BV-BFV Approach to General Relativity

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Abstract

This thesis is devoted to the study of different formulations of General Relativity[1,2] (GR) as a a fundamental theory of the gravitational interaction in the setting of Cattaneo, Mnev and Reshetikhin (CMR) on manifolds with boundary. The Batalin (Fradkin) Vilkovisky formalisms (BV and BFV) were joined by CMR[3,4,5] to associate to a BV gauge theory on a space-time manifold M a correspondent BFV structure on its boundary ∂M, and a set of axioms for general gauge theories was proposed in this context, in order to have a neat quantisation scheme.

The present work is aimed at testing the axioms on different, classically equivalent formulations of General Relativity, namely the Einstein Hilbert metric theory of gravity, the Palatini Holst[6,7] tetrad formulation of GR and two BF-like theories that go under the name of Plebanski action[8] and McDowell-Mansouri action[9].

We prove that only some of these formulations satisfy the CMR axioms, thus inducing a BV-BFV theory: the Einstein Hilbert theory, for all manifolds with boundary of dimension $d + 1 \neq 2$ with spacelike or timelke boundary components, and the BF-formulation of the McDowell-Mansoury action, under some natural regularity assumptions on the field B.

The classical canonical analysis for the Einstein Hilbert and the Palatini Holst actions is also discussed, and we show how the machinery is capable of recovering known results in a straightforward way, yielding in addition an explicit symplectic characterisation of the phase space of the theory.

This is a first step in the programme of CMR quantisation of gauge theories on manifolds with boundary, applied to the fundamental, and still open case of General Relativity.

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Zusammenfassung

Diese Dissertation ist dem Studium verschiedener Formulierungen der Allgemeinen Relativitätstheorie (AR) als Grundtheorie der Gravitationswechselwirkung, im Rahmen des Cattaneo, Mnev, Reshetikhin Formalismus über Mannifaltigkeiten mit Rand gewidmet. Um durch die Verbindung der Formalismen von Batalin (Fradkin) und Vilkovisky, einer BV-Theorie über einer Mannigfaltigkeit mit Rand M eine entsprechende BFV-Theorie über ihrem Rand zu assoziieren, haben CMR Axiome vorgeschlagen, die eine Eichtheorie erfüllen muss, um ein ordentliches Quantisierungsschema zu erlauben.

Diese Arbeit testet diese Axiome für verschiedene, klassisch äquivalente Formulierungen der AR: der Einstein-Hilbert-Theorie der Schwerkraft, der Palatini-Holst-Formulierung der AR, sowie zwei BF-ähnlichen Theorien, die Plebanksi, b.z.w. MacDowell-Mansouri-Theorie genannt werden.

Wir beweisen, dass nur manche dieser Formulierungen die CMR Axiome erfüllen, und somit eine BV-BFV Theorie induzieren: die Einstein-Hilbert-Theorie, für alle Mannigfaltigkeiten mit Rand in Dimension $d + 1 \neq 2$ mit Zeit/Raum-artigem Rand, und die MacDowell-Mansouri-Theorie, mit natürlichen Annahmen über die Felder.

Die klassische, kanonische Analysis für die Einstein-Hilbert und die Palatini-Holst Wirkungen wird auch diskutiert, und wir zeigen, dass der Formalismus in der Lage ist, bekannte Ergebnisse in einfacher Weise zu reproduzieren, und ausserdem eine explizite symplektische Beschreibung des Phasenraums ermöglicht.

Dies ist ein erster Schritt des CMR-Programmes zur Quantiesierung der Eichtheorien über Mannifaltigkeiten mit Rand, angewandt auf das fundamentale offene Problem der Allgemeinen Relativitätstheorie.

Contents

To my family.

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1
Introduction

One of the most important tasks of modern theoretical and mathematical physics is that of formalising the notion of quantisation of a physical theory, in such a way that established results can be recovered, and that the process of quantising classical field theories can be understood within a well defined mathematical framework.

Learning from the algebraic lesson taught by category theory, one can arguably think that, whatever the appropriate notion of quantisation eventually turns out to be, it should be understood as a functor between two adequately chosen categories, encoding and joining on one hand the data of classical field theories, and on the other their quantum counterpart. In this direction have been focused the efforts of axiomatic field theory of Atiyah and Segal [10,11], and of Costello [12] and Lurie [13], more recently. In this sense, the notion of cobordism with additional geometric data, i.e. manifolds with boundary and possibly corners, becomes the source category on which classical field theory is cast: higher codimension manifolds representing objects and lower codimension ones representing morphisms, and morphisms between them.

On the other hand, much has been learned by the mathematical physics' community from the successful endeavours of Becchi, Rouet, Stora, Tyutin [14] (BRST), as well as Batalin, (Fradkin) and Vilkovisky [15,16] (B(F)V), to understand the perturbative quantisation of gauge theories in the framework of cohomological resolutions of quotients. There, instead of making sense of integrals on some (generically non smooth) reduced space of fields, one is able to work with some appropriate *replacement*, at the price of dealing with non-physical fields to encode gauge symmetries. This has been hinting that physical states, and the *path integral* itself should be related to the cohomology of an appropriate complex, and this technology has been proven successful where other methods fail.

The natural step would be that of putting these ideas together: keeping the interpretation of path integration as some *algebraic* entity, while casting it in an axiomatic way and clarifying the *functoriality* of some process that we might hope to call quantisation.

A recent attempt in this direction came from Cattaneo, Mnëv and Reshetikhin[3,4,5] (CMR) that understood that the B(F)V formalism is flexible enough to treat BV structures on manifolds with boundaries in such a way that the gluing becomes a natural operation, thus setting the stage for a possible axiomatic approach to boundary BV quantisation (see also[17] for gluing of manifolds in the general framework of synthetic differential geometry). They were able to show[3] that a large class of physically relevant theories satisfy their BV-BFV axioms, i.e. they induce a BFV structure on the boundary, when the input data is that of a BV structure on the bulk manifold. Furthermore, introducing the notion of *extended theory*, they showed that this induction process is self-similar, in the sense that the induced boundary structure might furthermore induce a compatible theory on its boundary (when the bulk manifold has corners, indeed). The rich machinery of AKSZ (Alexandrov, Kontsevich, Schwarz and Zaboronsky[18]) satisfies rather naturally the BV-BFV axioms, and provides important examples of extended theories.

The key observation in CMR is that one needs some compatibility between the bulk BV structure and the boundary BFV structure for the state associated to the bulk to be a cocycle (and hence to define a physical state). At the semiclassical level, the fundamental compatibility condition is that the failure of the bulk action to be the Hamiltonian function for the BV operator should be given by the pullback of the boundary Noether 1-form. This can always be achieved in terms of a larger space of boundary fields on which the differential of the Noether 1-form is degenerate. The crucial assumption is that the symplectic reduction of this 2-form should be smooth.

The proposed quantisation procedure[5] is a modified version of the BV quantisation, whenever the bulk-to-boundary compatibility can be transferred to the quantum setting. This is different from the usual BFV *Hamiltonian* analysis of field theories, precisely due to this novel compatibility relations between bulk and boundary structures, which imply that the gauge fixing (and the associated bulk BV-quantisation) is controlled by the cohomological data coming from the quantisation of the boundary.

This is a first step in joining the two strands together, as one can view the higher co-dimension theories as the objects in some suitable category (encoding classical data), while the morphisms and higher transformations being interpreted as lower co-dimension manifolds, all enriched by the respective extended-BV-BFV structures.

This promising new way of looking at gauge theories has been tested already on a series of examples. For instance, in the first CMR paper[3] a long chapter is dedicated to casting fundamental theories like Yang Mills, Chern Simons and the broad class of AKSZ theories in the new formalism. The CMR quantisation of abelian BF theory has been worked out in detail[5] .

It is therefore expected that the formalism be challenged with, possibly, one of the most hard-to-handle gauge theories: General Relativity (GR). The aim of this PhD thesis is in fact that of initiating the journey of BV-BFV quantum gravity, by performing a *from scratch* analysis of different formulations of GR, from the point of view of the BV-BFV formalism of CMR.

A plethora of actions

In this work we consider General Relativity as the mathematical theory of the gravitational interaction, encoded in the fundamental equivalence principle and the Einstein field equations for a pseudo-Riemannian metric on a space-time manifold M, and we will make a tour through its different formulations. We start from the older formalism of Einstein and Hilbert[1,2], where the basic field is a pseudo-Riemannian metric, to more abstract formulations where the metric is seen as a derived quantity.

The Palatini-Holst formalism[6,7] is taken into account, where the basic fields are chosen to be a connection in the associated bundle of the frame bundle, and a section of the frame bundle itself. The Plebanski formalism[8] and other formulations of gravity[9], instead, consider even the frame field to be a derived entity, which comes into play only when the equations of motion are enforced.

The literature in the field is thick and immensely rich. Many attempts in view of a direct canonical quantisation are to be mentioned[19,20,21,22], and yet the community working in quantum gravity is profoundly divided into different schools of thought, *String Theory* and *Loop Quantum Gravity* being the two main lines of research.

From a sort of *super partes* view point, we decided to analyse the different possible theories without necessarily backing one or the other description of GR, and we have found some clear differences in applying our formalism to this example or the other. We believe that this phenomenon is of non-trivial relevance, if we have confidence in the process of falsification of scientific theories, as well as their refinement through conflicting theoretical or experimental outcomes.

In particular, we have shown that while the Eistein Hilbert formulation allows for an essentially straightforward BV-BFV treatment, the Palatini-Holst formalism does not, and in a robust way. Since the BV-BFV axioms essentially state the compatibility of the symmetries with the boundary data, a first mathematical interpretation of this result could tell us that the phase space of Palatini-Holst gravity is far from being a smooth space.

On the other hand, further generalisations that describe GR as *broken* BF theories, such as the Plebanski action or the McDowell Mansouri action in the BF formalism, exhibit a better behaved BV-BFV structure, even if still not enough to satisfy the CMR axioms.

We will see that satisfying the BV-BFV axioms when diffeomorphisms are into play is far less trivial than one would expect by looking at other fundamental theories such as Yang Mills. This somehow reflects the more complicated nature of gravity and its resistance in being treated in the same framework of *ordinary* gauge theories.

In particular, the very fact that *some* formulations of GR, which would otherwise be regarded as classically equivalent, fail to yield a well defined BV-BFV theory might be hinting that they are not truly suitable for the programme of (perturbative) quantisation. In this sense the CMR axioms might be taken as a criterion to decide which fundamental description should be chosen. The very way one theory or another fails to satisfy the axioms tells us something about which potentials can be chosen in the action functional, in order to grant the CMR compatibility with the boundary, and guides us in the selection of an appropriate action functional.

Einstein Hilbert action

In Chapter 3 and 4 the Einstein Hilbert formulation of General Relativity is taken into account as the classical input to construct an extended BV theory.

After some training with one dimensional examples in Chapter 3, we tackle the general problem of the BV-BFV structure of General Relativity for any $d+1 \neq 2$ spacetime dimensions. The case $d=1$ has to be treated separately, owing to the presence of a larger invariance, and it will be done in a further development of the present work. It is interesting to notice that our technique does not work in $d=1$, namely the BV-BFV theory fails if we do not consider the appropriate symmetries.

Owing to the complexity of the calculations, we resort to two procedures. One is the choice of a preferred set of coordinates (Section 4.2) that makes the metric block-diagonal on the boundary. The second one is the adoption of the ADM variables (Arnowitt Deser Misner[23,24], in Section 4.3). This is done only in a neighborhood of the boundary, which is required to be either space-like or time-like, and without asking global hyperbolicity, as opposed to what is usually done in the literature. What this means, more precisely, is that one has to impose some compatibility with the boundary of the allowed pseudo-Riemannian structures. In particular, we will restrict the space of classical fields to those Lorentzian metrics on M such that their residual signature when restricted to a boundary component is either space-like or time-like. We will eventually show that the two procedures yield equivalent results once some compatibility conditions are enforced, clarifying what it means to choose such adapted coordinates.

Notice that global hyperbolicity turns out to be an unnecessary restriction for what concerns the canonical structure. We work in a more general context where all the results can be carried out without requiring any global equal-time slicing. An improvement in generality for what concerns the canonical structure and the compatibility with symmetries goes in the direction of path integral quantisation, where contributions from outside the critical locus of the action must be taken into account. So even if global existence and uniqueness of solutions to the field equations is not granted for non globally hyperbolic structures, it is important that the statements about the canonical structure hold true more generally.

The main result of Chapter 4 is that the extended BV theory of gravity in the Einstein Hilbert formalism yields a BV-BFV structure when the above mentioned conditions on the boundary are met (Theorem 4.11). The explicit expressions for the relevant quantities are given in a local Darboux chart and we automatically recover the known Hamiltonian formulation of GR, including the algebra of constraints, from the boundary data (Theorem 4.12). As a byproduct we can show that the algebra of constraints yields a non-trivial example of a coisotropic structure that does not manifestly come from a Lie algebra action, and yet is linear in the ghost fields. Indeed, the structure constants are replaced by *structure functions*, depending on the metric on the boundary (c.f. Section 4.4.2). This addresses a question posed by Blohmann, Fernandes and Weinstein[25] .

We observe that the BV-BFV machinery yields interesting results already in the classical case. By interpreting the boundary terms as a 1-form on the space of (pre-)boundary fields (the Noether form) we are able to recover part of the Hamiltonian description of general relativity. In fact we are able to write down the symplectic form on the phase space - the symplectic reduction of the space of restriction of fields and normal jets to the boundary - explicitly (more on this in Section 2.1). We argue that this explicit description of the true phase space of the system through symplectic reduction (when possible) gives a cleaner understanding of the canonical relations among fields than the usual Poisson bracket formalism.

Turning on the symmetries by extending the space of fields according to the BV/BRST prescriptions (Chapter 2, Section 2.2) and performing the CMR boundary analysis, we are then able to recover the rest of the canonical data, i.e. the algebra of canonical constraints and the residual gauge symmetry on the boundary, in the form of an induced (degree 1) action functional. The Dirac analysis of constraints[26] is greatly simplified by the tools of symplectic geometry and the interpretation of the boundary action as embodying the resolution of the coisotropic submanifold defined by the constraints. As a matter of fact we believe that this procedure should become a standard one for the canonical analysis of classical field theories.

Moreover, all of the one-dimensional models we analysed also satisfy the CMR axioms (c.f. Chapter 3). We consider both pure gravity and scalar matter coupled with the gravitational field, as well as a Robertson Walker cosmological model coupled with scalar matter, following Hartle and Hawking[27]. In the latter case, we again find that the boundary action automatically encodes the algebra of constraints, in the form of (a reduced version of) the Wheeler DeWitt Hamiltonian constraint[24].

Palatini-Holst action

The first non-metric theory of the gravitational interaction that we approach in this thesis is the Palatini-Holst (PH) formulation[6,7], which has the interesting feature of embedding GR in the usual framework of gauge theories, by making it a theory of principal connections. Chapter 5 is devoted to this.

The main idea is that of relaxing the requirement that the connection be dependent on the metric (as it is the case of the Levi-Civita connection) by letting the compatibility become dynamical, i.e. encoded in an Euler Lagrange equation for a new variational problem. Instead of considering the pseudo-Riemannian metric as a fundamental field, in the Palatini-Holst theory it is regarded as a derived quantity constructed from a co-tetrad, i.e. a section of the frame bundle.

This means that the two theories are equivalent only *on shell*, that is on the critical locus of the action, i.e. on those field configurations that solve the equations of motion. As we will see, this *equivalence-on-shell* does not say much on the symmetries of the action and on their compatibility with the boundary, or on the structure of the reduced phase space.

In Chapter 5, Section 5.2, we will show how to describe the symplectic space of boundary fields - the (non reduced) phase space of the system - by providing an explicit expression for the symplectic structure, and clarifying the Hamiltonian picture of GR in the tetrad formalism.

We will then turn to the BV-framework, and to its BFV counterpart, by presenting a natural result that implements the diffeomorphisms as gauge symmetries in such setting, for all theories of differential forms on a principal bundle (like the present one). This is necessary to extend the classical theory to a BV theory on the bulk manifold M. The main result, to be found in Section 5.3, Theorem 5.10, will state that the BV Palatini-Holst theory does not satisfy the BV-BFV axioms, and therefore does not induce a well defined BFV structure on the boundary ∂M.

We stress that this is a strong deviation from the Einstein Hilbert theory, and it crucially implies that there is no way to retain the required compatibility conditions between bulk and boundary structures. More precisely, the failure is of the pre-boundary structure to be pre-symplectic and to allow for a smooth symplectic reduction. This means in particular that the reduced phase space (most likely non smooth) does not have a smooth BFV replacement induced from the bulk.

In Section 5.4 we will test the idea of dynamically implementing the *Half-Shell* constraint, i.e. compatibility between the connection and the frame field, that fixes the independent principal connection to the Levi Civita connection. The classical phase space and the canonical structure are found in a straightforward way. In the BV setting we prove that the said constraint is coisotropic in the space of bulk fields and symmetry-invariant, and yet we show that this will only worsen the singularity that is found when adopting the BV-BFV approach (Theorem 5.15).

This result poses an important question about what variational principles that describe the same Euler Lagrange equations should be considered truly equivalent, and which should be regarded as *better quantisable*. The BV-BFV axioms might then be used as a criterion to determine whether a given variational principle has better chances than others to yield a sensible quantisation theory, if we believe that whatever quantisation eventually turns out to be, it should essentially be a functorial association of a suitable target category of linear objects, to the source category of space-time cobordisms with structure.

In other words, the naturality of the requirement of a bulk theory to be compatible with its boundary data, makes it hard to think that a correct notion of quantisation can be developed without taking this requirement into account.

BF-like actions for General Relativity

In Chapter 6 we will go one step further in abstraction and consider the tetrad field as a derived quantity as well. Retaining the geometric data of a principal bundle with connection on the space time manifold M, the basic field will be chosen to be any two form B with values in the Lie algebra, and the action functional will be a modification of the topological BF action.

Topological theories of the BF kind exhibit a large group of symmetries under the action of which all solutions to the field equations are equivalent. The theory has no local degrees of freedom, and clearly some symmetry breaking must be taken into account if we want it to be a model for General Relativity, which instead has two local physical degrees of freedom[28]. Different ways of breaking this symmetry will generate different realisations of General Relativity as a BF theory[29,30,31].

We will analyse two of these realisations, first the (non-chiral) Plebanski theory[8,30,31], when the symmetry breaking is realised through the introduction of a suitable Lagrange multiplier, which can be considered as a singular quadratic potential term for the field B. Later we will turn to the BF version of the McDowell-Mansouri action for General Relativity, which achieves symmetry breaking at the level of the Lie algebra instead, in a sort of Higgs-like fashion.

The two theories have different interesting features and analysing their interaction with diffeomorphisms will tells us something more on the compatibility of such an algebra of symmetries with Lagrange multipliers and potential terms, in relation to the boundary.

As a matter of fact, the main result of Section 6.1 is that the non-chiral Plebanski formulation does not satisfy the CMR axioms, and therefore does not induce a globally smooth symplectic reduction, even though, under some regularity assumptions for the fields B its singularity is much more tame than the Palatini-Holst version presented in Chapter 5. The explicit implementation of diffeomorphisms (Proposition 5.8) yields the important observation that when considering Lagrange multipliers and relative constraints, their invariance with respect to the symmetries is not a trivial issue. Our result shows in fact that the main source of singularity in the BV Plebanski theory comes precisely from the terms needed to establish the invariance of the constraints under space-time diffeomorphisms.

This already suggests that the particular way we break the symmetry does indeed matter when we explicitly consider symmetries, even if the action functional has a critical locus which is diffeomorphic to that of Einstein and Hilbert. As a matter of fact, in the BF formulation of McDowell Mansouri theory, where the symmetry breaking is performed by introducing a regular potential term, the problem associated with the Lagrange multipliers is not present anymore.

In Section 6.2 we will show that under appropriate regularity conditions for the field B, the BV-extended BF formulation of the McDowell-Mansouri action for General Relativity does indeed satisfy the CMR axioms and therefore yields a BV-BFV structure.

2

Lagrangian field theories on manifolds with boundary

In this introductory Chapter we will expound the basic ideas and mathematical tools that will be fundamental throughout the rest of the work. We will present first the general picture of field theories on manifolds with boundary, and then we will dwell on the BV-BFV machinery, and the relationship with the BRST formalism for gauge theories.

This work is a first step in the programme of BV-BFV quantisation of General Relativity and we will be focusing mainly on the classical framework. Nonetheless, being quantisation the final goal, we shall briefly review the ideas underlying the boundary BV quantisation technique, as recently proposed by Cattaneo Mnëv and Reshetikhin[5].

2.1 Field Theory with boundary

It is customary to describe classical field theories by means of an action functional S on some space of fields $\mathcal{F}$, usually sections of vector bundles or sheaves, with the property of being *local*, i.e. dependent on the fields and a finite number of derivatives in the form

$$S = \int_M \mathcal{L}[\phi, \partial^I \phi, \partial^I \partial^J \phi, \dots] \tag{2.1}$$

with I, J multiindices and $\mathcal{L}$ a *Lagrangian density*.

The first example to bear in mind is given by classical mechanics, seen as a field theory where fields are paths in a target manifold. For simplicity we will choose $\mathbb{R}$ as a target, the space of fields being $\mathcal{F} = \mathrm{Maps}([a, b], \mathbb{R})$, and the action reads:

$$S[q] = \int_a^b \left(\frac{1}{2} m\dot{q}^2 - V(q) \right) dt$$

with $q \in \mathcal{F}$, and V a *potential*. The classical machinery then extracts information from this data by solving the associated variational problem, meaning that the *physical* dynamical content is encoded in the critical locus of the functional S, which yields the Euler Lagrange equations.

On the other hand, turning to the quantum formulation of the theory, one wishes to interpret the factor

$$e^{\frac{i}{\hbar} S[\phi]}$$

as a probability amplitude, to be integrated against the "to-be-made-sense-of-measure" $\mathcal{D}\phi$, with which one would like to endow the space of fields. Following Feynman, the physical content of the quantum theory is encoded in the above mentioned integrals on the space of fields, which can be made sense of perturbatively, i.e. expanding in $\hbar$ around the critical locus of the classical action.

If on the physical side this is understood by saying that the quantum behaviour should be a perturbation of the classical behaviour, adding *quantum contributions* of different orders, the mathematical problem becomes that of making sense of the (formal) neighbourhood of the critical locus of S in the space of fields.

In the mentioned example of classical mechanics, this procedure is understood and it can be fully carried out after analytical continuation, the outcome being the path integral formulation of quantum mechanics. In that case the measure on the space of fields makes sense and the integral is an *actual* integral (using the Wiener measure, see[32] for a recent account on the subject). In passing to field theory, we will assume that things work analogously, while waiting for a formal proof that the perturbative expansions through which we define the path integral really are an approximation of a well defined mathematical object, which behaves like an integral on the space of fields.

This involved picture is further complicated by possible extra data our theory might enjoy, such as boundaries, for which one has to handle possibly non vanishing conditions, and symmetries that make the critical locus degenerate.

Traditionally, the canonical/Hamiltonian analysis data (á la Dirac[26]) and the problem of symmetries and gauge fixings have been considered separately. A great deal of literature has tackled the problem of symmetries of field theories through the Faddeev Popov ghost method[33], later understood under the more general framework of BRST (Becchi Rouet Stora, Tyutin[14]) and ultimately generalised by Batalin (Fradkin) and Vilkovisky (B(F)V)[15,16] to treat also symmetries that do not come from Lie algebra actions. On the other hand, the Hamiltonian analysis of field theories has been independently developed in order to make sense of canonical quantisation.

It was only recently that a consistent treatment of the two problems has been developed, by allowing a gauge theory to be cast on manifolds with boundary, and understanding under which conditions the boundary structure is compatible with the bulk data. Theories whose symmetric data and boundary can be consistently treated together are called BV-BFV, or CMR (Cattaneo, Mnëv, Reshetikhin) theories[3,4,5] (see Section 2.2, Definition 2.5). This approach and technology will be the starting point of our analysis, and the basic framework we will refer to in this work. The main CMR technology will be presented in this background chapter.

This approach has many advantages, such as a compatible cutting-gluing procedure, which allows to break topologically nontrivial manifolds into pieces, for which one might expect the quantisation to be simpler, as well as a powerful understanding of the quantisation procedure itself as a suitable generalisation of the Atiyah-Segal axioms for (topological) gauge theories[5,10,11]. Moreover, this approach gives a clear handle on the classical theory, providing a much cleaner understanding of the *Hamiltonian* approach to classical gauge theories, after Dirac and his canonical constraint analysis[26] which is in fact a first step towards their quantisation.

Before we begin, we would like to outline here the general idea. Consider again the previous case of classical mechanics on an interval, Eq. (2.1). If we compute the variation of the action and integrate by parts we get

$$\delta S = -\int_I \underbrace{(m\ddot{q} + V')}_{EL}\delta q + \int_{\partial I} m\dot{q}\delta q$$

where the term EL will yield the Euler Lagrange equations, which in this case are Newton's equations: $m\ddot{q} = -V'$.

Before interpreting the remaining boundary term, notice that the space of Cauchy data, i.e. the information one needs to complement EL with, in order to uniquely solve the equations of motion, is given by $C = TM$, i.e. the assignment of a value of position and velocity at the boundary (at time a, or b). On the other hand, once we have a path q, i.e. a point in $\mathcal{F}$ we can find its initial and final position and velocity, i.e. we have

$$\pi_a : \begin{array}{ccc} \mathcal{F} & \longrightarrow & C \\ q & \longmapsto & (q(a), \dot{q}(a)) \end{array}$$

and equivalently for b.

Therefore, it is easy to gather that we can interpret the boundary term as the pullback of a one form on C:

$$\delta S = \mathrm{el} + \pi_b^*\alpha - \pi_a^*\alpha, \qquad \alpha = m\dot{q}\delta q \in \Omega^1(C)$$

and its differential $\omega := \delta\alpha$ is nothing less than the pullback of the canonical symplectic form on the cotangent bundle along the Legendre transform (enforcing $p = m\dot{q}$).

Alternatively, if we want to avoid facing the problem of the well definiteness of the E.L. equations at this stage, we can interpret the extra term on the boundary as the pullback of a one form on a different space, defined as the space of germs of functions on the boundary cross an infinitesimal interval $\partial I \times [0, \epsilon]$, computed at $\partial I \times \{0\}$. This procedure will generalise in a straightforward way to more general manifolds M with boundary ∂M and the space of such germs will be denoted by $\widetilde{\mathcal{F}}_{\partial M}$. The link between these two analogous descriptions is clear after the respective reduction is performed (C as a coisotropic submanifold and $\widetilde{\mathcal{F}}_{\partial M}$ as a presymplectic manifold).

The advantage of this point of view is already clear if we observe that for degenerate Lagrangians the Legendre transform will not be well defined on the whole phase space, and the Hamiltonian formalism is less trivially employed. In those situations, as it will be clear from our discussions, the boundary Lagrangian setting will be far more fruitful (on another approach to Lagrangian field theories see also Costello[12]).

Let us sketch the general construction for classical gauge theories on manifolds with boundary and the direction to go in order to tackle the quantum theory.

2.2 BV and BFV axioms

We will consider here a general framework for gauge field theories. First of all we fix the space dimension, say d, and assign to a d-dimensional manifold M (possibly with boundary, and other geometric data, like a Riemannian structure) a space of fields $\mathcal{F}_M$, i.e. a $\mathbb{Z}$-graded odd-symplectic manifold, with a symplectic form Ω_M of degree $|\Omega_M| = k$ together with a local, degree $k+1$ functional S_M of the fields and a finite number of their derivatives.

The equations of motion (i.e. the dynamical content of the theory) are encoded in the Euler Lagrange variational problem for the functional S_M. The $\mathbb{Z}$ grading is sometimes called *ghost number*, but it will be often replaced by the computationally friendly *total degree*, which takes into account the sum of different gradings when the fields belong to some graded vector space themselves (e.g. differential forms).

The symmetries are encoded by an odd vector field $Q_M \in \Gamma(T[1]M)$ such that $[Q_M, Q_M] = 0$. A vector field with such a property is said to be *cohomological*. Q_M is also referred to as the classical BRST operator.

Among these pieces of data some compatibility conditions are required. We give the following definitions for different values of k. In the convention we adopt ordinary symplectic manifolds are called (0)-symplectic in the graded setting. Our model for a bulk theory will be given by

Definition 2.1. *A BV-theory on a closed manifold M is the collection of data $(\mathcal{F}_M, S_M, Q_M, \Omega_M)$ with $(\mathcal{F}_M, \Omega_M)$ a $\mathbb{Z}$-graded (-1)-symplectic manifold, and S_M and Q_M respectively a degree 0 function and a degree 1 vector field on $\mathcal{F}_M$ such that*

1. $\iota_{Q_M}\Omega_M = \delta S_M$*, i.e.* S_M *is the Hamiltonian function of* Q_M
2. $[Q_M, Q_M] = 0$*, i.e.* Q_M *is cohomological.*

The symplectic structure defines an odd-Poisson bracket $(,)$ on $\mathcal{F}_M$ and the above conditions together imply

$$(S, S) = 0 \tag{2.2}$$

the Classical Master Equation *(CME).*

✠

On the other hand, the model for a boundary theory, *induced* in some sense to be explained, will be given by

Definition 2.2. *A BFV-theory on a closed manifold N is the collection of data $(\mathcal{F}_N, S_N, Q_N, \Omega_N)$ with $(\mathcal{F}_N, \Omega_N)$ a $\mathbb{Z}$-graded 0-symplectic manifold, and S_N and Q_N respectively a degree 1 function and a degree 1 vector field on $\mathcal{F}_N$ such that*

1. $\iota_{Q_N}\Omega_N = \delta S_N$*, i.e.* S_N *is the Hamiltonian function of* Q_N
2. $[Q_N, Q_N] = 0$*, i.e.* Q_N *is cohomological.*

This implies that S_N satisfies the CME.

✠

In general one starts from a classical theory, that is an action functional S_{cl} for some space of classical fields F_M and a distribution D_M in the bulk encoding the symmetries, i.e. $L_X(S_{\mathrm{cl}}) = 0$ for all $X \in \Gamma(D_M)$. The main requirement on D_M for the formalism to make sense is that D_M be involutive on the critical locus of S_{cl}. Notice that D_M can be the distribution induced by a Lie algebra (group) action, in which case it is involutive on the whole space of fields. When this is the case we will talk of the BRST formalism, even though the setting will be slightly different from the original one (for another account on the relationship between the BV and BRST formalism see, e.g.[34]). We will be mainly interested in these types of theories, but for the sake of completeness we will sketch the general construction.

To construct a BV theory on the bulk starting from classical data, and assuming that M has no boundary, we must first extend the space of fields to accommodate the symmetries: $F_M \rightsquigarrow \mathcal{F}_M = T^*[-1]D_M[1]$. Symmetries are considered with a degree shift of $+1$, whereas the dualisation introduces a different class of fields (called anti-fields) with opposite parity to their conjugate fields, owing to the -1 shift in the cotangent functor. This yields a (-1)-symplectic manifold, which is a good candidate to be the space of bulk fields we want to work with.

The classical action has to be extended as well to a new local functional on $\mathcal{F}_M$, and if we want this to satisfy the axioms of the BV theory we must impose the CME on the extended action. This process will a priori need the introduction of higher degree fields to the space of fields in order to resolve, under some regularity assumptions, the relations among degree 1 fields. This process of extension goes through co-homological perturbation theory[35,36,15,37,4] and it will ensure us to end up with a BV structure on the bulk. However, for a theory which is BRST-like, the extension is determined by the following straightforward result[15]:

Theorem 2.3. *If D_M comes from a Lie algebra action, the functional $S_{BV} = S_{cl} + \langle \Phi^\dagger, Q_M\Phi\rangle$ on the space of fields $\mathcal{F}_M = T^*[-1]D_M[1]$ satisfies the CME, where Φ is a multiplet of fields in $D_M[1]$, $\Phi^\dagger$ denotes the corresponding multiplet of conjugate (anti-)fields and Q_M is the degree 1 vector field encoding the symmetries of D_M.*

$\mathcal{F}_M$ is then a (-1)-symplectic manifold and together with S_{BV} and Q_M it yields a BV theory that (minimally) extends the classical theory.

More details on the BRST formalism and how it can be embedded in the BV framework will be given in Section 2.4.

2.3 BV-BFV axioms on manifolds with boundary

As we already mentioned, Definition 2.2 will be a boundary model for Definition 2.1. In what follows we will explain in which sense. Say that we start from the data defining a BV theory, but this time we allow M to have a boundary: the requirement that $\iota_{Q_M}\Omega_M = \delta S_M$ is (in general) no longer true. What will happen is that the integration by parts one usually has to take into account when computing δS will leave some non zero terms *on the boundary*. More precisely, consider the map

$$\widetilde{\pi} : \mathcal{F}_M \longrightarrow \widetilde{\mathcal{F}}_{\partial M} \tag{2.3}$$

that takes all fields and their transversal jets to their restrictions to the boundary (it is a surjective submersion). We can interpret the boundary terms as the pullback of a one form* $\widetilde{\alpha}$ on $\widetilde{\mathcal{F}}_{\partial M}$, namely

$$\iota_{Q_M}\Omega_M = \delta S_M + \widetilde{\pi}^*\widetilde{\alpha} \tag{2.4}$$

We will call $\widetilde{\alpha}$ the *pre-boundary* one form.

Notice that if we are given this data, we can interpret this as a *broken BV theory*, which induces some data on the boundary. We can in fact consider the pre-boundary two form $\widetilde{\omega} := \delta\widetilde{\alpha}$ and if it is pre-symplectic (i.e. its kernel has constant rank) then we can define the true space of boundary fields $\mathcal{F}^{\partial}_{\partial M}$ to be the symplectic reduction of the space of pre-boundary fields, namely:

$$\mathcal{F}^{\partial}_{\partial M} = \widetilde{\mathcal{F}}_M / \ker(\widetilde{\omega}) \tag{2.5}$$

with projection to the quotient denoted by $\pi : \widetilde{\mathcal{F}}_{\partial M} \longrightarrow \mathcal{F}^{\partial}_{\partial M}$. If all of the above assumptions are satisfied, the map $\pi_M := \pi \circ \widetilde{\pi}$ is a surjective submersion, the reduced two form $\omega^{\partial}_{\partial M} := \underline{\widetilde{\omega}}$ is a 0-symplectic form, and we have the following

*In full generality $\widetilde{\alpha}$ is a connection on a line bundle, yet when S_M is a function on the space of fields, $\widetilde{\alpha}$ is a globally well defined 1-form, since integration by parts is a local operation in the space of fields.

Proposition 2.4 (CMR[4]). *The cohomological vector field Q_M projects to a cohomological vector field $Q^\partial_{\partial M}$ on the space of boundary fields $\mathcal{F}^\partial_{\partial M}$. Moreover $Q^\partial_{\partial M}$ is Hamiltonian for a function $S^\partial_{\partial M}$, the boundary action.*

We can now summarise this as follows: we will call a pre-BV-BFV theory on a d-dimensional manifold M with boundary ∂M a collection of data $(\mathcal{F}_M, S_M, Q_M, \Omega_M)$ with $(\mathcal{F}_M, \Omega_M)$ a $\mathbb{Z}$-graded (-1)-symplectic manifold, and S_M and Q_M respectively a degree 0 local functional and a degree 1 vector field on $\mathcal{F}_M$ such that

1. $[Q_M, Q_M] = 0$, i.e. Q_M is cohomological,
2. The map $\widetilde{\pi}$ from the space of bulk fields $\mathcal{F}_M$ to the space of pre-boundary fields $\widetilde{\mathcal{F}}_{\partial M}$ is a surjective submersion.
3. Q_M is $\widetilde{\pi}$-projectable to a cohomological vector field $\widetilde{Q}$ on $\widetilde{\mathcal{F}}_{\partial M}$
4. The BV-BFV formula $\iota_Q \Omega_M = \delta S_M + \widetilde{\pi}^* \widetilde{\alpha}$ is satisfied.

Definition 2.5 (CMR[3]). *Whenever the pre-boundary 2-form $\widetilde{\omega}$ is pre-symplectic on $\widetilde{\mathcal{F}}_M$ and the symplectic reduction to the space of boundary fields $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M})$ can be performed, this induces the BFV theory $(\mathcal{F}^\partial_{\partial M}, S^\partial_{\partial M}, Q^\partial_{\partial M}, \omega^\partial_{\partial M})$ on ∂M.*

The composition of $\widetilde{\pi}$ with the symplectic reduction map $\pi\colon \widetilde{\mathcal{F}}_{\partial M} \longrightarrow \mathcal{F}^\partial_{\partial M}$ will yield another pre-BV-BFV theory, for the symplectic form $\omega^\partial_{\partial M}$ and the surjective submersion $\pi_M = \pi \circ \widetilde{\pi}\colon \mathcal{F}_M \longrightarrow \mathcal{F}^\partial_{\partial M}$ satisfying axioms from (1) *to* (3). *In this case we say that the theory is BV-BFV. Furthermore, if $\widetilde{\alpha}$ is basic, $\widetilde{\alpha} = \pi^* \alpha^\partial_{\partial M}$, we say that the BV-BFV theory is* exact *and we have the fundamental formula*

$$\iota_{Q_M} \Omega_M = \delta S_M + \pi_M^* \alpha^\partial_{\partial M} \tag{2.6}$$

✠

The advantage of such a point of view is at least twofold. First of all, as we just saw, the formalism is *large enough* to be able to describe consistently what happens both in the bulk and in the boundary. On the other hand it is flexible enough to allow for symmetries that are more general than a Lie group action. For instance it is possible to accomodate symmetries that close only on shell (e.g. Poisson sigma model) or symmetries whose generators are not linearly independent, where higher relations among the relations are required (e.g. BF theory or other theories involving ($d > 1$)-differential forms).

The BV theory that we have constructed in Theorem 2.3 starting from a gauge theory of the BRST-kind is sometimes called the *minimal BV extension* of the gauge theory. When a non trivial boundary is allowed, we will use this minimal extension as the starting point for the BV-BFV analysis.

What one aims to establish is whether this minimal BV theory on the bulk is indeed a BV-BFV theory. In this work we will analyse different source classical actions, all classically equivalent, and we will determine which of these do indeed satisfy the CMR axioms.

2.4 BV versus BRST

In our language, the data that has to be specified in order to define a gauge theory consists essentially of a space of fields $\mathcal{F}$ on which a *local* functional is given, the action functional S, and a distribution $\mathcal{D}$ in the tangent space such that S is invariant under the action of all the vector fields in the distribution: $S \in C^\infty(\mathcal{F})^{\mathcal{D}}$.

We have already mentioned that this is potentially a problem for setting up a perturbative quantisation scheme (even ahead of the well definiteness issue of the path integral), because the critical locus of S, of which formal power series in $\hbar$ represent a formal neighborhood, is degenerate due to the symmetries, and no perturbative expansions around classical solutions can be performed.

In other words, what is happening is that in summing over all field cofigurations we are summing over *physically equivalent* ones, where by equivalent we roughly mean that they lie in the same leaf of the distribution D.

Sometimes, i.e. most of the times in physically-relevant examples, the distribution is given by a (faithful) Lie algebra action, and it is therefore involutive on the whole space of fields. This is a strong condition that allows us to use a very particular technique, that goes under the name of BRST[14] formalism. In this section we will outline the generalities of this mechanism and explain how this is generalised to the BV setting.

To fix the ideas, using a simplified example, consider a function S in $\mathbb{R}^3$ that only depends on the distance from the center, that is the modulus of a vector. Such a function is clearly invariant under the action of the group of rotations $SO(3)$ and the distribution in this case is given by the Lie algebra action on $\mathbb{R}^3$ (adjoint or coadjoint action). If we interpret $\mathbb{R}^3$ to be the space of fields and the function S as the action, the problem now is to sum over all configurations, without counting redundant field configurations, whose total contributions equal the volume of the gauge group (finite in this simple example).

One would like to count each $SO(3)$ orbit only once by integrating on a submanifold which is transversal to all orbits (in this case a ray from the origin). This idea lies at the heart of the concept of gauge fixing. The difference, though, is that while in this simple example it is possible to integrate directly the space of leaves of the distribution, for the more general examples this will not be possible and one needs a way to characterise this space in a different way.

The BRST formalism allows us to do so by *cohomologically resolving* the functions on the space of fields that are invariant under the action of a Lie group, essentially by extending the space of fields with the Chevalley-Eilemberg complex. Let us see how, in greater detail.

2.4.1 BRST formalism and gauge fixing

To sketch a first generalisation of the previous discussion let us assume that we can encode the data of a submanifold N transversal to the G-orbits in a function $H\colon \mathcal{F} \longrightarrow \mathfrak{g}$, for which 0 is the regular value fixing $N = H^{-1}(0)$. Moreover we will denote by X^i a basis of fundamental vector fields coming from the lie algebra action $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(\mathcal{F})$, so that $X^i = \rho(\xi^i)$, with $\{\xi^i\}$ a basis in $\mathfrak{g}$.

Now, extend the space of fields to include a shifted copy of the Lie algebra:

$$\mathcal{F}_{min} = \mathcal{F} \oplus \mathfrak{g}[1] \ni (\phi, c) \tag{2.7}$$

We are now working in the setting of graded vector spaces (and manifolds). Shifting the Lie algebra by 1 means considering a graded vector space concentrated in degree -1. For practical matters, we are simply declaring the elements in $\mathfrak{g}$ to behave like Grassmann variables.

Consider the operator

$$Q_{min} = c_i X^i - \frac{1}{2} c_i c_j f_k^{ij} \frac{\partial}{\partial c_k} \tag{2.8}$$

as a vector field on $C^\infty(\mathcal{F}_{min}) = C^\infty(\mathcal{F}) \otimes \bigwedge^\bullet \mathfrak{g}^*$, which is the $\mathfrak{g}$-module given by the Chevalley-Eilemberg complex and the smooth functions on $\mathcal{F}$, with the c_i's representing coordinates on $\mathfrak{g}[1]$. It is easy to check that $[Q_{min}, Q_{min}] = 0$, of degree 1, yielding a differential on $C^\infty(\mathcal{F}_{min})$, and we have that

$$H^0(C^\infty(\mathcal{F}_{min}), Q_{min}) \simeq C^\infty(\mathcal{F})^{\mathfrak{g}} \simeq C^\infty\left(\mathcal{F}/\mathfrak{g}\right) \tag{2.9}$$

The idea behind the BRST formalism and gauge fixing, as we will see, is that one wants to interpret S as a cocycle in degree zero for some operator of the space of functions, encoding the symmetries as Q_{min} does. Since we are only interested in the cohomology, a cocycle can be shifted by an appropriate coboundary. The problem one has to face, at this stage, is that in order to change the representative degree-zero cocycle in a given class, one must be able to build degree -1 coboundaries, and to do so we must extend the space of fields once more, to be able to deal with negative degrees.

The choice of a different representative represents the *gauge fixing*, i.e. the choice of a particular transversal section to the orbits in the space of fields.

Notice that we need to enlarge the space of fields in a way that the equality (2.9) is not crucially spoiled. We do it by adding a contractible space with a deRham differential on it, so that its contribution to the cohomology will be trivial. Namely, the space

$$\mathcal{F}_{gf} := \mathfrak{g}^*[-1] \oplus \mathfrak{g}^* \ni (\bar{c}, \lambda) \tag{2.10}$$

together with the differential

$$Q_{gf} := \lambda^i \frac{\partial}{\partial \bar{c}^i} \tag{2.11}$$

which clearly squares to zero. Then the correct BRST space for gauge fixing is given by

$$\mathcal{F}_{BRST} = \mathcal{F}_{min} \oplus \mathcal{F}_{gf} \tag{2.12}$$

with differential $Q_{BRST} = Q_{min} + Q_{gf}$. The gauge fixed action will look like $S_{gf} = S + Q_{BRST}\Psi$ for some $|\Psi| = -1$ (sometimes called gauge fixing fermion).

In the simplified case we are outlining, considering the functions H_i to be those that define the submanifold $N = H^{-1}(0)$ we can choose the gauge fixing fermion to be $\Psi := H_i\bar{c}^i$. The resulting gauge fixed action in this case will represent the well known Faddeev-Popov action[33].

To summarise, the BRST formalism provides a resolution of the functions on the quotient, i.e. the space of leaves for the group action, in such a way that the action functional is interpreted as a class in the cohomology of the differential Q_{BRST}. Choosing a representative in the class is tantamount to the choice of a gauge fixing.

2.4.2 BRST in the BV formalism

As we said, the BV formalism is somehow an extension of the BRST formalism. This is true in many ways: on the one hand for it allows us to treat more general symmetries than Lie algebra actions, but also because it is potentially compatible with a BFV structure on the boundary. In this section we would like to understand how one can embed the BRST construction in the BV setting. We will do this in a rather general example where we consider theories of principal connections.

The space of fields that we shall consider is the space of connections $\mathcal{A}_P$ on a principal bundle $G \to P \to M$ and the action functional is a local functional of the gauge connection, that is invariant under infinitesimal gauge transformations (in the sense of principal bundle morphisms). For instance, Yang-Mills theory is specified by

$$S_{YM} = \int_M \mathrm{Tr}\,(F_A \wedge \star F_A)$$

where $\star$ is the hodge star induced by the choice of some Riemannian or Lorentzian metric g on the closed manifold M, whereas the trace comes from any bilinear invariant pairing in $\mathfrak{g} = \mathrm{Lie}(G)$. Another example is given by the Chern Simons action on a three manifold

M:
$$S_{CS} = \int_M \frac{1}{2} < A, dA > + \frac{1}{3} < A, [A, A] >$$

where again $<,>$ is an invariant, bilinear non degenerate inner product in $\mathfrak{g}$.

The basic fields in these examples are connections $A \in \mathcal{A}_P$ on a principal bundle P, that can be seen as one-forms on P with values in $\mathfrak{g}$, or as one-forms on M with values in the adjoint bundle ad$P$. The symmetries are encoded by the odd fields $c \in \Omega^0(M, \mathrm{ad}P)[1]$, that are nothing but the generators of the gauge transformations, with degree shifted by $+1$. As a matter of fact the gauge transformations read

$$\delta A = d_A c; \qquad \delta c = \frac{1}{2}[c, c]$$

It is possible to think of this δ-operator as a vector field on the space of fields

$$\mathcal{F}_{\min} = \Omega^1(P, \mathfrak{g}) \times \Omega^0(M, \mathrm{ad}P)[1]$$

and it clearly satisfies $[\delta, \delta] = 0$, the cohomological condition. What we are doing is again simply considering the Chevalley-Eilemberg complex to encode the symmetry degrees of freedom: so far, nothing new.

Now consider the total space of fields to be the shifted cotangent bundle

$$\mathcal{F}_{BV} = T^*[-1]\mathcal{F}_{min} \ni (A, c, A^\dagger, c^\dagger)$$

with
$$A^\dagger \in \Omega^{d-1}(P, \mathfrak{g}^*)[1], \quad c^\dagger \in \Omega^d(M, \mathrm{ad}^* P)[-2]$$

and define the BV symplectic form to be the canonical (-1)-symplectic two form on $\mathcal{F}_{BV}$

$$\Omega_{BV} = \int_M (\delta A^\dagger, \delta A) + (\delta c^\dagger, \delta c)$$

where we used the canonical pairing $(,)$ between $\mathfrak{g}$ and its dual.

Observe that if we are given a vector field δ on the space $\mathcal{F}_{\min}$, by taking its cotangent lift $\check{\delta}$ we get the vector field

$$Q_{BRST} \coloneqq \check{\delta} \in \Gamma(T^*\mathcal{F}_{\min})[1]$$

and its Hamiltonian function with respect to Ω_{BV}, denoted by S_{BRST} reads

$$S_{BRST} = \int_M (A^\dagger, d_A c) + \frac{1}{2}(c^\dagger, [c, c])$$

Now, denoting the new action by $S = S_{cl} + S_{BRST}$ we can check (if M is closed)

$$\{S, S\} = 0$$

where $\{\cdot, \cdot\}$ is the (odd) Poisson bracket induced by Ω_{BV}, whereas the gauge invariance enforces $\delta S = 0$ and the fact that Q_{BRST} squares to zero implies that $\{S_{BRST}, S_{BRST}\} = 0$ as well. Notice that this partly proves Theorem 2.3, stated in Section 2.2

Up until here we have rewritten the minimal BRST structure in the BV formalism, and this will agree with what was presented in 2.4.1, *mutatis mutandis*. Again, to perform the gauge fixing as we did before we would like to add the new fields

$$\lambda \in \Omega^d(M, \mathrm{ad}^* P), \quad \bar{c} \in \Omega^d(M, \mathrm{ad}^* P)[-1]$$

and their cotangent fibres:

$$\mathcal{F}^{\mathrm{ext}} = \mathcal{F} \times T^*[-1]\left(\Omega^0[-1] \times \Omega^0\right)(M, \mathrm{ad}^* P)$$

with additional fibre fields $\bar{c}^\dagger$ and $\lambda^\dagger$. The action gets extended to

$$S^{ext} = S_{BRST} + \int_M \bar{c}^\dagger \lambda$$

together with

$$\omega^{ext} = \omega + \int_M (\delta\bar{c}^\dagger, \delta\bar{c}) + (\delta\lambda^\dagger, \delta\lambda)$$

It is a matter of easy calculations to show that we still have

$$(S^{ext}, S^{ext}) = 0$$

and the extension of the action of the cohomological vector field reads

$$(S^{ext}, \bar{c}) = \lambda, \;\; (S^{ext}, \lambda^\dagger) = \bar{c}^\dagger, \;\; (S^{ext}, \lambda) = (S^{ext}, \bar{c}^\dagger) = 0$$

This can be interpreted as the De Rham differential, through the identification $\lambda = d\bar{c}$ and $\bar{c}^\dagger = d\lambda^\dagger$.

We can now choose a function ψ of degree -1 such that

$$A^\dagger = \frac{\delta\psi}{\delta A}, \; c^\dagger = \frac{\delta\psi}{\delta c}, \; \bar{c}^\dagger = \frac{\delta\psi}{\delta\bar{c}}, \; \lambda^\dagger = \frac{\delta\psi}{\delta\lambda}$$

that is to say $\Phi^\dagger = \frac{\delta\psi}{\delta\Phi}$ where Φ are the base fields and the dagger marks the cotangent fibre fields. Therefore one writes

$$\int_{\mathcal{F}^{ext}} e^{\frac{i}{\hbar} S^{ext}\big|_{\Phi^\dagger = \frac{\delta\psi}{\delta\Phi}}} \, [\mathcal{D}\Phi]$$

where the integral is taken over the base fields.

Typically one chooses $\psi = \int \bar{c} d^*_{A_0}(A - A_0) + \alpha\bar{c} * \lambda$ to obtain the gauge fixed action

$$S^{ext}\big|_{\mathcal{L}_\psi} = S_{cl} + \int \underbrace{\bar{c} d^*_{A_0} d_A c}_{\text{FP-det}} + \underbrace{\lambda d_{A_0}(A - A_0)}_{\text{Lorenz}} + \underbrace{\alpha\lambda * \lambda}_{\text{Extra}}$$

where the first term yields the usual Faddeev Popov determinant, the second term enforces the Lorenz gauge and the extra term depending on α is a non-necessary correction that helps in simplifying later computations.

If we chose $\alpha = 0$ we would have, eliminating λ by using its equation of motion, that $\bar{c}^\dagger = 0$. So in the space $\widetilde{\mathcal{F}}^{ext} := \mathcal{F}_{min} \times \{c, \bar{c}\}$ we have a coisotropic submanifold $C = \{\bar{c}^\dagger = 0\}$ together with the Lagrangian $\mathcal{L}_\psi \subset \widetilde{\mathcal{F}}^{ext}$. Reducing, we get the Lagrangian submanifold

$$\underline{\mathcal{L}_\psi} = \{(A, c, A^\dagger, c^\dagger) \in \mathcal{F} \;\big|\; d^*_{A_0}(A - A_0) = 0, \; c^\dagger = 0, \; A^\dagger \in \mathrm{Im}(d^*_{A_0})\} \subset \underline{\widetilde{\mathcal{F}}}^{ext} = \mathcal{F}$$

so that finally we have

$$\int_{\mathcal{L}_\psi \subset \mathcal{F}^{ext}} e^{\frac{i}{\hbar}S^{ext}} = \int_{\underline{\mathcal{L}_\psi}} e^{\frac{i}{\hbar}S}$$

Assume now that the principal bundle is trivial, $P = M \times G$ and the reference connection $A_0 = 0$. Then $\mathcal{A} = \Omega^1(M, \mathfrak{g})$, $d^* A = 0$ and $A^\dagger \in \mathrm{Im}(d^*)$. So the reduced Lagrangian reads:

$$\underline{\mathcal{L}}_\psi = \{(A, c, A^\dagger, c^\dagger) \,\big|\, d^* A = 0,\ c^\dagger = 0,\ A^\dagger \in \mathrm{Im}(d^*)\}$$

Gauge fixing in the BV formalism is interpreted as integration over Lagrangian submanifolds, and the main BV theorem states that when the extended action satisfies some particular non trivial condition, called Quantum Master Equation (cf. below) the BV path integral will not depend on the gauge fixing, i.e. on the particular Lagrangian submanifold. More on this can be found in Appendix B.

By reading the BRST formalism in the BV setting one is able to observe that when the gauge fixing is performed by means of a gauge fixing fermion, the Lagrangian submanifold is of a very peculiar kind: it is the graph of an exact one form. This is an unnecessary restriction, that we can conveniently get rid of in the BV framework, by simply requiring the gauge fixing to be given by the choice of a Lagrangian submanifold.

2.5 Quantum bulk-boundary correspondence

The CMR axioms of Definition 2.5 for BV-BFV theories provide a notion of compatibility between a bulk BV and a boundary BFV theory. Assume we are given an exact BV-BFV theory in the form of bulk data $(\mathcal{F}_M, S_M, Q_M, \Omega_M)$, and the respective boundary data $(\mathcal{F}^\partial_{\partial M}, S^\partial_{\partial M}, Q^\partial_{\partial M}, \omega^\partial_{\partial M})$, we have the fundamental CMR formula

$$\iota_{Q_M}\Omega_M = \delta S_M + \pi^*_M \alpha^\partial_{\partial M} \tag{2.13}$$

Now, assume we are given an operator Δ of degree -1 on $\mathcal{F}_M$ such that $\Delta^2 = 0$ and the following compatibility relation holds:

$$(a,b)_{\Omega_M} = (-1)^{|a|}\left(\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)\right) \tag{2.14}$$

where $(\cdot,\cdot)_{\Omega_M}$ is the odd-Poisson structure associated with the odd-symplectic structure Ω_M. Such an operator Δ is called BV-Laplacian. We will not go into details of the construction of such an operator, whose existence is not in general granted as soon as we work in an infinite dimensional setting†, and for the general theory of BV algebras and Gesternhaber brackets and their relationship with homological algebra we refer to the vast literature on the subject[35,38,39]. A brief expansion on this will be anyway considered in Appendix B.

In field theory we are interested in functions of the form $g = \exp\{\frac{i}{\hbar}S\}$ and one can check that

$$\Delta e^{\frac{1}{\hbar}S} = 0 \Longleftrightarrow \frac{1}{2}(S,S) - i\hbar\Delta S = 0$$

This is called *Quantum Master Equation* (QME), and since we are interested in integrating g on some Lagrangian submanifold, encoding the gauge fixing, the BV Theorem[16] tells us that if QME holds, we can conclude that the integral is independent of the choice of gauge fixing (cf. Lemmas B.2 and B.3, Appendix B).

Generally, though, we start from the classical counterpart $\hbar \to 0$ that would be (S_0, S_0) (the classical master equation, CME) in the hypothesis $S = \sum_n \hbar^n S_n$ and M a closed manifold. Then one can start computing the perturbative corrections, for instance

$$(S_0, S_1) = -i\Delta S_1$$

and this will produce corrections to the CME. It is clear that there are obstructions, since the above equations tell us that S_0 is a cocycle, that we want to write as a coboundary, and this is not possible in general.

Notice that the Classical master equation makes sense also in the ∞-dimensional setting, in that it relies only on the BV bracket $(\cdot,\cdot)$. We need to regularise the theory to make sense of the full QME. Obstructions to this perturbative approach are usually given by anomalies.

†On the contrary Δ does indeed always exists, in finite dimensions.

Turning back to our bulk-boundary correspondence, let us assume that such a BV-Laplacian is given, in addition to the BV-BFV theory, and let us require for simplicity that $\Delta S = 0$. We will need to assume a series of thing in what follows, in order to illustrate how the BV-BFV compatibility should carry over to the quantum setting. The general theory has been set forth by CMR in their work[5], to which we refer.

Assume that we are given a polarisation in $\mathcal{F}^\partial_{\partial M}$ such that we have the splitting in the space of bulk fields:

$$\mathcal{F}_M = \mathcal{Y} \times \mathcal{B} \tag{2.15}$$

where $\mathcal{B}$ is the space of leaves of the polarisation. Moreover, as it is customary in geometric quantisation, we will require that $\alpha^\partial_{\partial M}$ vanishes on the Lagrangian fibres of the polarisation. These requirements are somehow natural when working with affine spaces. Finally, we will require that Ω_M be concentrated in $\mathcal{Y}$, and yet is nondegenerate. Notice that this is possible only in infinite dimensions, as it is true that in finite dimensions the very requirement that both $\mathcal{F}_M$ and $\mathcal{F}^\partial_{\partial M}$ be symplectic is provably impossible to satisfy. It can be amended precisely by asking that Ω_M be symplectic on the fibres $\mathcal{Y}$ or, equivalently that $\mathcal{F}_M \longrightarrow \mathcal{B}$ be a symplectic fibration. As a matter of fact we should think of $\mathcal{F}_M$ as being essentially a BV-space $\mathcal{Y}$ that depends on parameters that live in the space $\mathcal{B}$.

Recalling (2.13) and splitting $\delta = \delta_{\mathcal{Y}} + \delta_{\mathcal{B}}$, we deduce that

$$\begin{cases} \delta_{\mathcal{Y}} S = \iota_{Q_{\mathcal{Y}}} \Omega_M \\ \delta_{\mathcal{B}} S = -\pi^* \alpha^\partial \end{cases} \tag{2.16}$$

Moreover we have

Lemma 2.6. *With the above assumptions, the following formula holds:*

$$\frac{1}{2}(S, S)_{\mathcal{Y}} = \pi^* S^\partial \tag{2.17}$$

Proof. Starting from (2.13) we compute

$$\iota_{Q_M}\iota_{Q_M}\Omega_M = L_{Q_M}S_M + \pi_M^*\iota_{Q^\partial}\alpha^\partial \tag{2.18}$$

$$L_{Q_M}\Omega_M = \pi_M^*\omega^\partial \tag{2.19}$$

then we need the following computation

$$\begin{aligned}
L_{Q_M}\iota_{Q_M}\Omega_M &= -\delta L_{Q_M}S_M + \pi_M^* L_{Q^\partial}\alpha^\partial \\
-\iota_{Q_M}L_{Q_M}\Omega_M &= -\delta(L_{Q_M}S) + \pi_M^*\iota_{Q^\partial}\delta\alpha^\partial - \pi_M^*\delta\iota_{Q^\partial}\alpha^\partial \\
-\pi_M^*\iota_{Q^\partial}\delta\alpha^\partial &= -\delta(L_{Q_M}S) + \pi_M^*\iota_{Q^\partial}\omega^\partial - \pi_M^*\delta\iota_{Q^\partial}\alpha^\partial
\end{aligned}$$

where we used $[L_{Q_M}, \iota_{Q_M}] = \iota_{[Q_M,Q_M]} \equiv 0$, equation (2.19) and Proposition 2.4. Rearranging the terms and using the BFV CME, i.e. $\iota_{Q^\partial}\omega^\partial = \delta S^\partial$, we get an equivalence of exact forms, hence

$$L_{Q_M}S = -\pi_M^*\iota_{Q^\partial}\alpha^\partial + \pi_M^*(2S^\partial) \tag{2.20}$$

Plugging this into (2.18) we get the failure of the Classical Master equation in the bulk:

$$\frac{1}{2}(S_M, S_M) = \pi^* S^\partial \tag{2.21}$$

and using the fact that Ω_M is concentrated on $\mathcal{Y}$ we obtain the result. ✓

The quantisation scheme, following CMR[5] is as follows. First of all we have to perform the quantisation of the symplectic manifold of boundary fields, for instance through geometric quantisation. It is in fact likely that the structure of $\mathcal{F}^\partial_{\partial M}$ be that of a cotangent bundle. Assuming that this is the case and that $\alpha^\partial = -p\delta q$, denoting by p the fiber coordinates and by q the base coordinates, we have that $\frac{\delta S^\partial}{\delta q} = p$.

The key step is that of quantising the boundary action S^∂ to an operator[‡] using the canonical quantisation rules and standard ordering, placing all derivatives to the right:

$$\mathsf{D} := S^\partial\left(q, -i\hbar\frac{\delta}{\delta q}\right) \tag{2.22}$$

[‡]In their paper[5] CMR call it Ω. We change the notation to avoid confusion with the BV form.

and this will give us the following quantum compatibility relation

Lemma 2.7 (CMR[5])**.** *Under all of the previous assumptions we have the fundamental formula*

$$\widehat{\mathsf{D}}\, e^{\frac{i}{\hbar}S} := \left(\hbar^2\Delta + \mathsf{D}\right) e^{\frac{i}{\hbar}S} = 0 \tag{2.23}$$

which we will call Modified Quantum Master Equation *(mQME).*

Proof. First of all we compute $\mathsf{D}e^{\frac{1}{\hbar}S} = \pi_M^* S^\partial e^{\frac{i}{\hbar}S}$ by using the splitting $\mathcal{F}_M = \mathcal{Y} \times \mathcal{B}$ and the linearity in p of the action. Then we can compute

$$\Delta e^{\frac{i}{\hbar}S} = \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 (S,S)_{\mathcal{Y}} e^{\frac{i}{\hbar}S} = \left(\frac{i}{\hbar}\right)^2 \mathsf{D}e^{\frac{i}{\hbar}S}$$

using Lemma 2.6. ✓

The quantisation scheme then carries over to the bulk, under certain assumptions. As we already mentioned, and accordingly to what happens in the closed BV case, we have to choose a Lagrangian submanifold $\mathcal{L} \subset \mathcal{Y}$ to fix the gauge. Then we can define a state for this choice of gauge fixing as a function on the base parameter space $\mathcal{B}$, namely:

$$\Psi_{\mathcal{L}} := \int_{\mathcal{L}} e^{\frac{i}{\hbar}S} \in \mathrm{Fun}(\mathcal{B}) \tag{2.24}$$

and the fundamental result is that a change in the Lagrangian submanifold results in a $\widehat{\mathsf{D}}$-exact error term, and that the state itself is $\widehat{\mathsf{D}}$-closed. The physical Hilbert space, when $\widehat{\mathsf{D}}^2 = 0$, is then interpreted as the cohomology of $\widehat{\mathsf{D}}$ in degree zero. A state is therefore a cocycle for $\widehat{\mathsf{D}}$ and gauge fixing is nothing but the choice of a representative in its cohomology class.

2.6 General remarks

In this work we will consider several action functionals for General Relativity and the variational problems associated to them, establishing whether they satisfy the CMR axioms. All of the functionals we consider have somehow diffeomorphic critical loci, since they describe the metric dynamics given by the Einstein Equation. At the same time it is assumed that the principle of general covariance lies at the foundations of all of these different formulations, thus requiring that the fundamental symmetries of the theory be given by the action of diffeomorphisms on the fields (plus possible internal symmetries, if the fundamental fields have additional internal degrees of freedom). This is essentially the notion of *classical equivalence* of field theories.

It is important to notice that we will assume that the basic symmetries of the system be determined for *closed manifolds*, i.e. when the boundary is empty. In the presence of a boundary we will employ the same distribution of symmetries and interpret the results so obtained. For instance we will not only consider diffeomorphisms that preserve the boundary, as this will allow us to encode important information. Moreover, this procedure is compatible with the interpretation of symmetries being given by the action of a cohomological vector field Q, which is defined on a cosed manifold, and the boundary structure is induced by the failure of the Classical Master Equation. As a matter of fact, from the fundamental BV-BFV formula

$$\iota_{Q_M}\Omega_M = \delta S_M + \pi_M^*\alpha_{\partial M}^{\partial} \tag{2.25}$$

we get that

$$L_{Q_M} S_M = \iota_{Q_M}\iota_{Q_M}\Omega_M - \pi_M^*\iota_{Q_{\partial M}^{\partial}}\alpha_{\partial M}^{\partial} \tag{2.26}$$

showing how the failure of the gauge invariance in the presence of boundaries is controlled by the boundary structure.

Strictly speaking, in fact, invariance under diffeomorphisms is broken when a boundary is taken into account, as the integration by part will produce boundary terms that must be compensated. This will anyway result in a canonical transformation of the symplectic space of boundary fields, without modifying the boundary BFV structure. Looking at the remarkable and well-known case of Chern-Simons theory, whose action is not gauge invariant in the presence of a boundary, one can construct a line bundle over the space of fields and recover the gauge invariance of the action by extending it to a functional on the line bundle. A similar feature can be similarly expected for the case of General Relativity, even though this issue will not be explicitly addressed here.

As a second remark, I would like to make a comment on the *functoriality* of the assignment of a space of fields and a BV structure to every bulk manifold, and of the boundary BFV structure to every boundary manifold. Consider the case of GR in the Einstein Hilbert formalism, where we have to apply the ADM decomposition in a neighborhood of the boundary, and thus require that the Lorentzian metrics we consider be compatible with such boundary geometry. This assignment is not functorial: the space of fields we associate to the manifold obtained by gluing two pieces does not coincide (it is indeed smaller) than the space of fields we could associate to the same manifold if we forget about the compatibility conditions along the gluing submanifold.

However, we argue that this is not a crucial problem, as the relevant functoriality properties should be guaranteed when passing from the category of classical BV-BFV theories to some appropriate category of quantum theories. At the level of cobordisms it is sufficient that the infinitesimal BV-BFV structure on cylinders $\Sigma \times [0, \epsilon]$ yields a Lagrangian submanifold of the space of boundary fields, given by the projection to the boundary of the solutions of the Euler Lagrange equations.

3
One dimensional examples

To start off with our analysis of the BV-BFV boundary structure for general relativity, we shall consider first some simple, yet instructive 1-dimensional examples. Although some of the results that we will find along the way can be recovered from the general picture presented in Chapter 4, it will be possible to go a little deeper in the analysis of the boundary structure and couple matter to pure gravity, due to the simplicity of the circumstances.

We will in fact consider the case of a pure-gravity model, to which we will add a scalar field, and finally a Robertson-Walker cosmological model.

All of the examples presented in this Chapter (and in the rest of the Thesis) will be of the BRST type, with *gauge group* given by space-time diffeomorphisms. We will therefore construct a BV theory by minimally-extending the classical theory (c.f. Theorem 2.3).

3.1 Pure 1-d gravity.

The simplest example of a model for gravity in one dimension is General Relativity without matter on an interval. Since in one dimension the Riemann-Ricci tensor vanishes, the classical action for pure gravity in the Einstein Hilbert formalism is given just by the cosmological term:

$$S^{cl}_{\text{pure}} = \Lambda \int_I \sqrt{g} dt \tag{3.1}$$

where $g \in \Gamma\left(S^2_+ T^* I\right)$ represents the nondegenerate metric (hence the + subscript) in one dimension $g(t)dt^2$ and I is some one dimensional interval $I \simeq [0, 1]$. In this case the system is invariant w.r.t any diffeomorphism of the interval, which means that the infinitesimal symmetries are encoded by the space of vector fields on M. The cohomological vector field Q is then described by the following action on the fields:

$$\boxed{\begin{aligned} \mathrm{Q}\, g &= \xi \dot{g} + 2 g \dot{\xi} \\ \mathrm{Q}\, \xi &= \xi \dot{\xi} \end{aligned}} \tag{3.2}$$

with $\xi \in \Gamma(T[1]M)$ an odd vector field of ghost number (degree) 1, encoding the action of infinitesimal diffeomorphisms, and it is simple to check that $Q^2 = 0$.

The BV extended action is then given by

$$\boxed{S^{BV}_{\text{pure}} = \Lambda \int \sqrt{g}\, dt - \int (\xi \dot{g} + 2 g \dot{\xi}) g^\dagger\, dt + \int \xi \dot{\xi} \xi^\dagger\, dt} \tag{3.3}$$

where we have introduced the degree -1 anti-field $g^\dagger \in \Gamma\left(S^2 T[-1] I\right)$ and the degree 0 anti-ghost field $\xi^\dagger \in \Omega^1(I)$. Altogether the space of fields reads

$$\mathcal{F}_{\text{pure}} = \Gamma\left(S^2_+ T^* I\right) \oplus \Gamma\left(T[1] I\right) \oplus \Gamma\left(S^2 T[-1] I\right) \oplus \Gamma\left(T^* I\right) \tag{3.4}$$

and it is endowed with a canonical odd-symplectic form Ω_{BV}.

Theorem 3.1. *The data given by $(\mathcal{F}_{pure}, Q, S^{BV}_{pure}, \Omega_{BV})$ yields an exact BV-BFV theory on the boundary ∂I.*

Proof. To prove the statement, we must first show that we can induce a pre-symplectic exact two-form $\widetilde{\omega}$ on the space of pre-boundary fields, and then show that the one-form is indeed horizontal with respect to the vertical distribution induced by its kernel, for this will ensure its basicity and the fact that the BV-BFV structure is exact.

To begin with, the variation of the BV-extended action is

$$\begin{aligned}\delta S^{BV}_{\text{pure}} &= \int_I \left\{ \left(\frac{\Lambda}{2\sqrt{g}} - \dot{\xi} g^\dagger + \xi \dot{g}^\dagger \right) \delta g + \left(\xi \dot{g} + 2g\dot{\xi} \right) \delta g^\dagger \right. \\ &\quad \left. + \left(\dot{g} g^\dagger + 2g\dot{g}^\dagger + 2\dot{\xi}\xi^\dagger + \xi\dot{\xi}^\dagger \right) \delta\xi + \xi\dot{\xi}\delta\xi^\dagger \right\} dt \\ &\quad + \left(-\xi g^\dagger \delta g - 2gg^\dagger \delta\xi - \xi\xi^\dagger \delta\xi \right) \Big|_{\partial I} \\ &= \text{EL} + \widetilde{\pi}_I^* \widetilde{\alpha}\end{aligned} \tag{3.5}$$

where $\widetilde{\pi}_I : \mathcal{F}_{\text{pure}} \longrightarrow \widetilde{\mathcal{F}}_{\text{pure}}$ is the surjective submersion that takes all the fields and jets to their restriction to the boundary . To simplify the notation, we will use the same symbols to denote the fields and their restrictions.

Taking into account the incoming boundary of I, we have that the boundary one-form $\widetilde{\alpha}$ reads:

$$\widetilde{\alpha} = \int_{\partial I} \xi g^\dagger \delta g + 2gg^\dagger \delta\xi + \xi\xi^\dagger \delta\xi \tag{3.6}$$

from which we compute the boundary two-form to be

$$\widetilde{\omega} = \delta\widetilde{\alpha} = \int_{\partial I} -g^\dagger \delta\xi\delta g - \xi\delta g^\dagger \delta g + 2g\delta g^\dagger \delta\xi + \xi^\dagger \delta\xi\delta\xi - \xi\delta\xi^\dagger \delta\xi \tag{3.7}$$

Contracting the two form ω with the general expression for a vector field X

$$X = X_g \frac{\delta}{\delta g} + X_{g^\dagger} \frac{\delta}{\delta g^\dagger} + X_\xi \frac{\delta}{\delta\xi} + X_{\xi^\dagger} \frac{\delta}{\delta\xi^\dagger}$$

it is straightforward to find the kernel to be such that

$$X_\xi = -X_g \frac{\xi}{2g}; \quad X_{g^\dagger} = X_g \left(\frac{\xi\xi^\dagger}{2g^2} - \frac{g^\dagger}{2g} \right) - X_{\xi^\dagger} \frac{\xi}{2g} \tag{3.8}$$

and a basis is given by the choice of the free parameters $X_g, X_{\xi^\dagger}$:

$$\begin{aligned} \mathbb{\Gamma} &:= \frac{\delta}{\delta g} + \left(\frac{\xi\xi^\dagger}{2g^2} - \frac{g^\dagger}{2g} \right) \frac{\delta}{\delta g^\dagger} - \frac{\xi}{2g} \frac{\delta}{\delta \xi} \\ \Xi^\dagger &:= \frac{\delta}{\delta \xi^\dagger} - \frac{\xi}{2g} \frac{\delta}{\delta g^\dagger} \end{aligned} \tag{3.9}$$

This proves that the two form $\widetilde{\omega}$ is presymplectic, for its kernel has constant dimension everywhere on the boundary. Moreover, with a simple computation one can check that

$$\iota_{\mathbb{\Gamma}} \widetilde{\alpha} = \iota_{\Xi^\dagger} \widetilde{\alpha} = 0$$

ensuring the horizontality of the pre-boundary one-form.

Performing symplectic reduction of the pre-symplectic manifold $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ one is left with a (0-)symplectic manifold, the space of boundary fields:

$$\left(\mathcal{F}^\partial := \widetilde{\mathcal{F}} / \mathrm{Ker}(\widetilde{\omega}), \omega^\partial := \underline{\widetilde{\omega}} \right)$$

✓

Now that we have ensured that the data $(\mathcal{F}_{\text{pure}}, Q, S^{BV}_{\text{pure}}, \Omega_{BV})$ yields a BV-BFV theory, the following result will tell us how the boundary structure looks like in local coordinates, and will give us a procedure to adopt also in the more involved examples to come. The surjective submersion from the space of bulk fields to the space of pre-boundary fields $\widetilde{\pi}_I : \mathcal{F}_{\text{pure}} \longrightarrow \widetilde{\mathcal{F}}_{\text{pure}}$ composed with the symplectic reduction map $\pi^\partial : \widetilde{\mathcal{F}}_{\text{pure}} \longrightarrow \mathcal{F}^\partial_{\text{pure}}$ is a surjective submersion that we will denote by $\pi_I = \pi^\partial \circ \widetilde{\pi}_I$.

Theorem 3.2. *The surjective submersion* $\pi_I : \mathcal{F}_{pure} \longrightarrow \mathcal{F}^{\partial}_{pure}$ *is given by*

$$\pi_I : \begin{cases} \widetilde{g}^{\dagger} = \frac{\sqrt{g}}{2} g^{\dagger} + \dfrac{\xi^{\dagger}\xi}{4\sqrt{g}} \\ \\ \widetilde{\xi} = \sqrt{g}\xi \end{cases} \tag{3.10}$$

Moreover, the boundary one-form $\alpha^{\partial} \in \Omega^1(\mathcal{F}^{\partial})$ *reads:*

$$\alpha^{\partial} = \int_{\partial I} \widetilde{g}^{\dagger} \delta \widetilde{\xi} \tag{3.11}$$

and is pulled back to $\widetilde{\alpha}$ *in* (3.6) *along the projection* π_I*, namely* $\widetilde{\alpha} = \widetilde{\pi}_I^* \alpha^{\partial}$*, whereas the boundary cohomological vector field* Q^{∂} *is given by:*

$$Q^{\partial} := \pi_{I*} Q = \int_{\partial I} \frac{\Lambda}{4} \frac{\delta}{\delta \widetilde{g}^{\dagger}} \ . \tag{3.12}$$

Finally, the boundary action reads

$$S^{\partial}_{pure} = \frac{\Lambda}{4} \int_{\partial I} \widetilde{\xi} \ . \tag{3.13}$$

Proof. We divide the proof in two parts. First we will find the explicit expression in coordinates of the projection map and find the one-form on the boundary. Then we will turn to the pushforward of the cohomological vector field.

- We would like to quotient the kernel of the form $\widetilde{\omega}$ in projecting on the space of boundary fields so to have a symplectic form on the reduction. One way to do this is to find an explicit global section by flowing along the vertical vector fields (3.9) in the kernel. For example, we can set $\xi^{\dagger} = 0$ using the flow of $\Xi^{\dagger}$, namely:

$$\theta \Xi^{\dagger} = \theta \frac{\delta}{\delta \xi^{\dagger}} - \theta \frac{\xi}{2g} \frac{\delta}{\delta g^{\dagger}} \Rightarrow (\xi^{\dagger})' = \theta$$

where $(\xi^\dagger)'$ means the derivative with respect to the flow parameter s. Then

$$\xi^\dagger(s) = \xi_0^\dagger + \theta s \rightsquigarrow \theta = -\xi_0^\dagger$$

by imposing $\xi^\dagger(1) = 0$. It follows that

$$(g^\dagger)' = +\frac{\xi_0^\dagger \xi_0}{2g_0} \Longrightarrow g^\dagger(s) = g_0^\dagger + \frac{\xi_0^\dagger \xi_0}{2g_0} s$$

and therefore the temporary new variable reads:

$$\widehat{g^\dagger} \coloneqq g^\dagger(1) = g_0^\dagger + \frac{\xi_0^\dagger \xi_0}{2g_0}$$

fixing the value of the variable $g^\dagger$ after flowing along $\Xi^\dagger$.

Now we use the other vector in the kernel $\mathbb{\Gamma}$ to set $g = 1$, since it cannot be set to zero due to nondegeneracy. Then

$$\mu\mathbb{\Gamma} = \mu\frac{\delta}{\delta g} + \mu\left(\frac{\xi\xi^\dagger}{2g^2} - \frac{g^\dagger}{2g}\right)\frac{\delta}{\delta g^\dagger} - \mu\frac{\xi}{2g}\frac{\delta}{\delta\xi} \Rightarrow (g)' = \mu$$

but this time we have

$$g(\tau) = g_0 + \mu\tau \longrightarrow \mu = 1 - g_0 \longrightarrow g(\tau) = g_0 + (1 - g_0)\tau$$

The remaining differential equations are the one for ξ

$$(\xi)' = -\frac{(1 - g_0)}{2(g_0 + (1 - g_0)\tau)}\xi =: \varphi(\tau)\xi$$

leading to*

$$\xi(\tau) = \frac{\sqrt{g_0}}{\sqrt{g_0 + (1 - g_0)\tau}}\xi_0$$

*The field ξ is odd, so it does not make sense to *divide by* ξ and recognise the derivative of a logarithm. Rather, we check that the given ansatz for $\xi(\tau)$ is indeed a solution.

from which we define

$$\underline{\xi} := \xi(1) = \sqrt{g_0}\xi_0,$$

and the other one for $g^\dagger$

$$(g^\dagger)' = \phi(t) g^\dagger$$

which is similarly solved to yield

$$\underline{g}^\dagger := \sqrt{g_0}\,\widehat{g}^\dagger = \sqrt{g_0} g_0^\dagger + \frac{\xi_0^\dagger \xi_0}{2\sqrt{g_0}}. \tag{3.14}$$

Notice that the vector field $\mathbb{\Gamma}$ has one more term in $\frac{\delta}{\delta g^\dagger}$, but since in $s = 1$ along the flow of $\Xi^\dagger$ we have imposed $\xi^\dagger = 0$, the term vanishes.

The projection map π to the symplectic reduction reads then as a map from the pre-boundary fields $g, g^\dagger, \xi, \xi^\dagger$ to some new variables $\widetilde{g}^\dagger, \widetilde{\xi}$ such that

$$\pi : \begin{cases} \widetilde{g}^\dagger = \frac{\sqrt{g}}{2} g^\dagger + \dfrac{\xi^\dagger \xi}{4\sqrt{g}} \\ \\ \widetilde{\xi} = \sqrt{g}\xi \end{cases} \tag{3.15}$$

where we set $\widetilde{g}^\dagger = \frac{1}{2}\underline{g}^\dagger$. Pre-composing π with the pre-boundary map $\widetilde{\pi}_I$ yields the formula in (3.10).

It is easy to check that the *ansatz*

$$\alpha^\partial = \int_{\partial I} \widetilde{g}^\dagger \delta \widetilde{\xi} \tag{3.16}$$

is pulled back to $\widetilde{\alpha}$ in (3.6) along the projection, namely $\widetilde{\alpha} = \pi^* \alpha^\partial$.

- We consider now the pushforward of the vector field Q along π_I. On the space of fields we have the canonical two form

$$\Omega = \int \{\delta g \delta g^{\dagger} + \delta\xi\delta\xi^{\dagger}\} dt$$

and considering the BV equation in the absence of boundary

$$i_Q \Omega = \delta S_{BV},$$

where $\boldsymbol{Q}$ is given by the generic expression

$$Q = \int dt \left\{ Q_g \frac{\delta}{\delta g} + Q_{g^{\dagger}} \frac{\delta}{\delta g^{\dagger}} + Q_{\xi} \frac{\delta}{\delta \xi} + Q_{\xi^{\dagger}} \frac{\delta}{\delta \xi^{\dagger}} \right\}$$

wherefrom

$$i_Q \Omega = \int dt \left\{ Q_g \delta g^{\dagger} + Q_{g^{\dagger}} \delta g + Q_{\xi} \delta \xi^{\dagger} + Q_{\xi^{\dagger}} \delta \xi \right\},$$

one obtains

$$Q = \int dt \left\{ \left(\xi \dot{g} + 2g\dot{\xi} \right) \frac{\delta}{\delta g} + \left(\frac{\Lambda}{2\sqrt{g}} - \dot{\xi} g^{\dagger} + \xi \dot{g}^{\dagger} \right) \frac{\delta}{\delta g^{\dagger}} \right.$$
$$\left. + \xi\dot{\xi} \frac{\delta}{\delta \xi} + \left(\dot{g} g^{\dagger} + 2g\dot{g}^{\dagger} + 2\dot{\xi}\xi^{\dagger} + \xi\dot{\xi}^{\dagger} \right) \frac{\delta}{\delta \xi^{\dagger}} \right\}.$$

Clearly the original symmetry relations are recovered and extended to the antifields:

$$\begin{aligned}
Q\, g &= \xi \dot{g} + 2g\dot{\xi} \\
Q\, \xi &= \xi\dot{\xi} \\
Q\, g^{\dagger} &= \frac{\Lambda}{2\sqrt{g}} - \dot{\xi} g^{\dagger} + \xi \dot{g}^{\dagger} \\
Q\, \xi^{\dagger} &= \dot{g} g^{\dagger} + 2g\dot{g}^{\dagger} + 2\dot{\xi}\xi^{\dagger} + \xi\dot{\xi}^{\dagger}
\end{aligned}$$

whereby one means, for instance

$$Qg(t) = \int dt'\, Q_g(t') \frac{\delta}{\delta g} g(t') \delta(t - t') = Q_g(t) = \xi \dot{g} + 2g\dot{\xi}$$

The general theory[4] guarantees that Q is projectable on the boundary, factoring through the pre-boundary, where the pre-boundary cohomological vector field $\widetilde{Q}$ is obtained by restricting fields and jets to ∂I. By adding to $\widetilde{Q}$ some combination of the vectors fields in the kernel of the boundary form $G, \Xi^\dagger$, one can express $\widetilde{Q}$ in a manifestly projectable way. More precisely, recalling the expressions (3.9), one may write:

$$\widetilde{Q}' = \widetilde{Q} - \widetilde{Q}_g \mathbb{\Gamma} - \widetilde{Q}_{\xi^\dagger} \Xi^\dagger$$

Now let the generators in the tangent space be transformed as

$$\frac{\delta}{\delta \phi} = \sum_{\widetilde{\phi}} \frac{\delta \widetilde{\phi}}{\delta \phi} \frac{\delta}{\delta \widetilde{\phi}}$$

for $\phi \in \{g, g^\dagger, \xi, \xi^\dagger\}$. Indeed, using (3.15) we have

$$\frac{\delta}{\delta g^\dagger} = \frac{\delta \widetilde{g}^\dagger}{\delta g^\dagger} \frac{\delta}{\delta \widetilde{g}^\dagger} = \frac{\sqrt{g}}{2} \frac{\delta}{\delta \widetilde{g}^\dagger}; \quad \frac{\delta}{\delta \xi} = \frac{\delta \widetilde{\xi}}{\delta \xi} \frac{\delta}{\delta \widetilde{\xi}} = \sqrt{g} \frac{\delta}{\delta \widetilde{\xi}}$$

Then, simplifying the resulting expression for $\widetilde{Q}'$ and using the explicit expression for π_I, one is left with the boundary vector field:

$$Q^\partial = \int_{\partial I} \frac{\Lambda}{4} \frac{\delta}{\delta \widetilde{g}^\dagger}$$

The (straightforward) details of the calculation can be found in Computation A.1, Appendix A.

Finally, to find the boundary action we use the formula[40]

$$S^\partial = \iota_{Q^\partial} \iota_{\mathbb{E}^\partial} \omega^\partial$$

where the vector field $\mathbb{E}^\partial$ is the graded Euler vector field on the boundary

$$\mathbb{E}^\partial = \int\limits_{\partial I} \widetilde{\xi}\frac{\delta}{\delta\widetilde{\xi}} - \widetilde{g}^\dagger \frac{\delta}{\delta\widetilde{g}^\dagger}$$

and it is simple to check that

$$S^\partial_{\text{pure}} = \frac{\Lambda}{4}\int\limits_{\partial I}\widetilde{\xi}$$

✓

Now that we have the full boundary structure, we can analyse the properties of the bulk critical locus EL, i.e. the solutions to the Euler Lagrange equations, and their projection to the space of boundary fields. This one dimensional example is easy enough to be fully computed. We have the following

Theorem 3.3. *Denote by EL the critical locus of the BV action (3.3) in the space of bulk fields $\mathcal{F}$ and by $EL^\partial := \pi_I(EL)$ its projection to the boundary. Then, when $\Lambda = 0$ one has that*

$$EL^\partial \subset \mathcal{F}^\partial \times \mathcal{F}^\partial \tag{3.17}$$

is the Lagrangian submanifold given by the graph of the identity. Otherwise $EL^\partial = \emptyset$.

Proof. We need to solve first the Euler Lagrange equations coming from the variational problem (3.5), that is to say

$$\xi\dot{g} + 2g\dot{\xi} = 0 \tag{3.18a}$$

$$\xi\dot{\xi} = 0 \tag{3.18b}$$

$$\frac{\Lambda}{2\sqrt{g}} - \dot{\xi}g^\dagger + \xi\dot{g}^\dagger = 0 \tag{3.18c}$$

$$\dot{g}g^\dagger + 2g\dot{g}^\dagger + 2\dot{\xi}\xi^\dagger + \xi\dot{\xi} = 0. \tag{3.18d}$$

From (3.18a) one gathers that

$$\xi(t) = \sqrt{\frac{g_0}{g(t)}}\xi_0$$

which means in particular that $\sqrt{g_1}\xi_1 = \sqrt{g_0}\xi_0$ and thus the projection variable $\widetilde{\xi} = \sqrt{g}\xi$ is preserved:

$$\frac{d}{dt}\widetilde{\xi} = 0 \tag{3.19}$$

Equation (3.18b) follows from (3.18a) and after some rewritings one can express equation (3.18d) as

$$\sqrt{g}\dot{g}^{\dagger} + \frac{\dot{g}g^{\dagger}}{2\sqrt{g}} + \frac{1}{2}\frac{d}{dt}\left(\frac{\xi^{\dagger}}{g}\right)\widetilde{\xi} = 0 \tag{3.20}$$

Now, taking into consideration the projection in (3.10) and deriving the definition of $\widetilde{g}^{\dagger}$ with respect to time, one obtains:

$$\frac{d}{dt}\widetilde{g}^{\dagger} = \sqrt{g}\dot{g}^{\dagger} + \frac{\dot{g}g^{\dagger}}{2\sqrt{g}} + \frac{1}{2}\frac{d}{dt}\left(\frac{\xi^{\dagger}}{g}\right)\widetilde{\xi} + \frac{1}{2}\left(\frac{\xi^{\dagger}}{g}\right)\frac{d}{dt}\widetilde{\xi}$$

which vanishes, since (3.19) enforces the vanishing of last term and the rest coincides with the left hand side of (3.20). So

$$\frac{d}{dt}\widetilde{g}^{\dagger} = 0. \tag{3.21}$$

We are left with equation (3.18c). Using (3.18a) to express $\dot{\xi} = -\frac{\dot{g}}{2g}\xi$ and inverting $\sqrt{g}$ in (3.20) to find an expression for $\dot{g}^{\dagger}$, it simplifies to

$$\frac{\Lambda}{2\sqrt{g}} = 0 \tag{3.22}$$

and it is only satisfied when $\Lambda = 0$, since $\sqrt{g}$ cannot be zero. When $\Lambda = 0$ one has that the coisotropic submanifold C defined by the constraint equation:

$$C\colon \frac{\delta S^{\partial}}{\delta\widetilde{g}^{\dagger}} = \Lambda = 0$$

on which the Euler Lagrange equations take place, coincides with the whole space of boundary fields. Moreover the submanifold

$$EL^{\partial} \subset \mathcal{F}^{\partial} \tag{3.23}$$

will yield the graph of the identity, i.e. a Lagrangian submanifold, in $\mathcal{F}^{\partial} \times \overline{\mathcal{F}^{\partial}}$.

✓

3.2 Coupling with matter

We will describe now a one dimensional theory where gravity is coupled to a matter field. First we will be considering the case where the cosmological contribution is turned off $\Lambda = 0$, and then we will see how the picture changes when we introduce a cosmological correction.

3.2.1 Pure matter, $\Lambda = 0$

The classical action is given by the following expression

$$S^{cl}_{\text{mat}} = \int_I \frac{1}{2} y \dot{\phi}^2 dt \tag{3.24}$$

where y is the reciprocal $y = g^{-1}$ of the metric component $g(t)$ and $\phi = \phi(t)$ is a scalar matter field.

Again, the cohomological vector field is a datum of the problem since the symmetries are given by any diffeomorphism (vector fields). The action on the variables is as follows:

$$\boxed{\begin{aligned} &\mathrm{Q}\, y = \xi \dot{y} - \dot{\xi} y \\ &\mathrm{Q}\, \phi = \xi \dot{\phi} \\ &\mathrm{Q}\, \xi = \xi \dot{\xi} \end{aligned}} \tag{3.25}$$

as it can be checked by computing $Qy = Qg^{-1} = -g^{-2}Qg$, using $Qg = \xi\dot{g} + \dot{\xi}g$, and the

BV-extended action reads

$$\boxed{S^{BV}_{\text{mat}} = \int_I \left\{ \frac{1}{2} y\dot{\phi}^2 - \xi\dot{\phi}\phi^\dagger - (\xi\dot{y} - \dot{\xi}y)y^\dagger + \xi\dot{\xi}\xi^\dagger \right\} dt} \tag{3.26}$$

The space of fields in this case is analogous to what we had in Section 3.1, plus the addition of the matter fields $\mathcal{F}_{\text{mat}} \coloneqq T^*[-1]C^\infty(I) \ni (\phi, \phi^\dagger)$. We have the following

Theorem 3.4. *The data* $(\mathcal{F}_{pure} \oplus \mathcal{F}_{mat}, Q, S^{BV}_{mat}, \Omega_{BV})$ *yields an exact BV-BFV theory on the boundary* ∂I.

Proof. By performing a variation of S^{BV}_{mat} with respect to all variables and integrating by parts we are left with

$$\begin{aligned}
\delta S^{BV}_{\text{mat}} = \int_I \Big\{ & \left(-\dot{y}\dot{\phi} - y\ddot{\phi} + \dot{\xi}\phi^\dagger + \xi\dot{\phi}^\dagger\right)\delta\phi + \xi\dot{\phi}\delta\phi^\dagger \\
& + \left(\frac{1}{2}\dot{\phi}^2 + 2\dot{\xi}y^\dagger + \xi\dot{y}^\dagger\right)\delta y + (\xi\dot{y} - \dot{\xi}y)\delta y^\dagger \\
& + \left(-\dot{\phi}\phi^\dagger - 2\dot{y}y^\dagger - y\dot{y}^\dagger + 2\dot{\xi}\xi^\dagger + \xi\dot{\xi}^\dagger\right)\delta\xi - \dot{\xi}\xi\delta\xi^\dagger \Big\} dt \\
+ & \left(y\dot{\phi}\delta\phi - \xi\phi^\dagger\delta\phi - \xi y^\dagger\delta y + yy^\dagger\delta\xi - \xi\xi^\dagger\delta\xi\right)\Big|_{\partial I}
\end{aligned} \tag{3.27}$$

where the boundary term defines a one-form on the space of pre-boundary fields $\widetilde{\mathcal{F}}_{\text{pure}} \oplus \widetilde{\mathcal{F}}_{\text{mat}}$ given once more by the fields' and jets' restrictions to the boundary:

$$\widetilde{\alpha} = \int_{\partial I} \left\{ yJ_\phi\delta\phi - \xi\phi^\dagger\delta\phi - \xi y^\dagger\delta y + yy^\dagger\delta\xi - \xi\xi^\dagger\delta\xi \right\}. \tag{3.28}$$

where we introduced a notation for the *normal* jet $J_\phi \coloneqq \dot{\phi}|_{\partial I}$. Notice that when the coupling is introduced, the relevant normal jet J_ϕ appears in the explicit expression of the pre-boundary one and two forms and it is to be considered as an independent field on the boundary. We compute $\widetilde{\omega} = \delta\widetilde{\alpha}$ as

$$\begin{aligned}\widetilde{\omega} = \int_{\partial I} \Big\{& y\delta J_\phi \delta\phi - J_\phi \delta y \delta\phi + \xi\delta\phi^\dagger \delta\phi - \phi^\dagger \delta\xi\delta\phi - 2y^\dagger \delta\xi\delta y \\ &+\xi\delta y^\dagger \delta y + y\delta y^\dagger \delta\xi - \xi^\dagger \delta\xi\delta\xi + \xi\delta\xi^\dagger \delta\xi \Big\}\end{aligned} \tag{3.29}$$

whose kernel is given by the following equations

$$\begin{aligned} X_\phi &= 0 \\ X_\xi &= X_y \frac{\xi}{y} \\ X_{J_\phi} &= -X_y \left(\frac{J_\phi}{y} - \frac{\xi\phi^\dagger}{y^2} \right) - X_{\phi^\dagger} \frac{\xi}{y} \\ X_{y^\dagger} &= -2X_y \left(\frac{y^\dagger}{y} - \frac{\xi\xi^\dagger}{y^2} \right) + X_{\xi^\dagger} \frac{\xi}{y}, \end{aligned}$$

and it is easy to gather that a basis of the kernel is given by

$$\begin{aligned} \mathbb{Y} &= \frac{\delta}{\delta y} - \left(\frac{J_\phi}{y} - \frac{\xi\phi^\dagger}{y^2} \right) \frac{\delta}{\delta J_\phi} + \frac{\xi}{y}\frac{\delta}{\delta\xi} - 2\left(\frac{y^\dagger}{y} - \frac{\xi\xi^\dagger}{y^2} \right) \frac{\delta}{\delta y^\dagger} \\ \Phi^\dagger &= \frac{\delta}{\delta\phi^\dagger} - \frac{\xi}{y}\frac{\delta}{\delta J_\phi} \\ \Xi^\dagger &= \frac{\delta}{\delta\xi^\dagger} + \frac{\xi}{y}\frac{\delta}{\delta y^\dagger} \end{aligned} \tag{3.30}$$

The dimension of the kernel of $\widetilde{\omega}$ is constant everywhere on the boundary, and the pre-boundary two form is therefore presymplectic.

It is straightforward to check that $\widetilde{\alpha}$ is horizontal, i.e.

$$\iota_{\mathbb{Y}}\widetilde{\alpha} = \iota_{\Phi^\dagger}\widetilde{\alpha} = \iota_{\Xi^\dagger}\widetilde{\alpha} = 0$$

and therefore the symplectic reduction

$$\left(\mathcal{F}^\partial := (\widetilde{\mathcal{F}}_{\text{pure}} \oplus \widetilde{\mathcal{F}}_{\text{mat}}) / \text{Ker}(\widetilde{\omega}), \omega^\partial := \underline{\widetilde{\omega}}\right)$$

is an exact symplectic manifold.

✓

The boundary structure is made explicit with the following

Theorem 3.5. *The surjective submersion* $\pi_I : \mathcal{F} \longrightarrow \mathcal{F}^\partial$ *is given by*

$$\pi_I : \begin{cases} \widetilde{x} = x \\ \widetilde{\xi} = \dfrac{\xi}{y} \\ \widetilde{y}^\dagger = y^\dagger y^2 - \xi^\dagger \xi y \\ \widetilde{J_\phi} = J_\phi y + \phi^\dagger \xi \end{cases} \tag{3.31}$$

Moreover, the boundary one-form α^∂ *reads*

$$\alpha^\partial = \int_{\partial I} \widetilde{J_\phi}\, \delta\widetilde{\phi} + \widetilde{y}^\dagger\, \delta\widetilde{\xi} \tag{3.32}$$

whereas the bulk cohomological vector field Q *projects to*

$$Q^\partial := \pi_{I*} Q = \int_{\partial I} \widetilde{J_\phi}\widetilde{\xi} \frac{\delta}{\delta\widetilde{\phi}} + \frac{1}{2}\widetilde{J_\phi}^2 \frac{\delta}{\delta\widetilde{y}^\dagger} \tag{3.33}$$

Finally, the boundary action reads

$$S^\partial_{mat} = \int_{\partial I} \frac{1}{2}\widetilde{J_\phi}^2 \widetilde{\xi}. \tag{3.34}$$

Proof. • Using the basis of the kernel of ω wisely it is possible to mimic the procedure used in Theorem 3.2 to find a global section, in order to perform the symplectic reduction to the space boundary fields. In particular we will use $\Phi^\dagger$ to set $\phi^\dagger = 0$ at a reference point $s = 1$ on the flow, $\Xi^\dagger$ to set $\xi^\dagger = 0$ and $\mathbb{Y}$ to set $y = 1$.

$$\theta\Phi^\dagger = \theta\frac{\delta}{\delta\xi^\dagger} + \theta\frac{\xi}{y}\frac{\delta}{\delta y^\dagger} \Rightarrow (\phi^\dagger)' = \theta$$

again this means that $\theta = -\phi_0^\dagger$ and therefore we have

$$\phi_0^\dagger \frac{\xi_0}{y_0} = (J_\phi)' \Longrightarrow J_\phi(s) = J_{\phi 0} + \frac{\phi_0^\dagger \xi_0}{y_0} s.$$

We can set at the reference point $s = 1$

$$\widehat{J_\phi} := J_\phi(1) = J_{\phi 0} + \frac{\phi_0^\dagger \xi_0}{y_0}$$

Analogously we proceed to set $\xi^\dagger = 0$.

$$\Lambda \Xi^\dagger = \Lambda \frac{\delta}{\delta \xi^\dagger} + \Lambda \frac{\xi}{y} \frac{\delta}{\delta y^\dagger} \Rightarrow (\xi^\dagger)' = \Lambda$$

from which $\Lambda = -\xi_0^\dagger$ and thus, solving the remaining part we have

$$-\frac{\xi_0^\dagger \xi_0}{y_0} = (y^\dagger)' \Longrightarrow y^\dagger(s) = y_0^\dagger - \frac{\xi_0^\dagger \xi_0}{y_0} s$$

and we define

$$\widehat{y}^\dagger \equiv y^\dagger(1) = y_0^\dagger - \frac{\xi_0^\dagger \xi_0}{y_0}$$

Now we are ready to act with $\mathbb{Y}$ and fix $y(\tau = 1) = 1$.

$$\mu \mathbb{Y} = \mu \frac{\delta}{\delta y} - \mu \frac{J_\phi}{y} \frac{\delta}{\delta J_\phi} + \mu \frac{\xi}{y} \frac{\delta}{\delta \xi} - 2\mu \frac{y^\dagger}{y} \frac{\delta}{\delta y^\dagger}$$

Notice that we have eliminated the terms in (3.30) containing $\xi^\dagger$ and $\phi^\dagger$ because we set them to zero at time $s = 1$. Now from $\mu = (y)'$ we conclude that $\mu = 1 - y_0$ and that $y(\tau) = y_0 + (1 - y_0)\tau$, therefore we have

$$(J_\phi)' = -\frac{1 - y_0}{y_0 + (1 - y_0)\tau} J_\phi =: f(\tau) J_\phi$$

that integrates to

$$J_\phi(\tau) = \widehat{J_\phi}\, \frac{y_0 + (1 - y_0)\tau}{y_0}$$

allowing us to define the first projection change of coordinates

$$\underline{J_\phi} \coloneqq J_\phi(1) = J_{\phi 0} y_0 + \phi_0^\dagger \xi_0$$

The third factor gives the equation

$$(\xi)' = \frac{1 - y_0}{y_0 + (1 - y_0)\tau}\xi = -f(\tau)\xi$$

whose solution reads

$$\xi(\tau) = \xi_0 \frac{y_0 + (1 - y_0)\tau}{y_0}$$

allowing us to define

$$\underline{\xi} = \frac{\xi_0}{y_0}.$$

Eventually, we are left to deal with the last equation, namely

$$(y^\dagger)' \equiv \xi(1) = 2f(\tau)y^\dagger$$

which by an analogous computation leads to

$$\underline{y}^\dagger \equiv y^\dagger(1) = y_0^\dagger y_0^2 - \xi_0^\dagger \xi_0 y_0.$$

Summing up, the projection from the pre-boundary fields $\widetilde{\mathcal{F}}$ to the space of boundary fields is given by:

$$\pi \colon \begin{cases} \widetilde{x} = x \\ \widetilde{\xi} = \dfrac{\xi}{y} \\ \widetilde{y}^\dagger = y^\dagger y^2 - \xi^\dagger \xi y \\ \widetilde{J_\phi} = J_\phi y + \phi^\dagger \xi \end{cases}$$

and pre-composing with $\widetilde{\pi}_I$ one obtains the similar expression (3.31).

It is once more a matter of straightforward computations to check that the boundary forms α^∂ and ω^∂ are given respectively by:

$$\alpha^\partial = \int_{\partial I} \widetilde{J}_\phi \delta\widetilde{\phi} + \widetilde{y}^\dagger \delta\widetilde{\xi}$$

and

$$\omega^\partial = \int_{\partial I} \delta\widetilde{J}_\phi \delta\widetilde{\phi} + \delta\widetilde{y}^\dagger \delta\widetilde{\xi}$$

- The standard two form Ω on the space of bulk fields is simply given by

$$\Omega = \int \left(\delta\phi\delta\phi^\dagger + \delta y\delta y^\dagger + \delta\xi\delta\xi^\dagger\right) dt.$$

and the strategy to find Q is analogous to the one used in Theorem 3.5:

$$\begin{aligned}
Q_\phi &= \xi J_\phi \\
Q_y &= \xi\dot{y} - \dot{\xi}y \\
Q_\xi &= \xi\dot{\xi} \\
Q_{\phi^\dagger} &= \dot{\xi}\phi^\dagger + \xi\dot{\phi}^\dagger \\
Q_{y^\dagger} &= \frac{1}{2}J_\phi^2 + 2\dot{\xi}y^\dagger + \xi\dot{y}^\dagger \\
Q_{\xi^\dagger} &= 2\dot{\xi}\xi^\dagger + \xi\dot{\xi}^\dagger - J_\phi\phi^\dagger - 2\dot{y}y^\dagger - y\dot{y}^\dagger
\end{aligned}$$

The pre-boundary cohomological vector field $\widetilde{Q}$ is found by restricting all of the above, although in this case we must add the component along $J_\phi = \dot{\phi}|_{\partial I}$, the first jet of ϕ, namely

$$\widetilde{Q}_{J_\phi} = \left(Q\dot{\phi}\right)\Big|_{\partial I} = \left(\frac{d}{dt}Q\phi\right)\Big|_{\partial I} = (\dot{\xi}\dot{\phi} + \xi\ddot{\phi})|_{\partial I} = \dot{\xi}J_\phi + \xi\dot{J}_\phi \tag{3.35}$$

Again, by adding some combinations of the vertical vector fields $\mathbb{Y}, \Xi^\dagger, \Phi^\dagger$ we obtain an expression of $\widetilde{Q}$ which is projectable. Namely

$$\widetilde{Q} = \widetilde{Q}_\phi \frac{\delta}{\delta\phi} + \widetilde{Q}_y \frac{\delta}{\delta y} + \widetilde{Q}_\xi \frac{\delta}{\delta\xi} + \widetilde{Q}_{\phi^\dagger} \frac{\delta}{\delta\phi^\dagger} + \widetilde{Q}_{y^\dagger} \frac{\delta}{\delta y^\dagger} + \widetilde{Q}_{\xi^\dagger} \frac{\delta}{\delta\xi^\dagger} + \\ + \widetilde{Q}_{J_\phi} \frac{\delta}{\delta J_\phi} - \widetilde{Q}_y \mathbb{Y} - \widetilde{Q}_{\phi^\dagger} \Phi^\dagger - \widetilde{Q}_{\xi^\dagger} \Xi^\dagger$$

After some computations, analogous to those of Theorem 3.5 one obtains the projected Q^∂ cohomological vector field on the boundary:

$$Q^\partial = \int_{\partial I} \widetilde{J_\phi}\widetilde{\xi} \frac{\delta}{\delta\widetilde{\phi}} + \frac{1}{2}\widetilde{J_\phi}^2 \frac{\delta}{\delta\widetilde{y^\dagger}}$$

The boundary action S^∂ is found using the boundary Euler vector

$$E^\partial = \int_{\partial I} \widetilde{\xi} \frac{\delta}{\delta\widetilde{\xi}} - \widetilde{y}^\dagger \frac{\delta}{\delta\widetilde{y^\dagger}}$$

and it reads

$$S^\partial_{\text{mat}} = \iota_{Q^\partial}\iota_{E^\partial}\omega^\partial = \int_{\partial I} \frac{1}{2}\widetilde{J_\phi}^2\widetilde{\xi}$$

✓

Remark 3.6. *Observe that from the expression of the boundary action, one can compute the coisotropic submanifold containing the solutions of the Euler-Lagrange equations to be given by*

$$\frac{\delta S^\partial}{\delta\widetilde{\xi}} = \frac{1}{2}\widetilde{J_\phi}^2 = 0 \tag{3.36}$$

◊

3.2.2 Cosmological correction.

Now we turn to a coupling of matter with gravity together with a cosmological contribution. To do this it turns out it is sufficient to *combine* the previous examples together in a suitable way.

Theorem 3.7. *The exact BV-BFV theory induced by the data*

$$(\mathcal{F}_{pure} \oplus \mathcal{F}_{mat}, Q, S^{BV}_{mat}, \Omega_{BV})$$

is stable under the replacement $S^{BV}_{mat} \mapsto S^{BV}_{mat} + S^{cl}_{pure}$. *The respective new cohomological vector field on the boundary reads*

$$Q^{\partial}_{\Lambda} = \widetilde{J}_{\phi}\widetilde{\xi}\frac{\delta}{\delta\widetilde{\phi}} + \left(\frac{1}{2}\widetilde{J}_{\phi}^{\,2} - \Lambda\right)\frac{\delta}{\delta\widetilde{y}^{\dagger}} \tag{3.37}$$

and the boundary action is

$$S^{\partial}_{\Lambda} = \left(\Lambda - \frac{1}{2}\widetilde{J}_{\phi}^{\,2}\right)\widetilde{\xi} \tag{3.38}$$

The projected critical locus $\pi_I(EL)$ *is Lagrangian in* $\mathcal{F}^{\partial}\times\overline{\mathcal{F}^{\partial}}$ *for* $\Lambda \neq 0$, *while the coisotropic submanifold* C_{Λ} *is defined by*

$$C_{\Lambda} : \frac{\delta S^{\partial}_{\Lambda}}{\delta\widetilde{\xi}} = 0 \Longrightarrow \frac{1}{2}\widetilde{J}_{\phi}^{\,2} = \Lambda \neq 0. \tag{3.39}$$

Proof. The replacement $S^{BV}_{\text{mat}} \mapsto S^{BV}_{\text{mat}} + S^{cl}_{\text{pure}}$ induces

$$\delta S^{BV}_{\text{mat}} \mapsto \delta S^{BV}_{mat}[y, y^{\dagger}] - \Lambda \int \frac{1}{y^2}\delta y dt \tag{3.40}$$

where we changed the fields in the pure action to be $y, y^{\dagger}$ instead of $g, g^{\dagger}$ and the coefficient along $y^{\dagger}$ of the cohomological vector field gets modified as

$$Q_{y^{\dagger}} \longmapsto Q_{y^{\dagger}} - \Lambda y^{-2}.$$

Therefore, after straightforward computations, the new boundary and pre-boundary vector field read

$$\widetilde{Q}_\Lambda = \widetilde{Q}_{\text{old}} - \frac{\Lambda}{2}\frac{\delta}{\delta y^\dagger} \Longrightarrow Q^\partial_\Lambda = \widetilde{J}_\phi \widetilde{\xi} \frac{\delta}{\delta \widetilde{\phi}} + \left(\frac{1}{2}\widetilde{J}_\phi^{\,2} - \Lambda\right)\frac{\delta}{\delta \widetilde{y}^\dagger} \tag{3.41}$$

from which the action on the boundary is found to be

$$S^\partial_\Lambda = \left(\Lambda - \frac{1}{2}\widetilde{J}_\phi^{\,2}\right)\widetilde{\xi} \tag{3.42}$$

Now, taking into account the variation (3.27) with the correction in (3.40) we obtain the following set of Euler Lagrange equations:

$$\xi\dot{\phi} = 0 \tag{3.43a}$$

$$\xi\dot{y} - \dot{\xi}y = 0 \tag{3.43b}$$

$$\xi\dot{\xi} = 0 \tag{3.43c}$$

$$\dot{\xi}\phi^\dagger - \xi\dot{\phi}^\dagger - \dot{y}\dot{\phi} - y\ddot{\phi} = 0 \tag{3.43d}$$

$$\frac{1}{2}\dot{\phi}^2 + 2\dot{\xi}y^\dagger + \xi\dot{y}^\dagger - \frac{\Lambda}{y^2} = 0 \tag{3.43e}$$

$$2\dot{\xi}\xi^\dagger + \xi\dot{\xi}^\dagger - \dot{\phi}\phi^\dagger - 2\dot{y}y^\dagger - y\dot{y}^\dagger = 0 \tag{3.43f}$$

From (3.43b) one immediately finds $\dot{\xi} = \xi\dot{y}y^{-1}$ and using it together with (3.43f) in (3.43e) one finds

$$\dot{\phi}y(\dot{\phi}y - 2\xi\phi^\dagger) = (\dot{\phi}y - 2\xi\phi^\dagger)^2 = 2\Lambda \tag{3.44}$$

If $\Lambda < 0$ it is clear that the solutions cannot be solved, and the critical locus collapses to the empty set. Assuming $\Lambda \geq 0$, one can use the projection (3.31) to obtain from (3.44)

$$\widetilde{J}_\phi = \pm\sqrt{2\Lambda} \tag{3.45}$$

or, multiplying (3.44) by ξ and using (3.43a)

$$0 \equiv \dot{\phi}\xi y - \phi^\dagger\xi^2 = \pm\sqrt{2\Lambda}\xi \overset{\Lambda\neq 0}{\Longrightarrow} \xi = 0 \tag{3.46}$$

Moreover, it is easy to check that in this case

$$\frac{d}{dt}\widetilde{\xi} \equiv \frac{d}{dt}\left(\frac{\xi}{y}\right) \equiv 0$$

and therefore

$$\widetilde{\xi} = \text{constant} = 0 \tag{3.47}$$

which will then be enforced in what follows. With a simple computation one can show now that (3.43d) is satisfied by simply deriving (3.44).

Using (3.43f) we compute

$$\frac{d}{dt}\widetilde{y}^{\dagger} = \frac{d}{dt}\left(y^{\dagger}y^{2}\right) = -\dot{\phi}yx^{\dagger} = \mp\sqrt{2\Lambda}x^{\dagger} \tag{3.48}$$

So altogether we conclude that when $\Lambda > 0$ the submanifold defined by the critical locus of the projected Euler Lagrange equations is given by

$$\pi_I(EL) = \left\{(\widetilde{J}_{\phi}, \widetilde{\xi}, \widetilde{\phi}, \widetilde{y}^{\dagger}) \,\middle|\, \widetilde{J}_{\phi} = \pm\sqrt{2\Lambda};\ \widetilde{\xi} = 0\right\} \tag{3.49}$$

We have already seen that the constrained coisotropic submanifold C_{Λ} on which motion takes place is described by the equation (3.39) and it is easy to gather that the submanifold $\pi_I(EL)$ is a Lagrangian submanifold in $\mathcal{F}^{\partial} \times \overline{\mathcal{F}^{\partial}}$.

Now let us analyse what happens when $\Lambda = 0$. This is compatible with (3.44), but we cannot conclude (3.46) anymore. Using $\dot{\phi}y = -\phi^{\dagger}\xi$ in (3.44) we can derive

$$-(\dot{\phi}y)^2 = 0 \Longrightarrow \dot{\phi} = 0 \Longrightarrow \phi = \text{constant}$$

Moreover, with a straightforward computation one checks that (c.f. Eq. (3.48))

$$\frac{d}{dt}\widetilde{y}^{\dagger} = \frac{d}{dt}(y^{\dagger}y^{2} - \xi^{\dagger}\xi y) = 0$$

Looking at $C_0 \times \overline{C_0}$ instead, the relation is then given by

$$\pi_I(EL) = \{J_{\phi_0} = J_{\phi_1} = 0;\ \widetilde{\xi}_0 = \widetilde{\xi}_1;\ \widetilde{\phi}_0 = \widetilde{\phi}_1;\ \widetilde{y}_0^\dagger = \widetilde{y}_1^\dagger\} \subseteq C_0 \times \overline{C_0} \tag{3.50}$$

where the indices $(0, 1)$ denote the two connected components of the boundary representing the two copies of C_0, and L is then not Lagrangian. ✓

Remark 3.8. *Recall that the constrained coisotropic submanifold on which motion takes place is described by the equation*

$$\frac{\delta S^\partial_\Lambda}{\delta \widetilde{\xi}} = 0 \Longrightarrow \frac{1}{2}\widetilde{J_\phi}^2 = \Lambda \tag{3.51}$$

For $\Lambda < 0$ *the coisotropic submanifold collapses to the empty set, and most interesting is the case* $\Lambda \geq 0$.

In this case the coisotropic submanifold is given by the set

$$C = \left\{(\widetilde{\phi}, \widetilde{J_\phi}, \widetilde{y}^\dagger, \widetilde{\xi}) \mid \widetilde{J_\phi} = \pm\sqrt{\Lambda}, \widetilde{\xi} = 0\right\} \tag{3.52}$$

and the foliation is given by the two vector fields: $\frac{\delta}{\delta\widetilde{\phi}}, \frac{\delta}{\delta\widetilde{y}^\dagger}$ *with the reduction being a single point* $\underline{C} = \{pt\}$.

$$\left[Q^\partial_\Lambda, \frac{\delta}{\delta\widetilde{\xi}}\right] = \widetilde{J_\phi}\frac{\delta}{\delta\widetilde{\phi}};\quad \left[Q^\partial_\Lambda, \frac{\delta}{\delta\widetilde{J_\phi}}\right] = -\dot{\widetilde{\phi}}\frac{\delta}{\delta\widetilde{y}^\dagger} - \widetilde{\xi}\frac{\delta}{\delta\widetilde{J_\phi}} \tag{3.53}$$

◊

3.3 Minisuperspace model. Robertson Walker metric and matter

In this section we would like to analyse the Lagrangian theory of gravity coming from the dynamics of a metric of the Robertson-Walker form

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2 \tag{3.54}$$

where $N(t)$ is usually referred to as the *lapse function* and $a(t)$ as the *scale factor*, and compare the results with[27]. By $d\Omega_3^2$ we denote the spherically symmetric 3 dimensional volume element. It is possible to couple a scalar matter field χ to gravity so that the classical action becomes

$$\boxed{S^{\text{cl}}_{RW} = \frac{1}{2}\int_I dt \left\{-\frac{a}{N}\dot{a}^2 + \frac{a}{N}\dot{\chi}^2 + Na - \frac{N}{a}\chi^2 - \Lambda N a^3\right\}} \tag{3.55}$$

with the obvious prescription that a, N be non vanishing.

Comparing with the already discussed cases, we may gather that the geometric nature of $N(t)$ is that of a 1-dimensional metric, in fact $N^2 dt^2 = g(t)dt^2$ when $N = \sqrt{g}$. We will interpret the pair (a, χ) as a map from the interval to $\mathbb{R}^2$, endowed with the pseudo euclidean metric $\eta_a = \text{diag}(-a, a)$. Therefore the kinetic and quadratic potential terms for a and χ in (3.55) are interpreted as

$$\int_I \eta_a(\dot{a}, \dot{\chi})\frac{dt}{\sqrt{g}} - \int_I \eta_a^{-1}(a, \chi)\sqrt{g}dt$$

and we will see at the end that this interpretation carries over to the boundary.

The symmetries of the action are given by

$$\boxed{\begin{aligned} Q N &= \xi\dot{N} + N\dot{\xi} \\ Q a &= \xi\dot{a} \\ Q \chi &= \xi\dot{\chi} \\ Q \xi &= \xi\dot{\xi} \end{aligned}} \tag{3.56}$$

since N transforms like a metric and $a, \chi \in C^\infty(I)$ transform like functions. The BV-extended action is then again given by the minimal BV extension

$$\boxed{S^{BV}_{RW} = S^{\text{cl}}_{RW} + \int dt \left\{-\left(\xi\dot{N} + N\dot{\xi}\right)N^\dagger - \xi\dot{a}a^\dagger - \xi\dot{\chi}\chi^\dagger + \xi\dot{\xi}\xi^\dagger\right\}} \tag{3.57}$$

The last piece of data we need is the space of fields, which in this case is given by

$$\mathcal{F}_{RW} := T^*[-1]\Big(\underbrace{\Gamma(S^2_+TI)}_{N} \oplus \underbrace{\mathrm{Maps}(I,\mathbb{R}^2)}_{(a,\chi)} \oplus \underbrace{\Gamma(T[1]I)}_{\xi}\Big) \tag{3.58}$$

endowed with the canonical (-1)-symplectic structure Ω_{BV}. We have the following

Theorem 3.9. *The data* $(\mathcal{F}_{RW}, S^{BV}_{RW}, Q, \Omega_{BV})$ *defines an exact BV-BFV theory.*

Proof. The variation of the action (we drop the specifications BV, RW) reads

$$\begin{aligned}
\delta S \;=\; \int_I dt\Bigg\{ &\left(\frac{N}{2} + \frac{\dot\chi^2}{2N} + \frac{\dot a^2}{2n} - \frac{a}{N^2}\dot a\dot N + \frac{a}{N}\ddot a - \frac{3}{2}\Lambda N a^2 + \frac{N\chi^2}{2a^2} + \dot\xi a^\dagger + \xi\dot a^\dagger\right)\delta a\\
&+\left(\frac{a}{2} + \frac{a}{2N^2}\dot a^2 - \frac{\Lambda}{2}a^3 - \frac{a}{2N^2}\dot\xi^2 - \frac{\xi^2}{2a} + \xi\dot N^\dagger\right)\delta N\\
&+\left(\frac{a}{N^2}\dot\chi\dot N - \frac{\dot a\dot N}{N} - \frac{a}{N}\ddot\chi - \frac{N\chi}{a} + \dot\xi\chi^\dagger + \xi\dot\chi^\dagger\right)\delta\chi\\
&+\left(2\dot\xi\xi + \xi\dot\xi^\dagger + N\dot N^\dagger - \dot a a^\dagger - \dot\chi\chi^\dagger\right)\delta\xi\\
&+\left(\xi\dot N + \dot\xi N\right)\delta N^\dagger + (\xi\dot a)\,\delta a^\dagger + (\xi\dot\chi)\,\delta\chi^\dagger + \left(\xi\dot\xi\right)\delta\xi^\dagger\\
&+\frac{d}{dt}\left(\frac{a\dot\chi}{N}\delta\chi - \frac{a\dot a}{N}\delta a - \xi\xi^\dagger\delta\xi - \xi N^\dagger\delta N - NN^\dagger\delta\xi - \xi a^\dagger\delta a - \xi\chi^\dagger\delta\chi\right)\Bigg\}
\end{aligned} \tag{3.59}$$

The last term above gives rise to the pre-boundary one- and two-forms that, denoting by $J_a := \dot a|_{\partial I}$, and $J_\chi := \dot\chi|_{\partial I}$ the *normal* jets on the boundary, read

$$\widetilde{\alpha} = \int_{\partial I}\left\{\frac{aJ_\chi}{N}\delta\chi - \frac{aJ_a}{N}\delta a - \xi\xi^\dagger\delta\xi - \xi N^\dagger\delta N - NN^\dagger\delta\xi - \xi a^\dagger\delta a - \xi\chi^\dagger\delta\chi\right\} \tag{3.60}$$

$$\begin{aligned}
\widetilde{\omega} = \int_{\partial I}\Bigg\{ &-\frac{a}{N}\delta J_a\delta a + \frac{aJ_a}{N^2}\delta N\delta a - a^\dagger\delta\xi\delta a + \xi\delta a^\dagger\delta a + \frac{J_\chi}{N}\delta a\delta\chi + \frac{a}{N}\delta J_\chi\delta\chi\\
&-\frac{aJ_\chi}{N^2}\delta N\delta\chi - \chi^\dagger\delta\xi\delta\chi + \xi\delta\chi^\dagger\delta\chi - \xi^\dagger\delta\xi\delta\xi + \xi\delta N^\dagger\delta N - N\delta N^\dagger\delta\xi + \xi\delta\xi^\dagger\delta\xi\Bigg\}
\end{aligned} \tag{3.61}$$

A vector field in the kernel of $\widetilde{\omega}$ turns out to be a combination of the basis

$$\begin{aligned}
\mathbb{N} &= \frac{\delta}{\delta N} + \left(\frac{J_a}{N} + \frac{\xi a^\dagger}{a}\right)\frac{\delta}{\delta J_a} + \left(\frac{J_\chi}{N} - \frac{\xi \chi^\dagger}{a}\right)\frac{\delta}{\delta J_\chi} + 2\frac{\xi\xi^\dagger}{N^2}\frac{\delta}{\delta N^\dagger} - \frac{\xi}{N}\frac{\delta}{\delta \xi}\\
\mathbb{A}^\dagger &= \frac{\delta}{\delta a^\dagger} + \frac{N\xi}{a}\frac{\delta}{\delta J_a}\\
\mathbb{X}^\dagger &= \frac{\delta}{\delta \chi^\dagger} - \frac{N\xi}{a}\frac{\delta}{\delta J_\chi}\\
\mathbb{\Xi}^\dagger &= \frac{\delta}{\delta \xi^\dagger} - \frac{\xi}{N}\frac{\delta}{\delta N^\dagger}
\end{aligned} \tag{3.62}$$

showing that $\widetilde{\omega}$ is pre-symplectic. It is again by means of a simple computation that one checks

$$\iota_{\mathbb{E}_i}\widetilde{\alpha} = 0$$

with $\mathbb{E}_i \in \{\mathbb{N}, \mathbb{A}, \mathbb{X}, \mathbb{\Xi}\}$, showing that $\widetilde{\alpha}$ is horizontal, and therefore $\omega^\partial = \delta\alpha^\partial$ is the exact symplectic form one obtains after the symplectic reduction of the space of fields:

$$\pi : \widetilde{\mathcal{F}}_{RW} \longrightarrow \mathcal{F}^\partial_{RW} \tag{3.63}$$

for $\widetilde{\alpha} = \pi^*\alpha^\partial$. The projection to the boundary fields is then given by composing π with the surjection to the space of pre-boundary fields $\widetilde{\pi}\colon \mathcal{F}_R W \longrightarrow \widetilde{\mathcal{F}_R} W$, namely $\pi_I = \pi \circ \widetilde{\pi}$. ✓

The procedure to obtain the explicit boundary structure for this case is totally analogous to the one used in Theorems 3.2 and 3.5. As a matter of fact we obtain

Theorem 3.10. *The surjective submersion $\pi_I : \mathcal{F}_{RW} \longrightarrow \mathcal{F}^{\partial}_{RW}$ is given by*

$$\pi_I : \begin{cases} \widetilde{a} = a \\ \widetilde{\chi} = \chi \\ \widetilde{\xi} = N\xi \\ \widetilde{J_a} = \dfrac{J_a}{N} - \dfrac{a^\dagger}{a}\xi \\ \widetilde{J_\chi} = \dfrac{J_\chi}{N} + \dfrac{\chi^\dagger}{a}\xi \\ \widetilde{N}^\dagger = N^\dagger + \dfrac{\xi^\dagger}{N}\xi \end{cases} \tag{3.64}$$

Moreover, the boundary one-form $\alpha^\partial \in \Omega^1(\mathcal{F}^\partial_{RW})$ reads:

$$\alpha^\partial = \int_{\partial I} -\widetilde{a}\widetilde{J_a}\delta\widetilde{a} + \widetilde{a}\widetilde{J_\chi}\delta\widetilde{\chi} - \widetilde{N}^\dagger\delta\widetilde{\xi} \tag{3.65}$$

Whereas the boundary cohomological vector field Q^∂ is given by:

$$\begin{aligned} Q^\partial = \int_{\partial I} \Bigg\{ & \widetilde{\xi}\,\widetilde{J_a}\frac{\delta}{\delta\widetilde{a}} + \widetilde{\xi}\,\widetilde{J_\chi}\frac{\delta}{\delta\widetilde{\chi}} - \left(\frac{\widetilde{J_a}\widetilde{J_\chi}}{\widetilde{a}} + \frac{\widetilde{\chi}}{\widetilde{a}}\right)\widetilde{\xi}\frac{\delta}{\delta\widetilde{J_\chi}} \\ & - \frac{1}{2}\left(\frac{1}{\widetilde{a}} + \left(\frac{\widetilde{J_\chi}^2}{\widetilde{a}} + \frac{\widetilde{J_a}^2}{\widetilde{a}} - 3\widetilde{a} + \frac{\widetilde{\chi}^2}{\widetilde{a}^3}\right)\widetilde{\xi}\right)\frac{\delta}{\delta\widetilde{J_a}} \\ & + \frac{1}{2}\left(\widetilde{a}\widetilde{J_a}^2 - \widetilde{a}\widetilde{J_\chi}^2 + \widetilde{a} - \Lambda\widetilde{a}^3 - \frac{\widetilde{\chi}^2}{\widetilde{a}}\right)\frac{\delta}{\delta\widetilde{N}^\dagger}\Bigg\} \end{aligned} \tag{3.66}$$

Finally, the boundary action reads

$$S^\partial = \int_{\partial I} -\frac{1}{2}\left(\widetilde{a}\widetilde{J_a}^2 - \widetilde{a}\widetilde{J_\chi}^2 + \widetilde{a} - \Lambda\widetilde{a}^3 - \frac{\widetilde{\chi}^2}{\widetilde{a}}\right)\widetilde{\xi}. \tag{3.67}$$

Proof.
- As we already mentioned, the procedure to find the explicit section goes through by solving the straightforward differential equations that arise from the procedure already discussed in Theorems 3.2 and 3.5.

- As for the projection of Q to the boundary, from the variation of the action (3.59) we find as usual that

$$
\begin{aligned}
Q_{a^\dagger} &= \frac{N}{2} + \frac{\dot{\chi}^2}{2N} + \frac{\dot{a}^2}{2n} - \frac{a}{N^2}\dot{a}\dot{N} + \frac{a}{N}\ddot{a} - \frac{3}{2}\Lambda N a^2 + \frac{N\chi^2}{2a^2} + \dot{\xi}a^\dagger + \xi\dot{a}^\dagger \\
Q_{\chi^\dagger} &= \frac{a}{N^2}\dot{\chi}\dot{N} - \frac{\dot{a}\dot{N}}{N} - \frac{a}{N}\ddot{\chi} - \frac{N\chi}{a} + \dot{\xi}\chi^\dagger + \xi\dot{\chi}^\dagger \\
Q_{N^\dagger} &= \frac{a}{2} + \frac{a}{2N^2}\dot{a}^2 - \frac{\Lambda}{2}a^3 - \frac{a}{2N^2}\dot{\xi}^2 - \frac{\xi^2}{2a} + \xi\dot{N}^\dagger \\
Q_{\xi^\dagger} &= 2\dot{\xi}\xi + \xi\dot{\xi}^\dagger + N\dot{N}^\dagger - \dot{a}a^\dagger - \dot{\chi}\chi^\dagger \\
Q_a &= \xi\dot{a} \\
Q_\chi &= \xi\dot{\chi} \\
Q_N &= \xi\dot{N} + \dot{\xi}N \\
Q_\xi &= \xi\dot{\xi}
\end{aligned}
\tag{3.68}
$$

then, from (3.56) we obtain the transformations for the relevant jets ($J_a = \dot{a}|_{\partial I}, J_\chi = \dot{\chi}|_{\partial I}$) to be

$$Q_{\dot{a}} \equiv Q(\dot{a}) = \dot{\xi}\dot{a} + \xi\ddot{a}\ ; \quad Q_{\dot{\chi}} \equiv Q(\dot{\chi}) = \dot{\xi}\dot{\chi} + \xi\ddot{\chi} \tag{3.69}$$

so that Q in the bulk is completed as

$$Q = \sum_{i\in I} Q_i \frac{\delta}{\delta i} \tag{3.70}$$

where the indices run over $I = \{a, \dot{a}, a^\dagger, \chi, \dot{\chi}, \chi^\dagger, N, N^\dagger, \xi, \xi^\dagger\}$. Again it is possible to combine Q with some multiples of the vectors in the kernel of ω, in particular we compute

$$\widetilde{Q} = Q - Q_N\mathbb{N} - Q_{a^\dagger}\mathbb{A}^\dagger - Q_{\chi^\dagger}\mathbb{X}^\dagger - Q_{\xi^\dagger}\Xi^\dagger \tag{3.71}$$

obtaining

$$\widetilde{Q} = \pi^* Q^\partial \tag{3.72}$$

where Q^∂ is given by the expression in Eq. (3.66) and the boundary action is found, as usual, via

$$S^{\partial} = \iota_{Q^{\partial}}\iota_{E^{\partial}}\omega^{\partial}$$

✓

Remark 3.11. *The coisotropic submanifold of the space of boundary fields containing the projection of the Euler Lagrange equations' set of solutions on the boundary is given by the derivative along* $\widetilde{\xi}$ *of the boundary action:*

$$\frac{\delta S^{\partial}}{\delta\widetilde{\xi}} = \widetilde{a}^2\,\widetilde{J}_a^2 - \widetilde{a}^2\,\widetilde{J}_\chi^2 + \widetilde{a}^2 - \Lambda\widetilde{a}^4 - \widetilde{\chi}^2 = 0 \tag{3.73}$$

and it yields a projected *version of the classical Wheeler deWitt equation.*

◊

Remark 3.12. *Notice that we can still interpret the boundary action as the pairing in* $\mathbb{R}^2$ *given by the pseudo-euclidean metric* $\eta_{\widetilde{a}} = \mathrm{diag}(-\widetilde{a},\widetilde{a})$ *for the fields' pairs* $(\widetilde{a},\widetilde{\chi})$ *and* $(\widetilde{J}_a,\widetilde{J}_\chi)$. *As a matter of fact:*

$$S^{\partial} = \frac{1}{2}\int_{\partial I}\left(\eta_{\widetilde{a}}(\widetilde{J}_a,\widetilde{J}_\chi) - \eta_{\widetilde{a}}^{-1}(\widetilde{a},\widetilde{\chi}) + \Lambda\widetilde{a}^3\right)\widetilde{\xi} \tag{3.74}$$

◊

4

Einstein Hilbert action in $d+1$ dimensions

General relativity in the formulation of Einstein and Hilbert[1,2] is a theory of the gravitational interaction where the dynamical field is a pseudo-Riemannian metric g of some (d+1) dimensional manifold M. The principle of general covariance requires that all the relevant expressions be invariant under the action of space-time diffeomorphisms and this makes General Relativity a *gauge theory* in the extended sense, that is, even if the basic field is not a principal connection. We will see in Chapter 5 and Chapter 6 how alternative descriptions of GR exist, casting it in a more standard way.

In this Chapter we will start from the said classical variational principle and symmetry distribution, and we will embed this data in the general framework we outlined in Section 2.2 as a BV theory, in order to understand whether such a theory satisfies the BV-BFV/CMR axioms (Definition 2.5), when the space-time manifold is allowed to have a boundary.

In Section 4.3.1 we will also analyse the classical canonical structure (i.e. the Hamiltonian formulation of GR[24]) and we will show how the bulk-to-boundary machinery yields a powerful and straightforward algorithm to perform such a canonical analysis, even when the BV extension is not performed. This will give us a clean grasp and understanding of the constraint algebra[26,35], providing a non trivial example of a coisotropic submanifold, which does not manifestly come from a Lie algebra action, in agreement with some recent observations[25].

Throughout the Chapter we will always assume that M has a non-empty boundary ∂M. In Section 4.2 we will analyse a simplified version, where we require that the metric be block diagonal in a neighbourhood of the boundary, which is tantamount to choosing a particular coordinate system. This is done in Section 4.2 and it will be rigorously justified from the results in Section 4.3, where this assumption will be relaxed.

The procedure that we will use to compute the boundary structure in Section 4.3 will be more efficient and more general. The assumption we will consider instead, namely that the boundary is entirely space/time-like and that it has a globally hyperbolic neighborhood, will allow us to adopt the ADM[23] coordinates (after Arnowitt, Deser and Misner), which will make the reduction to the boundary fields particularly straightforward.

The way we enforce this is by working only with those pseudo-Riemannian metrics on the manifold M which have space/time-like signature when restricted to the boundary. This space will be denoted by $\mathcal{PR}^{\partial M}_{(d,q)}$ to emphasise that the pseudo-Riemannian structures have to have some compatibility with the boundary in order to be acceptable.

Observe that in the literature[23,24] it is customary to require that the spacetime manifold M be *globally hyperbolic* or, equivalently, that it has the product structure $\Sigma \times \mathbb{R}$ for Σ an embedded space/time-like submanifold of M. This is indeed a much stronger requirement, and in fact we only ask that it be true in a neighborhood of the boundary.

Although it is true that this assumption will keep the results from being completely general, and that the extension to more general boundaries will require an adapted approach, this result represents a non trivial generalization of existing results on the canonical structure of GR. This is especially valuable in view of perturbative quantisation, since as many different space-time geometries as possible must be allowed for, in the spirit of integration over non extremal field configurations.

We will see how the results of Sections 4.2 and 4.3 agree when the appropriate consistency conditions are applied (cf. Subsection 4.4.1). In fact, one has to require that the off diagonal components of the metric vanish in a thin neighborhood of the boundary, rather than only on the boundary, when the block diagonal shape of the metric is enforced. This in particular implies that the normal jets of the metric vanish in the off diagonal direction, when restricted to the boundary, i.e. $\partial_n g_{na}\big|_{\partial M} = 0$. Notice, that even if the ADM procedure is significantly more general than the one presented in Section 4.2, the fact that we were able to solve the simplified version first was a non trivial step in the understanding of the general case.

4.1 Einstein Hilbert Formalism

The theory is formulated as a variational problem for the (second-order) Einstein-Hilbert action[1,2] (modulo multiplicative constants):

$$S^{cl}_{EH} = \int_M (R[g] - 2\Lambda)\ \sqrt{-\mathrm{g}}\, d^{d+1}x \tag{4.1}$$

where $R[g]$ is the Ricci scalar of the pseudo-Riemannian metric $g \in \mathcal{PR}_{(d,1)}(M)$ with signature $(d,1)$ in the bulk[*], $\mathrm{g} := -\det(g)$ and Λ is the cosmological constant. The dynamical equations for g are derived from the action as Euler Lagrange equations for the variational problem.

The symmetries of the action are given by the diffeomorphism of M and, generalizing to $(d+1)$ dimensions what has been done for the simpler 1-dimensional models, they can be implemented as follows:

$$\boxed{\begin{aligned} Qg &= L_\xi g \\ Q\xi &= \frac{1}{2}[\xi,\xi] \end{aligned}} \tag{4.2}$$

The theory is an example of what we called a BRST-like theory, since the symmetry distribution is involutive everywhere on the space of fields. Then we can find the BV action by

[*]We will require it to have space/time-like signature when restricted to the boundary, later on.

minimally extending the classical action; it reads

$$\boxed{S_{EH}^{BV} = \int\limits_M (R[g] - 2\Lambda)\sqrt{g}\, d^{d+1}x - \int\limits_M \left(L_\xi g\right) g^\dagger + \frac{1}{2}\int\limits_M \iota_{[\xi,\xi]}\xi^\dagger} \tag{4.3}$$

We introduced three new fields ξ, $g^\dagger$ and $\xi^\dagger$, respectively an odd vector field $\xi \in \Gamma(T[1]M)$ a section $g^\dagger \in \Gamma[-1]\left(S^2TM\right) \otimes \Omega^{\text{top}}(M)$, i.e. a symmetric tensor of type $(2,0)$ of ghost number -1 with values in top forms, and a one form with values in top forms $\xi^\dagger \in \Omega^1(M) \otimes \Omega^{\text{top}}(M)$. For the computations we will factor $\xi^\dagger = \chi \otimes v$ into its one form part χ and top form value v, which we may assume fixed. The space of fields we will consider is then given by

$$\mathcal{F}_{EH}^{\text{full}} = T^*[-1]\left[\mathcal{PR}_{(d,1)}(M) \oplus \Gamma(T[1]M)\right]. \tag{4.4}$$

As we already announced, we will first analyse a simplified version, where a certain particular shape of the metric on the boundary is assumed, and then we will proceed with a more general approach. This will require to consider some appropriate submanifold of $\mathcal{F}_{EH}^{\text{full}}$ as space of fields for the BV datum. The two strategies will yield the same results, when the appropriate compatibility is required. The hasty reader can skip directly to Section 4.3 for the general results.

4.2 Adapted coordinates

Since the action contains the derivatives of the fields, its variation is composed by a bulk term and a boundary term, the former will give us the Euler-Lagrange equations for the physical fields and the ghosts, while the latter will induce a theory on the boundary. With a straightforward computation from (4.3) one obtains

$$\begin{aligned} \delta S = &\int\limits_{\partial M} \left[\sqrt{g}\left(g^{\mu\nu}\delta\Gamma^n_{\nu\mu} - g^{\sigma n}\delta\Gamma^\rho_{\rho\sigma}\right) - \left(2g_{\rho\sigma}\delta\xi^{(\rho}g^{\dagger n\sigma)} - \xi^n\delta g_{\rho\sigma}g^{\dagger\rho\sigma}\right)\right] d^{d+1}x \\ &- \int\limits_{\partial M} \xi^n\delta\xi^\rho\chi_\rho\, v^\partial + \text{EL} \end{aligned} \tag{4.5}$$

By v^∂ we denote the fixed top form on the boundary coming from v. The normal coordinate is denoted by the special superscript ^{n}. The term EL denotes what will generate the Euler Lagrange equations, and it is interpreted as a one form on the space of fields. There is a surjective submersion on the space of pre-boundary fields

$$\widetilde{\pi}\colon \mathcal{F}_M \longrightarrow \widetilde{\mathcal{F}}_{\partial M} \tag{4.6}$$

where the target is defined as in Chapter 3 by the restrictions of the fields on the boundary plus the normal jets of fields computed at the boundary. As we shall see, the *relevant* jets of fields appearing in the pre-boundary one form are just the first jets $J_{\mu\nu} := \partial_n g_{\mu\nu}|_{\partial M}$.

The Euler Lagrange term reads:

$$\begin{aligned}\mathrm{EL} = \int_M \Big\{ & d^{d+1}x\Big[-\sqrt{\mathrm{g}}\Big(R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}\Big) + \partial_\rho\xi^\rho g^{\dagger\mu\nu} + \xi^\rho\partial_\rho g^{\dagger\mu\nu} - 2\partial_\rho\xi^{(\mu}g^{\dagger\nu)\rho}\Big]\delta g^{\mu\nu} + \\ &+ \Big\{2\partial_{(\rho}g_{\nu)\mu}g^{\dagger\rho\nu} + 2g_{\mu(\nu}\partial_{\rho)}g^{\dagger\rho\nu} + \partial_\rho\xi^\rho\chi_\mu + \xi^\rho\partial_\rho\chi_\mu - \partial_\mu g_{\rho\sigma}g^{\dagger\rho\sigma} + \partial_\mu\xi^\rho\chi_\rho\Big\}\delta\xi^\mu + \\ &+ \Big\{\xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho}\Big\}\delta g^{\dagger\mu\nu}\Big\} + \Big\{\xi^\rho\partial_\rho\xi^\mu\Big\}\delta\chi_\mu v\end{aligned} \tag{4.7}$$

with brackets around the indices standing for symmetrisation of indices, namely $A_{(\mu}B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$.

Let us simplify the computations by choosing an adapted coordinate system in which the normal vector is perpendicular to the boundary hypersurface. This, together with the requirement that g_{nn} be different from zero, requires that the metric $g_{\mu\nu}$ takes the following form:

$$g_{\mu\nu} = \begin{pmatrix} g_{nn} & 0 \\ 0 & g_{ab} \end{pmatrix} \tag{4.8}$$

with g_{ab} a non degenerate d-dimensional Riemannian metric. This means that $g_{na} = 0\ \forall a = 1\dots d$. For simplicity we assume the transversal direction to be timelike[†]. The space of pseudo-Riemannian metrics on M will be then restricted to those metrics for which the block-diagonal decomposition (4.8) is allowed. We will denote the resulting space of fields

[†]The computations in Section 4.3 will anyway cover the other possible cases.

by $\mathcal{F}^{B}_{EH} \subset \mathcal{F}^{\text{full}}_{EH}$. With this choice of space of fields we have that the pre-boundary one form simplifies to:

$$\begin{aligned}\widetilde{\alpha}^{B}_{EH} = &-\int_{\partial M} d^d x \left(\xi^n \delta g_{nn} g^{\dagger nn} + \xi^n \delta g_{ab} g^{\dagger ab} - 2\delta\xi^n g_{nn} g^{\dagger nn} - 2\delta\xi^{(a} g_{ab} g^{\dagger b)n}\right) \\ &+ \int_{\partial M} d^d x \frac{\sqrt{g}}{2}\left(g^{ab}\delta\left(g^{nn} J_{ab}\right) + g^{nn}\delta\left(g^{ab} J_{ab}\right)\right) - \xi^n \delta\xi^{\mu}\chi_{\mu} \nu^{\partial}\end{aligned} \tag{4.9}$$

where $J_{ab} = \partial_n g_{ab}|_{\partial M}$ is now an independent field on the space of pre-boundary fields $\widetilde{\mathcal{F}}^{B}_{EH}$.

Remark 4.1. *We will take care of the cohomological vector field later, but notice that the condition* $g_{na} = 0$ *on the boundary implies*

$$(Q_g)_{na}\Big|_{\partial M} = 0 \Longrightarrow \partial_n \xi^b = -g_{nn} g^{ba} \partial_a \xi^n \quad \text{on } \partial M \tag{4.10}$$

◊

We have that

Theorem 4.2. *For all* $d \neq 1$*, the data* $(\mathcal{F}^{B}_{EH}, Q, S^{BV}_{EH} + F, \Omega_{BV})$ *where*

$$F = -\frac{2}{d-1}\int_M L_{\xi}(g^{\dagger} g) \tag{4.11}$$

is a boundary term, induces an exact BV-BFV theory on the boundary ∂M*, provided that the latter is everywhere either space-like or time-like.*

Proof. From the expression (4.9) for $\widetilde{\alpha}^{B}_{EH}$ we can obtain the two form $\widetilde{\omega} = \delta\widetilde{\alpha}^{B}_{EH}$ and its kernel. After some straightforward computations, denoting a general vector field by

$$X = \sum_{\phi} (X_{\phi})\frac{\delta}{\delta\phi}; \quad \text{where } \phi \in \{g^{nn}, g_{ab}, J_{nn}, J_{ab}, g^{\dagger ab}, g^{\dagger nn}, g^{\dagger na}, \xi^n, \xi^a, \chi_n, \chi_a\}$$

with $a = 1 \dots d$, one gets to the result:

$$(X_g)_{ab} = 0 \ \forall a, b \tag{4.12a}$$

$$(X_\xi)^n = \frac{1}{2} g_{nn} (X_g)^{nn} \xi^n \tag{4.12b}$$

$$(X_\xi)^a = 0 \tag{4.12c}$$

$$(X_{g^\dagger})^{nn} = \frac{1}{2} g_{nn} (X_g)^{nn} g^{\dagger nn} + \frac{1}{2} (X_g)^{nn} \chi_n \xi^n - \frac{1}{2} g^{nn} (X_\chi)_n \xi^n \tag{4.12d}$$

$$(X_{g^\dagger})^{na} = -\frac{1}{4} g_{nn} (X_g)^{nn} g^{ab} \chi_b - \frac{1}{2} g^{ab} (X_\chi)_b \xi^n \tag{4.12e}$$

$$\begin{aligned} (X_J)_{ab} = & \frac{g_{nn}^2}{\sqrt{\mathrm{g}}} (X_g)^{nn} g^{\dagger cd} \left(g_{ca} g_{bd} - \frac{1}{d-1} g_{cd} g_{ab} \right) \xi^n + \\ & + \frac{2 g_{nn}}{\sqrt{\mathrm{g}}} (X_{g^\dagger})^{cd} \left(g_{ca} g_{bd} - \frac{1}{d-1} g_{cd} g_{ab} \right) \xi^n - \frac{1}{2} g_{nn} (X_g)^{nn} J_{ab} \end{aligned} \tag{4.12f}$$

The generators are easily found to be:

$$\mathbb{X}_n = \frac{\delta}{\delta \chi_n} - \frac{1}{2} g^{nn} \xi^n \frac{\delta}{\delta g^{\dagger nn}} \tag{4.13a}$$

$$\mathbb{X}_b = \frac{\delta}{\delta \chi_b} - \frac{1}{2} g^{ba} \xi^n \frac{\delta}{\delta g^{\dagger na}} \tag{4.13b}$$

$$\mathbb{G}^{\dagger cd} = \frac{\delta}{\delta g^{\dagger cd}} + \frac{2 g_{nn}}{\sqrt{\mathrm{g}}} \left(g_{ca} g_{bd} - \frac{1}{d-1} g_{cd} g_{ab} \right) \xi^n \frac{\delta}{\delta J_{ab}} \tag{4.13c}$$

$$\begin{aligned} \mathbb{G}^{nn} = & \frac{\delta}{\delta g^{nn}} - \left[\frac{1}{2} g_{nn} J_{ab} - \frac{g_{nn}^2}{\sqrt{\mathrm{g}}} \left(g^{\dagger cd} g_{ca} g_{bd} - \frac{1}{d-1} g^{\dagger cd} g_{cd} g_{ab} \right) \xi^n \right] \frac{\delta}{\delta J_{ab}} + \\ & + \frac{1}{2} \left(g_{nn} g^{\dagger nn} - \chi_n \xi^n \right) \frac{\delta}{\delta g^{\dagger nn}} - \frac{1}{4} g_{nn} g^{ab} \chi_b \xi^n \frac{\delta}{\delta g^{\dagger na}} + \frac{1}{2} g_{nn} \xi^n \frac{\delta}{\delta \xi^n} \end{aligned} \tag{4.13d}$$

and it is easy to gather that $\widetilde{\omega}$ is pre-symplectic.

It is now possible to check if the one-form $\widetilde{\alpha}$ is horizontal. As a matter of fact we have that

$$\iota_{\mathbb{G}^{nn}} \widetilde{\alpha}^B_{EH} = \frac{g_{nn}}{d-1} g^{\dagger cd} g_{cd} \xi^n = -\iota_{\mathbb{G}^{nn}} \delta F \tag{4.14}$$

together with

$$\iota_{\mathbb{G}^{\dagger cd}} \widetilde{\alpha}^B_{EH} = \frac{2}{d-1} g_{cd} \xi^n = -\iota_{\mathbb{G}^{\dagger cd}} \delta F$$

whereas

$$\iota_{\mathbb{X}^\rho}\widetilde{\alpha}^B_{EH} = \iota_{\mathbb{X}^\rho}\delta F = 0$$

Therefore, the corrected one-form $\overline{\alpha} := \widetilde{\alpha}_{EH} + \delta F$ is horizontal, and clearly $\widetilde{\omega} = \delta\widetilde{\alpha}^B_{EH} = \delta\overline{\alpha}$.

✓

Remark 4.3. *Notice that the* bulk-extended *boundary term F has to be added to the bulk action S^{BV}_{EH} in order to correct the horizontality of the pre-boundary one-form. Observe that the failure to horizontality is a function of degree 0, but it depends explicitly on the odd fields. We shall see later on how this is linked to the Gibbons-Hawking-York boundary term.*

◊

The previous result ensures that the symplectic reduction in the space of pre-boundary fields $\widetilde{\mathcal{F}}^B_{EH}$ can be performed. The explicit coordinate expression for the boundary structures can be computed in this case as well following a procedure similar to the one used for the examples in Chapter 3.

Theorem 4.4. *The surjective submersion $\pi_M : \mathcal{F}^B_{EH} \longrightarrow \mathcal{F}^\partial_{EH}$ is given by the local expression:*

$$\pi_M : \begin{cases} \widetilde{J}_{ab} & = J_{ab}\sqrt{|\mathrm{g}^{nn}|} - \frac{2}{\sqrt{\mathrm{g}^\partial}}\left(g^{\dagger cd}g_{ca}g_{bd} - \frac{1}{d-1}g^{\dagger cd}g_{cd}g_{ab}\right)\xi^n \\ \widetilde{g}^{\dagger nn} & = \left(g^{\dagger nn} + \frac{1}{2}g^{nn}\chi_n\xi^n\right)\sqrt{|\mathrm{g}_{nn}|} \\ \widetilde{g}^{\dagger na} & = g^{\dagger na} + \frac{1}{2}g^{ab}\chi_b\xi^n \\ \widetilde{\xi}^n & = \xi^n\sqrt{|\mathrm{g}_{nn}|} \\ \widetilde{\xi}^a & = \xi^a \\ \widetilde{g}_{ab} & = g_{ab} \end{cases} \tag{4.15}$$

Moreover, the boundary one-form α^∂ reads

$$\boxed{\alpha^\partial = -\int\limits_{\partial M}\left\{\frac{\sqrt{\mathrm{g}^\partial}}{2}\left(\delta\widetilde{g}^{ab}\widetilde{J}_{ab} + 2\widetilde{g}^{ab}\delta\widetilde{J}_{ab}\right) + 2\delta\widetilde{\xi}^n\widetilde{g}^{\dagger nn} - 2\widetilde{g}_{ab}\delta\widetilde{\xi}^a\widetilde{g}^{\dagger bn}\right\}} \tag{4.16}$$

and the two-form $\omega^\partial = \delta\alpha^\partial$.

The cohomological vector field Q projects to a vector field Q^∂ that is also cohomological if and only if one assumes that the normal jets $\partial_n g_{na}$ vanish on the boundary: $J_{na} = 0$. Such a vector field is Hamiltonian with respect to the boundary action

$$S^\partial = \int_{\partial M} \Bigg\{ \left(\frac{\sqrt{\mathrm{g}^\partial}}{4} \widetilde{g}^{ab} \left(\widetilde{J}_{ab} \widetilde{J}_{cd} - \widetilde{J}_{cb} \widetilde{J}_{ad} \right) \widetilde{g}^{cd} - \sqrt{\widetilde{\mathrm{g}}^\partial} \widetilde{R}^\partial - 2\partial_a \left(\widetilde{\xi}^a \widetilde{g}^{\dagger nn} \right) - 2\widetilde{g}^{\dagger na} \partial_a \widetilde{\xi}^n \right) \widetilde{\xi}^n + \\ - \sqrt{\widetilde{\mathrm{g}}^\partial} \partial_b \left(\widetilde{g}^{ad} \widetilde{J}_{ad} \right) \widetilde{\xi}^b + \partial_c \left(\sqrt{\widetilde{\mathrm{g}}^\partial} \widetilde{g}^{cd} \widetilde{J}_{ad} \right) \widetilde{\xi}^a + \frac{\sqrt{\widetilde{\mathrm{g}}^\partial}}{2} \partial_b \widetilde{g}^{ad} \widetilde{J}_{ad} \widetilde{\xi}^b - 2\partial_c \left(\widetilde{\xi}^c \widetilde{g}^{\dagger nb} \widetilde{g}_{ab} \right) \widetilde{\xi}^a \Bigg\} v^\partial \tag{4.17}$$

Proof. First, let us find the explicit expression for the map π_M. It is possible to adapt the procedure used in Chapter 3 to eliminate some variables and find an explicit section of the symplectic reduction $\pi \colon \widetilde{\mathcal{F}}^B_{EH} \longrightarrow \mathcal{F}^\partial_{EH}$. In particular we can flow along $\mathbb{X}_\rho$ to set $\chi_\rho|_{s=1} = 0$, and this will give us the temporary values of $g^{\dagger n\rho}$:

$$\hat{g}^{\dagger nn} := g_0^{\dagger nn} + \frac{1}{2} g_0^{nn} \chi_n^0 \xi_0^n;$$
$$\hat{g}^{\dagger na} := g_0^{\dagger na} + \frac{1}{2} g_0^{ab} \chi_b^0 \xi_0^n$$

The same can be done using $\mathbb{G}^{\dagger ab}$ to set $g^{\dagger ab}|_{s=1} = 0$ and this implies

$$\hat{J}_{ab} := J^0_{ab} - \frac{2}{\sqrt{\mathrm{g}}} g^0_{nn} \left(g_0^{\dagger cd} g^0_{ca} g^0_{bd} - \frac{1}{d-1} g_0^{\dagger cd} g^0_{cd} g^0_{ab} \right) \xi_0^n$$

Finally we can use $\mathbb{G}^{nn}$ to set $g^{nn}|_{s=1} = 1$. After some straightforward calculations one gets that the symplectic reduction map $\pi \colon \widetilde{\mathcal{F}}^B_{EH} \longrightarrow \mathcal{F}^\partial_{EH}$ is described in local coordinates by

$$\pi\colon \begin{cases} \widetilde{J}_{ab} &= J_{ab}\sqrt{|\mathrm{g}^{nn}|} - \frac{2}{\sqrt{\mathrm{g}^\partial}}\left(g^{\dagger cd}g_{ca}g_{bd} - \frac{1}{d-1}g^{\dagger cd}g_{cd}g_{ab}\right)\xi^n \\ \widetilde{g}^{\dagger nn} &= \left(g^{\dagger nn} + \frac{1}{2}g^{nn}\chi_n\xi^n\right)\sqrt{|\mathrm{g}_{nn}|} \\ \widetilde{g}^{\dagger na} &= g^{\dagger na} + \frac{1}{2}g^{ab}\chi_b\xi^n \\ \widetilde{\xi}^n &= \xi^n\sqrt{|\mathrm{g}_{nn}|} \\ \widetilde{\xi}^a &= \xi^a \\ \widetilde{g}_{ab} &= g_{ab} \end{cases}$$

showing the first claim after composing with $\widetilde{\pi}_M$. To find the boundary one-form, consider the ansatz

$$\alpha^\partial = -\int\limits_{\partial M} d^d x\left\{\frac{\sqrt{\mathrm{g}^\partial}}{2}\left(\delta\widetilde{g}^{ab}\widetilde{J}_{ab} + 2\widetilde{g}^{ab}\delta\widetilde{J}_{ab}\right) + 2\delta\widetilde{\xi}^n\widetilde{g}^{\dagger nn} - 2\widetilde{g}_{ab}\delta\widetilde{\xi}^a\widetilde{g}^{\dagger bn}\right\} \tag{4.18}$$

and, recalling the *bulk-extended* boundary term

$$F = -\frac{2}{d-1}\int\limits_M L_\xi(g^\dagger g)$$

we compute

$$\pi^*\alpha^\partial = \widetilde{\alpha}^B_{EH} + \delta F =: \bar{\alpha}$$

Clearly: $\delta\widetilde{\alpha}^B_{EH} = \delta\bar{\alpha} = \pi^*\delta\alpha^\partial = \pi^*\omega^\partial$, so we can safely compute the boundary two form by differentiating expression (4.16). Therefore also the second claim is proven.

Now we can move on to prove the statements concerning the rest of the boundary structure, namely Q^∂ and S^∂. Most of the computations have been removed from the proof and put in Appendix A, Computation A.2.

Let us take into account the bulk part of the variation, namely the Euler Lagrange term in (4.7). The defining equation of the cohomological vector field $\iota_Q\Omega_{BV} = \delta S^{EH}_{BV} + \pi^*_M\alpha^\partial$, where Ω_{BV} is the symplectic BV form in the bulk, yields:

$$\begin{aligned}
(Q_\chi)_\mu &= 2\partial_{(\rho}g_{\sigma)\mu}g^{\dagger\rho\sigma} + 2g_{\mu(\sigma}\partial_{\rho)}g^{\dagger\rho\sigma} - \partial_\mu g_{\rho\sigma}g^{\dagger\rho\sigma} + \partial_\mu\xi^\rho\chi_\rho + \partial_\rho\xi^\rho\chi_\mu + \xi^\rho\partial_\rho\chi_\mu \\
(Q_{g^\dagger})^{\mu\nu} &= \partial_\rho\xi^\rho g^{\dagger\mu\nu} + \xi^\rho\partial_\rho g^{\dagger\mu\nu} - 2\partial_\rho\xi^{(\mu}g^{\dagger\nu)\rho} - \sqrt{g}\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right) \\
(Q_g)_{\mu\nu} &= 2\partial_{(\mu}\xi^\rho g_{\nu)\rho} + \xi^\rho\partial_\rho g_{\mu\nu} \\
(Q_\xi)^\mu &= \xi^\rho\partial_\rho\xi^\mu = \frac{1}{2}[\xi,\xi]^\mu
\end{aligned}$$

Clearly the expressions for Q_g and Q_ξ agree with (4.2). Along the surjective submersion (4.6), $\widetilde{\pi}\colon \mathcal{F}^B_{EH} \longrightarrow \widetilde{\mathcal{F}}^B_{EH}$, the bulk vector field descends to a pre-boundary vector field $\widetilde{Q} = \widetilde{\pi}_*Q$, its components along a field ϕ being $(\widetilde{Q}_\phi) = (Q_\phi)\big|_{\partial M}$. Since the transversal jets $J_{\mu\nu} = \partial_n g_{\mu\nu}\big|_{\partial M}$ are relevant for the boundary structure, we must complete the cohomological pre-boundary vector field $\widetilde{Q}$ with their transformation law:

$$\widetilde{Q}J_{\mu\nu} = \partial_n Qg_{\mu\nu}\big|_{\partial M} = \partial_n\xi^\rho\partial_\rho g_{\mu\nu} + \xi^\rho\partial_\rho J_{\mu\nu} + 2\partial_{(\mu}\partial_n\xi^\rho g_{\nu)\rho} + 2\partial_{(\mu}\xi^\rho J_{\nu)\rho} \tag{4.19}$$

Remark 4.5. *Notice that requiring* $g_{na} = 0$ *in an arbitrarily thin neighborhood of the boundary automatically implies having* $J_{na} = 0$ *on the boundary, and this yields* $QJ_{na} = 0$ *with no extra restrictions on the fields, as it can be checked by direct computation from Eq.* (4.19). *Compare this with Remark 4.1.*

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The pre-boundary vector field $\widetilde{Q}$ then reads:

$$\widetilde{Q} = (\widetilde{Q}_\chi)_\mu\frac{\delta}{\delta\chi_\mu} + (\widetilde{Q}_{g^\dagger})^{\mu\nu}\frac{\delta}{\delta g^{\dagger\mu\nu}} + (\widetilde{Q}_J)_{\mu\nu}\frac{\delta}{\delta J_{\mu\nu}} + (\widetilde{Q}_g)_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + (\widetilde{Q}_\xi)^\mu\frac{\delta}{\delta\xi^\mu}$$

We can add to $\widetilde{Q}$ appropriate combinations of vertical vector fields to obtain one that is projectable. In particular we consider:

$$\widetilde{Q}' = \widetilde{Q} - (\widetilde{Q}_{g^\dagger})^{ab}\mathbb{G}^{\dagger ab} - (\widetilde{Q}_\chi)_\mu\mathbb{X}_\mu - (\widetilde{Q}_g)^{nn}\mathbb{G}^{nn}$$

whose expression can be found in (A.3).

Now we must write the above expression in terms of the boundary fields, and to do so we must first transform the basis of vector fields via the formula: $\frac{\delta}{\delta\Phi} = \sum_{\widetilde{\Phi}} \frac{\delta\widetilde{\Phi}}{\delta\Phi}\frac{\delta}{\delta\widetilde{\Phi}}$. This is done in Computation A.2, Equation (A.4), and it leads to the explicit expression for Q' given in (A.5).

Expanding the $\widetilde{J}_{ef}$ coefficient shown in Computation A.2, Equation (A.5), one is left with an expression that should depend solely on the boundary fields. Then, after some rewriting we get (cf. (A.8)):

$$Q'|_{\widetilde{J}} = Q^{\partial}_{\widetilde{J}_{ef}} \frac{\delta}{\delta \widetilde{J}_{ef}} + \hat{Q}'_{\widetilde{J}_{ef}}[J_{an}] \frac{\delta}{\delta \widetilde{J}_{ef}} \tag{4.20}$$

where $\hat{Q}'_{\widetilde{J}_{ef}}[J_{an}]$ is an expression that depends on the off diagonal jets J_{an}, and is therefore not projectable. Before elaborating any further on this, let us analyse the other coefficients. In Computation A.2 we find, analogously, that

$$Q'\Big|_{\widetilde{g}^{\dagger nn}} = (Q^{\partial}_{\widetilde{g}^{\dagger}})^{nn} \frac{\delta}{\delta g^{\dagger nn}} + \hat{Q}'_{\widetilde{g}^{\dagger nn}}[J_{na}] \frac{\delta}{\delta g^{\dagger nn}} \tag{4.21}$$

where again $\hat{Q}'_{\widetilde{g}^{\dagger nn}}[J_{na}]$ is a function of the non-projectable jets. A similar issue is encountered when computing $Q'_{\widetilde{g}^{\dagger nc}}$ since again we find

$$Q'\Big|_{\widetilde{g}^{\dagger nc}} = (Q^{\partial}_{\widetilde{g}^{\dagger}})^{nc} \frac{\delta}{\delta g^{\dagger nc}} + \hat{Q}'_{\widetilde{g}^{\dagger nc}}[J_{na}] \frac{\delta}{\delta g^{\dagger nc}} \tag{4.22}$$

Some of the expressions above contain non projectable coefficients, that are functions of the off diagonal normal jets J_{na}. To get rid of these terms we must (at least) assume that the off diagonal components of the metric g_{na} vanish on the boundary together with their first normal jets.

The rest of the coefficients of the cohomological vector field on the boundary do not share this projectability issue, and are computed in Computation A.2.

When this prescription is taken into account the cohomological vector field Q on the space of pre-boundary fields projects to a vector field Q^{∂} of degree 1 on the space of boundary fields, which is also cohomological, and whose components are given in (A.10).

From these expression of the boundary cohomological vector field we can compute the induced boundary action via

$$S^\partial = \iota_{Q^\partial}\iota_{E^\partial}\omega^\partial \tag{4.23}$$

where the vector field E^∂ is the boundary Euler vector field and reads:

$$E^\partial = \widetilde{\xi}^\rho \frac{\delta}{\delta\widetilde{\xi}^\rho} - \widetilde{g}^{\dagger n\rho}\frac{\delta}{\delta\widetilde{g}^{\dagger n\rho}}$$

The boundary action is given then by the following expression:

$$\begin{aligned} S^\partial = \int_{\partial M} \Bigg\{ \Bigg(& \frac{\sqrt{\widetilde{g}^\partial}}{4}\widetilde{g}^{ab}\left(\widetilde{J}_{ab}\widetilde{J}_{cd} - \widetilde{J}_{cb}\widetilde{J}_{ad}\right)\widetilde{g}^{cd} - \sqrt{\widetilde{g}^\partial}\widetilde{R}^\partial - 2\partial_a\left(\widetilde{\xi}^a\widetilde{g}^{\dagger nn}\right) - 2\widetilde{g}^{\dagger na}\partial_a\widetilde{\xi}^n\Bigg)\widetilde{\xi}^n \\ & - \sqrt{\widetilde{g}^\partial}\partial_b\left(\widetilde{g}^{ad}\widetilde{J}_{ad}\right)\widetilde{\xi}^b + \partial_c\left(\sqrt{\widetilde{g}^\partial}\widetilde{g}^{cd}\widetilde{J}_{ad}\right)\widetilde{\xi}^a + \frac{\sqrt{\widetilde{g}^\partial}}{2}\partial_b\widetilde{g}^{ad}\widetilde{J}_{ad}\widetilde{\xi}^b - 2\partial_c\left(\widetilde{\xi}^c\widetilde{g}^{\dagger nb}\widetilde{g}_{ab}\right)\widetilde{\xi}^a\Bigg\} \end{aligned}$$

✓

Remark 4.6. *Looking at the construction that Wheeler and DeWitt proposed for the Hamiltonian theory of gravity*[24] *we may gather that the formula relating their conjugate momenta* Π_{ij} *to our dynamical variables is as follows:*

$$\Pi_{ij} = \frac{\sqrt{g^\partial}}{2}\sqrt{g^{nn}}\left(J_{ij} - g_{ij}g^{cd}J_{cd}\right) = \pi^*\left(\frac{\sqrt{\widetilde{g}^\partial}}{2}\left(\widetilde{J}_{ij} - \widetilde{g}_{ij}\widetilde{g}^{cd}\widetilde{J}_{cd}\right)\right)\Bigg|_{g^\dagger=\xi=0} \tag{4.24}$$

It is a matter of a straightforward check to show that plugging (4.24) *into the Wheeler DeWitt Hamiltonian term:*

$$\begin{aligned} \frac{1}{2\sqrt{g^\partial}}&\left(2g^{a(c}g^{d)b} - g^{ab}g^{cd}\right)\Pi_{ab}\Pi_{cd} - \sqrt{g^\partial}R^\partial = \\ &= \pi_M^*\left(\frac{\sqrt{\widetilde{g}^\partial}}{4}\widetilde{g}^{ab}\left(\widetilde{J}_{ab}\widetilde{J}_{cd} - \widetilde{J}_{cb}\widetilde{J}_{ad}\right)\widetilde{g}^{cd} - \sqrt{\widetilde{g}^\partial}\widetilde{R}^\partial\right)\Bigg|_{g^\dagger=\xi=0} \end{aligned} \tag{4.25}$$

This is what we get from the induced boundary action after taking the $\widetilde{\xi}^n$*-derivative and setting to zero the remaining ghost fields (that is to say considering the degree-*0 *part of the remaining expression). For a non light-like boundary* ∂M *we obtain the Wheeler DeWitt constraint projected on the space of boundary* *fields, and we can consider $\widetilde{\xi}^n$ as a Lagrange multiplier.*

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4.3 ADM decomposition

Here we would like to generalise the previous analysis and establish whether the action functional (4.3) satisfies the CMR axioms of Definition 2.5, by using a different technique. The usual assumption taken into consideration in the literature is that the space-time manifold be globally hyperbolic, namely a direct product $\Sigma \times \mathbb{R}$, with Σ a space/time-like hypersurface. This condition is somehow natural when dealing with issues of existence and uniqueness of solutions to the Einstein equations, but from a general point of view it is somehow restrictive.

This gives us the idea to consider the case of a non-null boundary, described by a submanifold of the form $x^n = \text{const}$, such that it has a hyperbolic neighborhood. This is equivalent to (or rather means) asking that the space of pseudo-Riemannian structures on the manifold with boundary M be limited to those metrics whose restriction to the boundary has either time-like or space-like signature.

When that is the case, and when the transverse x^n component corresponds to a signature[‡] $-\epsilon$, it is customary[23,24] to write the metric and its inverse in the form:

$$\begin{aligned} g_{\mu\nu} &= \epsilon \begin{pmatrix} -(\eta^2 - \beta_a\beta^a) & \beta_b \\ \beta_a & \gamma_{ab} \end{pmatrix} \\ g^{\mu\nu} &= \epsilon\eta^{-2} \begin{pmatrix} -1 & \beta^b \\ \beta^a & \eta^2\gamma^{ab} - \beta^a\beta^b \end{pmatrix} \end{aligned} \tag{4.26}$$

where η and β_a are functions for all $a = 1, 2, 3$.

Notice that this decomposition is valid in a neighborhood of the boundary ∂M. This is a much weaker requirement than asking that M be globally hyperbolic, or globally foliated in spacelike (or timelike) slices.

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‡For simplicity $\epsilon = 1$ if x^n is a timelike direction, that is to say when the boundary is spacelike.

With this decomposition we have that $\sqrt{-g} = \eta\sqrt{|\gamma|_\epsilon}$, with $\gamma = \det\gamma_{ij}$, and $|\gamma|_\epsilon$ means that we consider the absolute value of the determinant when needed (if $\epsilon = -1$). We will understand this fact from now on and simply write $\sqrt{\gamma}$.

The classical Einstein Hilbert action gets rewritten as

$$S = \int_M \Big\{ \underbrace{\eta\sqrt{\gamma}(\epsilon(K_{ab}K^{ab} - K^2) + R^\partial - 2\Lambda)}_{L^{ADM}} + \\ - 2\partial_n(\sqrt{\gamma}K) + 2\partial_a(\sqrt{\gamma}K\beta^a - \sqrt{\gamma}\gamma^{ab}\partial_b\eta)\Big\} d^{d+1}x \tag{4.27}$$

where we define K_{ab}, the second fundamental form of the boundary submanifold and its trace K by means of the boundary covariant derivative ∇^∂ as follows

$$K_{ab} = \frac{1}{2}\eta^{-1}(2\nabla^\partial_{(a}\beta_{b)} - \partial_n\gamma_{ab}) = \frac{1}{2}\eta^{-1}T_{ab} \tag{4.28}$$

$$K = \gamma^{ab}K_{ab} = \frac{1}{2}\eta^{-1}\gamma^{ab}T_{ab} = \frac{1}{2}\eta^{-1}T \tag{4.29}$$

while T_{ab} and T are introduced for later convenience. We will redefine the ADM Lagrangian as

$$L_{ADM} := \eta\sqrt{\gamma}(\epsilon(K_{ab}K^{ab} - K^2) + R^\partial - 2\Lambda) \tag{4.30}$$

The classical space of fields in this case is then simply given by $\mathcal{F}_{\text{cl}} = \mathcal{PR}^{\partial M}_{(d,1)}(M)$ the space of pseudo Riemannian metrics on M with signaure (d,1), and space/time-like signature when restricted to the boundary.

The classical action we will consider from now on is the integral of the ADM Lagrangian: $S^{ADM} := \int_M L^{ADM}$.

Remark 4.7. *The total derivatives appearing in the new formulation of the theory build up the* Gibbons Hawking York *boundary term. In our framework it will change the one form on the boundary by an exact term, that will not interfere with the boundary structure. This will have an important effect however in connecting the BV-BFV theory for the Einstein Hilbert action in the adapted coordinates of Theorem 4.2 to the one we will get in what follows.*

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4.3.1 Classical boundary structure

To start off, we will consider first the classical (i.e. non-BV) structure that is induced on the boundary. This is often called *canonical analysis*, and one replaces the Lagrangian description with the Hamiltonian in the phase space of the system. The advantage in applying our variational approach to the classical case as well, is that we are able to perform the symplectic reduction of the space of classical pre-boundary fields, to find a well defined symplectic structure on the space of classical boundary fields, i.e. the phase space, encoding the canonical relations in a straightforward way.

Proposition 4.8. *The space of classical boundary fields for General Relativity in the ADM formalism for any dimension* $d+1 \neq 2$ *is a symplectic manifold. In a local chart the symplectic form reads*

$$\omega^\partial = \epsilon \int_{\partial M} \delta\boldsymbol{\gamma}^{ab}\delta\boldsymbol{\Pi}_{ab} \tag{4.31}$$

where the symplectic reduction map reads:

$$\pi\colon \begin{cases} \boldsymbol{\gamma}_{ij} = \gamma_{ij} \\ \boldsymbol{\Pi}_{lm} = \frac{\sqrt{\gamma}}{2}\left(\widetilde{J}_{lm} - \gamma_{lm}\gamma^{ij}\widetilde{J}_{ij}\right) \end{cases} \tag{4.32}$$

with

$$\widetilde{J}_{lm} = \eta^{-1}\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) \tag{4.33}$$

Proof. Consider the variation of the ADM action S_{ADM}, which splits in a bulk term and a boundary term. The latter is interpreted as a one-form $\widetilde{\alpha}$ on the space of pre-boundary fields $\widetilde{\mathcal{F}}_{\mathrm{cl}}$, which is given by restrictions of the bulk metric and its normal jets $J_{ab} := \partial_n\gamma_{ab}\big|_{\partial M}$ to ∂M:

$$\widetilde{\alpha} = 2\epsilon \int_{\partial M} \left\{\delta(\sqrt{\gamma}\gamma^{ab})K_{ab} - \frac{\sqrt{\gamma}}{2}\delta\gamma^{ab}K_{ab}\right\} d^d x \tag{4.34}$$

and we have that the two-form $\widetilde{\omega} = \delta\widetilde{\alpha}$, using the definitions (4.28) of K_{ab} and T_{ab}, reads

$$\widetilde{\omega} = \epsilon \int_{\partial M} \Big\{ \delta\eta^{-1}\delta(\sqrt{\gamma}\gamma^{ab})T_{ab} - \eta^{-1}\delta(\sqrt{\gamma}\gamma^{ab})\delta T_{ab} + \\ - \eta^{-1}\frac{\delta\sqrt{\gamma}\delta\gamma^{ab}}{2}T_{ab} - \delta\eta^{-1}\frac{\sqrt{\gamma}}{2}\delta\gamma^{ab}T_{ab} + \eta^{-1}\frac{\sqrt{\gamma}}{2}\delta\gamma^{ab}\delta T_{ab} \Big\} d^d x \quad (4.35)$$

The space of classical pre-boundary fields $\widetilde{\mathcal{F}}_{\mathrm{cl}}$ is then given by restrictions to ∂M of the bulk metric and its normal jets $J_{ab} := \partial_n \gamma_{ab}\big|_{\partial M}$. Both K and T are functions of g and J. Observe that the transversal jets J_{na} are not present because of the clever rewriting of the action, valid in a neighborhood of the boundary.

The kernel of the two form is found to be, for $d \neq 1$ by

$$(X_\gamma)^{ab} = 0 \quad (4.36)$$

$$(X_T)_{lm} = -\eta(X_{\eta^{-1}})T_{lm} \quad (4.37)$$

as it can be seen with a straightforward computation. It turns out that the (X_{β_m}) component of a vector field in the kernel is free, as well as the η^{-1} component. In fact, equation (4.37) can be unfolded to yield:

$$(X_J)_{lm} = -\eta(X_{\eta^{-1}})J_{lm} + 2\nabla_{(l}(X_\beta)_{m)} + 2\eta(X_{\eta^{-1}})\nabla_{(l}\beta_{m)} \quad (4.38)$$

The generators in the Kernel are

$$\mathbb{E}^{-1} = (X_{\eta^{-1}})\frac{\delta}{\delta\eta^{-1}} - \eta(X_{\eta^{-1}})J_{lm}\frac{\delta}{\delta J_{lm}} + 2\eta(X_{\eta^{-1}})\nabla_{(l}\beta_{m)}\frac{\delta}{\delta J_{lm}} \quad (4.39)$$

$$\mathbb{B}_l = (X_\beta)_l\frac{\delta}{\delta\beta_l} + 2\nabla_{(l}(X_\beta)_{m)}\frac{\delta}{\delta J_{lm}} \quad (4.40)$$

and thus, solving the differential equations given by the kernel vector fields together with

Equation (4.36) gives us the projection to the boundary fields:

$$\pi\colon \begin{cases} \widetilde{\gamma}_{ij} = \gamma_{ij} \\ \widetilde{J}_{lm} = \eta^{-1}\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) \end{cases} \tag{4.41}$$

It is a matter of a simple check to verify that the one form

$$\alpha^{\partial} = \epsilon \int\limits_{\partial M} \frac{\sqrt{\widetilde{\gamma}}}{2}\left(\delta\widetilde{\gamma}^{ij}\widetilde{\gamma}_{ij}\widetilde{\gamma}^{lm}\widetilde{J}_{lm} - \delta\widetilde{\gamma}^{lm}\widetilde{J}_{lm}\right) \tag{4.42}$$

is horizontal, i.e. $\iota_{\mathbb{E}^{-1}}\widetilde{\alpha} = \iota_{\mathbb{B}_l}\widetilde{\alpha} = 0$, and that it pulls back to the boundary one form $\widetilde{\alpha}$ along π:

$$\pi^*\alpha^{\partial} = \widetilde{\alpha}$$

This implies that the symplectic manifold $\left(\mathcal{F}^{\partial}, \delta\alpha^{\partial}\right)$ is exact.

Introducing the new variables $\boldsymbol{\gamma}^{ab} \equiv \widetilde{\gamma}^{ab}$ and $\mathbf{\Pi}_{lm} = \frac{\sqrt{\gamma}}{2}\left(\widetilde{J}_{lm} - \gamma_{lm}\gamma^{ij}\widetilde{J}_{ij}\right)$ we have

$$\alpha^{\partial} = -\epsilon \int\limits_{\partial M} \delta\boldsymbol{\gamma}^{ab}\mathbf{\Pi}_{ab} \Longrightarrow \omega^{\partial} = \epsilon \int\limits_{\partial M} \delta\boldsymbol{\gamma}^{ab}\delta\mathbf{\Pi}_{ab} \tag{4.43}$$

which is the symplectic form in the space of classical boundary fields, expressed in local Darboux coordinates. ✓

Remark 4.9. *We managed to recover the phase space description of General Relativity in the symplectic framework. Notice that in the non-BV setting the compatibility with the boundary structure is encoded in the boundary term $\pi_M^*\alpha^{\partial}_{\partial M}$, a failure of the variation of the action from being given by the Euler Lagrange equations alone. When turning to the BV theory we will see how this compatibility can be enriched to yield the full fundamental formula* (2.6).

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Remark 4.10. *Observe that we have performed a symplectic reduction that encodes the usual canonical analysis of General Relativity (this time explicitly in the ADM formalism). Our boundary field $\mathbf{\Pi}_{ab}$ is a projected version of the usual (i.e. literature)* momentum *coordinate conjugate to γ^{ab} (let us call it $p_{ab} = \pi^*\mathbf{\Pi}_{ab}$), with the difference that in the present case the* conjugacy *is in the symplectic sense, as we* *quotient by the kernel of the pre-symplectic form $\widetilde{\omega}$.*

◇

In what follows we will show how this can be extended to the BV setting, which explicitly encodes the symmetries. This will allow us to recover the usual energy and momentum constraints in a straightforward way, still holding on to the clean symplectic description of the phase space.

4.4 BV-BFV-ADM theory

Recalling the general theory we outlined in Section 2.2, in order to perform a consistent analysis of the theory including the symmetries, one has to find the correct BV data. The geometric information we need is the distribution in the space of fields that generates the symmetries.

In our case, General Relativity is invariant under the action of the whole diffeomorphism group of the space-time manifold M. The theory can be treated as a BRST-like theory since the symmetry algebra $\Gamma(TM)$ closes everywhere in the space of fields, and we can use Theorem 2.3 to extend the classical ADM action to its BV-extended counterpart. Indeed we consider the following action:

$$\boxed{S^{BV}_{ADM} = \int\limits_{M} \left\{\eta\sqrt{\gamma}(\epsilon(K_{ab}K^{ab} - K^2) + R^{\partial} - 2\Lambda)d^{d+1}x - \left(L_{\xi}g\right)g^{\dagger} + \frac{1}{2}\iota_{[\xi,\xi]}\xi^{\dagger}\right\}} \tag{4.44}$$

where we introduced the same symmetry terms we used in (4.2), i.e.

$$\boxed{\begin{aligned} Qg &= L_{\xi}g \\ Q\xi &= \frac{1}{2}[\xi,\xi] \end{aligned}} \tag{4.45}$$

and the space of fields is given by the shifted cotangent bundle:

$$\mathcal{F}_{ADM} \coloneqq T^*[-1]\left[\mathcal{PR}^{\partial M}_{d+1}(M) \oplus \Gamma\left(T[1]M\right)\right] \tag{4.46}$$

equipped with the canonical odd-symplectic form Ω_{BV}. Our first result in this setting is the following

Theorem 4.11. *For all* $d \neq 1$*, the data* $(\mathcal{F}_{ADM}, S^{BV}_{ADM}, Q, \Omega_{BV})$ *induces an exact BV-BFV theory. The induced data on the boundary will be denoted by* $(\mathcal{F}^{\partial}, S^{\partial}, Q^{\partial}, \omega^{\partial})$*. In particular we have that* $Q^{\partial} = \pi_{M*}Q$*, and* $\iota_{Q^{\partial}}\omega^{\partial} = \delta S^{\partial}$*.*

Proof. The variation of S^{BV}_{ADM} induces the following boundary one-form, where we fixed the volume form $v = dx^n \wedge v^{\partial}$:

$$\begin{aligned}
\widetilde{\alpha}_{ADM} = 2\epsilon \int_{\partial M} \eta \left\{ \delta(\sqrt{\gamma}\gamma^{ab})K_{ab} - \frac{\sqrt{\gamma}}{2}\delta\gamma^{ab}K_{ab} \right\} v^{\partial} - \int_{\partial M} \xi^n \delta\xi^{\rho}\chi_{\rho} v^{\partial} \\
+ 2\epsilon \int_{\partial M} \left((-\eta^2 + \beta_a\beta^a)\delta\xi^n g^{\dagger nn} + \beta_a\delta\xi^n g^{\dagger an} + \beta_a\delta\xi^a g^{\dagger nn} + \gamma_{ab}\delta\xi^{(a} g^{\dagger b)n} \right) v^{\partial} \\
- \epsilon \int_{\partial M} \left(\xi^n\delta(-\eta^2 + \beta_a\beta^a) g^{\dagger nn} + 2\xi^n\delta\beta_a g^{\dagger an} + \xi^n\delta\gamma_{ab} g^{\dagger ab} \right) v^{\partial}
\end{aligned} \tag{4.47}$$

and two-form $\widetilde{\omega} = \delta\widetilde{\alpha}_{ADM}$:

$$\begin{aligned}
\widetilde{\omega} = \int_{\partial M} \epsilon\delta \left\{ \eta^{-1}\delta(\sqrt{\gamma}\gamma^{ab})K_{ab} - \eta^{-1}\frac{\sqrt{\gamma}}{2}\delta\gamma^{ab}K_{ab} \right\} - \delta\xi^n\delta\xi^{\rho}\chi_{\rho} + \xi^n\delta\xi^{\rho}\delta\chi_{\rho} \\
+ \epsilon\Big(\delta(-\eta^2 + \beta_a\beta^a)\delta\xi^n g^{\dagger nn} + 2(-\eta^2 + \beta_a\beta^a)\delta\xi^n\delta g^{\dagger nn} + 2\beta_a\delta\xi^n\delta g^{\dagger an} \\
+ 2\delta\beta_a\delta\xi^a g^{\dagger nn} + 2\beta_a\delta\xi^a\delta g^{\dagger nn} + 2\delta\gamma_{ab}\delta\xi^{(a} g^{\dagger b)n} + 2\gamma_{ab}\delta\xi^{(a}\delta g^{\dagger b)n} \Big) \\
- \epsilon\Big(\xi^n\delta(-\eta^2 + \beta_a\beta^a)\delta g^{\dagger nn} + 2\xi^n\delta\beta_a\delta g^{\dagger an} + \delta\xi^n\delta\gamma_{ab} g^{\dagger ab} + \xi^n\delta\gamma_{ab}\delta g^{\dagger ab} \Big)
\end{aligned} \tag{4.48}$$

Recalling that K_{ab} is a function of $J_{ab} \coloneqq \partial_n\gamma|_{\partial M}$, it is just a matter of lengthy computations to show that $\widetilde{\omega}$ is presymplectic: indeed, excluding the case $d = 1$, the equations defining the kernel read:

$$
\begin{aligned}
(X_J)_{lm} =& + \eta^{-1}(X_\eta)J_{lm} + 2\nabla_{(l}(X_\beta)_{m)} - 2\eta^{-1}(X_\eta)\nabla_{(l}\beta_{m)} \\
&+ \frac{4}{\sqrt{\gamma}}(X_\eta)\left(\frac{1}{d-1}\gamma_{lm}\beta_a - \beta_{(l}\gamma_{m)a}\right)g^{\dagger an}\xi^n \\
&- \frac{4}{\sqrt{\gamma}}\eta\left(\frac{1}{d-1}\gamma_{lm}(X_\beta)_a - (X_\beta)_{(l}\gamma_{m)a}\right)g^{\dagger an}\xi^n \\
&+ \frac{2}{\sqrt{\gamma}}(X_\eta)\left(\frac{1}{d-1}\gamma_{lm}\gamma_{ab} - \gamma_{la}\gamma_{bm}\right)g^{\dagger ab}\xi^n \\
&- \frac{2}{\sqrt{\gamma}}\eta\left(\frac{1}{d-1}\gamma_{lm}\gamma_{ab} - \gamma_{al}\gamma_{bm}\right)(X_{g^\dagger})^{ab}\xi^n
\end{aligned} \tag{4.49}
$$

$$
\begin{aligned}
(X_g^\dagger)^{bn} =& - \gamma^{ab}(X_\beta)_a g^{\dagger nn} + \eta^{-1}\beta^b(X_\eta)g^{\dagger nn} + \epsilon\eta^{-3}\beta^b\beta^a\chi_a(X_\eta)\xi^n \\
&- \frac{\epsilon}{2}\eta^{-2}\beta^b\beta^a(X_\chi)_a\xi^n - \frac{\epsilon}{2}\eta^{-2}\beta^b(X_\beta)_c\gamma^{cd}\chi_d\xi^n - \epsilon\eta^{-3}\beta^b(X_\eta)\chi_n\xi^n \\
&+ \frac{\epsilon}{2}\eta^{-2}\beta^b(X_\chi)_n\xi^n - \frac{\epsilon}{2}\eta^{-1}\gamma^{ba}\chi_a(X_\eta)\xi^n + \frac{\epsilon}{2}\gamma^{ba}(X_\chi)_a\xi^n
\end{aligned} \tag{4.50}
$$

$$
\begin{aligned}
(X_{g^\dagger})^{nn} =& - \eta^{-1}(X_\eta)g^{\dagger nn} - \epsilon\eta^{-3}(X_\eta)\beta^a\chi_a\xi^n + \frac{\epsilon}{2}\eta^{-2}\beta^a(X_\chi)_a\xi^n \\
&+ \frac{\epsilon}{2}\eta^{-2}(X_\beta)_b\gamma^{ab}\chi_a\xi^n + \epsilon\eta^{-3}(X_\eta)\chi_n\xi^n - \frac{\epsilon}{2}\eta^{-2}(X_\chi)_n\xi^n
\end{aligned} \tag{4.51}
$$

$$
(X_\xi)^a = + \beta^a\eta^{-1}(X_\eta)\xi^n - \gamma^{ab}(X_\beta)_b\xi^n \tag{4.52}
$$

$$
(X_\xi)^n = - \eta^{-1}(X_\eta)\xi^n \tag{4.53}
$$

As a matter of fact, contracting $\widetilde{\omega}$ with a general vector field X and collecting the terms along the normal jet δJ_{ab} we have the equations

$$
\delta J_{ab}: \quad - \frac{\sqrt{\gamma}}{2}\eta^{-1}\gamma_{cd}(X_\gamma)^{cd}\gamma^{ab} + \frac{\sqrt{\gamma}}{2}\eta^{-1}(X_\gamma)^{ab} = 0
$$

for all $a, b = 1, \dots d$ and taking the trace of this results in the following condition:

$$
(d-1)\mathrm{Tr}(X_\gamma) = 0 \tag{4.54}
$$

Assuming $d \neq 1$ we can conclude that $\mathrm{Tr}(X_\gamma) = 0$ and consequently that $(X_\gamma) \equiv 0$. Collecting all terms along the other fields one gets in a lengthy but straightforward way the rest of the kernel equations.

The kernel is generated by the (vertical) vector fields:

$$\mathbb{X}_{(n)} =(X_\chi)_n \frac{\delta}{\delta \chi_n} - \frac{\epsilon}{2}\eta^{-2}(X_\chi)_n \xi^n \frac{\delta}{\delta g^{\dagger nn}} + \frac{\epsilon}{2}\beta^b \eta^{-2}(X_\chi)_n \xi^n \frac{\delta}{\delta g^{\dagger bn}} \tag{4.55a}$$

$$\begin{aligned}\mathbb{X}_{(a)} =&(X_\chi)_a \frac{\delta}{\delta \chi_a} + \frac{\epsilon}{2}\eta^{-2}\beta^a (X_\chi)_a \xi^n \frac{\delta}{\delta g^{\dagger nn}} \\ &- \left(\frac{\epsilon}{2}\eta^{-2}\beta^b\beta^a (X_\chi)_a \xi^n - \frac{\epsilon}{2}\gamma^{ba}(X_\chi)_a \xi^n\right)\frac{\delta}{\delta g^{\dagger bn}}\end{aligned} \tag{4.55b}$$

$$\begin{aligned}\mathbb{B}_{(a)} =&(X_\beta)_a \frac{\delta}{\delta \beta_a} - \gamma^{ab}(X_\beta)_a \xi^n \frac{\delta}{\delta \xi^b} + \frac{\epsilon}{2}\eta^{-2}\gamma^{ab}(X_\beta)_a \chi_b \xi^n \frac{\delta}{\delta g^{\dagger nn}} \\ &+ \left(2\nabla_{(l}(X_\beta)_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\eta\left((X_\beta)_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}(X_\beta)_a\right) g^{\dagger an}\xi^n\right)\frac{\delta}{\delta J_{lm}} \\ &+ \left(-\frac{\epsilon}{2}\eta^{-2}\beta^b\gamma^{cd}(X_\beta)_c \chi_d \xi^n - \gamma^{ab}(X_\beta)_a g^{\dagger nn}\right)\frac{\delta}{\delta g^{\dagger bn}}\end{aligned} \tag{4.55c}$$

$$\mathbb{G}^{\dagger(ab)} =(X_{g^\dagger})^{ab}\frac{\delta}{\delta g^{\dagger ab}} + \frac{2\epsilon}{\sqrt{\gamma}}\eta\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)(X_{g^\dagger})^{ab}\xi^n \frac{\delta}{\delta J_{lm}} \tag{4.55d}$$

$$\begin{aligned}\mathbb{E} =&(X_\eta)\frac{\delta}{\delta \eta} - \eta^{-1}(X_\eta)\xi^n \frac{\delta}{\delta \xi^n} + \beta^a \eta^{-1}(X_\eta)\xi^n \frac{\delta}{\delta \xi^a} \\ &- \eta^{-1}(X_\eta) g^{\dagger nn}\frac{\delta}{\delta g^{\dagger nn}} - \epsilon\eta^{-3}\left(\beta^a\chi_a - \chi_n\right)(X_\eta)\xi^n \frac{\delta}{\delta g^{\dagger nn}} \\ &- \left(\epsilon\eta^{-3}\beta^b\chi_n - \epsilon\eta^{-3}\beta^b\beta^a\chi_a + \frac{\epsilon}{2}\eta^{-1}\gamma^{ba}\chi_a\right)(X_\eta)\xi^n \frac{\delta}{\delta g^{\dagger bn}} \\ &- \frac{4\epsilon}{\sqrt{\gamma}}(X_\eta)\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right) g^{\dagger an}\xi^n \frac{\delta}{\delta J_{lm}} \\ &- \frac{2\epsilon}{\sqrt{\gamma}}(X_\eta)\left(\gamma_{la}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right) g^{\dagger ab}\xi^n \frac{\delta}{\delta J_{ab}} \\ &+ \eta^{-1}\beta^a (X_\eta) g^{\dagger nn}\frac{\delta}{\delta g^{\dagger an}} + \eta^{-1}(X_\eta)\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right)\frac{\delta}{\delta J_{lm}}\end{aligned} \tag{4.55e}$$

It is easy to check that the boundary one form (4.47) is annihilated by all vertical vector fields (4.55), and it is therefore basic, proving the exactness of the BV-BFV structure and concluding the proof. ✓

It is already clear from this result that the ADM decomposition of space and time makes the BV-BFV structure much better behaved than the block-diagonal EH version of Section 4.2, since the pre-boundary one-form is basic *on the nose*, not needing any correction term. Moreover, this result is far more general.

The explicit expression in a local chart is established by the following result:

Theorem 4.12. *The surjective submersion $\pi_M : \mathcal{F}_{ADM} \longrightarrow \mathcal{F}^{\partial}_{ADM}$ is given by the local expression:*

$$\pi_M : \begin{cases} \mathbf{\Pi}_{lm} &= \frac{\sqrt{\gamma}}{2}\left(\widetilde{J}_{lm} - \widetilde{\gamma}_{lm}\widetilde{\gamma}^{ij}\widetilde{J}_{ij}\right) \\ \boldsymbol{\varphi}_n &= -2\left\{\eta g^{\dagger nn} - \frac{\epsilon}{2}\eta^{-1}\left(\beta^a\chi_a - \chi_n\right)\xi^n\right\} \\ \boldsymbol{\varphi}_a &= 2\,\gamma_{ab}\left\{g^{\dagger bn} + \gamma^{ba}\beta_a g^{\dagger nn} - \frac{\epsilon}{2}\gamma^{ba}\chi_a\xi^n\right\} \\ \boldsymbol{\xi}^b &= \xi^b + \gamma^{ba}\beta_a\xi^n \\ \boldsymbol{\xi}^n &= \eta\,\xi^n \\ \boldsymbol{\gamma}_{ab} &= \gamma_{ab} \end{cases} \tag{4.56}$$

with

$$\begin{aligned} \widetilde{J}_{lm} = \Big\{&\eta^{-1}\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) - \frac{2\epsilon}{\sqrt{\gamma}}\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)g^{\dagger ab}\xi^n \\ &- \frac{4}{\sqrt{\gamma}}\,\epsilon\left(\beta_{(l}\gamma_{m)b} - \frac{1}{d-1}\gamma_{lm}\beta_b\right)g^{\dagger bn}\xi^n - \frac{2\epsilon}{\sqrt{\gamma}}\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_b\beta^b\right)g^{\dagger nn}\xi^n\Big\} \end{aligned}$$

The boundary symplectic structure on the space of boundary fields reads in these coordinates ($\rho = \{n, a\}$):

$$\omega^{\partial} = \epsilon\int_{\partial M} \delta\boldsymbol{\gamma}^{ab}\delta\mathbf{\Pi}_{ab} + \delta\boldsymbol{\xi}^{\rho}\delta\boldsymbol{\varphi}_{\rho}. \tag{4.57}$$

Moreover, the boundary action is given by the expression

$$\begin{aligned} S^{\partial} = &\int_{\partial M}\left\{\frac{\epsilon}{\sqrt{\boldsymbol{\gamma}}}\left(\mathbf{\Pi}^{ab}\mathbf{\Pi}_{ab} - \frac{1}{d-1}\mathbf{\Pi}^2\right) + \sqrt{\boldsymbol{\gamma}}\left(R^{\partial} - 2\Lambda\right) + \epsilon\partial_a\left(\boldsymbol{\xi}^a\boldsymbol{\varphi}_n\right) - \epsilon\boldsymbol{\gamma}^{ab}\boldsymbol{\varphi}_b\partial_a\boldsymbol{\xi}^n\right\}\boldsymbol{\xi}^n \\ &+ \int_{\partial M}\left\{-\partial_c\left(\boldsymbol{\gamma}^{cd}\mathbf{\Pi}_{da}\right) - (\partial_a\boldsymbol{\gamma}^{cd})\mathbf{\Pi}_{cd} + \epsilon\partial_c\left(\boldsymbol{\xi}^c\boldsymbol{\varphi}_a\right)\right\}\boldsymbol{\xi}^a. \end{aligned} \tag{4.58}$$

Proof. Using the vertical vector fields (4.55) to eliminate β_a, χ_ρ and $g^{\dagger ab}$ (see Computation A.3) one finds the section of the symplectic reduction to the space of boundary fields to be

$$\pi: \begin{cases} \widetilde{J}_{lm} & = \eta^{-1}\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) - \frac{2\epsilon}{\sqrt{\gamma}}\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right) g^{\dagger ab}\xi^n \\ & \quad -\frac{4\epsilon}{\sqrt{\gamma}}\left(\beta_{(l}\gamma_{m)b} - \frac{1}{d-1}\gamma_{lm}\beta_b\right) g^{\dagger bn}\xi^n - \frac{2\epsilon}{\sqrt{\gamma}}\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_b\beta^b\right) g^{\dagger nn}\xi^n \\ \widetilde{g}^{\dagger nn} & = \eta g^{\dagger nn} + \frac{\epsilon}{2}\eta^{-1}\left(\chi_n - \beta^a\chi_a\right)\xi^n \\ \widetilde{g}^{\dagger bn} & = g^{\dagger bn} + \gamma^{ba}\beta_a g^{\dagger nn} + \frac{\epsilon}{2}\gamma^{ba}\chi_a\xi^n \\ \widetilde{\xi}^b & = \xi^b + \gamma^{ba}\beta_a\xi^n \\ \widetilde{\xi}^n & = \eta\xi^n \\ \widetilde{\gamma}_{ab} & = \gamma_{ab} \end{cases} \tag{4.59}$$

The boundary one-form α^∂ will be given by the expression

$$\alpha^\partial = \epsilon \int_{\partial M} \left\{ \frac{\sqrt{\widetilde{\gamma}}}{2}\left(\delta\widetilde{\gamma}^{ab}\widetilde{\gamma}_{ab}\widetilde{\gamma}^{lm}\widetilde{J}_{lm} - \delta\widetilde{\gamma}^{lm}\widetilde{J}_{lm}\right) - 2\delta\widetilde{\xi}^n\widetilde{g}^{\dagger nn} + 2\gamma_{ab}\delta\widetilde{\xi}^a\widetilde{g}^{\dagger bn} \right\} \tag{4.60}$$

as it is straightforward to check that $\pi^*\alpha^\partial = \widetilde{\alpha}_{ADM}$. Introducing the new variables $\boldsymbol{\gamma}^{ab} \equiv \widetilde{\gamma}^{ab}$, $\mathbf{\Pi}_{ab} = \frac{\sqrt{\widetilde{\gamma}}}{2}\left(\widetilde{J}_{ab} - \widetilde{\gamma}_{ab}\widetilde{\gamma}^{ij}\widetilde{J}_{ij}\right)$ together with $\boldsymbol{\varphi}_n = -2\widetilde{g}^{\dagger nn}$, $\boldsymbol{\varphi}_a = 2\widetilde{\gamma}_{ab}\widetilde{g}^{\dagger bn}$ and $\boldsymbol{\xi}^\rho = \widetilde{\xi}^\rho$, we can write the symplectic boundary form as:

$$\omega^\partial = \epsilon \int_{\partial M} \delta\boldsymbol{\gamma}^{ab}\delta\mathbf{\Pi}_{ab} + \delta\boldsymbol{\xi}^\rho\delta\boldsymbol{\varphi}_\rho \tag{4.61}$$

and recover expression (4.56) and (4.58) for the projection and the boundary action in the Darboux coordinates.

We would like to compute the cohomological boundary vector field. First of all we must extract the analogous bulk vector field, encoding the equations of motion and the symmetries of the system, using the fundamental formula:

$$\iota_Q\Omega_{BV} = \delta S + \pi^*\alpha^\partial \tag{4.62}$$

A shortcut to do this in the ADM formalism, instead of computing cumbersome integrations by parts, consists in considering the classical Einstein Hilbert action, whose classical vacuum equations of motion are given by

$$\sqrt{\gamma}\left(R_{\mu\nu} - \left(\frac{1}{2}R - \Lambda\right)g_{\mu\nu}\right) \equiv G_{\mu\nu} = 0$$

and to express them using the ADM decomposition. This is done by projecting the above equation on the new field direction, with the help of the Gauss-Codazzi equations and the Ricci equations.

Doing so, one obtains the projection of the relevant Euler Lagrange terms in the ADM formalism, namely:

$$\epsilon G_{\eta} \coloneqq \epsilon\left(\frac{\delta S_{cl}}{\delta\eta}\right) = \sqrt{\gamma}\left(\epsilon\left(R^{\partial} - 2\Lambda\right) + K^2 - K_{ab}K^{ab}\right) \tag{4.63}$$

$$\epsilon G_{\beta_a} \coloneqq \epsilon\left(\frac{\delta S_{cl}}{\delta\beta_b}\right) = 2\gamma^{ba}\left[\partial_c(\sqrt{\gamma}\gamma^{cd}K_{da}) + \frac{1}{2}\partial_a\gamma^{cd}K_{cd} - \sqrt{\gamma}\partial_a K\right] \tag{4.64}$$

$$\epsilon G_{\gamma_{ab}} \coloneqq \epsilon\left(\frac{\delta S_{cl}}{\delta\gamma_{ab}}\right) = \sqrt{\gamma}\left(\partial_n K_{ab} - \beta^k\partial_k K_{ab} - 2K_{k(a}\partial_{b)}(g^{kc}\beta_c)\right) \tag{4.65}$$

Notice that the formula for G_{β_a} is only apparently different from the usual *momentum constraint* that can be found in the literature (see e.g.[24]):

$$\mathcal{H}_c \coloneqq \sqrt{\gamma}\gamma^{ba}\left(\gamma^{cd}\nabla^{\partial}_c K_{da} - \nabla^{\partial}_a K\right)$$

as it can be seen by manipulating the covariant derivatives.

Adding the BV part we have that the derivatives of the action with respect to the new fields read:

$$\begin{aligned}
\left(\frac{\delta S_{BV}}{\delta\eta}\right) &= -2\epsilon\eta\left(\partial_\rho\xi^\rho g^{\dagger nn} + \xi^\rho\partial_\rho g^{\dagger nn} - 2\partial_\rho\xi^n g^{\dagger n\rho}\right)\\
\left(\frac{\delta S_{BV}}{\delta\beta_a}\right) &= 2\epsilon\left(\partial_\rho\xi^\rho g^{\dagger an} + \xi^\rho\partial_\rho g^{\dagger an} - \partial_\rho\xi^a g^{\dagger n\rho} - \partial_\rho\xi^n g^{\dagger a\rho}\right) +\\
&\quad + 2\epsilon\beta^a\left(\partial_\rho\xi^\rho g^{\dagger nn} + \xi^\rho\partial_\rho g^{\dagger nn} - 2\partial_\rho\xi^n g^{\dagger n\rho}\right)\\
\left(\frac{\delta S_{BV}}{\delta\gamma_{ab}}\right) &= \epsilon\left(\partial_\rho\xi^\rho g^{\dagger ab} + \xi^\rho\partial_\rho g^{\dagger ab} - 2\partial_\rho\xi^{(a}g^{\dagger b)\rho}\right) +\\
&\quad - 2\epsilon\beta^a\beta^b\left(\partial_\rho\xi^\rho g^{\dagger nn} + \xi^\rho\partial_\rho g^{\dagger nn} - 2\partial_\rho\xi^n g^{\dagger n\rho}\right)\\
\left(\frac{\delta S_{BV}}{\delta g^{\dagger ab}}\right) &= \epsilon\left(\xi^\rho\partial_\rho\gamma_{ab} + 2\partial_{(a}\xi^n\beta_{b)} + 2\partial_{(a}\xi^c\gamma_{b)c}\right)\\
\left(\frac{\delta S_{BV}}{\delta g^{\dagger na}}\right) &= \epsilon\left(\xi^\rho\partial_\rho\beta_a + \partial_n\xi^n\beta_a + \partial_n\xi^b\gamma_{ab} + \partial_a\xi^n(-\eta^2+\beta_c\beta^c) + \partial_a\xi^b\beta_b\right)\\
\left(\frac{\delta S_{BV}}{\delta g^{\dagger nn}}\right) &= \epsilon\left(\xi^\rho\partial_\rho(-\eta^2+\beta_c\beta^c) + 2\partial_n\xi^n(-\eta^2+\beta_c\beta^c) + 2\partial_n\xi^a\beta_a\right)
\end{aligned}$$

In addition we have:

$$\begin{aligned}
\left(\frac{\delta S_{BV}}{\delta\xi^n}\right) &= \epsilon\left(\partial_n(-\eta^2+\beta_c\beta^c)g^{\dagger nn} + 2\partial_a(-\eta^2+\beta_c\beta^c)g^{\dagger na} + 2\partial_{(a}\beta_{b)}g^{\dagger ab}\right)\\
&\quad + \epsilon\left(2(-\eta^2+\beta_c\beta^c)\partial_n g^{\dagger nn} + 2\beta_a\partial_n g^{\dagger na} + 2(-\eta^2+\beta_c\beta^c)\partial_a g^{\dagger na}\right)\\
&\quad + \epsilon\left(2\beta_{(a}\partial_{b)}g^{\dagger ab} - J_{ab}g^{\dagger ab}\right) + \xi^\rho\partial_\rho\chi_n + \partial_\rho\xi^\rho\chi_n + \partial_n\xi^\rho\chi_\rho\\
\left(\frac{\delta S_{BV}}{\delta\xi^a}\right) &= 2\epsilon\left(\partial_n\beta_a g^{\dagger nn} + J_{ab}g^{\dagger nb} + \partial_b\beta_a g^{\dagger nb} + \partial_{(b}\gamma_{c)a}g^{\dagger bc}\right)\\
&\quad + 2\epsilon\left(\beta_a\partial_n g^{\dagger nn}\beta_a\partial_b g^{\dagger nb} + \gamma_{ab}\partial_n g^{\dagger nb} + \gamma_{a(b}\partial_{c)}g^{\dagger bc} - \frac{1}{2}\partial_a\gamma_{cd}g^{\dagger cd}\right)\\
&\quad - \epsilon\partial_a\left(-\eta^2+\beta_c\beta^c\right)g\dagger^{nn} - 2\epsilon\partial_a\beta_c c g^{\dagger cn} + \partial_\rho\xi^\rho\chi_a + \xi^\rho\partial_\rho\chi_a + \partial_a\xi^\rho\chi_\rho\\
\left(\frac{\delta S_{BV}}{\delta\chi_\mu}\right) &= \xi^\rho\partial_\rho\xi^\mu
\end{aligned}$$

Now we would like to use these derivatives to write down the components of the bulk vector field Q, by imposing (4.62). We are still using the antifields $g^{\dagger\mu\nu}$ and therefore we have to collect the terms as follows:

$$
\begin{aligned}
(Q_{g^\dagger})^{nn} =& -\frac{1}{2}\eta^{-1}\epsilon\left(\frac{\delta S}{\delta\eta}\right)\\
=& -\frac{1}{2}\eta^{-1}\epsilon G_\eta + \left(\partial_\rho\xi^\rho g^{\dagger nn} + \xi^\rho\partial_\rho g^{\dagger nn} - 2\partial_\rho\xi^n g^{\dagger n\rho}\right)\\
(Q_{g^\dagger})^{na} =& \frac{\epsilon}{2}\left(\frac{\delta S}{\delta\beta_a}\right) - \beta^a(Q_{g^\dagger})^{nn}\\
=& \epsilon G_{\beta_a} + \frac{\epsilon}{2}\eta^{-1}\beta^a G_\eta + \left(\partial_c\xi^c g^{\dagger an} + \xi^\rho\partial_\rho g^{\dagger an} - \partial_\rho\xi^a g^{\dagger n\rho} - \partial_c\xi^n g^{\dagger ac}\right)\\
(Q_{g^\dagger})^{ab} =& \epsilon\left(\frac{\delta S}{\delta\gamma_{ab}}\right) + \beta^a\beta^a(Q_{g^\dagger})^{nn}\\
=& \epsilon G_{\gamma_{ab}} - \frac{\epsilon}{2}\eta^{-1}\beta^a\beta^b G_\eta + \left(\partial_\rho\xi^\rho g^{\dagger ab} + \xi^\rho\partial_\rho g^{\dagger ab} - 2\partial_\rho\xi^{(a} g^{\dagger b)\rho}\right)\\
(Q_\gamma)_{ab} =& \epsilon\left(\frac{\delta S_{BV}}{\delta g^{\dagger ab}}\right)\\
=& \left(\xi^\rho\partial_\rho\gamma_{ab} + 2\partial_{(a}\xi^n\beta_{b)} + 2\partial_{(a}\xi^c\gamma_{b)c}\right)\\
(Q_\beta)_a =& \epsilon\left(\frac{\delta S_{BV}}{\delta g^{\dagger an}}\right)\\
=& \left(\xi^\rho\partial_\rho\beta_a + \partial_n\xi^n\beta_a + \partial_n\xi^b\gamma_{ab} + \partial_a\xi^n(-\eta^2 + \beta_c\beta^c) + \partial_a\xi^b\beta_b\right)\\
(Q_\eta) =& -\frac{\epsilon}{2}\eta^{-1}\left(\frac{\delta S_{BV}}{\delta g^{\dagger nn}}\right) + \epsilon\eta^{-1}\beta^a\left(\frac{\delta S_{BV}}{\delta g^{\dagger na}}\right) - \frac{\epsilon}{2}\eta^{-1}\beta^a\beta^b\left(\frac{\delta S_{BV}}{\delta g^{\dagger ab}}\right)\\
=& \left(\xi^\rho\partial_\rho\eta + \partial_n\xi^n\eta - \eta\beta^a\partial_a\xi^n\right)\\
(Q_\xi)^\rho =& \left(\frac{\delta S_{BV}}{\delta\chi_\rho}\right)\\
(Q_\chi)_n =& \left(\frac{\delta S_{BV}}{\delta\xi^n}\right)\\
(Q_\chi)_a =& \left(\frac{\delta S_{BV}}{\delta\xi^a}\right)
\end{aligned}
$$

Now, the bulk Q vector field is extended to the normal jets when projected to the pre-boundary vector field $\widetilde{Q}$:

$$
(\widetilde{Q}J)_{\mu\nu} = (\partial_n(Qg_{\mu\nu}))\Big|_{\partial M} = \left(\partial_n\xi^\rho\partial_\rho g_{\mu\nu} + \xi^\rho\partial_\rho\partial_n g_{\mu\nu} + 2\partial_{(\mu}\partial_n\xi^\rho g_{\nu)\rho} + 2\partial_{(\mu}\xi^\rho\partial_n g_{\nu)\rho}\right)\Big|_{\partial M}
$$

of which we will only need

$$(\widetilde{Q}J)_{ab} = \epsilon\left(\partial_n\xi^\rho\partial_\rho\gamma_{ab} + \xi^\rho\partial_\rho J_{ab} + 2\partial_{(a}\partial_n\xi^\rho g_{b)\rho} + 2\partial_{(a}\xi^c J_{b)c} + 2\partial_{(a}\xi^n\partial_n\beta_{b)}\right)$$

so that the full pre-boundary vector field reads:

$$\begin{aligned}\widetilde{Q} =&(\widetilde{Q}_\eta)\frac{\delta}{\delta\eta} + (\widetilde{Q}_\beta)_a\frac{\delta}{\delta\beta_a} + (\widetilde{Q}_\gamma)_{ab}\frac{\delta}{\delta\gamma_{ab}} + (\widetilde{Q}_{g^\dagger})^{ab}\frac{\delta}{\delta g^{\dagger ab}} + 2(\widetilde{Q}_{g^\dagger})^{na}\frac{\delta}{\delta g^{\dagger na}}\\ &+(\widetilde{Q}_{g^\dagger})^{nn}\frac{\delta}{\delta g^{\dagger nn}} + (\widetilde{Q}_\xi)^n\frac{\delta}{\delta\xi^n} + (\widetilde{Q}_\xi)^a\frac{\delta}{\delta\xi^a} + (\widetilde{Q}_\chi)_\mu\frac{\delta}{\delta\chi_\mu} + (\widetilde{Q}_J)_{lm}\frac{\delta}{\delta J_{lm}}\end{aligned}$$

Now there are two equivalent ways to obtain the rest of the boundary structure: either we compute the explicit projection of the Q vector field (see below) or we consider the following simplifying technique.

Produce a degree one function via[40]:

$$\widetilde{S} = \iota_{\widetilde{Q}}\iota_{\widetilde{E}}\widetilde{\omega}$$

where $\widetilde{E}$ is the Euler vector field on the space of pre-boundary fields, i.e.:

$$\widetilde{E} = \int\limits_{\partial M}\xi^\rho\frac{\delta}{\delta\xi^\rho} - g^{\dagger\mu\nu}\frac{\delta}{\delta g^{\dagger\mu\nu}} - 2\chi_\rho\frac{\delta}{\delta\chi_\rho}$$

Then the true boundary action S^∂ is such that $\widetilde{S} = \pi^* S^\partial$ for degree reasons and the surjectivity of the surjection π_M, which factors through $\widetilde{\mathcal{F}}_{ADM}$, and Q^∂ is its Hamiltonian vector field. The boundary action is then found to be:

$$\begin{aligned}S^\partial =&\int\limits_{\partial M}\left\{\sqrt{\widetilde{\gamma}}\left(\frac{\epsilon}{4}\left(\widetilde{J}^{ab}\widetilde{J}_{ab} - \widetilde{J}^2\right) + R^\partial - \Lambda\right) - 2\epsilon\partial_a\left(\widetilde{\xi}^a\widetilde{g}^{\dagger nn}\right) - 2\epsilon\widetilde{g}^{\dagger na}\partial_a\widetilde{\xi}^n\right\}\widetilde{\xi}^n\\ &+\int\limits_{\partial M}\left\{\sqrt{\widetilde{\gamma}}\partial_a\widetilde{J} - \partial_c\left(\sqrt{\widetilde{\gamma}}\widetilde{\gamma}^{cd}\widetilde{J}_{da}\right) - \frac{\sqrt{\widetilde{\gamma}}}{2}(\partial_a\widetilde{\gamma}^{cd})\widetilde{J}_{cd} + 2\epsilon\partial_c\left(\widetilde{\xi}^c\widetilde{g}^{\dagger nb}\widetilde{\gamma}_{ba}\right)\right\}\widetilde{\xi}^a \qquad (4.66)\end{aligned}$$

where by $\widetilde{J}$ we denote the trace $\widetilde{\gamma}^{ab}\widetilde{J}_{ab}$.

Even though the computations are simpler this way, it might be worthwhile to outline the alternative procedure as well. This can be found in Computation A.4. ✓

Remark 4.13. *Notice that this calculation was simplified by the ansatz given by the previous result: Theorem 4.4. Both the boundary action and symplectic form agree with what we had computed previously, as was also discussed in subsection 4.4.1. One interesting thing to notice, is that we do not need explicitly to assume that g_{na} vanish in a neighbourhood of the boundary. This is possibly explained by saying that the bare assumption of $g_{na}|_{\partial M} = 0$ on the boundary is not really appropriate. In subsection 4.4.1 we will see explicitly how to understand the equivalence of this induced BFV structure to the one obtained assuming a block diagonal metric tensor, with vanishing off diagonal normal jets on the boundary.*

◇

This result is a clean first step in the direction of BV-BFV quantisation of General Relativity as proposed by CMR in[5]. It states the compatibility of bulk and boundary structures, in relation with the symmetries. Notice that the BV-BFV axioms in Definition 2.5 need not be satisfied by a generic gauge theory and the statement is therefore nontrivial. Arguable as it might be to consider gauge theories with this property to be somehow *better quantisable*, it provides nevertheless a clear mean of distinction between different variational problems describing the same equations of motion (see Chapter 5 for a comparison with the Palatini-Holst formulation of GR, and Chapter 6 for BF-like theories of gravity).

The machinery is able to handle a more complex and sophisticated set of data, than the standard canonical analysis. When a theory on the boundary is induced, it encodes a number of characteristic features packing up relevant data in a very efficient way. As we will see in Section 4.4.2, the piece of data that carries all the relevant information on the boundary is, not surprisingly, the boundary action.

Finally, recall that in the $1+1$ dimensional case it is known that the Einstein equations are trivial, and the symmetry distribution has to be amended to take conformal transformations into account. The critical dimension $d = 1$ is however marked out by the equations for the kernel of the pre-boundary 2-form $\widetilde{\omega}$, both in the classical and the BV-extended case (cf. Theorem 4.11 and Proposition 4.8), confirming that the strategy has to be altered to analyse this specific example.

4.4.1 Recovering adapted coodinates

Using only the $\mathbb{B}$ vertical vector fields in (4.55) it is possible to set the β_a fields to zero. This means that it is possible to use some diffeomorphism (i.e. partially choose a gauge) to put the metric tensor in the block diagonal form (4.8) without affecting the canonical structure.

This is not only true for the classical theory, but as we will see in the following it carries the BV structure along in a consistent fashion.

Theorem 4.14. *There is a diffeomorphism of presymplectic manifolds*

$$\phi : (\widetilde{\mathcal{F}}^{B}_{EH}, \delta\widetilde{\alpha}_{EH}) \longrightarrow (\mathcal{F}_{red}, \delta\alpha_{red}) \tag{4.67}$$

where $\pi_{red} \colon \widetilde{\mathcal{F}}_{ADM} \longrightarrow \mathcal{F}_{red}$ *is a surjective submersion and* $\widetilde{\alpha}_{EH}$ *is given by Equation* (4.9) *.*

Proof. The proof goes through by only quotienting the span of the $\mathbb{B}$ vector field in the kernel of $\delta\widetilde{\alpha}$. This induces a projection to an intermediate space of ADM boundary fields $\pi_{\mathrm{red}} \colon \widetilde{\mathcal{F}}_{ADM} \longrightarrow \mathcal{F}_{\mathrm{red}}$. Solving the straightforward differential equations coming from the explicit expression (4.55c) (c.f. Computation A.3), it reads:

$$\pi_{\mathrm{red}} \colon \begin{cases} \overline{J}_{lm} = J_{lm} - 2\nabla_{(l}\beta_{m)} - \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right)\left(g^{\dagger an} + \gamma^{ac}\beta_c g^{\dagger nn}\right)\xi^n \\ \overline{g}^{\dagger nn} = g^{\dagger nn} + \frac{\epsilon}{2}\eta^{-2}\gamma^{ab}\beta_a\chi_b\xi^n \\ \overline{g}^{\dagger an} = g^{\dagger an} + \gamma^{ab}\beta_b g^{\dagger nn} \\ \overline{\xi}^a = \xi^a + \gamma^{ab}\beta_b\xi^n \\ \overline{\xi}^n = \xi^n \\ \overline{\gamma}_{ab} = \gamma_{ab} \end{cases} \tag{4.68}$$

and it is a simple check to show that $\widetilde{\alpha}_{ADM} = \pi^*_{\mathrm{red}}\alpha_{\mathrm{red}}$ if we set

$$\alpha_{\mathrm{red}} = \int_{\partial M} \frac{\sqrt{\gamma}}{2}\eta^{-1}\left(\delta\overline{\gamma}^{ab}\overline{\gamma}_{ab}\overline{\gamma}^{lm}\overline{J}_{lm} - \delta\overline{\gamma}^{lm}\overline{J}_{lm}\right) - 2\int_{\partial M}\eta^2\delta\overline{\xi}^n\overline{g}^{\dagger nn} + 2\int_{\partial M}\overline{\gamma}_{ab}\delta\overline{\xi}^a\overline{g}^{\dagger bn} \tag{4.69}$$

The map ϕ is then simply given by the obvious assignment of homologous coordinates in a local chart, with the prescription$\phi^*(\eta^2) = g_n n$.

Now, the pre-boundary structure $(\mathcal{F}_{\text{red}}, \omega_{\text{red}} = \delta\alpha_{\text{red}})$ is equivalent to the one we would find by taking the E.H. action and assuming that $g_{\mu\nu}$ is block diagonal. As a matter of fact the resulting boundary one-forms differ from one another by an exact term $2\delta(\sqrt{\gamma}\gamma^{ab}\overline{J}_{ab})$ that pulls back to the usual Gibbons-Hawking-York boundary term $2\delta(\sqrt{\gamma}K)$ plus the extra term δF that depends on the ghost fields and fixes the projectability of the boundary one form in the Einstein Hilbert formalism (c.f. Theorem 4.2). More formally we have that

$$\widetilde{\alpha}_{EH} = \phi^*(\alpha_{\text{red}} + 2\delta(\sqrt{\gamma}\gamma^{ab}\overline{J}_{ab})) \tag{4.70}$$

and the pullback of the correction term is precisely what one would expect:

$$2\pi^*_{\text{red}}\delta(\sqrt{\gamma}\gamma^{ab}\overline{J}_{ab}) = -2\delta(\sqrt{\gamma}K) - \delta F \tag{4.71}$$

✓

This means that, had we taken into account the Gibbons-Hawking-York term in computing the boundary structure induced by the Eistein Hilbert action, we would have found the additional exact term $-2\delta(\sqrt{\gamma}K)$ in the pre-boundary one-form (4.9), closing the circle.

4.4.2 Constraints algebra on the boundary

As we already announced, from the boundary action (4.58) it is possible to read the constraint structure of canonical gravity. As a matter of fact, the degree zero (ghost number, gh) part of the derivatives $\frac{\delta S^\partial}{\delta\xi^\mu}$ reads

$$\left.\frac{\delta S^\partial}{\delta\xi^n}\right|_{gh=0} = \frac{\epsilon}{\sqrt{\gamma}}\left(\boldsymbol{\Pi}^{ab}\boldsymbol{\Pi}_{ab} - \frac{1}{d-1}\boldsymbol{\Pi}^2\right) + \sqrt{\gamma}\left(R^\partial - 2\Lambda\right) \equiv \mathcal{H} \tag{4.72}$$

$$\left.\frac{\delta S^\partial}{\delta\xi^a}\right|_{gh=0} = -\partial_c\left(\boldsymbol{\gamma}^{cd}\boldsymbol{\Pi}_{da}\right) - (\partial_a\boldsymbol{\gamma}^{cd})\boldsymbol{\Pi}_{cd} \equiv \mathcal{H}_a \tag{4.73}$$

which are the symplectic-reduced versions of the standard constraints (4.63) and (4.64) respectively.

On the other hand, the residual gauge symmetries can be found by computing the relative components of the boundary cohomological vector field Q^∂, using the fact that $\iota_{Q^\partial}\omega^\partial = \delta S^\partial$:

$$
\begin{aligned}
(Q^\partial)_{\boldsymbol{\xi}^n} &= (\boldsymbol{\xi}^c\partial_c\boldsymbol{\xi}^n)\,\frac{\delta}{\delta\boldsymbol{\xi}^n}\\
(Q^\partial)_{\boldsymbol{\xi}^a} &= \left(\boldsymbol{\xi}^n\boldsymbol{\gamma}^{ab}\partial_b\boldsymbol{\xi}^n + \boldsymbol{\xi}^c\partial_c\boldsymbol{\xi}^a\right)\frac{\delta}{\delta\boldsymbol{\xi}^a}\\
(Q^\partial)_{\boldsymbol{\gamma}_{ab}} &= \left(\boldsymbol{\xi}^n\frac{2}{\sqrt{\boldsymbol{\gamma}}}(\boldsymbol{\Pi}_{ab} - \frac{\boldsymbol{\gamma}_{ab}}{d-1}\boldsymbol{\Pi}) + \boldsymbol{\xi}^c\partial_c\boldsymbol{\gamma}_{ab} + 2\partial_{(a}\boldsymbol{\xi}^c\boldsymbol{\gamma}_{b)c}\right)\frac{\delta}{\delta\boldsymbol{\gamma}_{ab}}
\end{aligned}
$$

It is interesting to notice that the symmetries above are a corrected version of the usual gauge symmetry for a d-dimensional metric on the boundary under the action of boundary diffeomorphisms $\boldsymbol{\xi}^\partial \in T[1]\partial M$. In fact they can be compactly rewritten as

$$
(Q^\partial)_{\boldsymbol{\gamma}} = \boldsymbol{\xi}^n\frac{2}{\sqrt{\boldsymbol{\gamma}}}(\boldsymbol{\Pi} - \frac{\boldsymbol{\gamma}}{d-1}\mathrm{Tr}\boldsymbol{\Pi}) + L_{\boldsymbol{\xi}^\partial}\boldsymbol{\gamma} \tag{4.74}
$$

$$
(Q^\partial)_{\boldsymbol{\xi}^\partial} = \boldsymbol{\xi}^n\boldsymbol{\gamma}^{-1}\nabla\boldsymbol{\xi}^n + \frac{1}{2}[\boldsymbol{\xi}^\partial,\boldsymbol{\xi}^\partial] \tag{4.75}
$$

$$
(Q^\partial)_{\boldsymbol{\xi}^n} = L_{\boldsymbol{\xi}^\partial}\boldsymbol{\xi}^n \tag{4.76}
$$

This means that they do not manifestly show a Lie algebra behaviour and the *structure functions* depend on $\boldsymbol{\gamma}^{-1}$. Yet the boundary BFV action (4.58) is at most linear in the antighosts $\boldsymbol{\varphi}$. This is in agreement with the observations in[25]. The BFV formalism provides for a cohomological resolution of symmetry-invariant coisotropic submanifolds[47,48,16,36,35], and in this case of the constraint submanifold of canonical gravity, modulo residual gauge symmetry.

The (cohomological) description of the the canonical, constrained phase space for General Relativity is then obtained from a simple variational problem in the bulk. This encompasses a number of classical results in the field while clarifying related issues at the same time. Moreover, we stress that on top of obtaining the expected BFV resolution of the canonical structure on the boundary, we are able to establish a connection with the boundary data through the explicit projection π, and the fundamental equation $\iota_Q\Omega = \delta S^{BV}_{ADM}\alpha^\partial$.

This is the starting point for the BV-BFV programme to quantisation of gauge theories on manifolds with boundary.

4.4.3 Extension to the boundary of the boundary

We would like to use our boundary action now as the new input for a theory on a manifold with boundary, and therefore we will ask $\partial\partial M \neq \emptyset$. Moreover, since we are left with a gravity-like term $\int_{\partial M} \sqrt{\gamma} R^{\partial} \xi^n$ in the boundary action, we will perform the ADM decomposition in a neighborhood of the boundary of the boundary. To do this we will require $\partial\partial M$ to have only light-like isolated points if $\epsilon = -1$, or no condition at all if $\epsilon = 1$ (euclidean boundary).

To fix the notation we will have

$$\begin{aligned} \widetilde{\gamma}_{ab} &= \underline{\phi}^{\frac{1-\epsilon}{2}} \begin{pmatrix} +\epsilon\alpha^2 + \mathsf{b}_i \mathsf{b}^i & \mathsf{b}_i \\ \mathsf{b}_j & h_{ij} \end{pmatrix} \\ \widetilde{\gamma}^{ab} &= \underline{\phi}^{\frac{1-\epsilon}{2}} \alpha^{-2} \begin{pmatrix} +\epsilon & \mathsf{b}^i \\ \mathsf{b}^j & \alpha^2 h^{ij} - \mathsf{b}^i \mathsf{b}^j \end{pmatrix} \end{aligned} \tag{4.77}$$

with roman indices $\{a, b, c, d, e, f, l, m, n, p, q\}$ denoting boundary directions and $\{i, j, k, l, r, s, w, v, u\}$ denoting boundary of the boundary directions. Notice that $\phi := \underline{\phi}^{\frac{1-\epsilon}{2}} = 1$ when $\epsilon = 1$ since there is no residual signature to be accounted for: the boundary metric has euclidean signature. In the case $\epsilon = -1$ we have $\phi = \underline{\phi} = +1$ when the new transversal direction $x^{\underline{n}}$ is timelike and $\phi = \underline{\phi} = -1$ when it is not.

Then the ADM decomposition of ∂M yields

$$\begin{aligned} \sqrt{|\widetilde{\gamma}|_\epsilon} R^{\partial} \widetilde{\xi}^n =& \alpha \sqrt{|h|_\phi} \left(\phi \left(H_{ij} H^{ij} - H^2 \right) \right) \widetilde{\xi}^n + \alpha \sqrt{|h|_\phi} R^{\partial\partial} \widetilde{\xi}^n \\ & - 2\partial_{\underline{n}} \left(\sqrt{|h|_\phi} H \right) \widetilde{\xi}^n + 2\partial_j \left(\sqrt{|h|_\phi} (H \mathsf{b}^j - h^{ji} \partial_i \alpha) \right) \widetilde{\xi}^n \end{aligned} \tag{4.78}$$

where again we highlighted the fact that one might consider taking the absolute value of the determinants of the various metrics involved, according to the spacetime signature and the values of ϵ and ϕ. We will drop this notation from now on. The tensor H_{ij} is the extrinsic curvature of the boundary of the boundary and reads

$$H_{ij} = \frac{1}{2}\alpha^{-1}(2\nabla^{\partial\partial}_{(i}\mathsf{b}_{j)} - \partial_{\underline{n}}h_{ij}), \quad H = h^{ij}H_{ij}$$

with $\nabla^{\partial\partial}$ being the Levi-Civita connection on $\partial\partial M$ w.r.t. the induced metric h_{ij}.

Notice that expression (4.78) differs from the previous ADM Lagrangian in that we cannot neglect the total derivatives anymore, owing to the presence of the ghost fields $\widetilde{\xi}^{\rho}$.

Claim 4.15. *The data $(\mathcal{F}^{\partial}_{\partial M}, S^{\partial}, Q^{\partial}, \omega^{\partial})$ on the boundary ∂M induces the data of a surjective submersion*

$$\widetilde{\pi}_{\partial M} \colon \mathcal{F}^{\partial}_{\partial M} \longrightarrow \widetilde{\mathcal{F}}^{\partial}_{\partial M} \tag{4.79}$$

with $\widetilde{\mathcal{F}}^{\partial}_{\partial M}$ the space of restrictions of the fields in $\mathcal{F}^{\partial}_{\partial M}$ and their jets to $\partial\partial M$, together with a pre symplectic form $\widetilde{\omega}^{\partial}$ on $\widetilde{\mathcal{F}}^{\partial}_{\partial M}$. Performing symplectic reduction and denoting $\mathcal{F}^{\partial\partial}_{\partial\partial M} := \underline{\widetilde{\mathcal{F}}^{\partial}_{\partial M}}$ we obtain the data $(\mathcal{F}^{\partial\partial}_{\partial\partial M}, S^{\partial\partial}, Q^{\partial\partial}, \omega^{\partial\partial})$, with $[Q^{\partial\partial}, Q^{\partial\partial}] = 0$, $\iota_{Q^{\partial\partial}}\omega^{\partial\partial} = \delta S^{\partial\partial}$ and $\pi_{\partial M} \colon \mathcal{F}^{\partial}_{\partial M} \longrightarrow \mathcal{F}^{\partial\partial}_{\partial\partial M}$ a surjective submersion.

5

General Relativity in the tetrad formalism

The main difference between the Einstein Hilbert theory of gravity and other theories for elementary forces is that the former does not strictly look like a gauge theory. In other words, it is not a theory of connections, unlike electromagnetism or chromodynamics or the standard model of particle physics. Nevertheless, there is a different formulation of GR as a gauge theory, in the sense that there is an action functional that produces the same physical data as EH, and yet it is different from a structural point of view.

Consider the principal fiber bundle of (co-)frames on M, with the natural action of $SO(3,1)$ on it. The dynamical fields are the co-frame field $e\colon TM \longrightarrow \mathcal{V}$, which we require to be an isomorphism with $\mathcal{V}$ being a vector bundle on M whose fiber is the Pseudo-Riemannian vector space (V,η), and a connection ω in the principal $SO(3,1)$ bundle $\omega|_U\colon U \longrightarrow \mathfrak{so}(3,1)$. The vector bundle $\mathcal{V}$ is the associated bundle to the bundle of $SO(3,1)$ frames, and it is isomorphic to TM, while η is a pseudo-Riemannian metric on the fibers.

Remark 5.1. *Notice that this implicitly requires that we fix a reference metric on TM to break down the group from $GL(4)$ to $SO(4,1)$, as TM is naturally the associated vector bundle to the principal bundle of $GL(4)$-frames. This metric has no relevance whatsoever in further computations and it is* not *regarded as a background metric.*

◊

Notice that one can consider $\bigwedge^2 V$-valued connections, using the isomorphism with the Lie algebra:

$$\eta\colon \bigwedge^2 V \xrightarrow{\sim} \mathfrak{so}(3,1) \tag{5.1}$$

which maps the basis $e_i \wedge e_j$ to the basis of matrices t^i_j of the Lie algebra by raising/lowering indices. Notice that we require ω to be η-compatible as a connection in $\mathcal{V}$. This implies that, when pulled back to a connection on the space-time using a non-degenerate e, we have that the curvature F_ω is antisymmetric in the internal indices and the connection turns out to be torsion free, in agreement with the general assumption of General Relativity, when spin matter is not coupled.

In this setting the theory is fully described by the *Palatini action*[6]:

$$S_{\text{Pal}} = \mathrm{Tr} \int_M e \wedge e \wedge F_\omega + \Lambda e^4 \tag{5.2}$$

By $\mathrm{Tr}\colon \bigwedge^4 V \longrightarrow \mathbb{R}$ we denote the volume form in $\bigwedge^4 V$ normalised such that $\mathrm{Tr}(u_i \wedge u_j \wedge u_k \wedge u_l) = \epsilon_{ijkl}$, where $\{u_i\}_{i=1}^4$ is an η-orthonormal basis in V, and Λ is the cosmological constant. The Euler Lagrange equations for the associated variational problem yield at the same time the Einstein's equation, and the compatibility condition of ω and e. The latter condition, together with the zero torsion requirement on the connection, requires that covariant derivatives be taken w.r.t. the Levi Civita connection. More explicitly, the Euler Lagrange equations for (5.2) read

$$d_\omega(e \wedge e) = 0 \Leftrightarrow d_\omega e = 0 \tag{5.3}$$

$$e \wedge F_\omega = 0 \Leftrightarrow [F_\omega]^{ij}_{ik} = 0 \tag{5.4}$$

with $i, j, k = 1 \dots 4$ indices in V. Notice that the double implication holds when we assume that e is an isomorphism, and in this case Equations (5.3) and (5.4) describe the same geometro-dynamics of the Einstein-Hilbert variational problem.

Remark 5.2. *Observe that, strictly speaking, the two theories are equivalent only when condition* (5.3) *is used to rewrite the Palatini-Holst action in terms of the curvature of Levi-Civita connection. Equivalence* on shell *of the effective action does not ensure that the rest of the relevant structure carries through from one description to another, as we will see. One should compare this with the option of dynamically implementing the mentioned constraint (cf. Theorem 5.12, and Section 5.4).*

$\diamond$

The *minimality* of the theory has been analysed by many authors, mainly in relation to the canonical formulation of Loop Quantum Gravity (LQG). As it happens, it is shown in[41] how one can easily consider the most general theory of gravity of this kind to be a topological modification of the Palatini action*. This modification goes under the name of Holst action[7], and it is still possible to add a finite number of boundary corrections. The most general shape of a Palatini-like theory of gravity is indeed given by

$$\begin{aligned} S_{\text{tot}} = & \int_M \mathrm{Tr}\,[\alpha_1(e \wedge e \wedge F_\omega) + \alpha_2 \star (e \wedge e) \wedge F_\omega] + \alpha_6(\Lambda)\mathrm{Tr}(e^4) \\ & + \int_M (\alpha_3 - i\alpha_4) dL_{CS}(\omega^-) + (\alpha_3 + i\alpha_4) dL_{CS}(\omega^+) + \alpha_5 d(d_\omega \star e \wedge e) \end{aligned} \tag{5.5}$$

A few comments are in order. The trace is induced by the orientation in V and we used the internal Hodge $\star$. The α_1, α_2 terms, with respect to a basis $\{u_i\}_{i=1}^4$ explicitly read:

$$\left(\alpha_1 \epsilon_{ijkl} e^i \wedge e^j \wedge F^{kl} + \alpha_2 e^i \wedge e^j \wedge F^{kl} \eta_{ik}\eta_{jl}\right) \in \Omega^{\text{top}}(M)$$

with η_{ij} being the given Lorenzian inner product, which is diagonal $\eta = \mathrm{diag}\{1, 1, 1, -1\}$ with respect to the basis $\{u_i\}_{i=1}^4$. This will be interpreted later on in Lemma 5.3 as a volume form in the top exterior power.

*The term topological being referred to the fact that it does not affect the dynamics.

The coefficient $\alpha_6(\Lambda)$ is proportional to the cosmological constant, whereas the components $\omega^{\pm}$ are respectively the (anti-)selfdual parts of the connection ω and the functionals L_{CS} are Chern-Simons forms. It can be seen[41], that the total derivative terms in (5.5) unfold to yield topological terms proportional to the Pontrjagin, Euler and Nieh-Yan classes.

Notice that the terms from α_3 to α_5 are relevant neither for the dynamical theory nor for the boundary structure. As a matter of fact they arise as exact corrections to the boundary 1-form, and therefore they will only induce canonical transformations in the (pre-) symplectic space of (pre-) boundary fields.

5.1 Palatini-Holst action

The α_2 term in (5.5) will have a non trivial effect in both the bulk and the boundary theory, and we shall retain it in what follows. The other topological boundary terms will be discarded in this analysis. In doing this we will rename our parameters as it is customary in the literature, namely by introducing the so-called Barbero-Immirzi[20,21] parameter $\gamma \in \mathbb{R}\backslash\{0\}$ and considering the (real) Holst action

$$S_{\text{Holst}} = \int_M \left(\text{Tr}(e \wedge e \wedge F_\omega) + \frac{1}{\gamma} \star (e \wedge e) \wedge F_\omega\right) + \alpha_6(\Lambda)\text{Tr}(e^4) \tag{5.6}$$

This theory is equivalent to the Palatini action only in the limit $\gamma \to \infty$, but it still describes the same (Einstein) equations, up to a rescaling factor γ. However, this apparently harmless shift turned out to be a source of *ambiguity* in the quantisation scheme[21,22].

The parameter itself was first introduced by Barbero[20] to generalise the construction of Ashtekar canonical quantum gravity[19] in terms of a real $SU(2)$ connection, later improved by Immirzi[21]. Ashtekar's formulation dealt with complex selfdual connections instead, which are recovered by fixing $\gamma = i$. This complexification can be avoided at the price of introducing some parameter-dependend canonical transformation of the phase space, mapping the Palatini fields to some γ-rescaled fields. This parameter dependence has been observed to be *non-quantisable*[22], in the sense that it cannot be unitarily implemented, which means that the quantisation of the theory without the γ parameter is not unitarily equivalent to the scaled one.

For the time being we will be interested in the semiclassical structure only, and the generalisation introduced by the Barbero-Immirzi parameter will be taken into account only for completeness. The standard Palatini descripion is obtained in the limit $\gamma \to \infty$ and, as we will see, as long as the classical theory is concerned, the boundary structure will not present any unexpected behaviour.

The introduction of the Barbero-Immmirzi parameter changes the pairing structure between $e \wedge e$ and F_ω. This can be understood in the following sense:

Lemma 5.3. *Consider the pseudo-Euclidean vector space* (V, η) *and the maps*

$$\widetilde{T}_\gamma \colon \begin{array}{ccc} \bigwedge^2 V & \longrightarrow & \bigwedge^2 V \\ \alpha & \longmapsto & \alpha + \frac{1}{\gamma} \star \alpha \end{array} \tag{5.7}$$

$$\hat{T}_\gamma \colon \begin{array}{ccc} \bigwedge^4 V & \longrightarrow & \mathbb{R} \\ \alpha \wedge \beta & \longmapsto & \mathrm{Tr}[\widetilde{T}_\gamma(\alpha) \wedge \beta] \end{array} \tag{5.8}$$

$$T_\gamma \colon \begin{array}{ccc} \bigwedge^2 V & \longrightarrow & \bigwedge^2 V^* \\ \alpha & \longmapsto & \hat{T}_\gamma(\alpha \wedge \cdot) \end{array} \tag{5.9}$$

for $\gamma \in \mathbb{R}\backslash\{0\}$. *Then all of the above are isomorphisms for all* $\gamma \neq \pm i$, *and they define a non-degenerate symmetric inner product in* $\bigwedge^2 V$. *Moreover,* $\widetilde{T}_\gamma$ *is symmetric with respect to the inner product induced by the trace, i.e.*

$$\mathrm{Tr}[\widetilde{T}_\gamma(\alpha) \wedge \beta] = \mathrm{Tr}[\alpha \wedge \widetilde{T}_\gamma(\beta)]$$

Remark 5.4. *Observe that we asked* $\gamma \in \mathbb{R}$, *so the condition* $\gamma \neq \pm i$ *should be automatic. As we already mentioned, one can make sense of the formalism in the complexification of* $\mathfrak{so}(3,1)$, *leading to the Ashtekar formulation of Palatini gravity, when global hyperbolicity and possibly a time gauge are enforced.* ◊

Proof. Consider the linear map $\hat{T}_\gamma \colon \bigwedge^4 V \longrightarrow \mathbb{R}$ and evaluate it on the basis $u_i \wedge u_j \wedge u_k \wedge u_l$, where $\{u_i\}$ is the basis of V that diagonalises η. It takes the value

$$\hat{T}_\gamma[u_i \wedge u_j \wedge u_k \wedge u_l] = \left[\epsilon_{ijkl} + \frac{2}{\gamma}\eta_{i(k}\eta_{l)j}\right]. \tag{5.10}$$

as can be easily checked by the fact that

$$\star u_i \wedge u_j = \frac{1}{2}\epsilon_{ijkl}\eta^{k\langle m}\eta^{n\rangle l}u_m \wedge u_n$$

and

$$\mathrm{Tr}(u_i \wedge u_j \wedge u_k \wedge u_l) = \epsilon_{ijkl}.$$

If we relabel the basis indices in $\bigwedge^2 V$ *via* $(12, 13, 14, 23, 24, 34) \to (\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6})$. It is simple to gather that the representative matrix of T_γ with respect to the canonical bases in $\bigwedge^2 V$ and $\bigwedge^2 V^*$, relabeled as just mentioned, is given by

$$[T_\gamma] = \begin{pmatrix} \gamma^{-1} & 0 & 0 & 0 & 0 & 1 \\ 0 & \gamma^{-1} & 0 & 0 & -1 & 0 \\ 0 & 0 & -\gamma^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma^{-1} & 0 & 0 \\ 0 & -1 & 0 & 0 & -\gamma^{-1} & 0 \\ 1 & 0 & 0 & 0 & 0 & -\gamma^{-1} \end{pmatrix}$$

and its determinant is $\det[T_\gamma] = -(1 + \gamma^{-2})^3$. Now, the combination

$$f_{ij}^{\alpha} := u_i \wedge u_j + \alpha\, \eta_{im}\eta_{jn}\epsilon^{mnkl}u_k \wedge u_l$$

for $\alpha \in \mathbb{R}$ is a basis of $\bigwedge^2 V$ for all $\alpha \neq \pm i$. In fact, the linear map F_α sending $\{u_i \wedge u_j\}$ to $\{f_{ij}^{\alpha}\}$ reads

$$[F_\alpha] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & -\alpha & 0 \\ 0 & 0 & 1 & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ -\alpha & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\det(F_\alpha) = (1 + \alpha^2)^3$. In particular, for $\alpha = \frac{\gamma}{2}$ we have $\widetilde{T}_\gamma \equiv F_{\frac{\gamma}{2}}$.

To prove the symmetry of $\widetilde{T}_\gamma$ we can compute

$$\begin{aligned}\mathrm{Tr}[\widetilde{T}_\gamma(\alpha)\wedge\beta] &= \alpha^{ij}\beta^{mn}\mathrm{Tr}\left[\left(u_iu_j+\frac{1}{2\gamma}\epsilon_{ij}^{kl}u_ku_l\right)\wedge u_mu_n\right]= \\ &= \alpha^{ij}\beta^{mn}\left(\epsilon_{ijmn}+\frac{1}{2\gamma}\epsilon_{ij}^{kl}\epsilon_{klmn}\right)=\alpha^{ij}\beta^{mn}\left(\epsilon_{ijmn}+\frac{1}{2\gamma}\epsilon_{ijpq}\epsilon_{mn}^{pq}\right)= \\ &= \alpha^{ij}\beta^{mn}\mathrm{Tr}\left[u_iu_j\wedge\left(u_mu_n+\frac{1}{2\gamma}\epsilon_{mn}^{pq}u_pu_q\right)\right]=\mathrm{Tr}[\alpha\wedge\widetilde{T}_\gamma(\beta)]\end{aligned}$$

or equivalently use the fact that $\mathrm{Tr}[\widetilde{T}_\gamma(\alpha)\wedge\beta]=\hat{T}_\gamma(\alpha\wedge\beta)$ and that $\hat{T}_\gamma$ is a manifestly symmetric bilinear map (c.f. (5.10)) on $\bigwedge^2 V$. ✓

We are now ready to start the analysis of the boundary structure for the Palatini-Holst theory in the classical and BV settings.

5.2 Classical boundary structure

Similarly to what we did for the classical Einstein Hilbert formulation of General Relativity, we can analyse the classical phase space for the Palatini-Holst formulation. We will compare this to the BV extension in Section 5.3.

According to the tetrad framework outlined at the begining of Chapter 5 the space of classical fields for the Palatini-Holst theory of gravity is given by

$$\mathcal{F}^{cl}_{\text{Holst}}=\underbrace{\Omega^1(M,\mathcal{V})}_{e}\oplus\underbrace{\mathcal{A}_P}_{\omega} \tag{5.11}$$

where $\mathcal{A}_P$ denotes the space of principal connections on a principal bundle $P\longrightarrow M$ with structure group $SO(3,1)$. A connection is locally described by a one-form ω (on a chart) with values in $\mathfrak{so}(3,1)\simeq\bigwedge^2 V$.

Remark 5.5. *Notice that, in the literature (e.g.[19,22]), globally hyperbolic structure of space-time is usually assumed for Palatini-Holst gravity. We will instead consider any* $3+1$*-dimensional manifold with boundary, without specifying the kind of boundaries we allow. This means we will not put any extra restriction* *on the fields (c.f. with Chapter 4, Section 4.2).*

◊

Theorem 5.6. *The classical phase space for the Palatini-Holst theory of gravity*

$$S_{PH} = \int_M \hat{T}_\gamma \left[e \wedge eF_\omega \right] + \Lambda \mathrm{Tr}(e^4) \tag{5.12}$$

is a symplectic manifold for all values of the Barbero-Immirzi parameter $\gamma \in \mathbb{R}$. Denoting by $\widetilde{\mathcal{F}}^{cl}_{PH}$ the space of pre-boundary classical fields, i.e. the space of restrictions of fields an their jets to the boundary, the symplectic reduction with respect to the kernel of the differential of the Noether 1-form $\pi : \widetilde{\mathcal{F}} \longrightarrow \mathcal{F}^{cl\partial}_{PH}$ can be performed and it reads :

$$\pi : \begin{cases} \widetilde{e} = e \\ T_\gamma \left[\widetilde{\omega} \right] = T_\gamma \left[\hat{\omega} \right]^{dn}_c e_d \wedge e_n dx^c \\ T_\gamma [\widetilde{\beta}] = \sum_c T_\gamma \left[\hat{\omega} \right]^{dc}_c e_d \wedge e_c dx^c \end{cases} \tag{5.13}$$

and the symplectic form is found to be

$$\varpi^\partial = 2 \int_{\partial M} \hat{T}_\gamma \left[\widetilde{e} \wedge \delta \widetilde{e} \wedge \delta(\widetilde{\omega} + \widetilde{\beta}) \right] \tag{5.14}$$

Proof. The variation of the Palatini-Holst action (5.6) splits into a bulk term, which we will not consider in what follows, and a boundary term. The latter is interpreted as a pre-boundary one-form on the space of pre-boundary fields $\widetilde{\mathcal{F}}^{cl}_{\mathrm{Holst}}$ of all restrictions of fields to the boundary, it reads

$$\widetilde{\alpha} = \int_{\partial M} \hat{T}_\gamma \left[e \wedge e \wedge \delta\omega \right] \tag{5.15}$$

and it gives rise to the pre-boundary two-form

$$\widetilde{\varpi} = \int_{\partial M} 2\hat{T}_\gamma \left[\delta e \wedge e \wedge \delta\omega \right] \tag{5.16}$$

The restrictions of fields to the boundary are denoted with the same symbols, but we understand ω as an $\mathfrak{so}(3,1)$-valued one form on the boundary, while e is again a V valued one-form on the boundary. In particular, in the basis $\{u_i\}_{i=1...4}$ of V we have $e = e^i_a u_i dx^a$ whereas $\omega = \omega^{ij}_a u_i \wedge u_j dx^a$ where we fix that the indices a, b, c run over the boundary directions $1, 2, 3$. Notice, however, that the vectors $e_a = e^i_a u_i$ are a basis of a three dimensional subspace $W \subset V$, and we can complete it to a basis of V by introducing a vector e_n orthogonal to all the e_a's.

We rewrite (5.16) as $\widetilde{\varpi} = 2\int_{\partial M} \mathrm{Tr}\left[\delta e \wedge e \wedge \delta T_\gamma[\omega]\right]$, and using Lemma 5.3 we can read the equations defining the kernel of $\widetilde{\varpi}$ from (5.16)

$$(X_e)e = 0 \tag{5.17a}$$

$$(X_{T_\gamma[\omega]})e = 0 \tag{5.17b}$$

Using the basis $\{e_\mu\} = \{e_a, e_n\}_{a=1...3}$ we can expand (X_e) and $(X_{T_\gamma[\omega]})$ in the basis and find, for (5.17a)

$$(X_e)^\mu_a e_\mu e_b \epsilon^{abc} = 0 \Longleftrightarrow (X_e) \equiv 0 \tag{5.18}$$

whereas, for (5.17b)

$$(X_{T_\gamma[\omega]})^{\mu\nu}_a e_\mu e_\nu e_b \epsilon^{abc} = 0 \Longleftrightarrow \begin{cases} (X_{T_\gamma[\omega]})^{bn}_a = 0 & \forall a, b = 1, 2, 3 \\ (X_{T_\gamma[\omega]})^{bc}_a \text{ free} & \forall a \neq b \neq c \\ \sum_a (X_{T_\gamma[\omega]})^{ab}_a = 0 & \forall b = 1, 2, 3 \end{cases} \tag{5.19}$$

Indeed, with a closer look at the pre-boundary two-form one can gather that the components $(X_{T_\gamma[\omega]})^{bc}_a$ do not appear explicitly, which means that they belong freely to the kernel and therefore can be put to zero using their associated vertical vector fields. We can perform a change of coordinates in the space of pre-boundary fields by writing

$$T_\gamma[\omega] = T_\gamma[\hat{\omega}]^{df}_c\, e_d \wedge e_f dx^c + T_\gamma[\hat{\omega}]^{dn}_c\, e_d \wedge e_n dx^c \tag{5.20}$$

moreover we can define $\beta := \sum_c \hat{\omega}^{dc}_c e_d \wedge e_c dx^c$ so that we can read the above kernel equations as $(X_{T_\gamma[\hat{\omega}]})^{bn}_a = (X_{T_\gamma[\beta]}) = 0$ and project to the symplectic reduction. Pre-composing the symplectic reduction with the restriction map $\widetilde{\pi}\colon \mathcal{F}^{cl}_{\text{Holst}} \longrightarrow \widetilde{\mathcal{F}}^{cl}_{\text{Holst}}$ we get the map to the space of boundary fields

$$\pi_M\colon \begin{cases} \widetilde{e} = e \\ T_\gamma\left[\widetilde{\omega}\right] = T_\gamma\left[\hat{\omega}\right]^{dn}_{c} e_d \wedge e_n dx^c \\ T_\gamma[\widetilde{\beta}] = \sum_c T_\gamma\left[\hat{\omega}\right]^{dc}_{c} e_d \wedge e_c dx^c \end{cases} \tag{5.21}$$

It is easy to check that $\widetilde{\alpha}$ is horizontal and that the one-form

$$\alpha^\partial = \int_{\partial M} \hat{T}_\gamma\left[\widetilde{e}\widetilde{e}\delta(\widetilde{\omega}+\widetilde{\beta})\right] \tag{5.22}$$

is the correct boundary one-form, namely: $\widetilde{\alpha} = \pi_M^*\alpha^\partial$. ✓

5.3 Covariant BV theory

We would like to extend the classical theory to a BV theory including the symmetries. In order to do this we must understand that the Palatini-Holst description of gravity is again a BRST-like gauge theory (as for the Einstein Hilbert version, Chapter 4) so that it admits a minimal BV extension.

Differently from the EH case, in the Palatini Holst theory one has to deal with a space-time symmetry and an internal gauge freedom, due to the $\mathfrak{so}(3,1)$ structure. For the results in this Chapter and the following ones we will need this:

Lemma 5.7. *Let $P \longrightarrow M$ be a G principal bundle and let A be a connection on it. Consider any degree 1 vector field ξ on M, and any associated vector bundle $\mathcal{V}$ with typical fiber the $\mathfrak{g}$ module $V_\mathfrak{g}$. For any differential form $\Phi \in \Omega^\bullet(M,\mathcal{V})$ define the covariant Lie derivative to be*

$$L^A_\xi = [\iota_\xi, d_A]\Phi \tag{5.23}$$

with d_A being the covariant derivative induced by the connection A. We have the formula:

$$L^A_{[\xi,\xi]}\Phi - [L^A_\xi, L^A_\xi]\Phi + [\iota_\xi\iota_\xi F_A, \Phi] = 0 \tag{5.24}$$

Proof. The proof is just a straightforward but lengthy computation:

$$\begin{aligned} L^A_{[\xi,\xi]}\Phi - [L^A_\xi, L^A_\xi]\Phi = L_{[\xi,\xi]}\Phi - [L_\xi, L_\xi]\Phi + \iota_{[\xi,\xi]}[A,\Phi] + [A, \iota_{[\xi,\xi]}\Phi] \\ - \iota_\xi d[\iota_\xi A, \Phi] - \iota_\xi[A, L^A_\xi\Phi] + d\iota_\xi[\iota_\xi A, \Phi] + [A, \iota_\xi L^A_\xi \Phi] = \\ = 2\iota_\xi d\iota_\xi[A,\Phi] - \iota_\xi\iota_\xi d[A, \Phi - d\iota_\xi\iota_\xi[A,\Phi] + [A, 2\iota_\xi d\iota_\xi\Phi - \iota_\xi\iota_\xi d\Phi] \\ - 2\iota_\xi d\iota_\xi[A,\Phi] + \iota_\xi d[A, \iota_\xi\Phi] - \iota_\xi[A, \iota_\xi d\Phi - \iota_\xi[A, [\iota_\xi A, \Phi]] \\ + \iota_\xi[A, d\iota_\xi\Phi] + d[\iota_\xi A, \iota_\xi\Phi] + [A, \iota_\xi\iota_\xi\Phi + [\iota_\xi A, \iota_\xi\Phi] - \iota_\xi d\iota_\xi\Phi] = 0 \end{aligned}$$

as it can be carefully checked by expanding all terms. We used the well known identity $L_{[\xi,\xi]}\Phi - [L_\xi, L_\xi]\Phi = 0$, of which this Lemma is some special generalisation. ✓

This will be used to prove the following

Proposition 5.8. *Consider the same assumptions of Lemma 5.7 and denote by ρ the representation on the $\mathfrak{g}$-module $V_\mathfrak{g}$. Let $c \in \Omega^0[1](M, \mathrm{ad}P)$ be a degree 1 function with $\mathrm{ad}P$ the adjoint bundle to the G-bundle $P \longrightarrow M$, and define Q a vector field on the graded manifold $\mathcal{A}_P \oplus \Omega^\bullet(M, \mathcal{V}) \oplus \mathfrak{X}[1](M) \oplus \Omega^0[1](M, \mathrm{ad}P)$ by the assignment:*

$$\boxed{\begin{array}{cc} Q\,A = \iota_\xi F_A - d_\omega c & Q\,\Phi = L^A_\xi\Phi - \rho(c)\Phi \\ \\ Q\,c = \frac{1}{2}\iota_\xi\iota_\xi F_A - \frac{1}{2}[c,c] & Q\,\xi = \frac{1}{2}[\xi,\xi] \end{array}} \tag{5.25}$$

Then $[Q,Q] = 0$.

Proof. It is chiefly a long and straightforward computation to check that Q *cohomological*, that is to say $Q^2 = 0$. We shall report the main steps of the various checks:

$$\begin{aligned}
Q^2 c = \frac{1}{4}([L_\xi,\iota_\xi]\iota_\xi F_A + \iota_\xi[L_\xi,\iota_\xi]F_A) - \frac{1}{2}(\iota_\xi\iota_\xi d(\iota_\xi F_A - d_A c) + \iota_\xi\iota_\xi[A,\iota_\xi F_A - d_A c] + [\iota_\xi\iota_\xi F_A, c]) \\
= \frac{1}{4}(\iota_\xi d\iota_\xi\iota_\xi F_A - \iota_\xi\iota_\xi d\iota_\xi F_A - \iota_\xi\iota_\xi\iota_\xi dF_A) - \frac{1}{2}\iota_\xi\iota_\xi[A,\iota_\xi F_A] \\
= \frac{1}{4}(\iota_\xi d\iota_\xi\iota_\xi dA - \iota_\xi\iota_\xi d\iota_\xi dA) + \frac{1}{8}(\iota_\xi d\iota_\xi\iota_\xi[A,A] - \iota_\xi\iota_\xi d\iota_\xi[A,A] - \iota_\xi\iota_\xi\iota_\xi d[A,A]) - [\iota_\xi A, \iota_\xi\iota_\xi dA] \\
= \frac{1}{4}(\iota_\xi d\iota_\xi\iota_\xi dA - \iota_\xi\iota_\xi d\iota_\xi dA) = -\frac{1}{2}\xi^\rho\xi^\mu(\partial_\rho\partial_\mu A_\nu)\xi^\nu = 0
\end{aligned} \tag{5.26}$$

with the last equality following from the contraction of a symmetric tensor by an antisymmetric one. The rest essentially follows from Lemma 5.7, as we have

$$\begin{aligned}
Q^2_{PL}\Phi =& \frac{1}{2}L^A_{[\xi,\xi]}\Phi - L^A_\xi L^A_\xi\Phi + L^A_\xi[c,\Phi] + \iota_\xi[\iota_\xi F_A, -d_A c, \Phi] \\
&-[\iota_\xi F_A, -d_A c, \iota_\xi\Phi] + [C, L^A_\xi\Phi] - [c,[c,\Phi]] = \\
=& \frac{1}{2}L^A_{[\xi,\xi]}\Phi - L^A_\xi L^A_\xi\Phi - \frac{1}{2}[\iota_\xi\iota_\xi F_A, \Phi] = 0
\end{aligned}$$

together with

$$\begin{aligned}
Q^2 A = \frac{1}{2}\iota_{[\xi,\xi]}F_A - \iota_\xi d_A\left(\iota_\xi F_A - d_A c\right) + \frac{1}{2}d_A\left(\iota_\xi\iota_\xi F_A - [c,c]\right) - \left[\iota_\xi F_A - d_A c, c\right] \\
= -\frac{1}{2}\iota_\xi\iota_\xi dF_A - \iota_\xi\left[A,\iota_\xi F_A\right] + \frac{1}{2}\left[A,\iota_\xi\iota_\xi F_A\right] = -\frac{1}{2}\iota_\xi\iota_\xi d_A F_A = 0
\end{aligned}$$

and $Q^2\xi = 0$ follows from the Jacobi identity. ✓

This result tells us how to implement diffeomorphism as gauge symmetries for different theories involving differential forms with values in some representation of the internal Lie algebra $\mathfrak{g}$. As we shall see below this is the case of the Palatini formulation of General Relativity.

In the literature, Piguet, following Moritsch, Schweda and Sorella[42,43], suggested a BRST operator s for the Palatini Holst theory of gravity that reads

$$\begin{aligned}
\mathrm{s}\, e &= L_\xi e + [\theta, e] \\
\mathrm{s}\, \omega &= L_\xi \omega + d_\omega \theta \\
\mathrm{s}\, \xi &= \frac{1}{2}[\xi, \xi] \\
\mathrm{s}\, \theta &= L_\xi \theta + \frac{1}{2}[\theta, \theta]
\end{aligned} \tag{5.27}$$

where ξ is a vector field with ghost number $\mathrm{gh}(\xi) = 1$ and θ is a function with values in $\Lambda^2 V$ and ghost number $\mathrm{gh}(\theta) = 1$. This operator takes into account *non global* fields, like ω, which is a connection on a non trivial bundle, and non covariant derivatives. We can now propose a covariant version as follows:

Proposition 5.9. *Define the new ghost variable* c*,*

$$c := \iota_\xi \omega - \theta \tag{5.28}$$

which is a function with values in $\Lambda^2 V$ *of ghost number* $\mathrm{gh}(c) = 1$*. The BV operator for the Palatini formalism is given by the cohomological vector field* Q*:*

$$\boxed{\begin{array}{cc}
Q\,\omega = \iota_\xi F_\omega - d_\omega c & Q\, e = L_\xi^\omega e - [c, e] \\[2em]
Q\, c = \frac{1}{2}\iota_\xi \iota_\xi F_\omega - \frac{1}{2}[c, c] & Q\, \xi = \frac{1}{2}[\xi, \xi]
\end{array}} \tag{5.29}$$

where F_ω *is the curvature of* ω*,* $L_\xi^\omega = [\iota_\xi, d_\omega]$ *is the covariant Lie derivative along* ξ *with connection* ω*, and* ξ *is a vector field with ghost number* $\mathrm{gh}(\xi) = 1$*. Then, The minimal BV extension of* S_{PH} *by* Q *in* (5.29) *defines then a BV theory on the space of fields*

$$\left(\mathcal{F}_{PH} := T^*[-1]\mathcal{F}_{min}, \Omega_{BV}^\gamma\right) \tag{5.30}$$

where $\mathcal{F}_{min}$ is defined as

$$\boxed{\mathcal{F}_{min} \coloneqq \underbrace{\Omega^1(M,\mathcal{V})}_{e} \oplus \underbrace{\mathcal{A}_P}_{\omega} \oplus \underbrace{\mathfrak{X}[1](M)}_{\xi} \oplus \underbrace{\Omega^0[1](M,\mathrm{ad}P)}_{c}} \tag{5.31}$$

and the (-1)*-symplectic form* Ω^{γ}_{BV} *depending on the pairing* T_γ*.*

Proof. First of all notice that the operator (5.29) involves only global fields and covariant operations. To prove that Q is indeed a symmetry of the action we check that $QS_{PH} = 0$:

$$\begin{aligned} QS_{PH} &= \int 2[\iota_\xi, d_\omega]eeF_\omega - 2[c,e]eF_\omega - eed_\omega(\iota_\xi F_\omega - d_\omega c) \\ &= -2d_\omega e\iota_\xi eF_\omega - 2d_\omega ee\iota_\xi F_\omega - 2d_\omega\iota_\xi eeF_\omega - eed_\omega\iota_\xi F_\omega + ee[F_\omega, c] - 2[c,e]eF_\omega \end{aligned}$$
$$2ed_\omega\iota_\xi eF_\omega + e\iota_\xi ed_\omega F_\omega - 2d_\omega ee\iota_\xi F_\omega - 2d_\omega\iota_\xi eeF_\omega + 2d_\omega ee\iota_\xi F_\omega - \langle ee, \mathrm{ad}_c F_\omega\rangle - \langle \mathrm{ad}_c(ee), F_\omega\rangle = 0$$

whereas the property of Q being cohomological follows from Proposition 5.8, where $\bigwedge^2 V \simeq \mathfrak{g}$, $A = \omega$ and V clearly bears a representation of $\mathfrak{g}$. ✓

Starting from the Holst action given in (5.6), and recalling the pairing T_γ coming from the *twisted volume* $\hat{T}_\gamma$ in $\bigwedge^4 V$ defined in Lemma 5.3

$$\hat{T}_\gamma(\alpha\beta) = \mathrm{Tr}\left(\left(\alpha + \frac{1}{\gamma}\star\alpha\right)\wedge\beta\right)$$

for all $\alpha, \beta \in \bigwedge^2 V$ the minimally-BV-extended Holst action is then given by the expression:

$$\boxed{\begin{aligned} S^{BV}_{PH} = &\int\limits_M \hat{T}_\gamma(e\wedge e\wedge F_\omega) + \mathrm{Tr}\left\{\left(\iota_\xi F_\omega - d_\omega c\right)\omega^\dagger - \left([\iota_\xi, d_\omega]e - [c,e]\right)e^\dagger\right\} \\ &+ \frac{1}{2}\int\limits_M \mathrm{Tr}\left\{\left(\iota_\xi\iota_\xi F_\omega - [c,c]\right)c^\dagger\right\} + \int\limits_M \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger \end{aligned}} \tag{5.32}$$

where the nature of the fields, anti-fields, ghosts and anti-ghosts in $\mathcal{F}_{PH} = T^*[-1]\mathcal{F}_{min}$ is summarised in the following table:

Field	$\Omega^\bullet(M)$	$\Lambda^\bullet V$	Ghost	Total Degree
ω	1	2	0	3
e	1	1	0	2
c	0	2	1	3
ξ	/	/	1	1
$\omega^\dagger$	3	2	-1	4
$e^\dagger$	3	3	-1	5
$c^\dagger$	4	2	-2	4
$\xi^\dagger$	$4 \otimes 1$	/	-2	3

(5.33)

The ghost field ξ is a vector field on M, and its dual anti-ghost is a one form with values in top forms. We will decompose it as follows:

$$\xi^\dagger = \chi v \tag{5.34}$$

with $\chi \in \Omega^1(M)[-2]$ and v a volume form.

We are now ready to establish whether the BV theory (5.32) obtained by minimally extending the Palatini Holst action does satisfy the BV-BFV axioms or not.

Theorem 5.10. *The BV data* $(\mathcal{F}_{PH}, S^{BV}_{PH}, Q, \Omega^\gamma_{BV})$ *on a* $(3+1)$ *Pseudo-Riemannian manifold* M *with boundary* ∂M *does not yield a BV-BFV theory. This is true for any value of* γ*, including the limiting case* $\gamma \to \infty$*, which yields the usual Palatini formulation of gravity.*

Proof. The full variation of S^{BV}_{PH} reads as follows:

$$\begin{aligned}\delta S^{BV}_{PH} = &\int\limits_{\partial M} -\hat{T}_\gamma(ee\delta\omega) + \mathrm{Tr}\left\{\delta\omega(\iota_\xi\omega^\dagger) + \delta c\omega^\dagger + \delta e(\iota_\xi e^\dagger) + (\iota_{\delta\xi}e)e^\dagger\right\} \\ &+ \int\limits_{\partial M} \mathrm{Tr}\left\{-(\iota_\xi\delta e)e^\dagger - \frac{1}{2}\delta\omega(\iota_\xi\iota_\xi c^\dagger)\right\} + \int\limits_{\partial M}(\iota_{\delta\xi}\chi)\iota_\xi v + \int\limits_M \text{Bulk Terms}\end{aligned} \tag{5.35}$$

The variation of the ξ-ghost part is computed as:

$$\delta \int_M \frac{1}{2}\iota_{[\xi,\xi]}\chi v = \int_M \left(\iota_{[\delta\xi,\xi]}\chi - \frac{1}{2}\iota_{[\xi,\xi]}\delta\chi\right) v = \int_M \left(\underbrace{[[\iota_{\delta\xi}, d], \iota_\xi]\chi}_{A} - \frac{1}{2}\iota_{[\xi,\xi]}\delta\chi\right) v$$
$$\int_M A v = \int_M \left(\iota_{\delta\xi} d\iota_\xi\chi - \iota_\xi\iota_{\delta\xi} d\chi\right) v + \int_M (\iota_{\delta\xi}\chi) d\iota_\xi v + \int_{\partial M} (\iota_{\delta\xi}\chi)\iota_\xi v \quad (5.36)$$

and thus

$$\delta \int_M \frac{1}{2}\iota_{[\xi,\xi]}\chi v = \int_M \left(\iota_{\delta\xi} d\iota_\xi\chi - \iota_\xi\iota_{\delta\xi} d\chi - \frac{1}{2}\iota_{[\xi,\xi]}\delta\chi\right) v - \int_M (\iota_{\delta\xi}\chi) d\iota_\xi v + \int_{\partial M} (\iota_{\delta\xi}\chi)\iota_\xi v \quad (5.37)$$

If we denote by ξ^n the transversal part of ξ with respect to the boundary, and with v^∂ a volume form on the boundary, we may rewrite $\iota_\xi v = -\xi^n v^\partial$.

To obtain the pre-boundary one form $\widetilde{\alpha}$ we must consider the restriction of the fields to the boundary and their possible residual transversal components. With an abuse of notation, the restriction of the fields to the boundary will be denoted by the same symbol, whereas an apex ^{n} will be assigned to the transversal components. For instance, we will write $\iota_\xi\phi\big|_{\partial M} = \iota_{\xi^\partial}\phi^\partial + \phi_n\xi^n \equiv \iota_\xi\phi + \phi_n\xi^n$ by renaming the restrictions to the boundary $\phi^\partial \equiv \phi$ where ϕ is any suitable field, and $\xi^\partial \equiv \xi$. We obtain

$$\begin{aligned}\widetilde{\alpha} = &\int_{\partial M} -\hat{T}_\gamma(ee\delta\omega) + \mathrm{Tr}\left\{\delta\omega(\iota_\xi\omega^\dagger) + \delta\omega\,\omega_n^\dagger\xi^n + \delta c\,\omega^\dagger - \delta e\, e_n^\dagger\xi^n - \delta e(\iota_\xi e^\dagger)\right\} \\ &+ \int_{\partial M} \mathrm{Tr}\left\{-\delta(e_n\xi^n)e^\dagger - \delta(\iota_\xi e)e^\dagger - \delta\omega\,(\iota_\xi c_n^\dagger)\xi^n\right\} - \xi^n\iota_{\delta\xi}\chi\, v^\partial\end{aligned} \quad (5.38)$$

and we may compute the pre-boundary 2-form $\widetilde{\varpi} = \delta\widetilde{\alpha}$ to be ($a = 1\dots3$)

$$\begin{aligned}\widetilde{\varpi} = \int\limits_{\partial M} & -2\hat{T}_\gamma(\delta e e\delta\omega) + \mathrm{Tr}\Big[\delta\omega(\omega_a^\dagger\delta\xi^a) + \delta\omega(\iota_\xi\delta\omega^\dagger) - \delta\omega\delta\xi^n\omega_n^\dagger \\ & + \delta\omega\delta\omega_n^\dagger\xi^n + \delta c\delta\omega^\dagger + \delta e(e_a^\dagger\delta\xi^a) + \delta e e_n^\dagger\delta\xi^n + \delta e\delta e_n^\dagger\xi^n \\ & + \delta(e_n\xi^n)\delta e^\dagger - e_a\delta\xi^a\delta e^\dagger + \delta\omega\,\delta\xi^n\iota_\xi c_n^\dagger - \delta\omega\,\xi^n\iota_{\delta\xi}c_n^\dagger - \delta\omega\,\xi^n\iota_\xi\delta c_n^\dagger\Big] + \\ & + (\xi^n\delta\xi^n\delta\chi_n - \delta\xi^n\delta\xi^n\chi_n - \delta\xi^n\chi_a\delta\xi^a + \xi^n\delta\chi_a\delta\xi^a)\,v^\partial\end{aligned} \tag{5.39}$$

The kernel of $\widetilde{\varpi}$ is defined by the equations:

$$(X_{\omega^\dagger}) = 0 \tag{5.40a}$$

$$(X_c) = \iota_\xi(X_\omega) \tag{5.40b}$$

$$(X_{\xi^\rho})e_\rho + (X_{e_n})\xi^n = 0 \tag{5.40c}$$

together with

$$\hat{T}_\gamma\{(X_\omega)\wedge e\wedge\delta e\} = \frac{1}{2}\mathrm{Tr}\{\Omega\wedge\delta e\} \tag{5.41}$$

$$\hat{T}_\gamma\{(X_e)\wedge e\wedge\delta\omega\} = \frac{1}{2}\mathrm{Tr}\{E\wedge\delta\omega\} \tag{5.42}$$

where

$$\Omega := \Big[(X_{\xi^n})e_n^\dagger + (X_{e_n^\dagger})\xi^n + \iota_{(X_\xi)}e^\dagger\Big] \tag{5.43}$$

$$E := \Big[(X_{\omega_n^\dagger})\xi^n - (X_{\xi^\rho})\omega_\rho^\dagger + (X_{\xi^n})\iota_\xi c_n^\dagger - \iota_{(X_\xi)}c_n^\dagger\xi^n - \iota_\xi(X_{c_n^\dagger})\xi^n\Big] \tag{5.44}$$

with $\Omega\in\Omega^2(\partial M)\otimes\bigwedge^3 V$ and $E\in\Omega^2(\partial M)\otimes\bigwedge^2 V$. In addition we have

$$\begin{aligned}\mathrm{Tr}\Big[(X_e)e_n^\dagger - (X_\omega)\omega_n^\dagger - (X_{e^\dagger})e_n + (X_\omega)c_{na}^\dagger\xi^a\Big] + \\ -\Big(2(X_{\xi^n})\chi_n + (X_{\chi_n})\xi^n + (X_{\xi^a})\chi_a\Big)\,v^\partial = 0\end{aligned} \tag{5.45}$$

$$\begin{aligned}\mathrm{Tr}\Big[(X_e)e_a^\dagger - (X_\omega)\omega_a^\dagger - (X_\omega)c_{na}^\dagger\xi^n - (X_{e^\dagger})e_a\Big] + \\ -\Big((X_{\xi^n})\chi_a + (X_{\chi_a})\xi^n\Big)\,v^\partial = 0\end{aligned} \tag{5.46}$$

where the latter is valid for all $a = 1, 2, 3$, and finally, for all $\rho = 1 \dots 4$

$$(X_\omega)\xi^n = 0 \tag{5.47a}$$

$$(X_e)\xi^n = 0 \tag{5.47b}$$

$$\iota_\xi(X_\omega)\xi^n = 0 \tag{5.47c}$$

$$(X_\xi)^\rho \xi^n = 0 \tag{5.47d}$$

Although the equations in (5.47) look singular, it is easy to check they are not†, since $(\xi^n)^2 = 0$. Crucially, though, equation (5.42) is highly singular. As a matter of fact, counting the number of unknowns (the $(X_e)^i_a$ are 12, independent fields) against the number of equations (the $\delta\omega_a^{ij}$ are 18 independent variations) it is easy to gather that the system admits solutions only when relations among the E coefficients (5.44) are imposed. On the other hand such relations are singular in that they involve polynomial expressions of odd fields only.

The kernel of $\widetilde{\varpi}$ has therefore a larger set of generators depending on the point on the space of fields, and the pre-boundary two form is therefore not pre-symplectic. ✓

This result is a no-go theorem, at least for what concerns the BV-BFV quantisation scheme. It is telling us that there is something that crucially fails when we try to induce symmetry-compatible data on the boundary. The space of boundary fields - i.e. the reduction by the kernel of $\widetilde{\omega}$ - is not smooth, and therefore it does not yield a smooth BFV resolution of the classical reduced phase space, compatible with the boundary in the sense of Definition (2.5), Section 2.2. The source of this degeneracy seems very much due to the fact that we have *too many free fields*.

Remark 5.11. *What fails in satisfying the BV-BFV axioms is the pre-symplecticity of the pre-boundary two form $\widetilde{\omega}$, as its kernel does not define a subbundle of the tangent bundle on the space of fields. This is a first, highly non-trivial example where this condition is not fulfilled.*

◊

Notice that the classical theory is well defined, since symplectic reduction is possible when the symmetries are omitted. The (homogeneous) system of kernel equations is triv-

†Here we are assuming that the rest of the equations for X_ω and X_e can be partially solved, even it it won't be possible to solve them fully. This is just to remark that the problem does not come from (5.47).

ially solved, when symmetries are switched off. At the classical level we expect the structure to be equivalent to the Eistein Hilbert action when the condition that ω be Levi Civita is imposed. This symplectic reduction, and the coisotropic submanifold of canonical constraints may also be independently formulated in terms of the BFV formalism, but this is not compatible with bulk BV structure.

What this result is hinting, though, is that if we want to encode symmetries in a consistent way we cannot consider the Palatini formalism *as it is*. Observe that in three dimension the ratio equations/unknowns becomes 1, and the problem is not present, in agreement with the fact that the theory is basically a topological BF theory, and the CMR axioms are satisfied for such theories.

Comparing this result with what we found in the case of the EH formalism (Theorems 4.11 and 4.12) we can understand that something goes wrong when extending the physical fields to two separate entities: the tetrad e and the spin connection (with trivial torsion) ω. The two theories are, in fact, equivalent only *on half shell*, that is to say, only when the equation of motion (5.3) is enforced, i.e. requiring that ω be the Levi-Civita connection.

To overcome this problem one could try to implement condition (5.3) in the BV machinery, or resort to other equivalent descriptions of the classical theory[8,29,30,50]. The former approach is considered in Section 5.4, whereas the latter is analysed in detail in Chapter 6.

5.4 Half-shell localisation

We have seen in the previous sections how the BV version of the Palatini Holst action does not satisfy the BV-BFV axioms. This is a deviation from the equivalence at the classical level with the Einstein-Hilbert formulation of General Relativity (cf. Chapter 4).

It is clear that the Palatini-Holst action is slightly more general than the Einstein-Hilbert formulation of GR. The fact that the (torsion free) connection is independent from the metric, and it is uniquely determined only when the Half-Shell constraint (5.3) is enforced marks a difference in the two theories. One question one could ask is whether there is a way to implement it consistently with the symmetries, while still holding on to the tetrad formalism.

In this Section we will consider such a localisation to the Half-Shell submanifold $d_\omega e = 0$

through a Lagrange multiplier $t \in \Omega^2(M, \mathcal{V}^*)$ where we may identify the fibers of $\mathcal{V}^*$ with $\bigwedge^3 V$:

$$\boxed{S_{HS} = \int\limits_M \frac{1}{2}\hat{T}_\gamma\left[e \wedge e \wedge F_\omega\right] + \mathrm{Tr}\left[t \wedge d_\omega e\right] + \frac{\Lambda}{4}\mathrm{Tr}\left[e \wedge e \wedge e \wedge e\right]} \tag{5.48}$$

Theorem 5.12. *The equations of motion for the action functional S_{HS} coincide with those of the Einstein-Hilbert theory and moreover, whenever M admits a boundary ∂M, it exhibits a symplectic space of boundary fields. The projection to this space is given by*

$$\pi_M : \begin{cases} \widetilde{t} = t + T_\gamma[\underline{\omega} - \omega] \wedge e \\ \widetilde{e} = e \end{cases} \tag{5.49}$$

where $\underline{\omega}$ is the Levi Civita e-compatible connection, and the (exact) symplectic form reads

$$\boxed{\varpi^\partial = \int\limits_{\partial M} \mathrm{Tr}\left[\delta\widetilde{t}\,\delta\widetilde{e}\right]} \tag{5.50}$$

Proof. First of all let us analyse the Euler Lagrange equations for the action. They read:

$$\delta\omega : d_\omega e \wedge e - t \overset{\circ}{\wedge} e = 0 \tag{5.51a}$$

$$\delta e : e \wedge T_\gamma\left[F_\omega\right] + d_\omega t + \Lambda e^3 = 0 \tag{5.51b}$$

$$\delta t : d_\omega e = 0 \tag{5.51c}$$

where $t \overset{\circ}{\wedge} e$ stands for $\frac{\delta}{\delta\omega}(t \wedge \omega \cdot e) = t^{ijk}_{\mu\nu} e^m_\rho \epsilon_{ijkl} \epsilon^{\mu\nu\rho\sigma}$ for all σ space-time index and l, m internal indices. The dot denotes indeed the action of ω on e. Enforcing the half shell constraint $d_\omega e = 0$, which implies that ω is the Levi-Civita connection, represented in the tetrad formalism by the special connection $\underline{\omega}$, we obtain $t = 0$ and the Einstein equation in the tetrad formalism

$$e \wedge F_{\underline{\omega}} + \Lambda e^3 = 0 \tag{5.52}$$

Starting from the computations in Theorem 5.6, we gather that the pre-boundary two-form $\widetilde{\varpi}_{HS}$ reads

$$\widetilde{\varpi}_{HS} = \int\limits_{\partial M} \hat{T}_\gamma \left[\delta e \wedge e \wedge \delta\omega\right] - \mathrm{Tr}\left[\delta t \wedge \delta e\right] \tag{5.53}$$

and the kernel of this two-form is easily found to be:

$$(X_t) = (X_{T_\gamma[\omega]}) \wedge e \tag{5.54a}$$

$$(X_e) = 0 \tag{5.54b}$$

This means that ω can be fixed using the vertical vector field

$$\Omega = (X_{T_\gamma[\omega]})\frac{\delta}{\delta T_\gamma\left[\omega\right]} + (X_{T_\gamma[\omega]}) \wedge e\frac{\delta}{\delta t} \tag{5.55}$$

and t is modified accordingly. Flowing along Ω we can set ω to be a background connection $\underline{\omega}$, which we may eventually choose to be the restriction to the boundary of the solution to the EL equations, and this fixes $(X_{T_\gamma[\omega]}) = T_\gamma\left[\underline{\omega} - \omega_0\right]$. Then, by solving the straightforward differential equation $\dot{t} = T_\gamma[\underline{\omega} - \omega_0] \wedge e_0$:

$$t(s) = t_0 + T_\gamma[\underline{\omega} - \omega_0] \wedge e_0\ s \tag{5.56}$$

we set $t(1) = t_0 + T_\gamma[\underline{\omega} - \omega_0] \wedge e_0$.

Notice, however, that the pre-boundary one-form is not horizontal with respect to the kernel foliation defined by equations (5.54), as the generator $\Omega = \frac{\delta}{\delta\omega}$ does not lie in the kernel of $\widetilde{\alpha}$. We can nevertheless modify $\widetilde{\alpha}$ by adding the exact term $\frac{1}{2}\int_M d\,\hat{T}_\gamma\left[e \wedge e \wedge (\underline{\omega} - \omega)\right] + \int_M d(e \wedge t)$ to the action (5.48), yielding

$$\overline{\alpha} = \widetilde{\alpha} + \frac{1}{2}\delta(e \wedge e \wedge T_\gamma[\omega]) + \delta(e \wedge t) \tag{5.57}$$

and it is easy to gather that the following one form on the space of boundary fields

$$\alpha^\partial = \int\limits_{\partial M} \widetilde{t}\delta\widetilde{e} \tag{5.58}$$

will be such that

$$\overline{\alpha} = \pi_M^* \alpha^\partial \tag{5.59}$$

where the projection to the space of boundary fields π_M is then clearly given by (5.49), as we can set $\widetilde{t} \coloneqq t + T_\gamma \left[\underline{\omega} - \omega\right] \wedge e$. ✓

Proposition 5.13. *The projection to the space of classical boundary fields of the Euler Lagrange equation for the action* (5.48) *is isotropic but not Lagrangian.*

Proof. Consider the Euler Lagrange equations for the Half-Shell-constrained Palatini action as given in (5.51). Their projection to the space of pre-boundary fields is given by their restrictions as differential forms:

$$\widetilde{\pi}(EL) \equiv EL\big|_{\partial M} : \begin{cases} \omega = \underline{\omega} \\ t = 0 \\ e \wedge F_{\underline{\omega}} + \Lambda e^3 = 0 \end{cases} \tag{5.60}$$

Notice that we substituted equation $d_\omega e$ with the equivalent condition on the connection $\omega = \underline{\omega}$. Taking into account the projection to the space of boundary fields (5.49) we can easily recognise the projected critical locus to be

$$EL^\partial \coloneqq \pi_M(EL) : \begin{cases} \widetilde{t} = 0 \\ \widetilde{e} \wedge F_{\underline{\omega}} + \Lambda \widetilde{e}^3 = 0 \end{cases} \tag{5.61}$$

since $\widetilde{t} = t + T_\gamma \left[\underline{\omega} - \omega\right] \wedge e$, the critical locus yields precisely $t = 0$ when $\omega = \underline{\omega}$. It is easy to check that EL^∂ is isotropic, as $\widetilde{t} = 0$ implies $\omega^\partial\big|_{EL^\partial} = 0$, confirming the general theory since $\delta S_{HS} = EL + \pi_M^* \alpha^\partial$.

Actually, $\widetilde{t} = 0$ defines a Lagrangian submanifold, which is then spoiled by equation $\widetilde{e} \wedge F_{\underline{\omega}} + \Lambda \widetilde{e}^3 = 0$. A way to see this is by explicitly checking that their Poisson bracket is not proportional to the constraints, and thus EL^∂ fails to be a coisotropic submanifold. ✓

Since the two theories are actually classically equivalent only when the constraint $d_\omega e = 0$ is imposed, it is not unreasonable to think that it precisely marks the deviation at the BV-

BFV level, meaning that the BV Palatini action might in fact become BV-BFV equivalent to the Einstein-Hilbert theory when the constraint is enforced in the right way.

The following result precisely hints at this direction:

Proposition 5.14. *The submanifold $H : d_\omega e = 0$ in the space of fields for the BV-extended Palatini-Holst theory $\mathcal{F}_{Holst}$ is coisotropic and Q-invariant.*

Proof. To verify that H defines a coisotropic submanifold we must consider the Poisson bracket of two local functions of the form

$$f_{\alpha_{1,2}} = \int_M \mathrm{Tr}\left[\alpha_{1,2} \wedge d_\omega e\right]$$

with $\alpha_{1,2} \in \Omega^2[-1](M, \mathcal{V}^*)$ (notice that we will consider its total parity to be odd), but since the local form of the (odd)-Poisson structure induced by Ω involves a derivation with respect to an anti-field for every derivation with respect to a field we can easily gather that

$$(f_{\alpha_1}, f_{\alpha_2}) = \left(\int_M \mathrm{Tr}\left[\alpha_1 \wedge d_\omega e\right], \int_M \mathrm{Tr}\left[\alpha_2 \wedge d_\omega e\right] \right)_\Omega = 0. \tag{5.62}$$

To prove the Q invariance we compute

$$\begin{aligned} Q \int_M \mathrm{Tr}\left[\alpha \wedge d_\omega e\right] &= - \int_M \mathrm{Tr}\left[\alpha \wedge (Q\omega) e\right] + \int_M \mathrm{Tr}\left[\alpha \wedge d_\omega (Qe)\right] \\ &= - \int_M \mathrm{Tr}\Big[\alpha \wedge (\iota_\xi F_\omega - d_\omega c) e\Big] + \int_M \mathrm{Tr}\Big[\alpha \wedge (d_\omega L_\xi e - d_\omega (ce))\Big] \\ = - \int_M \mathrm{Tr}\Big[\alpha \wedge (\iota_\xi F_\omega e)\Big] &+ \int_M \mathrm{Tr}\Big[\alpha \wedge d_\omega \iota_\xi d_\omega e\Big] - \int_M \mathrm{Tr}\Big[\alpha \wedge (F_\omega \iota_\xi e)\Big] + \int_M \mathrm{Tr}\left[\alpha \wedge (c d_\omega e)\right] \end{aligned} \tag{5.63}$$

where the terms containing $d_\omega c$ cancel out. Notice now that the expression $\iota_\xi (\alpha \wedge F_\omega e)$ is identically zero for we are contracting a 5-form, and therefore we can move the contraction at the price of a sign:

$$-\mathrm{Tr}\left[\alpha \wedge (F_\omega \iota_\xi e)\right] = \mathrm{Tr}\left[\iota_\xi \alpha \wedge (F_\omega e)\right] + \mathrm{Tr}\left[\alpha \wedge (\iota_\xi F_\omega e)\right]$$

Moreover, recalling that $F_\omega = d_\omega^2$ we can carry on the computation as

$$Q \int_M \mathrm{Tr}\left[\alpha \wedge d_\omega e\right] = \int_M \mathrm{Tr}\left[\iota_\xi \alpha \wedge d_\omega^2 e\right] + \int_M \mathrm{Tr}\left[\alpha \wedge d_\omega \iota_\xi d_\omega e\right] + \int_M \mathrm{Tr}\left[\alpha \wedge c d_\omega e\right] =$$
$$\int_M \mathrm{Tr}\left[d_\omega \iota_\xi \alpha \wedge d_\omega e\right] - \int_M \mathrm{Tr}\left[\iota_\xi d_\omega \alpha \wedge d_\omega e\right] + \int_M \mathrm{Tr}\left[c\alpha \wedge d_\omega e\right] \quad (5.64)$$

where we have integrated by parts d_ω and we have used the identity $\langle \alpha, \mathrm{ad}_c d_\omega e\rangle = \langle \mathrm{ad}_c \alpha, d_\omega e\rangle$ owing to the degrees of c and α. Understanding $\mathrm{ad}_c \alpha \equiv c\alpha$ we conclude that

$$Q \int_M \mathrm{Tr}\left[\alpha \wedge d_\omega e\right] = -\int_M \mathrm{Tr}\left[\left(L_\xi^\omega \alpha - c\alpha\right) \wedge d_\omega e\right] \quad (5.65)$$

which vanishes on H. ✓

This is a first necessary step to think that one can recover a BV-BFV theory for the Palatini-Holst formulation of GR, when the Half-Shell constraint is enforced. Remarkably, though, this is not the case.

To incorporate the constraint in the BV formalism what one has to do is to extend the Palatini-Holst action (5.6) by adding the term $\int_M \mathrm{Tr}\left[t \wedge d_\omega e\right]$ as we did in (5.48). This time, we have to take into account the explicit symmetry of t as well, which was spelled out explicitly in Proposition 5.14. The new constrained action we will consider then reads

$$\boxed{S_{HC} := S_{PH}^{BV} + \int_M \mathrm{Tr}\left[t \wedge d_\omega e\right] + \int_M \mathrm{Tr}\left[\left(L_\xi^\omega t - ct\right) t^\dagger\right]} \quad (5.66)$$

where $t^\dagger \in \Omega^2[-1](M, \mathcal{V})$ will be the field dual to t. The space of fields gets enlarged accordingly to

$$\mathcal{F}_{HC} := T^*[-1]\left(\mathcal{F}_{min} \oplus \left(\Omega^2(M) \otimes \bigwedge^3 V\right)\right) \tag{5.67}$$

with its modified canonical (-1)-symplectic form Ω^γ_{HC}.

Theorem 5.15. *The BV theory given by the constrained Palatini-Holst action* $(\mathcal{F}_{HC}, S_{HC}, Q, \Omega^\gamma_{HC})$ *does not satisfy the BV-BFV axioms.*

Proof. To prove this statement we will need to start from the computations of Theorem 5.10. The additional terms in S_{HC} will change the pre-boundary one-form to

$$\widetilde{\alpha}_{HC} = \widetilde{\alpha}_{PH} - \int_{\partial M} \mathrm{Tr}\,[t\delta e] + \int_{\partial M} \mathrm{Tr}\left[\delta t\left(\iota_\xi t^\dagger + t^\dagger_n \xi^n\right) + \delta\left(\iota_\xi t + t_n \xi^n\right) t^\dagger\right] \tag{5.68}$$

were $\widetilde{\alpha}_{PH}$ is like in (5.38). The addition to the pre-boundary two form $\widetilde{\varpi}$ of (5.39) is given by

$$\widetilde{\varpi}_{HC} = \widetilde{\varpi}_{PH} + \int_{\partial M} \mathrm{Tr}\left[-\delta t\delta e + \delta t\left(\iota_{\delta\xi} t^\dagger + \delta t^\dagger_n \xi^n - t^\dagger_n \delta\xi^n\right) + \iota_{\delta\xi} t \delta t^\dagger\right] \tag{5.69}$$

The kernel equations get modified such that on top of equations from (5.40) to (5.47) we have the new equation

$$(X_e) = \iota_{(X_\xi)} t^\dagger - t^\dagger_n (X_{\xi^n}) + (X_{t^\dagger_n})\xi^n \tag{5.70}$$

that will make equations (5.41) even more singular. As a matter of fact, using equation (5.40c), namely

$$(X_{\xi^\rho}) e_\rho + (X_{e_n})\xi^n = 0$$

to solve for (X_ξ) as a function of $(X_{e_n})\xi^n$ we find that plugging (5.70) into (5.41) we obtain a series of 18 unsolvable relations between odd fields, of the kind:

$$\begin{aligned} T_\gamma \left[\left(-t^\dagger_\rho (X_{e_n})^\rho + (X_{t^\dagger_n})\right) \wedge e\right]^{ij}_{ab} \xi^n = \\ = \left[(X_{\omega^\dagger_n})\xi^n + (X_{e_n})^\rho \xi^n \omega^\dagger_\rho - (X_{e_n})^n \xi^n \iota_\xi c^\dagger_n - \iota_{(X_\xi)} c^\dagger_n \xi^n - \iota_\xi (X_{c^\dagger_n})\xi^n\right]^{ij}_{ab} \end{aligned} \tag{5.71}$$

The modified pre-boundary two-form $\widetilde{\varpi}_{HC}$ is therefore not pre-symplectic, thus failing to satisfy the BV-BFV axioms. ✓

The above Theorem 5.15 significantly strengthens the failure in fitting the Palatini-Holst description of General Relativity in the CMR framework. The BV theory obtained by explicitly considering the Half Shell constraint is fully classically equivalent to the Einstein Hilbert formulation of GR. Yet the two theories differ in the BV setting, i.e. when symmetries are explicitly taken into account.

As we will see in Chapter 6, the situation improves ever so slightly when a further step in the *abstraction* is performed, and instead of a tetrad field e one considers a Lie algebra valued 2-form, by constructing a *broken* BF theory[‡]. Understanding this behaviour will possibly help us clarify which classically equivalent actions do indeed allow for a CMR description, letting us step further in the program of BV-BFV quantisation[5].

[‡]Broken here means that the theory one has to consider does not enjoy the full distribution of symmetries of the usual BF theory.

6

BF formulations of General Relativity

In this Chapter we will focus on alternative formulations of General Relativity[8,45,30]. The main common idea underlying these alternative formulations is that both the space-time metric (as in Chapter 4) and the (co-)tetrad field (as in Chapter 5) are to be considered as non-fundamental, derived quantities. These are replaced by a rather abstract field B, a two form with values in a Lie algebra, together with a connection A in a principal bundle over the space-time manifold M for the corresponding Lie group. Typically one chooses $SO(3,1)$ to be the structure group, but generalisations are taken into account[29], in view of a unification of fundamental forces.

Generally speaking, theories of this kind are called BF theories, when the action functional is taken to be of the form

$$S = \int_M \mathrm{Tr}\,[B \wedge F] \tag{6.1}$$

where Tr denotes some pairing in $\mathfrak{g}$, like the Cartan-Killing form.

BF theories are invariant under the action of a large symmetry distribution. As a matter of fact they are *topological* theories, in that there are no local degrees of freedom left when the symmetries are taken into account. In other words all solutions to the equations of motion are locally gauge equivalent and there is no residual *dynamics*.

It is nevertheless known that General Relativity is a gauge theory that retains two propagating degrees of freedom, up to gauge equivalence; therefore, if we want to describe GR, it is of prime importance that we device some mechanism to break the symmetry down to a basic diffeomorphism invariance, as we expect a theory of GR to enjoy.

In particular, on a closed manifold, a BF theory admits two different symmetry transformations: the internal gauge transformation $\delta^g A = d_A c$, $\delta^g B = [c, B]$ and the shift transformation $\delta^s B = d_A \tau$, where c is a $\mathfrak{g}$-valued function and τ a $\mathfrak{g}$-valued one-form.

It is possible to add to the BF action a potential term $V(B)$ depending solely on the B field. One very typical example of a potential is a quadratic coupling of the kind $V(B) = \frac{\Lambda}{2}\langle B, B\rangle$, with $\langle\cdot,\cdot\rangle$ being some possibly degenerate inner product, and Λ a constant, to be interpreted as the cosmological constant.

Sometimes, the potential can break the shift symmetry, and the equations of motion will yield an effective theory that recovers the Einstein Hilbert action of the Palatini action. In what follows we will analyse two different examples of symmetry breaking. First we will consider the *singular* potential given by the specification of a Lagrangian multiplier coupled to a quadratic BB term. This action, together with its modifications and extensions, goes under the name of Plebanski action[29] and will be analysed in Section 6.1.

Another possible way of describing General Relativity using a BF theory is through the BF version of the MacDowell-Mansouri action[9,49,30]. There, the main idea is to extend the Lie algebra $\mathfrak{so}(3,1)$ to $\mathfrak{so}(4,1)$ and then explicitly break the symmetry in the BF action by introducing a potential that will at the same time reduce the internal gauge symmetry back to the Lorentz group, and that will forbid the invariance under the shift symmetry. This action will be analysed in Section 6.2.

The infinitesimal transformations for the Lie group of space-time diffeomorphism, namely the Lie derivative along generic vector fields, can be recovered from the symmetries of the

BF theory[30]. As a matter of fact, for $\xi \in \Gamma(T[1]M)$ we can compute

$$\delta^d A := L_\xi A = \iota_\xi dA - d\iota_\xi + \iota_\xi[A,A] - \iota_\xi[A,A] = \iota_\xi F - d_A\iota_\xi A$$

and

$$\delta^d B := L_\xi B = \iota_\xi dB - d\iota_\xi B + \iota_\xi[A,B] - \iota_\xi[A,B] = \iota_\xi d_A B - d_A\iota_\xi B - [\iota_\xi A, B]$$

It is clear that the Lie derivatives above can be expressed *on shell* (i.e. when $F = 0, d_A B = 0$) using the symmetries of the BF theory, under the identification $c = \iota_\xi A$ and $\tau = \iota_\xi B$. This however doesn't keep a particular action from being symmetric with respect to diffeomorphism also off shell.

As a matter of fact in all BF theories we will consider in this Chapter, even though the shift symmetry for the B field will be broken either by a potential term or by a constraint, we will still retain the symmetry under space-time diffeomorphisms.

6.1 Plebanski action

The Plebanski action for GR is a BF-like action functional for the Lie algebra $\mathfrak{so}(3,1) \simeq \bigwedge^2 V$, with (V,η) a pseudo-Euclidean vector space, together with a dynamical constraint. We include a Lagrange multiplier in the action, in such a way that when the constraint is enforced, the Palatini(-Holst) formulation is recovered. This is done through the introduction of a function ϕ with values in the symmetric power $\left(\bigwedge^2 V^*\right)^{\otimes_s 2}$ coupled to the B field as

$$\phi_{ijkl} B^{ij} \wedge B^{kl} \tag{6.2}$$

Considering the symmetry of the indices of ϕ_{ijkl} we gather that it has 21 free component, which is one too many if we want to breakdown the symmetry of the BF action so, to be left with 2 degrees of freedom. The customary way to overcome this is by fixing the *trace*[31,44,30], by weighing the two different invariant volume forms in $\left(\bigwedge^2 V\right)^{\otimes_s 2}$, namely ϵ^{ijkl} and $\eta^{ijkl} = \eta^{i\langle j}\eta^{k\rangle l}$. One has to introduce a second Lagrange multiplier $\psi \in \Omega^4(M)$ to enforce the condition

$$\left(a\epsilon^{ijkl} + b\eta^{ijkl}\right)\phi_{ijkl}(x) = 0$$

at every point.

It is well known in the literature (e.g.[46] and references therein) that when such constraints are enforced there exist tetrads $e\colon TM \longrightarrow \mathcal{V}$, where $\mathcal{V}$ is a vector bundle with typical fibre the pseudo-Euclidean vector space (V, η), such that

$$B = \pm T_\gamma(e \wedge e) \tag{6.3}$$

where T_γ is the linear map defined in Lemma 5.3 of Chapter 5 and γ is recovered as $\gamma = \frac{b}{a}$.

Remark 6.1. *The constraint* (6.2) *is usually called* simplicity constraint *when the trace condition on ϕ is enforced. In the literature[31,44], different versions of this constraint are considered, basically depending on which volume form one uses to take the trace. The mixed version we are considering here goes under the name of* Non-Chiral Plebanski action*, and is the correct one to recover the Palatini-Holst formulation of General Relativity with Barbero-Immirzi parameter. Other choices of volume forms will yield either the standard Palatini formulation (when ϵ^{ijkl} is chosen), or a topological term (when η^{ijkl} is chosen). For a complete account on this topic we refer to the excellent paper[30].*

◊

In this framework it is possible to introduce a cosmological term as well[44,30], by adding a constant coupling of the form

$$\frac{1}{6}\left(\frac{\lambda}{2}\epsilon_{ijkl} + \mu\eta_{ijkl}\right)B^{ij} \wedge B^{kl} \tag{6.4}$$

The action we will consider in this chapter is therefore obtained by putting together all of the above modifications to (6.1)

$$\boxed{S_{PL} = \int_M (B, F)_\eta - \frac{1}{2}\langle\langle \varphi^{(\lambda,\mu)}, BB\rangle\rangle + \psi\langle\langle H_{a,b}, \phi\rangle\rangle} \tag{6.5}$$

where $\varphi^{(\lambda,\mu)}_{ijkl} = \left(\phi_{ijkl} + \frac{\lambda}{6}\epsilon_{ijkl} + \frac{\mu}{3}\eta_{ijkl}\right)$, $H^{ijkl}_{a,b} = (a\epsilon^{ijkl} + b\eta^{ijkl})$, while $(\cdot,\cdot)_\eta$ denotes the inner product in $\mathfrak{g} = \mathfrak{so}(3,1) \simeq \bigwedge^2 V$ induced by η, and $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical pairing in $(\mathfrak{g}^*)^{\otimes_s 2}$. Denote by $\bigwedge_S \mathcal{V}^*$ the vector bundle with fiber $(\mathfrak{g}^*)^{\otimes_s 2}$. To explicitly include the symmetries in the picture we will need the following result:

Proposition 6.2. *Let M be a 4 dimensional manifold together with an $SO(3,1)$ bundle $P \longrightarrow M$ over it. Consider the space of fields:*

$$\mathcal{F}_{PL} = T^*[-1]\mathcal{F}_{min} \tag{6.6}$$

where we define $\mathcal{F}_{min}$ to be[*]

$$\underbrace{\Omega^2(M, \bigwedge^2 \mathcal{V}) \oplus \Omega^0(M, \mathrm{ad}P)[1]}_{(B,c)} \oplus \underbrace{\mathcal{A}_P}_{A} \oplus \underbrace{\Omega^0(M, \bigwedge_S \mathcal{V}^*)}_{\phi} \oplus \underbrace{\Omega^4(M)}_{\psi} \oplus \underbrace{\Gamma(T[1]M)}_{\xi} \tag{6.7}$$

$\mathcal{A}_P$ denotes connections on the principal bundle P. Consider the degree 1 vector field Q_{PL} on $\mathcal{F}_{PL}$ defined by the assignment (we drop the PL subscript)

$$\boxed{\begin{array}{ccc} QB = L_\xi^A B - [c,B]; & QA = \iota_\xi F_A - d_A c; & Q\phi = L_\xi^A \phi - \rho_c \phi; \\ Qc = \frac{1}{2}\left(\iota_\xi \iota_\xi F_A - [c,c]\right); & Q\xi = \frac{1}{2}[\xi,\xi]; & Q\psi = L_\xi \psi \end{array}} \tag{6.8}$$

where ρ_c denotes the representation of $\mathfrak{so}(3,1)$ in $(\mathfrak{g}^*)^{\otimes_s 2}$.

If we denote by S_{PL}^{BV} the minimally extended BV action (cf. Theorem 2.3, Section 2.2) $S_{PL}^{BV} = S_{PL} + (Q\Phi, \Phi^\dagger)$, with Φ being the base fields in $\mathcal{F}_{min}$ and $\Phi^\dagger$ the respective cotangent fields, then the data $(\mathcal{F}_{PL}, S_{PL}^{BV}, Q_{PL}, \Omega_{PL})$ defines a BV theory. The action explicitly reads:

$$\begin{aligned} S_{PL}^{BV} = \int_M \Big\{ &(B,F)_\eta - \frac{1}{2}\langle\langle \varphi^{(\lambda,\mu)}, BB\rangle\rangle + \psi\langle\langle H_{a,b}, \phi\rangle\rangle - L_\xi \psi \psi^\dagger + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger \\ &- \left\langle \left(L_\xi^A B - [c,B]\right), B^\dagger \right\rangle - \left\langle \left(L_\xi^A - \rho_c \phi\right), \phi^\dagger \right\rangle + \left\langle \left(\iota_\xi F - d_A c\right), A^\dagger \right\rangle \Big\} \end{aligned} \tag{6.9}$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing between $\mathfrak{g}$ and its dual.

[*]We use the shorthand notations $V^{\wedge 2}$ and $\left(V^{*\wedge 2} \otimes_s V^{*\wedge 2}\right)$ for typographic reasons.

Proof. First of all we prove that Q_{PL} annihilates S_{PL}

$$
\begin{aligned}
Q_{PL}S_{PL} = \int_M &[(L_\xi^A B - [c,B], F_A)_\eta - (B, d_A(\iota_\xi F_A - d_A c))_\eta - \frac{1}{2}\langle\langle L_\xi^A \phi - \rho_c\phi, BB\rangle\rangle \\
&- \langle\langle \phi, (L_\xi^A B - [c,B])B\rangle\rangle + L_\xi\psi\langle\langle a\epsilon + b\eta, \phi\rangle\rangle + \psi\langle\langle a\epsilon + b\eta, L_\xi^A\phi - \rho_c\phi\rangle\rangle = \\
&= \int_M -(B, L_\xi^A F_A)_\eta - ([c,B], F_A)_\eta + (B, d_A\iota_\xi F_A - \iota_\xi d_A F_A)_\eta + \langle B, [F,c]\rangle \\
&+ \langle\langle \phi, L_\xi BB\rangle\rangle + \frac{1}{2}\langle\langle \rho_c\phi, BB\rangle\rangle - \langle\langle \phi, L_\xi BB\rangle\rangle + \langle\langle \phi, [c,B]B\rangle\rangle - \psi\langle\langle H_{a,b}, \rho_c\phi\rangle\rangle = 0
\end{aligned}
\tag{6.10}
$$

where we used $H_{a,b} = (a\epsilon + b\eta)$, the Bianchi identity $d_A F_A = 0$, we integrated by parts the Lie derivatives and used that $\langle [c,X], Y\rangle = \langle X, [Y,c]\rangle$. The same holds for the representation ρ_c, namely $\langle\langle \rho_c\phi, BB\rangle\rangle = -2\langle\langle \phi, [c,B]B\rangle\rangle$.

To check that Q_{PL} is cohomological we essentially resort to Proposition 5.8, Chapter 5, Section 5.3, repeatedly, for Φ being B, ϕ and ψ. ✓

Given the facts above, we can proceed to the analysis of the boundary structure for this BV theory. It turns out that we will encounter obstructions also in this case, in fact:

Theorem 6.3. *The BV data given in Proposition 6.2, when M is allowed to have a boundary, does not satisfy the BV-BFV axioms.*

Proof. Compute the variation of the action (6.9) to obtain the pre-boundary one-form $\widetilde{\alpha}$:

$$
\begin{aligned}
\widetilde{\alpha} = \int_{\partial M} &B\delta A + \delta A\iota_\xi A^\dagger + \delta A A_n^\dagger \xi^n - \delta B\iota_\xi B^\dagger - \delta B B_n^\dagger \xi^n - \delta(\iota_\xi B)B^\dagger - \delta(B_n\xi^n)B^\dagger \\
&- \delta\phi\iota_\xi\phi^\dagger - \delta\phi\phi_n^\dagger\xi^n - \delta(\iota_\xi\psi)\psi^\dagger - \delta(\psi_n\xi^n)\psi^\dagger + \delta c A^\dagger - \frac{1}{2}\delta A\iota_\xi c_n^\dagger \xi^n - \xi^n\delta\xi^\rho\xi_\rho^\dagger
\end{aligned}
\tag{6.11}
$$

where we omitted the obvious pairing symbols to keep the notation clean. The pre-boundary two-form reads

$$
\begin{aligned}
\widetilde{\omega} = \int_{\partial M} &\delta B \delta A + \delta A \iota_{\delta\xi} A^\dagger + \delta A \iota_\xi \delta A^\dagger - \delta A \delta \xi^n A_n^\dagger + \delta A \xi^n \delta A_n^\dagger + \delta B \iota_{\delta\xi} B^\dagger \\
&+ \delta B \delta \xi^n B_n^\dagger + \iota_{\delta\xi} B \delta B^\dagger - \delta B \xi^n \delta B_n^\dagger - \delta \xi^n B_n \delta B^\dagger + \xi^n \delta B_n \delta B^\dagger + \delta c \delta A^\dagger \\
&+ \delta\phi \iota_{\delta\xi}\phi^\dagger + \delta\phi\iota_\xi\delta\phi^\dagger + \delta\phi\delta\xi^n\phi_n^\dagger - \delta\phi\xi^n\delta\phi_n^\dagger + \iota_{\delta\xi}\psi\delta\psi^\dagger + \iota_\xi\delta\psi\delta\psi^\dagger - \frac{1}{2}\delta A \iota_{\delta\xi} c_n^\dagger \xi^n \\
&+ \frac{1}{2}\delta A \iota_\xi c_n^\dagger \delta\xi^n - \frac{1}{2}\delta A \iota_\xi \delta c_n^\dagger \xi^n - \delta\xi^n \psi_n \delta\psi^\dagger + \xi^n \delta\psi_n \delta\psi^\dagger - \delta\xi^n \delta\xi^\rho \xi_\rho^\dagger + \xi^n \delta\xi^\rho \delta\xi_\rho^\dagger
\end{aligned} \tag{6.12}
$$

The kernel of the two-form $\widetilde{\omega}$ is defined by the following groups of relations:

$$
(X_B) = (X_{\xi^n}) A_n^\dagger - \iota_{(X_\xi)} A^\dagger - (X_{A_n^\dagger})\xi^n + \frac{1}{2}\left(\iota_{(X_\xi)} c_n^\dagger + \iota_\xi (X_{c_n^\dagger})\right)\xi^n - \frac{1}{2}(X_{\xi^n})\iota_\xi c_n^\dagger \tag{6.13a}
$$

$$
(X_A) = -\iota_{(X_\xi)} B^\dagger - (X_{B_n^\dagger})\xi^n - (X_{\xi^n}) B_n^\dagger \tag{6.13b}
$$

$$
\iota_{(\widehat{X_\xi})} \hat{B} = -(X_{B_n})\xi^n \tag{6.13c}
$$

$$
(X_c) = \iota_\xi (X_A) \tag{6.13d}
$$

$$
(X_{A^\dagger}) = 0 \tag{6.13e}
$$

where by the hat we mean that all components transverse and parallel to the boundary of the respective field appear in the expression, that is to say $\iota_{(\widehat{X_\xi})}\hat{B} = -(X_\xi)^\rho B_\rho$ with $\rho = 1, 2, 3, n$.

In addition to equations (6.13) we have

$$
\begin{aligned}
(X_{B^\dagger}) B_\rho = (X_B) B_n^\dagger - (X_A) A_n^\dagger + (X_\phi)\phi_n^\dagger - (X_{\psi^\dagger})\psi_n + \\
+ (X_A)\iota_\xi c_\rho^\dagger - (X_{\xi^n})\xi_\rho^\dagger - (X_{\xi_\rho^\dagger})\xi^n - \delta_\rho^n (X_{\xi^\mu})\xi_\mu^\dagger
\end{aligned} \tag{6.14}
$$

and the set of (critical) equations

$$
\iota_{\hat{\xi}}(\widehat{X_{\phi^\dagger}}) = \iota_{(\widehat{X_\xi})}\hat{\phi}^\dagger \tag{6.15a}
$$

$$
\iota_{\hat{\xi}}(\widehat{X_\psi}) = \iota_{(\widehat{X_\xi})}\hat{\psi} \tag{6.15b}
$$

Notice that these equations are singular in that they depend on generically not invertible fields, such as $\hat{\psi}$ and $\hat{\phi}^\dagger$ (which is addition is an odd field). Finally

$$(X_B)\xi^n = 0 \tag{6.16a}$$
$$(X_A)\xi^n = 0 \tag{6.16b}$$
$$(\widehat{X}_\xi)\xi^n = 0 \tag{6.16c}$$
$$\iota(X_A)\xi^n = 0 \tag{6.16d}$$
$$(X_{B^\dagger})\xi^n = 0 \tag{6.16e}$$
$$(X_\phi)\xi^\rho = 0 \ \ \forall\rho = 1,2,3,n \tag{6.16f}$$
$$(X_{\psi^\dagger})\xi^\rho = 0 \ \ \forall\rho = 1,2,3,n \tag{6.16g}$$

Equations (6.13a), (6.13b), (6.13d) and (6.13e) are easily solved. Analysing equation (6.13c), we might require that $B(x)$ is non degenerate as a map from $\bigwedge^2(T_xM)$ to $\bigwedge^2 V$, in order to solve for $(\widehat{X}_\xi)$. Anyway this does not solve the of singularity of equations (6.15). Equations (6.16f) and (6.16g) are also singular: they imply that either ξ is the zero vector field, or we have that $(X_\phi) = (X_{\psi^\dagger}) = 0$.

The kernel of $\widetilde{\omega}$ does not have constant rank, and symplectic reduction cannot be performed. Therefore, the Plebanski formalism for General Relativity does not yield a BV-BFV theory. ✓

From the proof of the Theorem some interesting things emerge. First of all the role of the Lagrange multipliers. The main source of singularity comes, in fact, from equations involving ϕ, ψ, their dual antifilelds and the respective vector field coefficients. Moreover, it is crucial to observe that we enforced the constraint $\langle\langle H_{a,b}, \phi\rangle\rangle = 0$ by means of a Lagrange multiplier, but we could have done it as it is usually done in the literature, i.e. by simply *requiring* ϕ to have null trace. This means that the among the simplicity constraints coming from the variation of the term $\langle\langle \varphi^{(\lambda,\mu)}, BB\rangle\rangle$ with respect to ϕ, some relations would have had to be enforced. Classically this is allowed, but additional care is required when dealing with symmetries and non vanishing boundary conditions, as one should make sure that the constraint be invariant under the action of symmetries, possibly up to boundary terms.

This is indeed the case for what concerns internal gauge symmetries, as the term $\langle\langle H_{a,b}, \phi\rangle\rangle$ vanishes when acted upon by the generator of Lie algebra trasformations, for both ϵ^{ijkl} and

η^{ijkl} are invariant volume-forms. It is nevertheless not true when considering diffeomorphisms as explicit gauge transformations. As a matter of fact, to keep the action invariant one has to compensate for the transformation $\delta^d \phi = L^A_\xi \phi$ with the obvious transformation of the Lagrange multiplier $\delta^d \psi = L_\xi \psi$ (cf. Proposition 6.2).

The interaction between Lagrange multipliers and spacetime diffeomorphisms as gauge symmetries appears to be incompatible, in the sense that the constraints spoil the regularity of the B(F)V theory on the boundary. Part of the problem comes from the fact that multipliers are not fully *dynamical*, in the sense that they are not allowed to have a kinetic term. Such a term would otherwise make the critical locus different from that of General Relativity, and the two theories would not even be classically equivalent anymore. Indeed, one can compare the results in Chapter 5, Section 5.4, to see that in the Palatini-Holst formalism the observation applies, when we try to enforce the Half-Shell constraint with a Lagrange multiplier.

We can probably say that the symmetry breaking is better achieved when a different mechanism is taken into account, which doesn't require localisation on a constraint submanifold. In what follows we will see how this can be done for a theory of general Relativity, by rewriting the MacDowell-Mansouri action[9,50] as a BF action[49,30] for a suitable extension of the Lie algebra.

6.2 MacDowell Mansouri action

The second action of the BF-kind that we would like to approach in the BV-BFV framework was introduced by MacDowell and Mansouri[9], later understood by Wise in the Cartan formalism[50] and rewritten as a BF theory, as reported by Freidel and Speziale[30]. The main idea is to consider the splitting $\mathfrak{so}(4,1) \simeq \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$ and violate the shift symmetry of the BF term while reducing the internal Lie symmetry back to $SO(3,1)$ at the same time by introducing a potential term that explicitly breaks the symmetry. Such a symmetry breaking term is chosen to be

$$S_{SB} = \frac{1}{2} \int\limits_M \epsilon_{ijklm} v^m B^{ij} \wedge B^{kl} = \frac{1}{2} \int\limits_M \epsilon_{ijkl} v \hat{B}^{ij} \wedge \hat{B}^{kl} \tag{6.17}$$

where $v^i = (0,0,0,0,v)$ is a fixed vector in $\mathfrak{so}(4,1)$ and $\hat{B}$ is the projection of B to the $\mathfrak{so}(3,1)$ subalgebra.

Borrowing the notation from Section 6.1, the BF-formulation of the MacDowell-Mansouri action is given by

$$S_{MM} = \int_M \mathrm{Tr}\left[B \wedge F_A - \frac{\beta}{2} B \wedge B - \frac{\alpha}{2}\hat{B} \wedge \hat{B}\right] \tag{6.18}$$

where now A is an $\mathfrak{so}(4,1) = \mathfrak{g}$ connection and B is a two-form with values in the extended algebra $\mathfrak{g}$. Its subalgebra restriction $\hat{B}$ takes values in $\mathfrak{h} = \mathfrak{so}(3,1)$.

The symmetry breaking statement is clarified by the following proposition:

Proposition 6.4. *Let P be a principal $G = SO(4,1)$ bundle over M and let $H = SO(3,1)$ be a subgroup. Consider the fields $c \in \Omega(M, \mathrm{ad}P_H) = \Omega(P, \mathfrak{h})$ where $\mathrm{ad}P_H = P \underset{H}{\times} \mathfrak{h}$. Let $\mathcal{F}_{MM} = T^*[-1]\mathcal{F}^0_{MM}$, endowed with the canonical (-1)-symplectic form Ω, with*

$$\mathcal{F}^0_{MM} = \underbrace{\Omega^2(M, \textstyle\bigwedge^2 \mathcal{V})}_{B} \oplus \underbrace{\Omega^0(M, \mathrm{ad}P_H)[1]}_{c} \oplus \underbrace{\mathcal{A}_P}_{A} \oplus \underbrace{\Gamma(T[1]M)}_{\xi} \tag{6.19}$$

and define a vector field on $\mathcal{F}_{MM}$ by

$$\boxed{\begin{array}{ll} QB = L^A_\xi B - [c,B]; & QA = \iota_\xi F_A - d_A c; \\ Qc = \frac{1}{2}\left(\iota_\xi \iota_\xi F_A - [c,c]\right); & Q\xi = \frac{1}{2}[\xi,\xi]; \end{array}} \tag{6.20}$$

If we denote by S^{BV}_{MM} the minimally extended BV action given by $S^{BV}_{MM} = S_{MM} + (Q\Phi, \Phi^\dagger)$ with Φ being the base fields in $\mathcal{F}_{min}$ and $\Phi^\dagger$ the respective cotangent fields, then the data $(\mathcal{F}_{MM}, S^{BV}_{MM}, Q, \Omega)$ define a BV theory.

Proof. To see that Q is cohomological we resort once more to Proposition 5.8, Chapter 5, Section 5.3, for the field B, and observing that $c \in \Omega(P, \mathfrak{h})$ implies that $c \in \Omega(P, \mathfrak{g})$.

The only nontrivial part in checking that $QS_{MM} = 0$ is to realise that the symmetry breaking term (6.17) is invariant only under the action of the subalgebra $\mathfrak{h}$, and that the subalgebra $\mathfrak{h}$ is invariant under the action of $\mathfrak{g}$ as the splitting $\mathfrak{so}(4,1) = \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$ is

$\mathfrak{so}(4,1)$-invariant. ✓

Now that we have the necessary preparation we can face the main question of this Section.

Claim 6.5. *The BV theory of Proposition 6.4 defined by the data* $(\mathcal{F}_{MM}, S^{BV}_{MM}, Q, \Omega)$ *defines a BV-BFV theory on* M *when it has a boundary,* $\partial M \neq \emptyset$.

Partial Proof. The computations are identical to those of Theorem 6.3, with the difference that all terms containing ϕ, ψ and their relative anti-fields are set to zero. The equations in the kernel of the pre-boundary one form $\widetilde{\omega}$ are grouped as follows:

$$(X_{A^\dagger}) = 0 \tag{6.21a}$$
$$(X_c) = \iota_\xi (X_A) \tag{6.21b}$$
$$(X_A) = \iota_{(X_\xi)} B + B^\dagger_n (X_{\xi^n}) + (X_{B^\dagger_n})\xi^n \tag{6.21c}$$
$$(X_B) = \iota_{(X_\xi)} A^\dagger - (X_{\xi^n}) A^\dagger_n + X A^\dagger_n \xi^n - \frac{1}{2}\iota_\xi (X_{c^\dagger_n})\xi^n + \frac{1}{2}(X_{\xi^n})\iota_\xi c^\dagger_n - \frac{1}{2}\iota_{(X_\xi)} c^\dagger_n \xi^n \tag{6.21d}$$

together with

$$\iota_{(\underline{X_\xi})}\underline{B} = -(X_{B_n})\xi^n \tag{6.22a}$$
$$\mathrm{Tr}\left[(X_{B^\dagger})B\right] = \mathrm{Tr}\left[(X_B)B^\dagger - (X_A)A^\dagger + \frac{1}{2}(X_A)\iota_\xi c^\dagger\right] - (X_{\xi^n})\chi - \mathbb{E}^n \iota_{(\underline{X_\xi})}\underline{\chi} \tag{6.22b}$$

with $\underline{\phi}$ meaning that we include both component transverse and tangent to the boundary, and

$$(X_A)\xi^n = 0 \tag{6.23a}$$
$$(X_B)\xi^n = 0 \tag{6.23b}$$
$$(X_{B^\dagger})\xi^n = 0 \tag{6.23c}$$
$$\iota_\xi (X_A)\xi^n = 0 \tag{6.23d}$$

Equations in group (6.21) are regular. Using the fact that it is possible to define a metric with the combination

$$g_{\mu\nu} = \epsilon^{\alpha\beta\gamma\delta}\eta_{ab}f^{a}_{cd}B^{c}_{\mu\alpha}B^{d}_{\nu\beta}B^{b}_{\gamma\delta} \tag{6.24}$$

when f^{a}_{bc} are the structure constants of the lie algebra $\mathfrak{so}(4,1)$, we can invert equations (6.22), yielding $(\underline{X}_{\xi}) \propto \xi^n$ which then will also imply $(X_{B^\dagger}), (X_A), (X_B) \propto \xi^n$, and equations (6.23) will be automatically satisfied. ✓

Conclusion

This Thesis was devoted to the analysis of different formulations of the theory of General Relativity from the point of view of the (semiclassical) Batalin-Vilkovisky formalism on manifolds with boundary.

The machinery that allows one to induce boundary data starting from an action functional on the boundary, which essentially is analysis of the Noether form, turned out to be useful to describe the classical theory in a manifestly symplectic fashion, even when the symmetries are not taken into account dynamically (see Proposition 4.8 and Theorem 5.6). As a matter of fact, this allows us to simplify the canonical Dirac analysis, by explicitly describing the symplectic space of boundary fields (which contains the coisotropic subalgebra of canonical constraints).

The rest of the BV-BFV procedure, i.e. when symmetries are included, completes the canonical description of the theory by yielding an explicit resolution of the said coisotropic submanifold modulo gauge symmetries. Such information is encoded in the boundary action S^{∂}, the Hamiltonian function of the boundary cohomological vector field Q^{∂}, whenever it can be computed and the axioms are satisfied.

The comparison of different formulations of General Relativity as a fundamental theory of the gravitational interaction highlighted some fundamental differences in the BV structures that the theories enjoy, marking a deviation with respect to the classical (i.e. non BV) behaviours, which are considered to be *equivalent*. The very fact that (e.g.) the Einstein Hilbert formulation of GR does satisfy the CMR axioms, while the Palatini Holst formulation does not, unequivocally tells us that the BV-BFV criterion *refines* the notion of equivalence between gauge field theories.

This has crucial consequences in understanding to which extent we can replace theories with one another and could possibly suggest that a particular formulation is more suitable than others in view of quantisation. In any case, a clarification of what can be legitimately considered *equivalent* is fundamental for the development of quantum field theory.

Moreover, it is interesting to notice that the failure of some action functionals to extend to a BV-BFV theory gives us a better understanding of the use of Lagrange multipliers and their relationship with diffeomorphisms as a gauge symmetry. In fact, comparing the

Plebanski action and the MacDowell-Mansouri actions for GR, both seen as broken BF theories, the specific way we choose to break the symmetry does indeed matter, when deciding if the BV data satisfies the CMR axioms.

The first challenge then is that of *fixing* the PH theory of gravity in the BV setting, that is find the correct formulation under which it satisfies the CMR axioms, and interpret why the theory does not work with the natural assumption we have considered here.

Secondly, it will be important to understand the profound meaning of the failure of the CMR axioms and the possible way of fixing them. Our guess is that the BV-BFV construction and the *naturality* of the way the boundary data is induced should indicate the correct prescriptions to explicitly treat symmetries. More explicitly, requiring that *some* BV-extension of the theory satisfies the CMR axioms and resolves the reduced phase space, which in the end is classical data and it is well defined before the BV analysis is taken into account, will tell us what are the correct symmetry prescriptions of the theory.

In the future it will be important to make contact with the physics literature and language, translating problems and results from theoretical physics to test this formalism further, and to exploit the power of the BV formalism in the quantum gravity communities' daily routine.

Eventually, once the tetradic and/or BF formulation of GR will be understood as a BV-BFV theory, it will be of prime importance to attack the problem of its (perturbative) quantisation, for instance by applying (and possibly adapting) the CMR prescription for the quantisation of BV-BFV theories.

All in all, this is a first step in that direction that has the good feature of highlighting an important *caveat* when regarding alternative theories as equivalent, and this will actively help us understanding which path should be chosen when treading towards a quantum theory of gravity.

A

Cumbersome or lengthy computations

Computation A.1 (Theorem 3.2, Chapter 3). *We want to compute the explicit expression of the projectable pre-boundary vector field* $\widetilde{Q}' = \widetilde{Q} - \widetilde{Q}_g \mathbb{\Gamma} - \widetilde{Q}_{g^\dagger} \Xi^\dagger$*. Recalling that*

$$\begin{aligned}
\widetilde{Q}\, g &= \xi \dot{g} + 2g\dot{\xi} \\
\widetilde{Q}\, \xi &= \xi\dot{\xi} \\
\widetilde{Q}\, g^\dagger &= \frac{\Lambda}{2\sqrt{g}} - \dot{\xi} g^\dagger + \xi \dot{g}^\dagger \\
\widetilde{Q}\, \xi^\dagger &= \dot{g} g^\dagger + 2g\dot{g}^\dagger + 2\dot{\xi}\xi^\dagger + \xi\dot{\xi}^\dagger
\end{aligned}$$

and that

$$\begin{aligned}
\mathbb{\Gamma} &:= \frac{\delta}{\delta g} + \left(\frac{\xi\xi^\dagger}{2g^2} - \frac{g^\dagger}{2g}\right)\frac{\delta}{\delta g^\dagger} - \frac{\xi}{2g}\frac{\delta}{\delta \xi} \\
\Xi^\dagger &:= \frac{\delta}{\delta \xi^\dagger} - \frac{\xi}{2g}\frac{\delta}{\delta g^\dagger}
\end{aligned}$$

we expand

$$\widetilde{Q}' = (\xi\dot{\xi})\frac{\delta}{\delta\xi} + \left(\frac{\Lambda}{2\sqrt{g}} - \dot{\xi}g^{\dagger} + \xi\dot{g}^{\dagger}\right)\frac{\delta}{\delta g^{\dagger}} - (\xi\dot{g} + 2g\dot{\xi})\left(\frac{\xi\xi^{\dagger}}{2g^2} - \frac{g^{\dagger}}{2g}\right)\frac{\delta}{\delta g^{\dagger}}$$
$$+ (\xi\dot{g} + 2g\dot{\xi})\frac{\xi}{2g}\frac{\delta}{\delta\xi} + (\dot{g}g^{\dagger} + 2g\dot{g}^{\dagger} + 2\dot{\xi}\xi^{\dagger} + \xi\dot{\xi}^{\dagger})\frac{\xi}{2g}\frac{\delta}{\delta g^{\dagger}}$$
$$= \left(\frac{\Lambda}{2\sqrt{g}} - \dot{\xi}g^{\dagger} + \xi\dot{g}^{\dagger} + \frac{\xi\dot{g}g^{\dagger}}{2g} - \frac{\dot{\xi}\xi\xi^{\dagger}}{g} + \dot{\xi}g^{\dagger} + \frac{\dot{g}g^{\dagger}\xi}{2g} + \dot{g}^{\dagger}\xi + \frac{\dot{\xi}\xi^{\dagger}\xi}{g}\right)\frac{\delta}{\delta g^{\dagger}} = \frac{\Lambda}{2\sqrt{g}}\frac{\delta}{\delta g^{\dagger}} \tag{A.1}$$

using now the transformation law $\frac{\delta}{\delta g^{\dagger}} = \frac{\sqrt{g}}{2}\frac{\delta}{\delta \widetilde{g}^{\dagger}}$ *we obtain the result:*

$$Q^{\partial} = \frac{\Lambda}{4}\frac{\delta}{\delta\widetilde{g}^{\dagger}} \tag{A.2}$$

Computation A.2 (Theorem 4.4, Chapter 4)**.** *Expanding the expression for the vector field:*

$$Q' = Q - (Q_{g^{\dagger}})^{ab}\mathbb{G}^{\dagger ab} - (Q_{\chi})_{\mu}\mathbb{X}_{\mu} - (Q_g)^{nn}\mathbb{G}^{nn}$$

we get

$$Q' = (Q_g)^{ab}\frac{\delta}{\delta g_{ab}} + (Q_{\xi})^{\mu}\frac{\delta}{\delta\xi^{\mu}} + (Q_{g^{\dagger}})^{n\nu}\frac{\delta}{\delta g^{\dagger n\nu}} - \frac{2}{\sqrt{\mathrm{g}}}(Q_{g^{\dagger}})^{ab}g_{nn}\left(g_{ac}g_{bd} - \frac{1}{d-1}g_{ab}g_{bd}\right)\xi^n\frac{\delta}{\delta J_{cd}}$$
$$+ \left(\frac{1}{2}(Q_g)^{nn}g_{nn}J_{cd} - \frac{1}{\sqrt{\mathrm{g}}}(Q_g)^{nn}g_{nn}^2\left(g^{\dagger ab}g_{ac}g_{bd} - \frac{1}{d-1}g^{\dagger ab}g_{ab}g_{bd}\right)\xi^n\right)\frac{\delta}{\delta J_{cd}}$$
$$+ \frac{1}{2}(Q_{\chi})_n g^{nn}\xi^n\frac{\delta}{\delta g^{\dagger nn}} + \frac{1}{2}(Q_{\chi})_a g^{ab}\xi^n\frac{\delta}{\delta g^{\dagger bn}} - \frac{1}{2}\left((Q_g)^{nn}g_{nn}g^{\dagger nn} - (Q_g)^{nn}\chi_n\xi^n\right)\frac{\delta}{\delta g^{\dagger nn}}$$
$$+ \frac{1}{4}(Q_g)^{nn}g_{nn}g^{ba}\chi_a\xi^n\frac{\delta}{\delta g^{\dagger nb}} - \frac{1}{2}(Q_g)^{nn}g_{nn}\xi^n\frac{\delta}{\delta\xi^n} + (Q_J)_{ef}\frac{\delta}{\delta J_{ef}} \tag{A.3}$$

We transform the generators $\frac{\delta}{\delta\Phi}$ deriving the following formulas:

$$
\begin{aligned}
\frac{\delta}{\delta \xi^n} &= \sqrt{g_{nn}}\frac{\delta}{\delta\widetilde{\xi^n}} + \frac{2}{\sqrt{\mathrm{g}^{\partial}}} g^{\dagger cd}\left(g_{ce}g_{bf} - \frac{1}{d-1} g_{cd}g_{ef}\right)\frac{\delta}{\delta\widetilde{J}_{ef}} + \frac{1}{2}\sqrt{g^{nn}}\chi_n\frac{\delta}{\delta\widetilde{g}^{\dagger nn}} + \frac{1}{2} g^{ab}\chi_b\frac{\delta}{\delta\widetilde{g}^{\dagger na}}\\
\frac{\delta}{\delta g_{ab}} &= \frac{\delta}{\delta\widetilde{g}_{ab}} - \frac{1}{2} g^{ca} g^{db}\chi_d\xi^n\frac{\delta}{\delta\widetilde{g}^{\dagger nc}} + \frac{1}{\sqrt{\mathrm{g}^{\partial}}} g^{ab} g^{\dagger cd}\left(g_{ce}g_{df} - \frac{1}{d-1} g_{cb}g_{df}\right)\xi^n\frac{\delta}{\delta\widetilde{J}_{ef}} +\\
&\quad - \frac{2}{\sqrt{\mathrm{g}^{\partial}}} g^{\dagger cd}\left(\delta^a_c\delta^b_e g_{df} + g_{ce}\delta^a_d\delta^b_f - \frac{1}{d-1}\left(\delta^a_c\delta^b_d g_{ef} + g_{cd}\delta^a_e\delta^b_f\right)\right)\xi^n\frac{\delta}{\delta\widetilde{J}_{ef}}\\
\frac{\delta}{\delta J_{ef}} &= \sqrt{g^{nn}}\frac{\delta}{\delta\widetilde{J}_{ef}}\\
\frac{\delta}{\delta g^{\dagger nn}} &= \sqrt{g_{nn}}\frac{\delta}{\delta\widetilde{g}^{\dagger nn}}\\
\frac{\delta}{\delta g^{\dagger na}} &= \frac{\delta}{\delta\widetilde{g}^{\dagger na}}\\
\frac{\delta}{\delta \xi^a} &= \frac{\delta}{\delta\widetilde{\xi}^a}
\end{aligned}
\tag{A.4}
$$

which lead Q' to the have following expression:

$$
\begin{aligned}
Q' = &\left\{ \frac{1}{\sqrt{g^\partial}} (Q_g)_{ab} g^{ab} g^{\dagger cd} \left(g_{ce} g_{df} - \frac{1}{d-1} g_{cd} g_{ef} \right) \xi^n + \frac{2}{\sqrt{g^\partial}} (Q_\xi)^n g^{\dagger cd} \left(g_{ce} g_{df} - \frac{1}{d-1} g_{cd} g_{ef} \right) \right. \\
&- \frac{2}{\sqrt{g^\partial}} \left((Q_g)_{ce} g^{\dagger cd} g_{df} + (Q_g)_{df} g^{\dagger cd} g_{ce} - \frac{1}{d-1} (Q_g)_{cd} g^{\dagger cd} g_{ef} - \frac{1}{d-1} (Q_g)_{ef} g^{\dagger cd} g_{cd} \right) \xi^n \\
&\left. - \frac{2}{\sqrt{g^\partial}} (Q_{g^\dagger})^{cd} \left(g_{ce} g_{df} - \frac{1}{d-1} g_{cd} g_{ef} \right) \xi^n - \frac{1}{2} (Q_g)_{nn} g^{nn} \sqrt{g^{nn}} J_{ef} \right\} \frac{\delta}{\delta \widetilde{J}_{ef}} \\
&+ \left\{ (Q_{g^\dagger})^{nc} - \frac{1}{2} (Q_g)_{ab} g^{ac} g^{bd} \chi_d \xi^n + \frac{1}{2} (Q_\xi)^n g^{cd} \chi_d + \frac{1}{2} (Q_\chi)_d g^{cd} \xi^n \right\} \frac{\delta}{\delta \widetilde{g}^{\dagger nc}} \\
&+ \left\{ (Q_{g^\dagger})^{nn} + \frac{1}{2} (Q_g)_{nn} \sqrt{g^{nn}} g^{\dagger nn} + \frac{1}{2} (Q_\xi)^n \sqrt{g^{nn}} \chi_n \right. \\
&\left. + \frac{1}{2} (Q_\chi)_n \sqrt{g^{nn}} \xi^n - \frac{1}{4} (Q_g)_{nn} (g^{nn})^{\frac{3}{2}} \chi_n \xi^n \right\} \frac{\delta}{\delta \widetilde{g}^{\dagger nn}} \\
&+ \left\{ (Q_\xi)^n \sqrt{g_{nn}} + \frac{1}{2} (Q_g)_{nn} \sqrt{g^{nn}} \xi^n \right\} \frac{\delta}{\delta \widetilde{\xi}^n} \\
&+ \sqrt{g^{nn}} (Q_J)_{ef} \frac{\delta}{\delta \widetilde{J}_{ef}} + (Q_\xi)^a \frac{\delta}{\delta \widetilde{\xi}^a} + (Q_g)_{ab} \frac{\delta}{\delta \widetilde{g}_{ab}}
\end{aligned}
\tag{A.5}
$$

Expanding and computing the coefficient of $\frac{\delta}{\delta \widetilde{J}_{ef}}$ one obtains the following intermediate expression:

$$
\begin{aligned}
&\pi^* \left\{ \widetilde{\xi}^s \partial_s \widetilde{J}_{ef} + 2 \partial_{(e} \widetilde{\xi}^a \widetilde{J}_{f)a} - \frac{4}{\sqrt{\widetilde{g}^\partial}} \left(\partial_{(e} \widetilde{\xi}^n \widetilde{g}^{\dagger dn} \widetilde{g}_{f)d} - \frac{1}{d-1} \partial_d \widetilde{\xi}^n \widetilde{g}^{\dagger nd} \widetilde{g}_{ef} \right) \widetilde{\xi}^n \right\} \\
&+ \left\{ - \sqrt{g_{nn}} g^{ab} \partial_b \xi^n \partial_a g_{ef} + \xi^n \partial_n (J_{ef} \sqrt{g^{nn}}) - 2 \partial_{(e} (g_{nn} g^{ab} \partial_b \xi^n) g_{f)a} \sqrt{g^{nn}} \right. \\
&\qquad \left. + 2 \sqrt{g^{nn}} \partial_{(e} \xi^n J_{f)n} + 2 \sqrt{g_{nn}} \underbrace{\left(R_{(ef)} + \frac{R^n - R^\partial}{2(d-1)} g_{ef} \right)}_{\text{Einstein term}} \xi^n \right\}
\end{aligned}
$$

The goal is clearly to write the whole coefficient as the pullback of some function on $\mathcal{F}^\partial$. Notice that the Einstein term *comes from the following computation:*

$$
E.T. = \left(R^{cd} - \frac{R}{2} g^{cd} \right) \left(g_{c(e} g_{f)d} - \frac{1}{d-1} g_{cd} g_{ef} \right)
$$

where $R = g^{\mu\nu}R_{\mu\nu} = g^{nn}R_{nn} + g^{ab}R_{ab} =: R^n + \overline{R}$*. Hence:*

$$E.T. = R_{(ef)} - \frac{\overline{R}}{d-1}g_{ef} - \frac{R^n + \overline{R}}{2}g_{ef} + \frac{R^n + \overline{R}}{2}\frac{d}{d-1}g_{ef} = R_{(ef)} + \frac{R^n - \overline{R}}{2(d-1)}g_{ef}$$

Now, the Ricci tensor and the difference between the two traces read:

$$\begin{aligned} 2\sqrt{g_{nn}}R_{ef}\xi^n =& - \partial_n\left(J_{ef}\sqrt{g^{nn}}\right)\xi^n - \frac{1}{2}\sqrt{g_{nn}}\partial_{(e}g^{nn}\partial_{f)}g_{nn}\xi^n \\ &- \sqrt{g^{nn}}\partial_{(e}\partial_{f)}g_{nn}\xi^n + 2\partial_{(e}g_{f)b}g^{ba}\partial_a\sqrt{g_{nn}}\xi^n \\ &+ \pi^*\left\{\widetilde{J}_{b(e}\widetilde{g}^{ba}\widetilde{J}_{f)a}\widetilde{\xi}^n - \frac{1}{2}\widetilde{J}_{ef}\widetilde{g}^{ba}\widetilde{J}_{ab}\widetilde{\xi}^n + 2\widetilde{R}^\partial_{ef}\widetilde{\xi}^n\right\} \\ &- \partial_b g_{ef}g^{ba}\partial_a\sqrt{g_{nn}}\xi^n \end{aligned} \tag{A.6}$$

$$\begin{aligned} \sqrt{g_{nn}}(R^n - \overline{R})\xi^n =& \xi^n\partial_a\left(g^{ab}J_{bn}\sqrt{g^{nn}}\right) + \frac{1}{2}\xi^n\sqrt{g^{nn}}J_{nb}g^{ba}g^{cd}\partial_a g_{cd} \\ &+ \pi^*\left\{\frac{1}{4}\widetilde{g}^{ab}\widetilde{J}_{ab}\widetilde{g}^{cd}\widetilde{J}_{cd}\widetilde{\xi}^n - \frac{1}{4}\widetilde{g}^{ab}\widetilde{J}_{cb}\widetilde{g}^{cd}\widetilde{J}_{ad}\widetilde{\xi}^n - \widetilde{R}^\partial\widetilde{\xi}^n\right\} \end{aligned} \tag{A.7}$$

Where $\widetilde{R}^\partial_{ef}$ *denotes the Ricci tensor of the* d*-dimensional manifold* ∂M*, and* $\widetilde{R}^\partial$ *its trace. Notice that* $R_{ef} \neq \widetilde{R}^\partial_{ef}$ *and* $\overline{R} \neq \widetilde{R}^\partial$*.*

Putting all together, after some rewriting we get:

$$\begin{aligned} \pi^*\Bigg\{&\widetilde{\xi}^s\partial_s\widetilde{J}_{ef} + 2\partial_{(e}\widetilde{\xi}^a\widetilde{J}_{f)a} - 2\partial_{(e}\partial_{f)}\widetilde{\xi}^n - \widetilde{g}^{ab}\partial_a\widetilde{g}_{ef}\partial_b\widetilde{\xi}^n + 2\partial_{(e}\widetilde{g}_{f)b}\widetilde{g}^{ab}\partial_a\widetilde{\xi}^n \\ &- \frac{4}{\sqrt{\widetilde{g}^\partial}}\left(\partial_{(e}\widetilde{\xi}^n\widetilde{g}^{\dagger dn}\widetilde{g}_{f)d} - \frac{1}{d-1}\partial_d\widetilde{\xi}^n\widetilde{g}^{\dagger nd}\widetilde{g}_{ef}\right)\widetilde{\xi}^n + \left(2\widetilde{R}^\partial_{ef} - \frac{\widetilde{R}^\partial}{d-1}\widetilde{g}_{ef}\right)\widetilde{\xi}^n \\ &+ \widetilde{J}_{b(e}\widetilde{g}^{ba}\widetilde{J}_{f)a}\widetilde{\xi}^n - \frac{1}{2}\widetilde{J}_{ef}\widetilde{g}^{ba}\widetilde{J}_{ab}\widetilde{\xi}^n + \frac{1}{d-1}\left(\frac{1}{4}\widetilde{g}^{ab}\widetilde{J}_{ab}\widetilde{g}^{cd}\widetilde{J}_{cd} - \frac{1}{4}\widetilde{g}^{ab}\widetilde{J}_{cb}\widetilde{g}^{cd}\widetilde{J}_{ad}\right)\widetilde{g}_{ef}\widetilde{\xi}^n\Bigg\}\frac{\delta}{\delta\widetilde{J}_{ef}} \\ &+ \left\{2\sqrt{g^{nn}}\partial_{(e}\xi^n J_{f)n} + \xi^n\partial_a\left(g^{ab}J_{bn}\sqrt{g^{nn}}\right) + \frac{1}{2}\xi^n\sqrt{g^{nn}}J_{nb}g^{ba}g^{cd}\partial_a g_{cd}\right\}\frac{\delta}{\delta\widetilde{J}_{ef}} \end{aligned} \tag{A.8}$$

Let us turn now to the $\widetilde{g}^{\dagger nn}$ *coefficient. We first notice that the term* $\sqrt{g}\left(R^{nn} - \frac{R}{2}g^{nn}\right)\sqrt{g^{nn}}$ *coming from the* $(Q_{g^\dagger})^{nn}\sqrt{g^{nn}}$ *factor is computed as:*

$$\sqrt{g}\left(R^{nn} - \frac{R}{2}g^{nn}\right)\sqrt{g_{nn}} = \sqrt{g^\partial}g_{nn}\left(R^n g^{nn} - \frac{1}{2}R^n g^{nn} - \frac{1}{2}R^\partial g^{nn}\right) = \frac{\sqrt{g^\partial}}{2}\left(R^n - R^\partial\right)$$

Using this, we can expand the $g^{\dagger nn}$ *and obtain:*

$$
\begin{aligned}
Q'\Big|_{\widetilde{g}^{\dagger nn}} = \pi^*\Bigg\{&\partial_a\left(\widetilde{\xi}^a\widetilde{g}^{\dagger nn}\right) + \partial_a\widetilde{g}^{\dagger an}\widetilde{\xi}^n + 2\widetilde{g}^{\dagger na}\partial_a\widetilde{\xi}^n \\
&- \frac{\sqrt{\widetilde{g}^\partial}}{8}\widetilde{g}^{ab}\left(\widetilde{J}_{ab}\widetilde{J}_{cd} - \widetilde{J}_{cb}\widetilde{J}_{ad}\right)\widetilde{g}^{cd} + \frac{\sqrt{\widetilde{g}^\partial}}{2}\widetilde{R}^\partial\Bigg\}\frac{\delta}{\delta g^{\dagger nn}} \\
&\left\{\frac{1}{2}J_{an}g^{\dagger an}\sqrt{g^{nn}}\xi^n - \partial_a\left(\frac{\sqrt{g^\partial}}{2}g^{ab}J_{bn}g^{nn}\right)\right\}\frac{\delta}{\delta g^{\dagger nn}}
\end{aligned} \tag{A.9}
$$

For the $g^{\dagger nc}$ *coefficient we will need to compute the following combination:*

$$
\begin{aligned}
-\sqrt{g}R^{nc} &= -\sqrt{g^\partial}\sqrt{g^{nn}}R_{nb}g^{bc} = \\
&-\partial_a\left(\frac{\sqrt{g^\partial}}{2}g^{ad}J_{db}\sqrt{g^{nn}}\right)g^{bc} + \frac{\sqrt{g^\partial}}{2}\partial_b\left(g^{ad}J_{ad}\sqrt{g^{nn}}\right)g^{bc} - \frac{\sqrt{g^\partial}}{4}\sqrt{g^{nn}}g^{cb}\partial_b g^{ad}J_{ad}
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\pi^*\Bigg\{&\frac{\sqrt{\widetilde{g}^\partial}}{2}\partial_b\left(\widetilde{g}^{ad}\widetilde{J}_{ad}\right)\widetilde{g}^{bc} - \partial_a\left(\frac{\sqrt{\widetilde{g}^\partial}}{2}\widetilde{g}^{ad}\widetilde{J}_{db}\right)\widetilde{g}^{bc} - \frac{\sqrt{\widetilde{g}^\partial}}{4}\widetilde{g}^{cb}\partial_b\widetilde{g}^{ad}\widetilde{J}_{ad} \\
&+ \partial_s\left(\widetilde{\xi}^s\widetilde{g}^{\dagger nc}\right) + \widetilde{g}^{cb}\partial_b\widetilde{\xi}^n\widetilde{g}^{\dagger nn} - \partial_b\widetilde{\xi}^c\widetilde{g}^{\dagger bn} + \widetilde{J}_{ad}\widetilde{g}^{\dagger an}\widetilde{g}^{dc}\widetilde{\xi}^n\Bigg\}\frac{\delta}{\delta\widetilde{g}^{\dagger nc}} \\
&+\left\{J_{nd}g^{\dagger nn}g^{dc}\xi^n - \frac{\sqrt{g^\partial}}{2}(g^{nn})^{\frac{3}{2}}J_{nd}g^{db}J_{ba}g^{ac}\right\}\frac{\delta}{\delta\widetilde{g}^{\dagger nc}}
\end{aligned}
$$

Finally, the remaining terms for $Q'\Big|_{\widetilde{\xi}^n}$, $Q'\Big|_{\widetilde{\xi}^a}$ *and* $Q'\Big|_{\widetilde{g}_{ab}}$ *are easily computed:*

$$
\begin{aligned}
\left\{(Q_\xi)^n\sqrt{g_{nn}} + \frac{1}{2}(Q_g)_{nn}\sqrt{g^{nn}}\xi^n\right\}\frac{\delta}{\delta\widetilde{\xi}^n} &= \pi^*\left\{\widetilde{\xi}^a\partial_a\widetilde{\xi}^n\right\}\frac{\delta}{\delta\widetilde{\xi}^n} \\
(Q_\xi)^a\frac{\delta}{\delta\widetilde{\xi}^a} &= \pi^*\left\{\widetilde{\xi}^b\partial_b\widetilde{\xi}^a - \widetilde{\xi}^n\widetilde{g}^{ab}\partial_b\widetilde{\xi}^n\right\}\frac{\delta}{\delta\widetilde{\xi}^a} \\
(Q_g)_{ab}\frac{\delta}{\delta\widetilde{g}_{ab}} &= \pi^*\left\{\widetilde{\xi}^n\widetilde{J}_{ab} + \widetilde{\xi}^c\partial_c\widetilde{g}_{ab} + 2\partial_a\widetilde{\xi}^c\widetilde{g}_{bc}\right\}\frac{\delta}{\delta\widetilde{g}_{ab}}
\end{aligned}
$$

where, in the second line, we have used (4.10)*. Altogether, the components of* Q^∂ *read*

$$(Q^{\partial}_{\widetilde{J}})_{ef} = \Bigg\{ \widetilde{\xi}^s \partial_s \widetilde{J}_{ef} + 2\partial_{(e} \widetilde{\xi}^a \widetilde{J}_{f)a} - \frac{4}{\sqrt{\widetilde{g}^{\partial}}} \left(\partial_{(e} \widetilde{\xi}^n \widetilde{g}^{\dagger dn} \widetilde{g}_{f)d} - \frac{1}{d-1} \partial_d \widetilde{\xi}^n \widetilde{g}^{\dagger nd} \widetilde{g}_{ef} \right) \widetilde{\xi}^n$$

$$+ \left(2\widetilde{R}^{\partial}_{ef} - \frac{\widetilde{R}^{\partial}}{d-1} \widetilde{g}_{ef} \right) \widetilde{\xi}^n - 2\partial_{(e} \partial_{f)} \widetilde{\xi}^n - \widetilde{g}^{ab} \partial_a \widetilde{g}_{ef} \partial_b \widetilde{\xi}^n + 2\partial_{(e} \widetilde{g}_{f)b} \widetilde{g}^{ab} \partial_a \widetilde{\xi}^n$$

$$+ \widetilde{J}_{b(e} \widetilde{g}^{ba} \widetilde{J}_{f)a} \widetilde{\xi}^n - \frac{1}{2} \widetilde{J}_{ef} \widetilde{g}^{ba} \widetilde{J}_{ab} \widetilde{\xi}^n + \frac{1}{d-1} \left(\frac{1}{4} \widetilde{g}^{ab} \widetilde{J}_{ab} \widetilde{g}^{cd} \widetilde{J}_{cd} - \frac{1}{4} \widetilde{g}^{ab} \widetilde{J}_{cb} \widetilde{g}^{cd} \widetilde{J}_{ad} \right) \widetilde{g}_{ef} \widetilde{\xi}^n \Bigg\} \tag{A.10a}$$

$$(Q^{\partial}_{\widetilde{g}^{\dagger}})^{nn} = \Bigg\{ \partial_a \left(\widetilde{\xi}^a \widetilde{g}^{\dagger nn} \right) + \partial_a \widetilde{g}^{\dagger an} \widetilde{\xi}^n + 2\widetilde{g}^{\dagger na} \partial_a \widetilde{\xi}^n - \frac{\sqrt{\widetilde{g}^{\partial}}}{8} \widetilde{g}^{ab} \left(\widetilde{J}_{ab} \widetilde{J}_{cd} - \widetilde{J}_{cb} \widetilde{J}_{ad} \right) \widetilde{g}^{cd} + \frac{\sqrt{\widetilde{g}^{\partial}}}{2} \widetilde{R}^{\partial} \Bigg\} \tag{A.10b}$$

$$(Q^{\partial}_{\widetilde{g}^{\dagger}})^{nc} = \Bigg\{ \frac{\sqrt{\widetilde{g}^{\partial}}}{2} \partial_b \left(\widetilde{g}^{ad} \widetilde{J}_{ad} \right) \widetilde{g}^{bc} - \partial_a \left(\frac{\sqrt{\widetilde{g}^{\partial}}}{2} \widetilde{g}^{ad} \widetilde{J}_{db} \right) \widetilde{g}^{bc} - \frac{\sqrt{\widetilde{g}^{\partial}}}{4} \widetilde{g}^{cb} \partial_b \widetilde{g}^{ad} \widetilde{J}_{ad}$$

$$+ \partial_s \left(\widetilde{\xi}^s \widetilde{g}^{\dagger nc} \right) + \widetilde{g}^{cb} \partial_b \widetilde{\xi}^n \widetilde{g}^{\dagger nn} - \partial_b \widetilde{\xi}^c \widetilde{g}^{\dagger bn} + \widetilde{J}_{ad} \widetilde{g}^{\dagger an} \widetilde{g}^{dc} \widetilde{\xi}^n \Bigg\} \tag{A.10c}$$

$$(Q^{\partial}_{\widetilde{\xi}})^n = \left\{ \widetilde{\xi}^a \partial_a \widetilde{\xi}^n \right\} \tag{A.10d}$$

$$(Q^{\partial}_{\widetilde{\xi}})^a = \left\{ \widetilde{\xi}^b \partial_b \widetilde{\xi}^a - \widetilde{\xi}^n \widetilde{g}^{ab} \partial_b \widetilde{\xi}^n \right\} \tag{A.10e}$$

$$(Q^{\partial}_{\widetilde{g}})_{ab} = \left\{ \widetilde{\xi}^n \widetilde{J}_{ab} + \widetilde{\xi}^c \partial_c \widetilde{g}_{ab} + 2\partial_{(a} \widetilde{\xi}^c \widetilde{g}_{b)c} \right\} \tag{A.10f}$$

Computation A.3 (Theorem 4.12, Chapter 4). *We use the vertical vector fields of Theorem 4.11 to find an explicit section of the surjective submersion $\pi : \mathcal{F}_{ADM} \longrightarrow \mathcal{F}^{\partial}_{ADM}$. First of all, use $\mathbb{B}$ to set $\beta_a = 0$, this implies $(X_\beta)_a = -\beta^0_a$ together with $\beta_a(t) = (1-t)\beta^0_a$ and we have the first two differential equations:*

$$\dot{\xi}^b = +\gamma^{ab} \beta^0_a \xi^n \tag{A.11a}$$

$$\dot{g}^{\dagger nn} = -\frac{\epsilon}{2} \eta^{-2} \gamma^{ab} \beta^0_a \chi_b \xi^n \tag{A.11b}$$

$$\tag{A.11c}$$

that are easily solved to yield

$$\xi(t) = \xi^b_0 + \gamma ba \beta^0_a \xi^n t \tag{A.12a}$$

$$g^{\dagger nn}(t) = g^{\dagger nn}_0 - \frac{\epsilon}{2} \eta^{-2} \gamma^{ab} \beta^0_a \chi_b \xi^n t \tag{A.12b}$$

we use (A.12b) and the time rule for $\beta_a(t)$ to solve the third and fourth equations

$$
\begin{aligned}
\dot{g}^{\dagger nb} =& +\frac{\epsilon}{2}\eta^{-2}\beta^b\gamma^{cd}\beta^0_c\chi_d\xi^n + \gamma^{ab}\beta^0_a g^{\dagger nn} \\
=& \gamma^{ba}\beta^0_a g^{\dagger nn}_0 - \frac{\epsilon}{2}\eta^{-2}\gamma^{ba}\beta^0_a\gamma^{cd}\beta^0_c\chi_d\xi^n(2t-1) \\
g^{\dagger nb}(t) =& g^{\dagger nb}_0 + \gamma^{ba}\beta^0_a g^{\dagger nn}_0 t - \frac{\epsilon}{2}\eta^{-2}\gamma^{ba}\beta^0_a\gamma^{cd}\beta^0_c\chi_d\xi^n(t^2-t)
\end{aligned}
$$

$$
\begin{aligned}
\dot{J}_{lm} =& -\left(2\nabla_{(l}\beta^0_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta^0_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta^0_a\right)g^{\dagger an}\xi^n\right) \\
=& -2\nabla_{(l}\beta^0_{m)} - \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta^0_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta^0_a\right)g^{\dagger an}_0\xi^n + \\
& -\frac{2\epsilon}{\sqrt{\gamma}}\left(\beta^0_{(l}\beta^0_{m)} - \frac{1}{d-1}\gamma_{lm}\beta^0_b\beta^b_0\right)g^{\dagger nn}_0\xi^n t \\
J_{lm} =& -2\nabla_{(l}\beta^0_{m)}t - \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta^0_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta^0_a\right)g^{\dagger an}_0\xi^n t + \\
& -\frac{2\epsilon}{\sqrt{\gamma}}\left(\beta^0_{(l}\beta^0_{m)} - \frac{1}{d-1}\gamma_{lm}\beta^0_b\beta^b_0\right)g^{\dagger nn}_0\xi^n t^2
\end{aligned}
$$

So we can fix the temporary value of our fields at $t = 1$ to

$$
\begin{aligned}
\hat{J}_{lm} =& -2\nabla_{(l}\beta^0_{m)} - \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta^0_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta^0_a\right)g^{\dagger an}_0\xi^n + \\
& -\frac{2\epsilon}{\sqrt{\gamma}}\left(\beta^0_{(l}\beta^0_{m)} - \frac{1}{d-1}\gamma_{lm}\beta^0_b\beta^b_0\right)g^{\dagger nn}_0\xi^n \qquad \text{(A.13a)} \\
\hat{g}^{\dagger nb} =& g^{\dagger nb} + \gamma^{ba}\beta_a g^{\dagger nn} \qquad \text{(A.13b)} \\
\hat{g}^{\dagger nn} =& g^{\dagger nn} - \frac{\epsilon}{2}\eta^{-2}\gamma^{ab}\beta_a\chi_b\xi^n \qquad \text{(A.13c)}
\end{aligned}
$$

Now we can turn to the vector fields $\mathbb{X}_\rho$ and use them to set $\chi_\rho = 0$ at some value of the internal evolution parameter s. As usual we impose $(X_\chi)_\rho = -\chi^0_\rho$ and $\chi_\rho(t) = (1-t)\chi^0_\rho$. The new equations are

$$
\begin{aligned}
\dot{g}^{\dagger nn} =& +\frac{\epsilon}{2}\eta^{-2}\chi^0_n\xi^n \\
\dot{g}^{\dagger nb} =& -\frac{\epsilon}{2}\gamma^{ba}\chi^0_a\xi^n
\end{aligned}
$$

which will yield an additional correction to the temporary value of our fields:

$$\hat{\hat{g}}^{\dagger nn} = g^{\dagger nn} + \frac{\epsilon}{2}\eta^{-2}\left(\chi_n - \gamma^{ab}\beta_a\chi_b\right)\xi^n \tag{A.14a}$$

$$\hat{\hat{g}}^{\dagger nb} = g^{\dagger nb} + \gamma^{ba}\beta_a g^{\dagger nn} - \frac{\epsilon}{2}\gamma^{ba}\chi_a\xi^n \tag{A.14b}$$

Similar is what happens when we use $\mathbb{G}^{\dagger ab}$*, for we get the equation*

$$\dot{J}_{lm} = -\frac{2\epsilon}{\sqrt{\gamma}}\eta\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{ab}\gamma_{lm}\right)g_0^{\dagger ab}\xi^n$$

that will correct the temporary value of $\hat{J}_{lm}$ *to*

$$\begin{aligned}\hat{\hat{J}}_{lm} = &-2\nabla_{(l}\beta^0_{m)} - \frac{4\epsilon}{\sqrt{\gamma}}\eta\left(\beta^0_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta^0_a\right)g_0^{\dagger an}\xi^n + \\ &- \frac{2\epsilon}{\sqrt{\gamma}}\left(\beta^0_{(l}\beta^0_{m)} - \frac{1}{d-1}\gamma_{lm}\beta^0_b\beta^b_0\right)g_0^{\dagger nn}\xi^n \\ &- \frac{2\epsilon}{\sqrt{\gamma}}\eta\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{ab}\gamma_{lm}\right)g_0^{\dagger ab}\xi^n\end{aligned} \tag{A.15}$$

Finally, we use the vector field $\mathbb{E}$ *to set* $\eta = 1$*. This implies that the time law for* η *be given by* $\eta(t) = (1-\eta_0)t + \eta_0$ *and* $(X_\eta) = 1 - \eta_0$*. The associated equations read*

$$\dot{\xi}^n = -\frac{1-\eta_0}{(1-\eta_0)t+\eta_0}\xi^n \tag{A.16a}$$

$$\dot{g}^{\dagger nn} = -\frac{1-\eta_0}{(1-\eta_0)t+\eta_0}g^{\dagger nn} \tag{A.16b}$$

$$\dot{J}_{lm} = \frac{1-\eta_0}{(1-\eta_0)t+\eta_0}J_{lm} \tag{A.16c}$$

yielding, at time $t = 1$*, the following corrections to the fields:* $\widetilde{\xi}^n = \eta\xi^n$*,* $\widetilde{g}^{\dagger nn} = \eta\hat{\hat{g}}^{\dagger nn}$ *and* $\widetilde{J}_{lm} = \eta^{-1}\hat{\hat{J}}_{lm}$*.*

Computation A.4 (Theorem 4.12, Chapter 4)**.** *We will remove the vertical vector fields from* $\widetilde{Q}$ *in the following combination:*

$$\widetilde{Q}' = \widetilde{Q} - (Q_{g^\dagger})^{ab}\mathbb{G}^{\dagger ab} - (Q_\beta)_a\mathbb{B}_a - (Q_\eta)\mathbb{E} - (Q_\chi)_\mu\mathbb{X}_\mu$$

yielding

$$
\begin{aligned}
Q' =& (Q_\gamma)_{ab}\frac{\delta}{\delta\gamma_{ab}} + 2(Q_{g^\dagger})^{na}\frac{\delta}{\delta g^{\dagger na}} + (Q_{g^\dagger})^{nn}\frac{\delta}{\delta g^{\dagger nn}} + (Q_\xi)^n\frac{\delta}{\delta\xi^n} + (Q_\xi)^a\frac{\delta}{\delta\xi^a}\\
&+(Q_J)_{lm}\frac{\delta}{\delta J_{lm}} - \frac{2\epsilon}{\sqrt{\gamma}}\eta(Q_{g^\dagger})^{ab}\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)\xi^n\frac{\delta}{\delta J_{lm}}\\
&+\gamma^{ab}(Q_\beta)_a\xi^n\frac{\delta}{\delta\xi^b} + \frac{\epsilon}{2}\eta^{-2}\gamma^{ab}(Q_\beta)_a\chi_b\xi^n\frac{\delta}{\delta g^{\dagger nn}}\\
&-\left(2\nabla_{(l}(Q_\beta)_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\eta\left((Q_\beta)_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}(Q_\beta)_a\right)g^{\dagger an}\xi^n\right)\frac{\delta}{\delta J_{lm}}\\
&-\left(\frac{\epsilon}{2}\eta^{-2}\beta^b\gamma^{cd}(Q_\beta)_c\chi_d\xi^n - \gamma^{ab}(Q_\beta)_a g^{\dagger nn}\right)\frac{\delta}{\delta g^{\dagger bn}}\\
&+\eta^{-1}(Q_\eta)\xi^n\frac{\delta}{\delta\xi^n} - \beta^a\eta^{-1}(Q_\eta)\xi^n\frac{\delta}{\delta\xi^a} + \eta^{-1}(Q_\eta)g^{\dagger nn}\frac{\delta}{\delta g^{\dagger nn}}\\
&-\epsilon\eta^{-3}\left(\beta^a\chi_a - \chi_n\right)(Q_\eta)\xi^n\frac{\delta}{\delta g^{\dagger nn}} - \eta^{-1}\beta^a(Q_\eta)g^{\dagger nn}\frac{\delta}{\delta g^{\dagger an}}\\
&-\left(\epsilon\eta^{-3}\beta^b\chi_n - \epsilon\eta^{-3}\beta^b\beta^a\chi_a + \frac{\epsilon}{2}\eta^{-1}\gamma^{ba}\chi_a\right)(Q_\eta)\xi^n\frac{\delta}{\delta g^{\dagger bn}}\\
&+\frac{4\epsilon}{\sqrt{\gamma}}(Q_\eta)\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right)g^{\dagger an}\xi^n\frac{\delta}{\delta J_{lm}}\\
&+\frac{2\epsilon}{\sqrt{\gamma}}(Q_\eta)\left(\gamma_{la}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)g^{\dagger ab}\xi^n\frac{\delta}{\delta J_{ab}}\\
&-\eta^{-1}(Q_\eta)\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right)\frac{\delta}{\delta J_{lm}} - \frac{\epsilon}{2}\eta^{-2}(Q_\chi)_n\xi^n\frac{\delta}{\delta g^{\dagger nn}}\\
&+\frac{\epsilon}{2}\eta^{-2}\beta^a(Q_\chi)_a\xi^n\frac{\delta}{\delta g^{\dagger nn}} + \frac{\epsilon}{2}\beta^b\eta^{-2}(Q_\chi)_n\xi^n\frac{\delta}{\delta g^{\dagger bn}}\\
&-\left(\frac{\epsilon}{2}\eta^{-2}\beta^b\beta^a(Q_\chi)_a\xi^n - \frac{\epsilon}{2}\gamma^{ba}(Q_\chi)_a\xi^n\right)\frac{\delta}{\delta g^{\dagger bn}}
\end{aligned}
$$

collecting:

$$
\begin{aligned}
Q' =& \left\{ (Q_J)_{lm} - \frac{2\epsilon}{\sqrt{\gamma}} \eta (Q_{g^\dagger})^{ab} \left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab} \right) \xi^n - \eta^{-1}(Q_\eta)\,(J_{lm} - 2\nabla_{(l}\beta_{m)}) + \right. \\
& + \frac{4\epsilon}{\sqrt{\gamma}}(Q_\eta) \left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a \right) g^{\dagger an}\xi^n + \frac{2\epsilon}{\sqrt{\gamma}}(Q_\eta) \left(\gamma_{la}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab} \right) g^{\dagger ab}\xi^n + \\
& \left. - \left(2\nabla_{(l}(Q_\beta)_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\eta \left((Q_\beta)_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}(Q_\beta)_a \right) g^{\dagger an}\xi^n \right) \right\} \frac{\delta}{\delta J_{lm}} \\
& + \left\{ (Q_{g^\dagger})^{nn} + \frac{\epsilon}{2}\eta^{-2}\gamma^{ab}(Q_\beta)_a\chi_b\xi^n + \eta^{-1}(Q_\eta)g^{\dagger nn} - \epsilon\eta^{-3}\,(\beta^a\chi_a - \chi_n)\,(Q_\eta)\xi^n \right. \\
& \left. - \frac{\epsilon}{2}\eta^{-2}(Q_\chi)_n\xi^n + \frac{\epsilon}{2}\eta^{-2}\beta^a(Q_\chi)_a\xi^n \right\} \frac{\delta}{\delta g^{\dagger nn}} \\
& + \left\{ 2(Q_{g^\dagger})^{na} - \left(\frac{\epsilon}{2}\eta^{-2}\beta^a\gamma^{bc}\chi_c\xi^n - \gamma^{ab}g^{\dagger nn} \right)(Q_\beta)_b - \eta^{-1}\beta^a(Q_\eta)g^{\dagger nn} + \right. \\
& - \left(\epsilon\eta^{-3}\beta^a\chi_n - \epsilon\eta^{-3}\beta^a\beta^b\chi_b + \frac{\epsilon}{2}\eta^{-1}\gamma^{ab}\chi_b \right)(Q_\eta)\xi^n + \frac{\epsilon}{2}\beta^b\eta^{-2}(Q_\chi)_n\xi^n \frac{\delta}{\delta g^{\dagger bn}} \\
& \left. - \left(\frac{\epsilon}{2}\eta^{-2}\beta^b\beta^a(Q_\chi)_a\xi^n - \frac{\epsilon}{2}\gamma^{ba}(Q_\chi)_a\xi^n \right) \frac{\delta}{\delta g^{\dagger bn}} \right\} \frac{\delta}{\delta g^{\dagger na}} + (Q_\gamma)_{ab}\frac{\delta}{\delta\gamma_{ab}} \\
& + \left\{ (Q_\xi)^a + \gamma^{ab}(Q_\beta)_b\xi^n - \beta^a\eta^{-1}(Q_\eta)\xi^n \right\} \frac{\delta}{\delta\xi^a} + \left\{ (Q_\xi)^n + \eta^{-1}(Q_\eta)\xi^n \right\} \frac{\delta}{\delta\xi^n}
\end{aligned}
$$

Then, pushing forward the basis in the tangent space along the projection map we have:

$$
\begin{aligned}
(\widetilde{Q})_{\widetilde{J}_{lm}} = \Bigg\{ & \eta^{-1}(Q_J)_{lm} - \frac{2\epsilon}{\sqrt{\gamma}}(Q_{g^\dagger})^{ab}\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)\xi^n - \eta^{-2}(Q_\eta)\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) + \\
& + \eta^{-1}\frac{4\epsilon}{\sqrt{\gamma}}(Q_\eta)\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right)g^{\dagger an}\xi^n + \eta^{-1}\frac{2\epsilon}{\sqrt{\gamma}}(Q_\eta)\left(\gamma_{la}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)g^{\dagger ab}\xi^n + \\
& - \left(2\eta^{-1}\nabla_{(l}(Q_\beta)_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\left((Q_\beta)_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}(Q_\beta)_a\right)g^{\dagger an}\xi^n\right) + \\
& + \frac{\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\gamma_{cl}\gamma_{dm} - \frac{1}{d-1}\gamma_{lm}\gamma_{cd}\right)g^{\dagger cd}\xi^n + \frac{2\epsilon}{\sqrt{\gamma}}\frac{(Q_\gamma)_{lm}}{d-1}\beta_c\beta^c g^{\dagger nn}\xi^n \\
& - \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_\gamma)_{cl}\gamma_{dm} + \gamma_{cl}(Q_\gamma)_{dm} - \frac{1}{d-1}\gamma_{cd}(Q_\gamma)_{lm} - \frac{1}{d-1}\gamma_{cd}(Q_\gamma)_{lm}\right)g^{\dagger cd}\xi^n + \\
& + \frac{2\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\beta_{(l}\gamma_{m)c} - \frac{1}{d-1}\gamma_{lm}\beta_c\right)g^{\dagger cn}\xi^n - \frac{4\epsilon}{\sqrt{\gamma}}\left(\beta_{(l}(Q_\gamma)_{m)c} - \frac{1}{d-1}(Q_\gamma)_{lm}\beta_c\right)g^{\dagger cn}\xi^n \\
& + \frac{\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_c\beta^c\right)g^{\dagger nn}\xi^n \\
& - \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_{g^\dagger})^{nn} + \eta^{-1}(Q_\eta)g^{\dagger nn}\right)\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_c\beta^c\right)\xi^n \\
& - \frac{4\epsilon}{\sqrt{\gamma}}\left(2(Q_{g^\dagger})^{na} + \gamma^{ab}g^{\dagger nn}(Q_\beta)_b - \eta^{-1}\beta^a(Q_\eta)g^{\dagger nn}\right)\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right)\xi^n \\
& + \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_\xi)^n + \eta^{-1}(Q_\eta)\xi^n\right)\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)g^{\dagger ab} \\
& + \frac{4\epsilon}{\sqrt{\gamma}}\left((Q_\xi)^n + \eta^{-1}(Q_\eta)\xi^n\right)\left(\beta_{(l}\gamma_{m)b} - \frac{1}{d-1}\gamma_{lm}\beta_b\right)g^{\dagger bn} \\
& + \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_\xi)^n + \eta^{-1}(Q_\eta)\xi^n\right)\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_b\beta^b\right)g^{\dagger nn}\Bigg\}\frac{\delta}{\delta\widetilde{J}_{lm}}
\end{aligned}
$$

Simplifies to:

$$
\begin{aligned}
(\widetilde{Q})_{\widetilde{J}_{lm}} = \Bigg\{ & \eta^{-1}(Q_J)_{lm} - \frac{2\epsilon}{\sqrt{\gamma}}(Q_{g^\dagger})^{ab}\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)\xi^n - \eta^{-2}(Q_\eta)\left(J_{lm} - 2\nabla_{(l}\beta_{m)}\right) + \\
& - \left(2\eta^{-1}\nabla_{(l}(Q_\beta)_{m)} + \frac{4\epsilon}{\sqrt{\gamma}}\left((Q_\beta)_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}(Q_\beta)_a\right)g^{\dagger an}\xi^n\right) + \\
& + \frac{\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\gamma_{cl}\gamma_{dm} - \frac{1}{d-1}\gamma_{lm}\gamma_{cd}\right)g^{\dagger cd}\xi^n + \frac{2\epsilon}{\sqrt{\gamma}}\frac{(Q_\gamma)_{lm}}{d-1}\beta_c\beta^c g^{\dagger nn}\xi^n + \\
& - \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_\gamma)_{cl}\gamma_{dm} + \gamma_{cl}(Q_\gamma)_{dm} - \frac{1}{d-1}\gamma_{cd}(Q_\gamma)_{lm} - \frac{1}{d-1}\gamma_{cd}(Q_\gamma)_{lm}\right)g^{\dagger cd}\xi^n + \\
& + \frac{2\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\beta_{(l}\gamma_{m)c} - \frac{1}{d-1}\gamma_{lm}\beta_c\right)g^{\dagger cn}\xi^n - \frac{4\epsilon}{\sqrt{\gamma}}\left(\beta_{(l}(Q_\gamma)_{m)c} - \frac{1}{d-1}(Q_\gamma)_{lm}\beta_c\right)g^{\dagger cn}\xi^n + \\
& + \frac{\epsilon}{\sqrt{\gamma}}(Q_\gamma)_{ab}\gamma^{ab}\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_c\beta^c\right)g^{\dagger nn}\xi^n - \frac{2\epsilon}{\sqrt{\gamma}}\left((Q_{g^\dagger})^{nn}\right)\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_c\beta^c\right)\xi^n + \\
& - \frac{4\epsilon}{\sqrt{\gamma}}\left(2(Q_{g^\dagger})^{na} + \gamma^{ab}g^{\dagger nn}(Q_\beta)_b\right)\left(\beta_{(l}\gamma_{m)a} - \frac{1}{d-1}\gamma_{lm}\beta_a\right)\xi^n + \frac{2\epsilon}{\sqrt{\gamma}}(Q_\xi)^n\left(\gamma_{al}\gamma_{bm} - \frac{1}{d-1}\gamma_{lm}\gamma_{ab}\right)g^{\dagger ab} + \\
& + \frac{4\epsilon}{\sqrt{\gamma}}(Q_\xi)^n\left(\beta_{(l}\gamma_{m)b} - \frac{1}{d-1}\gamma_{lm}\beta_b\right)g^{\dagger bn} + \frac{2\epsilon}{\sqrt{\gamma}}(Q_\xi)^n\left(\beta_{(l}\beta_{m)} - \frac{1}{d-1}\gamma_{lm}\beta_b\beta^b\right)g^{\dagger nn}\Bigg\}\frac{\delta}{\delta\widetilde{J}_{lm}}
\end{aligned}
$$

together with

$$
\begin{aligned}
(\widetilde{Q})_{\widetilde{g}^{\dagger nn}} = & \Big\{ \eta (Q_{g^\dagger})^{nn} + \frac{\epsilon}{2}\eta^{-1}\gamma^{ab}(Q_\beta)_a\chi_b\xi^n + (Q_\eta)g^{\dagger nn} - \frac{\epsilon}{2}\eta^{-2}\left(\beta^a\chi_a - \chi_n\right)(Q_\eta)\xi^n + \\
& - \frac{\epsilon}{2}\eta^{-1}(Q_\gamma)_{ab}\gamma^{ac}\gamma^{bd}\beta_c\chi_d\xi^n + \frac{\epsilon}{2}\eta^{-1}(Q_\xi)^n\left(\beta^c\chi_c - \chi_n\right) + \\
& - \frac{\epsilon}{2}\eta^{-1}(Q_\chi)_n\xi^n + \frac{\epsilon}{2}\eta^{-1}\beta^a(Q_\chi)_a\xi^n \Big\} \frac{\delta}{\delta\widetilde{g}^{\dagger nn}} \\
(\widetilde{Q})_{\widetilde{g}^{\dagger na}} = & \Big\{ 2(Q_{g^\dagger})^{na} - \left(\frac{\epsilon}{2}\eta^{-2}\beta^a\gamma^{bc}\chi_c\xi^n - \gamma^{ab}g^{\dagger nn}\right)(Q_\beta)_b - \eta^{-1}\beta^a(Q_\eta)g^{\dagger nn} + \\
& - \left(\epsilon\eta^{-3}\beta^a\chi_n - \epsilon\eta^{-3}\beta^a\beta^b\chi_b + \frac{\epsilon}{2}\eta^{-1}\gamma^{ab}\chi_b\right)(Q_\eta)\xi^n + \beta^a(Q_{g^\dagger})^{nn} + \beta^a\frac{\epsilon}{2}\eta^{-2}\gamma^{cd}(Q_\beta)_c\chi_d\xi^n + \\
& + \eta^{-1}\beta^a(Q_\eta)g^{\dagger nn} - \epsilon\eta^{-3}\beta^a\left(\beta^b\chi_b - \chi_n\right)(Q_\eta)\xi^n + \frac{\epsilon}{2}\gamma^{ab}\chi_b\left((Q_\xi)^n + \eta^{-1}(Q_\eta)\xi^n\right) + \\
& - (Q_\gamma)_{cd}\gamma^{ca}\gamma^{bd}\beta_b g^{\dagger nn} - \frac{\epsilon}{2}(Q_\gamma)_{cd}\gamma^{ac}\gamma^{bd}\chi_b\xi^n + \frac{\epsilon}{2}\beta^a\eta^{-2}(Q_\chi)_n\xi^n \\
& - \left(\frac{\epsilon}{2}\eta^{-2}\beta^b\beta^a(Q_\chi)_b\xi^n - \frac{\epsilon}{2}\gamma^{ab}(Q_\chi)_b\xi^n\right) - \frac{\epsilon}{2}\eta^{-2}\beta^a(Q_\chi)_n\xi^n + \frac{\epsilon}{2}\eta^{-2}\beta^a\beta^b(Q_\chi)_b\xi^n \Big\} \frac{\delta}{\delta\widetilde{g}^{\dagger na}}
\end{aligned}
$$

The simpler terms are easily computed:

$$
\begin{aligned}
(\widetilde{Q}')_{\widetilde{\xi}^n} = & \Big\{ \eta(Q_\xi)^n + (Q_\eta)\xi^n \Big\} \frac{\delta}{\delta\widetilde{\xi}^n} \\
\equiv & \left\{ \widetilde{\xi}^c\partial_c\widetilde{\xi}^n \right\} \frac{\delta}{\delta\widetilde{\xi}^n} \\
(\widetilde{Q}')_{\widetilde{\xi}^a} = & \Big\{ (Q_\xi)^a + \gamma^{ab}(Q_\beta)_b\xi^n + (Q_\xi)^n\beta^a - (Q_\gamma)_{cd}\gamma^{ca}\gamma^{bd}\beta_b\xi^n \Big\} \frac{\delta}{\delta\widetilde{\xi}^a} \\
\equiv & \left\{ \widetilde{\xi}^n\widetilde{\gamma}^{ab}\partial_b\widetilde{\xi}^n + \widetilde{\xi}^c\partial_c\widetilde{\xi}^a \right\} \frac{\delta}{\delta\widetilde{\xi}^a} \\
(\widetilde{Q}')_{\widetilde{\gamma}_{ab}} = & (Q_\gamma)_{ab}\frac{\delta}{\delta\widetilde{\gamma}_{ab}} \\
\equiv & \left\{ \widetilde{\xi}^n\widetilde{J}_{ab} + \widetilde{\xi}^c\partial_c\widetilde{\gamma}_{ab} + 2\partial_{(a}\widetilde{\xi}^c\widetilde{\gamma}_{b)c} \right\} \frac{\delta}{\delta\widetilde{\gamma}_{ab}}
\end{aligned}
$$

In principle, it would be possible to proceed in the same way to compute all the other coefficients. To simplify the calculations, though the remaining components of Q^∂ can be recovered from $\delta S^\partial = \iota_{Q^\partial}\omega^\partial$.

B

Generalities on the BV formalism

In this appendix we will review the basic concepts underlying the BV framework for gauge theories. We will first work out some of the details in the local, finite dimensional case to move on to manifolds and the globalisation of the BV machinery.

B.1 Finite dimensional local model

The finite dimensional BV formalism can be cast in local coordinates as follows. Let A be a finitely generated, graded commutative algebra, with $2n$ generators q^i, p_i and $i = 1 \dots n$. We shall ask that half of the generators have even parity, and that the p_i's have opposite parity with respect to the q^i's. In particular, for a $\mathbb{Z}$-grading we have $|p_i| = -|q^i| - 1$.

We consider a completion $\hat{A}$ to allow for C^∞ functions in the even generators, and so we may encode the grading in a vector field E, the graded Euler vector field, as follows:

$$E = \sum_i |q^i| q^i \frac{\partial}{\partial q^i} + |p_i| p_i \frac{\partial}{\partial p_i} \Longrightarrow E(f) = |f| f$$

From now on we will use the shorthand notation $\partial_i = \frac{\partial}{\partial q^i}, \partial^j = \frac{\partial}{\partial p_j}$.

Definition B.1. *We define the BV Laplacian to be the second order operator:*

$$\Delta = \sum_i (-1)^{|q^i|}\partial_i \partial^i$$

✠

Notice that if we endow the local model with a symplectic structure, given a function f we can retrieve its Hamiltonian vector field X_f. We can write

$$\Delta f = -\frac{1}{2}\mathrm{div}(X_f)$$

where $\mathrm{div}(X)$ is the divergence of the vector field X, and it can be shown to be equal to the supertrace $\mathrm{Str}\left[X\overleftarrow{\partial}\right]$ of the right-Jacobian matrix of X (we take derivatives from the right).

It is easy to show that such an operator satisfies

$$\Delta^2 = 0$$

$$\Delta(fg) = (\Delta f)g + (-1)^{|f|}f(\Delta g) + (-1)^{|f|}(f, g) \tag{B.1}$$

where $(f, g)_\Delta$ is an odd-Poisson bracket, called BV-bracket. Notice that, using derivatives from the right, since

$$\frac{\partial f}{\partial z^\mu} = (-1)^{|z^\mu|(|f|+1)}\frac{f\overleftarrow{\partial}}{\partial z^\mu}$$

we can explicitly write

$$(f, g)_\Delta = \sum_i \frac{f\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial} g}{\partial p_i} - \frac{f\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial} g}{\partial q^i}$$

which is the canonical Poisson bracket associated with the canonical symplectic form $\omega = \sum_i dp_i dq^i$.

Now, one important application of this, which will generalise to the gauge-fixing procedure, is the following: consider some smooth function $f(p, q)$ and an odd function $\psi(q)$. Define

$$\int_{\mathcal{L}_\psi} f := \int f\Big|_{p_i = \partial_i \psi} dq^1 \dots dq^n$$

where the integration measure is a Grassmann integral along the possibly odd q^i coordinates and the Lebesgue integral along the even ones. Then we have

Lemma B.2. *If* $f = \Delta g$ *then* $\int\limits_{\mathcal{L}_\psi} f = 0$ *with vanishing boundary conditions.*

Proof. Compute the total derivative

$$\sum_i (-1)^{|q^i|}\partial_i \left(\partial^i g\right)_{p=d\psi} = \sum_i (-1)^{|q^i|}\left(\partial_i\partial^i g\right)_{p=d\psi} + \sum_{ij}(-1)^{|q^i|}\left(\partial_i\partial_j\psi\partial^j\partial^i g\right)_{p=d\psi}$$

where the second term in the r.h.s. vanishes by an anti-symmmetry argument and the first one is Δg. Integrating we have

$$0 = \int\limits_{\mathcal{L}_\psi} f + 0$$

since the left hand side is a total derivative and we have assumed vanishing boundary conditions. ✓

Lemma B.3. *Assume* $\Delta f = 0$*, then* $\int\limits_{\mathcal{L}_\psi} f$ *is invariant under deformations of* ψ*.*

Proof. Let ψ_t be a family of odd functions and define $I_t = \int\limits_{\mathcal{L}_{\psi_t}} f$. Then

$$\frac{dI_t}{dt} = \int \left(\partial_i\dot{\psi}_t\partial^i f\right)_{p=d\psi} dq^1 \dots dq^n = \pm\int \left(\Delta(\dot{\psi}_t f)\right)_{p=d\psi} dq^1 \dots dq^n = 0$$

where we used the Leibniz property (B.1) of Δ, together with the fact that $\psi = \psi(q)$, $\Delta f = 0$ and Lemma B.2. ✓

Typically this series of two lemmas is extended to the case where Δf is not integrable on some initial Lagrangian submanifold $\mathcal{L}_0$. If it is integrable on a close enough lagrangian submanifold $\mathcal{L}_\psi$, then we define the ill defined integral to be this regularised version on the new lagrangian submanifold.

BV-pushforward fixing a number of variables, say k, in $\{q^i, p_i\}$ we can split the variables in four sets: $q' = \{q^1 \dots q^k\}$, $q'' = \{q^{k+1} \dots q^n\}$, $p' = \{p_1 \dots p_k\}$ and $p'' = \{p_{k+1}, \dots p_n\}$. Then the BV laplacian splits in

$$\Delta = \Delta' + \Delta''$$

and given $f(q, p)$ and an odd function off the q'' variables only, one defines

$$F(q'p') = \int f\Big|_{p''=d\psi} dq'' =: \int_{\mathcal{L}''_\psi} f$$

and we have

Lemma B.4. *Under the above assumptions*

1. $\Delta' F = \int_{\mathcal{L}''_\psi} \Delta f$ *i.e. the pushforward is a chain map for the laplacians.*

2. *If* $F_t = \int_{\mathcal{L}''_{\psi_t}} f$ *with* $\Delta f = 0$ *then* $\dot{F}_t$ *is* Δ'*-exact.*

Proof. Analogous to the previous proofs. ✓

B.2 Global BV formalism

From now on we will consider odd-symplectic manifolds (M, ω), and we will restrict our scope to the finite dimensional case. For a more exhaustive review on supemanifolds and graded manifolds see, e.g.[51,52]. Notice that for odd-symplectic manifolds there is the following strong structural theorem:

Theorem B.5 (Schwarz, Batchelor)**.** *Any odd symplectic manifold* (M, ω) *is symplectomorphic to the shifted cotangent bundle* ΠT^*N *for some even manifold* N*. More generally a* C^∞ *supermanifold is always (non-canonically) superdiffeomorphic to a* C^∞ *supermaniffold of the form* ΠE*, with* E *an even vector bundle.*

The above Theorem provides a global version of Darboux theorem. Moreover, given a Berezinian ρ on M we may define

$$\Delta_\rho f := -\frac{1}{2}\mathrm{div}_\rho X_f$$

and this will automatically come satisfy the Leibniz identity (B.1). What may fail is the nilpotency, which we shall ask as a separate requirement:

Definition B.6. (M, ω, ρ) *is called an SP-manifold iff* $\Delta_\rho^2 = 0$.

✠

Proposition B.7. *Let* $M \simeq \Pi T^*N$ *with N an even manifold. Let* v *be a volume form for* N*, then* v^2 *is a Berezinian such that*

$$\Delta_{v^2}^2 = 0$$

Proof. Using that $C^\infty(\Pi T^*N) \simeq \mathcal{V}(N)$, the space of multivector fields, we consider

$$\begin{aligned} V^\bullet(N) &\xrightarrow[\sim]{\phi_v} \Omega^{\dim N - \bullet}(N) \\ X &\longrightarrow \iota_X v \end{aligned}$$

Then, showing that the formula

$$\Delta_{\nu^2} = \phi_\nu^{-1} \circ d \circ \phi_\nu$$

holds, we immediately show $\Delta_{\nu^2}^2 = 0$. ✓

Remark B.8. *If $C \subset N$ is a submanifold of N, then ΠN^*C will be Lagrangian in ΠT^*N. Therefore, if $X \in C^\infty(\Pi T^*N)$*

$$\int_{\Pi N^*C} X\sqrt{\nu^2} = \int_C \phi_\nu(X)$$

Given another choice of berezinian, say $\rho = \phi^2\nu^2$ with $\phi \in C^\infty(\Pi T^*N)$, we have

$$\int_{\Pi N^*C} X\sqrt{\rho} = \int_C \phi_\nu(X\phi)$$

So we are allowed to change either C to some homologous submanifold, and take its conormal bundle, or we can use an Hamiltonian flow on ΠN^*C:

$$\mathcal{L} \xrightarrow{\text{deform}} \Pi N^*C \xrightarrow{\text{homologous}} \Pi N^*C' \xrightarrow{\text{deform}} \mathcal{L}'$$

and thus

$$\int_{\mathcal{L}} X\sqrt{\rho} = \int_{\mathcal{L}'} X\sqrt{\rho}$$

◊

Now let (M, ω) be odd symplectic. Consider the action of both the deRham differential d and of $\delta := \omega \wedge \cdot$ on $\omega^\bullet(M)$. Clearly $\delta^2 = 0$. In the $\mathbb{Z}$-grading we have $|\omega| = -1$, but being it a 2-form we get $|\delta| = 1$. Notice that $d\omega = 0$ implies $d\delta + \delta d = 0$.

The following is by P. Ševera,[53].

Lemma B.9. $H^\bullet(\Omega^i(M), \delta) \simeq \{\frac{1}{2}\text{−densities}\}$ *canonically.*

Lemma B.10. *The induced differential on $E_1 = H^\bullet(\Omega^1, \delta)$ is zero. Therefore $E_2 = E_1$. The induced differential on E_2 is given by Δ, the canonical Laplacian on half densities.*

If ρ is a Berezinian on M, define the map

$$\sqrt{\rho}\cdot : \begin{array}{ccc} C^\infty(M) & \longrightarrow & \left\{\frac{1}{2}\text{-densities}\right\} \\ \widetilde{\Delta}_\rho & \longleftarrow & \Delta_{can} \end{array}$$

so that

Lemma B.11. $\widetilde{\Delta}_\rho = \Delta_\rho$ *if and only if* $\Delta_\rho^2 = 0$.

So finally, given $\mathcal{L} \subset M$, consider $[\alpha] \in H^\bullet(\Omega^\bullet, \delta)$, i.e. $\omega \wedge \alpha = 0$, then the restriction of α to a tubular neighborhood of $\mathcal{L}$ is

$$[\alpha_{\mathcal{L}}] \in H^\bullet\left(\Omega^\bullet(\Pi T^*\mathcal{L}, \delta)\right) = \mathrm{Ber}(\mathcal{L}).$$

where $[\alpha_{\mathcal{L}}]$ means $\omega_{\text{can}} \wedge \alpha_{\mathcal{L}} = 0$. Therefore

$$\int\limits_{\mathcal{L}} [\alpha] := \int\limits_{\mathcal{L}} \alpha_{\mathcal{L}}$$

is well defined.

To give a reasonable integration theory one needs the notion of integral forms:

Definition B.12. *Let N be a supermanifold, we define the set of integral forms on N to be*

$$\mathrm{Int}(N) := \left\{ \frac{1}{2}\text{-densities on } \Pi T^*N \right\} \overset{\phi}{\simeq} H^\bullet(\Omega^\bullet(\Pi T^*N), \delta_{can})$$

and therefore they form a complex with differential given by Δ_{can}.

✠

Then if $C \subset N$ is a submanifold, $\alpha \in \mathrm{Int}(N)$

$$\int\limits_{C} [\alpha] := \int\limits_{\Pi N^* C} \phi[\alpha]$$

and Stokes theorem carries over with Δ as deRham differential.

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Bibliography

[1] A. Einstein, *Zur allgemeinen Relativätstheorie*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin, 831-839 (1915).

[2] D. Hilbert, *Die Grundlagen der Physik*, Konigl. Gesell. d. Wiss. Göttingen, Nachr. Math. Phys. Kl. 395-407 (1915).

[3] A.S. Cattaneo, P. Mnëv, N. Reshetikhin, *Classical BV theories on manifolds with boundary*, Comm. Math. Phys. 332 (2): 535-603 (2014).

[4] A.S. Cattaneo, P. Mnëv, N. Reshetikhin, *Semiclassical quantization of Lagrangian field theories*, Mathematical Aspects of Quantum Field Theories, in Mathematical Physics Studies 2015, pp 275-324.

[5] A.S. Cattaneo, P. Mnëv, N. Reshetikhin, *Perturbative quantum gauge theories on manifolds with boundary*, arXiv:1507.01221

[6] A. Palatini, *Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton*, Rend. Circ. Mat. Palermo **43**, 203 (1919).

[7] S. Holst, *Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action*, Phys. Rev. **D 53**, 5966 (1996).

[8] J.F. Plebanski, J. Math. Phys. **18**, 2511 (1997).

[9] S. W. MacDowell and F. Mansouri, *unified geometric theory of gravity and supergravity*, Phys. Rev. Lett. 38, 739 - Published 4 April 1977; Erratum Phys. Rev. Lett. 38, 1376 (1977).

[10] M. Atiyah, *Topological quantum field theories*, Inst. Hautes Etudes Sci. Publ. Math. **68**. 175-186 (1988).

[11] G. Segal, *The definition of conformal field theory*, in: Differential geometrical methods in theoretical physics, Springer Netherlands, 165–171 (1988).

[12] K. Costello, *Renormalization and effective field theories*, Mathematical Surveys and Monographs 170, AMS (2011).

[13] J. Lurie *On the classification of topological field theories*, Current Developments in Mathematics, Vol 2008, 129-280 (2009).

[14] C. Becchi, A. Rouet and R. Stora, Phys. Lett. B52 (1974) 344.
C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42 (1975) 127.
C. Becchi, A. Rouet and R. Stora, Ann. Phys. 98, 2 (1976).
I. V. Tyutin, Lebedev Physics Institute preprint 39 (1975), arXiv:0812.0580.

[15] I. A. Batalin, G. A. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. B, 102 27 (1981).

[16] I. A. Batalin, E. S. Fradkin, *A generalized canonical formalism and quantization of reducible gauge theories*, Phys. Lett. B 122 2, 157–164 (1983).
I. A. Batalin, G. A. Vilkovisky, *Relativistic S-matrix of dynamical systems with boson and fermion costraints*, Phys. Lett. B 69 3, 309–312 (1977).

[17] V.S. Schlegel, *Gluing manifolds in the Cahiers topos*, arXiv:1503.07408.

[18] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the Master equation and topological quantum field theory*, Int. J. Modern Phys. **A 12**(7): 1405-1429, (1997).

[19] A. Ashtekar, Phys. Rev. Lett. **57**, 18 (1986).

[20] J. F. Barbero G., Phys. Rev. **D 51** 5507 (1995).

[21] G. Immirzi, Class. Quant. Grav. **14**, L177 (1997).

[22] C. Rovelli and T. Thiemann, Phys. Rev. **D 57**, 1009 (1998).

[23] R. Arnowitt, S. Deser, C. Misner, *Dynamical Structure and Definition of Energy in General Relativity*. Physical Review 116 (5): 1322-1330 (1959).

[24] B. S. DeWitt, *Quantum Theory of Gravity. I. The Canonical Theory*, Phys. Rev. 160, 1113, (1967).

[25] C. Blohmann, M.C. Barbosa Fernandes and A. Weinstein, Commun. Contemp. Math. 15, 1250061 (2013).

[26] P.A.M. Dirac, *Generalized Hamiltonian dynamics*, Canad. J. Math. **2**, 129-148 (1950).

[27] J.B. Hartle and S.W. Hawking, *Wave function of the Universe*, Phys. Rev. D 28, 2960 (1983).

[28] J.W. York, *Role of Conformal Three-Geometry in the Dynamics of Gravitation*, Phys. Rev. Lett. **28** (16), 1082 (1972).

[29] L.Smolin, Phys. Rev. **D80**, 124017 (2009).

[30] L. Freidel and S. Speziale, *On the Relations between Gravity and BF Theories*, SIGMA **8**, 032 (2012).

[31] R. Capovilla, M. Montesinos, V.A. Prieto and E. Rojas, *BF gravity and the Immirzi parameter*, Class. Quant. Grav., **18**, 6 (2001).

[32] I. Doubechies and J.R. Klauder, J. Math. Phys. 26, 2239 (1985).

[33] L. D. Faddeev and V. N. Popov, *Feynman Diagrams for the Yang-Mills Field*, Phys. Lett. B25 29 (1967).

[34] P. Mnev, *Discrete BF theory*, arXiv:0809.1160 (2008).

[35] J. Stasheff, *Homological reduction of constrained Poisson algebras*, J. Diff. Geom. **45**, 221–240 (1997).

[36] J.Stasheff, *Deformation Theory and the Batalin-Vilkovisky Master Equation*, Deformation theory and symplectic geometry, proceedings, Meeting, Ascona, Switzerland, June 16-22, 1996, arXiv:q-alg/9702012.

[37] G. Felder and D. Kazhdan. *The classical master equation*, in Contemporary Mathematics, Vol **610** (2014).

[38] VV.AA., A.S. Cattaneo, A. Giaquinto and P. Xu Editors of *Higher structures in Geometry and Physics*, Progress in Mathematics, Vol. 287 (2011), Birkäuser Basel.

[39] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** 2, 267-288 (1963).

[40] D. Roytenberg, *AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories*, Lett. Math. Phys. 79: 143-159 (2007).

[41] D. J. Rezende and A. Perez, Phys. Rev. **D 79**, 064026 (2009).

[42] O. Piguet, Class. Quant. Grav. **17**, 3799-3806 (2000).

[43] O. Moritsch, M. Schweda, S. P. Sorella, Class. Quant. Grav. **11** 1225 (1994).

[44] L. Smolin and S. Speziale, *A note on the Plebanski action with cosmological constant and an Immirzi parameter*, Phys. Rev. **D 81**, 024032 (2010).

[45] R. Capovilla, J. Dell and T. Jackobson, Class. Quant. Grav **8** 59 (1991).

[46] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2007.

[47] F. Schaetz, *BFV-complex and higher homotopy structures*, Comm. Math. Phys. **286** (2), 399-443 (2009).

[48] F. Schaetz, *Invariance of the BFV complex*, Pac. J. Math. **248** (2), (2010).

[49] L. Freidel and A. Starodubstev, *Quantum Gravity in terms of Topological observables*, arXiv:hep-th/0501191.

[50] D. Wise, *MacDowell-Mansouri Gravity and Cartan Geometry*, Class. Quant. Grav. **27** 15 (2010).

[51] A.S. Cattaneo and F. Schaetz, *Introduction to Supergeometry*, Rev. Math. Phys. **23** 6, 669-606 (2014).

[52] C. Carmeli, L. Caston and R. Fioresi, *Mathematical Foundations of Supersymmetry*, European Mathematical Society (2010).

[53] P. Ševera, *On the Origin of the BV Operator on Odd Symplectic Supermanifolds*, Lett. Math. Phys. **78** 1, 55-59 (2006).