as follows. First, let us shift the variables so that time start at 0 and ends at T. We Fourier expand $z(t)$ as

$$z(t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi t}{T}. \tag{3.17}$$

This choice satisfies the (3.9). The integral in the exponential gives

$$\int_0^T dt(\dot{z}^2 - \omega^2 z^2) = \frac{T}{2}\sum_{n=1}^{\infty} a_n^2 \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]. \tag{3.18}$$

In order to have a well-defined transformation we must check that the number of variables is the same before and after the transformation. Indeed, the Fourier transformation from $y(t)$ to a_n may be thought of as a change of variables in the integration. To check this, take the number of the time slice to be $N+1$, including both $t=0$ and $t=T$, for which there exist $N-1$ independent z_k variables. Therefore, we must set $a_n = 0$ for all $n > N-1$. Next, we compute the corresponding Jacobian. Denote by t_k the kth time slice when the interval $[0,T]$ is split into N infinitesimal parts, then

$$J_N = \det \frac{\partial z_k}{\partial a_n} = \det\left(\sin\frac{n\pi t_k}{T}\right). \tag{3.19}$$

We evaluate the Jacobian for the easiest possible case, which is that of the free particle. Therefore, let us make a digression to evaluate the above mentioned probability amplitude.

3.2 Solution of the free particle

For a free particle, the Lagrangian is $L = \frac{1}{2}m\dot{x}^2$. Carefully derived solutions to the free particle can be found in Chapter 2 of Kleinert [1] as well as in Chapter 3 of Grosche and Steiner [4]. The amplitude is computed by first noting that the Hamiltonian is given

$$H = p\dot{x} - L = \frac{p^2}{2m}, \tag{3.20}$$

so that

$$\begin{aligned}\langle x_f, t_f \,|\, x_i, t_i\rangle &= \left\langle x_f | e^{-i\hat{H}T} | \, x_i \right\rangle = \int dp \left\langle x_f | \exp(-i\hat{H}T) | \, p \right\rangle \langle p | \, x_i\rangle \\ &= \int \frac{dp}{2\pi} e^{ip(x_f - x_i)} e^{-iTp^2/(2m)} = \sqrt{\frac{m}{2\pi i T}} \exp\left(\frac{im(x_f - x_i)^2}{2T}\right)\end{aligned} \tag{3.21}$$

where $T = t_f - t_i$ as noted before and ε here denotes the discretization of time $\varepsilon = T/N$. Similarly to the theory of functional integration we have amplitude

$$\langle x_f, t_f \,|\, x_i, t_i\rangle = \lim_{n\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{n/2} \int dx_1 \cdots dx_{n-1} \exp\left[i\varepsilon \sum_{k=1}^{n} \frac{m}{2}\left(\frac{x_k - x_{k-1}}{\varepsilon}\right)^2\right]. \tag{3.22}$$

We now change the coordinates to $z_k = (m/(2\varepsilon))^{1/2} x_k$ so that the amplitude becomes

$$\langle x_f, t_f \,|\, x_i, t_i\rangle = \lim_{n\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{n/2} \left(\frac{2\varepsilon}{m}\right)^{(n-1)/2} \int dz_1 \cdots dz_{n-1} \exp\left[i \sum_{k=1}^{n} (z_k - z_{k-1})^2\right]. \tag{3.23}$$

In the appendix, we prove by induction that

$$\int dz_1 \cdots dz_{n-1} \exp\left[i \sum_{k=1}^{n} (z_k - z_{k-1})^2\right] = \left[\frac{(i\pi)^{n-1}}{n}\right]^{1/2} e^{i(z_n - z_0)^2/n}. \tag{3.24}$$

The expression reduces to

$$
\begin{aligned}
\langle x_f, t_f \,|\, x_i, t_i \rangle &= \lim_{n\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{n/2} \left(\frac{2\pi i\varepsilon}{m}\right)^{(n-1)/2} \frac{1}{\sqrt{n}} e^{im(x_f-x_0)^2/(2n\varepsilon)} \\
&= \sqrt{\frac{m}{2\pi iT}} \exp\left[\frac{im}{2T}(x_f-x_i)^2\right].
\end{aligned} \tag{3.25}
$$

Taking (3.25) into account we arrive at

$$
\langle x_f, T \,|\, x_i, 0 \rangle = \left(\frac{1}{2\pi iT}\right)^{1/2} \exp\left[\frac{im}{2T}(x_f-x_i)^2\right] = \left(\frac{1}{2\pi iT}\right)^{1/2} e^{iS[x_c]}. \tag{3.26}
$$

When we write this in terms of a path integral we obtain

$$
e^{iS[x_c]} \int\limits_{z(0)=z(T)=0} Dz \, \exp\left[i\frac{m}{2}\int_{t_i}^{t_f} dt\dot{z}^2\right]. \tag{3.27}
$$

Now, from (3.18) we have

$$
\frac{m}{2}\int_0^T dt\,\dot{z}^2 \to m\sum_{n=1}^{N} \frac{a_n^2 n^2 \pi^2}{4T} \tag{3.28}
$$

and when we compare both path integral expressions one has the equality

$$
\begin{aligned}
\left(\frac{1}{2\pi iT}\right)^{1/2} &= \int\limits_{z(0)=z(T)=0} Dz \, \exp\left[i\frac{m}{2}\int_{t_i}^{t_f} dt\dot{z}^2\right] \\
&= \lim_{N\to\infty} J_N \left(\frac{1}{2\pi i\varepsilon}\right)^{1/2} \int da_1 \dots da_{N-1} \exp\left(im\sum_{n=1}^{N-1} \frac{a_n^2 n^2\pi^2}{4T}\right).
\end{aligned} \tag{3.29}
$$

Now comes the process of evaluating the Gaussian integrals

$$
\begin{aligned}
\left(\frac{1}{2\pi iT}\right)^{1/2} &= \lim_{N\to\infty} J_N \left(\frac{1}{2\pi i\varepsilon}\right)^{N/2} \prod_{n=1}^{N-1} \frac{1}{n}\left(\frac{4\pi iT}{\pi^2}\right)^{1/2} \\
&= \lim_{N\to\infty} J_N \left(\frac{1}{2\pi i\varepsilon}\right)^{N/2} \frac{1}{(N-1)!}\left(\frac{4\pi iT}{\pi^2}\right)^{(N-1)/2}
\end{aligned} \tag{3.30}
$$

and from this we obtain a formula for the Jacobian

$$
J_N = 2^{-(N-1)/2} N^{-N/2} \pi^{N/1} (N-1)! \underset{N\to\infty}{\to} \infty \tag{3.31}
$$

The Jacobian is divergent, but this divergence is not relevant because J_N is combined with other divergent factors.
Let us now return to the original problem of the probability amplitude of a harmonic oscillator. The amplitude was

$$
\begin{aligned}
\langle x_f, T \,|\, x_i, 0 \rangle &= \lim_{N\to\infty} J_N \left(\frac{1}{2\pi iT}\right)^{N/2} e^{iS[x_c]} \\
&\times \int da_1 \dots da_{N-1} \exp\left[i\frac{mT}{4}\sum_{n=1}^{N-1} a_n^2 \left\{\left(\frac{n\pi}{T}\right)^2 - \omega^2\right\}\right].
\end{aligned} \tag{3.32}
$$

As we did with the free particle, we carry out the computation of the Gaussian integrals using the formula

$$\int da_n \exp\left[\frac{imT}{4}a_n^2\left\{\left(\frac{n\pi}{T}\right)^2-\omega^2\right\}\right]=\left(\frac{4iT}{\pi n^2}\right)^{1/2}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}. \tag{3.33}$$

This breaks the amplitude (3.32) into smaller parts

$$\begin{aligned}\langle x_f,t_f\,|\,x_i,t_i\rangle &= \lim_{N\to\infty} J_N\left(\frac{N}{2\pi iT}\right)^{N/2} e^{iS[x_c]}\prod_{k=1}^{N-1}\left[\frac{1}{k}\left(\frac{4iT}{\pi}\right)^{1/2}\right]\prod_{n=1}^{N-1}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}\\ &= \lim_{N\to\infty}\left(\frac{1}{2\pi iT}\right)^{1/2} e^{iS[x_c]}\prod_{n=1}^{N-1}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}.\end{aligned} \tag{3.34}$$

It can be shown that this product is equal to

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]=\frac{\sin\omega T}{\omega T}. \tag{3.35}$$

The divergence of J_n cancels the divergence of the other terms and therefore we are left with a finite value confirming what we stated above concerning the irrelevance of J_∞. When we insert the value of the product we arrive at the final result (2.36)

$$\begin{aligned}\langle x_f,t_f\,|\,x_i,t_i\rangle &= \left(\frac{\omega}{2\pi i\sin\omega T}\right)^{1/2} e^{iS[x_c]}\\ &= \left(\frac{\omega}{2\pi i\sin\omega T}\right)^{1/2}\exp\left[\frac{i\omega}{2\sin\omega T}\left\{(x_f^2+x_i^2)\cos\omega T-2x_ix_f\right\}\right].\end{aligned} \tag{3.36}$$

3.3 The bosonic partition function

If we have a Hamiltonian $\hat{H}$ whose spectrum is bounded from below then, by adding a positive constant to the Hamiltonian, we can make $\hat{H}$ positive definite, i.e.

$$\mathrm{spec}(\hat{H})=\{0<E_0\le E_1\le\cdots\le E_n\le\cdots\}. \tag{3.37}$$

Also we assume that the ground state is not degenerate. The spectral decomposition of $e^{-i\hat{H}t}$ is

$$e^{-i\hat{H}t}=\sum_n e^{-iE_nt}\,|n\rangle\,\langle n| \tag{3.38}$$

and this decomposition is analytic in the lower half-plane of t, where we have $\hat{H}\,|n\rangle=E_n\,|n\rangle$. As we have done when evaluating the Gaussian integrals we introduce the Wick rotation $t=-i\tau$ where τ is real and positive, this gives us $\dot{x}=idx/d\tau$ and $e^{-i\hat{H}t}=e^{-\hat{H}\tau}$ so that

$$i\int_{t_i}^{t_f}dt\left[\frac{1}{2}m\dot{x}^2-V(x)\right]=i(-i)\int_{\tau_i}^{\tau_f}dt\left[-\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2-V(x)\right]=-\int_{\tau_i}^{\tau_f}dt\left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2+V(x)\right] \tag{3.39}$$

Consequently, the path integral becomes

$$\begin{aligned}\langle x_f, t_f \,|\, x_i, t_i\rangle &= \left\langle x_f, t_f \,| e^{-\hat{H}(\tau_f-\tau_i)} |\, x_i, t_i \right\rangle \\ &= \int \bar{D}x \exp\left[-\int_{\tau_i}^{\tau_f} d\tau \left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]\right],\end{aligned} \tag{3.40}$$

where $\bar{D}x$ is the integration measure in the imaginary time τ. Equation (3.40) shows the connection between the functional approach and statistical mechanics.
Let us now define partition function [1], [2], [4], [5] of a Hamiltonian $\hat{H}$ as

$$Z(\beta) = \mathrm{Tr} e^{-\beta\hat{H}}, \tag{3.41}$$

where β is a positive constant and the trace is over the Hilbert space associated with $\hat{H}$. This partition can be written in terms of eigenstates of energy $\{|E_n\rangle\}$ with

$$\hat{H}\,|E_n\rangle = E_n\,|E_n\rangle\,, \qquad \langle E_m|\, E_n\rangle = \delta_{mn}. \tag{3.42}$$

In this case

$$Z(\beta) = \sum_n \left\langle E_n|e^{-\beta\hat{H}}|\, E_n\right\rangle = \sum_n \langle E_n|e^{-\beta E_n}|\, E_n\rangle = \sum_n e^{-\beta E_n}, \tag{3.43}$$

or in terms of the eigenvector $|x\rangle$ of the position operator $\hat{x}$,

$$Z(\beta) = \int dx \left\langle x|e^{-\beta\hat{H}}|\, x\right\rangle. \tag{3.44}$$

Initially we had an arbitrary β but if we set it to be $\beta = iT$ we find that

$$\left\langle x_f|e^{-i\hat{H}T}|\, x_i\right\rangle = \left\langle x_f|e^{-\beta\hat{H}}|\, x_i\right\rangle, \tag{3.45}$$

and from this we have the path integral expression of the partition function

$$\begin{aligned}Z(\beta) &= \int dz \int_{x(0)=x(\beta)=z} \bar{D}x \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{x}^2 + V(x)\right)\right] \\ &= \int_{\text{periodic}} \bar{D}x \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{x}^2 + V(x)\right)\right],\end{aligned} \tag{3.46}$$

where the periodic integral indicates that the integral is over all paths which are periodic in the interval $[0, \beta]$.
When we apply this to the harmonic oscillator, the partition function is simply

$$\mathrm{Tr} e^{-\beta\hat{H}} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\omega}. \tag{3.47}$$

Although there are a number of ways of evaluating the partition function, here we choose one where the use of the ζ regularization is illustrated. Proceed as follows. Set imaginary time $\tau = iT$ to obtain the path integral

$$\int_{z(0)=z(T)=0} Dz \exp\left[\frac{i}{2}\int dt\, z\left(-\frac{d^2}{dt^2} - \omega^2\right) z\right] \to \int_{z(0)=z(\beta)=0} \bar{D}z \exp\left[-\frac{1}{2}\int d\tau\, z\left(-\frac{d^2}{dt^2} + \omega^2\right) z\right], \tag{3.48}$$

in this case $\bar{D}z$ indicates the path integration measure with imaginary time. Suppose we have an $n \times n$ Hermitian matrix $\mathbf{M}$ with positive-definite eigenvalues λ_k where $1 \leq k \leq n$ then we show in the appendix that (see Rajantie [3])

$$\prod_{k=1}^{n} \left(\int_{-\infty}^{\infty} dx_k \right) \exp\left[-\frac{1}{2} \sum_{p,q} x_p M_{pq} x_q \right] = \pi^{n/2} \prod_{k=1}^{n} \frac{1}{\sqrt{\lambda_k}} = \frac{\pi^{n/2}}{\det \mathbf{M}}. \tag{3.49}$$

This is a matrix generalization of the scalar Gaussian integral

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2} \lambda x^2 \right) = \sqrt{\frac{2\pi}{\lambda}}, \quad \lambda > 0.$$

The next task is to define the determinant of an operator O by the infinite product of its eigenvalues λ_k. This is accomplished by setting $\mathrm{Det}O = \prod_k \lambda_k$. Note that Det with capital d denotes the determinant of an operator, whereas with a small d it denotes the determinant of a matrix, same applies for Tr and tr.

3.4 Zeta regularization solution of the bosonic partition function

Using this we can we can write the integral over imaginary time as

$$\int_{z(0)=z(\beta)=0} \bar{D}z \exp\left[-\frac{1}{2} \int d\tau\, z \left(-\frac{d^2}{dt^2} + \omega^2 \right) z \right] = \left[\mathrm{Det}_D \left(-\frac{d^2}{d\tau^2} + \omega^2 \right) \right]^{-1/2} \tag{3.50}$$

here the D denotes the Dirichlet boundary condition $z(0) = z(\beta) = 0$.
Similarly to what we did with (3.18) we see that the general solution

$$z(\tau) = \frac{1}{\sqrt{\beta}} \sum_{n=1}^{\infty} z_n \sin \frac{n\pi\tau}{\beta}. \tag{3.51}$$

We are restricted to having the coefficients z_n real as z is a real function. We are now in a position to write formal expressions. Knowing that the eigenvalues of the eigenfuction $\sin(n\pi\tau/\beta)$ are $\lambda_n = (n\pi/\beta)^2 + \omega^2$ we may write the determinant of the operator as

$$\mathrm{Det}_D \left(-\frac{d^2}{d\tau^2} + \omega^2 \right) = \prod_{n=1}^{\infty} \lambda_n = \prod_{n=1}^{\infty} \left[\left(\frac{n\pi}{\beta} \right)^2 + \omega^2 \right] = \prod_{n=1}^{\infty} \left(\frac{n\pi}{\beta} \right)^2 \prod_{m=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{m\pi} \right)^2 \right]. \tag{3.52}$$

It is now time to identify the first infinite product with the functional determinant, i.e.

$$\mathrm{Det}_D \left(-\frac{d^2}{d\tau^2} \right) \leftrightarrow \prod_{n=1}^{\infty} \left(\frac{n\pi}{\beta} \right)^2. \tag{3.53}$$

Here is where the ζ function comes into play. Suppose O is an operator with positive-define eigenvalues λ_n. Following [1], [4], [5] in this case, we take the log

$$\log \mathrm{Det}O = \log \prod_n \lambda_n = \mathrm{Tr} \log O = \sum_n \log \lambda_n. \tag{3.54}$$

Now we define the spectral ζ function as

$$\zeta_O(s) := \sum_n \lambda_n^{-s}. \tag{3.55}$$

The sum converges for sufficiently large Re(s) and $\zeta_O(s)$ is analytic in s in this region. Additionally, it can be analytically continued to the whole s plane except at a possible finite number of points. The derivative of the spectral ζ function is linked to functional determinant by

$$\left.\frac{d\zeta_O(s)}{ds}\right|_{s=0} = -\sum_n \log \lambda_n. \tag{3.56}$$

And therefore the expression for Det$O$ is

$$\mathrm{Det}O = \exp\left[-\left.\frac{d\zeta_O(s)}{ds}\right|_{s=0}\right]. \tag{3.57}$$

The operator we are interested in is $O = -d^2/d\tau^2$ so this yields

$$\zeta_{-d^2/d\tau^2}(s) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{\beta}\right)^{-2s} = \left(\frac{\beta}{\pi}\right)^{2s}\zeta(2s). \tag{3.58}$$

As we proved in Chapter 1, the ζ function is analytic over the whole complex s plane except at the simple pole at $s = 1$. The values (2.30) and (2.33)

$$\zeta(0) = -\frac{1}{2} \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi)$$

were also calculated, and we can use them now to obtain

$$\zeta'_{-d^2/d\tau^2}(0) = 2\log\left(\frac{\beta}{\pi}\right)\zeta(0) + 2\zeta'(0) = -\log(2\beta). \tag{3.59}$$

Putting this into the expression for the determinant with Dirichlet conditions we have

$$\mathrm{Det}_D\left(-\frac{d^2}{d\tau^2}\right) = e^{\log(2\beta)} = 2\beta \tag{3.60}$$

and finally

$$\mathrm{Det}_D\left(-\frac{d^2}{d\tau^2} + \omega^2\right) = 2\beta\prod_{p=1}^{\infty}\left[1 + \left(\frac{\beta\omega}{p\pi}\right)^2\right]. \tag{3.61}$$

Note how the infinite product now becomes finite due to $\zeta(0)$ and $\zeta'(0)$.
Let us go back to the partition function

$$\begin{aligned}\mathrm{Tr}e^{-\beta\hat{H}} &= \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\omega} = \left[2\beta\prod_{p=1}^{\infty}\left\{1+\left(\frac{\beta\omega}{p\pi}\right)^2\right\}\right]^{-1/2}\left[\frac{\pi}{\omega\tanh(\beta\omega/2)}\right]^{1/2} \\ &= \left(2\beta\frac{\pi}{\beta\omega}\sinh\beta\omega\right)^{-1/2}\left[\frac{\pi}{\omega\tanh(\beta\omega/2)}\right]^{1/2} = \frac{1}{2\sinh(\beta\omega/2)}.\end{aligned} \tag{3.62}$$

3.5 Alternative solution

It is important to note that there is a more direct way of computing this partition function and which is more satisfying for solvable cases such as the harmonic oscillator but which fails under more complicated Lagrangians.
This can be done by computing

$$Z(\beta) = \mathrm{Tr} e^{-\beta \hat{H}} = \int dx \left\langle x \right| \exp(-\beta \hat{H}) \left| x \right\rangle.$$

Recall that

$$\langle x_f, t_f \,|\, x_i, t_i \rangle = \left(\frac{\omega}{2\pi i \sin \omega T} \right)^{1/2} \exp \left[\frac{i\omega}{2 \sin \omega T} \left\{ (x_f^2 + x_i^2) \cos \omega T - 2 x_i x_f \right\} \right],$$

so that

$$\begin{aligned} Z(\beta) &= \left(\frac{\omega}{2\pi i(-i \sinh \beta\omega)} \right)^{1/2} \int dx \exp i \left[\frac{\omega}{-2i \sinh \beta\omega} (2x^2 \cosh \beta\omega - 2x^2) \right] \\ &= \left(\frac{\omega}{2\pi \sinh \beta\omega} \right)^{1/2} \left(\frac{\pi}{\omega \tanh \beta\omega/2} \right)^{1/2} = \frac{1}{2 \sinh(\beta\omega/2)}. \end{aligned}$$

3.6 The Fermionic partition function

The quantisation of bosonic particles is done by using commutation relations, however, the quantisation of fermionic particles require a more different approach, namely that of anti-commutation relations. This in turn requires anti-commuting numbers which are called Grassmann numbers. We continue the presentation from [1], [4] and [5].
In analogy with the bosonic harmonic oscillator which was described by the Hamiltonian,

$$H = \tfrac{1}{2}(a^\dagger a + a a^\dagger), \tag{3.63}$$

where a and $a^\dagger$ satisfy the commutation relations

$$[a, a^\dagger] = 1 \qquad [a, a] = [a^\dagger, a^\dagger] = 0. \tag{3.64}$$

The Hamiltonian has eigenvalues $(n + \frac{1}{2})\omega$ where n is an integer with eigenvector $|n\rangle$

$$H\,|n\rangle = (n + \tfrac{1}{2})\omega\,|n\rangle\,. \tag{3.65}$$

From now we drop the hat notation in the operators whenever there is no risk for confusion with the eigenvalue.
The prescription for the fermionic Hamiltonian is to set

$$H = \tfrac{1}{2}(c^\dagger c - c c^\dagger)\omega. \tag{3.66}$$

This may be thought of as a Fourier component of the Dirac Hamiltonian, which describes relativistic fermions. However, it is evident that if c and $c^\dagger$ were to satisfy commutation relations then the Hamiltonian would be a constant, and therefore it is more appropriate to consider anti-commutation relations

$$\left\{c, c^\dagger\right\} = 1 \qquad \{c, c\} = \left\{c^\dagger, c^\dagger\right\} = 0. \tag{3.67}$$

In this case the Hamiltonian becomes

$$H = \tfrac{1}{2}\left[c^\dagger c - (1 - c^\dagger c)\right]\omega = \left(N - \tfrac{1}{2}\right)\omega, \tag{3.68}$$

where $N = c^\dagger c$. The eigenvalues of N are either 0 or 1 since

$$N^2 = c^\dagger c c^\dagger c = N \Leftrightarrow N(N-1) = 0.$$

Next we need a description of the Hilbert space of the Hamiltonian. To this end, let $|n\rangle$ be an eigenvector of H with eigenvalue n (necessarily $n = 0, 1$). Then the following equations hold

$$H|0\rangle = -\frac{\omega}{2}|0\rangle \qquad H|1\rangle = \frac{\omega}{2}|1\rangle \tag{3.69}$$

For the sake of convenience we introduce the spin-notation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.70}$$

$$c^\dagger|0\rangle = |1\rangle \quad c^\dagger|1\rangle = 0 \quad c|0\rangle = 0 \quad c|1\rangle = |0\rangle.$$

When the basis vectors of the space have this form then the operators take the matrix representations

$$c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad H = \frac{\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.71}$$

Instead of having the bosonic commutation relation $[x,p] = i$ we now have $[x,p] = 0$. The anti-commutation relation $\{c, c^\dagger\} = 1$ is replaced by $\{\theta, \theta^*\} = 0$ where θ and θ^* are anti-commuting numbers, i.e. Grassmann numbers which we proceed to develop in further detail in the appendix.
The Hamiltonian of the fermionic harmonic oscillator is $H = (c^\dagger c - 1/2)\omega$, with eigenvalues $\pm\omega/2$. From our previous discussion of the partition function and Grassmann numbers we know that

$$Z(\beta) = \mathrm{Tr}e^{-\beta H} = \sum_{n=0}^{1}\langle n|e^{-\beta H}|\,n\rangle = e^{\beta\omega/2} + e^{-\beta\omega/2} = 2\cosh(\beta\omega/2). \tag{3.72}$$

As with the bosonic case, we can evaluate $Z(\beta)$ in two different ways using the path integral formalism. Let us start with some preliminary results. Let H be the Hamiltonian of a fermionic harmonic oscillator, its partition function is written as

$$\mathrm{Tr}e^{-\beta H} = \int d\theta^* d\theta \,\langle -\theta|e^{-\beta H}|\,\theta\rangle\, e^{-\theta^*\theta}. \tag{3.73}$$

We can show this by the inserting (A.60) into the partition function (3.72), i.e.

$$\begin{aligned} Z(\beta) &= \sum_{n=0,1}\langle n|e^{-\beta H}|\,n\rangle = \sum_n \int d\theta^* d\theta\, |\theta\rangle\langle\theta|\, e^{-\theta^*\theta}\langle n|e^{-\beta H}|\,n\rangle \\ &= \sum_n \int d\theta^* d\theta e^{-\theta^*\theta}\langle n\,|\,\theta\rangle\langle\theta|e^{-\beta H}|\,n\rangle \\ &= \sum_n \int d\theta^* d\theta(1 - \theta^*\theta)\left(\langle n\,|\,0\rangle + \langle n\,|\,1\rangle\,\theta\right)\left(\langle 0|e^{-\beta H}|\,n\rangle + \theta^*\langle 1|e^{-\beta H}|\,n\rangle\right) \end{aligned}$$

Consequently we have

$$Z(\beta)=\sum_n\int d\theta^* d\theta(1-\theta^*\theta)$$
$$\times\left[\left\langle 0|e^{-\beta H}|\,n\right\rangle \langle n\,|\,0\rangle-\theta^*\theta\left\langle 1|e^{-\beta H}|\,n\right\rangle\langle n\,|\,1\rangle+\theta\left\langle 0|e^{-\beta H}|\,n\right\rangle\langle n\,|\,1\rangle+\theta^*\left\langle 1|e^{-\beta H}|\,n\right\rangle\langle n\,|\,0\rangle\right]. \tag{3.74}$$

Note now that the last term of the integrand does not contribute to the integral and therefore we may substitute θ^* to $-\theta^*$ which implies that

$$\begin{aligned}Z(\beta)&=\sum_n\int d\theta^* d\theta(1-\theta^*\theta)\\&\times\left[\left\langle 0|e^{-\beta H}|\,n\right\rangle \langle n\,|\,0\rangle-\theta^*\theta\left\langle 1|e^{-\beta H}|\,n\right\rangle\langle n\,|\,1\rangle+\theta\left\langle 0|e^{-\beta H}|\,n\right\rangle\langle n\,|\,1\rangle-\theta^*\left\langle 1|e^{-\beta H}|\,n\right\rangle\langle n\,|\,0\rangle\right]\\&=\int d\theta^* d\theta e^{-\theta^*\theta}\left\langle -\theta|e^{-\beta H}|\,\theta\right\rangle.\end{aligned} \tag{3.75}$$

Unlike the bosonic case, we have to impose an anti-periodic boundary over $[0,\beta]$ in the trace since the Grassmann variable is θ when $\tau=0$ and it is $-\theta$ when $\tau=\beta$.
By invoking the expression

$$e^{-\beta H}=\lim_{N\to\infty}(1-\beta H/N)^N \tag{3.76}$$

and inserting the completeness relation (A.60) at each step one has the following expression for the partition function

$$\begin{aligned}Z(\beta)&=\lim_{N\to\infty}\int d\theta^* d\theta e^{-\theta^*\theta}\left\langle -\theta|(1-\beta H/N)^N|\,\theta\right\rangle\\&=\lim_{N\to\infty}\int d\theta^* d\theta\prod_{k=1}^{N-1}d\theta_k^* d\theta_k\exp\left[-\sum_{n=1}^{N-1}\theta_n^*\theta_n\right]\\&\times\langle -\theta|(1-\varepsilon H)|\,\theta_{N-1}\rangle\langle\theta_{N-1}|\cdots|\,\theta_1\rangle\langle\theta_1|(1-\varepsilon H)|\,\theta\rangle\\&=\lim_{N\to\infty}\int\prod_{k=1}^{N}d\theta_k^* d\theta_k\exp\left[-\sum_{n=1}^{N}\theta_n^*\theta_n\right]\\&\times\langle\theta_N|(1-\varepsilon H)|\,\theta_{N-1}\rangle\langle\theta_{N-1}|\cdots|\,\theta_1\rangle\langle\theta_1|(1-\varepsilon H)|-\theta_N\rangle,\end{aligned} \tag{3.77}$$

where we have the usual conventions and we have been using all along

$$\varepsilon=\beta/N\quad\text{and}\quad\theta=-\theta_N=\theta_0,\quad\theta^*=-\overset{*}{\theta}_N=-\overset{*}{\theta}_0. \tag{3.78}$$

Matrix elements are evaluated (up to first order) as

$$\begin{aligned}\langle\theta_k|(1-\varepsilon H)|\,\theta_{k-1}\rangle&=\langle\theta_k|\,\theta_{k-1}\rangle\left[1-\varepsilon\frac{\langle\theta_k|H|\,\theta_{k-1}\rangle}{\langle\theta_k|\,\theta_{k-1}\rangle}\right]\\&=\langle\theta_k|\,\theta_{k-1}\rangle\exp(-\varepsilon\langle\theta_k|H|\,\theta_{k-1}\rangle/\langle\theta_k|\theta_{k-1}\rangle)\\&=e^{\theta_k^*\theta_{k-1}}e^{-\varepsilon\omega(\theta_k^*\theta_{k-1}-1/2)}=e^{\varepsilon\omega/2}e^{(1-\varepsilon\omega)\theta_k^*\theta_{k-1}}.\end{aligned} \tag{3.79}$$

In terms of the path integral the partition function becomes (2.113)

$$\begin{aligned}Z(\beta)&=\lim_{N\to\infty}e^{\beta\omega/2}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k\exp\left[-\sum_{n=1}^{N}\theta_n^*\theta_n\right]\exp\left[(1-\varepsilon\omega)\sum_{n=1}^{N}\theta_n^*\theta_n\right]\\&=e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k\exp\left[-\sum_{n=1}^{N}\left\{\theta_n^*(\theta_n-\theta_{n-1})+\varepsilon\omega\theta_n^*\theta_{n-1}\right\}\right],\end{aligned}$$

and upon simplification of the exponential [1]

$$Z(\beta) = e^{\beta\omega/2} \lim_{N\to\infty} \prod_{k=1}^{N} \int d\theta_k^* d\theta_k \exp(-\theta^{\dagger} \cdot B \cdot \theta), \tag{3.80}$$

where we have the following vector and matrix elements

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}, \quad \theta^{\dagger} = \begin{pmatrix} \theta_1^* & \cdots & \theta_N^* \end{pmatrix}, \quad B_N = \begin{pmatrix} 1 & 0 & \cdots & 0 & -y \\ y & 1 & 0 & \cdots & 0 \\ 0 & y & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y & 1 \end{pmatrix}, \tag{3.81}$$

where $y = -1 + \varepsilon\omega$.
The computation is ended by recalling that from the definition of the Grassmann Gaussian integral one had

$$Z(\beta) = e^{\beta\omega/2} \lim_{N\to\infty} \det B_N = e^{\beta\omega/2} \lim_{N\to\infty} \left[1 + (1-\beta\omega/N)^N\right] = e^{\beta\omega/2}(1+e^{\beta\omega}) = 2\cosh\frac{\beta\omega}{2}. \tag{3.82}$$

As with the bosonic partition function, we can arrive to the same result using the ζ function (a generalization of it), and this will prove useful later.
Recall that we showed that

$$\begin{aligned} Z(\beta) &= e^{\beta\omega/2} \lim_{N\to\infty} \prod_{k=1}^{N} \int d\theta_k^* d\theta_k \exp(-\theta^{\dagger} \cdot B \cdot \theta) \\ &= e^{\beta\omega/2} \int D\theta_k^* D\theta_k \exp\left[-\int_0^{\beta} d\tau \theta^* \left((1-\varepsilon\omega)\frac{d}{d\tau} + \omega\right)\theta\right] \\ &= e^{\beta\omega/2} \mathrm{Det}_{\theta(\beta)=-\theta(0)} \left((1-\varepsilon\omega)\frac{d}{d\tau} + \omega\right). \end{aligned} \tag{3.83}$$

The subscript $\theta(\beta) = -\theta(0)$ indicates that the eigenvalue should be evaluated for the solutions of the anti-periodic boundary condition $\theta(\beta) = -\theta(0)$.
First, we expand the orbit $\theta(\tau)$ in the Fourier modes. The eigenmodes and the corresponding eigenvalues are

$$\exp\left(\frac{\pi i(2n+1)\tau}{\beta}\right), \quad (1-\varepsilon\omega)\frac{\pi i(2n+1)}{\beta} + \omega, \tag{3.84}$$

where n runs as $n = 0, \pm1, \pm2, \cdots$. The number of degrees of freedom is $N(= \beta/\varepsilon)$ so the coherent states are (over)complete. The presence ε in operator will account for the fact that the infinite contribution of the eigenvalues is finite.

3.7 Zeta regularization solution of the fermionic partition function

Since one complex variable has two real degrees of freedom, we need to truncate the product at $-N/4 \le k \le N/4$. Following this prescription, one has

$$Z(\beta) = e^{\beta\omega/2} \lim_{N\to\infty} \prod_{k=-N/4}^{N/4} \left[i(1-\varepsilon\omega)\frac{\pi(2n-1)}{\beta} + \omega\right]$$

$$
\begin{aligned}
&= e^{\beta\omega/2} e^{-\beta\omega/2} \prod_{k=1}^{\infty} \left[\left(\frac{2\pi(n-1/2)}{\beta} \right)^2 + \omega^2 \right] \\
&= \prod_{k=1}^{\infty} \left[\frac{\pi(2k-1)}{\beta} \right]^2 \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{\pi(2n-1)} \right)^2 \right]
\end{aligned} \tag{3.85}
$$

The trouble comes from the first infinite product, Ξ, which is divergent and as such it is in need of ζ values to become finite. This can be accomplished as follows

$$
\log \Xi = 2 \sum_{k=1}^{\infty} \log \left[\frac{2\pi(k-1/2)}{\beta} \right], \tag{3.86}
$$

and we define the corresponding ζ function by (which is the Hurwitz ζ function)

$$
\zeta_{\text{fermion}}(s) = \sum_{k=1}^{\infty} \left[\frac{2\pi(k-1/2)}{\beta} \right]^{-s} = \left(\frac{\beta}{2\pi} \right)^s \zeta(s, 1/2), \tag{3.87}
$$

where (see Chapter 2, Eq 2.11)

$$
\zeta(s,a) = \sum_{k=0}^{\infty} (k+a)^{-s},
$$

where $0 < a < 1$. This gives

$$
\Xi = \exp(-2\zeta'_{\text{fermion}}(0)). \tag{3.88}
$$

So now we are left with the issue of differentiating the ζ_{fermion} function at $s = 0$ which is done as follows [1]

$$
\zeta'_{\text{fermion}}(0) = \log \left(\frac{\beta}{2\pi} \right) \zeta(0, 1/2) + \zeta'(0, 1/2) = -\frac{1}{2} \log 2, \tag{3.89}
$$

since we showed in Chapter 2 (Theorem 10) that

$$
\zeta(0, \tfrac{1}{2}) = 0, \quad \zeta'(0, \tfrac{1}{2}) = -\tfrac{1}{2} \log 2.
$$

Putting all of this together, we obtain the surprising result

$$
\Xi = \exp(-2\zeta'_{\text{fermion}}(0)) = e^{\log 2} = 2. \tag{3.90}
$$

This result indicates that Ξ is independent of β once the regularization is performed. Finally, the partition function is evaluated using all these facts

$$
Z(\beta) = 2 \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{\pi(2n-1)} \right)^2 \right] = 2 \cosh \frac{\beta\omega}{2}, \tag{3.91}
$$

by the virtue of the formula

$$
\prod_{n=1}^{\infty} \left[1 + \left(\frac{x}{\pi(2n-1)} \right)^2 \right] = \cosh \frac{x}{2}. \tag{3.92}
$$

Note the similarity between (3.35) and (3.92) as well as (3.62) and (3.91) for the bosonic and fermionic cases respectively.
Richard Feynman was an advocate of using solutions of known problems in unknown problems, quoting him '*The same equations have the same solutions*'. The rationale behind this statement is once we solve a mathematical problem, we can re-use the solution in another physical situation. Feynman was skilled in transforming a problem into one that he could solve. This is precisely what we have done in this case. We can prove (3.92) using path integrals.

REFERENCES
The discussion of the role of the ζ function in quantum mechanics and zeta function regularization as explained in the above chapter can be traced back to
- [1] Hagen Kleinert's *Path Integrals in Quantum Mechanics* [pages 81 to 83, 161 to 163 and 600 to 614]
- [2] Peskin and Schroeder [pages 299 to 301] and
- [3] *Advanced Quantum Field Theory* course by Rajantie for the development of Grassmann numbers.

It is also complemented from
- [4] C. Grosche and F. Steiner *Handbook of Feynman Path Integrals* [pages 37 to 44, 55 to 59 for the fermionic case] and
- [5] EOR's [p 65 to 67] for the fermionic case.

4 Dimensional Regularization

4.1 Generating functional and probability amplitudes in the presence of a source J

From the quantum mechanical case we can build a generalization with several degrees of freedom: a field theory. We will exclusively deal with ϕ^4, where $\phi(x)$ is a real scalar field. Let us summarize standard results from field theory from Peskin and Schroder [2] as well as Rajantie [3]. The action is built from the Lagrangian

$$S = \int dx\, \mathscr{L}(\phi(x), \partial_\mu \phi(x)), \tag{4.1}$$

where it is understood that L is the Lagrangian density. The equations of motion (EOM) are given by the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi(x))} = -\frac{\partial \mathscr{L}}{\partial \phi(x)}. \tag{4.2}$$

From the free scalar field Lagrangian

$$\mathscr{L}_0(\phi(x), \partial_\mu \phi(x)) = -\frac{1}{2}(\partial_\mu \partial^\mu \phi + m^2 \phi^2) \tag{4.3}$$

we can derive the Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi = 0. \tag{4.4}$$

When there is a source J present the vacuum amplitude has functional representation

$$\langle 0, \infty | 0, -\infty \rangle_J = Z[J] = N \int \mathscr{D}\phi \exp\left[i \int dx \left(\mathscr{L}_0 + J\phi + \frac{i}{2}\varepsilon \phi^2 \right) \right], \tag{4.5}$$

with the artificial $i\varepsilon$ is added to make sure the integral converges. We can think of J as driving force, i.e. at any time we are allowed to drive the system in any arbitrary way and measure the response. Integrating by parts we obtain

$$\begin{aligned} Z[J] &= \int \mathscr{D}\phi \exp\left[i \int dx \left(\mathscr{L}_0 + J\phi + \frac{i}{2}\varepsilon\phi^2 \right) \right] \\ &= \int \mathscr{D}\phi \exp\left[i \int dx \left(\frac{1}{2} \left\{ \phi(\partial_\mu \partial^\mu \phi - m^2)\phi + i\varepsilon\phi^2 \right\} + J\phi \right) \right]. \end{aligned} \tag{4.6}$$

In this case, the Klein-Gordon equation becomes the slightly more generalized equation

$$(\partial_\mu \partial^\mu - m^2 + i\varepsilon)\phi_c = -J. \tag{4.7}$$

Working in d dimensions and defining the Feynman propagator as

$$\Delta(x - y) = \frac{-1}{(2\pi)^d} \int d^d k \frac{e^{ik(x-y)}}{k^2 + m^2 - i\varepsilon} \tag{4.8}$$

the solution to the generalized Klein-Gordon equation becomes

$$\phi_c(x) = -\int dy \Delta(x - y) J(y). \tag{4.9}$$

The Feynman propagator obeys

$$(\partial_\mu \partial^\mu - m^2 + i\varepsilon)\Delta(x-y) = \delta^d(x-y). \tag{4.10}$$

Hence the vacuum amplitude can be written in terms of the source J as

$$\langle 0, \infty | 0, -\infty \rangle_J = N \exp\left[-\frac{i}{2}\int dxdy J(x)\Delta(x-y)J(y)\right] \tag{4.11}$$

or

$$Z_0[J] = Z_0[0] \exp\left[-\frac{i}{2}\int dxdy J(x)\Delta(x-y)J(y)\right]. \tag{4.12}$$

by setting $\langle 0, \infty | 0, -\infty \rangle_J := Z_0[J]$. The Feynman propagator is also computed by the functional derivative of $Z_0[J]$

$$\Delta(x-y) = \left.\frac{i}{Z_0[0]}\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)}\right|_{J=0}. \tag{4.13}$$

In order to evaluate $Z_0[0]$ (which is the vacuum to vacuum amplitude when there is no source) we need to introduce imaginary time $x^4 = t = ix^0$ and operator $\bar{\partial}_\mu \bar{\partial}^\mu = \partial_\tau^2 + \nabla^2$ so that

$$Z_0[0] = \int \bar{\mathscr{D}}\phi \exp\left[\frac{1}{2}\int dx \phi(\bar{\partial}_\mu \bar{\partial}^\mu - m^2)\phi\right] = \frac{1}{\sqrt{\mathrm{Det}(\bar{\partial}_\mu \bar{\partial}^\mu - m^2)}}, \tag{4.14}$$

with capital d, the determinant is the product of eigenvalues with corresponding boundary condition. With term sources, the Lagrangian of the free complex scalar field takes the form

$$\mathscr{L}_0 = -\partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 + J\phi^* + J^*\phi, \tag{4.15}$$

and consequently the generating functional becomes

$$\begin{aligned} Z_0[J, J^*] &= \int \mathscr{D}\phi \mathscr{D}\phi^* \exp\left[i\int dx(\mathscr{L}_0 - i\varepsilon |\phi|^2)\right] \\ &= \int \mathscr{D}\phi \mathscr{D}\phi^* \exp\left[i\int dx \phi^*(\partial_\mu \partial^\mu - m^2 - i\varepsilon)\phi + J\phi^* + J^*\phi\right], \end{aligned} \tag{4.16}$$

differentiating we obtain the propagator

$$\Delta(x-y) = \left.\frac{i}{Z_0[0,0]}\frac{\delta^2 Z_0[J, J^*]}{\delta J^*(x)\delta J(y)}\right|_{J=J^*=0}. \tag{4.17}$$

We may split the function by virtue of the Klein-Gordon equations $(\partial_\mu \partial^\mu - m^2)\phi = -J$ and $(\partial_\mu \partial^\mu - m^2)\phi^* = -J^*$

$$Z_0[J, J^*] = Z_0[0,0] \exp\left[-i\int dxdy J^*(x)\Delta(x-y)J(y)\right] \tag{4.18}$$

and by using another Wick rotation we have

$$Z_0[0,0] = \int \mathscr{D}\phi \mathscr{D}\phi^* \exp\left[-i\int dx \phi^*(\partial_\mu \partial^\mu - m^2 - i\varepsilon)\phi\right] = \frac{1}{\mathrm{Det}(\bar{\partial}_\mu \bar{\partial}^\mu - m^2)}. \tag{4.19}$$

The presence of a potential in the Lagrangian

$$\mathscr{L}(\phi, \partial_\mu \phi) = \mathscr{L}_0(\phi, \partial_\mu \phi) - V(\phi) \tag{4.20}$$

comes at a double price: the form of the potential is limited by symmetry and renormalization of the theory and this theory needs to be handled perturbatively. The potential is usually of the form $V(\phi) = \frac{\alpha}{n!}\phi^n$ where α is a real number that sets the strength of the interaction and $n > 2$ is an integer. As with the free theory, the generating functional is [2], [4]

$$\begin{aligned} Z[J] &= \int \mathscr{D}\phi \exp\left[i \int dx \left(\frac{1}{2}\phi(\partial_\mu \partial^\mu - m^2)\phi - V(\phi) + J\phi\right)\right] \\ &= \int \mathscr{D}\phi \exp\left[-i \int dx V(\phi)\right] \exp\left[i \int dx \left(\mathscr{L}_0(\phi, \partial_\mu \phi) + J\phi\right)\right] \\ &= \exp\left[-i \int dx V\left(-i\frac{\delta}{\delta J(x)}\right)\right] \int \mathscr{D}\phi \exp\left[i \int dx \left(\mathscr{L}_0(\phi, \partial_\mu \phi) + J\phi\right)\right] \\ &= \sum_{k=0}^{\infty} \int dx_1 \cdots \int dx_k \frac{(-i)^k}{k!} V\left(-i\frac{\delta}{\delta J(x_1)}\right) \cdots V\left(-i\frac{\delta}{\delta J(x_k)}\right) Z_0[J]. \end{aligned} \tag{4.21}$$

The Green function (which is the vacuum expectation of the order time product of field operators)

$$G_n(x_1, \cdots, x_n) := \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle = \left.\frac{(-i)^n \delta^n}{\delta J(x_1)\cdots\delta J(x_n)} Z[J]\right|_{J=0} \tag{4.22}$$

is generated by the generating functional $Z[J]$.
However, we can see that this is the nth functional derivative of $Z[J]$ around $J = 0$ and therefore we may plug it into the Taylor expansion of the exponential above and we obtain

$$Z[J] = \sum_{k=1}^{\infty} \frac{1}{k!} \left[\prod_{i=1}^{n} \int dx_i J(x_i)\right] \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle = \left\langle 0 \left| T \exp \int dx J(x)\phi(x) \right| 0 \right\rangle. \tag{4.23}$$

Connected n-point functions are generated by

$$Z[J] = \exp(-W[J]), \tag{4.24}$$

and the effective action is defined by a Legendre transformation as follows

$$\Gamma[\phi_{\text{cl}}] := W[J] - \int d\tau dx^i J \phi_{\text{cl}} \tag{4.25}$$

where

$$\phi_{\text{cl}} := \langle \phi \rangle_J = \frac{\delta W[J]}{\delta J}. \tag{4.26}$$

We will also see that $\Gamma[\phi_{\text{cl}}]$ generates 1-particle irreducible diagrams (see Bailin and Love [1] and Ramond [4]).
It is convenient now to derive the above discussion in a formal manner and with a closer

analogy to statistical mechanics. The generating functional of correlation functions for a field theory with Lagrangian L is given by (4.5)

$$Z[J] = \int \mathscr{D}\phi \exp\left[i \int d^4x(\mathscr{L} + J\phi)\right], \tag{4.27}$$

where the time variable is contained between -T and T, with $T \to \infty(1 - i\varepsilon)$. Furthermore we have the following

$$\begin{aligned}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathscr{D}\phi\phi(x_1)\phi(x_2)\exp\left[i\int_{-T}^{T} d^4x\mathscr{L}\right]}{\int \mathscr{D}\phi \exp\left[i\int_{-T}^{T} d^4x\mathscr{L}\right]} \\ &= Z[J]^{-1}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\bigg|_{J=0}.\end{aligned} \tag{4.28}$$

Let us do some manipulations on the time variable; when we derived the path integral formulation of quantum mechanics (see AQFT [3]) it was shown that the time integration was tilted into the complex plane in the direction that would allow the contour of integration to be rotated clockwise onto the imaginary axis. We assumed that the original infinitesimal rotation gives the correct imaginary infinitesimal to produce the Feynman propagator.
Now, the wick rotation of the time coordinate $t \to -ix^0$ yields a Euclidean 4-vector product

$$x^2 = t^2 - |\mathbf{x}|^2 \to -(x^0)^2 - |\mathbf{x}|^2 = -|x_E|^2, \tag{4.29}$$

and similarly we assume that the analytic continuation of the time variables in any Green's function of a quantum field theory produces a correlation function *invariant* under the rotational symmetry of four-dimensional Euclidean space.

4.2 Functional energy, action and potential and the classical field $\phi_{\text{cl}}(x)$

Let us now apply this to the ϕ^4 theory. As we know the action in this case is

$$S = \int d^4x(\mathscr{L} + J\phi) = \int d^4x\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right), \tag{4.30}$$

and performing the Wick rotation

$$i\int d^4x_E(\mathscr{L}_E - J\phi) = i\int d^4x_E\left(\frac{1}{2}(\partial_{E\mu}\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - J\phi\right), \tag{4.31}$$

which in turn gives the Wick-rotated generating functional

$$Z[J] = \int \mathscr{D}\phi \exp\left[-\int d^4x_E(\mathscr{L}_E - J\phi)\right]. \tag{4.32}$$

The functional $\mathscr{L}_E[\phi]$ is bounded from below and when the field ϕ has large amplitude or large gradient the functional becomes large. These two facts would imply that $L_E[\phi]$ has the form of an energy and consequently it is a possible candidate for a statistical weight for the fluctuations of ϕ.
Within this light, the Wick rotated functional $Z[J]$ is the partition function describing the statistical mechanics of a macroscopic system when approximating the fluctuating variable

as a field.
Finally, let us push this analogy between field theory and statistical mechanics further by presenting the Green's function of $\phi(x_E)$ [2]

$$\langle\phi(x_{E1})\phi(x_{E2})\rangle = \int \frac{d^4k_E}{(2\pi)^4}\frac{e^{ik_E\cdot(x_{E1}-x_{E2})}}{k_E^2+m^2}, \tag{4.33}$$

which is in fact the Feynman propagator evaluated in the spacelike region and this falls off as $\exp(-m|x_{E1}-x_{E2}|)$. This correspondence between quantum field theory and statistical mechanics plays an important part in understanding ultraviolet divergences.
Recalling the generating functional of correlation functions, we define an energy functional $E[J]$ by

$$Z[J] = \exp(-iE[J]) = \int \mathscr{D}\phi \exp\left[i\int d^4x(\mathscr{L}+J\phi)\right] = \left\langle\Omega|e^{-iHT}|\Omega\right\rangle, \tag{4.34}$$

with the constraints on time explained above. Note the $-i$ factor in the exponential in contrast to (4.24). The functional $E[J]$ is, as we have said before, the vacuum energy as a function of the external source J. Let us perform the functional derivative of $E[J]$ with respect to $J(x)$

$$\begin{aligned}\frac{\delta}{\delta J(x)}E[J] &= i\frac{\delta}{\delta J(x)}\log Z\\ &= -\left(\int \mathscr{D}\phi \exp\left[i\int d^4x(\mathscr{L}+J\phi)\right]\right)^{-1}\int \mathscr{D}\phi\phi(x)\exp\left[i\int d^4x(\mathscr{L}+J\phi)\right]\end{aligned} \tag{4.35}$$

and set

$$\frac{\delta}{\delta J(x)}E[J] = -\langle\Omega|\phi(x)|\Omega\rangle_J \tag{4.36}$$

the vacuum expectation value in the presence of a source J.
Next, we define
- the classical field as

$$\phi_{\mathrm{cl}}(x) = \langle\Omega|\phi(x)|\Omega\rangle_J\,, \tag{4.37}$$

a weighted average over all possible fluctuations, and dependent on the source J.
- the effective action as the Legendre transformation of $E[J]$ i.e. as in (4.25)

$$\Gamma[\phi_{\mathrm{cl}}] := -E[J] - \int d^4x' J(x')\phi_{\mathrm{cl}}(x'). \tag{4.38}$$

By virtue of (4.36) we have the following

$$\begin{aligned}\frac{\delta}{\delta\phi_{\mathrm{cl}}(x)}\Gamma[\phi_{\mathrm{cl}}] &= -\frac{\delta}{\delta\phi_{\mathrm{cl}}(x)}E[J] - \int d^4x'\frac{\delta J(x')}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(x') - J(x)\\ &= -\int d^4x'\frac{\delta J(x')}{\delta\phi_{\mathrm{cl}}(x)}\frac{\delta E[J]}{\delta J(x')} - \int d^4x'\frac{\delta J(x')}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(x') - J(x) = -J(x)\end{aligned} \tag{4.39}$$

which means that when the source is set to zero, the equation

$$\frac{\delta}{\delta\phi_{\mathrm{cl}}(x)}\Gamma[\phi_{\mathrm{cl}}] = 0 \tag{4.40}$$

is satisfied by the effective action. This equation has solutions which are the values of $\langle\phi(x)\rangle$ in the stable states of the theory. It will be assumed that the possible vacuum states are invariant under translation and Lorentz transformations. This implies a substantial simplification of (4.40) as for each possible vacuum state the corresponding solution ϕ_{cl} will be independent of x, and hence it is just solving an ODE of one variable.
Thermodynamically, Γ is proportional to the volume of the spacetime region over which the functional integral is taken, and therefore, it can be a large quantity. Consequently, in terms of volume V and associated T of the region we may write

$$\Gamma[\phi_{\text{cl}}] = -(VT)V_{\text{eff}}(\phi_{\text{cl}}), \tag{4.41}$$

where V_{eff} is the effective *potential*. In order that $\Gamma[\phi_{\text{cl}}]$ have an extremum we need the following to hold

$$\frac{\delta}{\delta\phi_{\text{cl}}}V_{\text{eff}}(\phi_{\text{cl}}) = 0. \tag{4.42}$$

Each solution of (4.42) is a translation invariant state without source, i.e $J = 0$. Therefore, the effective action (4.38) is $-E$ in this case ($\Gamma = -E$) and consequently $V_{\text{eff}}(\phi_{\text{cl}})$ evaluated at the solution of (4.42) is the energy density of the corresponding state.
The effective potential defined by (4.41) and (4.42) yields a function whose minimization defines the exact vacuum sate of the field theory including all effects of quantum corrections. The evaluation of $V_{\text{eff}}(\phi_{\text{cl}})$ will follow from the path integral formulation. In order to accomplish this we will follow Peskin and Schroeder's method which in turn follows from R. Jackiw [5] and dates back to 1974. The idea is to compute the effective action Γ directly from its path integral definition and then obtain V_{eff} by focusing on constant values of ϕ_{cl}. Because we are using renormalized perturbation theory, the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \tag{4.43}$$

ought to be split as

$$\mathscr{L} = \mathscr{L}_1 + \delta\mathscr{L}, \tag{4.44}$$

which is analogous to the split of the Lagrangian in renormalized ϕ^4 theory done in AQFT, see Rajantie [3] and Bailin and Love Chp 7.4 [1], i.e. rescaling the field $\phi = Z^{1/2}\phi_r$ where Z is the residue in the LSZ reduction formula

$$\int d^4x\,\langle\Omega \mid T\phi(x)\phi(0)|\,\Omega\rangle\, e^{ip\cdot x} = \frac{iZ}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2), \tag{4.45}$$

with m being the physical mass. This rescale changes the Lagrangian into

$$L = \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m_0^2\phi_r^2 - \frac{\lambda_0}{4!}\phi_r^4 + \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4, \tag{4.46}$$

by the use of $\delta_Z = Z - 1$, $\delta_m = m_0^2 Z - m^2$ and $\delta_\lambda = \lambda^2 Z - \lambda$ where m and λ are physically measured. The last three terms are known as the counter terms and they take into account the infinite and unobservable shifts between the bare parameters and the physical parameters.
At the lowest order in perturbation theory the relationship between the source and the classical field is

$$\left.\frac{\delta\mathscr{L}}{\delta\phi}\right|_{\phi=\phi_{\text{cl}}} + J(x) = 0. \tag{4.47}$$

Because the functional $Z[J]$ depends on $\phi_{\rm cl}$ through its dependence on J our goal is to compute Γ as a function of $\phi_{\rm cl}$. This will be the starting point.
Next, we define J_1 to be the function that exactly satisfies the classical field equation above for higher orders, i.e. when $\mathscr{L} = \mathscr{L}_1$

$$\left.\frac{\delta \mathscr{L}_1}{\delta \phi}\right|_{\phi=\phi_{\rm cl}} + J_1(x) = 0, \tag{4.48}$$

and the difference between both sources J and J_1 will be written as (see Peskin and Schroeder 11.4)

$$J(x) = J_1(x) + \delta J(x), \tag{4.49}$$

where δJ has to be determined, order by order in perturbation theory by use of (4.37), that is by using the equation $\langle \phi(x) \rangle_J = \phi_{\rm cl}(x)$. We may now write (4.34) as

$$e^{-iE[J]} = \int \mathscr{D}\phi \exp\left(i \int d^4x (\mathscr{L}_1[\phi] + J_1 \phi)\right) \exp\left(i \int d^4x (\delta\mathscr{L}[\phi] + \delta J \phi)\right), \tag{4.50}$$

where all the counter terms are in the second exponential. Let us concentrate on the first exponential first. Expanding the exponential about $\phi(x) = \phi_{\rm cl}(x) + \eta(x)$ yields

$$\begin{aligned}\int d^4x (\mathscr{L}_1[\phi] + J_1\phi) = & \int d^4x (\mathscr{L}_1[\phi_{\rm cl}] + J_1\phi_{\rm cl}) + \int d^4x \eta(x) \left(\frac{\delta \mathscr{L}_1}{\delta\phi} + J_1\right) \\ & + \frac{1}{2!} \int d^4x d^4y \eta(x)\eta(y) \left(\frac{\delta^2 \mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right) \\ & + \frac{1}{3!} \int d^4x d^4y d^4z \eta(x)\eta(y)\eta(z) \left(\frac{\delta^3 \mathscr{L}_1}{\delta\phi(x)\delta\phi(y)\delta\phi(z)}\right) + \cdots \end{aligned} \tag{4.51}$$

and it is understood that the functional derivatives of $\mathscr{L}_1$ are evaluated at $\phi_{\rm cl}$.
The second integral on the RHS vanishes by (4.48) and therefore the integral over η is a Gaussian integral, where the perturbative corrections are given by the cubic and higher terms.
Let us assume that the coefficients of (4.51) (i.e. the successive functional derivatives of $\mathscr{L}_1$) give well-defined operators. If we keep only terms up to quadratic order in η and we only focus of the first integral of (4.50) we find that there is a pure Gaussian integral which can be evaluated in terms of a functional determinant as we have computed in the Appendix

$$\begin{aligned}& \int \mathscr{D}\eta \exp\left[i \left(d^4x (\mathscr{L}_1[\phi_{\rm cl}] + J_1\phi_{\rm cl}\right) + \frac{1}{2}\int d^4x d^4y \eta \frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\eta\right] \\ & = \exp\left[i \int d^4x (\mathscr{L}_1[\phi_{\rm cl}] + J_1\phi_{\rm cl})\right] \left(\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right]\right)^{-1/2}, \end{aligned} \tag{4.52}$$

the lowest-order quantum correction to the effective action is given by the determinant. If we now consider the second integral of (4.50) which consists of the counter terms of the Lagrangian and expanding as we have done before we have

$$(\delta\mathscr{L}[\phi_{\rm cl}] + \delta J\phi_{\rm cl}) + (\delta\mathscr{L}[\phi_{\rm cl} + \eta] - \delta\mathscr{L}[\phi_{\rm cl}] + \delta J\eta). \tag{4.53}$$

The cubic and higher order terms in η in (4.51) produce Feynman diagram expansion of the functional integral in (4.50) in which the propagator is the inverted operator

$$-i\left(\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right)^{-1}, \tag{4.54}$$

and hence when the second term in (4.53) is expanded as a Taylor series in η the successive terms give counter term vertices which can be included in the above mentioned Feynman diagrams. The first term is a constant with respect to the integral over η thus it gives additional terms in the exponent of (4.52).

Taking (4.52) with the contributions from higher order vertices and counter terms together we obtain an expression for the functional integral (4.50). Feynman diagrams representing the higher order terms can be arranged in such a way that they yield the exponential of the sum of the connected diagrams, obtaining the expression for $E[J]$

$$\begin{aligned}-iE[J] = i\int d^4x(\mathscr{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}}) - \frac{1}{2}\log\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right] + (\text{connected diagrams})\\ + i\int d^4x(\delta\mathscr{L}[\phi_{\text{cl}}] + \delta J\phi_{\text{cl}}),\end{aligned} \tag{4.55}$$

and finally by virtue of (4.49) and (4.38) we finally have [2]

$$\Gamma[\phi_{\text{cl}}] = \int d^4x\mathscr{L}_1[\phi_{\text{cl}}] + \frac{i}{2}\log\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right] - i(\text{connected diagrams}) + \int d^4x(\delta\mathscr{L}[\phi_{\text{cl}}]), \tag{4.56}$$

or by (4.44)

$$\Gamma[\phi_{\text{cl}}] = \int d^4x\mathscr{L}[\phi_{\text{cl}}] + \frac{i}{2}\log\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right] - i(\text{connected diagrams}).$$

This is indeed the expression we were seeking since Γ is a function of ϕ_{cl}, taking away the J dependence. The Feynman diagrams in the expression for Γ have no external lines and they all contain at least two loops. The last term of (4.56) gives the counter terms that are needed for the renormalization conditions on Γ and cancel the divergences that appear in the evaluation of the determinant and the diagrams. We shall ignore any one-particle irreducible one-point diagram (these diagrams are cancelled by the adjustment of δJ).

As a side point, note how a calculation of the sort

$$\begin{aligned}\int\frac{d^dk}{(2\pi)^d}\log(-k^2+m^2) &= i\int\frac{d^dk_{\text{Euc}}}{(2\pi)^d}\log(k_{\text{Euc}}^2+m^2) = -i\frac{\partial}{\partial x}\int\frac{d^dk_{\text{Euc}}}{(2\pi)^d}\,(k_{\text{Euc}}^2+m^2)^{-x}\Big|_{x=0}\\ &= -i\frac{\partial}{\partial x}\left(\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(x-d/2)}{\Gamma(x)}\frac{1}{m^{2(x-d/2)}}\right)\Bigg|_{x=0} = -im^d\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\end{aligned} \tag{4.57}$$

yields

$$\begin{aligned}\log\det(\partial^2+m^2) = \operatorname{tr}\log(\partial^2+m^2) = \sum_k\log(-k^2+m^2)\\ = (VT)\int\frac{d^4k}{(2\pi)^4}\log(k^2+m^2) = -i(VT)\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}m^d\end{aligned}$$

with V (volume) and T (time) as explained before. We could call this Γ renormalization. More will be said about it later on.

4.3 Derivation of φ^4 potential at $\phi_{\rm cl}(x)$

In the Unification course, it was shown that spontaneous symmetry breaking requires the development of a vacuum expectation value (VEV) from the scalar field. Furthermore, the VEV is determined by the minimisation of the effective potential,

$$\frac{dV}{d\varphi_{\rm cl}} = 0. \tag{4.58}$$

The effective potential is given exclusively by the potential V of the Lagrangian if no quantum effects are taken into account. Perturbation theory allows us to place the quantum terms, however this would clash with the non-perturbative nature of spontaneous symmetry breaking. The alternative parameter is the loop expansion which we now describe.
Let us re-write the theory around (4.24) as follows. With a generating functional X of the connected Green functions in a scalar field theory,

$$Z[J] = \exp(i\hbar^{-1}X[J]) = N'\int \mathscr{D}\varphi \exp\left(i\hbar^{-1}\int d^4x(\mathscr{L} + J\varphi)\right) \tag{4.59}$$

with normalization constant N' chosen so that

$$Z[0] = 1 \quad X[0] = 0. \tag{4.60}$$

One-particle-irreducible Green functions $\Gamma^{(n)}$ are generated by the effective action $\Gamma[\varphi_{\rm cl}]$ by the use of [1], [2], [3], [4]

$$\Gamma[\varphi_{\rm cl}] = \sum_{n=1}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots\int d^4x_n\Gamma^{(n)}(x_1,\cdots,x_n)\varphi_{\rm cl}(x_1)\cdots\varphi_{\rm cl}(x_n), \tag{4.61}$$

and we note the equation

$$G^{(N)}(x_1,x_2,\cdots,x_N) = \left.\frac{1}{i^N}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N}Z[J]\right|_{J=0}.$$

In (4.59) the factor $\hbar^{-1}$ multiplies the whole Lagrangian (not just the interaction part) each of the V vertices in any diagram will carry a $\hbar^{-1}$ factor and each of the I internal lines will carry a $\hbar$ factor. Each of the E external lines in Green functions $\tilde{G}^{(E)}$ has a propagator. With this information, there overall factor is

$$\hbar^{-V+I+E} = \hbar^{L+1-E}, \tag{4.62}$$

by virtue of $L = I - V + 1$ and noting that E is the number of external lines. Furthermore, there is a factor of $\hbar^{L-E}$ in any diagram in expansion of $\hbar^{-1}X$. The one-particle-irreducible Green functions $\tilde{\Gamma}^{(E)}$ have no propagators, hence the multiplying factor is only $\hbar^L$. This means that the power of the $\hbar$ indicates the number of *loops*. When there are no loops ($L = 0$) the only non-vanishing $\tilde{\Gamma}^{(E)}$ are

$$\tilde{\Gamma}^{(2)}(p,-p) = p^2 - \mu^2 \tag{4.63}$$

and

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = -\lambda, \tag{4.64}$$

which gives the approximation to the potential (see Bailin and Love 4.4)

$$V(\varphi_{\rm cl}) = -\sum_{n=1}^{\infty} \frac{i^n}{n!}\tilde{\Gamma}^{(n)}(0,\cdots,0)\varphi_{\rm cl}^{\rm n} \Rightarrow V_0(\varphi_{\rm cl}) = \frac{1}{2}\mu^2\varphi_{\rm cl}^2 + \frac{1}{4!}\lambda\varphi_{\rm cl}^4, \tag{4.65}$$

as classically expected without quantum corrections. Note, however, that the role of $\hbar$ is not central to the discussion as no assumptions about its size have been made, it is only an expansion parameter. Bailin and Love [201] give same effective action $\Gamma[\varphi_{\rm cl}]$ with

$$\begin{aligned}\Gamma[\varphi_{\rm cl}] = -i\log N - \frac{1}{8}\lambda\int dx[i\Delta_F(0)]^2 - \frac{1}{2}\int dx\varphi_{\rm cl}(x)(\partial^\mu\partial_\mu + \mu^2)\varphi_{\rm cl}(x)\\ - \frac{1}{4}\lambda i\Delta_F(0)\int dx\varphi_{\rm cl}^2(x) - \frac{1}{24}\lambda\int dx\varphi_{\rm cl}^4(x) + O(\lambda^2).\end{aligned} \tag{4.66}$$

The terms that contain a Feynman propagator $\Delta_F(0)$ come from divergent loop integrals and hence they do not contribute in zeroth order, consequently we may write

$$\Gamma_0[\varphi_{\rm cl}] = -\frac{1}{2}\int dx\varphi_{\rm cl}(x)(\partial^\mu\partial_\mu + \mu^2)\varphi_{\rm cl}(x) - \frac{1}{4!}\lambda\int dx\varphi_{\rm cl}^4(x) \tag{4.67}$$

again taking into account that $N = 1$ implying $\Gamma[0] = 0$.
The first order accuracy of $\hbar$, the loop expansion that is, of the effective potential V and effective action Γ can be computed by writing

$$\varphi(x) = \varphi_0(x) + \tilde{\varphi}(x), \tag{4.68}$$

with φ_0 being the zeroth order approximation to $\varphi_{\rm cl}$. Hence this shift in the functional integration variable must satisfy the following EOM

$$(\partial^\mu\partial_\mu + \mu^2)\varphi_0(x) + \frac{\lambda}{6}\varphi_0^3(x) = J(x). \tag{4.69}$$

When we plug this change into the Lagrangian density

$$\mathscr{L}(\varphi) = \frac{1}{2}(\partial^\mu\varphi)(\partial_\mu\varphi) - \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \tag{4.70}$$

we have the following integral

$$\begin{aligned}\int d^4x(\mathscr{L} + J\varphi) = &\int d^4x(\mathscr{L}(\varphi_0(x)) + J\varphi_0)\\ &+ \int d^4x\left((\partial^\mu\tilde{\varphi})(\partial_\mu\varphi_0) - \mu^2\tilde{\varphi}\varphi_0 - \frac{1}{6}\lambda\tilde{\varphi}\varphi_0^3 + J\tilde{\varphi}\right)\\ &+ \int d^4x\left(L_2(\tilde{\varphi},\varphi_0) - \frac{1}{6}\lambda\tilde{\varphi}^3\varphi_0 - \frac{1}{4}\lambda\tilde{\varphi}^4\right)\end{aligned} \tag{4.71}$$

with $\mathscr{L}_2$ accounting for all the quadratic leftovers in $\tilde{\varphi}$ when the shift is performed, i.e.

$$\mathscr{L}_2 = \frac{1}{2}(\partial^\mu\tilde{\varphi})(\partial_\mu\tilde{\varphi}) - \frac{1}{2}\mu^2\tilde{\varphi}^2 - \frac{1}{4}\lambda\varphi_0^2\tilde{\varphi}^2. \tag{4.72}$$

Now, since $\tilde{\varphi}$ minimises the classical action then the linear term in $\tilde{\varphi}$ of (4.71) disappears. Our next step is to re-scale the integration variable as follows

$$\tilde{\varphi} = \hbar^{1/2}\varphi \tag{4.73}$$

And plug this back into (4.59) yielding

$$Z[J] = N' \exp i\hbar^{-1} \int d^4x[\mathscr{L}(\varphi_0) + J\varphi_0] \\ \times \int \mathscr{D}\varphi \exp i \int d^4x \left(\mathscr{L}_2(\varphi, \varphi_0) - \frac{\lambda}{6}\hbar^{1/2}\varphi^3\varphi_0 - \frac{1}{24}\lambda\hbar\varphi^4 \right) \tag{4.74}$$

We can recover a Gaussian integral out of this by noting that we are only interested in the first-order corrections, hence terms proportional to $\hbar^{1/2}$ and $\hbar$ may be discarded. This procedure gives

$$\int d^4x \mathscr{L}_2(\varphi, \varphi_0) = -\frac{1}{2} \int d^4x d^4x' \varphi(x') A(x', x, \varphi_0) \varphi(x) \tag{4.75}$$

with

$$A(x', x, \varphi_0) = \left[-\partial_{x'\mu}\partial_x^\mu + \mu^2 + \frac{1}{2}\lambda\varphi_0^2 \right] \delta(x' - x), \tag{4.76}$$

finally obtaining for the value of Z

$$Z[J] \approx N' \exp i\hbar^{-1} \int d^4x[\mathscr{L}(\varphi_0) + J\varphi_0] \exp\left[-\frac{1}{2} \mathrm{tr} \log A(x', x, \varphi_0) \right]. \tag{4.77}$$

By our conditions (4.60) and $\varphi_0[0] = 0$ we then obtain

$$Z[0] = 1 \approx N' \exp\left[-\frac{i}{2} \mathrm{tr} \log A(\mathbf{x}', \mathbf{x}, 0) \right]. \tag{4.78}$$

Here is where we can choose method. We will, instead of using ζ regularization, work out the terms in the expansion.
Comparison with (4.59) shows that by keeping the same definition of A we then have

$$X_0[J] = \int d^4x[\mathscr{L}(\varphi_0) + J\varphi_0] \tag{4.79}$$

and

$$X_1[J] = \frac{i}{2} \mathrm{tr} \log \frac{A(x', x, \varphi_0)}{A(x', x, 0)}. \tag{4.80}$$

The effective action is computed by expanding [1], [4]

$$\Gamma[\varphi_{\mathrm{cl}}] = \Gamma_0[\varphi_{\mathrm{cl}}] + \hbar\Gamma_1[\varphi_{\mathrm{cl}}] + O(\hbar^2). \tag{4.81}$$

Now, since φ_{cl} is a functional of J, then

$$\varphi_{\mathrm{cl}}(x) = \frac{\delta X[J]}{\delta J(x)}, \tag{4.82}$$

we have the following expansion

$$\Gamma_0[\varphi_{\mathrm{cl}}] = X_0[J] - \int d^4x J\varphi_0 = \int d^4x \mathscr{L}(\varphi_0) = \int d^4x \left(\frac{1}{2}(\partial^\mu\varphi_{\mathrm{cl}})(\partial_\mu\varphi_{\mathrm{cl}}) - \frac{1}{2}\mu^2\varphi_{\mathrm{cl}}^2 - \frac{1}{4!}\lambda\varphi_{\mathrm{cl}}^4 \right), \tag{4.83}$$

which is the same as (4.67). The additional term in the expanded effective action is

$$\begin{aligned}\hbar\Gamma_1[\varphi_{\rm cl}] &= X_0[J] - \Gamma_0[\varphi_{\rm cl}] - \int d^4x J\varphi_{\rm cl} + \hbar X_1[J] \\ &= \int d^4x[\mathscr{L}(\varphi_0) + J\varphi_0] - \int d^4x[\mathscr{L}(\varphi_{\rm cl}) + J\varphi_{\rm cl}] + \frac{i\hbar}{2}\mathrm{tr}\log\frac{A(x',x,\varphi_0)}{A(x',x,0)}. \quad (4.84)\end{aligned}$$

Because φ_0 is a solution of (4.69) the difference of the two integrals in (4.84) is of order $(\varphi_0 - \varphi_{\rm cl})^2 = O(\hbar^2)$, and we can interchange φ_0 and $\varphi_{\rm cl}$ at this level of accuracy. Therefore

$$\Gamma_1[\varphi_{\rm cl}] = \frac{i}{2}\mathrm{tr}\log\frac{A(x',x,\varphi_{\rm cl})}{A(x',x,0)}. \quad (4.85)$$

Next, the effective potential $V(\varphi_{\rm cl})$ can be derived from $\Gamma[\varphi_{\rm cl}]$ by setting $\varphi_{\rm cl}$ constant, in which case

$$\Gamma[\varphi_{\rm cl}] = -\int d^4x V(\varphi_{\rm cl}). \quad (4.86)$$

Delta functions allow us to diagonalise $A(x',x,\varphi_{\rm cl})$ which is a prerequisite to properly define the logarithmic part of (4.85). This is done as follows

$$\begin{aligned}A(x',x,\varphi_{\rm cl}) &= \left(-\partial_{x'\mu}\partial_x^\mu + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right)\delta(x'-x) \\ &= \int \frac{d^4k}{(2\pi)^4}\left(-\partial_{x'\mu}\partial_x^\mu + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right)e^{ik(x'-x)} \\ &= \int \frac{d^4k}{(2\pi)^4}\left(-k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right)e^{ik(x'-x)} \\ &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}e^{ix'k'}\left(-k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right)\delta(k'-k)e^{-ikx} \quad (4.87)\end{aligned}$$

Hence performing log and trace operation

$$\log A(x',x,\varphi_{\rm cl}) = \int d^4k d^4k' \frac{e^{ix'k'}}{(2\pi)^2}\log\left(-k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right)\delta(k'-k)\frac{e^{-ixk}}{(2\pi)^2} \quad (4.88)$$

so that

$$\mathrm{tr}\log A = \int d^4x d^4x' \delta(x'-x)\log A(x',x,\varphi_{\rm cl}) = \int d^4x \int \frac{d^4k}{(2\pi)^4}\log\left(-k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2\right). \quad (4.89)$$

This the formula for the determinant of A since

$$\mathrm{Tr}\log A = \log\det A. \quad (4.90)$$

Summarizing we can now state the following: the one-loop order contribution to the effective potential is

$$V_1(\varphi_{\rm cl}) = \frac{-i}{2}\int \frac{d^4k}{(2\pi)^4}\log\left(\frac{-k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_{\rm cl}^2}{-k^2 + \mu^2}\right) \quad (4.91)$$

and while at this order, the effective potential is approximated by

$$V(\varphi_{\rm cl}) \approx V_0(\varphi_{\rm cl}) + V_1(\varphi_{\rm cl}) = \frac{1}{2}\mu^2\varphi_{\rm cl}^2 + \frac{1}{4!}\lambda\varphi_{\rm cl}^4 - \frac{i}{2}\hbar\int\frac{d^4k}{(2\pi)^4}\log\left(1-\frac{1}{2}\frac{\lambda\varphi_{\rm cl}^2}{k^2-\mu^2}\right), \quad (4.92)$$

where, the Greek parameters are bare parameters, i.e. the ones present in the Lagrangian. The integral term in the potential is ultraviolet divergent and it requires dimensional regularization to deal with i.e. (see for instance Bailin and Love, p.80)

$$\begin{aligned} I(\omega,\mu_B) &= \int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2-\mu_B^2+i\varepsilon)^{-1} = \frac{1}{i}\int\frac{d^{2\omega}\ \not{k}}{(2\pi)^{2\omega}}(\not{k}^2+\mu_B^2)^{-1} \\ &= \frac{i\mu_B^2}{16\pi^2}(M^2)^{\omega-2}\left(\frac{1}{2-\omega}+\Gamma'(1)+1-\log\frac{\mu_B^2}{4\pi M^2}+O(\omega-2)\right). \end{aligned} \quad (4.93)$$

By the use of the following transformations for the bare parameters

$$\varphi_B(x) = Z^{1/2}\varphi(x) \quad Z\mu_B^2 = \mu^2+\delta\mu^2 \quad Z^2\lambda_B = \lambda+\delta\lambda, \quad (4.94)$$

we can transform the above potential (4.92) to

$$V(\varphi_{\rm cl}) = \frac{1}{2}\mu^2\varphi_{\rm cl}^2 + \frac{1}{4!}\lambda\varphi_{\rm cl}^4 + \frac{1}{2}\delta\mu^2\varphi_{\rm cl}^2 + \frac{1}{4!}\delta\lambda\varphi_{\rm cl}^4 - \frac{i}{2}\hbar\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\log\left(1-\frac{1}{2}\frac{\lambda\varphi_{\rm cl}^2}{k^2-\mu^2}\right), \quad (4.95)$$

with the new Greek parameters now being the renormalized parameters rather than the bare. It is important to note, however, that these parameters have constraints imposed by the $\overline{\rm MS}$ scheme. In terms of Green functions, these develop singularities in $\omega-2$ and these are neutralized by the counter terms which are [1]

$$\delta\lambda = M^{4-2\omega}\left(a_0(\hat\lambda, M/\mu,\omega) + \sum_{k=1}^{\infty}\frac{a_k(\hat\lambda, M/\mu)}{(2-\omega)^k}\right) \quad (4.96)$$

$$\delta\mu^2 = \mu^2\left(b_0(\hat\lambda, M/\mu,\omega) + \sum_{k=1}^{\infty}\frac{b_k(\hat\lambda, M/\mu)}{(2-\omega)^k}\right) \quad (4.97)$$

$$\delta Z = c_0(\hat\lambda, M/\mu,\omega) + \sum_{k=1}^{\infty}\frac{c_k(\hat\lambda, M/\mu)}{(2-\omega)^k} \quad (4.98)$$

with a_0, b_0 and c_0 regular as $\omega\to 2$ and $\hat\lambda = \lambda M^{2\omega-4}$. Because we have only simple poles, the k-integrals only have simple poles at $2-\omega$. This means that $\forall k>1 \Rightarrow a_k = b_k = 0$. The one-point-irreducible Green functions of the renormalized theory behave as

$$\tilde\Gamma^2(p,-p) = p^2(1+\delta Z_1) - \mu^2 - \delta\mu_1^2 + \frac{\hat\lambda\mu^2}{32\pi^2}\left(\frac{1}{2-\omega}-\gamma+1-\log\frac{\mu^2}{4\pi M}+O(\omega-2)\right). \quad (4.99)$$

This can be shown by using

$$I_4(\mu_B) = \int\frac{d^4k}{(2\pi)^4}(k^2-\mu_B^2+i\varepsilon)^{-1} \quad (4.100)$$

and (4.93) on the Feynman propagator

$$\Delta_F(0) = \int \frac{d^4k}{(2\pi)^4}(k^2 - \mu_B^2 + i\varepsilon)^{-1} = M^{2\omega-4}\frac{i\mu^2}{16\pi^2}\left(\frac{1}{2-\omega} - \gamma + 1 - \log\frac{\mu^2}{4\pi M} + O(\omega-2)\right). \tag{4.101}$$

This propagator, in turn, is present in the Green function as

$$\tilde{\Gamma}^2(p,-p) = p^2(1+\delta Z_1) + \left(\mu^2 + \frac{1}{2}\lambda i\Delta_F(0) + \delta\mu_1^2\right) + O(\lambda^2), \tag{4.102}$$

as it can be seen by writing the Feynman diagrams to leading order.
The computation of $\tilde{\Gamma}^4$ is lengthier but follows the same lines, eventually, since $\tilde{\Gamma}^4$ is finite in any renormalisation scheme we have that as $\omega \to 2$ the following holds

$$\frac{3\lambda\hat{\lambda}}{32\pi^2}\frac{1}{2-\omega} - \delta\lambda_2 \to \text{const}, \tag{4.103}$$

with

$$\delta\lambda = \sum_{k=2}^{\infty}\delta\lambda_k. \tag{4.104}$$

The finite part of $\delta\lambda_2$ is arbitrary.
Going back to our discussion of (4.96) to (4.98) now we see that the poles in $2-\omega$ are neutralized by setting

$$a_1 = \frac{3}{32\pi^2}\hat{\lambda}^2, \quad b_1 = \frac{1}{32\pi^2}\hat{\lambda}. \tag{4.105}$$

Therefore putting this back together in the potential one has

$$\begin{aligned} V(\varphi_{\text{cl}}) = {} & \frac{1}{2}\mu^2\varphi_{\text{cl}}^2\left[1 + b_0 - \frac{\lambda}{32\pi^2}\left(\frac{3}{2} - \gamma + \log 4\pi\right)\right] \\ & + \frac{1}{4!}\varphi_{\text{cl}}^4\left[\lambda + a_0 - \frac{3\lambda^2}{32\pi^2}\left(\frac{3}{2} - \gamma + \log 4\pi\right)\right] \\ & + \frac{1}{64\pi^2}\left[\left(\mu^2 + \frac{1}{2}\lambda\varphi_{\text{cl}}\right)^2\log\frac{\mu^2 + \frac{1}{2}\lambda\varphi_{\text{cl}}^2}{M^2} - \mu^4\log\frac{\mu^2}{M^2}\right]. \end{aligned} \tag{4.106}$$

In turns out that in the $\overline{\text{MS}}$ scheme we may choose (see Rajantie or Bailin and Love)

$$\delta\lambda^{\overline{\text{MS}}} = \frac{3\lambda\hat{\lambda}}{32\pi^2}\left(\frac{1}{2-\omega} - \gamma + \log 4\pi\right) + O(\lambda^3) \tag{4.107}$$

$$a_0^{\overline{\text{MS}}} = \frac{3\hat{\lambda}^2}{32\pi^2}[-\gamma + \log 4\pi] + O(\lambda^3), \quad a_1^{\overline{\text{MS}}} = \frac{3\hat{\lambda}^2}{32\pi^2} + O(\lambda^3), \tag{4.108}$$

This will get rid of the $-\gamma + \log 4\pi$ factors.
However, and here is the main objective of this comparison with the ζ function regularization, we could also renormalize $V(\varphi_{\text{cl}})$ by writing it as a function of physical mass and

coupling constant. This can be accomplished by setting a_0^{phys} and b_0^{phys} such that the following hold

$$\left.\frac{d^2V}{d\varphi_{\text{cl}}^2}\right|_{\varphi_{\text{cl}}=0} = \mu^2, \qquad \left.\frac{d^4V}{d\varphi_{\text{cl}}^4}\right|_{\varphi_{\text{cl}}=M} = \lambda, \tag{4.109}$$

In the case that μ^2 is small there could be radiative corrections that generate spontaneous symmetry breaking. Performing the second and fourth derivatives (4.109) and using

$$\log \frac{\lambda M^2}{2\mu^2} = -\frac{8}{3}$$

leads to the potential [1], [4]

$$V(\varphi_{\text{cl}}) = \frac{1}{2}\mu^2\varphi_{\text{cl}}^2 + \frac{\lambda}{4!}\varphi_{\text{cl}}^4 + \frac{\lambda^2\varphi_{\text{cl}}^4}{256\pi^2}\left(\log\frac{\varphi_{\text{cl}}^2}{M^2} - \frac{25}{6}\right). \tag{4.110}$$

4.4 Γ-evaluation of dimensional loop integrals

Let us make a brief digression to see how one would compute integrals of the form

$$\Upsilon_\beta(k) = \int_{-\infty}^{\infty} d^\beta\ell F(\ell,k) \tag{4.111}$$

where F behaves as ℓ^{-2} or ℓ^{-4} for large ℓ. The key component behind dimensional regularization is that by lowering the number of dimensions over which one integrates the divergences trivially disappear [4]. For example as $F \to \ell^{\beta=-4}$ in 2-D the integral above converges at the ultraviolet end.
The precise technique runs as follows, let

$$\Upsilon(\omega,k) = \int_{-\infty}^{\infty} d^{2\omega}\ell F(\ell,k), \tag{4.112}$$

where ω is a complex variable. We have to choose a domain where Υ has no singularities in the ω plane. Carefully choose a function Υ' which has well-defined singularities outside the domain of convergence. By analytic continuation (yet again) Υ and Υ' are the same function. In order to accomplish this, we first establish a domain of convergence for the loop integral in the ω plane. Then we construct a function which overlaps with the loop integral in its domain of convergence but is defined in a large domain which encloses the point $\omega = 2$ and then take limit $\omega \to 2$.
We shall follow the steps of 't Hooft and Veltman [6] which are simplified in [4]. We start by splitting up the domain of integration as

$$d^{2\omega}\ell \to d^4\ell d^{2\omega-4}\ell. \tag{4.113}$$

By introducting polar coordinates and letting L be the length of the $2\omega-4$ dimensional ℓ vector the integral becomes

$$\Upsilon = \int d^4\ell \int d\Omega_{2\omega-4} \int_0^{\infty} dL \frac{L^{2\omega-5}}{L^2+\ell^2+m^2}. \tag{4.114}$$

By using the familiar result from AQFT [3]

$$\int d\Omega_{2\omega-4} = \frac{2\pi^{\omega-2}}{\Gamma(\omega-2)} \tag{4.115}$$

the integral now reads

$$\Upsilon = \frac{2\pi^{\omega-2}}{\Gamma(\omega-2)} \int d^4\ell \int_0^\infty dL \frac{L^{2\omega-5}}{L^2+\ell^2+m^2}. \tag{4.116}$$

Note that our critieria is not met. This is not well-defined because it is divergent for $\omega \geq 1$ and integrating over L diverges at the lower, i.e. when $\omega \leq 2$. Therefore, we do not have an overlapping region where Υ is well-defined. However,

$$L^{2\omega-6} = \frac{1}{\omega-2}\frac{d}{dL^2}(L^2)^{\omega-2}. \tag{4.117}$$

If we use this to integrate by parts Υ yields

$$\Upsilon = \pi^{\omega-2}\frac{\omega-2}{\Gamma(\omega-1)} \int d^4\ell \int_0^\infty dL^2 \frac{1}{\omega-2}(L^2)^{\omega-2}\left(-\frac{d}{dL^2}\right)\frac{1}{L^2+\ell^2+m^2} + \text{surface term} \tag{4.118}$$

ignoring the surface term and simplying

$$\Upsilon = \frac{\pi^{\omega-2}}{\Gamma(\omega-1)} \int d^4\ell \int_0^\infty dL^2 (L^2)^{\omega-2}\left(-\frac{d}{dL^2}\right)\frac{1}{L^2+\ell^2+m^2}. \tag{4.119}$$

This is still too weak as there is no overlap: we have a divergence for $\omega \leq 1$ and another one for $\omega \geq 1$. It would have been sufficient if the divergence has been *logarithmic*. This can be cured by performing the same procedure again, that is lower the dimension

$$\Upsilon = \frac{\pi^{\omega-2}}{\Gamma(\omega)} \int d^4\ell \int_0^\infty dL^2 (L^2)^{\omega-1}\left(-\frac{d}{dL^2}\right)\left(-\frac{d}{dL^2}\right)\frac{1}{L^2+\ell^2+m^2} \tag{4.120}$$

and this equation is well defined for $0 < \omega < 1$. The process of analytic continuation to the point $\omega = 2$ requires the using the trick

$$1 = \frac{1}{5}\left(\frac{\partial L}{\partial L} + \frac{\partial \ell_\mu}{\partial \ell_\mu}\right), \tag{4.121}$$

and it is no suprise that integration by parts is the next step

$$\begin{aligned}
\Upsilon &= -\frac{2\pi^{\omega-2}}{5\Gamma(\omega)} \int d^4\ell \int_0^\infty dL^2 \left(\ell_\mu \frac{\partial}{\partial \ell_\mu} + 2L^2\frac{\partial}{\partial L^2} + 1\right)\frac{(L^2)^{\omega-1}}{(L^2+\ell^2+m^2)^3} \\
&= -\frac{3m^2}{\omega-1}\frac{2\pi^{\omega-2}}{\Gamma(\omega)} \int d^4\ell \int_0^\infty dL^2 \frac{(L^2)^{\omega-1}}{(L^2+\ell^2+m^2)^4}.
\end{aligned}$$

Let us note the pole at $\omega = 1$. In this case, Υ diverges when $\omega \geq 2$ therefore repeating this again to increase the exponent of the denominator yields

$$\Upsilon = -\frac{2 \cdot 3 \cdot 4 \cdot m^4}{(\omega-1)(\omega-2)} \frac{\pi^{\omega-2}}{\Gamma(\omega)} \int d^4\ell \int_0^\infty dL^2 \frac{(L^2)^{\omega-1}}{(L^2+\ell^2+m^2)^5}. \tag{4.122}$$

This is exactly what we aimed for. The (simple) pole is now at $\omega = 2$ and Υ now converges. By the means of another well known loop integral indetity (see AQFT, Peskin and Schroder, Bailin and Love, Ramond)

$$\int \frac{d^{2\omega}\ell}{\ell^2+m^2} = \pi^\omega \frac{\Gamma(1-\omega)}{\Gamma(1)} \frac{1}{(m^2)^{1-\omega}} \tag{4.123}$$

and expanding around the Γ function around the poles $-n$ where n is an integer

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi(n+1) + O(\varepsilon)\right] \tag{4.124}$$

so that

$$\Gamma(1-\omega) = \frac{1}{2}\left[\frac{1}{2-\omega} + \psi(2) + O(2-\omega)\right] \tag{4.125}$$

and consequently,

$$\lim_{\omega\to 2} \int \frac{d^{2\omega}\ell}{\ell^2+m^2} = -\pi^2 m^2 \left[\frac{1}{2-\omega} + \psi(2) + O(2-\omega)\right]. \tag{4.126}$$

Finally, we note that

$$\psi(n) = \sum_{k=1}^{n} \frac{1}{k} - \gamma \tag{4.127}$$

hence

$$\psi(2) = \frac{3}{2} - \gamma \tag{4.128}$$

and therefore

$$\lim_{\omega\to 2} \int \frac{d^{2\omega}\ell}{\ell^2+m^2} = -\pi^2 m^2 \left[\frac{1}{2-\omega} + \frac{3}{2} - \gamma + O(2-\omega)\right]. \tag{4.129}$$

The general integral is

$$\int \frac{d^{2\omega}\ell}{(2\pi)^{2\omega}} (\ell^2 - \mu^2 + i\varepsilon)^{-n} = i(-1)^n \frac{\mu^{2\omega-2n}}{(4\pi)^\omega} \frac{\Gamma(n-\omega)}{\Gamma(n)}. \tag{4.130}$$

The RHS is regular at $\omega = 2$ and the LHS is convergent in four dimensions. Therefore dimensional regularization has renormalised the divergent integral while leaving convergent integrals unaffected when $\omega \to 2$. Naturally, this is due to analytic continuation.

4.5 Expansion of Z in Eucliean spacetime

We now go back to the Euclidean space definition of the generating functional

$$Z_E[J] = N_E \int \mathscr{D}\phi \exp\left\{-\int d^4\bar{x}\left[\frac{1}{2}\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) - J\phi\right]\right\}. \tag{4.131}$$

This will be evaluated by expanding the action.
In order to do this, set ϕ_0 to be a field configuration, then

$$\begin{aligned} S_{\mathrm{Euc}}[\phi, J] &:= \int d^4\bar{x}\left[\frac{1}{2}\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) - J\phi\right] \\ &= S_{\mathrm{Euc}}[\phi_0, J] + \int d^4\bar{x}\left[\frac{\delta S_{\mathrm{Euc}}}{\delta\phi}(\phi - \phi_0)\right] \\ &+ \frac{1}{2}\int d^4\bar{x}_1 d^4\bar{x}_2\left[\frac{\delta^2 S_{\mathrm{Euc}}}{\delta\phi(\bar{x}_1)\delta\phi(\bar{x}_2)}(\phi(\bar{x}_1) - \phi_0(\bar{x}_1))(\phi(\bar{x}_2) - \phi_0(\bar{x}_2))\right] + \cdots . \end{aligned} \tag{4.132}$$

It is understood that the functional derivatives are evaluated at ϕ_0. We know from AQFT [3] that the classical limit can be recovered, by taking S_{Euc} to be stationary at ϕ_0 and this implies that ϕ_0 satisfies the classical EOM with the source term

$$\left.\frac{\delta S_{\mathrm{Euc}}}{\delta\phi}\right|_{x_0} = -\bar{\partial}_\mu\bar{\partial}^\mu\phi(x_0) + m^2\phi(x_0) + V'[\phi(x_0)] - J = 0. \tag{4.133}$$

Integrating by parts gives

$$S_{\mathrm{Euc}}[\phi(x_0), J] = \frac{1}{2}\int d^4\bar{x}\left[2 - \phi(x_0)\frac{d}{d\phi(x_0)}\right][-J\phi(x_0) + V'[\phi(x_0)]], \tag{4.134}$$

whereas the second derivative is an operator

$$\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)}S_{\mathrm{Euc}} = \delta(\bar{x}_1 - \bar{x}_2)\left[-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''[\phi(x_1)]\right]. \tag{4.135}$$

We need to make regression now on how to evaluate integrals of the sort

$$I := \int dx \exp(-\alpha(x)) \tag{4.136}$$

This can be evaluated by expanding the exponential around a point x_0 where α is stationary

$$\alpha(x) = \alpha(x_0) + \frac{1}{2}(x - x_0)^2\alpha''(x_0) + O(x^3). \tag{4.137}$$

The I integral is then approximated

$$I = \exp(-\alpha(x_0))\int dx \exp\left\{-\frac{1}{2}(x - x_0)^2\alpha''(x_0)\right\}, \tag{4.138}$$

and we can recognize this as a Gaussian integral (when the higher derivatives are ignored). The degree of this approximation depends obviously on α. When α is smallest then the integrand is largest and the points away from the minimum do not add a substantial contribution.

Equipped with this method we can apply it to the functional we have just arrived at a crucially important result [4]

$$
\begin{aligned}
Z_E[J] &= N_E \exp\left\{-S_{\text{Euc}}[\phi(x_0), J]\right\} \int \mathscr{D}\phi \exp\left\{-\frac{1}{2}\int d^4\bar{x}_1 d^4\bar{x}_2 \left(\phi(\bar{x}_1)\frac{\delta^2 S_{\text{Euc}}}{\delta\phi(\bar{x}_1)\delta\phi(\bar{x}_2)}\phi(\bar{x}_2)\right)\right\} \\
&= N_E' \frac{\exp\left\{-S_{\text{Euc}}[\phi(x_0), J]\right\}}{\sqrt{\det\left[(-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''[\phi(x_0)])\delta(x_1 - x_2)\right]}},
\end{aligned}
\tag{4.139}
$$

where we have ignored the higher order terms. The precise derivation is in the Appendix. Let us do some re-write to make this expression easier to handle, first the determinant, which we will call $\mathbf{M}$ can be taken care of by using the identity

$$
\det\mathbf{M} = \exp(\operatorname{tr}\log\mathbf{M}), \tag{4.140}
$$

so that

$$
Z_E[J] = N_E' \exp\left\{-S_{\text{Euc}}[\phi(x_0), J] - \frac{1}{2}\operatorname{tr}\log(-\bar{\partial}_\mu\bar{\partial}^\mu + m^2 + V''[\phi(x_0)])\delta(x_1 - x_2)\right\}, \tag{4.141}
$$

the delta term accounts for the quantum perturbations (or corrections) to $Z[J]$, whereas the first term accounts for the classical contribution. Also note that we set by convention the determinant of an operator to be the product of its eigenvalues and that because ϕ_0 satisfies $\delta S_{\text{Euc}}/\delta\phi|_{x_0} = 0$ then it is a functional of J.
The concluding remark is to find the equivalence of these results in terms of the classical field ϕ_{cl} and explore the corresponding effective action. When we work in Euclidean space, the classical field was defined as

$$
\phi_{\text{cl}}(\bar{x}) = -\frac{\delta Z_E}{\delta J(\bar{x})} \approx -\frac{\delta S_E}{\delta J(\bar{x})} + O(\hbar), \tag{4.142}
$$

and by using (3.57) and (3.58) we can have ϕ_{cl} as a function (functional, rather) of J, however at the cost of doing it order by order in λ. The term $O(\hbar)$ stands for quantum corrections. This relationship can be inverted and we can find $J(\bar{x})$ as a function of ϕ_{cl}. This inversion can be carried out to give (see Ramond)

$$
J(\bar{x}) = (\bar{\partial}^2 - m^2)\phi_{\text{cl}}(\bar{x}) - \frac{\lambda}{3!}\phi_{\text{cl}}^3(\bar{x}). \tag{4.143}
$$

An attractive (and indispensable) feature is that there are no higher terms in λ, comparison with

$$
0 = \left.\frac{\delta S_{\text{Euc}}}{\delta\phi}\right|_{x_0} = -\bar{\partial}_\mu\bar{\partial}^\mu\phi(x_0) + m^2\phi(x_0) + V'[\phi(x_0)] - J \tag{4.144}
$$

gives

$$
\phi_{\text{cl}}(\bar{x}) = \phi_0(\bar{x}) + O(\hbar). \tag{4.145}
$$

By integrating $J(x) = -\delta\Gamma[\phi_{\text{cl}}]/\delta\phi_{\text{cl}}$ we see that to this order that

$$
\Gamma_{\text{eff}}[\phi_{\text{cl}}] = -\int d^4\bar{x}\left[\frac{1}{2}\phi_{\text{cl}}(\bar{\partial}^2 - m^2)\phi_{\text{cl}} - \frac{\lambda}{4!}\phi_{\text{cl}}^4(\bar{x})\right], \tag{4.146}
$$

which is indeed an effective action.

REFERENCES

The above summary on quantum field theory has its roots several sources. Overall it follows the presentation given by

- [1] Bailin and Love *Introduction to Gauge Field Theory* section 4.4,
- [2] Peskin and Schroeder [275 to 294, 306 to 308 and 365 to 372].

At times it is complemented by the courses of QED and AQFT from the MSc QFFF

- [3] Rajantie's AQFT
- [4] Ramond for the expansion of the generating functional integrals in Eucliean spacetime.

However, it is broad enough to be explained in most QFT books. The papers quoted are

- [5] R. Jackiw, *Phys. Rev.* D9 1686 (1974)
- [6] 't Hooft and Veltman, *Nucl. Phys.* 44B 189

5 Zeta Regularization in Field Theory

5.1 Heat kernels and Mellin transforms

At this point we can put together some of the concepts acquired in the second and third chapters. Firstly we will further develop the theory of quantum corrections by evaluating the determinant of the **M** matrix and then see how this is related to the use of the ζ function in quantum theory as we explained in Chapter 3. In what follows, we will extrapolate the results found in the quantum mechanical section to field theory, this section borrows some its contents from Harfield [2], Hawking [3] and Ramond [5]. As we mentioned earlier, the determinant in the expression (4.139)

$$Z_E[J] = \frac{N'_E \exp\{-S_{\mathrm{Euc}}[\phi(x_0), J]\}}{\sqrt{\det\left[(-\bar{\partial}_\mu \bar{\partial}^\mu + m^2 + V''[\phi(x_0)])\delta(x_1 - x_2)\right]}} \tag{5.1}$$

must be interpreted as the product of the eigenvalues of the operator. In order to discretize these eigenvalues we truncate the space (by use of a box). We then multiply the resulting eigenvalues and then let the size of the box increase to infinity.
First we state some preliminary results that will become useful later on. As it is shown in a course on partial differential equations, the heat function

$$G(\bar{x}, \bar{y}, t) := \sum_n \exp(-\lambda_n t)\psi_n(\bar{x})\psi_n^*(\bar{y}) \tag{5.2}$$

satisfies the heat equation

$$A_{\bar{x}} G(\bar{x}, \bar{y}, t) = -\frac{\partial}{\partial t} G(\bar{x}, \bar{y}, t), \tag{5.3}$$

where $A_{\bar{x}}$ is taken to act on the first argument of G and initial condition

$$G(\bar{x}, \bar{y}, t = 0) = \delta(\bar{x} - \bar{y}). \tag{5.4}$$

The key step, the relationship between the heat kernel G and the ζ function is the following result

$$\zeta_A(s)\Gamma(s) = \int_0^\infty dt\, t^{s-1} \int d^4\bar{x} G(\bar{x}, \bar{x}, t), \tag{5.5}$$

which we proceed to explain. Note the similarity with (1.15), this should already give us a hint on how to proceed. The eigenfunctions $\psi_n(x)$ of A satisfy the following orthogonality relations

$$\langle \psi_n, \psi_m^* \rangle = \int dx \psi_n^*(x)\psi_m(x) = \delta_{nm}, \quad \sum_n \psi_n^*(x)\psi_n(x') = \delta(x - x') \tag{5.6}$$

We can explain (5.5) by taking the trace of G

$$\mathrm{Tr}(G) = \int dx G(x, x, t) = \sum_n e^{-\lambda_n t} \tag{5.7}$$

(remembering $\mathrm{Tr}(G)$ is a function of t) multiply it by $e^{-\lambda t}$ and integrate it with respect to t and then with respect to λ to obtain

$$\int d\lambda \int_0^\infty dt \mathrm{Tr}(G) e^{-\lambda t} = \sum_n \int d\lambda \frac{1}{\lambda_n + \lambda} = \sum_n \log(\lambda_n + \lambda). \tag{5.8}$$

The determinant of A then is written as

$$\det A = \exp\left[\int d\lambda \int_0^\infty dt\ \mathrm{Tr}(G) e^{-\lambda t}\big|_{\lambda=0}\right] \tag{5.9}$$

ignoring the factor that would show from the λ integration. Swapping the integrals yields

$$\det A = \sum_n e^{-\lambda_n t} = -\int_0^\infty dt t^{-1} \mathrm{Tr}(G) \tag{5.10}$$

The relationship between the trace of the heat kernel $\mathrm{Tr}(G)$ and the ζ function is given by a Mellin transformation, see Hawking [3]

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \mathrm{Tr}(G), \tag{5.11}$$

which justifies (5.5). A solution of the heat equation

$$-\bar{\partial}_x^2 G_0(\bar{x}, \bar{y}, t) = -\frac{\partial}{\partial t} G_0, \tag{5.12}$$

with the boundary condition $G_0(\bar{x}, \bar{y}, t = 0) = \delta(\bar{x} - \bar{y})$ is

$$G_0(\bar{x}, \bar{y}, t) = \frac{1}{16\pi^2 r^2} \exp\left(-\frac{1}{4r}(\bar{x} - \bar{y})^2\right), \tag{5.13}$$

also a classic result from partial differential equations.
Let us generalize some of the techniques we used in Chapter 3. Consider an operator A with positive real discrete eigenvalues λ_i where i runs from 1 to n and its eigenfunctions are $\psi_n(x)$ i.e. $A\psi_i(x) = \lambda_i \psi_i(x)$. From here we set

$$\zeta_A(s) = \sum_n \lambda_n^{-s}, \tag{5.14}$$

which we call the zeta function associated to the operator A. The sum is over all the eigenvalues and A is a real variable. If the operator A were the one-dimensional harmonic oscillator Hamiltonian, then ζ_A would be the Riemann zeta function (excluding the singular zero-point energy).
The first observation is that [1], [2], [3], [4] and [5]

$$\frac{d}{ds}\zeta_A(s)\bigg|_{s=0} = -\sum_n \log \lambda_n^s \exp(-s \log \lambda_n^s)|_{s=0} = -\log \prod_n \lambda_n, \tag{5.15}$$

which gives the determinant

$$\det A = \prod_n \lambda_n = \exp(-\zeta'_A(0)). \tag{5.16}$$

As we know from our discussion of chapter 3. The zeta-function ζ_A is not always singular at $s = 0$ for physically interesting operators and hence the convenience of writing this representation for $\det A$.
Algorithmically we have a procedure to compute the determinant of A and it can be done as follows. First we need to find the solution to the heat equation subject to the delta-function initial condition (5.4). Second, once this is done we can insert the solution into the ζ representation above and we have ζ_A. Finally evaluate at $s = 0$ and compute $\exp(-\zeta'_A(0))$.

5.2 Derivation of φ^4 potential at $\varphi_{\rm cl}$ using ζ regularization

In our case, our operator is

$$A = -\bar{\partial}^2 + m^2 + \frac{\lambda}{2}\phi_0^2(\bar{x}), \tag{5.17}$$

where $\phi_0(\bar{x})$ is a solution of the classical equations with source J as we discussed in Chapter 4. Also note that in (5.1) we have a V'' factor in the determinant which takes the form $V'' = (\lambda/2)\phi_0^2$ in the A operator.
As we have pointed out a solution of the heat equation with the boundary condition $G_0(\bar{x}, \bar{y}, t = 0) = \delta(\bar{x} - \bar{y})$ is

$$G_0(\bar{x}, \bar{y}, t) = \frac{1}{16\pi^2 r^2} \exp\left(-\frac{1}{4r}(\bar{x} - \bar{y})^2\right). \tag{5.18}$$

However, this is only a part of the operator as we want to find $G_0(\bar{x}, \bar{y}, t)$ subject to the initial condition (5.4) which obeys the whole A operator

$$\left[-\bar{\partial}^2 + m^2 + \frac{\lambda}{2}\phi_0^2(\bar{x})\right] G(\bar{x}, \bar{y}, t) = -\frac{\partial}{\partial t} G(\bar{x}, \bar{y}, t). \tag{5.19}$$

For arbitrary fields ϕ_0 we need to expand the effective action as (see (4.81) or Bailin and Love, [5])

$$\Gamma_E[\phi_{\rm cl}] = \Gamma_E^{(0)}[\phi_{\rm cl}] + \hbar\Gamma_E^{(1)}[\phi_{\rm cl}] + \cdots, \tag{5.20}$$

the $\hbar$ indicates quantum terms. Given the fact that the effective potential at 1 loop is of the form [2]

$$V_{\rm eff}[\phi_{\rm cl}] \int d^4x = \frac{1}{2}\log\det(-\partial^2 + m_0^2 + \tfrac{\lambda_0}{2}\phi_{\rm cl}^2) - \frac{1}{2}\log\det(-\partial^2 + m_0^2), \tag{5.21}$$

as we also argued in general terms in Chapter 4 (4.56) and (4.85), allows us to write

$$\Gamma_E^{(1)}[\phi_{\rm cl}] = -\frac{1}{2}\zeta'_{[-\bar{\partial}^2 + m^2 + \frac{\lambda}{2}\phi_0^2(\bar{x})]}(0), \tag{5.22}$$

by use of (5.1) and (5.16) combined with (5.17). Note that replacing ϕ_0 by $\phi_{\rm cl}$ is not a problem as there are no new quantum errors, that is errors up to $O(\hbar)$. By the expansion of the effective potential we can swap m and m_0. From the fact that the effective action can be written as

$$\Gamma_E[\phi_{\text{cl}}] = \int d^4\bar{x} \left[V(\phi_{\text{cl}}(\bar{x})) + F(\phi_{\text{cl}})\bar{\partial}_\mu \phi_{\text{cl}}(\bar{x})\bar{\partial}^\mu \phi_{\text{cl}}(\bar{x}) + \cdots\right] \tag{5.23}$$

we can compute the quantum $O(\hbar)$ contribution to $V(\phi_{\text{cl}}(\bar{x}))$ by considering a *constant* field configuration. That is, suppose we take $\phi_{\text{cl}}(\bar{x}) = v$ where v is a constant independent of $\bar{x}$, then

$$\Gamma_E[\phi_{\text{cl}}] = \int d^4\bar{x} V(v), \tag{5.24}$$

and Γ_E is proportional to the infinite volume element $\int d^4\bar{x}$, since the Euclidean space R_4 is unbounded. This can be temporarily solved by taking the space to be the sphere S_4 then the volume is just that of the 5-dimensional sphere and hence finite. While the radius is finite we need not worry about the infrared divergence. We then let the radius of the sphere tend to infinity.
Taking V out of the integral we have [2], [5]

$$V(v) \int d^4\bar{x} = -\frac{1}{2}\zeta'_{[-\bar{\partial}^2 + m^2 + \frac{\lambda}{2}v^2]}(0). \tag{5.25}$$

We can proceed to integrate (5.19) when v is constant this yields

$$G(\bar{x}, \bar{y}, t) = \frac{\mu^4}{16\pi^2 t^2} \exp\left[-\frac{\mu^2(\bar{x} - \bar{y})^2}{4t}\right] \exp\left[\left(-m^2 + \frac{\lambda}{2}v^2\right)\frac{t}{\mu^2}\right], \tag{5.26}$$

where the μ factor needs to be explained: it has dimensions of mass so that t is dimensionless. Using the Mellin transform (5.5) we obtain

$$\begin{aligned}
\zeta_A(s) &= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int d^4\bar{x} \frac{\mu^4}{16\pi^2 t^2} \exp\left[\left(-m^2 + \frac{\lambda}{2}v^2\right)\frac{t}{\mu^2}\right] \\
&= \frac{\mu^4}{16\pi^2 t^2} \left(\frac{m^2 + \frac{\lambda}{2}v^2}{\mu^2}\right)^{2-s} \frac{\Gamma(s-2)}{\Gamma(s)} \int d^4\bar{x},
\end{aligned} \tag{5.27}$$

the volume element $\int d^4\bar{x}$ is present because it is in (5.25) and here t has been rescaled since the integration over t is valid when $s > 2$ however the ζ function is defined everywhere by analytic continuation as we know from Chapter 2. Differentiating and comparing these two equations we have

$$\begin{aligned}
V(v) &= -\frac{\mu^4}{32\pi^2} \frac{d}{ds} \left\{ \frac{1}{(s-2)(s-1)} \left(\frac{m^2 + \frac{\lambda}{2}v^2}{\mu^2}\right)^{2-s} \right\}\Bigg|_{s=0} \\
&= \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}v^2\right)^2 \left(\log\frac{m^2 + \frac{\lambda}{2}v^2}{\mu^2} - \frac{3}{2}\right).
\end{aligned} \tag{5.28}$$

Note that we have used

$$\frac{\Gamma(s-2)}{\Gamma(s)} = \frac{1}{(s-2)(s-1)}. \tag{5.29}$$

Also note that applying (5.16) and working with a non-constant field configuration $\phi_{\rm cl}$ yields

$$\log\det A = \frac{1}{32\pi^2}\left(m_0^2+\frac{\lambda_0\phi_{\rm cl}^2}{2}\right)^2\left[\log\left(m_0^2+\frac{\lambda_0\phi_{\rm cl}}{2}\right)-\frac{3}{2}\right]\int d^4\bar{x}. \tag{5.30}$$

Equipped with this functional form of V the effective potential can be written as

$$V(\phi_{\rm cl}) = \frac{1}{2}m^2\phi_{\rm cl}^2(\bar{x})+\frac{\lambda}{4!}\phi_{\rm cl}^4(\bar{x})+\frac{\hbar}{64\pi^2}\left(m^2+\frac{\lambda}{2}\phi_{\rm cl}^2(\bar{x})\right)^2\left(\log\frac{m^2+\frac{\lambda}{2}v^2}{\mu^2}-\frac{3}{2}\right), \tag{5.31}$$

ignoring terms of order $\hbar^2$. We make a pause now to examine this result. Superficially the first striking observation is that there is a strong dependence on the unknown and arbitrary scale μ^2. This would seem to imply that the potential is therefore arbitrary. However V depends on the parameters m^2 and λ which are undefined, except for the fact that they are included in the classical Lagrangian.
Let us take the special massless case. This yields the following

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=0} = 0. \tag{5.32}$$

Now we *define* the mass squared as the coefficient of the terms ϕ^2 in the Lagrangian evaluated at $\phi=0$. To first quantum corrections the coefficient is zero, if it is classically zero. The λ term is defined to be the coefficient of the fourth derivative of V evaluated at some constant point $\phi=M$, i.e.

$$\lambda := \left.\frac{d^4V}{d\phi^4}\right|_{\phi=M}. \tag{5.33}$$

We cannot take $\phi=0$ as with the mass squared factor because of the infrared divergence coming from the logarithm. When we differentiate (5.31), set $m^2=0$ and use (5.33) we see that the above condition requires

$$\log\frac{\lambda M^2}{2\mu^2} = -\frac{8}{3}. \tag{5.34}$$

In this case, we may use M^2 instead of $2\mu^2/\lambda$ and write the result as (Bailin and Love and [5])

$$V(\phi_{\rm cl}) = \frac{\lambda}{4!}\phi_{\rm cl}^4(\bar{x})+\frac{\lambda^2\phi_{\rm cl}^4}{256\pi^2}\left(\log\frac{\phi_{\rm cl}^2}{M^2}-\frac{25}{6}\right), \tag{5.35}$$

which is exactly the same result we found in Section (4.3). This was proved by Coleman and Weinberg in 1973 [5]. The main result that can be extracted is that we need to be careful with how we define the input parameters in the Lagrangian if we are to take into account quantum corrections. Again, superficially it seems that (5.35) depends on another arbitrary scale M^2, but in fact it does not. Given the normalization condition, if we change the scale from M^2 to M'^2 we simultaneously have to change at the same time λ to λ' by use of (5.33)

$$\lambda' = \lambda+\frac{3\lambda^2}{16\pi^2}\log\frac{M'}{M}. \tag{5.36}$$

Therefore the potential (see Ramond)

$$V(\phi_{\rm cl}) = \frac{\lambda'}{4!}\phi_{\rm cl}^4(\bar{x}) + \frac{\lambda'^2\phi_{\rm cl}^4}{256\pi^2}\left(\log\frac{\phi_{\rm cl}^2}{M'^2} - \frac{25}{6}\right) \tag{5.37}$$

is indeed invariant under the representation $V(\lambda', M') = V(\lambda, M)$. This proves that the physics behind remains unchanged but our way of interpreting the coefficients changes.

5.3 Coupling constants

Let us now look more closely to the scaling of determinants and the coupling constants. The ζ function technique just used allows us derive scaling properties for determinants. First, we will need the computation of ζ function $\zeta_{[-\bar{\partial}^2+\frac{\lambda}{2}\phi_{\rm cl}^2]}(0)$. This can be accomplished by taking the asymptotic expansion of $G(\bar{x},\bar{y},t)$ at $\mu^2=1$,

$$G(\bar{x},\bar{y},t) = e^{-\varepsilon t}\frac{e^{-(\bar{x}-\bar{y})^2/(4t)}}{16\pi^2 t^2}\sum_{n=0}^{\infty} a_n(\bar{x},\bar{y})t^n, \tag{5.38}$$

with $\varepsilon > 0$ as a convergence factor. The boundary condition (5.4) sets the condition

$$a_0(\bar{x},\bar{x}) = 1. \tag{5.39}$$

Additionally, when we insert (5.38) into the PDE (5.19) we find recursion relations for the a_n coefficients

$$(\bar{x}-\bar{y})_\mu \frac{\partial}{\partial \bar{x}_\mu} a_0(\bar{x},\bar{y}) = 0 \tag{5.40}$$

and for $n = 0, 1, 2, \cdots$

$$\left[(n+1) + (\bar{x}-\bar{y})_\mu\frac{\partial}{\partial\bar{x}_\mu}\right] a_{n+1}(\bar{x},\bar{y}) = \left(\bar{\partial}_x^2 - \frac{\lambda}{2}\phi_{\rm cl}^2 + \varepsilon\right) a_n(\bar{x},\bar{y}). \tag{5.41}$$

When we compute the first terms we have

$$a_1(\bar{x},\bar{x}) = -\frac{\lambda}{2}\phi_{\rm cl}^2 + \varepsilon \tag{5.42}$$

$$a_2(\bar{x},\bar{x}) = \frac{\lambda^2}{8}\phi_{\rm cl}^4(\bar{x}) - \frac{\lambda}{4}\bar{\partial}_x^2\phi_{\rm cl}(\bar{x}) - \frac{\varepsilon}{2}\lambda\phi_{\rm cl}^2(\bar{x}) + \frac{\varepsilon^2}{2}. \tag{5.43}$$

Let us now use these results. We work under a scale change $A \to A' = e^{ad}A$, where d is the natural dimension of A. By the definition of the zeta function we have

$$\zeta_{A'}(s) = e^{-ads}\zeta_A(s) \tag{5.44}$$

which implies that

$$\det(e^{ad}A) = e^{ad\zeta'_A(0)}\det A. \tag{5.45}$$

Let us illustrate this with an example. Under the transformation

$$x_\mu \to x'_\mu = e^a x_\mu \qquad \phi_{\rm cl} \to \phi'_{\rm cl} = e^{-a}\phi_{\rm cl} \tag{5.46}$$

the massless classical action

$$S_E[\phi_{\mathrm{cl}}] = -\int d^4\bar{x} \left[\frac{1}{2}\phi_{\mathrm{cl}}\bar{\partial}^2\phi_{\mathrm{cl}} - \frac{\lambda}{4!}\phi_{\mathrm{cl}}^4\right] \tag{5.47}$$

is unchanged. However, the path integral corresponding to this action is not scale invariant since in the steepest descent approximation the change in the effective action to quantum order is as follows

$$S_E^{\mathrm{eff}}[\phi_{\mathrm{cl}}] \to S_E^{\mathrm{eff}'}[\phi_{\mathrm{cl}}] = S_E^{\mathrm{eff}}[\phi_{\mathrm{cl}}] - \hbar a \zeta_{[-\bar{\partial}^2 + \frac{\lambda}{2}\phi_{\mathrm{cl}}^2]}(0). \tag{5.48}$$

Plugging (5.38) into (5.3) and with the assistance of the first two terms a_1 and a_2 and integrating out the $\bar{\partial}^2$ with the divergence theorem yields

$$\zeta(0) = \frac{1}{16\pi^2}\int d^4\bar{x}\frac{\lambda^2}{8}\phi_{\mathrm{cl}}^4(\bar{x}). \tag{5.49}$$

A small digression is now required to further explain this. In 4 dimensions and in the presence of mass the heat kernel is

$$G(\bar{x},\bar{y},t) = \frac{e^{-(\bar{x}-\bar{y})^2/(4t)}}{16\pi^2 t^2}\exp\left(-\frac{1}{4t}|\bar{x}-\bar{y}|^2\right)\exp\left[-\left(m_0^2 + \frac{\lambda_0\phi_{\mathrm{cl}}^2}{2}\right)t\right] \tag{5.50}$$

and like in (5.27) and (5.28)

$$\zeta(s) = \frac{1}{16\pi^2}\left(m_0^2 + \frac{\lambda_0\phi_{\mathrm{cl}}^2}{2}\right)^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)}\int d^4\bar{x}. \tag{5.51}$$

Furthermore, note that in the more restricted case $B = -\partial^2 + \omega^2$ we have kernel

$$G(x,x',t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-x')^2}{4t}\right)\exp(-\omega^2 t) \tag{5.52}$$

and the determinant of B is

$$\begin{aligned}\log\det B &= -\frac{1}{\sqrt{4\pi}}\int_0^\infty dt t^{-3/2}\exp(-\omega^2 t)\int d^4\bar{x} = -\frac{\omega}{\sqrt{4\pi}}\int d^4\bar{x}\int_0^\infty dt e^{-t}t^{-3/2} \\ &= -\frac{\omega}{\sqrt{4\pi}}\int d^4\bar{x}\left[t^{-1/2}e^{-t}\Big|_0^\infty - 2\pi^{1/2}\right] \end{aligned} \tag{5.53}$$

integration by parts was used in the last line. Ignoring the divergent factor we have the simplification

$$\det B = \exp\left(\omega\int d^4\bar{x}\right). \tag{5.54}$$

We can justify the disregard of the infinite in the integral for $\log\det B$ by expressing it in terms of Γ functions in this divergence zone by analytic continuation.

$$\log\det B = -\frac{\omega}{\sqrt{4\pi}}\left(\int d^4\bar{x}\right)\Gamma\left(-\frac{1}{2}\right) \tag{5.55}$$

where the Γ function is defined in this region by analytical continuation.
Finally, the divergence of $\Gamma(s-2)$ at $s=0$ cancels that the divergence of $\Gamma(0)$, and this process makes $\zeta(0)$ regular.
In terms of the effective action (5.48)

$$S_E^{\text{eff}'}[\phi_{\text{cl}}] = S_E^{\text{eff}}[\phi_{\text{cl}}] - \hbar a \frac{\lambda^2}{128\pi^2} \int d^4\bar{x}\phi_{\text{cl}}^4(\bar{x}). \tag{5.56}$$

Consequently the effect of the transformation to quantum order has been to change the coupling constant λ by the following

$$\frac{\lambda}{4!} \to \frac{\lambda'}{4!} = \frac{\lambda}{4!} - \hbar a \frac{\lambda^2}{128\pi^2} \Leftrightarrow \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\hbar a. \tag{5.57}$$

What this is means is that *the dimensionless coupling constant λ evolves as a result of quantum effects*. This evolution is in term of scale dependence. At large scales the coupling constant decreases indicating that the non-interaction theory is a good approximation for the asymptotic states. On the other hand, if the scale decreases, the coupling increases. Independently of how small λ was at the beginning this increment might throw away the results obtained in the perturbation of λ. Moreover, this scaling law is like the one we found earlier, and they are both correct to quantum orders.

5.4 Partition functions in field theory

We recall that for any given time t a quantum mechanical system with one degree of freedom, q and canonically conjugate momentum p, is described in terms of the spectrum of its Hamiltonian $H(p,q)$. From the path integral formulation we know that if the system at an initial time t_i is measured to be in the state $\left|q^i\right\rangle$, then the probability that the system will be found in the state $\left|q^f\right\rangle$, at a final time t_f is exactly

$$\left\langle q^f_{t_f} \;\middle|\; q^i_{t_i} \right\rangle = \left\langle q^f \;\middle|\; e^{-i(t_f-t_i)H} \middle|\, q^i \right\rangle, \tag{5.58}$$

and it can be written in terms of path integrals as

$$\left\langle q^f \;\middle|\; e^{-i(t_f-t_i)H} \middle|\, q^i \right\rangle = \int \mathscr{D}q \int \mathscr{D}p \exp\left[i \int_{t_i}^{t_f} dt[p\dot{q} - H(p,q)] \right], \tag{5.59}$$

the factor Dq denotes integration between the initial and final configurations q^i and q^f; the dot over q denotes the derivative of q with respect to time.
Quantum field theory and statistical mechanics share certain common elements and precisely this analogy will allow us to apply path integrals to the description of dynamical systems at finite temperature (see [1] and [4]). The first step, as we did in Chapter 3, is to compute the partition function

$$Z = \text{Tr}[e^{-\beta H}], \tag{5.60}$$

where the constant is

$$\beta = (kT)^{-1}, \tag{5.61}$$

taking into account that the trace is taken to be the sum over all the possible configurations the system is allowed to take. Note that the time is singled out. Now, the probability for the system to be in state of energy E is *identified* with

$$P = Z^{-1} e^{-\beta H}. \tag{5.62}$$

The value of any function for the dynamical variable $f(q, p)$ is given by

$$\langle f \rangle = \mathrm{Tr}(fP) = Z^{-1}\mathrm{Tr}(f e^{-\beta H}). \tag{5.63}$$

This does show a special similarity with zero temperature quantum mechanics and QFT but the degree of this similarity is not fully illustrated. However, we can push the analogy to calculate partition functions, specially this one. Let us start with a system which can be regarded as a field theory in zero space dimensions.
We can compare this expression to the partition function for the same system at temperature β^{-1}

$$Z = \mathrm{Tr}[e^{-\beta H}] = \sum_q \langle q | e^{-\beta H} | q \rangle. \tag{5.64}$$

Let us draw comparisons between (5.59) and (5.64). If we set $i(t_f - t_i) = \beta$ or alternatively set $t_i = 0$ and then $it_f = \beta$ since the origin of time is arbitrary. Next, set $q^f = q^i$ which means that the initial and final configurations are the same, and since the difference is a β factor, the only requirement is that the relevant configuration is periodic in the functional integrals

$$q(\beta) = q(0). \tag{5.65}$$

Thus the functional integration Dq is over the space of *periodic* functions as stressed in Chapter 3. In this case, the sum over q in (5.59) is implicit. When we do the comparison we can write

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int \mathscr{D}q \int \mathscr{D}p \exp\left[\int_0^\beta d\tau \left(ip\frac{dq}{d\tau} - H\right)\right], \tag{5.66}$$

bearing in mind again that Dq is over periodic functions.
If we take a well-behaved potential $V(q)$ we could scale the temperature dependence purely into the q integral. To do this, we make the following transformations

$$\bar{\tau} = \tau\beta^{-1}, \quad \bar{p} = p\beta^{1/2}, \quad \bar{q} = q\beta^{-1/2}, \tag{5.67}$$

then the exponent of the integrand becomes

$$\int_0^1 d\bar{\tau} \left[i\bar{p}\frac{d\bar{q}}{d\bar{\tau}} - \frac{\bar{p}^2}{2} - \beta V(\beta^{1/2}\bar{q})\right]. \tag{5.68}$$

Furthermore we can drop all the bars because the path integral measure is invariant under the changes (5.67) thus we can write the partition function as

$$Z = \int \mathscr{D}q \int \mathscr{D}p \exp\left[\int_0^1 d\tau \left(ip\dot{q} - \frac{1}{2}p^2 - \beta V(q\beta^{1/2})\right)\right]. \tag{5.69}$$

Next, set $p' = p - i\dot{q}$, so that the measures are equal

$$\mathscr{D}p' = \mathscr{D}p, \tag{5.70}$$

which in turn, by competing the square in the exponent, allows us to write

$$Z = \int \mathscr{D}p' \exp\left[-\frac{1}{2}\int_0^1 d\tau p'^2\right] \int \mathscr{D}q \exp\left[-\int_0^1 d\tau \left(\frac{1}{2}\dot{q}^2 + \beta V(q\beta^{1/2})\right)\right]. \tag{5.71}$$

As we have done repeatedly in AQFT we can ignore the p' integral (even though it is infinite) because it is independent of β. We call this integral N, and the reason why it can be ignored is because N is usually always present in the numerator and denominators of correlation functions.
The only example we could tackle is that of an integral that can be evaluated, i.e. a Gaussian integral. The integral becomes Gaussian when we take the harmonic oscillator potential $V(q) = \frac{1}{2}\omega^2 q^2$, the partition function is

$$Z = N \int \mathscr{D}q \exp\left[-\int_0^1 d\tau \left(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\beta^2\omega^2 q^2\right)\right]. \tag{5.72}$$

It is one of the very few types that can actually be integrated. By virtue of (5.65) we have

$$\frac{1}{2}\int_0^1 d\tau \left(\frac{dq}{d\tau}\right)^2 = -\frac{1}{2}\int_0^1 d\tau q \frac{d^2}{d\tau^2} q, \tag{5.73}$$

since the extra surface term is eliminated and therefore

$$Z = N \int \mathscr{D}q \exp\left[-\frac{1}{2}\int_0^1 d\tau q \left(-\frac{d^2}{d\tau^2} + \omega^2\beta^2\right) q\right]. \tag{5.74}$$

If we proceed by analogy with the discrete case we have

$$\int \mathscr{D}q \exp\left[-\frac{1}{2}\int_0^1 d\tau q \left(-\frac{d^2}{d\tau^2} + \omega^2\beta^2\right) q\right] = \frac{N'}{\sqrt{\mathrm{Det} A}}, \tag{5.75}$$

where N' is a constant, and A is the operator [1], [4], [5]

$$A = -\frac{d^2}{d\tau^2} + \omega^2\beta^2 \tag{5.76}$$

with positive definite eigenvalues (it must not contain zero eigenvalues as these would create infinities which have to be removed). In order to prove (5.75) as we did in Chapter 3 we need to express $q(\tau)$ in terms of its Fourier components (QFT course) then transform into the normal modes of A and integrate each one using

$$\int_0^\infty dq_n \exp\left(-\frac{1}{2}a_n q_n^2\right) = \sqrt{\frac{2\pi}{a_n}}. \tag{5.77}$$

The operator A operates on periodic functions with unit period, which can all be expanded in terms of the complete Fourier set $\{e^{2\pi i n\tau}\}$. The eigenvalues of A are

$$(4\pi^2 n^2 + \omega^2\beta^2); \quad n \in \mathbb{Z}. \tag{5.78}$$

Note the analogy with the eigenvalues of the quantum *mechanical* operator (3.84).
Hence multiplying we have the determinant of the operator A,

$$\det A = \prod_{n\in\mathbb{Z}} (4\pi^2 n^2 + \omega^2\beta^2). \tag{5.79}$$

Setting $x^2 = \omega^2\beta^2$ yields the following

$$\frac{d}{dx^2}\log\det A = \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2 n^2 + x^2} = \frac{1}{x^2} + 2\sum_{n\geq 1}\frac{1}{4\pi^2 n^2 + x^2}. \tag{5.80}$$

Substituting the formula that was shown in the Appendix

$$\coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{n\geq 1}\frac{1}{x^2 + n^2} \tag{5.81}$$

$$\frac{d}{dx^2}\log\det A = \frac{1}{2x}\coth\frac{x}{2} \tag{5.82}$$

and from here we can integrate to find

$$\log\frac{\det A}{C} = \int dx \coth\frac{x}{2} = 2\log\sinh\frac{x}{2} = 2\log\sinh\frac{\omega\beta}{2}. \tag{5.83}$$

Removing the logs we have the formula for the determinant

$$\det A = C\sinh^2\frac{\omega\beta}{2}. \tag{5.84}$$

When we clean the expression we arrive to

$$F = -\frac{1}{\beta}\log Z = -\frac{D}{\beta} + \frac{1}{2}\omega + \frac{1}{\beta}\log(1 - e^{-\omega\beta}), \tag{5.85}$$

where D is another constant, the zero-point energy is identified at $1 = e^{-\omega\beta}$. This formula is called the thermodynamic potential.
Let us go back to our discussion of the ζ function. The heat equation associated with the A operator (5.76) is (see [1], [2], [5])

$$G(t,t',\sigma) = \sum_{n\in\mathbb{Z}} \exp\left\{2\pi i n(t-t') - (\omega^2\beta^2 + 4\pi^2 n^2)\sigma\right\}, \tag{5.86}$$

and recalling our Mellin transform (5.5) for the ζ function

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\sigma\sigma^{s-1}\int_0^1 dt \sum_{n\in\mathbb{Z}}\exp\left\{-(\omega^2\beta^2 + 4\pi^2 n^2)\sigma\right\}. \tag{5.87}$$

Scaling σ by $\omega^2\beta^2 + 4\pi^2 n^2$ leads nowhere as we simply come back to the expression

$$\zeta_A(s) = \sum_{n\in\mathbb{Z}} \frac{1}{(\omega^2\beta^2 + 4\pi^2 n^2)^s}. \tag{5.88}$$

The technique we need to use is somewhat messier, it involves expanding in powers of $\omega\beta$ then integrating and re-arranging the sums. With this in mind we have

$$\sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 \sigma} = 1 + 2\sum_{n\geq 1} e^{-4\pi^2 n^2\sigma}, \tag{5.89}$$

and integrating we have

$$\zeta_A(s) = (\omega\beta)^{-2s} + \frac{2}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(\omega\beta)^{2k}}{k!}(-1)^{-k} \sum_{n=1}^{\infty} \int_0^\infty d\sigma \sigma^{s+k-1} e^{-4\pi^2 n^2 \sigma}. \tag{5.90}$$

Now it is the time when we can rescale σ by $4\pi^2 n^2$ and we can also identify the sum over n with the Riemann ζ function $\zeta(2s) = \sum_{n=1}^\infty n^{-2s}$, which gives

$$\zeta_A(s) = (\omega\beta)^{-2s} + \frac{2}{(4\pi^2)^s}\zeta(2s) + \frac{2}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{(\omega\beta)^{2k}}{k!} \frac{(-1)^k}{(4\pi^2)^{s+k}} \Gamma(s+k)\zeta(2s+2k) \tag{5.91}$$

In order to differentiate ζ_A at $s = 0$ we note that the sum is well behaved at $s = 0$ and as $s \to 0$ a non zero term arises from the derivative of $\Gamma^{-1}(s)$. Recalling our formulas from Chapter 2 (2.30) and (2.33),

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log 2\pi, \tag{5.92}$$

and making use of the following expansions

$$(\omega\beta)^{-2s} + \frac{2}{(4\pi^2)^s}\zeta(2s) = 2(\log 2 + \log\pi - \log 2\pi - \log\omega\beta)s + O(s^2) \tag{5.93}$$

as well as

$$\frac{2}{\Gamma(s)} \frac{(\omega\beta)^{2k}}{k!} \frac{(-1)^k}{(4\pi^2)^{s+k}} \Gamma(s+k)\zeta(2s+2k) = \frac{(-1)^k}{k!} \frac{2^{1-2k}}{\pi^{2k}} (\omega\beta)^{2k}\Gamma(k)\zeta(2k)s + O(s^2) \tag{5.94}$$

it follows (by differentiating with respect to s) that

$$\zeta_A'(s) = -2\log\omega\beta + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(\omega\beta)^{2k}}{(4\pi^2)^k} \zeta(2k) + O(s). \tag{5.95}$$

Consequently we arrive at a neat expression for $\zeta_A'(0)$

$$\zeta_A'(0) = -2\log(\omega\beta) + 2\sum_{k=1}^{\infty} \frac{(\omega\beta)^{2k}(-1)^k\zeta(2k)}{k(4\pi^2)^k}. \tag{5.96}$$

Recalling the formula for the even values of the Riemann ζ function in terms of Bernoulli numbers (2.31)

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}, \tag{5.97}$$

we can continue simplifying (5.96)

$$\zeta_A'(0) = -2\log(\omega\beta) - \sum_{k=1}^{\infty} \frac{(\omega\beta)^{2k}}{(2k)!}\frac{1}{k}B_{2k}. \tag{5.98}$$

Using another formula from the appendix

$$\coth x = \frac{1}{x} + 2\sum_{k=1}^{\infty} \frac{(2x)^{2k-1}}{(2k)!}B_{2k}, \tag{5.99}$$

and integrating yields

$$\int dx \coth x = \log x + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k(2k)!}B_{2k}, \tag{5.100}$$

and finally by comparing with (5.99) and setting $x = \frac{1}{2}\omega\beta$, we obtain

$$\zeta_A'(0) = -2\log(\omega\beta) + 2\log\frac{\omega\beta}{2} - 2\log\sinh\frac{\omega\beta}{2} = -\omega\beta - 2\log(1 - e^{-\omega\beta}). \tag{5.101}$$

Solving for Z yields

$$\log Z = \frac{1}{2}\zeta_A'(0) = -\frac{1}{2}\omega\beta - \log(1 - e^{-\omega\beta}), \tag{5.102}$$

which is the same result we found earlier by evaluating the determinant with the eigenvalue method. Although this technique is lengthy it will enable us to show the connection between the ζ function and interacting quantum field theories in the high temperature limit as $\beta \to 0$.

5.5 High temperature limit

We consider a scalar field $\phi(x) = \phi(t, x^i)$, and canonical conjugate $\Pi(t, x^i)$, interacting with itself. The Hamiltonian in this case is

$$H = \int d^3x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) = \int d^3x \mathscr{H}. \tag{5.103}$$

Generalizing from the quantum mechanical case of Chapter 3, we immediately have

$$Z = \int \mathscr{D}\pi \int \mathscr{D}\phi \exp\left\{\int_0^\beta dt \int d^3x \left(i\Pi\frac{\partial\phi}{\partial t} - \mathscr{H}\right)\right\}, \tag{5.104}$$

again taking the ϕ integral over fields periodic in time $\phi(t, x^i) = \phi(t + \beta, x^i)$ and keeping the space variables unbounded. Rescaling the temperature $\phi \to \phi\beta^{1/2}$ and introducing the change

$$\Pi' = \Pi - i\frac{\partial\phi}{\partial t} \tag{5.105}$$

allows us to perform the Π integral

$$\begin{aligned} Z &= \int \mathscr{D}\pi \int \mathscr{D}\phi \exp\left\{\int_0^1 dt \int d^3x \left(i\Pi\frac{\partial\phi}{\partial t} - \frac{1}{2}\Pi^2 - \frac{1}{2}\beta^2(\nabla\phi)^2 - \beta V(\phi\beta^{1/2})\right)\right\} \\ &= N \int \mathscr{D}\phi \exp\left\{-\int_0^1 dt \int d^3x \left(\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\beta^2(\nabla\phi)^2 + \beta V(\phi\beta^{1/2})\right)\right\}. \end{aligned} \tag{5.106}$$

Note that we have a Euclidean integral. It is essential to note a feature of quantum theory: the temperature dependence also appears in $(\nabla\phi)^2$. We are, of course, working in ϕ^4 theory:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4.$$

The technique for approximate integrals of this sort at second order developped in Chapter 4 Section 4.5 readily gives

$$S[\phi, J] = S[\phi_0, J] + \int dt \int d^3x \left((\phi - \phi_0) \left.\frac{\delta S}{\delta\phi}\right|_{\phi_0} \right) + \frac{1}{2}\int dt \int d^3x \left((\phi - \phi_0)^2 \left.\frac{\delta^2 S}{\delta\phi^2}\right|_{\phi_0} \right) + \cdots, \tag{5.107}$$

where ϕ_0 satisfies the classical equation of motion

$$\left.\frac{\delta S}{\delta\phi}\right|_{\phi_0} = -\frac{\partial^2\phi_0}{\partial t^2} - \beta^2\nabla^2\phi_0 + m^2\beta^2\phi_0 + \frac{\lambda}{3!}\beta^3\phi_0^3 + J\beta^{3/2} = 0. \tag{5.108}$$

By taking the new operator B to be

$$B = \frac{\delta^2 S}{\delta\phi_0^2} = -\frac{\partial^2}{\partial t^2} - \beta^2\nabla^2 + m^2\beta^2 + \frac{\lambda}{2}\beta^2\phi_0^2 \tag{5.109}$$

and shifting the integration variable from ϕ to $\phi - \phi_0$ we obtain

$$\begin{aligned} Z &= Ne^{-S[\phi_0,J]}\int \mathscr{D}\phi \exp\left[-\frac{1}{2}\int_0^1 dt \int d^3x\phi \left(-\frac{\partial^2}{\partial t^2} - \beta^2\nabla^2 + m^2\beta^2 + \frac{\lambda}{2}\beta^3\phi_0^2\right)\phi\right] \\ &= \frac{N'e^{-S[\phi_0,J]}}{\sqrt{\det B}} = N'e^{-S[\phi_0,J]}(\det B)^{-1/2} = N'e^{-S[\phi_0,J]}\exp\left(\frac{1}{2}\zeta_B'(0)\right), \end{aligned} \tag{5.110}$$

with N' unknown. As before, with the non-interacting massless case, we will only consider constant ϕ_0 which will give information about the part of the one loop correction which does not depend on derivatives of ϕ_0 [Ramond]. Scaling backwards the classical equation of motion gives $\phi_0(t, x^i) = \beta^{1/2}\bar{\phi}_0(\beta t, x^i)$ where $\bar{\phi}_0$ is indepedent of β. In order to do this, we need to split up the operator B in two parts

$$B = -\frac{\partial^2}{\partial t^2} + \beta^2 C^2 \tag{5.111}$$

with C being β *independent*

$$C = -\nabla^2 + m^2 + \frac{\lambda}{2}\bar{\phi}_0^2. \tag{5.112}$$

The first step is to go back to the C-operated heat equation

$$C_x G_C(x^i, y^i, \sigma) = -\frac{\partial}{\partial\sigma}G_C(x^i, y^i, \sigma) \tag{5.113}$$

where C_x indicates that we are operating on x. The boundary condition is as usual

$$G_C(x^i, y^i, \sigma = 0) = \delta(x^i - y^i). \tag{5.114}$$

The solution we had for four dimensions can easily be generalized to d dimensions yielding

$$G_C(x^i, y^i, \sigma) = \frac{\mu^d}{(4\pi\sigma)^{d/2}} \exp\left\{-\frac{\mu}{4\sigma}(x^i - y^i)^2\right\} \exp\left\{-\left(m^2 + \frac{\lambda}{2}\bar{\phi}_0^2\right)\frac{\sigma}{\mu^2}\right\}, \tag{5.115}$$

where μ has the purpose of making σ dimensionless, i.e. it has mass dimension. As we argued for the quantum mechanical case, the eigenvalues for $-\frac{\partial^2}{\partial t^2}$ over periodic functions are $4\pi^2 n^2$ hence the heat kernel is essentially the same as (5.86)

$$\begin{aligned} G_B(t, x^i, t', y^i, \sigma) = {} & \frac{\mu^d}{(4\pi\sigma)^{d/2}} \exp\left\{-\frac{\mu^2}{4\sigma}(x^i - y^i)^2 - M^2\beta^2\frac{\sigma}{\mu^2}\right\} \\ & \times \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{4\pi^2 n^2}{\mu^2\beta^2}\sigma + 2\pi i n(t - t')\right\} \end{aligned} \tag{5.116}$$

where

$$M^2 = m^2 + \frac{\lambda}{2}\bar{\phi}_0^2. \tag{5.117}$$

The trace of the kernel then becomes

$$\begin{aligned} \mathrm{Tr} G_B &= \int d^\mu x G_B(t, x, t', x, \sigma) = \int_0^1 dt \int d^d x G_B(0, x, 0, x, \sigma) \\ &= \frac{\mu^d}{(4\pi\sigma)^{d/2}} e^{-M^2\beta^2\sigma/\mu^2} \int_0^1 dt \int d^d x \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 \sigma/(\mu^2\beta^2)}. \end{aligned} \tag{5.118}$$

Finally, putting this in the Mellin transform

$$\begin{aligned} \zeta_B(s) &= \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \mathrm{Tr} G_B(\sigma) \\ &= \frac{1}{\Gamma(s)} \frac{\mu^d}{(4\pi)^{d/2}} e^{-M^2\beta^2\sigma/\mu^2} \int_0^\infty d\sigma \sigma^{s-1-d/2} \int_0^1 dt \int d^d x \sum_{n=-\infty}^{\infty} e^{-(4\pi^2 n^2/(\mu^2\beta^2))\sigma} \end{aligned} \tag{5.119}$$

We note the following two points: (1) $d = 0$ is the quantum mechanical result and (2) the volume $V = \int d^d x$ can be regularized by constraining the system to be a finite box. Scaling by $\mu^2\beta^2\sigma \to \sigma$ simplifies the exponential in the sum

$$\zeta_B(s) = \frac{V}{\Gamma(s)} \frac{(\mu\beta)^{2s}}{(4\pi\beta^2)^{d/2}} \int_0^\infty d\sigma \sigma^{s-1-d/2} \sum_{n=-\infty}^{\infty} e^{-(4\pi^2 n^2 + M^2\beta^2)\sigma}. \tag{5.120}$$

As we did in (5.89) we split up the infinite sum as

$$\sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 \sigma} = 1 + 2\sum_{n=1}^{\infty} e^{-4\pi^2 n^2 \sigma}, \tag{5.121}$$

and then we have for $\zeta_B(s)$

$$\begin{aligned}\zeta_B(s) &= \frac{V(\mu\beta)^{2s}}{(4\pi\beta^2)^{d/2}}\frac{1}{\Gamma(s)}\left(\int_0^\infty d\sigma\sigma^{s-1-d/2}e^{-M^2\beta^2\sigma} + 2\int_0^\infty d\sigma\sigma^{s-1-d/2}e^{-M^2\beta^2\sigma}\sum_{n=1}^\infty e^{-4\pi^2n^2}\right) \\ &= \frac{V(\mu\beta)^{2s}}{(4\pi\beta^2)^{d/2}}\frac{1}{\Gamma(s)}\left(\Gamma(s-d/2) + 2\int_0^\infty d\sigma\sigma^{s-1-d/2}\sum_{n=1}^\infty e^{-(4\pi^2n^2+M^2\beta^2)\sigma}\right) \end{aligned} \tag{5.122}$$

Specialising to $d = 3$ we set the task of finding the limit of high temperature $\beta \to 0$.

$$\begin{aligned}\zeta_B(s) &= \frac{VM^3}{8\pi^{3/2}}\left(\frac{\mu}{M}\right)^{2s}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)} \\ &+ 2V\left(\frac{\pi}{\beta^2}\right)^{3/2}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\sum_{k=0}^\infty\frac{(\beta M)^{2k}}{k!}\frac{(-1)^k}{(2\pi)^{2k+2s}}\Gamma(s-\tfrac{3}{2}+k)\zeta(2s+2k-3)\end{aligned} \tag{5.123}$$

Ramond [5] leaves the evaluation of these terms as an exercise and I welcome the opportunity to provide my solution. We shall make use of

$$\Gamma(s)^{-1} = s + O(s^2), \tag{5.124}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{5.125}$$

$$\Gamma\left(s-\frac{u}{2}\right) = \Gamma\left(-\frac{u}{2}\right) + \Gamma\left(-\frac{u}{2}\right)\psi^{(0)}\left(-\frac{u}{2}\right)s + O(s^2) \tag{5.126}$$

and

$$\frac{VM^3}{8\pi^{3/2}}\left(\frac{\mu}{M}\right)^{2s} = \frac{VM^3}{8\pi^{3/2}} + \left(\frac{M^3V}{4\pi^{3/2}}\log\frac{\mu}{M}\right)s + O(s^2), \tag{5.127}$$

$$\frac{VM^3}{8\pi^{3/2}}\left(\frac{\mu}{M}\right)^{2s}\frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)} = \frac{VM^3}{6\pi}s + O(s^2), \tag{5.128}$$

$$\frac{d}{ds}\frac{VM^3}{8\pi^{3/2}}\left(\frac{\mu}{M}\right)^{2s}\frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)} = \frac{VM^3}{6\pi} + O(s). \tag{5.129}$$

The first term of the sum is very similar, except that it needs

$$\zeta(2s-3) = \zeta(-3) + 2\zeta'(-3)s + O(s^2) \tag{5.130}$$

$$(\alpha\beta)^{\pm s} = 1 \pm s\log(\alpha\beta) + O(s^2) \tag{5.131}$$

$$\zeta(-3) = \frac{1}{120} \tag{5.132}$$

yielding

$$\frac{\pi^2 V}{45\beta^3}s + \frac{\pi^2 V}{45\beta^3}\left[2\log\mu\beta + 240\zeta(-3) + \pi^2\psi^{(0)}\left(-\frac{3}{2}\right) - 2\log\pi - 2\log 2 + \gamma\right]s + O(s^3) \tag{5.133}$$

differentiating with respect to s gives

$$\frac{\pi^2 V}{45\beta^3}. \tag{5.134}$$

The third term (the second term in the sum, that is) requires $\zeta(-1) = -\frac{1}{12}$ and the same technique gives

$$-\frac{VM^2}{12\beta}. \tag{5.135}$$

The term at $k = 2$

$$\zeta_B(s) = \frac{VM^3}{8\pi^{3/2}}\left(\frac{\mu}{M}\right)^{2s}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)} + 2V\left(\frac{\pi}{\beta^2}\right)^{3/2}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\frac{(\beta M)^{2\cdot 2}}{k!}\frac{(-1)^2}{(2\pi)^{2\cdot 2+2s}} \times \Gamma(s-\tfrac{3}{2}+2)\zeta(2s+2\cdot 2-3) \tag{5.136}$$

is more delicate as at $s = 0$ we have a singularity coming from $\zeta(1)$. This can be evaluated as follows by expanding each factor separately

$$2V\pi^{3/2}\beta^{-3}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\frac{(\beta M)^4}{2}\frac{1}{(2\pi)^{4+2s}}\Gamma(s+\tfrac{1}{2}) = \frac{M^4V\beta}{16\pi^2}s + \frac{M^4V\beta}{16\pi^2} \times \left[\gamma - 2\log 2 - 2\log\pi + \log\mu\beta + \psi^{(0)}\left(0, \frac{1}{2}\right)\right]s^2 + O(s^3). \tag{5.137}$$

We make use of the Laurent expansion of $\zeta(s)$ instead of the Taylor expansion to account for the simple pole at $s = 0$

$$\zeta(2s+1) = \frac{1}{2s} + \gamma - 2\gamma_1 s + O(s^2), \tag{5.138}$$

where γ_1 is a constant. Then multiplying these two expansions together

$$2V\pi^{3/2}\beta^{-3}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\frac{(\beta M)^4}{2}\frac{1}{(2\pi)^{4+2s}}\Gamma(s+\tfrac{1}{2})\zeta(2s+1) = \frac{M^4V\beta}{32\pi^2} + \frac{M^4V\beta}{32\pi^2}\left(\gamma + \log\frac{\mu\beta}{4\pi}\right)s + O(s^2). \tag{5.139}$$

Differentiating with respect to s we have

$$\frac{d}{ds}\left(2V\pi^{3/2}\beta^{-3}\frac{(\mu\beta)^{2s}}{\Gamma(s)}\frac{(\beta M)^4}{2}\frac{1}{(2\pi)^{4+2s}}\Gamma(s+\tfrac{1}{2})\zeta(2s+1)\right) = \frac{M^4V\beta}{16\pi^2}\left(\gamma + \log\frac{\mu\beta}{4\pi}\right) + O(s). \tag{5.140}$$

Therefore we have

$$\log Z = \frac{1}{2}\zeta_B'(0) = V\left[\frac{M^3}{12\pi} + \frac{\pi^2}{90\beta^3} - \frac{M^2}{24\beta} + \frac{\beta M^4}{32\pi^2}\left(\gamma + \log\frac{\mu\beta}{4\pi}\right) + \cdots\right] \tag{5.141}$$

which amounts to a free energy per unit volume [5]

$$\frac{F}{V} = -\frac{M^3}{12\pi\beta} - \frac{\pi^3}{90\beta^4} + \frac{M^2}{24\beta^2} - \frac{M^4}{32\pi^2}\left(\gamma + \log\frac{\mu\beta}{4\pi}\right) + \cdots \tag{5.142}$$

in the limit of high temperature $\beta \to 0$.

5.6 Equivalence of ζ and dimensional regularization in φ^4 theory

Let us now first summarize the features of the ζ regularization technique. We shall start with with situations where we do not know explicitly the eigenvalues of a non-negative self-adjoint operator A

$$A\psi_n(x) = \lambda_n \psi_n(x). \tag{5.143}$$

In these cases, we consider the heat equation

$$\frac{\partial}{\partial t} G_A(x,y,t) + AG_A(x,y,t) = 0 \tag{5.144}$$

where A is taken to act on the first argument of G_A. The initial condition is

$$G_A(x,y,0) = \delta(x-y). \tag{5.145}$$

The expression G_A is the heat kernel and it accounts for the diffusion over a region of spacetime of a unit of heat placed at y at $t=0$, i.e.

$$G_A(x,y,t) = \langle x|e^{-tA}|y\rangle = \sum_n e^{-t\lambda_n}\psi_n(x)\psi_n^*(y). \tag{5.146}$$

When we set $x=y$ and we integrate over spacetime we obtain

$$\int d^4x G_A(x,x,t) = \sum_n e^{-\lambda_n t} = \mathrm{Tr}[G_A(t)]. \tag{5.147}$$

The ζ function of A is connected to $\mathrm{Tr}G_A(t)$ by the Mellin transform

$$\zeta_A(s) = \sum_n \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int\limits_0^\infty dt t^{s-1} \mathrm{Tr} G_A(t), \tag{5.148}$$

with the understanding that zero modes are not taken in the sum.
If we make the transformation $A \to A' = \alpha^{-1}A$. The eigenvalues of A become $\lambda_n \to \alpha^{-1}\lambda_n$ and also the scale μ becomes μ'. The new ζ_A is

$$\zeta_{A'/(\mu')^2}(s) = \alpha^{2s}\left(\frac{\mu'}{\mu}\right)^{2s} \zeta_{A/\mu^2}(s). \tag{5.149}$$

Accordingly we obtain

$$\begin{aligned}\log\det\left(A'/(\mu')^2\right) &= -\frac{d}{ds}\zeta_{A'/(\mu')^2}(s)\Big|_{s=0} \\ &= \log\det(A/\mu^2) - \log\alpha^2\zeta_{A/\mu^2}(0) - \log\left(\frac{\mu'}{\mu}\right)^2 \zeta_{A/\mu^2}(0),\end{aligned}$$

the presence of $\zeta_{A/\mu^2}(0)$ indicates that $\det A$ is modified by the transformation.
Going back to the heat kernel [1], [2] and [5] $\mathrm{Tr}G_A(t) = \sum_n e^{-t\lambda_n}$ and multplying it by e^{-tm^2} and then integrating over t

$$\int\limits_0^\infty dt e^{-tm^2} \mathrm{Tr} G_A(t) = \sum_n (\lambda_n + m^2)^{-1}. \tag{5.150}$$

Performing another integration, this time with respect to m^2, and swapping the integrals yields

$$\int_0^\infty dm^2 \int_0^\infty dt e^{-tm^2} \mathrm{Tr} G_A(t) = \int_0^\infty dt t^{-1} \mathrm{Tr} G_A(t) = \sum_n \log(\lambda_n + m^2)\Big|_{m=0}^{m=\infty}. \tag{5.151}$$

Ignoring the upper limit we have arrived at

$$\log\det A = \sum_n \log \lambda_n = -\int_0^\infty dt t^{-1} \mathrm{Tr} G_A(t). \tag{5.152}$$

We need aslo, a formality, the introduction of a cutoff ε for evaluating $\log\det A$, i.e.

$$-\lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty dt t^{-1} \mathrm{Tr} G_A(t), \tag{5.153}$$

this is called the proper-time cutoff. This procedure necessitates the evaluation of the above integral by asympotitic expansion of $\mathrm{Tr} G_A(t)$.
However, if the determinant needs ζ regularization then

$$\log\det A = -\zeta_A'(0) = -\frac{d}{ds}\left[\frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \mathrm{Tr} G_A(t)\right]_{s=0} \tag{5.154}$$

An expansion around $s = 0$ of $1/\Gamma(s)$ using Weierstrass product

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n} = s(1 + \gamma s) + O(s^2) \tag{5.155}$$

and the polygamma function $\psi^{(0)}(s)$

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{d}{ds} \log\Gamma(s) = \psi^{(0)}(s) \tag{5.156}$$

$$\begin{aligned} \frac{d}{ds}\frac{1}{\Gamma(s)} = \frac{\psi^{(0)}(s)}{\Gamma(s)} &= s(1+\gamma s)\left(-\frac{1}{s} - \gamma + \zeta(2) s + \frac{1}{2}\psi^{(2)}(1) s^2 + O(s^3)\right) \\ &= -1 - 2\gamma s + O(s^2) \end{aligned} \tag{5.157}$$

by the use of

$$\psi^{(0)}(s+1) = \log\Gamma(s+1) = -\gamma s + \sum_{n=2}^\infty \frac{(-1)^n}{n} s^n \zeta(n) \tag{5.158}$$

gives

$$\log\det A = \lim_{s\to 0}\left[(1 + 2\gamma s)\int_0^\infty dt t^{s-1} \mathrm{Tr} G_A(t) + s(1+\gamma s)\int_0^\infty dt t^{s-1} \log t \mathrm{Tr} G_A(t)\right]. \tag{5.159}$$

Note how the above expression for $\log\det A$ without ζ function would follow if we ignored the divergent intrgral for $s = 0$.
The comparision begins by now trying to obtain the same answer for $\log\det A$ from dimensional regularization. To this end, we generalize the heat equation to $2\omega + 1$ dimensions noting that pole is now shifted to $2\omega = 4$. One way to do this is to take the product of the initial 4 dimensional spacetime with $2\omega - 4$ flat dimensions. In this case, the heat kernel $\mathrm{Tr}G_A(t)$ is changed by a factor of $(4\pi t)^{2-\omega}$ hence

$$\log\det A = -\frac{1}{(4\pi)^{\omega-2}}\int_0^\infty dt t^{1-\omega}\mathrm{Tr}G_A(t) = -(4\pi)^{2-\omega}\Gamma(2-\omega)\zeta_A(\omega-2). \tag{5.160}$$

A final expansion around $\omega = 2$ by the use of

$$-(4\pi)^{2-\omega}\Gamma(2-\omega) = \frac{1}{\omega-2} + \gamma - \log 4\pi + O(\omega-2) \tag{5.161}$$

and

$$\zeta_A(\omega-2) = \zeta_A(0) - \zeta_A'(0)(\omega-2) + O((\omega-2)^2) \tag{5.162}$$

yields

$$\log\det A = \frac{\zeta_A(0)}{\omega-2} + (\gamma - \log 4\pi)\zeta_A(0) - \zeta_A'(0) + O(\omega-2). \tag{5.163}$$

This encapsulates a fundamental result which has been at the core of the ζ and dimensional regularizations [1], namely there is a pole at $2\omega = 4$ with residue $\zeta_A(0)$ and finite part $-\zeta_A'(0) + (\gamma - \log 4\pi)\zeta_A(0)$. Consequently, *there is an equivalence (agreement is a better word) between the values of* $\log Z$ *derived by* ζ *and dimensional regularization.* This equivalence is up to a multiple of $\zeta_A(0)$ which can be absorbed in the normalization constant.
For the sake of completeness we finish the summary of the ζ technique by following the technique described in [1] but omitting their use of the spacetime metric. When we evaluate path integrals in curved spacetimes we compute exressions of the form

$$Z[\phi] = \int \mathscr{D}\phi \exp(iS[\phi]), \tag{5.164}$$

where $\mathscr{D}\phi$ is a measure on the space of matter field and $S[\phi]$ is the classical action. Certain boundary (or periodicity) conditions are satisfied by ϕ. For example, for temperature $T = 1/\beta$ the boson fields are periodic in imaginary time on some boundary at large distance, with period β. Then Z is the partition function from statistical mechanics. The leading contriubtion to the path integral will come from field configurations near the background ϕ_0 which satisfies the classical equations in addition to the boundary conditions. Setting $\phi = \phi_0 + \tilde{\phi}$ the action can be expanded about the background fields

$$S[\phi] = S[\phi_0] + S_{22}[\tilde{\phi}] + \cdots \tag{5.165}$$

where $S_{22}[\tilde{\phi}]$ is quadratic in the fluctuations of ϕ. Therefore we have

$$\log Z = iS[\phi_0] + \log\int \mathscr{D}\tilde{\phi}\exp(iS_{22}[\tilde{\phi}]) + \cdots \tag{5.166}$$

The quadratic term is also of the form

$$S_{22}[\tilde{\phi}] = -\frac{1}{2}\int d^4x\tilde{\phi}A_2\tilde{\phi} \tag{5.167}$$

with A_2 a second order differential operator constructed from the background field. (Note that in fermionic fields the operator would be of first order). The condition on the metric for A_2 to be real (elliptic) and self-adjoint is that the background metric be Euclieam. These attributes of A_2 will guarantee the existence of a complete set of eigenfuction ψ_n and spectrum λ_n such that

$$A_2\phi_n = \lambda_n\phi_n, \tag{5.168}$$

with orthogonality

$$\int d^4x\phi_n\phi_m = \delta_{nm}. \tag{5.169}$$

The field fluctuation $\tilde{\phi}$ can be expressed as

$$\tilde{\phi} = \sum_n \theta_n\phi_n \tag{5.170}$$

where the measure on the field can be written in terms of these θ_n coefficients

$$\mathscr{D}\phi = \prod_n \mu d\theta_n. \tag{5.171}$$

Here μ is a normalization constant with mass dimension. Putting all of this together yields

$$\begin{aligned} Z[\tilde{\phi}] &:= \int \mathscr{D}\phi\exp(iS_{22}[\tilde{\phi}]) = \prod_n\int \mu d\theta_n \exp\left(-\frac{1}{2}\lambda_n\theta_n^2\right) \\ &= \prod_n \mu\left(\frac{2\pi}{\lambda_n}\right)^{1/2} = \left[\det\left(\frac{1}{2\pi\mu^2}A_2\right)\right]^{-1/2}. \end{aligned} \tag{5.172}$$

This means that the quadratic contribution in the field fluctuations is computed by evaluating a determinant. The issue is that the convergence of the product is not obvious, let alone guaranteed, therefore making this expression sensible is a difficult problem. Finally, the free energy is proportional to the log of Z which by the use of

$$\det A_2 = \exp(\zeta'_{A_2}(0)) \tag{5.173}$$

becomes

$$\log Z[\tilde{\phi}] = \frac{1}{2}\zeta'_{A_2}(0) + \frac{1}{2}\log\left(\frac{1}{2\pi\mu^2}\right)\zeta_{A_2}(0). \tag{5.174}$$

Finally, let us summarize how the effective Lagrangian is affected by ζ regularization through the effective action for scalar fields φ^4. Following [1], let us consider a 2-dimensional spacetime area S with a constant electromagnetic field with field strength

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{5.175}$$

and potential

$$(A_x, A_y) = (0, Bx). \tag{5.176}$$

The quadratic differential operator we are interested in is

$$-\Delta^2 = -(\partial_\mu - iA_\mu)(\partial^\mu - iA^\mu) = -\partial_x^2 - (\partial_y - iBx)^2, \tag{5.177}$$

$-\Delta^2$ commutes with the second component of the momentum $\hat{p}_y = -i\partial_y$, hence taking the eigenvalues p_y of $\hat{p}_y$ we have

$$-\Delta^2 = -\partial_x^2 + B^2 \left(x - \frac{p_y}{B}\right)^2. \tag{5.178}$$

We can already see the form of the Hamiltonian of the harmonic oscillator shapping up. In fact, the change of variables $x \to x' = x - p_y/B$ gives twice this Hamiltonian with frequency $|B|$. Consequently, in this case, the eigenvalues are

$$\lambda_{p_y,n} = 2|B| \left(n + \frac{1}{2}\right). \tag{5.179}$$

Note that the independence of p_y indicates that all the levels are degenerate. Let us now produce the heat kernel

$$\mathrm{Tr}\left[G_{-\Delta^2+m^2}(t)\right] = \sum_{\lambda \in (-\Delta^2)} e^{-t(\lambda+m^2)} = \frac{SB}{4\pi} e^{-tm^2} \sum_{n\geq 0} e^{-tB(2n+1)} = \frac{SB}{4\pi} e^{-tm^2} \mathrm{csch}(tB) \tag{5.180}$$

because the degeneracy is $\frac{SB}{2\pi}$ and we take B to be positive. Note that when $B \to 0$ we have $\mathrm{csch}(tB) \sim (tB)^{-1}$ so that the free heat kernel becomes

$$\mathrm{Tr}\left[G_{-\Delta^2+m^2}(t)\right] \underset{B\to 0}{\sim} \frac{SB}{4\pi} e^{-tm^2} (tB)^{-1} = \frac{1}{4\pi t} e^{-tm^2} S. \tag{5.181}$$

The $\zeta_{-\Delta^2+m^2}$ then becomes

$$\zeta_{-\Delta^2+m^2}(s) = \frac{SB}{2\pi} \sum_{n=0}^{\infty} \left[2B\left(n + \frac{1}{2}\right) + m^2\right]^{-s} = \frac{SB}{2\pi} (2B)^{-s} \zeta\left(s, \frac{1}{2} + \frac{m^2}{2B}\right), \tag{5.182}$$

where we have a Hurwitz ζ function. The 1-loop effective Lagrangian is

$$\mathscr{L}_{\mathrm{eff}}^{(1)} = \frac{1}{S} S_{\mathrm{eff}}^{(1)} = \frac{1}{2S} \log \mathrm{Det}(-\Delta^2 + m^2) = -\frac{1}{2S} \zeta'_{-\Delta^2+m^2}(0). \tag{5.183}$$

A similar argument as that used in the proof of Theorem 9 of Chapter 2 shows that

$$\zeta\left(0, v\right) = \frac{1}{2} - v \tag{5.184}$$

and

$$\left.\frac{d}{ds}\zeta(s, v)\right|_{s=0} = \zeta'(0, v) = \log\Gamma(v) - \frac{1}{2}\log 2\pi, \tag{5.185}$$

in fact, note how at $v = 1$ we have the formula of Theorem 9. Hence

$$\begin{aligned}\mathscr{L}_{\text{eff}}^{(1)} &= \pi^{-1}2^{-2}B\left[\log(2B)\zeta\left(0, \frac{m^2+B}{2B}\right) - \zeta'\left(0, \frac{m^2+B}{2B}\right)\right] \\ &= \pi^{-1}2^{-2}B\left[\log(2B)\left(\frac{1}{2} - \frac{m^2+B}{2B}\right) - \log\Gamma\left(\frac{m^2+B}{2B}\right) + \frac{1}{2}\log 2\pi\right] \\ &= \frac{B}{8\pi}\left[\log 2\pi - 2\log\Gamma\left(\frac{m^2+B}{2B}\right) - m^2\log 2B\right] \end{aligned} \tag{5.186}$$

The physically interesting limits occur as $m \to 0$ and $B \to 0$ for which we have

$$\lim_{m\to 0}\mathscr{L}_{\text{eff}}^{(1)} = \frac{B}{8\pi}\log 2. \tag{5.187}$$

The limit as $B \to 0$ necessitates the Stirling formula (A.88) for $\log\Gamma(s+1)$ which is proved in the Appendix

$$\log\Gamma(s+1) = \frac{1}{2}\log 2\pi + \left(s + \frac{1}{2}\right)\log s - s + \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^s} - \cdots \tag{5.188}$$

so that

$$\lim_{B\to 0}\mathscr{L}_{\text{eff}}^{(1)} = -\frac{1}{16\pi}(m^2\log m^2 - m^2). \tag{5.189}$$

This concludes our summary of the ζ technique.

REFERENCES

The discussion of the role of the ζ function in quantum field theory follows from

- [1] EOR's *Zeta Regularization Techniques with Applications* (first three chapters)
- [2] Hartfield's *Quantum Field Theory of Point Particles and Strings* [611 to 616]
- [3] Hawking's paper was a quoted in the context of the determinant of a general operator A.
- [4] Kleinert's *Path Integrals*
- [5] Ramond's *Field Theory* [81 to 93]

Additionally, the exposition is the progression from the ideas presented in Chapter 3 (ζ functions in quantum mechanics) combined with Chapter 4 (path integrals in quantum fields). Finally, the second quoted paper is

- [6] S. Coleman and E. Weinberg *Phys. Rev.* D7 1888 (1973).

6 Casimir Effect

6.1 Experimental setup

Hendrik Casimir and Dirk Polder discovered the existence of the Casimir effect in 1948.
The Casimir effect is a force arising from a quantized field. For instance two uncharged metallic plates in a vacuum, placed a few micrometers apart, without any external electromagnetic field, affect the virtual photons which constitute the field, and generate a net force [1]: either an attraction or a repulsion depending on the specific arrangement of the two plates.

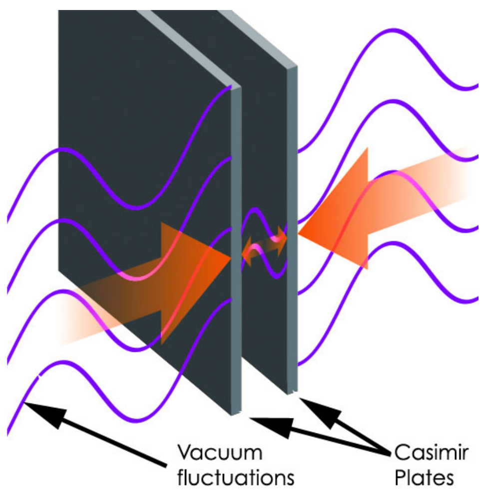

Figure 6.1: Two uncharged metallic plates in a vacuum, placed a few micrometers apart

The strength of the force falls off rapidly with distance thus it is only measurable when the distance between the objects is extremely small. We shall describe and compute the Casimir effect in terms of the zero-point energy of a quantized field in the intervening space between the objects instead of expressing it in terms of virtual particles interacting with the objects.
The Casimir effect can be understood by the idea that the presence of conducting metals alters the vacuum expectation value of the energy of the second quantized electromagnetic field [2].

6.2 ζ regulator

Using $E = \omega/2$ we can determine the vacuum expectation value of the energy of the electromagnetic field in the cavity to be

$$\langle E \rangle = \frac{1}{2} \sum_n E_n \tag{6.1}$$

with the sum running over all possible values of n accounting for the standing waves. Note that this sum is divergent. Each energy level E_n depends on the shape consequently $E_n(s)$ is the energy level, and $\langle E(s) \rangle$ is the vacuum expectation value.

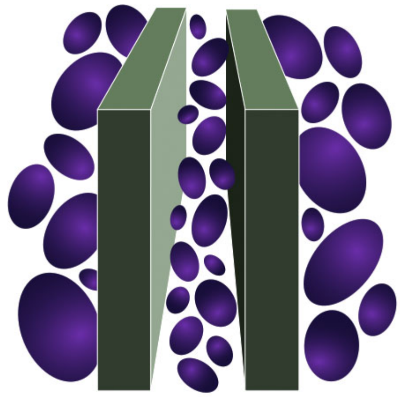

Figure 6.2: Virtual particles interacting with the plates

The force at point P on the wall of the cavity is equal to the change in the vacuum energy if the shape Ω of the wall is perturbed infinitesimally, say by $\delta\Omega$, at point P, i.e.

$$F(\mathrm{P}) = - \left. \frac{\delta \langle E(\Omega) \rangle}{\delta \Omega} \right|_{\mathrm{P}} \tag{6.2}$$

Casimir considered the space between a pair of conducting metal plates at distance r apart. In this case, the standing waves can be calculated, since the transverse component of the electric field and the normal component of the magnetic field must vanish on the surface of a conductor.
Ignoring the polarization and the magnetic components and assuming the parallel plates lie in the x-y plane, the standing waves are

$$\psi_n(x^i, t) = e^{-i\omega_n t} e^{i(k_x x + k_y y)} \sin(k_n z) \tag{6.3}$$

where ψ stands for the electric component of the electromagnetic field. Here, k_x and k_y are the wave vectors in directions parallel to the plates, and

$$k_n = \frac{n\pi}{a} \tag{6.4}$$

is the wave-vector perpendicular to the plates. Here, n is an integer, resulting from the requirement that ψ vanish on the metal plates. The energy of this wave is

$$\omega_n = \sqrt{k_x^2 + k_y^2 + \frac{n^2\pi^2}{r^2}}. \tag{6.5}$$

The vacuum energy is then the sum over all possible excitation modes

$$\langle E \rangle = \frac{1}{2} \int \frac{dk_x dk_y}{(2\pi)^2} 2 \sum_{n=1}^{\infty} A\omega_n \tag{6.6}$$

where A is the area of the metal plates, and a factor of 2 is introduced for the two possible polarizations of the wave. This expression is divergent, so we introduce a ζ regulator to make the expression finite, and in the end will be remove this regulator. The ζ regulated version of the energy per unit-area of the plate is

$$\frac{\langle E(s)\rangle}{A} = \int \frac{dk_x dk_y}{(2\pi)^2} \sum_{n=1}^{\infty} \omega_n \left|\omega_n\right|^{-s}. \tag{6.7}$$

This integral is finite for $s > 3$. The sum has a pole at $s = 3$, however it may be analytically continued to $s = 0$, where the expression is finite. We have

$$\begin{aligned} \frac{\langle E(s)\rangle}{A} &= \int \frac{dk_x dk_y}{(2\pi)^2} \sum_{n=1}^{\infty} \left(k_x^2 + k_y^2 + \frac{n^2\pi^2}{r^2}\right)^{1/2} \left|k_x^2 + k_y^2 + \frac{n^2\pi^2}{r^2}\right|^{-s/2} \\ &= \frac{1}{4\pi^2} \int dk_x dk_y \sum_{n=1}^{\infty} \left|k_x^2 + k_y^2 + \frac{n^2\pi^2}{r^2}\right|^{(1-s)/2}. \end{aligned} \tag{6.8}$$

We now introduce polar coordinates $\kappa^2 = k_x^2 + k_y^2$ and $dk_x dk_y = \kappa d\kappa d\theta$

$$\begin{aligned} \frac{\langle E(s)\rangle}{A} &= \frac{1}{4\pi^2} \sum_n \int_0^{\infty} \int_0^{2\pi} d\kappa d\theta \kappa \left|\kappa^2 + \frac{n^2\pi^2}{r^2}\right|^{(1-s)/2} = \frac{1}{4\pi^2} \sum_n \int_0^{\infty} 2\pi d\kappa\kappa \left|\kappa^2 + \frac{n^2\pi^2}{r^2}\right|^{(1-s)/2} \\ &= \frac{1}{2\pi} \sum_n \left(\frac{|n|^2}{r^2}\right)^{(3-s)/2} \frac{\pi^{3-s}}{s-3} = -\frac{1}{2r^{3-s}} \frac{\pi^{2-s}}{3-s} \sum_n |n|^{3-s}. \end{aligned} \tag{6.9}$$

At $s = 0$ we have the Riemann ζ function and $\zeta(-3) = \frac{1}{120}$ from Chapter 1

$$\frac{\langle E\rangle}{A} = \lim_{s\to 0} \frac{\langle E(s)\rangle}{A} = -\frac{1}{2r^3}\frac{\pi^2}{3}\zeta(-3) = -\frac{\pi^2}{6r^3}\frac{1}{120} = -\frac{\pi^2}{720r^3}. \tag{6.10}$$

Note that in terms of the Planck constant and the speed of light the above translates to [3]

$$\frac{\langle E\rangle}{A} = -\frac{\hbar c\pi^2}{720r^3}. \tag{6.11}$$

The Casimir force per unit area F_{Cas}/A for idealized, perfectly conducting plates with vacuum between them is

$$\frac{F_{\text{Cas}}}{A} = -\frac{d}{dr}\frac{\langle E\rangle}{A} = -\frac{\hbar c\pi^2}{240r^4}. \tag{6.12}$$

The minus sign indicates that the force is attractive, also it decreases faster than gravity due to the r^4 in the denominator.

6.3 Experimental evidence

The original form of the experiment, described above, successfully demonstrated the force to within 15% of the value predicted by the theory [4].

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.4. A Force from Empty Space: The Casimir Effect http://apod.nasa.gov/apod/ap061217.html

7 Conclusion

We have seen that the task of computing the generating functional $Z[J]$

$$Z[J] = \exp(-iE[J]) = \int \mathscr{D}\phi \exp\left[i\int d^4x(\mathscr{L} + J\phi)\right] = \langle\Omega|e^{-iHT}|\Omega\rangle, \qquad (7.1)$$

of a scalar field theory with source J is reduced to

$$\begin{aligned}-iE[J] = i\int d^4x(\mathscr{L}_1[\phi_{\text{cl}}] + J_1\phi_{\text{cl}}) - \frac{1}{2}\log\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right] + (\text{connected diagrams})\\ + i\int d^4x(\delta\mathscr{L}[\phi_{\text{cl}}] + \delta J\phi_{\text{cl}}),\end{aligned} \qquad (7.2)$$

where the classical field is taken to be

$$\phi_{\text{cl}}(x) = \langle\Omega|\phi(x)|\Omega\rangle_J = -\frac{\delta}{\delta J(x)}E[J]. \qquad (7.3)$$

The lowest order quantum corrections to to the effective potential $\Gamma[\phi_{\text{cl}}]$ is given by the functional determinant because the Feynman diagrams contributing to it have no external lines and the simplest ones turn out to have two loops hence [3]

$$\Gamma[\phi_{\text{cl}}] := -E[J] - \int d^4x' J(x')\phi_{\text{cl}}(x') = \Gamma_E^{(0)}[\phi_{\text{cl}}] + \hbar\Gamma_E^{(1)}[\phi_{\text{cl}}] + \cdots = \frac{i}{2}\log\det\left[-\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)}\right]. \qquad (7.4)$$

For the φ^4 scalar theory, the operator in the integral above is the differential quadratic operator [1], [2] and [3]

$$A = -\frac{\delta^2\mathscr{L}_1}{\delta\phi(x)\delta\phi(y)} = -\bar{\partial}^\mu\bar{\partial}_\mu + m^2 + V''[\phi_{\text{cl}}] \qquad (7.5)$$

where the bar indicates that we have passed to Eucliean spacetime coordinates. This is an essential step since it guarantees that A will be a real and self-adjoint operator.
The determinant of this operator can be achieved through dimensional regularization by evaluating integrals of the type [1]

$$\begin{aligned}I(\omega,\mu_B) &= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2 - \mu_B^2 + i\varepsilon)^{-1}\\ &= \frac{i\mu_B^2}{16\pi^2}(M^2)^{\omega-2}\left(\frac{1}{2-\omega} + \Gamma'(1) + 1 - \log\frac{\mu_B^2}{4\pi M^2} + O(\omega-2)\right).\end{aligned} \qquad (7.6)$$

which are computed by extensive use of the Γ function and in paricular using clever techniques of analytic continuation. This is a generalization of the integral

$$\lim_{\omega\to 2}\int\frac{d^{2\omega}\ell}{\ell^2+m^2} = -\pi^2 m^2\left[\frac{1}{2-\omega} + \frac{3}{2} - \gamma + O(2-\omega)\right]. \qquad (7.7)$$

However, the underlying mathematical restructure of these computations required the use of $\Gamma'(1) = -\gamma$ which is a signal that the Riemann ζ function is behind the scenes.
By the ζ technique we can show that

$$Z = Ne^{-S[\phi_0,J]}\exp\left(\frac{1}{2}\zeta_A'(0)\right) \qquad (7.8)$$

on in terms of the effective action

$$\Gamma_E^{(1)}[\phi_{\rm cl}] = -\frac{1}{2}\zeta_A'(0), \tag{7.9}$$

Furthermore the equivalence between dimensional and ζ regularization is manifest in [2]

$$\log\det A = \frac{\zeta_A(0)}{\omega-2} + (\gamma - \log 4\pi)\zeta_A(0) - \zeta_A'(0) + O(\omega-2). \tag{7.10}$$

Note that the last term in the last two equations is not important as it can be absorbed in the normalization constant.
This should not be surprising as the analytic continuation of both the Γ and ζ function are intrinsically linked through the functional equation. This has enabled us to show in two very different techniques (yet, necessarily equivalent from a mathematical point of view) for instance that

$$V(\phi_{\rm cl}) = \frac{\lambda}{4!}\phi_{\rm cl}^4(\bar{x}) + \frac{\lambda^2\phi_{\rm cl}^4}{256\pi^2}\left(\log\frac{\phi_{\rm cl}^2}{M^2} - \frac{25}{6}\right). \tag{7.11}$$

The latter technique has certain - conceputal and computational - advantages over the former which we proceed to explain now. According to Elizalde, Odintsov, and Romeo there is somewhat of a distaste in using ζ function regulazation in important scientific journals and prefer to use dimensional regularization because the former procedure seems ambiguous and ill defined.

Quoting these authors [2]:
The situation is such that, what is in fact a most elegant, well defined, and unique - in many aspects - regularization method, may look now to the non-specialist as just one more among many possible regularization procedures, plagued with difficulties and illdefiniteness.

The rest of this conclusion is based on their defense of the ζ function regularization procedure.
Let us suppose we have a proper-time Hamiltonian H of a quantum system with boundary conditions in a background field. This is equivalent to a differential operator A with corresponding boundary conditions. Irrespective of whether the spectrum of A may be computed explictily or not, to any such operator, we can define ζ_A rigorously as

$$\zeta_A(s) = \mathrm{Tr}A^{-s}. \tag{7.12}$$

As we have seen several times, when the eigenvalues λ_n of A form a discrete set and can be computed explicitly (i.e. the eigenvalues of H with boundary conditions and background field) we obtain

$$\zeta_A(s) = \sum_n \lambda_n^{-s}. \tag{7.13}$$

Next, comes the classification of the eigenvalues. If the are of the form an then we consider the Riemann ζ function and if there are of the form $a(n+b)$ we consider the Hurwitz ζ function.
Depending on our physical magnitude of interest, we have to compute the ζ function at a particular value of s. In field theory and quantum mechanics we have used $s = 0$ but for example in the vacuum energy of the Casimir effect, which is the sum over the spectrum

$$E_{\rm Cas} = \frac{1}{2}\sum_n \lambda_n \tag{7.14}$$

we have used $s = -3$ yielding

$$E_{\text{Cas}} = -\frac{1}{2r^3} A \frac{\pi^2}{3} \zeta(-3). \tag{7.15}$$

In general, series of the form (7.14) are divergent and this will call for analytic continuation through the ζ function. In this view, as we stressed in the Introduction, this regularization is a special case of the mathematical concept analytic continuation. Since this concept is defined uniquely and rigorously then so is the regularization procedure.

Let us work out a final example taken from Lang's *Elliptic Functions* [5]. When we compute the Casimir effect of a piecewise uniform closed string, inevitably we will run into a clearly infinite sum

$$\sum_{n=0}^{\infty} (n + \beta). \tag{7.16}$$

The eigenvalues in the sum $\lambda_n = n + \beta$ are the transverse oscillatons of the string. As we have pointed out above, this will necessitate the Hurwitz zeta function,

$$\zeta_A(s) = \sum_{n=0}^{\infty} (n + \beta)^{-s}, \tag{7.17}$$

which is valid for $\text{Re}(s) > 1$ but can be analytically continued as a meromorphic function to the whole complex plane. This was the jewel result of Chapter 2. Having said this, the ζ regularization procedure assigns unambiguously the value

$$\sum_{n=0}^{\infty} (n + \beta) = \zeta(-1, \beta) \tag{7.18}$$

to our sum (7.16). A mistake would be to write

$$\sum_{n=0}^{\infty} (n + \beta) = \zeta(-1) + \beta\zeta(0) \tag{7.19}$$

which yields a different result. X. Li, X. Shi and J. Zhang showed [6] in 1991 the necessity of using the Hurwitz ζ function instead of the Riemann ζ function.

The ζ regularization method can be viewed as one of many possibilities of analytic continuation in order to make sense infinite sums. When considered under this light, it shares some similarities with dimensional regularization. It has been argued that the two methods also share similar faults. However, we introduce ζ regularization to solve the problem of the dependence of the regularized result on the kind of extra dimensions added in dimensional regularization. It is a fact that a function may not have two different analytic continuations but the number of ways of defining different analytic continuations in endless. What remains to be studied is the use that one can make of them. This does not imply, however, that ζ regularization suffers from the same problem as dimensional regularization.

There exist endless analytical regularization procedures and *both* ζ and dimensional methods are but two examples. In the latter one may change any exponent at any place with the condition that he recovers the starting expression for a particular value of exponent.

A problem that arises in ζ regularization, however, is that the point at which the ζ function must be evaluated is (precisely) a pole of the analytic continuation. According to Elizalde, Odintsov, and Romeo [2] one eventually has to use renormalization group techniques to solve this issue.

These authors also ask the rethoric question: *which regularization does Nature use?* The elegance and uniqueness of the ζ technique makes it a plausible candidate.

Analogies
The numbers $N(T)$ of zeros in the critical strip $0 \le \sigma \le 1$ is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{7.20}$$

and the behaviour of the 1-loop effective Lagrangian as $B \to 0$ is

$$\lim_{B\to 0} \mathscr{L}^{(1)}_{\text{eff}} = -\frac{1}{16\pi}(m^2 \log m^2 - m^2). \tag{7.21}$$

The Laurent expansion of $\log\det A$ around $\omega = 2$ is

$$\log\det A = \frac{\zeta_A(0)}{\omega - 2} + (\gamma - \log 4\pi)\zeta_A(0) - \zeta_A'(0) + O(\omega - 2). \tag{7.22}$$

where as the sum of the zeros ρ of the Riemann ζ function is

$$\sum_{\text{Im}\rho > 0} \left(\frac{1}{\rho} + \frac{1}{1-\rho} \right) = \frac{1}{2}[2 + \gamma - \log 4\pi]. \tag{7.23}$$

This similar equations indicate that there might be a connection between the distribution of the zeros of the ζ function and the behaviour of the quantum field theories worth exploring.

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- [4] H. B. G. Casimir and D. Polder *The Influence of Retardation on the London-van der Waals Forces*
- [5] Serge Lang *Elliptic Functions*
- [6] X. Li, X. Shi and J. Zhang *Phys. Rev.* D44 (1991) 560

A Appendix

A.1 Generalized Gaussian integrals

Claim 1: The following holds

$$\int dx_1 \cdots \int dx_n \exp\left[i\lambda\left\{(x_1-a)^2+(x_2-x_1)^2+\cdots+(b-x_n)^2\right\}\right] \tag{A.1}$$

$$= \sqrt{\frac{i^n\pi^n}{(n+1)\lambda^n}} \exp\left[\frac{i\lambda}{n+1}(b-a)^2\right] \tag{A.2}$$

Proof. (by induction)
Assume it is true for n and show it is true for $n+1$

$$\int dx_1 \cdots \int dx_{n+1} \exp\left[i\lambda\left\{(x_1-a)^2+(x_2-x_1)^2+\cdots+(b-x_n)^2\right\}\right]$$

$$\begin{aligned}
&= \sqrt{\frac{i^n\pi^n}{(n+1)\lambda^n}} \int dx_{n+1} \exp\left[\frac{i\lambda}{n+1}(x_{n+1}-a)^2\right] \exp[i\lambda(b-x_{n+1})^2] \\
&= \left(\frac{i^n\pi^n}{(n+1)\lambda^n}\right)^2 \int dx_{n+1} \exp\left[i\lambda\left\{\frac{1}{n+1}(x_{n+1}-a)^2+(b-x_{n+1})^2\right\}\right],
\end{aligned} \tag{A.3}$$

the exponential in the integrand can be worked out as followsb by setting $x_{n+1}-a=y$

$$\begin{aligned}
\frac{1}{n+1}(x_{n+1}-a)^2+(b-x_{n+1})^2 &= \frac{n+2}{n+1}y^2-2y(b-a)+(b-a)^2 \\
&= \frac{n+2}{n+1}\left[y-\frac{n+1}{n+2}(b-a)\right]^2+\frac{1}{n+2}(b-a)^2.
\end{aligned}$$

Finally, let $\lambda-((n+1)/(n+2))(b-a)=z$ so that the integral becomes

$$\sqrt{\frac{i^n\pi^n}{(n+1)\lambda^n}} \int dz \exp\left[i\lambda\frac{n+2}{n+1}z^2+\frac{i\lambda}{n+2}(b-a)^2\right] = \sqrt{\frac{i^{n+1}\pi^{n+1}}{(n+1+1)\lambda^{n+1}}} \exp\left[\frac{i\lambda}{n+2}(b-a)^2\right], \tag{A.4}$$

and this concludes the proof by induction.

Claim 2: If $\mathbf{M}$ is a symmetric $N \times N$ matrix with real-valued elements M_{ij} and $\mathbf{q}$ and $\mathbf{J}$ are N component vectors with components q_i and J_i respectively, then

$$Z(J) = \int d^N q \exp\left(-\frac{1}{2}\mathbf{q}^T\mathbf{M}\mathbf{q}+\mathbf{J}^T\mathbf{q}\right) = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{M}}} \exp\left(\frac{1}{2}\mathbf{J}^T\mathbf{M}^{-1}\mathbf{J}\right) \tag{A.5}$$

Proof. The process is to diagonalize the matrix $\mathbf{M}$ as $\mathbf{M} = \Lambda\tilde{\mathbf{M}}\Lambda^T$ where the following relations hold

$$\Lambda^T\Lambda = 1 \quad \det\Lambda = 1 \quad \tilde{\mathbf{M}} = \begin{pmatrix} \tilde{m}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{m}_N \end{pmatrix}, \tag{A.6}$$

the integral becomes

$$Z(J) = \int d^N q \exp\left(-\frac{1}{2}\mathbf{q}^T \Lambda \tilde{\mathbf{M}} \Lambda^T \mathbf{q} + \mathbf{J}^T \mathbf{q}\right), \tag{A.7}$$

next we define the following $\tilde{\mathbf{q}} = \Lambda^T \mathbf{q}$ and $\tilde{\mathbf{J}} = \Lambda^T \mathbf{J}$ from which it follows that $d^N q = d^N \tilde{q} \det \Lambda = d^N \tilde{q}$ and

$$\begin{aligned} Z(J) &= \int d^N q \exp\left(-\frac{1}{2}\tilde{\mathbf{q}}^T \tilde{\mathbf{M}} \tilde{\mathbf{q}} + \tilde{\mathbf{J}}^T \tilde{\mathbf{q}}\right) = \int d^N \tilde{q}_i \exp\left[\sum_i -\frac{1}{2}\tilde{m}_i \tilde{q}_i^2 + \tilde{J}_i \tilde{q}_i\right] \\ &= \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} d^N \tilde{q}_i \exp\left[\sum_i -\frac{1}{2}\tilde{m}_i \tilde{q}_i^2 + \tilde{J}_i \tilde{q}_i\right]\right) = \prod_{i=1}^{N}\left(\sqrt{\frac{2\pi}{\tilde{m}_i}} \exp\left[\frac{\tilde{J}_i^2}{2\tilde{m}_i}\right]\right) \\ &= (2\pi)^{N/2}\left(\prod_i \tilde{m}_i\right)^{-1/2} \exp\left(\sum_i \frac{J_i^2}{2\tilde{m}_i}\right). \end{aligned} \tag{A.8}$$

Finally we note that the inverse of the diagonal matrix is

$$\tilde{\mathbf{M}}^{-1} = \begin{pmatrix} \tilde{m}_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{m}_N^{-1} \end{pmatrix} \tag{A.9}$$

and therefore the last product is the determinant of the matrix

$$\prod_i \tilde{m}_i = \det \tilde{\mathbf{M}} = \det \mathbf{M}, \tag{A.10}$$

and consequently the result follows

$$Z(J) = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{M}}} \exp\left(\frac{1}{2}\tilde{\mathbf{J}}^T \mathbf{M}^{-1} \tilde{\mathbf{J}}\right) = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{M}}} \exp\left(\frac{1}{2}\mathbf{J}^T \mathbf{M}^{-1} \mathbf{J}\right). \tag{A.11}$$

Claim 3: One has the following

$$\begin{aligned} Z_E[J] &= N_E \exp\{-S_{\text{Euc}}[\phi(x_0), J]\} \int D\phi \exp\left\{-\frac{1}{2}\int d^4x_1 d^4x_2 \left(\phi(x_1)\frac{\delta^2 S_{\text{Euc}}}{\delta\phi(x_1)\delta\phi(x_2)}\phi(x_2)\right)\right\} \\ &= N_E' \frac{\exp\{-S_{\text{Euc}}[\phi(x_0), J]\}}{\sqrt{\det\left[(-\bar{\partial}_\mu \bar{\partial}^\mu + m^2 + V''[\phi(x_0)])\delta(x_1 - x_2)\right]}}. \end{aligned} \tag{A.12}$$

Proof.
This follows from Claim 2 with the action

$$S_E[\phi, J] = \int d^4\bar{x}\left[\frac{1}{2}\bar{\partial}_\mu \phi \bar{\partial}^\mu \phi + \frac{1}{2}m^2\phi^2 + V(\phi) - J\phi\right] \tag{A.13}$$

as expanded on (4.136) and using

$$\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)} S_{\text{Euc}} = \delta(\bar{x}_1 - \bar{x}_2)\left[-\bar{\partial}_\mu \bar{\partial}^\mu + m^2 + V''[\phi(x_1)]\right] \tag{A.14}$$

$$Z_E[J] = N_E \exp\{-S_{\text{Euc}}[\phi(x_0), J]\} \int D\phi \exp\left\{-\frac{1}{2}\int d^4x_1 d^4x_2 \left(\phi(x_1)\frac{\delta^2 S_{\text{Euc}}}{\delta\phi(x_1)\delta\phi(x_2)}\phi(x_2)\right)\right\} \tag{A.15}$$

and this proves the claim.

The formulas

$$\coth x = \frac{1}{x} + 2\sum_{k=1}^{\infty}\frac{(2x)^{2k}}{(2k)^k}B_{2k}, \quad \coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{n\geq 1}\frac{1}{x^2+n^2} \tag{A.16}$$

are almost always quoted as Gradshteyn and Ryzhik, p.35. They represent the Laurent series. There is very little added value in reproducing the proofs here.

A.2 Grassman Numbers

Let us introduce some notation first. The following presentation about Grassmann numbers follows the notes of A. Rajantie and Peskin and Schroeder. Ordinary commuting numbers will be denoted c-numbers (these can be real or complex). Now let n generators $\{\theta_1, \cdots, \theta_n\}$ satisfy the anti-commutation relations

$$\{\theta_i, \theta_j\} = 0 \quad \forall i, j. \tag{A.17}$$

Then the set of the linear combinations of $\{\theta_i\}$ with the c-number coefficient is called the Grassmann number and the algebra generated by $\{\theta_i\}$ is called the Grassmann algebra, denoted by Λ^n. Let us taken an arbitrary element g of this algebra expand it as

$$g(\theta) = g_0 + \sum_{i=1}^{n} g_i\theta_i + \sum_{i<j} g_{ij}\theta_i\theta_j + \cdots = \sum_{0\leq k\leq n}\frac{1}{k!}\sum_{\{i\}} g_{i_1,\cdots,i_k}\theta_{i_1}\cdots\theta_{ik}, \tag{A.18}$$

where $g_0, g_i, g_{ij}, \cdots$ and $g_{i_1,\cdots,i_k}$ are c-numbers that are anti-symmetric under the exchange of two indices. Additionally, we can write g as

$$g(\theta) = \sum_{k_i=0,1}\tilde{g}_{k_1,\cdots,k_n}\theta_1^{k_1}\cdots\theta_n^{k_n}. \tag{A.19}$$

It is impossible for the set of Grassmann numbers to be an ordered set because the generator θ_k does not have a magnitude. The only number that is both c-number and Grassmann number is zero, moreover, a Grassmann number commutes with a c-number. From the discussion above it follows that

$$\theta_k^2 = 0 \tag{A.20}$$

$$\theta_{k_1}\theta_{k_2}\cdots\theta_{k_n} = \varepsilon_{k_1k_2\cdots k_n}\theta_1\theta_2\cdots\theta_n \tag{A.21}$$

$$\theta_{k_1}\theta_{k_2}\cdots\theta_{k_m} = 0 \quad (m > n). \tag{A.22}$$

The tensor $\varepsilon_{k_1k_2\cdots k_n}$ is the Levi-Civita symbol, defined as

$$\varepsilon_{k_1k_2\cdots k_n} = \begin{cases} +1 & \text{if}\{k_1\cdots k_n\} \text{ is an even permutation of } \{1\cdots n\} \\ -1 & \text{if}\{k_1\cdots k_n\} \text{ is an odd permutation of } \{1\cdots n\} \\ 0 & \text{otherwise} \end{cases} \tag{A.23}$$

Functions of Grassmann numbers are defined in terms of Taylor expansions of the function. If $n = 1$ we have the simple expression

$$e^{\theta} = 1 + \theta \tag{A.24}$$

since terms $O(\theta^2)$ are zero.
Our next step is to develop the theory of differentiation and integration of Grassmann variables, this theory has a few surprising facts, for instance differentiation is the same process as integration.
We assume that the differential operator acts on a function from the left, let θ_i and θ_j be two Grassmann variables then

$$\frac{\partial \theta_j}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}\theta_j = \delta_{ij}. \tag{A.25}$$

Similarly, we assume that the differential operator anti-commutes with θ_k. The product rule has the slightly different form

$$\frac{\partial}{\partial \theta_i}(\theta_j \theta_k) = \frac{\partial \theta_j}{\partial \theta_i}\theta_k - \theta_j \frac{\partial \theta_k}{\partial \theta_i} = \delta_{ij}\theta_k - \delta_{ik}\theta_j. \tag{A.26}$$

Moreover, the following properties hold

$$\frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} = 0 \Rightarrow \frac{\partial^2}{\partial \theta_i^2} = 0, \tag{A.27}$$

the last equation is termed nil-potency and finally

$$\frac{\partial}{\partial \theta_i}\theta_j + \theta_j \frac{\partial}{\partial \theta_i} = \delta_{ij}. \tag{A.28}$$

Let us now move to integration. To this end, we adopt the notation D for differentiation with respect to a Grassmann variable and $\int$ for integration. Let us suppose that these operations satisfy the relations

$$\int D = D \int = 0, \quad D(A) = 0 \Rightarrow \int(BA) = \int(B)A, \tag{A.29}$$

where A and B are arbitrary functions of Grassmann variables. The first part of the first equation implies that the integral of a derivative gives a surface term and it is set to zero, whereas the second part implies that the derivative of an integral vanishes. The last equation implies that if the derivative of the function is zero then it can be taken out the integral.
The relations are satisfied when D is proportional to $\int$ and for normalization purposes we set $\int = D$ and write

$$\int d\theta g(\theta) = \frac{\partial f(\theta)}{\partial \theta}. \tag{A.30}$$

From the previous definition it follows that

$$\int d\theta = \frac{\partial 1}{\partial \theta} = 0, \quad \int d\theta\, \theta = \frac{\partial \theta}{\partial \theta} = 1. \tag{A.31}$$

$$\int d\theta_1 d\theta_2 \cdots d\theta_n \; g(\theta_1 \theta_2 \cdots \theta_n) = \frac{\partial}{\partial \theta_1}\frac{\partial}{\partial \theta_2} \cdots \frac{\partial}{\partial \theta_n} g(\theta_1 \theta_2 \cdots \theta_n). \tag{A.32}$$

In a theory where differentiation is equivalent to integration we would expect some strange behaviour. In order to see this behaviour we consider the simpler case when we have only one generator and we change variables $\theta' = a\theta$ where a is a complex number, then one has

$$\int d\theta g(\theta) = \frac{\partial g(\theta)}{\partial \theta} = \frac{\partial g(\theta'/a)}{\partial \theta'/a} = a \int d\theta' g(\theta'/a) \tag{A.33}$$

which implies that $d\theta' = (1/a)d\theta$. The extension to the general case with n generators yields $\theta_i \to \theta'_i = a_{ij}\theta_j$ and hence

$$\begin{aligned}
\int d\theta_1 d\theta_2 \cdots d\theta_n \; g(\theta) &= \frac{\partial}{\partial \theta_1}\frac{\partial}{\partial \theta_2}\cdots\frac{\partial}{\partial \theta_n} g(\theta) \\
&= \sum_{k_i=1}^{n} \frac{\partial \theta'_{k_1}}{\partial \theta_1}\cdots\frac{\partial \theta'_{k_n}}{\partial \theta_n}\frac{\partial}{\partial \theta'_{k_1}}\cdots\frac{\partial}{\partial \theta'_{k_n}} g(a^{-1}\theta') \\
&= \sum_{k_i=1}^{n} \varepsilon_{k_1\cdots k_n} a_{k_1 1}\cdots a_{k_n n}\frac{\partial}{\partial \theta'_{k_1}}\cdots\frac{\partial}{\partial \theta'_{k_n}} g(a^{-1}\theta') \\
&= \det a \int d\theta'_1 \cdots d\theta'_n g(a^{-1}\theta'). \end{aligned} \tag{A.34}$$

Consequently the measure has the Jacobian

$$d\theta_1 d\theta_2 \cdots d\theta_n = (\det a) d\theta'_1 \cdots d\theta'_n. \tag{A.35}$$

In the case of a single variable, the delta function of a Grassmann variable is defined in a similar fashion as with c-numbers defined as

$$\int d\theta \, \delta(\theta - z) g(\theta) = g(z). \tag{A.36}$$

However, in the case of Grassmann variables we can obtain a closed expression for the delta function. If we set $g(z) = a + bz$ in the definition of the delta function we have

$$\int d\theta \, \delta(\theta - z)(a + b\theta) = a + bz \tag{A.37}$$

and this means that

$$\delta(\theta - z) = \theta - z. \tag{A.38}$$

Again, we can extend this to n generators if we are careful about the order of the variables

$$\delta^n(\theta - z) = (\theta_n - z_n)\cdots(\theta_2 - z_2)(\theta_1 - z_1). \tag{A.39}$$

We can find the integral of the delta function by considering complex Grassmann variables which we proceed to develop later. Consider

$$\int d\xi \, e^{i\xi\theta} = \int d\xi \, (1 + i\xi\theta) = i\theta \tag{A.40}$$

so that we have

$$\delta(\theta) = \theta = -i \int d\xi \, e^{i\xi\theta}. \tag{A.41}$$

One of the most crucial developments of the Grassmann variables is the Grassmann Gaussian integral which will be fundamental when developing the path integral formalism of fermions. Let us evaluate the following integral

$$I = \int d\theta_1^* d\theta_1 \cdots d\theta_n^* d\theta_n \exp\left(\sum_{ij} \theta_i^* M_{ij} \theta_j\right) \tag{A.42}$$

where it is important to stress that $\{\theta_i\}$ and $\{\theta_i^*\}$ are two independent sets of Grassmann variables. Since Grassmann variables θ_i and θ_i^* anti-commute we can take the $n \times n$ c-number matrix $\mathbf{M}$ to be anti-symmetric. The formula for the transformation of the measure solves the problem of the computation. Set $\theta_i' = \sum_j M_{ij}\theta_j$ this yields

$$I = \det M \int d\theta_1^* d\theta_1' \cdots d\theta_n^* d\theta_n' \exp\left(-\sum_i \theta_i^* \theta_i'\right) = \det M \left[\int d\theta^* d\theta (1 + \theta'\theta^*)\right]^n = \det M. \tag{A.43}$$

Complex conjugation is defined as

$$(\theta_i)^* = \theta_i^*, \quad (\theta_i^*)^* = \theta_i. \tag{A.44}$$

In the case of Grassmann variables we have

$$(\theta_i\theta_j)^* = \theta_j^*\theta_i^*. \tag{A.45}$$

The reasoning behind (2.94) is that the real c-number $\theta_i\theta_i^*$ does not satisfy $(\theta_i\theta_i^*)^* = \theta_i\theta_i^*$. Let us recall that the annihilation and creation operators c and $c^\dagger$ satisfy the anti-commutation relations $\{c, c^\dagger\} = 1$ and $\{c, c\} = \{c^\dagger, c^\dagger\} = 0$ and that the number operator $N = c^\dagger c$ has eigenvectors $|0\rangle$ and $|1\rangle$. We are now in a position to study the Hilbert space Ω spanned by these vectors, i.e.

$$\Omega = \text{span}\left\{|0\rangle, |1\rangle\right\}. \tag{A.46}$$

An arbitrary vector $|\omega\rangle \in \Omega$ can be written in the form

$$|\omega\rangle = |0\rangle\,\omega_0 + |1\rangle\,\omega_1, \tag{A.47}$$

with $\omega_i \in C$ where $i = 1, 2$.
Next we define the coherent states

$$|\theta\rangle = |0\rangle + |1\rangle\,\theta \quad \langle\theta| = \langle 0| + \theta^*\,\langle 1| \tag{A.48}$$

where θ and θ^* are Grassmann numbers.
The coherent states are eigenstates of c and $c^\dagger$ respectively, that is

$$c\,|\theta\rangle = |0\rangle\,\theta = |\theta\rangle\,\theta, \quad \langle\theta|\,c^\dagger = \theta^*\,\langle 0| = \theta^*\,\langle\theta|. \tag{A.49}$$

It can be shown fairly easily that the following identities hold

$$\langle\theta' \mid \theta\rangle = 1 + \theta'^*\theta = e^{\theta'^*\theta}, \tag{A.50}$$

$$\langle\theta \mid g\rangle = g_0 + \theta^* g_1, \tag{A.51}$$

$$\left\langle \theta \left| c^\dagger \right| g \right\rangle = \langle \theta \mid 1 \rangle g_0 = \theta^* g_0 = \theta^* \langle \theta \mid g \rangle , \tag{A.52}$$

$$\langle \theta \left| c \right| g \rangle = \langle \theta \mid 0 \rangle g_1 = \frac{\partial}{\partial \theta^*} \langle \theta \mid g \rangle . \tag{A.53}$$

Finally, we show how matrix elements are represented and the completeness relation. Let

$$p(c, c^\dagger) = p_{00} + p_{10} c^\dagger + p_{01} c + p_{11} c^\dagger c, \quad p_{ij} \in C \tag{A.54}$$

be an arbitrary function of c and $c^\dagger$.
The complex matrix elements of p can be written in terms of scalar products as

$$\langle 0 \left| p \right| 0 \rangle = p_{00}, \quad \langle 0 \left| p \right| 1 \rangle = p_{01}, \quad \langle 1 \left| p \right| 0 \rangle = p_{10}, \quad \langle 1 \left| p \right| 1 \rangle = p_{00} + p_{11}. \tag{A.55}$$

From these scalar products we can form the more general product

$$\langle \theta \left| p \right| \theta' \rangle = (p_{00} + \theta^* p_{10} + p_{01} \theta' + \theta^* \theta' p_{11}) e^{\theta^* \theta'} . \tag{A.56}$$

Moreover, one has

$$\int d\theta^* d\theta \left| \theta \right\rangle \left\langle \theta \right| e^{-\theta^* \theta} = \int d\theta^* d\theta \left(\left| 0 \right\rangle + \left| 1 \right\rangle \theta \right) \left(\left\langle 0 \right| + \theta^* \left\langle 1 \right| \right) \left(1 - \theta^* \theta \right) \tag{A.57}$$

$$\left\langle \theta \left| c^\dagger \right| g \right\rangle = \langle \theta \mid 1 \rangle g_0 = \theta^* g_0 = \theta^* \langle \theta \mid g \rangle , \tag{A.58}$$

$$\langle \theta \left| c \right| g \rangle = \langle \theta \mid 0 \rangle g_1 = \frac{\partial}{\partial \theta^*} \langle \theta \mid g \rangle . \tag{A.59}$$

and therefore the completeness relation is

$$\int d\theta^* d\theta \left| \theta \right\rangle \left\langle \theta \right| e^{-\theta^* \theta} = I. \tag{A.60}$$

A.3 The Mellin Transform and the series expansion of $\log \Gamma(s+1)$

We shall follow Titchmarsh's *Theory of Functions* and Withaker and Watson's *Modern Analysis*. The Mellin transform connects two functions $f(x)$ and $\Psi(s)$ in the following way

$$\Psi(s) = \int_0^\infty dx f(x) x^{s-1}, \quad f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds \Psi(s) x^{-s}. \tag{A.61}$$

For example if we take f to be $f(x) = e^{-x}$ then clearly $\Psi(s) = \Gamma(s)$ for $\sigma > 0$. We can also recover our formula relating the Γ and ζ functions by using $f(x) = (e^x - 1)^{-1}$ in which case $\Psi(s) = \Gamma(s)\zeta(s)$ for $\sigma > 1$. From the Weierstrass product we can write

$$\left(1 + \frac{z}{a}\right) \prod_{n=1}^\infty \left\{ e^{-z/n} \left(1 + \frac{z}{a+n}\right) \right\} = e^{-\gamma z} \frac{\Gamma(a)}{\Gamma(z+a)} \tag{A.62}$$

and take the principal values of the logarithms to be

$$\log\left(1 + \frac{z}{a}\right) + \log \prod_{n=1}^\infty \left\{ e^{-z/n} \left(1 + \frac{z}{a+n}\right) \right\} = \sum_{n=1}^\infty \left\{ e^{-z/n} \left(1 + \frac{z}{a+n}\right) \right\}$$

$$= \sum_{n=1}^{\infty}\left\{\left(\frac{-az}{n(a+n)}\right)+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z}{(a+n)^m}\right\}+\sum_{n=1}^{\infty}\left\{\left(\frac{-az}{n(a+n)}\right)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{a^m}\right\}, \tag{A.63}$$

and the absolute convergence of the series is assured for $|z|<a$ since

$$\sum_{n=1}^{\infty}\left\{\frac{a|z|}{n(a+n)}-\log\left(1-\frac{|z|}{a+n}\right)+\frac{|z|}{a+n}\right\} \tag{A.64}$$

converges. Now, taking logarithms we have

$$\log\frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)}=\frac{z}{a}-\sum_{n=1}^{\infty}\frac{az}{n(a+n)}+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}z^m\zeta(m,a). \tag{A.65}$$

Next, we need to consider

$$-\frac{1}{2\pi i}\oint_C ds\frac{\pi z^s}{s\sin\pi s}\zeta(s,a) \tag{A.66}$$

where the contour C is like Figure 2.4 except that it encloses the points $s=2,3,4,\cdots$ but not the points $1,0,-1,-2,\cdots$. The residue of this integral at $s=m\geq 2$ is given by

$$\begin{aligned}\operatorname*{res}_{s=m}\frac{\pi z^s}{s\sin\pi s}\zeta(s,a) &= \lim_{s\to m}(s-m)\frac{\pi z^s}{s\sin\pi s}\zeta(s,a)\\ &=\frac{\pi z^m}{m}\zeta(m,a)\lim_{s\to m}\frac{s-m}{\sin\pi s}=\frac{(-1)^m}{m}z^m\zeta(m,a).\end{aligned} \tag{A.67}$$

Since $\zeta(s,a)=O(1)$ as $\sigma\to\infty$ then the integral converges if $|z|<1$. By Cauchy's residue theorem we can change the sum involving the $\zeta(m,a)$ term for the integral

$$\log\frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)}=\frac{z}{a}-\sum_{n=1}^{\infty}\frac{az}{n(a+n)}-\frac{1}{2\pi i}\oint_C ds\frac{\pi z^s}{s\sin\pi s}\zeta(s,a). \tag{A.68}$$

Taking the exponential outside the logarithm and using the well known formula

$$\frac{d}{dz}\log\Gamma(z+1):=\psi^{(0)}(z+1)=\sum_{n=1}^{\infty}\frac{z}{n(n+x)}-\gamma \tag{A.69}$$

we see that

$$\log\frac{\Gamma(a)}{\Gamma(z+a)}=-z\frac{\Gamma'(a)}{\Gamma(a)}-\frac{1}{2\pi i}\oint_C ds\frac{\pi z^s}{s\sin\pi s}\zeta(s,a) \tag{A.70}$$

We now let D be a semicircle of large radius N with center at $s=\frac{3}{2}$, the semicircle lying on the right of the line $\sigma=\frac{3}{2}$. On this semicirle, $\zeta(s,a)=O(1)$ as well as $|z|<1$ and $-\pi+\delta\leq\arg z\leq\pi-\delta$ where $\delta>0$ and

$$\frac{\pi z^s}{s\sin\pi s}=O(|z|^{\sigma}e^{-\delta|t|}) \tag{A.71}$$

therefore

$$\lim_{N\to\infty} \int\limits_{D(N)} ds \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) = 0. \tag{A.72}$$

It follows immediately that if $|\arg z| \leq \pi - \delta$, and $|z| < 1$ then

$$\log \frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{2\pi i} \int\limits_{3/2-i\infty}^{3/2+i\infty} ds \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) \tag{A.73}$$

However, the above integral defines an analytic function of z for all values of $|z|$ if $|\arg z| \leq \pi - \delta$ then by analytic continuation is valid for all values of $|z|$ when $|\arg z| \leq \pi - \delta$. Let us consider a cutoff on this integral of the sort

$$\log \frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{2\pi i} \int\limits_{-n-1/2\pm iR}^{3/2\pm iR} ds \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) \tag{A.74}$$

where n is a fixed integer and $R \to \infty$. As we have seeen the integrand is $O(|z|^\sigma e^{-\delta R} R^{\tau(\sigma)})$ where $-n - \frac{1}{2} \leq \sigma \leq \frac{3}{2}$ and thus irrespective of which sign we take in the limits of the integral, the integral goes to zero as $R \to \infty$.
Applying Cauchy's residue theorem again yields

$$\log \frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{2\pi i} \int\limits_{-n-1/2-i\infty}^{-n-1/2+i\infty} ds \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) + \sum_{m=-1}^{n} \operatorname*{res}_{s=-m} \frac{\pi z^s}{s \sin \pi s} \zeta(s,a). \tag{A.75}$$

On this new path of integration we have

$$\left| \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) \right| < K z^{-n-1/2} e^{-\delta |t| \tau(-n-1/2)} |t|, \tag{A.76}$$

with K independent of both z and t and $\tau(\sigma)$ the function used above. Moreover, since

$$\int\limits_{-\infty}^{\infty} dt e^{-\delta|t|\tau(-n-1/2)} t < \infty \tag{A.77}$$

we obtain

$$\log \frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \sum_{m=-1}^{n} \operatorname*{res}_{s=-m} \frac{\pi z^s}{s \sin \pi s} \zeta(s,a) + O(z^{-n-1/2}) \tag{A.78}$$

provided that $|z|$ is large. We next need to compute the residues at $s = 0$ and $s = -1$. Both computations require the expansions around $s = 0$ and $s = -1$ as follows

$$\zeta(s,a) = \zeta(0,a) + \zeta'(0,a)s + O(s^2) = \frac{1}{2} - a + \zeta'(0,a)s + O(s^2) \tag{A.79}$$

$$z^s = 1 + s \log z + O(s^2) \tag{A.80}$$

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \zeta(2)s + O(s^2) \tag{A.81}$$

so that

$$\frac{1}{s}(1 + \zeta(2)^2 + \cdots)(1 + s \log z + \cdots)\left(\frac{1}{2} - a + s\zeta'(0,a) + \cdots\right) \tag{A.82}$$

and the residue at $s = 0$ is given by

$$\operatorname*{res}_{s=0} \frac{\pi z^s}{s \sin \pi s}\zeta(s,a) = \left(\frac{1}{2} - a\right)\log z + \zeta'(0,a) = \left(\frac{1}{2} - a\right)\log z + \log\Gamma(a) - \frac{1}{2}\log 2\pi. \tag{A.83}$$

For the residue at $s = -1$ we use the following expansions

$$\zeta(s,a) = \frac{1}{s-1} - \frac{\Gamma'(a)}{\Gamma(a)} + O(s-1) \tag{A.84}$$

$$z^s = z + (s-1)z\log z + O((s-1)^2) \tag{A.85}$$

$$\frac{\pi}{\sin \pi s} = -\frac{1}{s-1} - \zeta(2)(s-1) + O((s-1)^2) \tag{A.86}$$

so that the same technique yields

$$\operatorname*{res}_{s=-1} \frac{\pi z^s}{s \sin \pi s}\zeta(s,a) = -z\log z + z + z\frac{\Gamma'(a)}{\Gamma(a)}. \tag{A.87}$$

Finally, if $|z|$ is large and $|\arg z| \leq \pi - \delta$ we have

$$\log\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \sum_{k=1}^{n}(-1)^{k-1}\frac{B'_{k+2}(a)}{k(k+1)(k+2)}z^{-k} + O(z^{-n-1/2}). \tag{A.88}$$

where we have used (1.25)

$$\operatorname*{res}_{s=-m} \frac{\pi z^s}{s \sin \pi s}\zeta(s,a) = \frac{(-1)^m z^{-m}}{-m}\zeta(-m,a) = \frac{(-1)^m z^{-m}}{m}\frac{B_{m+1}(a)}{m+1} \tag{A.89}$$

and assumed a result on Bernoulli numbers

$$B_{m+1}(a) = \frac{B'_{m+2}(a)}{(m+1)(m+2)}. \tag{A.90}$$

As a side remark, note that specializing for the case $a = 1$ of the Riemann ζ function and replacing s by $1 - s$ we have

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x = -\frac{1}{2i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds \frac{\zeta(1-s)}{\sin \pi s}x^s, \tag{A.91}$$

Using an inverse Mellin transform with

$$f(x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x, \quad \Psi(s) = -\frac{\pi\zeta(1-s)}{\sin \pi s} \tag{A.92}$$

we obtain (again swapping $1-s$ for s)

$$\zeta(s) = -\frac{\sin \pi s}{\pi} \int_0^\infty ds x^{-s} \left\{ \frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x \right\} \tag{A.93}$$

valid for $0 < \sigma < 1$, a formula known as Kloosterman's equation. With the assistance of

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - 2 \int_0^\infty dt \frac{t}{(t^2+x^2)(e^{2\pi t}-1)} \tag{A.94}$$

we can write

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x} - \log x \tag{A.95}$$

$$= \frac{1}{2x} - 2 \int_0^\infty dt \frac{t}{(t^2+x^2)(e^{2\pi t}-1)} = -2 \int_0^\infty dt \frac{t}{t^2+x^2} \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} \right). \tag{A.96}$$

Following Titchmarsh we have

$$\begin{aligned}
\zeta(s) &= \frac{2\sin \pi s}{\pi} \int_0^\infty dx x^{-s} \int_0^\infty dt \frac{t}{t^2+x^2} \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} \right) \\
&= \frac{2\sin \pi s}{\pi} \int_0^\infty dt \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} \right) t \int_0^\infty \frac{x^{-s}}{t^2+x^2} \\
&= \frac{\sin \pi s}{\cos \frac{\pi s}{2}} \int_0^\infty dt \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} \right) t^{-s} \\
&= 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \int_0^\infty dt \left(\frac{1}{e^u-1} - \frac{1}{u} \right) u^{-s} \\
&= 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s),
\end{aligned} \tag{A.97}$$

i.e. the functional equation of the Riemann ζ function.

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