Topological Defects and Classifying Local Topological Field Theories in Low Dimensions

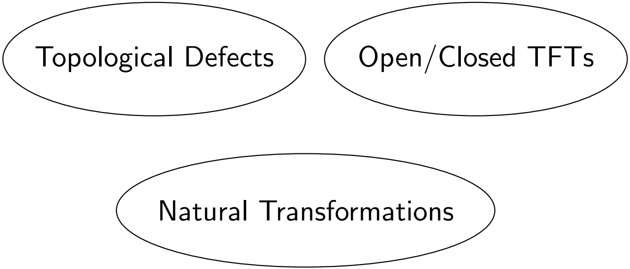
Chris Schommer-Pries

Departments of Mathematics:

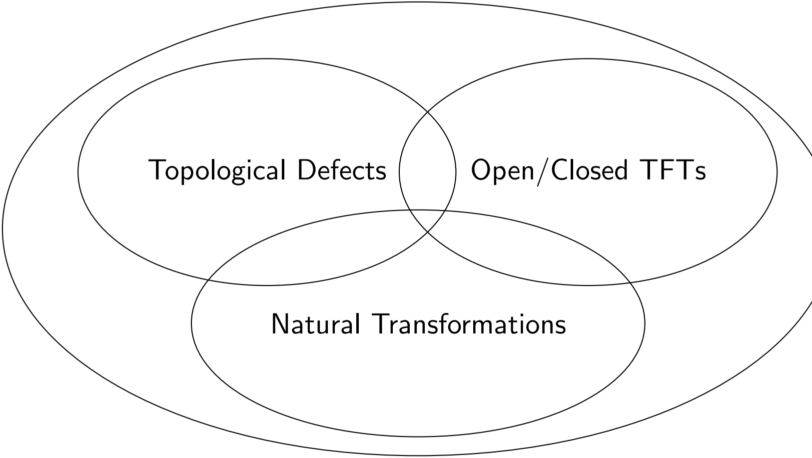
University of California, Berkeley
Max Planck Institute for Mathematics
Harvard University
MIT

Talk at the Mathematisches Forschungsinstitut Oberwolfach

Three Concepts in TFT



Three Concepts in Local TFT



What is a Topological Field Theory?

Atiyah-Segal Axioms:

Definition

A TFT is a symmetric monoidal functor:

$$\underbrace{\text{Bord}}_{\text{Bordism Category}} \to \underbrace{\text{C}}_{\text{Target Category}}$$

Usually C is $(\mathit{Vect}, \otimes)$

Algebra from Geometry

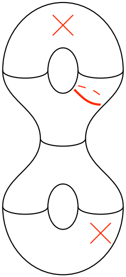
Theorem (Folklore)

2D TFTs are Commutative Frobenius Algebras

[R. Dijkgraaf, L. Abrams, S. Sawin, B. Dubrovin, Moore-Segal, . . .]

unit multiplication comultiplication counit

The 1st Two Talks (my impression)



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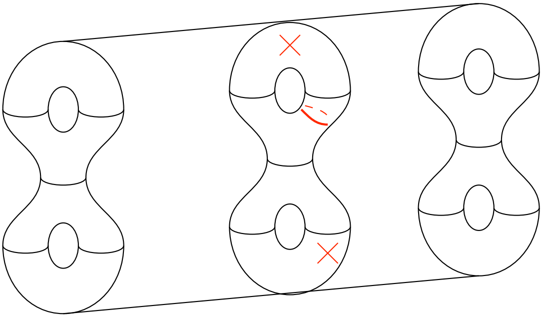

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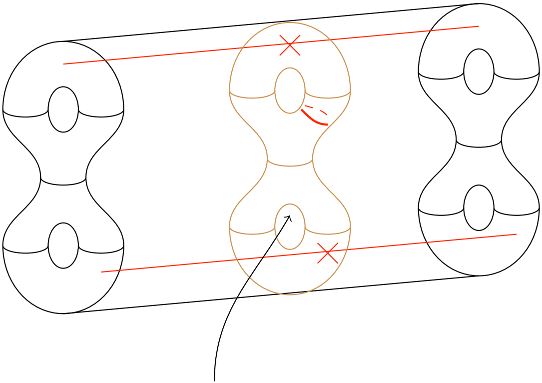

The 1st Two Talks (my impression)

$$\mathrm{Cob}_3^{\mathrm{decorated}} \xrightarrow{\mathrm{RT}} \mathrm{Vect}$$

The 1^{st} Two Talks (my impression)

$$\mathrm{Cob}_{2+}^{\mathrm{dec}} \xrightarrow{\;\emptyset\;} \mathrm{Cob}_{3}^{\mathrm{decorated}} \xrightarrow{\;\mathrm{RT}\;} \mathrm{Vect}$$

decorated surfaces + isotopy classes of diffeo.

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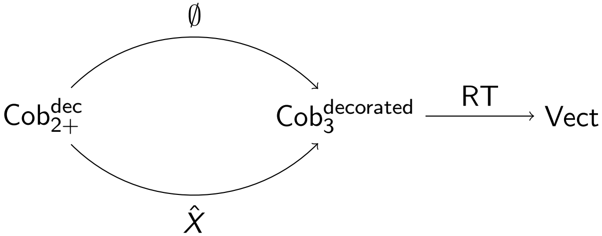

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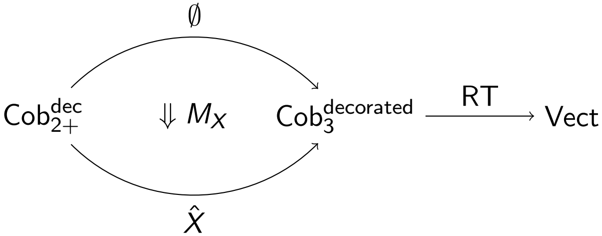
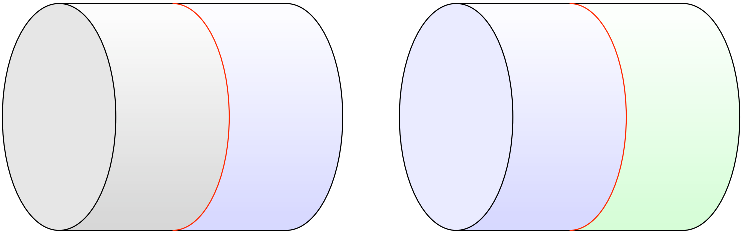
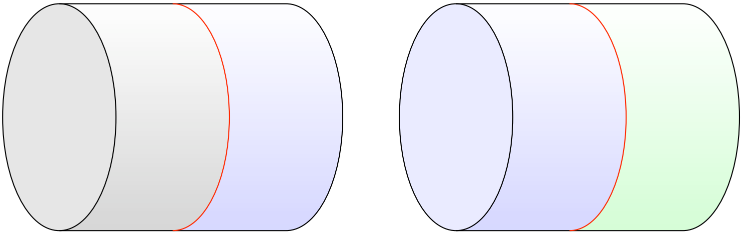


Defects $\longrightarrow$ Operators



$\mathcal{H}_a \longrightarrow \mathcal{H}_b$ $\qquad$ $\mathcal{H}_b \longrightarrow \mathcal{H}_c$

Defects $\longrightarrow$ Operators

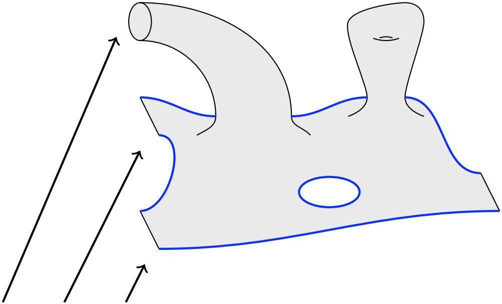


$$\mathcal{H}_a \longrightarrow \mathcal{H}_b \qquad \mathcal{H}_b \longrightarrow \mathcal{H}_c$$

Composition = "Fusion"

Open/Closed TQFTs

$$Z : Bord^{o/c} \to Vect$$



$$\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_o \xrightarrow{Z(\Sigma)} \mathcal{H}_o$$

Open/Closed TQFTs

Theorem (Moore-Segal, Cardy-Lewellen, Lazaroiu, Lauda-Pfeiffer)

Open/Closed TFTs ⇔ "Knowledgeable Frobenius Algberas"

$$i_! : A \rightleftarrows C : i^*$$

Cardy Condition!
Non-semisimple examples.

Local Topological Field Theories

- Higher Categorical Bordism Category: $(\mathrm{Bord}, \sqcup)$
- A Local TFT is a symmetric monoidal *n*-functor:

$$\mathrm{Bord} \to \mathrm{C}$$

Easiest Case: Bicategory, Bord_2

- Objects are zero manifolds.
- 1-Morphisms are 1-bordisms.
- 2-Morphisms are 2-bordisms (between 1-bordisms) (up to isomorphism)

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Bicategories

- Objects: $a, b, c, \ldots$

a b

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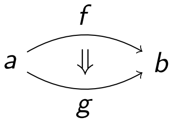

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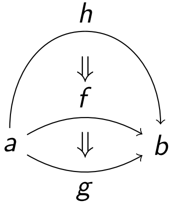
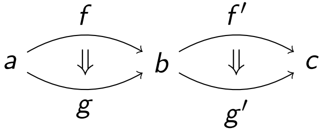


Bicategories

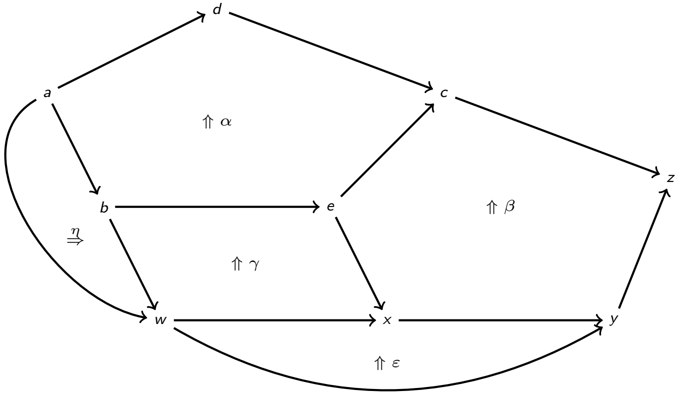
- Objects: $a, b, c, \ldots$
- Categories $\mathrm{B}(a, b)$
- Horizontal composition functors: Strict

$$\mathrm{B}(a, b) \times \mathrm{B}(b, c) \to \mathrm{B}(a, c)$$

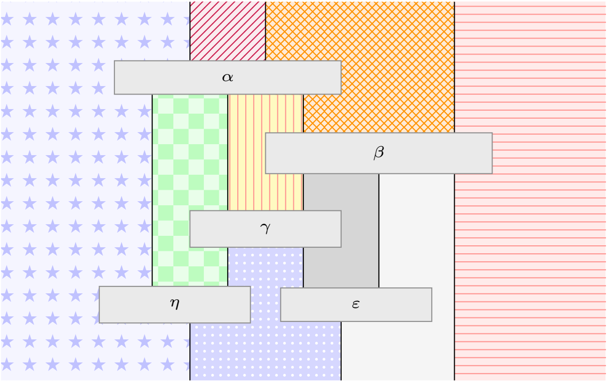


Pasting Diagrams vs String Diagrams

Pasting Diagram

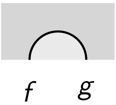


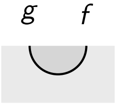
String Diagram



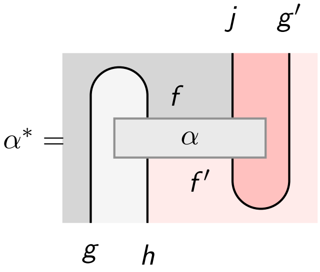
Adjunctions

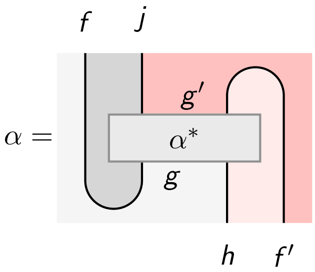
$$f : a \rightleftarrows b : g$$

$\eta =$ 

$\varepsilon =$ 

Mates





Homomorphisms of Bicategories

- $F : B_i \to C_i$,
- $\phi : F(g) \circ F(f) \to F(gf)$, (2-morphisms)

Such that:

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \longrightarrow & F(h \circ g) \circ Ff \\ \downarrow & & \downarrow \\ Fh \circ F(g \circ f) & \longrightarrow & F(h \circ g \circ f) \end{array}$$

Transformations of Homomorphisms

Ordinary Transformation

- $\sigma_a : Fa \to Ga$

$$\begin{array}{ccc} F(a) & \xrightarrow{\sigma_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\sigma_b]{} & G(b) \end{array}$$

Transformations of Homomorphisms

Bicategory Transformations

- $\sigma_a : Fa \to Ga$
- $\sigma_f : G(f) \circ \sigma_a \to \sigma_b \circ F(f)$

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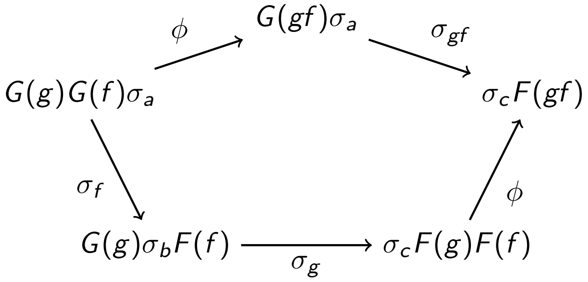
Such that...

Transformations of Homomorphisms

1. σ_f is natural in f,
2. and...



Classification Theorem

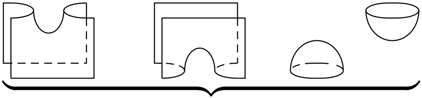
Theorem (SP)

The Oriented Bordism Bicategory has the following Generators and Relations as a Symmetric Monoidal Bicategory:

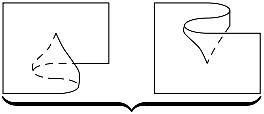
Generating Objects: $+\bullet$ $-\bullet$

Generating 1-Morphisms: 

Generating 2-Morphisms:



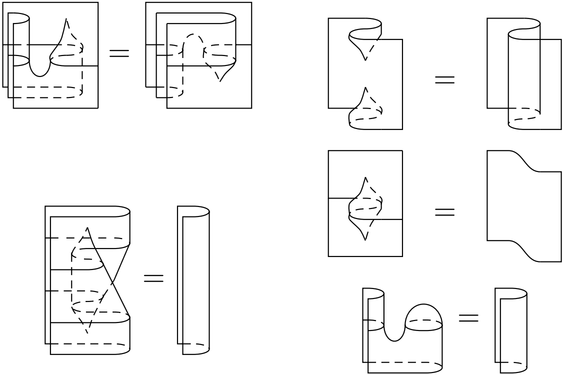
2D Morse generators



Cusp generators

The Relations

Relations among 2-Morphisms:



Proof: Step 1

Make Sense of Generators/Relations for Symmetric Monoidal Bicats.

Theorem (SP)

Given Generators/Relations Data: (G, R)

$\exists$ *symmetric monoidal bicat.* $\mathrm{F}_{(G,R)}$ *s.t.*

$$\mathrm{SymBicat}(\mathrm{F}_{(G,R)}, \mathrm{C}) \xrightarrow{\simeq} (G, R)\text{-}\mathit{data\ in}\ \mathrm{C}.$$

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(Part of) a Universal Property

Corollary (SP)

Homomorphisms, Transformations, Modifications determined by

$$h : (G_0, G_1, G_2) \longrightarrow (M_0, M_1, M_2)$$
$$t : (G_0, G_1) \longrightarrow (M_1, M_2)$$
$$m : G_0 \longrightarrow M_2$$

(Any target symmetric monoidal bicat M)

Statement of Classification Theorem

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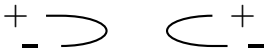
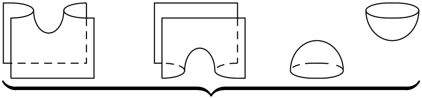
Generating 1-Morphisms: 

Generating 2-Morphisms: 

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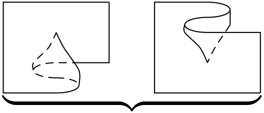

Cusp generators

Statement of Classification Theorem

Theorem (SP)

$$F_{(G,R)} \to \mathrm{Bord}_2$$

is an equivalence of symmetric monoidal bicategories.

Skip proof

Proof: Step 2

Theorem (SP)

$F_{(G,R)} \to \mathrm{Bord}_2$ *is an equivalence of symmetric monoidal bicategories iff*

- *Essentially surjective on objects*
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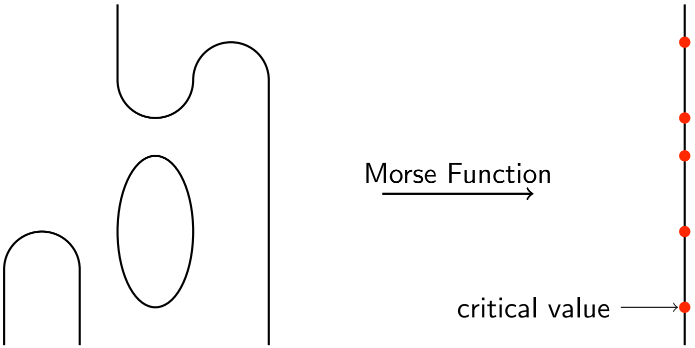

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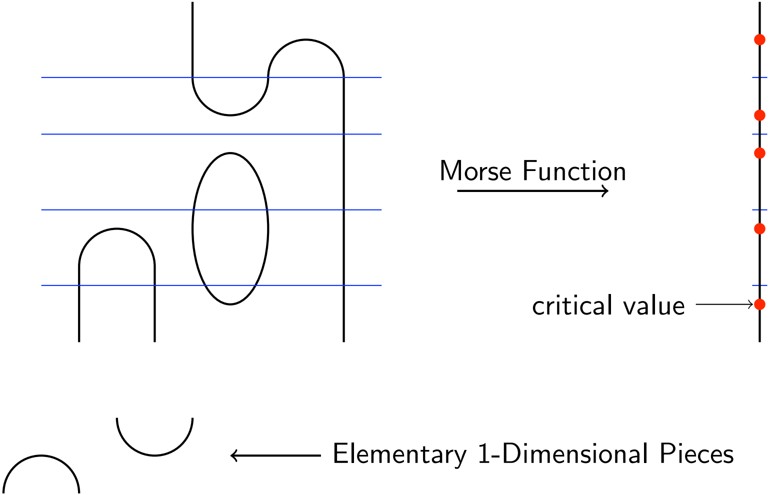

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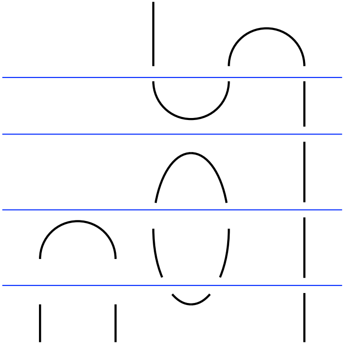
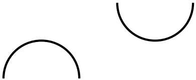
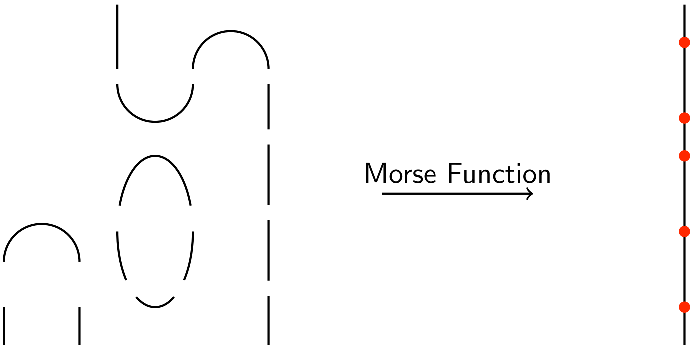


Morse Function →



← Elementary 1-Dimensional Pieces

Proof: Step 3 (1D Warmup)



Cerf Theory Gives Relations!

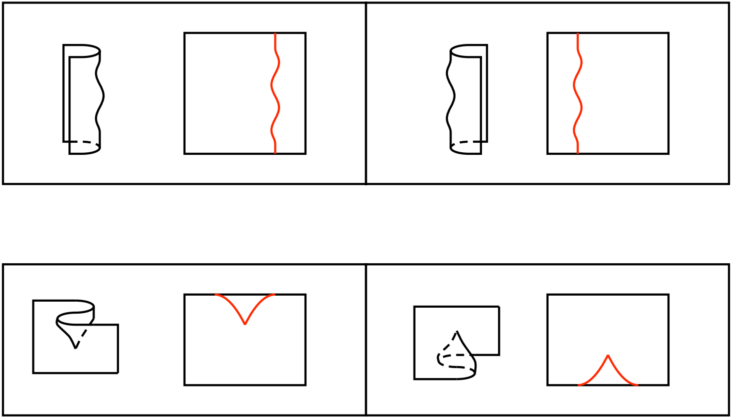
Singularities of Generic maps $\Sigma^2 \to \mathbb{R}^2$

Use the projection $\Sigma \to \mathbb{R}^2 \to \mathbb{R}$.

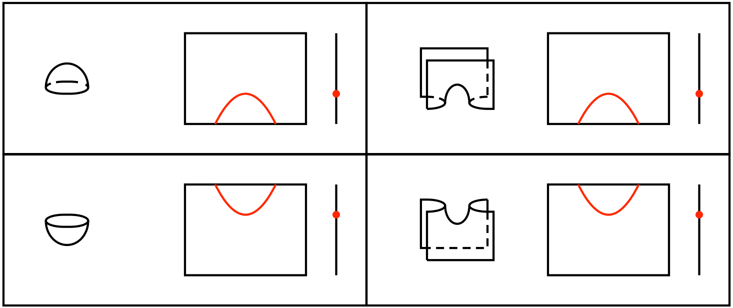
3 Kinds of Singularities:

- Folds
- Cusps
- 2D Morse

Folds and Cusps

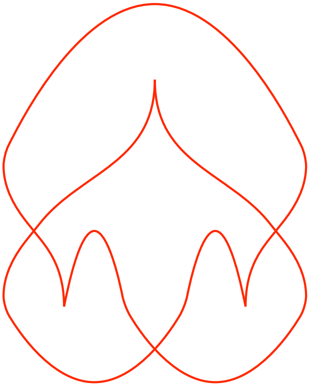


2D Morse



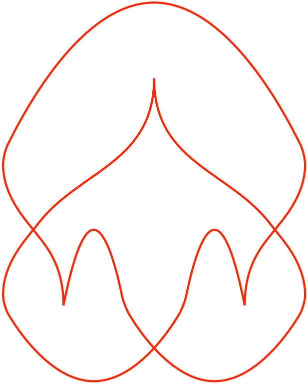
Example

What is it?



Example

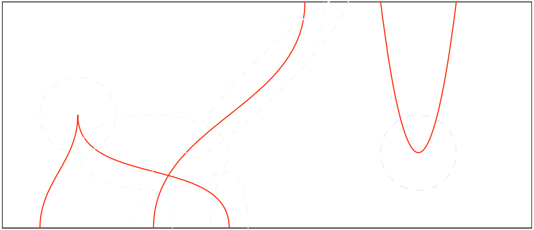
What is it? $\mathbb{RP}^2$!



Planar Decompositions

Skip rest of proof

Generic Maps to $\mathbb{R}^2$ Decompose Surfaces! (String Diagram!)



Question:

When are two Decompositions Equivalent?

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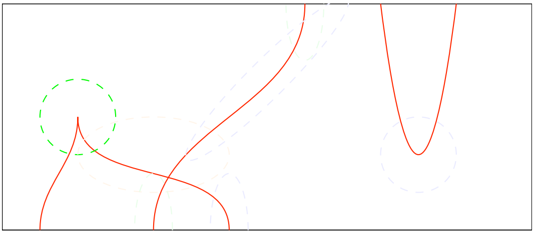

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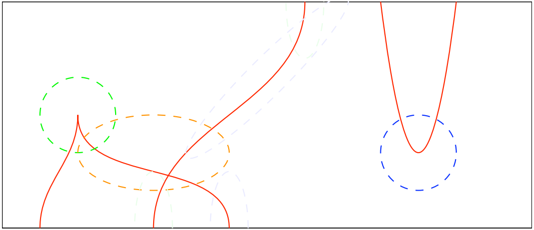

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Chris Schommer-Pries

Outline

Review of TFTs

Topological Field Theories

Defects

Open/Closed TFTs

Locality and Bicategories

Classifying Local 2D TFTs

Trans. and Defects

Planar Decompositions

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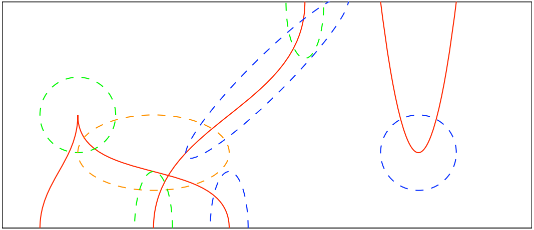

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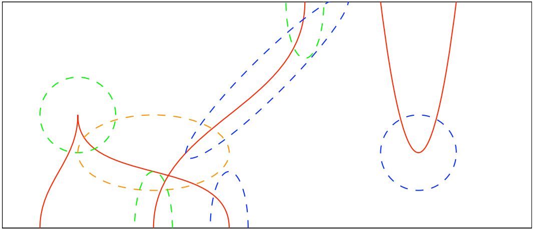


Question:

When are two Decompositions Equivalent?

Singularities of Generic Maps $\Sigma \times I \to \mathbb{R}^2 \times I$

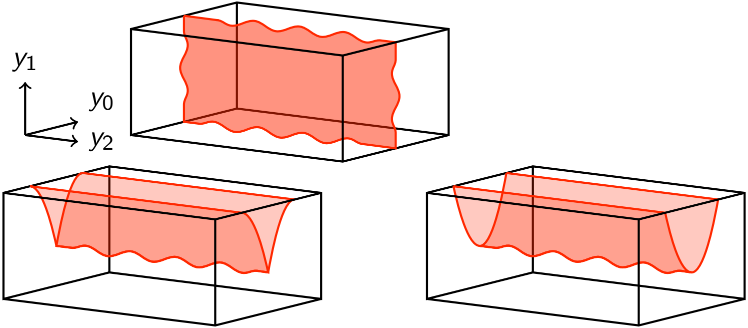
Paths of...

- Folds
- Cusps
- 2D Morse

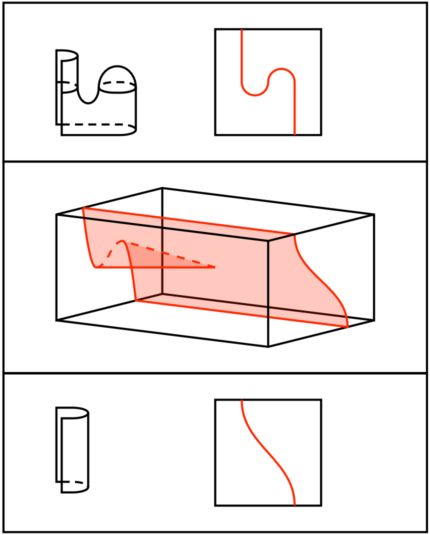
And...

- 2D Morse Relation
- Cusp Inversion
- Cusp Flip
- Swallowtails

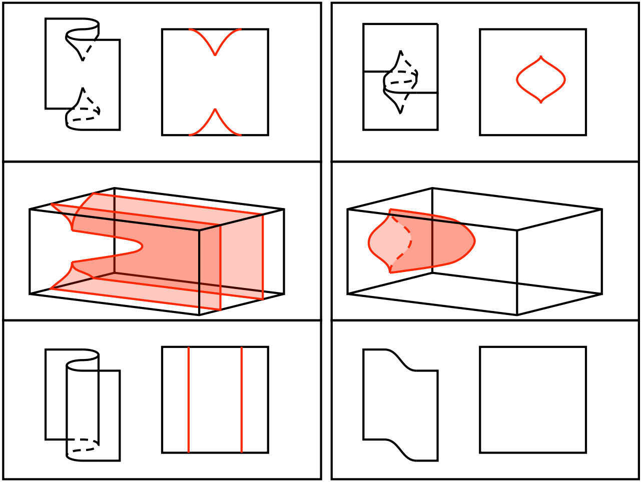
Paths of Folds, Cusps, and 2D Morse



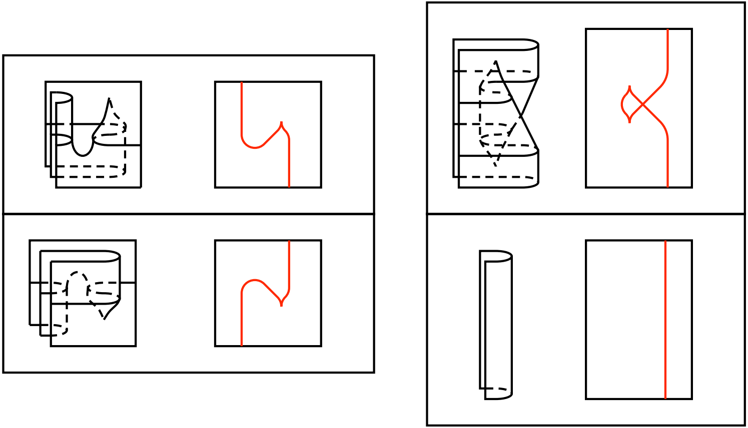
2D Morse Relation



Cusp Inversion



Cusp Flip and Swallowtail



Planar Decomposition

Theorem (SP)

Maps for surfaces give elementary generators:

- *Folds,*
- *Cusps,*
- *2D Morse, and*
- *"Gluing Data"*

Maps for $\Sigma \times I$ *give elementary relations.*

Planar Decomposition

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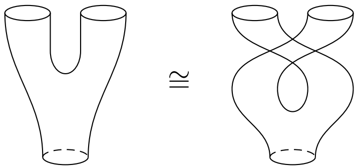
$\Rightarrow$ Step 3 ✓

Exercise

Prove:



Application

Alg = Bicat. of Algebras, Bimodules, Intertwiners.

Corollary (SP)

$\mathrm{TFT}_2(\mathrm{Alg}) \simeq$ *the bicategory of Separable Symmetric Frobenius Algebras with Morita equivalences as 1-morphisms, isomorphisms as 2-morphisms.*

Open/Closed $\neq$ Local

Transformations of Local 2D TFTs

Recall:

Corollary (SP)

Homomorphisms, Transformations, Modifications determined by

$$h : (G_0, G_1, G_2) \longrightarrow (M_0, M_1, M_2)$$
$$t : (G_0, G_1) \longrightarrow (M_1, M_2)$$
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(Any target symmetric monoidal bicat M)

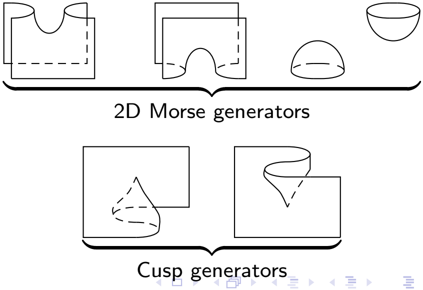
Transformations of Local 2D TFTs

Theorem (SP)

Bord_2 *has the following generators as a symmetric monoidal bicategory:*

Generating Objects: $+\,\bullet \quad -\,\bullet$

Generating 1-Morphisms: 

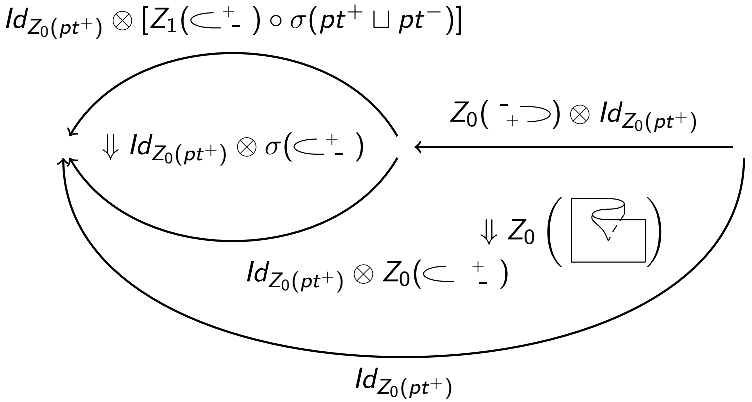
Generating 2-Morphisms: 

Transformations of Local TFTs

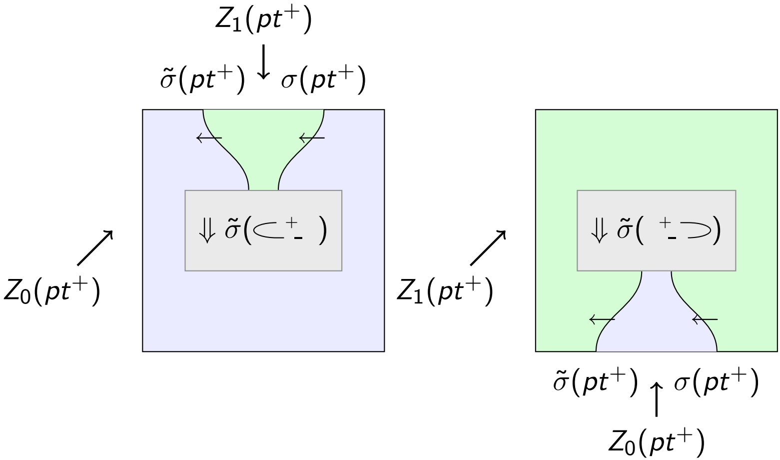
$$\sigma : Z_0 \Rightarrow Z_1$$

- $\sigma(pt^+) : Z_0(pt^+) \to Z_1(pt^+)$
- $\sigma(pt^-) : Z_0(pt^-) \to Z_1(pt^-)$
- $\sigma(\subset^{+}_{-}) : Z_1(\subset^{+}_{-}) \circ \sigma(pt^+ \sqcup pt^-) \to \sigma(\emptyset) \circ Z_0(\subset^{+}_{-})$
- $\sigma({}^{+}_{-}\supset) : Z_1({}^{+}_{-}\supset) \circ \sigma(\emptyset) \to \sigma(pt^+ \sqcup pt^-) \circ Z_0({}^{+}_{-}\supset)$

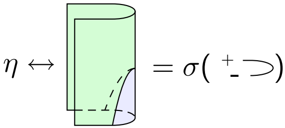
Write in terms of pt^+...

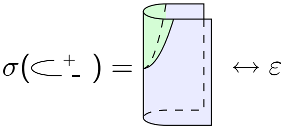


In String Diagrams...

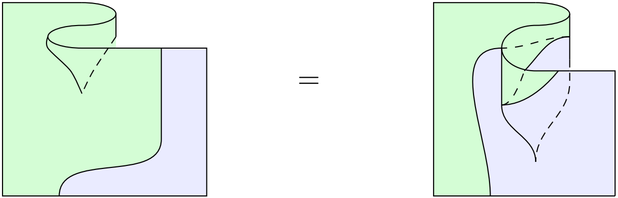


Graphical Notation

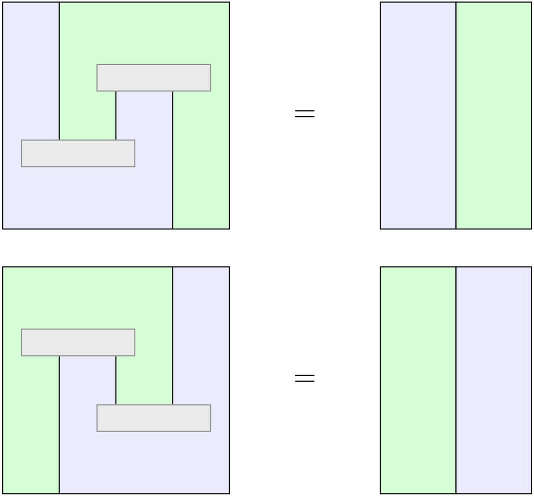




Naturality w.r.t. Cusps



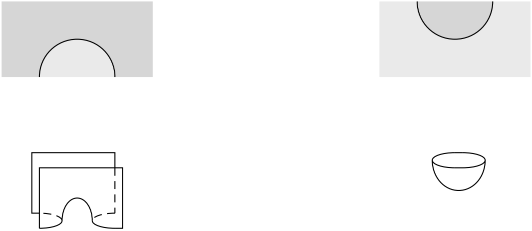
Graphical Notation



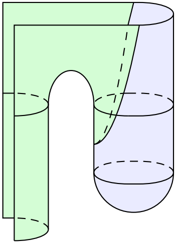
Duality in Bord_2

$\supset$ is left and right adjoint to $\subset$.

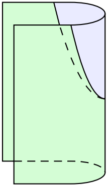
Ambidextrous Adjoints



Mates

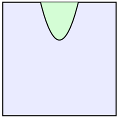


=:

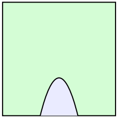


Ambidextrous Adjunction!

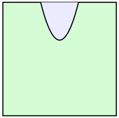
$\varepsilon =$



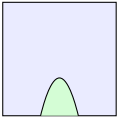
$\eta =$



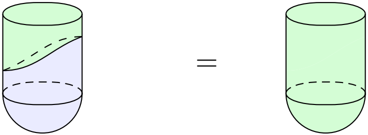
$\varepsilon^* =$



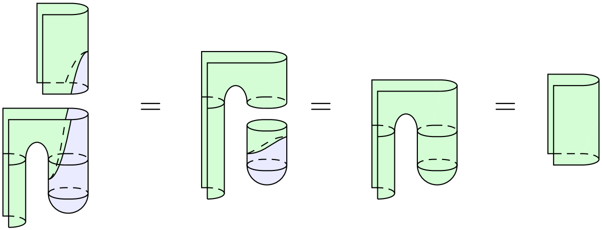
$\eta^* =$

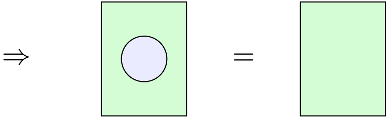


Naturality w.r.t. Cup

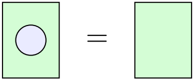
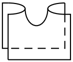
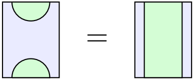


Implied Relation

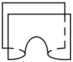
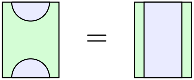
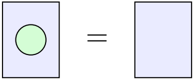




Naturality Relations

Generator	Naturality Relation
	
	

Naturality Relations

Generator	Naturality Relation
	
	

TFTs are a Space!

Corollary

Transformations between TFTs $\leftrightarrow$ *Adjoint Equivalences*

Note: Does not apply to Positive Boundary TFTs.

Supernatural Transformations

Definition

Supernatural Transformation = transformation data: σ_a, σ_f only natural w.r.t. invertible 2-morphisms.

$$i_0, i_1 : \mathrm{Bord}_2 \rightrightarrows \mathrm{Bord}_2^{\mathrm{dec}}$$

Theorem

$$\underbrace{(Z : \mathrm{Bord}_2^{dec} \to \mathrm{C})}_{\Leftrightarrow \textit{TFT with defects}} = \textit{Supernatural Trans. } Zi_0 \Rightarrow Zi_1$$

Remark: point defects ↔ "supernatural modifications"

Examples of Supernatural Transformations

- Natural Transformations. ✓
- Local Topological Defects. ✓
- Local Open-Closed TFTs...