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# On the formal definition of categories*

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This paper is an attempt to modify set theory in such a way as to provide for the existence of objects (like the category of groups) indispensable in modern mathematics. Guided by the ideas of TARSKI, VON NEUMANN, BERNAYS and GÖDEL, and inspired by the rapid development of category theory, we are going to describe axiomatically a generalized set theory $\mathscr{T}_1$ which is stronger than the set theory $\mathscr{T}_0$ (in the sense of [3]). It follows that every theorem of $\mathscr{T}_0$ is a theorem of $\mathscr{T}_1$, a fact which is very convenient since most mathematical theories are tacitly assumed to be stronger than $\mathscr{T}_0$. For this end, we consider sets called universes which are stable with respect to the elementary set-theoretical operations. The proposed set theory $\mathscr{T}_1$ differs from the usual one in a single axiom: Instead of the existence of an infinite set we require the existence of arbitrarily large universes. This in turn guarantees the existence of an infinite set. Recently, the author was told that Grothendieck uses a similar approach. He (reputedly) employs an axiom which is equivalent to our axiom A 5′. Each axiom of $\mathscr{T}_1$ is introduced with the understanding that one risks having to abandon this axiom one day if it leads to a contradiction. (Compare this standpoint with the famous address of N. BOURBAKI delivered at the eleventh meeting of the Association for Symbolic Logic, 1948 [2]). The reader will find axioms and further references on the TARSKI, VON NEUMANN, BERNAYS, GÖDEL system in [1], [4], and [9]. The logical difficulties encountered in dealing with categories are exhibited in [5] and [14] together with literature on the subject.

The paper is intended for the working mathematician who is familiar with Bourbaki's approach to formal mathematics (see [3]; also [11], appendix). The reader is not required to know about "non-simple applied first-order calculus" in order to understand its contents. The proposed set theory is treated like any other mathematical theory (e.g. group theory), its axioms and schemes being given explicitly. They comprise all the rules which are necessary for logical deductions (logical theory), for defining quantifiers (quantified theory) and for working with the equality sign (equalitary theory). These rules are usually separated from set theory and thrown into first-order calculus (see [4]).

To facilitate the reading of this note we review formal mathematics in the *first section*; the emphasis is on collectivizing relations. Universes are

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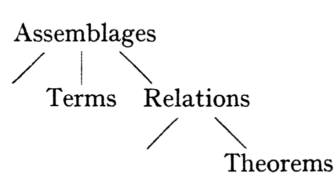
defined and studied in *section two.* They are closely related to Tarski's inaccessible sets and Lévy's standard complete models (see [*13*], [*15*] and the discussion at the end of the paper). In conformity with general definitions we describe, in *section three,* a theory $\mathcal{T}_1$, called the strong set theory, by giving its signs, writing down the axioms and schemes, and hoping that it is non-contradictory (otherwise every statement would automatically be true). For the sake of completeness, the definitions of categories and functors are recalled in *section four,* and a systematic terminology is suggested. Examples which show how the axiom of the existence of arbitrarily large universes can be used to construct categories (including a category of functors), are established in *section five.* We do not succeed in guaranteeing "the" category of groups, say, but rather "a" category of groups which, however, may be chosen as large as one pleases.

One might ask whether it is really necessary to invent stronger set theories. Since every assemblage starting with «let $X \in \mathcal{G}$ where $\mathcal{G}$ denotes a category of groups ...», could easily be replaced by «let $X$ be a group ...», the answer is "no". However, our minds like to "collect" objects which possess the same property. In addition, categories shorten otherwise lengthy statements and permit the mathematician to think in familiar terms. It is for this reason that we searched for a grammar to an already widely used language which would guarantee the existence of categories without endangering the results of contemporary mathematics and the methods employed for deriving it.

Throughout this paper, the terminology of [*3*] is employed. No effort is made to show that the strong set theory $\mathcal{T}_1$ is non-contradictory. I am indebted to Dr. HELMUT RÖHRL who experimented with the axioms in a course on categories given at the University of Minnesota, and who encouraged me to publish this paper.

**1. Collectivizing relations.** Recall that formal mathematics is a language built up by logical signs ($\square$, $\tau$, $\vee$, $\neg$), letters ($a, a', \ldots; A, A', \ldots$) and specific signs ($=, \in, \supset, \ldots$) all together referred to as signs. Assemblages are defined as juxtapositions of the signs; they represent the words in our language. Terms and relations are distinguished by formative constructions; they represent the objects and the statements made about the objects. Starting from axioms, theorems are introduced by means of proofs; they represent the true statements. The relationship is illustrated by the diagram below.




Definitions serve to abbreviate cumbersome assemblages using shorter symbols or phrases of the common language; the assemblages in question are not required to be meaningful. There are two types of axioms: explicit ones and implicit ones which are derived from schemes by specialization. In particular, the set theory is a mathematical theory in the fullest sense, defined by axioms and schemes. If, in a theory, a relation and its negation are true, then the theory is called contradictory. (In this case, every relation is true.) Frequently — as in the theory of groups — we tacitly assume that the theory we are working in is stronger than the theory of sets. A letter which appears in an explicit axiom is called a constant of the theory. Note that the theory of sets is without a constant, while the theory of groups possesses two constants (the group and the composition law).

Rules which simplify proof procedures and which strictly speaking belong to metamathematics are called criteria.

One of the most important features of the set theory is the provision for forming certain sets. In order to illustrate this point, consider the subsets of $X$. We are faced with the question of whether there exists an object whose elements are precisely the subsets of $X$. If this is true (as in the set theory) we say the relation $Y \subset X$ is collectivizing in $Y$.

More precisely, for every relation $R$, the term

$$(*) \qquad \tau_y\left((\forall x)\ (x \in y \Leftrightarrow R)\right)$$

is denoted by $\mathscr{E}_x(R)$ and called «the set of the $x$ such that $R$».

Furthermore, the relation

$$(**) \qquad (\exists y)\left((\forall x)\ (x \in y \Leftrightarrow R)\right)$$

is denoted by $\mathrm{Coll}_x R$ and called «$R$ is collectivizing in $x$». ($\tau_y(S)$ may be interpreted as «the privileged $y$ such that $S$»). We emphasize that nothing prevents us from forming the set $\mathscr{E}_x(R)$ whether $R$ is collectivizing in $x$ or not.

Criterion 1. *If, in some theory $\mathscr{T}$ stronger than the set theory, $R$ is a relation collectivizing in $x$, then the relations*

$$x \in \mathscr{E}_x(R) \quad \textit{and} \quad R$$

*are equivalent for all $x$.*

Indeed, $\mathrm{Coll}_x R$ is an abbreviation of $(**)$, which in virtue of $(*)$ and the definition of the existential quantifier can be written in the form

$$\left(\mathscr{E}_x(R) \mid y\right)\left((\forall x)\ (x \in y \Leftrightarrow R)\right).$$

Hence, the relation

$$(\forall x)\left(x \in \mathscr{E}_x(R) \Leftrightarrow R\right)$$

is a theorem of $\mathscr{T}$. (Conversely, the last relation implies that $R$ is collectivizing in $x$.)

Examples. 1) The relation $x \in y$ is collectivizing in $x$, because $(\forall x)$ $(x \in y \Leftrightarrow x \in y)$ hence $(\exists z)$ $(\forall x)$ $(x \in z \Leftrightarrow x \in y)$ are true. Therefore, the relations

$$x \in \mathscr{E}_x(x \in y) \quad \text{and} \quad x \in y$$

are equivalent for all $x$. In general, the relation $x \in \mathscr{E}_x(R)$ is typographically simpler than the original relation $R$.

2) Let $R$ be a relation, $A$ a term, $x$ a letter which does not occur in $A$. If the relation $R \Rightarrow (x \in A)$ is a theorem, then $R$ is collectivizing in $x$ (criterion C 52 of [3]).

**2. Universes.** In what follows, we are reasoning in a theory $\mathscr{T}$ stronger (i.e. based on more axioms) than the theory of sets deprived of the axiom A 5 (see [3]).

DEFINITION 1. We say that $M$ is a universe if the following conditions are fulfilled:

($U_I$) $(\forall X)\big((X \in M) \Rightarrow (X \subset M)\big)$.

($U_{II}$) $(\forall X)\big((X \in M) \Rightarrow (\mathfrak{P}(X) \in M)\big)$.

($U_{III}$) $(\forall X)(\forall Y)\big((X \in M \wedge Y \in M) \Rightarrow (\{X, Y\} \in M)\big)$.

($U_{IV}$) $(\forall X)(\forall Y)\big((X \in M \wedge Y \in M) \Rightarrow (X \times Y \in M)\big)$.

($U_V$) $(\forall I)(\forall X)\big((I \in M \wedge X \in M^I) \Rightarrow (\bigcup X \in M)\big)$.

EXAMPLE. The empty set $\emptyset$ is a universe.

PROPOSITION 1. *Subsets and quotient sets of elements of a universe $M$ belong to $M$.*

Indeed, $X \subset Y \in M$ implies $X \in \mathfrak{P}(Y) \in M$ by ($U_{II}$), hence, $X \in M$ by ($U_I$). The second part of the proposition follows immediately from ($U_{II}$) and the first part.

PROPOSITION 2. *Let $M$ be a universe. Then* $M = \bigcup_{Y \in M} Y$.

In virtue of ($U_{III}$), $X \in M$ implies $X \in \{X\} \in M$, hence $X \in Y$ for some $Y \in M$. On the other hand, $X \in Y$ for some $Y \in M$ results in $X \in M$ due to ($U_I$).

PROPOSITION 3. *Let $M$ be a universe. $X \in M$ implies $M \not\subset X$.*

Assume for a moment that $M \subset X$ for some $X \in M$. Then $A \subset M$ would imply $A \in M$ by prop. 1, hence $\mathfrak{P}(M) \subset M$. Write $\mathfrak{a}$ for $\operatorname{Card}(M)$. It would follow that $2^{\mathfrak{a}} \leqq \mathfrak{a}$, which is absurd, the Cantor theorem being true in $\mathscr{T}$.

COROLLARY. *For every universe $M$, $M \notin M$.*

THEOREM 1. *Let $M$ be a universe. $X \in M$ implies* $\operatorname{Card}(X) < \operatorname{Card}(M)$.

Indeed, $Y \subset X \in M$ implies $Y \in M$ (prop. 1), hence $\mathfrak{P}(X) \subset M$. Let $\mathfrak{a}$ be $\operatorname{Card}(X)$. Then $\mathfrak{a} < 2^{\mathfrak{a}} \leqq \operatorname{Card}(M)$, again using Cantor's theorem.

COROLLARY. *If the empty set $\emptyset$ belongs to a universe $M$, then $M$ is infinite.*

Apply ($U_{II}$) and theorem 1.

REMARK. ($U_I$) and theorem 1 together read: $X \in M \Rightarrow X \subset M \wedge \operatorname{Card}(X) < \operatorname{Card}(M)$. With the implication sign reversed: $X \subset M \wedge \operatorname{Card}(X) < \operatorname{Card}(M) \Rightarrow X \in M$, we obtain precisely Tarski's axiom ($A_4$) (see [15]) which we shall need only in the much weaker version ($U_{III}$).

Proposition 4. *Let $M$ be a universe. Assume that $I \in M$ and $X \in M^I$. Then:*

a) $\cap X \in M$, *provided* $I \neq \emptyset$.

b) $\cup X \in M$.

c) $\Sigma X \in M$.

d) $\Pi X \in M$.

We remark that $\cap X \subset X_i$ for some $i \in I$, which settles a) by prop. 1. Case b) is merely a repetition of $(U_V)$. As for c), write $\Sigma X$ in the form $\bigcup_{i \in I} (X_i \times \{i\})$, and apply prop. 1, $(U_{IV})$ and $(U_V)$. Finally d) follows from the fact that $\Pi X \subset \mathfrak{P}(I \times \cup X)$; use $(U_V)$, $(U_{IV})$ and $(U_{II})$ to conclude.

Corollary. *Let $M$ be a universe. Assume that $X \in M$ and $Y \in M$. Then:*

a) $X \cap Y \in M$.

b) $X \cup Y \in M$.

c) $X * Y \in M$ (*sum*).

d) $X \times Y \in M$.

e) $(X, Y) \in M$.

For a) to d), note that $\{X, Y\} \in M$ $(U_{III})$; for e) observe that $(X, Y) \in \{X\} \times \{Y\} \in M$.

Proposition 5. *Let $M$ be a universe. Assume that $X \in M$ and $Y \in M$. Then:*

a) $Y^X \in M$.

b) $\mathscr{F}(X, Y) \in M$.

Indeed, $Y^X$ is a subset of $\mathfrak{P}(X \times Y) \in M$, while $\mathscr{F}(X, Y)$ is a subset of $\mathfrak{P}(X \times Y) \times \mathfrak{P}(X) \times \mathfrak{P}(Y) \in M$.

**3. The strong set theory: Axioms.** Axiomatic set theory in the sense of [*3*] or [*11*], appendix, does not provide any means for distinguishing sets, classes, super-classes etc. (In other words, it is not a type-theory.) Although an intuitive interpretation of certain signs can be achieved through the axioms, no intrinsic meaning is attached to them.

We call strong set theory the theory whose relational signs are $=, \in$, and whose substantific sign is $\tau$. As far as axioms and schemes are concerned, we keep the axioms A 1 to A 4 and schemes S 1 to S 8 of [*3*], and we replace A 5 by the axiom

$$\text{A 5}'. \ (\forall X)\ (\exists M)\ \big((X \in M) \wedge (M \text{ is a universe})\big).$$

Remark 1. Axiom A 5′ does not make axioms A 2 and A 4 superfluous. One could, for example, try to argue as follows: There exists a universe $M$

such that $X \in M$. Hence, $Y < X$ implies $Y \in \mathfrak{P}(X) \in M$, hence $Y \in M$; apply C 52 of [*3*]. However, $Y < X$ does not imply $Y \in \mathfrak{P}(X)$ except if we know in advance that the relation $Y < X$ is collectivizing in $Y$. A slight change in the definition of universes would remedy the situation; simply replace $(\mathrm{U_{II}})$ by

$$(\forall X)\big((X \in M) \Rightarrow (\exists Z)(Z \in M \wedge (\forall Y)(Y \in Z \Leftrightarrow Y < X))\big).$$

For methodical reasons, we like to keep A 2 and A 4 in the present form.

From now on, we are reasoning in a theory $\mathcal{T}$ stronger than the strong set theory $\mathcal{T}_1$.

THEOREM 2. *There exists an infinite set.*

Indeed, the axiom A 5′ guarantees the existence of a universe $M$ such that $\emptyset \in M$. Moreover, $M$ is infinite in virtue of the cor. to theorem 1.

CRITERION 2. *The strong set theory $\mathcal{T}_1$ is stronger than the set theory $\mathcal{T}_0$. In particular, every theorem of $\mathcal{T}_0$ is a theorem of $\mathcal{T}_1$.*

This follows immediately from theorem 2.

PROPOSITION 6. *Let $M$ be a universe. If $\boldsymbol{N} \in M$, then $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$, and $\boldsymbol{K}$ belong to $M$. In particular, there exists a universe $M$, which contains $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$, and $\boldsymbol{K}$ as elements. ($\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{Z}, \boldsymbol{K}$, denote resp. the set of the natural integers, the rational integers, the rational numbers, the real numbers, the complex numbers, the quaternions).*

$\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ are subsets of $\mathfrak{P}(\boldsymbol{N} \times \boldsymbol{N})$, $\mathfrak{P}(\boldsymbol{Z} \times \boldsymbol{Z})$, $\mathfrak{P}(\mathfrak{P}(\boldsymbol{Q}))$ respectively, which belong to $M$ in turns. $\boldsymbol{C}$ is a quotient set of $\boldsymbol{R}^{(\boldsymbol{N})}$ which is a subset of $\boldsymbol{R}^{\boldsymbol{N}} \in M$. Similarly for $\boldsymbol{K}$. The second part of the proposition is a consequence of axiom A 5′ and the first part.

REMARK 2. The second part of prop. 6 could be proved independently in the following way: By A 5′, there exists a universe $M$ such that $\{\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{K}\} \in M$. Then apply property $(\mathrm{U_I})$ of universes.

**4. Categories.** For the sake of completeness, we recall in this section the definitions of categories and functors in a form which is in conformity with [*5*], [*7*], [*10*]. We introduce a systematic terminology which applies not only to abelian categories but also to non-abelian categories. The reader will notice that this section is independent of the preceeding ones, the reason being that axiom A 5′ can be dispensed with in the *definition* of categories, but not for showing the *existence* of categories as will be done in the next number.

A. MORPHISMS. Let $(x, y) \to x \top y$ be a (not necessarily everywhere defined) internal composition law between the elements of $X$, with graph $G$. We say the pair $(x, y)$ is composable (or, by abuse of the language, $x \top y$ is defined) if $(x, y) \in \mathrm{pr}_1 \langle G \rangle$. Note that, if for instance $(x \top y) \top z$ is defined, then $(x, y)$ and $(x \top y, z)$ are composable. $e$ is said to be a neutral element if $e \top x = x$ and $x \top e = x$ whenever the left side is defined. Note that the composite of neutral elements is neutral.

DEFINITION 2. We say $(\mathscr{C}, \mathscr{F})$ is a *category* if $\mathscr{C}$ is the graph of an internal composition law $(f, g) \to f \circ g$ between the elements of $\mathscr{F}$ verifying the following axioms:

($MO_I$) If one of the elements $(f \circ g) \circ h$, $f \circ (g \circ h)$ is defined, then so is the other, and they are equal.

($MO_{II}$) If $f \circ g$ and $g \circ h$ are defined, then so is $(f \circ g) \circ h$.

($MO_{III}$) For all $f \in \mathscr{F}$, there exist neutral elements $e$ and $e'$ such that $f \circ e$ and $e' \circ f$ are defined.

The elements of a category are called *morphisms* (or maps; jections in [5]); the neutral elements are referred to as *objects* (or identity maps). The set of the neutral elements of a category $\mathscr{F}$ is denoted by $\mathscr{F}_0$.

REMARK 1. Designate by $\mathscr{C}^0$ the opposite structure on $\mathscr{F}$, i.e., the set of the triples $(f, g, h)$ such that $(g, f, h) \in \mathscr{C}$. If $\mathscr{C}$ is the graph of the composition law $(f, g) \to f \circ g$, then $\mathscr{C}^0$ is the graph of the composition law $(f, g) \to g \circ f$. Obviously, $(\mathscr{C}^0, \mathscr{F})$ is a category called the *opposite* of $(\mathscr{C}, \mathscr{F})$.

If $(f, e)$ and $(f, e')$ are composable, then $f \circ e$, which equals $f$, is composable with $e'$. We infer from ($MO_I$) that $(e, e')$ is composable, hence $e = e \circ e' = e'$. Similarly for composable pairs $(e, f)$, $(e', f)$. The privileged neutral element $e$ such that $f \circ e = f$ (resp. $e \circ f = f$) is called the *right* (resp. *left*) *unit* associated with $f$, and is denoted by $\alpha(f)$ (resp. $\beta(f)$). Some authors (e.g. [5]) call it *source* (resp. *sink*). Note that, if $(f, g)$ is composable, then $\alpha(f \circ g) = \alpha(g)$ and $\beta(f \circ g) = \beta(f)$. In fact, if for example $(f \circ g, e)$ is composable, then so is $(g, e)$ by virtue of ($MO_I$).

PROPOSITION 7. *In a category $\mathscr{F}$, the pair $(f, g)$ is composable if and only if $\alpha(f) = \beta(g)$.*

Let $e$ be neutral. If $f \circ e$ and $f \circ g$, which equals $(f \circ e) \circ g$, are defined, then so is $e \circ g$ by ($MO_I$). On the other hand, if $f \circ e$ and $e \circ g$ are defined, then so is $(f \circ e) \circ g$, which equals $f \circ g$, in view of ($MO_{II}$).

For the remainder of the paper, we denote, for each category $\mathscr{F}$, by $\mathscr{F}(a, b)$ the set of the morphisms of $\mathscr{F}$ whose source is $a$ and whose sink is $b$. Instead of $f \in \mathscr{F}(a, b)$ one frequently writes $f: a \to b$.

In order to facilitate the intuitive interpretation, the following terminology is employed in connection with categories:

We say $f$ is a *monomorphism* (resp. *epimorphism*) if there exists $g$ such that $g \circ f$ (resp. $f \circ g$) is defined and neutral. In this case, $g$ is called a retraction (resp. section) associated with $f$; if $g$ is both a retraction and a section associated with $f$, then $f$ is called an *isomorphism*. (The corresponding algebraic terminology is "symmetrizable".)

We say $f$ is an *injective* (resp. *surjective*) morphism if $f\circ g=f\circ h$ (resp. $g\circ f=h\circ f$) implies $g=h$; $f$ is a *bijective* morphism, if it is both injective and surjective. (The corresponding algebraic terminology is "regular".)

It is clear that the composite of monomorphisms (resp. epimorphisms, isomorphisms, injective surjective, bijective morphisms) is of the same type.

PROPOSITION 8. *Let $f$ be an element of a category $\mathscr{F}$. $f$ is an isomorphism if and only if $f$ is a monomorphism and an epimorphism.*

The necessity is clear. For the sufficiency, let $r$ and $s$ be such that $r\circ f$ and $f\circ s$ are defined and neutral. By $(MO_{II})$, the pair $(r\circ f, s)$ is composable, which due to $(MO_I)$, yields $s=(r\circ f)\circ s=r\circ(f\circ s)=r$.

The privileged element $g$ such that $g\circ f$ and $f\circ g$ are defined and neutral, is called the *reciprocal* to $f$, and is denoted by $\overset{-1}{f}$. In particular, $f$ is reciprocal to $\overset{-1}{f}$, and we have $\alpha(\overset{-1}{f})=\beta(f)$ and $\beta(\overset{-1}{f})=\alpha(f)$. Indeed, $f\circ\alpha(f)$ which equals $f$ is composable with $\overset{-1}{f}$. We note from $(MO_I)$ that $\left(\alpha(f),\overset{-1}{f}\right)$ is composable, hence $\alpha(f)=\beta(\overset{-1}{f})$. Similar in the other case.

PROPOSITION 9. *If $f$ is a monomorphism (resp. epimorphism, isomorphism) and $g$ a retraction (resp. section, reciprocal) associated with $f$, then $f$ is injective (resp. surjective, bijective) and $g$ surjective (resp. injective, bijective).*

If $f$ is a monomorphism and $g$ a retraction associated with $f$, then $g\circ f$ is defined and neutral. Assume that $f\circ u=f\circ v$. Then by $(MO_{II})$, $g\circ(f\circ u)$ and $g\circ(f\circ v)$ are defined, yielding $u=(g\circ f)\circ u=g\circ(f\circ u)=g\circ(f\circ v)=(g\circ f)\circ v=v$ in virtue of $(MO_I)$. Similar in the other cases.

REMARK 2. Most authors use monomorphism (resp. epimorphism) synonymously with injective (resp. surjective) morphism. Compare proposition 8 with the example of a continuous function which is both injective and surjective without being an isomorphism of topological structures. On the other hand, the canonical embeddings $\mathrm{em}_i$ of $a_i$ into the direct sum $\sum_{i\in I} a_i$ and the canonical projections $\mathrm{pr}_i$ of the direct product $\prod_{i\in I} a_i$ into $a_i$ are monomorphisms and epimorphisms respectively, provided that for each pair $(i, k)$ of indices, there exists a morphism $f_{ki}: a_i\to a_k$. (For the definition of direct sums and products, compare [*6*]. Unfortunately, direct sums are called inverse products by ECKMANN and HILTON, which is not in conformity with general usage. Note, however the use of "free product" in categories of (not necessarily commutative) groups.)

DEFINITION 3. We say $(u, v, w)$ is a *standard* decomposition of $f$ if $f=u\circ v\circ w$ and if $u$ is injective and $w$ surjective. We say $(u, v, w)$ is *canonical* decomposition of $f$ if it is a standard decomposition, and if for every standard decomposition $(u', v', w')$ of $f$, there exist morphisms $\varphi$ and $\psi$ such that $u=u'\circ\varphi$ and $w=\psi\circ w'$.

PROPOSITION 10. *If* $(u, v, w)$ *is a canonical decomposition of* $f$, *and* $(u', v', w')$ *a standard decomposition of* $f$, *then there exists an injective morphism* $\varphi$ *and a surjective morphism* $\psi$ *such that* $u = u' \circ \varphi$, $w = \psi \circ w'$ *and* $v' = \varphi \circ v \circ \psi$. *If, in addition,* $(u', v', w')$ *is canonical, then* $\varphi$ *and* $\psi$ *are isomorphisms.*

Assume that $\varphi \circ g = \varphi \circ h$. Since $(u', \varphi)$ is composable, we conclude $u \circ g = u \circ h$, hence $g = h$: $\varphi$ is injective. Similarly, $g \circ \psi = h \circ \psi$ implies $g = h$: $\psi$ is surjective. By the same token, $u' \circ v' \circ w' = u' \circ \varphi \circ v \circ \psi \circ w'$ results in $v' = \varphi \circ v \circ \psi$. If, in addition, $(u', v', w')$ is canonical, then $u = u' \circ \varphi$, $u' = u \circ \varphi'$, $w = \psi \circ w'$, $w' = \psi' \circ w$ for some morphisms $\varphi$, $\varphi'$, $\psi$, $\psi'$. The first two relations yield $u = u \circ \varphi' \circ \varphi$, $u' = u' \circ \varphi \circ \varphi'$, hence, $\alpha(u) = \varphi' \circ \varphi$, $\alpha(u') = \varphi \circ \varphi'$. To finish the proof, start from the last two relations.

From here on, *additive* and *abelian categories* $\mathscr{F}$ may be defined by making each $\mathscr{F}(a, b)$ into a commutative group. See [*10*] and, for a generalization, [*12*].

B. FUNCTORS. In slight modification of general definitions (functor: representation of the category structures), we propose:

DEFINITION 4. *We say* $T$ *is a covariant functor defined in* $(\mathscr{C}, \mathscr{F})$ *with values in* $(\mathscr{C}', \mathscr{F}')$ *if* $T$ *is a quintuple* $(\mathscr{G}, \mathscr{C}, \mathscr{F}, \mathscr{C}', \mathscr{F}')$ *such that*

($FU_I$) $(\mathscr{C}, \mathscr{F})$ *and* $(\mathscr{C}', \mathscr{F}')$ *are categories, and* $(\mathscr{G}, \mathscr{F}, \mathscr{F}')$ *is a function;*

($FU_{II}$) $e \in \mathscr{F}_0$ *implies* $T(e) \in \mathscr{F}_0'$;

($FU_{III}$) *If* $f \circ g$ *is defined, then so is* $T(f) \circ T(g)$ *and one has* $T(f \circ g) = T(f) \circ T(g)$.

We say $T$ is a contravariant functor defined in $(\mathscr{C}, \mathscr{F})$ with values in $(\mathscr{C}', \mathscr{F}')$ if $T$ is a covariant functor defined in $(\mathscr{C}^0, \mathscr{F})$ with values in $(\mathscr{C}', \mathscr{F}')$ where $\mathscr{C}^0$ indicates the opposite structure. (By $T(f)$, we mean the privileged element $f'$ such that $(f, f') \in \mathscr{G}$.)

PROPOSITION 11. *Let* $T$ *be a functor. If* $e$ *is a right (resp. left) unit associated with* $f$, *then* $T(e)$ *is a right (resp. left) unit associated with* $T(f)$, *and one has* $T(\alpha(f)) = \alpha(T(f))$, $T(\beta(f)) = \beta(T(f))$.

Indeed, if $f$ is composable with the neutral element $e$, then $T(f)$ is composable with the neutral element $T(e)$, due to ($FU_{II}$) and ($FU_{III}$).

PROPOSITION 12. *Let* $T$ *be a functor. If* $f$ *is a monomorphism (resp. epimorphism, isomorphism) and* $g$ *a retraction (resp. section, reciprocal) associated with* $f$ *then* $T(f)$ *and* $T(g)$ *share the same property. In particular one has* $T(\overset{-1}{f}) = \widehat{T(f)}^{-1}$.

Consider for example the case where $g \circ f = e$ with $e$ being neutral. Due to ($FU_{III}$), $T(g) \circ T(f)$ is defined and we have $T(g) \circ T(f) = T(g \circ f) = T(e)$, which belongs to $\mathscr{F}_0'$ in virtue of ($FU_{II}$).

Let $S$ (resp. $T$) be the term $(\mathscr{G}, \mathscr{C}, \mathscr{F}, \mathscr{C}', \mathscr{F}')$ (resp. $(\mathscr{H}, \mathscr{C}^*, \mathscr{F}^*, \mathscr{C}'', \mathscr{F}'')$). By abuse of the language, the term $(\mathscr{H} \circ \mathscr{G}, \mathscr{C}, \mathscr{F}, \mathscr{C}'', \mathscr{F}'')$ is denoted by $T \circ S$ and is called the *composite* of $T$ and $S$. We say the pair $(T, S)$ is *composable* if $(\mathscr{C}^*, \mathscr{F}^*) = (\mathscr{C}', \mathscr{F}')$. In this case, $T \circ S$ is a functor provided $T$ and $S$ are functors.

DEFINITION 5. We say $\Phi$ is a *natural morphism* of $T$ into $S$ if the following axioms are verified.

($NA_I$) $T$ and $S$ are functors defined in $(\mathscr{C}, \mathscr{F})$ with values in $(\mathscr{C}', \mathscr{F}')$, and $\Phi$ is a function defined in $\mathscr{F}_0$ with values in $\mathscr{F}'$.

($NA_{II}$) For all $f \in \mathscr{F}$, the elements $\Phi(\beta(f)) \circ T(f)$ and $S(f) \circ \Phi(\alpha(f))$ are defined and equal.

If for all $e \in \mathscr{F}_0$, $\Phi(e)$ is a monomorphism (resp. epimorphism, isomorphism), then $\Phi$ is called a natural monomorphism (resp. epimorphism, isomorphism).

PROPOSITION 13. *Let $\Phi$ be a natural morphism of $T$ into $S$. For every $e \in \mathscr{F}_0$, we have $\alpha(\Phi(e)) = T(e)$ and $\beta(\Phi(e)) = S(e)$.*

Note that $\Phi(e) \circ T(e)$ (resp. $S(e) \circ \Phi(e)$) is defined in virtue of ($NA_{II}$).

**5. The strong set theory: Applications.** We discuss three illustrative examples.

1. Recall that a group is a pair $(V, X)$ such that $V$ (the group structure) belongs to $\mathfrak{P}(X \times X \times X)$ and that $(V, X)$ verifies the group axioms (associativity, existence of a neutral element, existence of symmetric elements). In particular, $(V, X \times X, X)$ is a function, called composition law between the elements of $X$. Recall further that a homomorphism of a group $(V, X)$ into a group $(V', X')$ is a function $(G, X, X')$ preserving composites. For our purposes, it is convenient to introduce quintuples $(G, V, X, V', X')$ (see [3], Chap. II, 2, n⁰ 2; if $x$ is a quintuple $(x_1, \ldots, x_5)$, write $\mathrm{pr}_i x$ for $x_i$) and to call them group morphisms, provided that $(V, X)$ and $(V', X')$ are groups, and that $(G, X, X')$ is a homomorphism of $(V, X)$ into $(V', X')$. Obviously, a group $(V, X)$ can be identified with the group morphism $(\Delta_X, V, X, V, X)$ where $\Delta_X$ stands for the diagonal of $X \times X$.

Because $(X, X') \in M \times M$ implies

$$\text{(1)} \quad \begin{cases} (G, V, X, V', X') \in \bigcup_{(Y,Y') \in M \times M} (\mathfrak{P}(Y \times Y') \times \mathfrak{P}(Y \times Y \times Y) \times \{Y\} \times \\ \qquad \times \mathfrak{P}(Y' \times Y' \times Y') \times \{Y'\}), \end{cases}$$

the relation «$x$ is a group morphism $\wedge$ $(\mathrm{pr}_3 x, \mathrm{pr}_5 x) \in M \times M$» is collectivizing in $x$ (criterion C 52 of [3]), resulting in a set $\mathscr{G}_M$, called *the set of the group morphism of type $M$*. In other words, $\mathscr{G}_M$ is the set of the group morphisms $(G, V, X, V', X')$ such that $X \in M$ and $X' \in M$.

If $M$ happens to be a universe, then (1) can be replaced by

$$\text{(2)} \quad \begin{cases} (G, V, X, V', X') \in \mathfrak{P}(X \times X') \times \mathfrak{P}(X \times X \times X) \times \\ \qquad \times \{X\} \times \mathfrak{P}(X' \times X' \times X') \times \{X'\} \in M, \end{cases}$$

resulting in $\mathscr{G}_M \in \mathfrak{P}(M)$.

Evidently $\mathcal{G}_M$ can be endowed with a composition law $(f, g) \to f \circ g$, defined for all composable pairs $(f, g)$ of group morphisms. Let $\mathcal{C}_M$ be the graph of the composition law. Then $(\mathcal{C}_M, \mathcal{G}_M)$ is a category called *the category of the group morphisms of type M*. Its objects are the morphisms $(\Delta_X, V, X, V, X)$ which can be identified with the groups $(V, X)$. Note that: $\alpha(G, V, X, V', X') = (\Delta_X, V, X, V, X)$, and $\beta(G, V, X, V', X') = (\Delta_{X'}, V', X', V', X')$.

While, up to now, all the constructions were possible in $\mathcal{T}_0$, it is a special feature of the strong set theory $\mathcal{T}_1$ that $M$ may be *chosen as large as one pleases*. For instance, there exists $M$ such that the classical groups (the additive group of the rational integers, the multiplicative group of the rational numbers $\neq 0$, etc.) belong to $\mathcal{G}_M$: Take a universe $M$ satisfying $\boldsymbol{N} \in M$, and apply prop. 6.

If the preceding construction is repeated with structures of species $\Sigma$ and $\sigma$-morphisms in the sense of [3], chap. 4, one eventually arrives at *the category of the $(\Sigma, \sigma)$-morphisms of type M*, each $(\Sigma, \sigma)$-morphism being a quintuple $((G_1, \ldots, G_n), V, (X_1, \ldots, X_n), V', (X'_1, \ldots, X'_n))$, where $V$ and $V'$ are structures of species $\Sigma$ on $(X_1, \ldots, X_n)$ and $(X'_1, \ldots, X'_n)$ resp., and where $((G_1, X_1, X'_1), \ldots, (G_n, X_n, X'_n))$ is a $\sigma$-morphism of $(X_1, \ldots, X_n)$, endowed with $V$, into $(X'_1, \ldots, X'_n)$, endowed with $V'$. Again, the set of the $(\Sigma, \sigma)$-morphisms of type $M$ can be made as large as one pleases by choosing $M$ sufficiently large.

2. Consider now different categories as well as functors of one category into another. Note that functors were already defined in such a way as to uniquely determine the category of departure and the category of arrival, i.e. the functors are category morphisms. Obviously, a category $(\mathcal{V}, \mathcal{X})$ can be identified with the functor $(\Delta_{\mathcal{X}}, \mathcal{V}, \mathcal{X}, \mathcal{V}, \mathcal{X})$.

As in example 1., the relation «$x$ is a functor $\wedge$ $(\mathrm{pr}_3 x, \mathrm{pr}_5 x) \in M' \times M'$» is collectivizing in $x$, yielding a set $\mathcal{K}_{M'}$, called *the set of the functors of type $M'$**. In other words, $\mathcal{K}_{M'}$ is the set of the functors $(\mathcal{G}, \mathcal{V}, \mathcal{X}, \mathcal{V}', \mathcal{X}')$ such that $\mathcal{X} \in M'$ and $\mathcal{X}' \in M'$. If $M'$ happens to be a universe, then $\mathcal{K}_{M'} \in \mathfrak{P}(M')$.

Evidently, $\mathcal{K}_{M'}$ can be endowed with a composition law $(T, S) \to T \circ S$, defined for all composable pairs $(T, S)$ of functors. Denote the graph of the composition law by $\mathcal{C}_{M'}$. Then $(\mathcal{C}_{M'}, \mathcal{K}_{M'})$ is a category, called *the category of the functors of type $M'$*. The objects are the functors $(\Delta_{\mathcal{F}}, \mathcal{C}, \mathcal{F}, \mathcal{C}, \mathcal{F})$ which can be identified with the categories $(\mathcal{C}, \mathcal{F})$. If $M$ and $M'$ are universes such that $\boldsymbol{N} \in M \in M'$, then $\mathcal{K}_{M'}$ contains the classical categories of type $M$. This answers the first question posed by MacLane in [14]. However, we do not know whether $\mathcal{K}_{M'}$ belongs to $M'$.

3. For a category $(\mathcal{V}, \mathcal{X})$ and an element $f$ of $\mathcal{X}$, let $\alpha(f)$ (resp. $\beta(f)$) be the right (resp. left) unit associated with $f$. It is clear that a long exact sequence $0 \to e_0 \xrightarrow{f_1} e_1 \xrightarrow{f_2} e_2 \xrightarrow{f_3} e_3 \to 0$ of morphisms of an Abelian category $(\mathcal{V}, \mathcal{X})$

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* In conformity with examples 1 and 2, one could call $\mathcal{X}$ a set of type $M$ if $M$ is a universe and if each element of $\mathcal{X}$ belongs to $M$.

can be represented by a triple $(f_1, f_2, f_3) \in \mathscr{X} \times \mathscr{X} \times \mathscr{X}$ such that $\alpha(f_1) = e_0$, $\beta(f_3) = e_3$, together with the conditions for exactness. The set $\mathscr{E}$ of those long exact sequences is a subset of $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$, which yields $\mathrm{Ext}^2(e_3, e_0) \subset \mathfrak{P}(\mathscr{E}) \subset \mathfrak{P}(\mathscr{X} \times \mathscr{X} \times \mathscr{X})$. If $(\mathscr{V}, \mathscr{X})$ is a category of type $M'$, where $M'$ is a universe, then $\mathrm{Ext}^2(e_3, e_0) \in M'$. Again, one can choose $M'$ as large as one pleases. For example, let $M$ and $M'$ be two universes with $M \in M'$. Take for $\mathscr{X}$ the set $\mathscr{G}_M^C$ of the morphisms of commutative groups of type $M$. Then $\mathscr{G}_M^C$ and $\mathrm{Ext}^2(e_3, e_0)$ belong to $M'$. One can easily extend $\mathrm{Ext}^2$ to a function mapping $\mathscr{G}_M^C \times \mathscr{G}_M^C$ into the set $\mathscr{G}_{M'}^C$ of the morphisms of commutative groups of type $M'$. This settles the second problem in MacLane's paper [*14*]. The remaining two problems can be treated in a similar fashion.

REMARK. 1. In example 2., we mentioned that we do not know whether $\mathscr{K}_{M'}$ belongs to $M'$. The following criterion clarifies this situation:

CRITERION 3. *Let $\mathscr{T}$ be a theory stronger than the set theory $\mathscr{T}_0$. Let $R$ be a relation of $\mathscr{T}$, and $x$, $x'$, $y$ distinct letters, $y$ not occuring in $R$. Assume, that the relations*

(I) $(x' \subset x \wedge R\{x\}) \Rightarrow R\{x'\}$;

(II) $R\{\mathscr{E}_x(R)\}$;

(III) $\mathrm{Coll}_x R$

*are theorems of $\mathscr{T}$. Then $\mathscr{T}$ is contradictory.*

Due to (I) and (II), $A \subset \mathscr{E}_x(R)$ implies $R\{A\}$, hence $A \in \mathscr{E}_x(R)$ by (III). We have $\mathfrak{P}(\mathscr{E}_x(R)) \subset \mathscr{E}_x(R)$. Writing $\mathfrak{a}$ for $\mathrm{Card}(\mathscr{E}_x(R))$, we obtain $2^{\mathfrak{a}} \leq \mathfrak{a}$. On the other hand, $\mathfrak{a} < 2^{\mathfrak{a}}$ is a theorem of $\mathscr{T}_0$, hence of $\mathscr{T}$. Therefore, $\mathscr{T}$ is contradictory.

**Historical Note.** A survey of the pertinent literature reveals three approaches to modified set theories.

The most widely used logical system distinguishes between sets and classes. It was invented by ZERMELO and FRAENKEL, and refined by VON NEUMANN, BERNAYS and GÖDEL. Unfortunately, this *two-type system* does not provide for the existence of classes of classes. We mention, however, that the Zermelo-Fraenkel system gives us an excellent motivation for the choice of the definition of universes. Indeed, if one formalizes the statement «$X$ is a set» by «$X \in M$» where $M$ is a constant, then the relations $(U_{II})$, $(U_{III})$, and $(U_V)$ of definition 1 turn out to be translations of the axioms (IV), (III), and (VI) of [*1*], pp. 6, 7, 8. (Note that $(U_I)$ is tacitly assumed in [*1*], while the introduction of $(U_{IV})$ depends largely on the concept of (ordered) pairs; compare [*3*], Chap. II. §2, exerc. 2.)

If one iterates the idea of ZERMELO and FRAENKEL, one arrives with necessity at super-classes. An interesting, but highly technical, version (which is beyond the scope of most practical mathematicians) of such a procedure was recently described by A. LÉVY in [*13*]. His model-theoretic approach to inaccessible cardinals uses a definition of standard complete models which not only involves the signs of a set theory $\mathcal{T}$ in question but also its axioms. More precisely, one obtains the definition of a standard complete model with respect to $\mathcal{T}$ by relativizing the quantifiers in the axioms and schemes of $\mathcal{T}$. By adjoining, to $\mathcal{T}$, an axiom which requires the existence of a standard complete model with respect to $\mathcal{T}$, one arrives at a theory $\mathcal{T}'$, say. By adjoining, to $\mathcal{T}'$, an axiom which requires the existence of a standard complete model with respect to $\mathcal{T}'$, one arrives at a theory $\mathcal{T}''$, say. This process can be continued by transfinite induction, and leads to a *hierarchy of set theories*, which, for mathematical purposes, is inconvenient and confusing. (For example, on which set theory should one base group theory?) It is easy to see, that a standard complete model with respect to the set theory $\mathcal{T}_0$ (in the sense of [*3*]) differs only slightly from a universe. Indeed, relativizing A 1 to A 4 results (essentially) in A 1, $(U_{III})$, A 3, and $(U_{II})$, while S 8 is responsible for $(U_V)$ and the criteria C 51, C 52, and C 53 of [*3*]. (Note that $(U_I)$ is incorporated in the definition of standard complete models.)

In order to avoid a hierarchy of set theories, and yet be able to climb higher and higher to sets which are sufficiently large for applications, one has only to go back to TARSKI, and revive his ideas with respect to the existence of *arbitrarily large inaccessible sets* within the framework of *a single set theory*. The meaning of the phrase "sufficiently large for applications" being ambiguous, we replaced in this paper Tarski's inaccessible sets by universes. The difference is only slight: Tarski's relations $(A_2')$ and $(A_3)$ in [*15*] are equivalent to $(U_I)$ and $(U_{II})$ respectively; $(A_4)$ implies $(U_{III})$ provided $M$ is infinite. As before, the introduction of $(U_{IV})$ depends on the concept of pairs which, in the case of TARSKI, is taken from [*8*]. The important axiom $(U_V)$ is missing.

Let me resume, that, even if axiom A 5′ is not quite original, and somewhat narrow for the logician, it is the simplest device available to construct effectively categories. To the best of my knowledge, this idea was never pursued in the literature.

## References


[*1*] BACHMANN, H.: Transfinite Zahlen. In: Ergebnisse der Math. Wiss. I. Berlin-Göttingen-Heidelberg: Springer 1955.

[*2*] BOURBAKI, N.: Foundations of Mathematics for the working mathematician. J. Symb. Logic **14**, 1–8 (1949).

[*3*] — Théorie des ensembles. Paris: Hermann chap. 1,2: 1960, chap. 4: 1957.

[*4*] CHURCH, A.: Introduction to Mathematical logic. Princeton 1956.

[*5*] DEDECKER, P.: Introduction aux structures locales, Colloque de Géométrie différentielle globale (CBRM), Bruxelles 1958, p. 103–135.

[*6*] ECKMANN, E., and P. HILTON: Group-like structures in general categories I. Math. Annalen **145**, 227–255 (1962).

[*7*] EHRESMANN, C.: Gattungen von lokalen Strukturen. Jber. d. Math.-Ver. **60**, 49–77 (1957).

[8] FRAENKEL, A.: Einleitung in die Mengenlehre. Berlin 1928.
[9] —, and Y. BAR-HILLEL: Foundations of set theory. Amsterdam 1958.
[10] GODEMENT, R.: Topologie algébrique et théorie des faisceaux. Paris: Hermann 1958.
[11] KELLEY, J.: General Topology. New York: D. van Nostrand 1953.
[12] KLEISLI, H.: Homotopy theory in abelian categories. Canad. J. of Math. **14**, 139—169 (1962).
[13] LÉVY, A.: Axiom Schemata of strong infinity in axiomatic set theory. Pac. J. Math. **10**, 223—238 (1960).
[14] MACLANE, S.: Locally small categories and the foundations of set theory. Symposium, Warsaw 1959, p. 25—43.
[15] TARSKI, A.: Über unerreichbare Kardinalzahlen. Fundamenta Mathem. **30**, 68—89 (1938).

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