Parallel transport, holonomy and all that - a homotopy point of view

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Abstract

This is a greatly revised version of the talk in the Deformation Theory Seminar at Penn Jan 19, 2011. In a homotopy setting, i.e. of fibrations = maps $p : E \to B$ with the homotopy lifting property, parallel transport and holonomy can be defined without a connection and in terms of morphisms from the space of paths or based loops without passing to homotopy. Closely related is the notion of (strong or ∞) homotopy action, which has variants under a variety of names. My aim is to impose some order on this zoo of concepts and names with major emphasis on the examples coming from fibrations.

Inspired by recent extensions in the smooth setting of parallel transport to representations of $Sing_{smooth}(B)$ on a smooth fibre bundle, I revisit the development of a notion of 'parallel' transport in the topological setting of fibrations with the homotopy lifting property and then extend it to representations of $Sing(B)$ on such fibrations.

Contents

1 Introduction/History

In clasical differential geometry (a language the muse did not sing at my cradle - see below), *parallel transport* is defined in the context of a *connection* on a smooth bundle $p : E \to B$. The latter can mean a covariant derivative operator, a differential 1-form or a set of horizontal subspaces in the tangent bundle $Tp : TE \to TB$. The corresponding *parallel transport* $\tau : E \times B^I \to E$ is constructed by lifting a path in B to a *unique!* path in E with specified starting point. The *holonomy* is given by the evaluation of τ on ΩB, the space of based loops in B. The *holonomy group* is the image as a subgroup of the structure group of the bundle. That it is a group follows from the uniqueness of the lifting. It is well defined up to conjugation depending on the choice of base point.

If $p : E \to B$ is only a fibration of topological spaces, the situation is different: we still can lift paths but not uniquely.

Perhaps the oldest treatment in algebraic topology (I learned it as a grad student from Hilton's Introduction to Homtopy Theory [Hil53] - the earliest textbook on the topic) is to consider the long exact sequence, where F is the fibre over a chosen base point in B,

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

ending with

$$\cdots \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

Of course, exactness is very weak at the end since the last three are in general only sets, but exactness at $\pi_0(F)$ is in terms of the action of $\pi_1(B)$ on $\pi_0(F)$. This passage to homotopy classes obscures the 'action' of ΩB on F. Initially, this was referred to as a *homotopy action* [**?**], meaning only that $f\lambda)\mu$ was homotopic to $f(\lambda\mu)$

In those days, at least at Princeton, there was no differential geometry until Milnor gave an undergrad course my final year there. Notice that Characteristic Classes consider differential forms only in Appendix C, added much later. I think this was the results of Serre's thesis which triumphed over characteristic 0, cf. choux de bruxelles.

It was also not 'til years later that I learned of the notiion of *thin homotopy* which quotients ΩB to a group without losing so much information. Just recently, Johannes Huebschman led me to a paper of Kobayashi (from 1954!) where he is already using what is now called thin homotopy in terms of plarallel transport and holonomy for smooth bundles with connection.

Back in 1966, in the Mexican Math Bulletin [Sta66], a journal not readily available, I showed that in the topological setting of fibrations the homotopy lifting property gave not only the above homotopy action, but in fact an sh(or A_∞)-action, which is to say the adjoint $\Omega B \to End(F)$ was an A_∞-map.

For my purposes, it was sufficient to consider transport along based loops in the base, though the arguments allow for transport along any path in the base.

2 Review of the construction of an A_∞-action

We first recall what are rightly known as Moore paths [Moo55] on a topological space X.

Definition 1. Let $R^+ = [0, \infty)$ be the nonnegative real line. For a space X, let $Moore(X)$ be the subspace of *Moore paths* $\subset X^{R^+} \times R^+$ of pairs (f, r) such that f is constant on $[r, \infty)$. There are two maps

- $\partial^-, \partial^+ : Moore(X) \to X$,
- $\partial^-(f, r) = f(0)$,
- $\partial^+(f, r) = f(r)$.

Recall composition $\circ$ of Moore paths in $Moore(X)$ is given by sending pairs $(\lambda, r), (\mu, s) \in Moore(X)$ such that $\lambda(r) = \mu(0)$ to $\lambda\mu \in Moore(X)$ which is constant on $[r + s, \infty)$, $\lambda\mu|[0, r] = \lambda|[0, r]$ and $\lambda\mu(t) = \mu(t - r)$ for $t \geq r$. An identity function $\epsilon : X \to Moore(X)$ is given by $\epsilon(x) = (\hat{x}, 0)$ where $\hat{x}$ is the constant map on R^+ with value x.

Composition is continuous and gives, as is well known, a category/groupoid structure on $Moore(X)$. If we had used the 'ancient' Poincaré paths $I \to X$, we would have had to work with an A_∞-structure on X^I. Indeed, it was working with that standard parameterization which led to A_∞-structures [Sug57, Sta63].

For a category C, we denote by $C_{(n)}$ the set of n-tuples of composable morphisms. In partcular, we will be concerned with $Moore(B)_{(n)}$. We will write $\mathbf{t}$ for $(t_1, \cdots, t_n)$ and $\hat{t}_i$ for $(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$, Back in 1988 [Sta88], I referred to strong homotopy representations, but today I will use the representation up to homotopy terminology, having in mind the generalization that comes next. Because $\lambda\mu$denotes travelling along λ first and then along μ, the actions will be written as right actions: $(e, \lambda) \mapsto e\lambda$.

Definition 2. A *representation up to homotopy* of $Moore(B)$ on a fibration $E \to B$ is an A_∞-morphism (or shm-morphism [Sug61]) from $Moore(B)$ to $End_B(E)$; that is, a collection of maps

$$\theta_n : I^{n-1} \times E \times_B Moore(B)_{(n)} \to E$$

(where $E \times_B Moore(B)_{(n)}$ consists of $n+1$-tuples $(e, \lambda_1, \dots, \lambda_n)$ where the λ_i are composable paths, constant on $[r_i, \infty)$, and $p(e) = \lambda_1(0)$) such that

$$p(\theta_n(\mathbf{t}, e, \lambda_1, \dots, \lambda_n)) = \lambda_n(r_n),$$

$$\theta_n(\mathbf{t}, --, \lambda_1, \dots, \lambda_n)$$

is a fibre homotopy equivalence and satisfies the usual/standard relations:

- $$\theta_n(t_1, \cdots, t_i = 0, \cdots, t_{n-1}, e, \lambda_1, \dots, \lambda_n)) = \theta_{n-1}(\hat{t}_i, e, \cdots, \lambda_i\lambda_{i+1}, \cdots))$$

- $$\theta_n(t_1, \cdots, t_i = 1, \cdots, t_{n-1}, e, \lambda_1, \dots, \lambda_n)) =$$
$$\theta_i(\cdots, t_{i-1}, \theta_{n-i}(t_{i+1}, \cdots, t_{n-1}, \lambda_i, \cdots, \lambda_n, e), \lambda_1, \dots, \lambda_{i-1},)$$

Remark 3. *That the parameterization is by cubes, as for Sugawara's strongly homotopy multiplicative maps rather than more general polytopes, reflects the fact that $Moore(X)$ and $End_B(E)$ are strictly associative. Strictly speaking, referring to $Moore(B) \to End_B(E)$ as an A_∞-map raises issues about a topology on $End_B(E)$; the adjoint formulas above avoid this difficulty.*

Since our construction uses in a crucial way the homotopy lifting property, we first construct maps

$$\Theta_n : I^n \times E \times_B Moore(B)_{(n)} \times_B \to E$$

such that the desired θ_n are then recovered at $t_1 = 1$.

The idea is that if Θ_j has been defined satisfying these relations for all $j < n$, the Θ_{n-1} will fit together to define Θ_n on all faces of the cube except for the face where $t_1 = 1$. In analogy with the horns of simplicial theory, we will talk about filling an *open box*, meaning the boundary of the cube minus the open face, called a *lid*, where $t_i = 1$ (compare horn-filling in the simplicial setting). Use the homotopy lifting property to 'fill in the box' after filling in the trivial image box in B.That box is B is trivial box since it is just the composite path $\lambda_1 \cdots \lambda_n$.

It might help to consider the cases $n = 1, 2$. Consider $(\lambda, r) \in Moore(B)$. Lift λ to a path $(\bar{\lambda}, r)$ starting at $e \in E$. Define $\Theta_1 : I \times E \to E$ by

$$\Theta_1(t, e, (\lambda, r)) = (\bar{\lambda}, r)(tr) \in E$$

and $\theta_1(e, (\lambda, r)) = \Theta_1(1, e, (\lambda, r)) =: e(\lambda, r)$.

Now lift (μ, s) to a path $(\bar{\mu}, s)$ starting at $e(\lambda, r) \in E$ and lift $(\lambda, r)(\mu, s)$ to a path $(\overline{\lambda\mu}, r{+}s)$ starting at e. These lifts fit together to define a map to E, which will be the restriction of the desired map on the open 2-dimensional box of the desired map Θ_2. This open box has an image in B which can trivially be filled in. Regarding the filling as a homotopy, the map to E on the open 2-dimensional box can be filled in by lifting that homotopy.

Theorem 4. (cf. Theorem A in [Sta66]) For any fibration $p : E \to B$, there is an A_∞-action $\{\theta_n\}$ of Moore(B) on E such that θ_1 is a fibre homotopy equivalence. This action is unique up to homotopy in the A_∞-sense.

In Theorem B in [Sta66], I proved further:

Theorem 5. Given an A_∞-action $\{\theta_n\}$ of the Moore loops ΩB on a space F, there is a fibre space $p_\theta : E_\theta \to B$ such that, up to homotopy, the A_∞-action $\{\theta_n\}$ can be recovered by the above procedure. If the A_∞-action $\{\theta_n\}$ was originally obtained by the above procedure from a fibre space $p : E \to B$, then p_θ is fibre homotopy equivalent to p.

This construction gave rise to the slightly more general (re)construction below. It can also be generalized to give an ∞-version of the Borel construction/homotopy quotient: $G \to X \to X_G = X//G$ for an sh-action [IM89].

3 Upping the ante to *Sing*

Inspired by Block-Smith [BS] and Igusa (arXiv:0912.0249), Abad and Schaetz [AS] look not at just composable paths, but rather look at the singular complex $Sing(B)$, which is also referred to as. For a singular k-simplex $\sigma : \Delta^k \to B$, there are several k-tuples of composable paths from vertex 0 to vertex k by restriction to edges, in fact, $k!$ such. Given σ, we denote by F_i the fibre over vertex $i \in \sigma$.

Following e.g. Abad-Schaetz [AS] (based on Abad's thesis and his earlier work with Crainic), we make the following definition of a *representation up to homotopy*, where we take a singular k-simplex σ to be (the image of) $< 0, 1, \cdots, k >$ with the p-th face $\partial_p \sigma$ being $< 0, \cdots, p-1, p+1, \cdots, k >$. However, we keep much of the notation above rather than switch to theirs.

Remark 6. *Again, in contrast to the smooth bundle case, the fibration case is considerably more subtle since horn filling in the base need not lift to horn filling in the total space*

Definition 3.1. A *representation up to homotopy* of $Sing(B)$ on a fibration $E \to B$ is a collection of maps $\{\theta_k\}_{k \geq 0}$ which assign to any k-simplex σ :

$\Delta^k \to B$ a map $\theta_k(\sigma) : I^{k-1} \times F_0 \to F_k$ satisfying the relations for any $e \in F_0$:

θ_0 is the identity on F_0

For any $(t_1, \cdots, t_{k-1})$,

$\theta_k(\sigma)(t_1, \cdots, t_{k-1}, -) : F_0 \to F_k$ is a homotopy equivalence.

For any $1 \leq p \leq k-1$ and $e \in F_0$,

$$\theta_k(\sigma)(\cdots, t_p = 0, \cdots, e) = \theta_{k-1}(\partial_p \sigma)(\cdots, \hat{t}_p, \cdots, , e)$$

$$\theta_k(\sigma)(\cdots, t_p = 1, \cdots, e) =$$
$$\theta_p(< 0, \cdots, p >)(t_1, \cdots, t_{p-1}, \theta_q(< p, \cdots, k >)(t_{p+1}, \cdots, t_k, e)).$$

Remark 7. *In definition 4, we worked with Moore paths so that the* A_∞*-map was between strictly associative spaces. Here instead the compatible 1-simplices compose just as e.g. a pair of 1-simplices and are related to a single 1-simplex only by an intervening 2-simplex. Associativity is trivial; the subtlety is in handling the 2-simplices and higher ones for multiple compositions. The idea of constructing a representation up to homotopy is very much like that of Theorem 1, the major difference being that instead of comparing two different liftings of the composed paths which are necessarily homotopic, we are comparing a lifting e.g. of a path from 0 to 1 to 2 with a lifting of a path from 0 to 2 IF there is a singular 2-simplex* $< 012 >$. *However, note that* $< 02 >$ *plays the role of* $\lambda_1\lambda_2$ *of Moore paths in the above formulas.*

Theorem 8. For any fibration $p : E \to B$, there is a representation up to homotopy of $Sing(B)$ on E.

The essence of the proof is in essence the same as that for Theorem 4. The desired θ_n will appear as the missing lid on an open box (defined inductively) which is filled in by homotopy liftings Θ_n of a *coherent* set of maps

$$p_n : I^n \to \Delta^n,$$

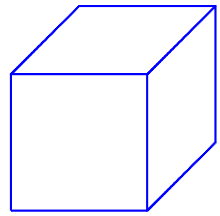
where Δ^n is the set

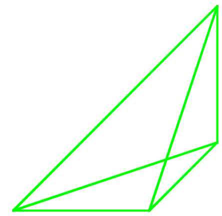
$$\{(t_1, \cdots, t_n) | 0 \leq t_1 \leq t_2 ... \leq 1\},$$

are given in terms of iterated convex linear functions. The basic example is

$$c : (x, y) \mapsto (x \cdot 1 + (1 - x)y, y).$$

Write $t_1 = t,\ t_2 = s,\ t_3 = r.$





For $n = 1$, define $q_1 : t \mapsto t_1$.
For $n = 2$, define $q_2 = c : (t, s) \mapsto (t \cdot 1 + (1 - t)s, s)$ and then

$$q_3 : (t, s, r) \mapsto (c(c(t, s), r), c(s, r), r) = (c(t \cdot 1 + (1 - t)s), r), c(s, r), r).$$

These have probably been written else; if you find them, let me know.

By coherent I mean respecting the facial structure of the cubes and simplices.

Closely related are coherent maps

$$\gamma_n : I^{n-1} \to P\Delta^n$$

where P denotes the set of paths, i.e. $P\Delta^n = Map(I, \Delta^n)$ and $\gamma_1 : I \to \Delta^1$ is the 'identity'. Such maps were first produced by Adams [Ada56] in the topological context by induction using the contractability of Δ^n. Later specific formulas were introduced by Chen [Che73,] and, most recently, equivalently but more transparently, by Igusa [Igu].

By coherent I mean precisely

$\gamma_1(0)$ is the trivial path, constant at 0.

For any $1 \le p \le k-1$,

$$\gamma_k(\cdots, t_p = 0, \cdots) = \gamma_{k-1}(\cdots, \hat{t}_p, \cdots)$$

and

$$\gamma_k(\sigma)(\cdots, t_p = 1, \cdots) =$$

$$\gamma_p(t_1, \cdots, t_{p-1})\gamma_q(t_{p+1}, \cdots, t_{k-1}).$$

One way to describe the relation between the p_n and the γ_n in words is: travel from vertex 0 partway to vertex 1 then straight partway to vertex 2 then straight partway to vertex 3 etc.

See file transport-figure.pdf

Note that these are slighty different from the version of γ_n given by Igusa; see the next figure taken from [Igu].

Hopefully the pattern is clear.

Correspondingly, the liftings $\Theta_n : I^n \times E \to E$ form a collection of maps which assign to any k-simplex $\sigma : \Delta^k \to B$ a map $\Theta_k(\sigma) : I^k \times F_0 \to F_k$ satisfying the relations for any $e \in F_0$:

$\Theta_0(0)$ is the identity on F_0

For any $(t_1, \cdots, t_k)$,

$\Theta_k(\sigma)(t_1, \cdots, t_k, -) : F_0 \to F_k$ is a homotopy equivalence.

For any $1 \le p \le k-1$,

$$\Theta_k(\sigma)(\cdots, t_p = 0, \cdots, e) = \Theta_{k-1}(\partial_p\sigma)(\cdots, \hat{t}_p, \cdots, e)$$

$$\Theta_k(\sigma)(\cdots, t_p = 1, \cdots, e) =$$

$$\Theta_p(< 0, \cdots, p >)(t_1, \cdots, t_{p-1}, \theta_q(< p, \cdots, k >)(t_{p+1}, \cdots, t_k, e)).$$

The desired θ_n is again recovered at $t_1 = 1$.

The maps p_n can be interpreted as homotopies $q_n : I \to (\Delta^n)^{I^{n-1}}$ and so subject to the homotopy lifting property. For example, $\gamma_1 : 0 \to P\Delta^1$ is a path which can be lifted as in Theorem 1 to give $\Theta_1 : I \times E \to E$. Then $\gamma_2 : I \to P\Delta^2$ such that 0 maps to the 'identity' path $I \to < 02 >$ while 1 maps to the concatenated path $< 01 >< 12 >$. (Henceforth, we will assume paths have been normalized to length 1 where appropriate.) Now lift the homotopy γ_2 to a homotopy $\Theta_2(< 012 >) : I \times I \times E \to E$ between $\Theta_1(<$

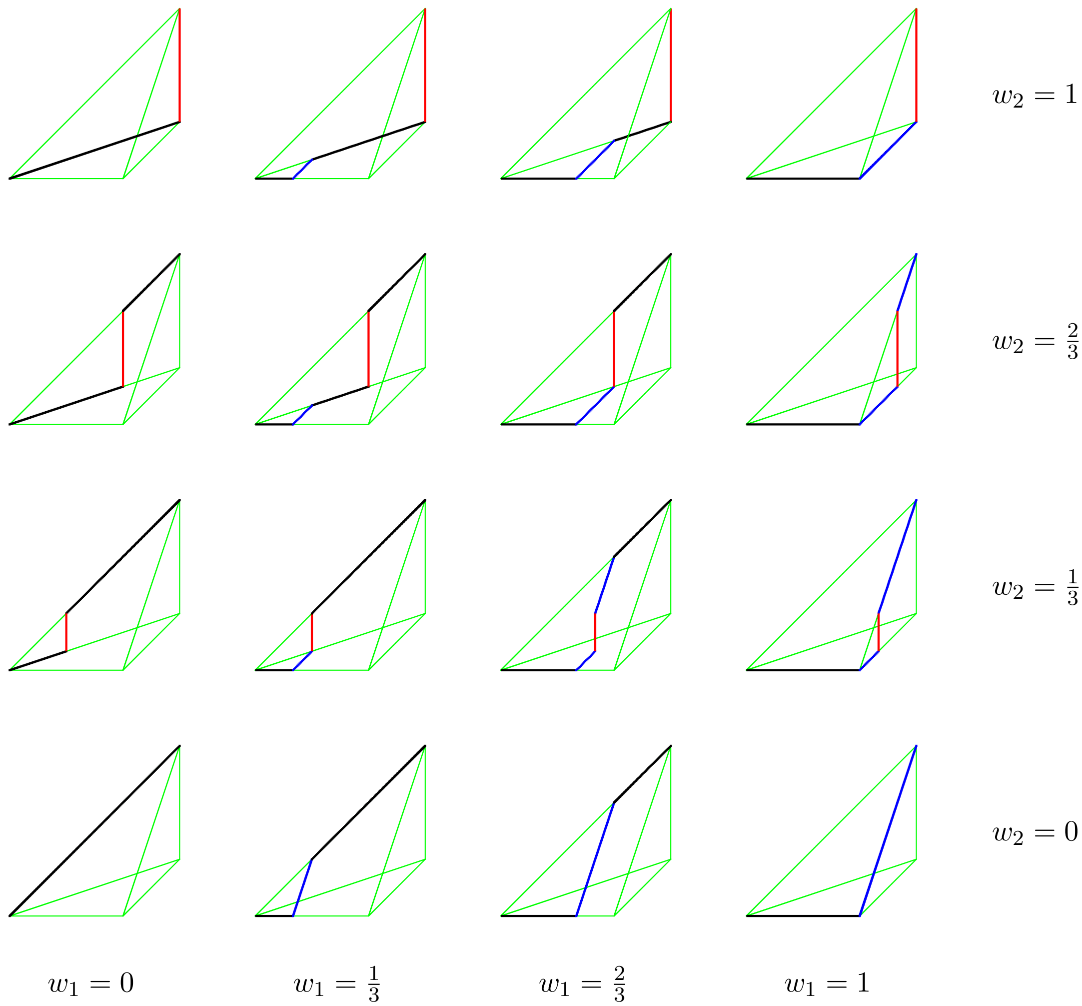


Figure 1: Igusa's Figure 3

$02 >$) and $\Theta_1(< 01 >< 12 >)$. In particular, $\Theta_2(< 012 >) : 1 \times I \times E \to E$ gives the desired homotopy $\theta_2(\sigma) : I \times F_0 \to F_k$.

The situation becomes slightly more complicated as we increase the dimension of σ. The case Δ^3 is illustrative. The faces $< 023 >$ and $< 013 >$ lift just as $< 012 >$ had via Θ_2, but that lift must then be 'whiskered' by a rectangle over $< 23 >$ which glues onto $\Theta_3(< 012 >$. In a less complicated way $< 123 >$ is lifted so that vertex 1 agrees with the end of the 'whisker' which is the lift of $< 01 >$. Thus the total lift of < 0123 ends with the desired $\theta_3 : I^2 \times F_0 \to F_3$. The needed whiskering (of various dimensions) is prescribed by the $t_p = 1$ relations of Definition 2 to be satisfied.

See file Theta 3.pdf

4 (Re)-construction of fibrations

In [Sta66], I showed how to construct a fibration from the data of an strong homotopy action of ΩB on a 'fibre' F. If the action came from a given fibration $F \to E \to B$, the constructed fibration was fibre homotopy equivalent to the given one. For *representations up to homotopy*, a similar result applies using analogous techniques, with some additional subtlety.

First we try to construct a fibration naively. Over each 1-simplex σ of $Sing(B)$, we take $\sigma \times F_0$ and attempt to glue these pieces appropriately. For the one simplices $< 01 >$ and $< 12 >$, we have $\theta_1 : F_0 \to F_1$ which tells us how to glue $< 01 > \times F_0$ to $< 12 > \times F_1$ at vertex 1, but, since $\theta_1 : F_0 \to F_2$ is not the composite of $\theta_1 : F_0 \to F_1$ and $\theta_1 : F_1 \to F_2$, we can not simply plug in $< 012 > \times F_0$ over $< 012 >$. However, we can plug in $I^2 \times F_0$ since $\theta_2 : I \times F_0 \to F_2$ will supply the glue over vertex 2.

To describe the fibration (or at least a quasi-fibration), we use the special maps $p_n : I^n \to \Delta^n$. Return to the description of the fibration $\bar{p}_2 : E_2 \to \Delta^2$ above. In greater precision,

$$E_2 = < 01 > \times F_0 \cup_1 < 12 > \times F_1 \cup_0 < 02 > \times F_0 \ \cup I^2 \times F_0.$$

The attaching maps over the vertices 0 and 1 are obvious as are the projections to the edges of Δ^2. On $I^2 \times F_0$, the attaching maps are obvious except for the face t_1 where it is given by $\theta_2 : I \times F_0 \to F_2$, so as to be compatible with the projection $I^2 \times F_0 \to \Delta^2$.

The result is at least a quasi-fibration $q : E_\theta \to B$ and can be replaced up to fibre homotopy equivalence by a true fibration.

Notice that although the definition of representation up to homotopy was in terms of a fibrations, in fact it really needs only the collection of fibres

F_σ for the 0-simplices of $Sing(B)$. The equivalence in the appropriate sense between *representations up to homotopy of* $Sing(B)$ and *fibrations over* B follows as for Theorem B in [Sta66].

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