AN INTRODUCTION TO STABILITY CONDITIONS

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Abstract. In these notes we introduce the notion of stability condition on the category of coherent sheaves on a smooth projective variety. These notes are meant to be a guide for someone approaching the subject for the first time, they are focused on examples and motivation to help intuition.

Contents

1. Introduction and motivation

The classification problem which permeates many areas of study in algebraic geometry is probably the main motivation for the study of moduli spaces. The ideas behind a moduli space are very simple, and represent attempts to answer questions like: is there a way to parametrize the set of objects I'm interested in? How far are these two objects from being isomorphic to each other? What does "far" in the previous question even mean?

One encounters moduli spaces very soon in algebraic geometry, often without even noticing. A simple example of a moduli space is a linear series: we can regard $|\mathcal{O}_{\mathbb{P}^n}(k)|$ as a projective space whose points represent a hypersurface of degree k in $\mathbb{P}^n$. In some sense, a moduli space is a geometric object whose points represent isomorphism classes of elements of some set. To be more precise, let's focus on moduli spaces of coherent sheaves on a variety. Given a smooth projective variety X over a field k, we want to construct a scheme which parametrizes isomorphism classes of coherent sheaves on X.

The case of line bundles is well understood: there exists a scheme Pic_X, called the *Picard scheme of* X, whose k-rational points form the Picard group $\operatorname{Pic}(X)$. This scheme is neither projective nor finite type, but it decomposes as

$$\operatorname{Pic}_X = \bigsqcup \operatorname{Pic}_X^P$$

and each component is projective and parametrizes line bundles with a given Hilbert polynomial with respect to a very ample class $\mathcal{O}(1)$. The Hilbert polynomial of a line bundle L is $\chi(X, L(m)) = \int \mathrm{ch}(L)\mathrm{ch}(\mathcal{O}(m))\mathrm{td}(S)$ (recall the Hirzebruch-Riemann-Roch formula). It only depends on the choice of the ample class and $\mathrm{c}_1(L)$. This suggests that when we try to generalize to higher rank we will need to make assumptions on numerical invariants, if we want to obtain components which are reasonably well-behaved. Unfortunately, fixing the Hilbert polynomial won't be enough to get the nice geometric properties we are interested in, as illustrated by the next examples.

Example 1.1. Consider the family of rank 2 vector bundles $E_n = \mathcal{O}(n) \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$. These are not isomorphic (for example, their spaces of global sections have different dimensions, $h^0(E_n) = n+1$). However, their Hilbert polynomials agree: $\mathrm{ch}(E_n) = \mathrm{rk}E_n + \mathrm{c}_1(E_n) = (2, 0)$ for all E_n. If a moduli space representing in particular the E_n existed, even indexing its components with the Hilbert polynomial wouldn't give a scheme of finite type: the condition $h^0(E_n) \geq m$ is closed, so we would get an infinite, stricly descending chain of closed subschemes.

Example 1.2. On $\mathbb{P}^1$ the group $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1})$ has dimension one. This means that non-trivial extensions

$$0 \to \mathcal{O}(-1) \to E_\lambda \to \mathcal{O}(1) \to 0$$

are isomorphic to $\mathcal{O} \oplus \mathcal{O}$ (the only other rank 2 vector bundle with 2 global sections and trivial first Chern class). This yields a rank 2 vector bundle over $\mathbb{A}^1 \times \mathbb{P}^1$ resctricting to $\mathcal{O} \oplus \mathcal{O}$ above $\lambda \times \mathbb{P}^1$ for $\lambda \neq 0$ and to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ above zero. Again, assume that a moduli space M existed. By its universal property, the vector bundle above corresponds to a map $\mathbb{A}^1 \to M$ which sends the line minus the origin in a point x corresponding to $\mathcal{O} \oplus \mathcal{O}$ and the origin to a different point y. This gives a way to extend the constant map to x defined on $\mathbb{A}^1 - 0$ which is different from the obvious assignment $0 \mapsto x$. In other words, M is not a separated scheme.

It is evident at this point that we need to add some extra condition to hope for the existence a well-behaved moduli space. To make a more precise statement, if X is a smooth projective variety over k, and P is a fixed Hilbert polynomial, denote by $(*)$ the aforementioned extra condition. We are interested in the functor

$$\begin{aligned} \mathcal{M} :& (Sch/k) \to Sets \\ & S \mapsto \{E \in \mathbf{Coh}(S \times X) | E \text{ S-flat}, \forall s \in S : P(E_s) = P, E_s \text{ satisfies } (*)\} / \sim \end{aligned}$$

where the equivalence relation is given by $E \sim E \otimes p^* M$ for some $M \in \mathrm{Pic}(S)$.

Definition 1.3. We say that a functor $\mathcal{M}$ is *corepresented* by a scheme M if there exists a natural transformation $\alpha : \mathcal{M} \to \mathrm{Hom}(-, M)$ such that every other transformation $\mathcal{M} \to \mathrm{Hom}(-, N)$ factors through a unique $M \to N$, i.e.:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\alpha} & \mathrm{Hom}(-, M) \\ & \searrow & \downarrow \exists! \\ & & \mathrm{Hom}(-, N) \end{array}$$

In this case, M is said to be a *moduli space* for $\mathcal{M}$. It is a *coarse* moduli space if we have a bijection $\mathcal{M}(k) \to M(k)$. It is a *fine* moduli space if α is an isomorphism. In this case we say that M *represents* $\mathcal{M}$. This last condition is equivalent to the existence of a universal family over $M \times X$

Our hope is to find a notion of stability which can guarantee the existence of a moduli space for $\mathcal{M}$. In these notes, rather than explaining why an efficient notion of stability is the one presented below, we want to illustrate how this notion generalizes to a much broader framework.

2. Stability for vector bundles on curves

Consider a smooth projective curve C over an algebraically closed field, and denote by $\mathbf{Coh}(C)$ the cateogry of coherent sheaves on C.

Definition 2.1. A coherent sheaf $E \in \mathbf{Coh}(C)$ is said to be *μ-stable* (resp. *μ-semistable*) if E is torsion free (i.e. locally free), and all proper torsion free subsheaves $0 \neq F \subset E$ satisfy

$$\mu(F) < \mu(E) \quad \text{(resp. } \mu(F) \leq \mu(E) \text{)}$$

where $\mu(E) = \dfrac{\deg(E)}{\mathrm{rk}(E)}$ is called the *slope* of E.

The notion of μ-stability, or the more general definition of stability à la Gieseker, turns out to be a suitable notion of stability for the construction of a moduli space of sheaves, indeed:

Theorem 2.2. *Fix a smooth projective curve C, and a Hilbert polynomial P. Then the functor*

$$\begin{aligned} \mathcal{M} :& (Sch/k) \to Sets \\ & S \mapsto \{E \in \textit{\textbf{Coh}}(S \times C) | E \text{ } S\text{-flat}, \forall s \in S : P(E_s) = P, E_s \text{ } \mu\text{-stable } \} / \sim_{iso} \end{aligned}$$

is corepresented by a projective scheme M. The closed points of M parametrize isomorphism classes of stable sheaves with Hilbert polynomial P.[1]

We shall rewrite stability in a way which is better suited to the generalizations we want to make. Define

$$Z(E) = -deg(E) + i\mathrm{rk}(E)$$

and let the *phase* $\phi(E) \in (0, 1]$ of a nonzero sheaf E be uniquely defined by

$$Z(E) \in \exp(i\pi\phi(E)) \cdot \mathbb{R}_{>0}$$

Then, the *stability function* Z defines

$$Z : \mathbf{Coh}(C) - \{0\} \to \overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}_{<0}.$$

We can talk about phases rather than slopes, because of the following:

Lemma/Definition 2.3. *Suppose $E \in$ **Coh**(C) is locally free. Then E is μ-stable iff for all proper subsheaves $0 \neq F \subset E$ we have the following inequality:*

$$\phi(F) < \phi(E). \tag{1}$$

Likewise for μ-semistability. Moreover, observe the formulae $\mu(E) = -\cot(\pi\phi(E))$ and $\pi\phi(E) = \cot^{-1}(-\mu(E))$.

Proof. Write $\dfrac{Z(E)}{\mathrm{rk}(E)} = -\mu(E) + i$. □

[1] For more on this part, see [2].

It seems natural to extend the definition of (semi)stability to the whole $\mathbf{Coh}(C)$ in terms of the (weak) inequality (1). This allows to treat torsion sheaves and vector bundles with the same machinery. The following properties will guide us in generalizing the notion of stability.

Proposition 2.4. The following semi-orthogonality properties hold:

- Let $0 \to K \to E \to G \to 0$ be a short exact sequece in $\mathbf{Coh}(C)$. Then
$$\phi(K) < \phi(E) \Leftrightarrow \phi(E) < \phi(G);$$
- Let $F \neq E \in \mathbf{Coh}(C)$ be semistable (resp. stable) such that $\phi(E) > \phi(F)$ (resp. $\phi(E) \geq \phi(F)$). Then
$$\mathrm{Hom}(E, F) = 0;$$
- If E, F are stable and $\phi(E) \geq \phi(F)$, then either $E \simeq F$ or $\mathrm{Hom}(E, F) = 0$;
- if E is stable, then $\mathrm{End}(E) \simeq k$.

Proof. For the first assertion: write equivalent inequalities for slopes and use additivity of degree and rank. The second assertion is proved as follows: let a be a nontrivial map $a : E \to F$, let $K \subset E$ be the kernel. Then by semistability and the first statement:
$$\phi(E/K) \geq \phi(E) > \phi(F) \geq \phi(E/K)$$
a contradiction. Similarly one proves the other statements. □

Proposition 2.5. Every $E \in \mathbf{Coh}(C)$ admits a *Harder-Narasimhan* filtration $0 = E_0 \subset E_1 \subsetneq \ldots \subsetneq E_n = E$ such that the quotients $A_i := E_i/E_{i-1}$ are semistable sheaves of phase $\phi_1 > \ldots > \phi_n$. The A_i are called semistable factors of E, they're unique.

Example 2.6. Let's illustrate the above notions on $\mathbb{P}^1$. Suppose a coherent sheaf on $\mathbb{P}^1$ has the form $E = E_1 \oplus (\bigoplus_{a_1 > \ldots > a_n} \mathcal{O}(a_i))$, where E_1 is the torsion part and also the first semistable factor. The Harder-Narasimhan filtration of E is given by
$$E_l = E_1 \oplus \left(\bigoplus_{i<l} \mathcal{O}(a_i) \right)$$
for $l \geq 2$. Indeed, E_1 is semistable since all its subobjects are torsion, hence of phase 1, and the subsequent quotients $A_i \simeq \mathcal{O}(a_i)$ are semistable sheaves of decreasing degree (hence of decreasing slope and of increasing phase). The only other possibility is that 2 or more a_i coincide. In that case, arrange the filtration to group them tohether in the same quotient. Indeed, the direct sum of two stable vector bundles of the same slope is again semistable. A hint on how to prove this: consider $F \subset E \oplus G$, and let $F_1 := F \cap (E \oplus \{0\})$ and F_2 be the image of the projection of F to G. Then $F = F_1 \oplus F_2$, and...

We can also introduce the following subcategories:

Definition 2.7. For $\phi \in (0, 1]$, let
$$\mathcal{P}(\phi) := \{E \in \mathbf{Coh}(X) | E \text{ is semistable of phase } \phi\}$$
considered as full linear subcategories of $\mathbf{Coh}(C)$.

3. Stability conditions

In this section we generalize the definition of μ-stability as presented above to the broader framework where $\mathcal{A}$ is an abelian category. Simultaneously we will also give definitions for a triangulated category $\mathcal{D}$, a reason why this will be needed is illustrated in example [4.2].

Definition 3.1. A *slicing* of $\mathcal{A}$ (resp. of $\mathcal{D}$) is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{A}$ for $\phi \in (0,1]$ (resp. $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$) such that:

(1) Semi-orthogonality: $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ if $\phi_1 > \phi_2$;
(2) Harder-Narasimhan: for all $0 \neq E \in \mathcal{A}$ there exists a filtration $0 = E_0 \subset E_1 \subsetneq \ldots \subsetneq E_n = E$ such that the quotients $A_i := E_i/E_{i-1} \in \mathcal{P}(\phi_i)$ and $\phi_1 > \ldots > \phi_n$ (For $E \in \mathcal{D}$, replace the inclusions with morphisms $E_{i-1} \to E_i$ and the quotients A_i with the respective cones).
(3) In the triangulated case, we require $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$.

We will call *semistable* of phase ϕ the objects in $\mathcal{P}(\phi)$, and *stable* the minimal ones among those, i.e. those without any proper subobject.

We also need to introduce the general version of the function Z in the previous section:

Definition 3.2. A *stability function* or *central charge* on an abelian category $\mathcal{A}$ is a linear map $Z : K(\mathcal{A}) \to \mathbb{C}$ such that $Z(E) = Z([E]) \in \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}_{<0}$ for all $0 \neq E \in \mathcal{A}$. Given a slicing $\mathcal{P}$ of $\mathcal{A}$, we say that Z is *compatible* with $\mathcal{P}$ if for all $0 \neq E \in \mathcal{P}(\phi)$ we have

$$Z(E) \in \exp(i\pi\phi(E)) \cdot \mathbb{R}_{>0}. \tag{2}$$

Similarly, but without restrictions on the image, a stability function on a triangulated category is a linear map $Z : K(\mathcal{D}) \to \mathbb{C}$. Compatibility is defined in the same way, just notice that in this case not all objects $E \in \mathcal{D}$ have a well defined slope (since the kernel of Z is nontrivial: it contains for example all objects of the form $F \oplus F[1]$).

Definition 3.3. A *stability condition* on an abelian category $\mathcal{A}$ is a pair $\sigma = (\mathcal{P}, Z)$ of a slicing of $\mathcal{A}$ and a compatible stability function. Likewise for a triangulated category[2].

Remark 3.4. Suppose we consider the abelian category $\mathbf{Coh}(X)$ of a smooth projective variety.

We are particularly interested in stability conditions which are *geometric*, i.e. such that point like objects (e.g. skyscrapers $k(x)$) are semistable of phase 1.

Moreover, we can restrict our attention on a particular class of stability functions. Recall that the motivation for this study is the construction of moduli spaces, hence our interest goes to the numeric properties of the sheaves. In particular, we can focus our attention on functions $Z : K(\mathbf{Coh}(X)) \to \mathbb{C}$ which factor through the Chern character $\mathrm{ch} : K(\mathbf{Coh}(X)) \to H^*(X, \mathbb{C})$. Thus, we can regard the central charge as an element of $H^*(X, \mathbb{C})^*$. Stability conditions of this kind are called *numerical*.

[2] In this case, an additional finiteness requirement on the slicing is needed.

4. Examples

4.1. **Numerical stability conditions on curves of positive genus.** All numerical stability condition on $D^b(C)$ for a smooth curve C, $g > 0$, induce[3] μ-stability on $\mathbf{Coh}(C)$. The argument, due to Bridgeland [6] and Macrì [5], goes as follows. First, one asseses the following:

Lemma 4.1. *Suppose $\sigma = (\mathcal{P}, Z)$ is a numerical stability condition on $D^b(C)$, where C is as above. Then all point sheaves $k(x)$ and all line bundles $L \in Pic(C)$ are stable with respect to σ.*

Now consider a line bundle L and a point sheaf $k(x)$. Say that L is stable of phase θ, by assumption $k(x)$ has phase 1. Since $\mathrm{Hom}(L, k(x)) \neq 0$, we deduce $\theta < 1$. On the other hand,

$$\mathrm{Hom}(k(x), L[1]) \simeq \mathrm{Ext}^1(k(x), L) \simeq \mathrm{Hom}(L, k(x))^* \neq 0$$

so $\theta + 1 > 1$ and $\theta > 0$. This shows that all line bundles lie in the upper half plane $\mathbb{H}$. Now, consider the sheaf $\mathcal{O}_C$: suppose that $Z(\mathcal{O}_C) = a + ib$ where $b > 0$. The function Z must have the form

$$Z(r, d) = r(a + ib) - d = -d + ra + rbi$$

since it's numerical, linear, and it must fit what we know for $k(x)$ and $\mathcal{O}_C$. Now it's easy to check that, for two numerical classes (r, d) and (r', d'), if ϕ denotes the phase induced by Z, we have

$$\phi(r, d) < \phi(r', d') \Leftrightarrow \frac{r}{d} < \frac{r'}{d'}.$$

If we translate back our work in the classical language, this is the stability condition induced by μ-stability.

Since on a curve every vector bundle can be filtered using only point sheaves or line bundles, this shows that the whole category $\mathbf{Coh}(C)$ is mapped in $\overline{\mathbb{H}}$. Another fact we need to recall to conclude is that, on a curve, studying coherent sheaves is enough to understand the derived category: any object of $D^b(C)$ splits as a sum of shifts of coherent sheaves (see [3, Cor. 3.15]).

Remark 4.2. The case for $\mathbb{P}^1$ is more complicated. This is due, roughly speaking, to the presence of many full exceptional sequences (cfr. [3, Sec. 1.4]). For a more complete survey on this, see [7] or [8].

4.2. **A first approach at surfaces.** Fix a smooth projective surface S and a curve $C \subset S$. It is not possible to build a geometric stability condition on $\mathbf{Coh}(S)$ in which the sheaf $\mathcal{O}_C$ is semistable of phase $\theta < 1$. Given these assumptions the image of $\mathbf{Coh}(S)$ does not lie in the upper half plane. Indeed, consider the sequences

$$0 \to \mathcal{O}_C(-C) \to \mathcal{O}_C \to \delta \to 0 \tag{3}$$

where δ is supported on a finite lenght subscheme. By geometricity, $\delta \in \mathcal{P}(1)$, hence $Z(\mathcal{O}_C(-nC))$ lies on the same horizontal line, distinct from the real axis (since $\theta < 1$) for all n. Then the sequence

$$0 \to \mathcal{O}_S(-nC) \to \mathcal{O}_S(-(n-1)C) \to \mathcal{O}_C(-(n-1)C) \to 0 \tag{4}$$

[3] We are being sloppy here. For a precise statement, see for example [1]

implies that $Z(\mathcal{O}_S(-nC))$ will eventually fall off the region $\overline{\mathbb{H}}$.

This example illustrate the necessity of the more general definitions for the category $D^b(S)$, in which we allow the central charge to have bigger image. The solution to the problem presented in the example will be to *tilt* the category $\mathbf{Coh}(S)$ within $D^b(S)$[4].

4.3. **Perspectives.** Stability conditions on surfaces provide a broad and interesting landscape. Indeed, moduli spaces of Bridgeland stable sheaves tell us a lot about the birational geometry of the classical space of Gieseker stable sheaves. A precise study of these problems is performed by Bayer and Macrì for K3 surfaces (see [9],[10]) and by Bertram, Martinez and Wang for Del Pezzo surfaces (in bigger detail for $\mathbb{P}^2$) in [11]. Bridgeland stability conditions were also successfully applied to the study of the minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ in [12].

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[4]We are deliberately hiding the interplay between a stability condition on an abelian category $\mathcal{A}$ and a stability condition on its derived category $D^b(\mathcal{A})$. One would need to say the words *torsion theory*, *t-structure*, and *heart*. For a survey on those notions, see [1]