SCHUR FUNCTIONS

Editorial comments. The Schur functions s_λ are a special basis for the algebra of symmetric functions Λ. They are also intimately connected with representations of the symmetric and general linear groups. In what follows we will give two alternative definitions of these functions, show how they are related to other symmetric function bases, explicitly describe their connection with representation theory, and state some of their properties. Three of the standard references for this material are [3, 6, 8, 9].

Definitions. Let $\mathbf{x} = \{x_1, \ldots, x_l\}$ be a set of variables and let Λ be the algebra of symmetric functions in $\mathbf{x}$. Bases for this algebra are indexed by *partitions* $\lambda = (\lambda_1, \ldots, \lambda_l)$, i.e., λ is a weakly decreasing sequence of l nonnegative integers λ_i called *parts*. Associated with any partition is an *alternant* which is the $l \times l$ determinant

$$a_\lambda = \det(x_i^{\lambda_j})$$

In particular for the partition $\delta = (l-1, l-2, \ldots, 0)$ we have Vandermonde's determinant $a_\delta = \prod_{i<j}(x_i - x_j)$. In his thesis [11], Schur defined the functions which bear his name as

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

where addition of partitions is component-wise. It is clear from this equation that s_λ is a symmetric homogeneous polynomial of degree $|\lambda| = \sum_i \lambda_i$.

There is a more combinatorial definition of a Schur function. A partition λ can be viewed as a *Ferrers shape* obtained by placing dots or cells in l left-justified rows with λ_i boxes in row i. One obtains a *semistandard Young Tableaux (SSYT)*, T, of shape λ by replacing each dot by a positive integer so that rows weakly increase and columns strictly increase. For example, if $\lambda = (4, 2, 1)$ then its shape and a possible tableau are

$$\lambda = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}, \quad T = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 3 & & \\ 4 & & & \end{array}.$$

Each tableau determines a monomial $\mathbf{x}^T = \prod_{i \in T} x_i$, e.g., in our example $\mathbf{x}^T = x_1^3 x_2 x_3^2 x_4$. Our second definition of the Schur function is then

$$s_\lambda = \sum_T \mathbf{x}^T$$

where the sum is over all SSYT of shape λ with entries between 1 and l.

Change of basis. The Schur functions can also be written in terms of the other standard bases for Λ. A *monomial symmetric function* m_λ is the sum of all monomials whose exponent sequence is some permutation of λ. Also define the *Kostka number* [5] $K_{\lambda\mu}$ as the number of SSYT T of shape λ and *content* $\mu = (\mu_1, \ldots, \mu_l)$, i.e., T contains μ_i entries equal to i for $1 \le i \le l$. The combinatorial definition of s_λ immediately gives the following.

Theorem 1 (Young's Rule)

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

Now consider the *complete homogeneous symmetric functions* $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$ and the *elementary symmetric functions* $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$ where h_{λ_i} (respectively, e_{λ_i}) is the sum of all (respectively, all square-free) monomials of degree λ_i. Also let λ' denote the partition *conjugate to* λ whose parts are the column lengths of λ's shape. In the preceding example, $\lambda' = (3, 2, 1, 1)$. For the two bases under consideration the s_λ can be described as a determinant.

Theorem 2 (Jacobi-Trudi Identity [2, 12] & dual)

$$s_\lambda = \det(h_{\lambda_i - i + j}) \quad \textit{and} \quad s_{\lambda'} = \det(e_{\lambda_i - i + j}).$$

Note that this theorem immediately implies

$$s_{(l)} = h_l \quad \text{and} \quad s_{(1^l)} = e_l$$

where (1^l) is the partition with l parts all equal to 1. These specializations also follow directly from the combinatorial definition of s_λ.

Representations. The description of s_λ in terms of the power sum symmetric functions brings in the representation theory of the symmetric group $\mathfrak{S}_n$. The irreducible representations of $\mathfrak{S}_n$ are indexed by partitions λ such that $|\lambda| = n$. Given a conjugacy class of $\mathfrak{S}_n$ corresponding to a partition μ let k_μ denote its size and let χ^λ_μ be the value of the λth irreducible character on the class. Now consider the *power sum symmetric function* $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$ where $p_{\lambda_i} = x_1^{\lambda_i} + \cdots + x_l^{\lambda_i}$.

Theorem 3 *If* $|\lambda| = n$ *then*

$$s_\lambda = \frac{1}{n!} \sum_{|\mu|=n} k_\mu \chi^\lambda_\mu p_\mu.$$

In other words, s_λ is the cycle-indicator generating function (in the sense of Polyá-Redfield enumeration) for the irreducible character of $\mathfrak{S}_n$ corresponding to λ.

Now consider the complex general linear group GL_l. A representation $\rho : GL_l \to GL_m$ is *polynomial* if for every $X \in GL_l$ the entries of $\rho(X)$ are polynomials in the entries of X. The polynomial representations of GL_l are indexed by the partitions λ with l nonnegative parts. Let χ be the character of a polynomial representation ρ and let X have eigenvalues $x_1, \ldots, x_l$. Then χ is a polynomial function of the x_i (because this is true for diagonal X which are dense in GL_l) and is symmetric (because χ is a class function). In fact more is true.

Theorem 4 *The irreducible polynomial characters of GL_l are precisely the s_λ for λ with l nonnegative parts.*

Properties. We can use the connection with representations of $\mathfrak{S}_n$ to construct an isomorphism of algebras. Let R^n denote the vector space of all class functions on $\mathfrak{S}_n$ and let $R = \sum_{n\geq 0} R^n$. The irreducible characters form a basis for R and endow it with a multiplication by induction of the tensor product. Frobenius' *characteristic map* [1] is $\mathrm{ch} : R \to \Lambda$ defined on $\chi \in R^n$ by

$$\mathrm{ch}(\chi) = \frac{1}{n!} \sum_{|\mu|=n} k_\mu \chi_\mu p_\mu$$

where χ_μ is the value of χ on the class corresponding to μ.

Theorem 5 *The map* $\mathrm{ch} : R \to \Lambda$ *is an isomorphism of algebras.*

In fact there are natural inner products on R and Λ that make ch an isometry.

A number of identities involving Schur functions have interesting bijective proofs using the combinatorial definition. Among the most famous are the following in which we assume we also have a set of variables $\mathbf{y} = \{y_1, \ldots, y_l\}$.

Theorem 6 (Cauchy Identity & dual)

$$\sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \prod_{i,j=1}^{l} \frac{1}{1 - x_i y_j}$$

and

$$\sum_\lambda s_\lambda(\mathbf{x}) s_{\lambda'}(\mathbf{y}) = \prod_{i,j=1}^{l} (1 + x_i y_j).$$

Knuth [4] has given algorithmic bijections between matrices and SSYT that prove these identities. It is a generalization of a map of Schensted [10] for *standard Young tableaux*, i.e., SSYT where the entries are precisely $1, \ldots, |\lambda|$.

We can also describe the structure constants for the algebra Λ in the basis s_λ combinatorially. If $\mu \subseteq \lambda$ as Ferrers shapes, then we have a *skew shape* λ/μ consisting of all dots or cells that are in λ but not in μ. Skew SSYT are defined in the obvious way. The *reverse row word* for a SSYT T is π_T obtained by reading the entries in each row from right to left, starting with the top row and working down. For our example tableau $\pi_T = 3111324$. Also a sequence of positive integers $\pi = w_1 \ldots w_n$ is a *lattice permutation* or *ballot sequence* if in every prefix $w_1 \ldots w_k$ the number of i's is at least as big as the number of $i+1$'s for all $i \geq 1$.

Theorem 7 (Littlewood-Richardson Rule [7]) *If*

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

then

$$c^\nu_{\lambda\mu} = \text{number of SSYT } T \text{ of shape } \nu/\lambda \text{ and content } \mu \text{ such that } \pi_T \text{ a ballot sequence.}$$

Via the characteristic map, the Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$ can also be viewed as giving the multiplicities of the character product $\chi^\lambda \chi^\mu$ when decomposed into irreducibles. Equivalently one can consider the decomposition of the inner tensor product of two irreducible polynomial representations of GL_l.

In conclusion we should mention that there are many generalizations of Schur functions, one of the most notable being the Hall-Littlewood functions. See Macdonald's book [8] for more information about them.

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