LINEAR LOGIC, $*$-AUTONOMOUS CATEGORIES AND COFREE COALGEBRAS[1]

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ABSTRACT. A brief outline of the categorical characterisation of Girard's linear logic is given, analagous to the relationship between cartesian closed categories and typed λ-calculus. The linear structure amounts to a $*$-autonomous category: a closed symmetric monoidal category $\mathbf{G}$ with finite products and a closed involution. Girard's exponential operator, ! , is a cotriple on $\mathbf{G}$ which carries the canonical comonoid structure on A with respect to cartesian product to a comonoid structure on $!A$ with respect to tensor product. This makes the Kleisli category for ! cartesian closed.

0. INTRODUCTION. In "Linear logic" [1987], Jean-Yves Girard introduced a logical system he described as "a logic behind logic". Linear logic was a consequence of his analysis of the structure of qualitative domains (Girard [1986]): he noticed that the interpretation of the usual conditional "$\Rightarrow$" could be decomposed into two more primitive notions, a linear conditional "$\multimap$" and a unary operator "!" (called "of course"), which is formally rather like an interior operator:

$$X \Rightarrow Y = !X \multimap Y \tag{1}$$

The purpose of this note is to answer two questions (and perhaps pose some others.) First, if "linear category" means the structure making valid the proportion

linear logic : linear category = typed λ-calculus : cartesian closed category

then what is a linear category? This question is quite easy, and in true categorical spirit, one finds that it was answered long before being put, namely by Barr [1979]. Our intent here is mainly to supply a few details to make the matter more precise (though we leave many more details to the reader), to point out some similarities with work of Lambek [1987] (see these proceedings), and to appeal for a change in some of the notation of Girard [1987].

Second, what is the meaning of Girard's exponential operator ! ? Since Girard has in fact offered several variants of ! in [1987], and another in Girard and Lafont [1987], one cannot be too dogmatic here, but some certainty as to the minimal demands ! makes is possible — in particular we show that ! ought to be a cotriple, and its Kleisli category ought to be cartesian closed, in order to capture the initial motivation of the exponential. (This is already implicit in equation (1).)

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ACKNOWLEDGEMENT. This note should be regarded as a "gloss" on GIRARD [1987], providing the categorical context and terminology for that work; I think the categorical setting provides a genuine improvement, and in particular, indicates how the notation may be made clearer. Others have come to similar conclusions: elsewhere in this volume DE PAIVA [1987] considers these matters, giving a fuller discussion of the interpretation of "$!A$" as "the cofree commutative comonoid over A", in the context of Dialectica categories. I would like to thank Michael Barr for pointing out that he had considered the essence of linear categories in [1979], thus giving further evidence of "the unreasonable influence of category theory in mathematics".

1. LINEAR LOGIC. There are several variations in the style Girard uses to present linear logic: *eg.* one sided sequents in [1987] and traditional sequents in GIRARD and LAFONT [1987]. I think the essence of the structure, especially its symmetry, is clearest when sequents in the style of Szabo's and Lambek's polycategories (SZABO[1975]) are used; here a sequent has the form

$$A_1, A_2, \ldots, A_n \to B_1, B_2, \ldots, B_m$$

(Of course, formally this is just an ordered pair of finite sequences — actually sets would do — of formulas.) The commas on the left should be thought of as some kind of conjunction, those on the right disjunction. (Better, think of the A_i on the left as data each to be used exactly once, and of the B_j on the right as possible alternate responses.)

1.1 DEFINITION. A (propositional) linear logic consists of formulas and sequents. Formulas are generated by the binary connectives $\otimes, \odot, \times, +$, and $-\!\circ$, and by the unary operation $\neg$, from a set of constants including I, ϕ, 1, and 0, and from variables.

Sequents consist of ordered pairs of finite sequences of formulas, as above; actually, finite sets of formulas would be better, in view of $(perm)$ below, but let us pass over this point. The sequents are generated by the following rules from "initial sequents" (*i.e.* axioms), which include the following. (Greek capitals represent finite sequences (sets) of formulas.)

AXIOMS.

$$\begin{array}{ll}
(id_A) : A \to A & \\
(IR) : \quad \to I & (\phi L) : \quad \phi \to \\
(1R) : \Gamma \to 1, \Delta & (0L) : \quad \Gamma, 0 \to \Delta \\
(d) : A \to \neg\neg A & (d^{-1}) : \neg\neg A \to A
\end{array}$$

RULES.

$$(perm) : \frac{\Gamma \to \Delta}{\sigma\Gamma \to \tau\Delta} \quad \text{for any permutations } \sigma, \tau.$$

$$(cut) : \frac{\Gamma \to A, \Delta \qquad A, \Theta \to \Psi}{\Gamma, \Theta \to \Psi, \Delta}$$

$$(\neg var) : \frac{\Gamma, A \to B, \Delta}{\Gamma, \neg B \to \neg A, \Delta}$$

$$(IL) : \quad \frac{\Gamma \to \Delta}{\Gamma, I \to \Delta} \qquad (\phi R) : \quad \frac{\Gamma \to \Delta}{\Gamma \to \phi, \Delta}$$

$$(\otimes L) : \frac{\Gamma, A, B \to \Delta}{\Gamma, A \otimes B \to \Delta} \qquad (\otimes R) : \frac{\Gamma \to A, \Delta \qquad \Theta \to B, \Psi}{\Gamma, \Theta \to A \otimes B, \Delta, \Psi}$$

$$(\odot L) : \frac{\Gamma, A \to \Delta \qquad \Theta, B \to \Psi}{\Gamma, \Theta, A \odot B \to \Delta, \Psi} \qquad (\odot R) : \frac{\Gamma \to A, B, \Delta}{\Gamma \to A \odot B, \Delta}$$

$$(\multimap L) : \frac{\Gamma \to A, \Delta \qquad \Theta, B \to \Psi}{\Gamma, \Theta, A \multimap B \to \Psi, \Delta} \qquad (\multimap R) : \frac{\Gamma, A \to B, \Delta}{\Gamma \to A \multimap B, \Delta}$$

$$(\times L) : \frac{\Gamma, A \to \Delta}{\Gamma, A \times B \to \Delta} \qquad \frac{\Gamma, B \to \Delta}{\Gamma, A \times B \to \Delta} \qquad (\times R) : \frac{\Gamma \to A, \Delta \qquad \Gamma \to B, \Delta}{\Gamma \to A \times B, \Delta}$$

$$(+L) : \frac{\Gamma, A \to \Delta \qquad \Gamma, B \to \Delta}{\Gamma, A + B \to \Delta} \qquad (+R) : \frac{\Gamma \to A, \Delta}{\Gamma \to A + B, \Delta} \qquad \frac{\Gamma \to B, \Delta}{\Gamma \to A + B, \Delta}$$

1.2 REMARKS. (1) Concerning notation: In GIRARD [1987] a somewhat different notation is used. I have made changes so as to use wherever possible notation that is standard from a categorical viewpoint. This table summarises the changes:

Girard	:	$A^\perp$	1	$\perp$	$\top$	⅋ or $\sqcup$	& or $\sqcap$	$\oplus$
Here	:	$\neg A$	I	ϕ	1	$\odot$	$\times$	$+$

(Symbols not changed: $0, \otimes, \multimap$.)

I believe it is more important to pair the connectives by de Morgan duality ($\otimes$with $\odot$, $\times$ with $+$) than by "distributivity considerations" (as would justify Girard's $\sqcup$ with $\sqcap$.) Furthermore, $\times$ and $+$ seem to really be cartesian product and categorical sum, so those symbols seem more appropriate than Girard's (particularly his $\oplus$.) I must confess to being unable to find an entirely satisfactory notation for the de Morgan dual to tensor product, either in words ("dual tensor" seems preferable to "cotensor" or "tensor sum", or to Girard's "par") or in symbols ($\odot$ has been chosen for its neutrality; $\oplus$ might have been better were it not already so widely in use elsewhere.)

(2) The following sequents may be derived:

$$\begin{array}{rlrl}
(m_{AB}) : & A, B \to A \otimes B & (\omega_{AB}) : & A \odot B \to A, B \\
(e_{AB}) : & A, A \multimap B \to B & & \\
(\pi^1_{AB}) : & A \times B \to A & (\varpi^1_{AB}) : & A \to A + B \\
(\pi^2_{AB}) : & A \times B \to B & (\varpi^2_{AB}) : & B \to A + B \\
(s_{AB}) : & A \otimes B \to B \otimes A & & \\
(a_{ABC}) : & (A \otimes B) \otimes C \to A \otimes (B \otimes C) & (a^{-1}_{ABC}) : & A \otimes (B \otimes C) \to (A \otimes B) \otimes C
\end{array}$$

(and similar sequents s', a' for $\odot$,)

$$(\neg_{AB}) : A \multimap B \to \neg B \multimap \neg A$$

It is easy to see that (m_{AB}) is equivalent to $(\otimes R)$, (w_{AB}) to $(\odot L)$, (e_{AB}) to $(\multimap L)$, (π_{AB}) to $(\times L)$, (ϖ_{AB}) to $(+R)$, **in the presence of** (cut). (Indeed, the rules amount to building in the required amount of (cut) to allow cut-elimination to go through.) As for symmetry and associativity, these follow from the rule $(perm)$ and the (implicit) associativity of concatenation. We give (a_{ABC}) as an illustration:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B, C \xrightarrow{(m)} B \otimes C \qquad A, B \otimes C \xrightarrow{(m)} A \otimes (B \otimes C)}{A, B, C \longrightarrow A \otimes (B \otimes C)}\ (cut)}{C, A, B \longrightarrow A \otimes (B \otimes C)}\ (perm)}{C, A \otimes B \longrightarrow A \otimes (B \otimes C)}\ (\otimes L)}{A \otimes B, C \to A \otimes (B \otimes C)}\ (perm)}{(A \otimes B) \otimes C \xrightarrow{(a)} A \otimes (B \otimes C)}\ (\otimes L) \quad .$$

As for $(\neg_{AB})$, it is given by

$$\dfrac{\dfrac{A, A \multimap B \xrightarrow{(e)} B}{\neg B, A \multimap B \to \neg A}\ (\neg var, \quad perm)}{A \multimap B \to \neg B \multimap \neg A}\ (\multimap R, perm)$$

1.3 If we are to characterize the notion of a "linear category", we must complete the description of linear logic as a "deductive system" (in the sense of LAMBEK and SCOTT [1986]). First we must add the equations between derivations of sequents needed to get the structure of a polycategory (SZABO [1975]); these equations essentially make (cut) into a "polycomposition" of "polyarrows" which is associative, "partially commutative", and has units (id_A). (Analagous equations for multicategories may be found in this volume in LAMBEK [1987]; for this reason I will not go into detail here for these or the remaining equations.) Next, we must account for the monoidal structure of I, $\otimes$ (and their duals $\phi, \odot$) by adding equations which make sequents $A_1, \ldots, A_n \to B_1, \ldots, B_m$ equivalent to sequents $A_1 \otimes \cdots \otimes A_n \to B_1 \odot \cdots \odot B_m$. (Clearly there are maps, using the evident "hom" notation

$$[A_1, \ldots, A_n; B_1, \ldots, B_m] \leftrightarrow [A_1 \otimes \cdots \otimes A_n; B_1 \odot \cdots \odot B_m]$$

given by the rules $(\otimes L, R)$, $(\odot L, R)$, (cut); the point is that these maps be isomorphisms and inverse to each other.)

Similarly, it is likely that we want the structure to be symmetric monoidal, closed, and have finite products and coproducts — each of these adds to the list of equations in the evident way. For instance, (a) and (a^{-1}) must be inverse, as must (s_{AB}) and (s_{BA}). (This last could be weakened, if we only want a braided monoidal category, as in JOYAL and STREET [1986]. However, this would complicate the rest of the structure, so we shall not pursue this further.) Moreover, $(\multimap R)$ should give a bijection

$$[\Gamma, A; B, \Delta] \xrightarrow{\sim} [\Gamma; A \multimap B, \Delta]$$

whose inverse is

$$\frac{\Gamma \to A \multimap B, \Delta \qquad A, A \multimap B \xrightarrow{(e)} B}{\Gamma, A \to B, \Delta} \quad (cut,perm).$$

Finally, it seems that $\neg$ is a contravariant functor (in view of $(\neg var)$), that it is strong (in view of $(\neg_{AB})$), and that it is an involution (in view of (d) — which thus must be inverse to (d^{-1}).) These yield further equations, including the following, (if we are to have a $*$-autonomous category, as defined in BARR [1979]): for any A, B, these derivations of the sequent $A \multimap B \to \neg\neg A \multimap \neg\neg B$ are equal: $(d^{-1} \multimap d) = (\neg_{\neg B \neg A})(\neg_{AB})$. Here $(d^{-1} \multimap d)$ is a case of the schema

$$\frac{C \xrightarrow{(f)} A \qquad B \xrightarrow{(g)} D}{A \multimap B \xrightarrow{(f \multimap g)} C \multimap D}$$

given by

$$\cfrac{\cfrac{C \xrightarrow{(f)} A \qquad \cfrac{A, A \multimap B \xrightarrow{(e)} B \qquad B \xrightarrow{(g)} D}{A, A \multimap B \to D}\ (cut)}{C, A \multimap B \to D}\ (cut)}{A \multimap B \to C \multimap D}\ (\multimap R)$$

and $(\neg_{\neg B \neg A})(\neg_{AB})$ is a case of *(cut), viz.* in general:

$$\frac{A \xrightarrow{(f)} B \quad B \xrightarrow{(g)} C}{A \xrightarrow{(g)(f)} C} \quad (cut)$$

Since the required equations may be easily generated from the above recipe (and are in essence to be found in the references given, for the most part), and since this process is familiar (for instance, to that of LAMBEK and SCOTT [1986] for λ-calculus), I shall avoid the messy notational baggage needed to make all the details explicit, by stating boldly and without discussion:

1.4 DEFINITION. A linear category $\mathbf{G}$ is a $*$-autonomous category with finite products.

REMARKS. For a fuller discussion of $*$-autonomous categories, see BARR [1979]. Here just let me say that $\mathbf{G}$ is a closed symmetric monoidal category $\mathbf{G}$ with an involution $\neg: \mathbf{G}^{op} \to \mathbf{G}$ given by a dualising object ϕ: in our notation this means $\neg A = A \multimap \phi$ and the canonical arrow $A \to ((A \multimap \phi) \multimap \phi)$ is an isomorphism. (Barr uses $*$ for our $\neg$.) In such a category the existence of finite coproducts follows from finite products by de Morgan duality.

1.5 PROPOSITION. Given any linear logic $\mathcal{L}$, a linear category $\mathbf{G}(\mathcal{L})$ may be constructed (whose objects are formulas and whose morphisms $A \to B$ are equivalence classes of derivations of sequents $A \to B$); given any linear category $\mathbf{G}$, a linear logic $\mathcal{L}(\mathbf{G})$ may be constructed (whose constants are the objects of $\mathbf{G}$ and whose axioms are the morphisms of $\mathbf{G}$). Furthermore $\mathbf{G} \simeq \mathbf{G}(\mathcal{L}(\mathbf{G}))$ and (in a suitable sense) $\mathcal{L}$ is equivalent to $\mathcal{L}(\mathbf{G}(\mathcal{L}))$.

2. THE EXPONENTIAL OPERATOR ! . In GIRARD [1987], these rules are given (in our notation) for the operator ! :

$$(der): \quad \frac{\Gamma, A \to \Delta}{\Gamma, !A \to \Delta} \qquad \text{(“dereliction”)}$$

$$(thin): \quad \frac{\Gamma \to \Delta}{\Gamma, !A \to \Delta} \qquad \text{(“thinning” or “weakening”)}$$

$$(contr): \quad \frac{\Gamma, !A, !A \to \Delta}{\Gamma, !A \to \Delta} \qquad \text{(“contraction”)}$$

$$(!): \quad \frac{!\Gamma \to A}{!\Gamma \to !A}$$

In $(!)$, $!\Gamma$ means $!A_1, !A_2, \ldots, !A_n$. Girard actually gives the rules for the de Morgan dual ? ; we shall not discuss ? .

2.1 It is perhaps worth simplifying these rules:

PROPOSITION. In the presence of linear logic:

(1) (der) is equivalent to the scheme $(\epsilon_A) : !A \to A$,
(2) $(thin)$ is equivalent to the scheme $(\epsilon'_A) : !A \to I$,
(3) $(contr)$ is equivalent to the scheme $(\delta'_A) : !A \to !A \otimes !A$,
(4) if $(der), (thin), (contr)$, then $(!)$ is equivalent to:

$$(\delta_A) : !A \to !!A \qquad \text{and } (fun) : \frac{A \to B}{!A \to !B}$$

$$(\Delta\ iso) : !A \otimes !B \leftrightarrow !(A \times B) : (\Delta\ iso)^{-1}$$

$$(i\ iso) : I \leftrightarrow !1 : (i\ iso)^{-1}$$

REMARKS. (1) $(\delta_A), (fun)$ arise from the case $n = 1$ of $(!)$, (Δiso) from the $n > 1$ case, and $(i\ iso)$ from the $n = 0$ case.

(2) Notice these rules seem to imply that we should regard ! as a functor (by (fun)), indeed a cotriple (or comonad) (by $(\epsilon_A), (\delta_A)$), and each $!A$ seems to be a comonoid (with respect to the monoidal structure $I, \otimes$), in view of $(\epsilon'_A), (\delta'_A)$. Furthermore, this comonoid structure seems to be the image under ! of the canonical comonoidal structure $(1 \leftarrow A \to A \times A)$ with respect to the cartesian structure $1, \times$, in view of $(\Delta\ iso)$, $(i\ iso)$. (These comments will take us straight to Definition 2.2.)

PROOF OF 2.1:

(1) For (ϵ_A), apply (der) to (id_A).For (der), apply (cut) to (ϵ_A).
(2) For (ϵ'_A), apply $(thin)$ to (IR). For $(thin)$, use (IL) and (cut) with (ϵ'_A).
(3) For (δ'_A), apply $(contr)$ to $(m_{!A!A})$. For $(contr)$, use $(\otimes L)$ and (cut) with (δ'_A).
(4) Given $(!)$, (δ_A) is $(!)$ applied to $(id_{!A})$.(fun) is the derived rule

$$\dfrac{\dfrac{A \to B}{!A \to B}\;(der)}{!A \to !B}\;(!)$$

$(\Delta\ iso)^{-1}$ is

$$\dfrac{\dfrac{\dfrac{A \times B \xrightarrow{(\pi^1)} A}{!(A \times B) \to !A}\;(fun) \qquad \dfrac{A \times B \xrightarrow{(\pi^2)} B}{!(A \times B) \to !B}\;(fun)}{!(A \times B), !(A \times B) \to !A \otimes !B}\;(\otimes R)}{!(A \times B) \to !A \otimes !B}\;(contr)$$

$(\Delta\ iso)$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{!A \to A}{!A, !B \to A}\;(thin) \qquad \dfrac{!B \to B}{!A, !B \to B}\;(thin, perm)}{!A, !B \to A \times B}\;(\times R)}{!A, !B \to !(A \times B)}\;(!)}{!A \otimes !B \to !(A \times B)}\;(\otimes L)$$

Conversely, given $(\delta_A), (fun)$, and the Δ, i iso's, we derive $(!)$ as follows.

First treat the $n = 0$ case as $I \to A$, *i.e.* as a special case of the $n = 1$ case. This is possible because of $(i\ iso)$; $I \to A$ may be thought of as $!1 \to A$. Then for $n = 1$, $(!)$ becomes the rule

$$\dfrac{!A_1 \to A}{!A_1 \to !A}$$

given by

$$\dfrac{!A_1 \xrightarrow{(\delta)} !!A_1 \qquad \dfrac{!A_1 \to A}{!!A_1 \to !A}\;(fun)}{!A_1 \to !A}\;(cut)$$

$(\Delta\ iso)$ allows the case when $n > 1$ to be reduced to the $n = 1$ case in view of the evident induced bijections (using the hom notation of 1.3):

$$[!A_1, \ldots, !A_n; A] \cong [!A_1 \otimes \cdots \otimes !A_n; A] \cong [!(A_1 \times \cdots \times A_n); A].$$

2.2 If we impose the appropriate equations on derivations, it is clear that we shall arrive at the following structure.

Definition. A Girard category consists of a linear category $\mathbf{G}$ together with a cotriple $!{:}\mathbf{G} \to \mathbf{G}$ satisfying the following:

(i) for each A of $\mathbf{G}$, $!A$ is a comonoid with respect to the tensor structure: $I \xleftarrow{\epsilon'_A} !A \xrightarrow{\delta'_A} !A \otimes !A$;

(ii) there are natural isomorphisms $!A \otimes !B \xrightarrow{\sim} !(A \times B)$, $I \xrightarrow{\sim} !1$; moreover $!$ takes the comonoid structure $(1 \xleftarrow{A} A \xrightarrow{\Delta} A \times A)$ with respect to the cartesian structure, to the comonoid structure in (i): *i.e.* these diagrams commute:

$$\begin{array}{ccc} !A & \xrightarrow{\delta'} & !A \otimes !A \\ \| & & \downarrow \wr \\ !A & \xrightarrow{!\Delta} & !(A \times A) \end{array} \qquad\qquad \begin{array}{ccc} !A & \xrightarrow{\epsilon'} & I \\ \| & & \downarrow \wr \\ !A & \xrightarrow{!A} & !1 \end{array}$$

Remark. In fact it is easy to note that (i) follows from (ii), the diagrams defining ϵ' and δ'. However, in view of the "uncertainty" surrounding (!), it seems best to keep all the rules seperate.

2.3 As before, we claim without further ado:

Proposition. Given a linear logic $\mathcal{L}$ with exponential operator !, a Girard category may be constructed on $\mathbf{G}(\mathcal{L})$; given a Girard category $\mathbf{G}$, !, the linear logic $\mathcal{L}(\mathbf{G})$ can be equipped with an exponential operator !. These constructions extend the equivalences of Proposition 1.5 in the evident way.

2.4 The essence of Girard's translation of intuitionistic logic into linear logic is the following:

Proposition. If $\mathbf{G}$, ! is a Girard category, then the Kleisli category $\mathbf{K}(\mathbf{G})$ is cartesian closed.

Proof. (For a basic reference on the categorical notions of cotriple, comonoid, and Kleisli category, see Mac Lane [1971].) Recall that the objects of $\mathbf{K}(\mathbf{G})$ are those of $\mathbf{G}$ itself, while the morphisms are given by

$$Hom_{\mathbf{K}(\mathbf{G})}(A, B) = Hom_{\mathbf{G}}(!A, B).$$

Writing $A \Rightarrow B$ for exponentiation B^A in $\mathbf{K}(\mathbf{G})$, $X \multimap Y$ for the internal hom in $\mathbf{G}$, it seems likely that

$$A \Rightarrow B = \,!A \multimap B$$

will do the trick. It is an easy matter to verify that

(i) the terminal object of $\mathbf{K}(\mathbf{G})$ is 1, the terminal object of $\mathbf{G}$;
(ii) the product $A \times B$ of $\mathbf{K}(\mathbf{G})$ is the same as in $\mathbf{G}$;
(iii) $A \Rightarrow B$ is $!A \multimap B$.

Appropriate bijections are given by

$$\begin{array}{c}
C \to A \times B \text{ in } \mathbf{K}(\mathbf{G}) \\ \hline
!C \to A \times B \text{ in } \mathbf{G} \\ \hline
!C \to A, \quad !C \to B \text{ in } \mathbf{G} \\ \hline
C \to A, \quad C \to B \text{ in } \mathbf{K}(\mathbf{G})
\end{array}
\qquad
\begin{array}{c}
C \to (A \Rightarrow B) \text{ in } \mathbf{K}(\mathbf{G}) \\ \hline
!C \to (!A \multimap B) \text{ in } \mathbf{G} \\ \hline
!A \otimes !C \to B \text{ in } \mathbf{G} \\ \hline
!(A \times C) \to B \text{ in } \mathbf{G} \\ \hline
A \times C \to B \text{ in } \mathbf{K}(\mathbf{G})
\end{array}$$

2.5 REMARKS. (1) In general $\mathbf{K}(\mathbf{G})$ does not have coproducts; Girard's interpretation of disjunction

$$A \vee B = !A + !B$$

is mysterious from this point of view, for the appropriate maps do not lie in $\mathbf{K}(\mathbf{G})$, though we do have a glimmer of the correct coproduct structure — *viz.* the bijections

$$\begin{array}{c}
!A + !B \to C \quad \text{in } \mathbf{G} \\ \hline
!A \to C \quad !B \to C \text{ in } \mathbf{G} \\ \hline
A \to C \quad B \to C \text{ in } \mathbf{K}(\mathbf{G})
\end{array}$$

(2) In GIRARD and LAFONT [1987], a stronger structure is considered for $!$, with the intention that $!A = I \times A \times (!A \otimes !A)$. What Girard and Lafont seem to require (again, since they do not give a deductive system, but only a logic, we are left to supply appropriate equations between derivations) amounts to $!A$ being the cofree commutative comonoid over A. This condition implies (and is stronger than) that $\mathbf{G}$, $!$ is a Girard category; the question is whether it is too strong. (The structure of coherent spaces and linear maps, in the next section, does not have this extra property, for instance, nor is the structure of DE PAIVA [1987] an example since it is not $*$-autonomous.) It seems to me that what is really wanted is that $\mathbf{K}(\mathbf{G})$ be cartesian closed, so the question is: what is the minimal condition on $!$ that guarantees this — *i.e.* can we axiomatise this condition satisfactorily? (L. Román asked a related question: what condition on $!$ makes the Eilenberg-Moore category cartesian closed?)

(3) In GIRARD [1987], the propositional system is extended to a predicate linear logic by the addition of free variables of appropriate types and quantifiers $\bigwedge, \bigvee$, subject to the rules

$$(\textstyle\bigwedge L) \quad \frac{\Gamma, A[x := t] \to \Delta}{\Gamma, \bigwedge x A \to \Delta} \qquad\qquad (\textstyle\bigwedge R) \quad \frac{\Gamma \to A, \Delta}{\Gamma \to \bigwedge x A, \Delta}$$

(with the usual restriction on $(\bigwedge R)$: x not free in Γ, Δ).

(The rules for $\bigvee$ are given by de Morgan duality, as is $\bigvee$ itself.)

Girard is not specific about the nature of the types here — we may suppose, for example, that the above amounts to the following categorical structure:

An indexed linear category consists of a category $\mathbf{S}$ with finite products, and an indexed category $\mathbf{G}$ over $\mathbf{S}$; for each S of $\mathbf{S}$, the fibre $\mathbf{G}^S$ is a linear category, whose structure is preserved by t^*, t any morphism of $\mathbf{S}$; furthermore, each π^* has both adjoints $\bigvee_\pi \dashv \pi^* \dashv \bigwedge_\pi$, where π is a projection morphism of $\mathbf{S}$. The idea here, of course, is that $\mathbf{G}^S$ consists of the linear formulas with free variable of type S and (equivalence classes of) derivations of such formulas. (To be certain the logic is properly fibred in this way we ought to add conditions to the rules of inference to ensure that in any derivation of propositional linear logic, the same variables appear throughout, and in the quantifier rules, the only variables lost are those explicitly indicated — such restrictions are analogous to those of SEELY [1983] for first order intuitionistic logic and [1987] for polymorphic λ–calculus, and cause no loss of expressive power, (with a liberal use of dummy free variables.)

As with the logic, the adjoints $\bigvee_\pi, \bigwedge_\pi$ are dual, and so one only need assume one exists. (This is analogous to the situation for cartesian product and sum in $*$-autonomous categories.)

In this context, we would define an indexed Girard category as an indexed linear category $\mathbf{G}$ over $\mathbf{S}$ so that each $\mathbf{G}^S$ was a Girard category (i.e., had a ! cotriple with the usual properties), and that each t^* preserved this structure also. For such an indexed cotriple, one can define the indexed Kleisli category $\mathbf{K}(\mathbf{G})$ over $\mathbf{S}$ $(\mathbf{K}(\mathbf{G})^S$ will be $\mathbf{K}(\mathbf{G}^S))$; we already know $\mathbf{K}(\mathbf{G})$ will be (indexed) cartesian closed, and a similar analysis will easily show that in $\mathbf{K}(\mathbf{G})$, each π^* (π a projection of $\mathbf{S}$) will have a right adjoint $\pi^* \dashv \Pi_\pi$ (given, on objects, by $\bigwedge_\pi$.)

In general, we won't have a $\sum_\pi \dashv \pi^*$; the situation is similar to that for coproducts, as the following bijections show:

$$\frac{\dfrac{A \to \pi^* B \text{ in } \mathbf{K}(\mathbf{G})^{S\times T}}{!A \to \pi^* B \text{ in } \mathbf{G}^{S\times T}}}{\bigvee_\pi !A \to B \text{ in } \mathbf{G}^S}$$

(Girard set $\sum_\pi A = \bigvee_\pi !A$.)

The corresponding bijections for Π show why $\Pi_\pi A = \bigwedge_\pi A$ works:

$$\begin{array}{c}
\pi^* B \to A \text{ in } \mathbf{K}(\mathbf{G})^{S\times T} \\ \hline
!\pi^* B \to A \text{ in } \mathbf{G}^{S\times T} \\ \hline
\pi^* !B \to A \text{ in } \mathbf{G}^{S\times T} \\ \hline
!B \to \bigwedge_\pi A \text{ in } \mathbf{G}^S \\ \hline
B \to \bigwedge_\pi A \text{ in } \mathbf{K}(\mathbf{G})^S
\end{array}$$

3. AN EXAMPLE: COHERENT SPACES. In [1987] GIRARD gives an example of a model of linear logic; I shall briefly summarise how that example may be presented in

this set-up. (This section will not be self-contained; I assume the reader has a copy of Girard [1987] in front of her.)

A coherent space is an atomic Scott domain closed under sups of families of pairwise compatible (or consistent) elements. Such a space X may be represented as a subdomain of the powerset $\mathcal{P}(|X|)$, where $|X|=$ the set of atoms of X; by this representation, the atoms are singletons. In fact, the structure of X is entirely given by the graph on $|X|$ defined by compatability: $x \quad y(mod X)$iff $x \vee y \in X$ (iff x, y are compatible in X.)

A linear map $f\colon X \to Y$ of coherent spaces must preserve sups of families of pairwise compatible elements and binary infs of compatible elements, as well as the order. Such a map is entirely determined by its trace: $\{< x, y >| \ x$ an atom of X, y an atom of $Y, y \leq f(x)\}$, since, for $a \in X$,

$$f(a) = \bigvee_{x \leq a} \bigvee_{y \leq f(x)} y.$$

The category **COHL** of coherent spaces and linear maps is a linear category; this is essentially proven in Girard [1987]. Furthermore, !: **COHL** $\to$ **COHL** makes **COHL** a Girard category; this is implicit in Girard [1987], but some of the details might be useful. Given a coherent space X, $!X$ is given by $|!X| = \mathcal{P}_{cfin}(|X|) =$ compatible finite subsets of $|X|$, with compatibility in $!X$ canonically induced by compatibility in X. (Viewing a space X as a subdomain of $\mathcal{P}(|X|)$, this would be written $|!X| = X_{fin} =$ finite elements of X.) Given a linear map $f\colon X \to Y$, $!f\colon !X \to !Y$ is characterised by the following: given an atom $\{x_1, \ldots, x_n\}$ of $!X$, an atom $\{y_1, \ldots, y_m\}$ of $!Y$, then $\{y_1, \ldots, y_m\} \leq (!f)(\{x_1, \ldots, x_n\})$ iff $n = m$ and $y_i \leq f(x_i)$ for each $i \leq n$. (!f is the "direct image made linear".)

For X in **COHL**, the map $\epsilon_X\colon !X \to X$ is given by: for a an atom of $!X$, x an atom of X, $x \leq \epsilon_X(a)$ iff $x \in a$, (*i.e.* $\{x\} \leq a$ in $!X$.) $\delta_X\colon !X \to !!X$ is given by: for a an atom of $!X$, b an atom of $!!X, b \leq \delta_X(a)$ iff $\bigvee b \leq a$.

The trace of $\epsilon'_X\colon !X \to I$ is the singleton $\{< \emptyset, 1 >\}$, where 1 is the unique atom of I. And $\delta'_X : !X \to !X \otimes !X$ is given by: for a, b, c atoms of $!X$, so that $< b, c >$ is an atom of $!X \otimes !X, < b, c > \leq \delta'_X(a)$ iff $b \vee c \leq a$.

It is a matter of straightforward calculation to show that these maps satisfy the equations for a cotriple and comonoid, and that the natural isomorphisms of Definition 2.2(ii) have the stated properties. The Kleisli category **K(COHL)** is **COHS**, the category of coherent spaces and *stable* maps, originally introduced (as "binary qualitative domains and stable maps") in Girard [1986].

COHS is well known to be cartesian closed, and does not have finite coproducts. (Girard discusses his treatment of sums in [1986].) Furthermore, **COHL** does not model the stronger axiom for ! in Girard and Lafont [1987] $!A \cong I \times A \times (!A \otimes !A)$, nor does ! create cofree comonoids. (Girard [1987] mentions varying the notion of coherent space, using trees, in order to model this situation.)

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