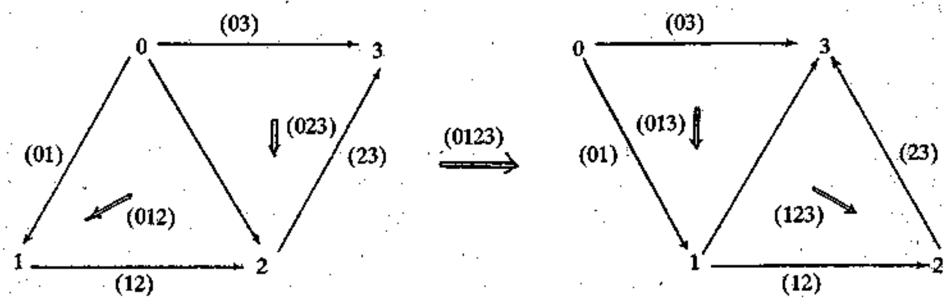
Journal of Pure and Applied Algebra 49 (1987) 283–335
North-Holland

THE ALGEBRA OF ORIENTED SIMPLEXES

Ross STREET*
School of Mathematics and Physics, Macquarie University, N.S.W. 2109, Australia

Communicated by S. Eilenberg
Received May 1985

An m-simplex x in an n-category A consists of the assignment of an r-cell $x(u)$ to each $(r+1)$-element subset u of $\{0,1,\ldots,m\}$ such that the source and target $(r-1)$-cells of $x(u)$ are appropriate composites of $x(v)$ for v a proper subset of u. As m increases, the appropriate composites quickly become hard to write down. This paper constructs an m-category $\mathcal{O}_m$ such that an m-functor $x:\mathcal{O}_m \to A$ is precisely an m-simplex in A. This leads to a simplicial set ΔA, called the nerve of A, and provides the basis for cohomology with coefficients in A. Higher order equivalences in A as well as free n-categories are carefully defined. Each $\mathcal{O}_m$ is free.



Introduction

History

The nerve of a category A is the simplicial set ΔA whose elements of dimension n are abutting n-tuples of arrows in A [10]. That this process generalizes to r-categories A, where now an element of dimension n is an n-simplex with an m-cell in each face of dimension m, I learned in conversation with John E. Roberts in 1979.

This informal description of A turns out to be harder to make precise than one would expect and it is the principal purpose of the present paper to give a simple accurate definition. The central idea (which I had a few months after the conversation with Roberts) is to describe "the free ω-category $\mathcal{O}_\omega$ on the ω-simplex".

* The author is grateful to Linda Harris for her inventive and professional typing of the diagrams in this paper.

Roberts [8] had a precise description of the nerve of a 3-category but remarked that no amount of staring at the low dimensional cocycle conditions would reveal the pattern for higher dimensions. However he had worked on a program aimed at solving this problem by characterizing nerves as simplicial sets with extra structure (*hollowness*) plus exactness conditions; he called them *complicial sets* but was unable to complete the program.

On hearing that Roberts was coming again to Australia, I began working seriously on his program (April–May 1982) obtaining an alternative characterization of complicial sets and a proof that complicial sets, whose elements of dimension greater than 2 are all hollow, are the isomorphs of nerves of 2-categories. In June 1982, Roberts gave me a copy of his old handwritten notes on complicial sets in which he had the alternative characterization and many general constructions [9].

With Jack Duskin at Macquarie University enthusiastic about the project, I returned to study $\mathcal{O}_\omega$ and circulated a handwritten conjecture in July 1982; the well-formedness notion of Section 2 below appeared there. The conjecture seemed hard to verify and I worked instead on enriched categories for 18 months. This work on enriched categories, sheaves and stacks led me to realize the importance of describing $\mathcal{O}_\omega$: apart from containing the higher cocycle conditions and the coherence information for multiple compositions, it seems to be related to the notion of space itself.

Jack Duskin returned to Australia in December 1983 and I returned to the attempts at my conjecture. In my office I had two posters of diagrams illustrating $\mathcal{O}_n$ for $n \leq 5$ which were made for the Macquarie University Open Day, August 1982. Jack encouraged me to draw $\mathcal{O}_6$ which took a weekend of working with rules which I could not make precise. Meanwhile, he worked on extracting higher dimensional figures from oriented simplexes.

The clue which put me on the track to a solution was that the well-formedness of a union occurring in a composite depends not only on the well-formedness of the separate cells at the level of that union but also on the well-formedness of all the lower dimensional cells which are involved in the sources and targets. I then tediously began characterizing the 2-cells, then the 3-cells, then the 4-cells, and even the 5-cells of $\mathcal{O}_\omega$ before coming up with a general argument which here appears in the proofs of Lemma 3.4(d) and Lemma 3.9.

The algorithm which gives the cocycle conditions as equations appears here in the proof of Corollary 3.14 (*excision of extremals*). I had written this algorithm down for Jack on 25 January 1984 but I was unable to justify its working. This depended on the present Lemma 3.13 whose proof eluded me until this month (August 1984).

Finally, I should say that, although I had been referring to $\mathcal{O}_\omega$ as the free ω-category on the ω-simplex $\Delta\omega$, I did not see how to make this universal property precise until last month. I had thought that an n-dimensional version of the notion of *computad* [12] (which is appropriate for $n = 2$) would be much more difficult to describe than $\mathcal{O}_\omega$ itself. This is not the case: free ω-categories can be characterized universally in a straightforward manner (see Section 4).

Motivation

Algebraic topology develops from the facial relationships between model spaces for each dimension. The usual choice for dimension n is the *standard n-simplex*

$$\Delta_n = \{x \in \mathbb{R}^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

If we put $[n] = \{0, 1, \ldots, n\}$, then Δ_{n+1} consists of the order-preserving functions $[n] \to I$ where I is the unit interval. The category Δ of ordered sets $[n]$ and order-preserving functions has some distinguished functions $\partial_i : [n] \to [n+1]$ where the image of ∂_i contains all elements of $[n+1]$ except i. These satisfy the *simplicial identities*

$$\partial_j \partial_i = \partial_i \partial_{j-1} \quad \text{for } i < j.$$

There is a functor $\Delta \to \mathrm{Sp}$ into the category of spaces which takes $[n]$ to Δ_n; the information on maps captures the facial relationships.

Abelian cohomology is obtained using a space X and an abelian group A. Put $X_n = \mathrm{Sp}(\Delta_n, X)$, the set of maps in Sp from Δ_n to X; the ∂_i induce functions $X_{n+1} \to X_n$. Let C^n denote the set of functions $X_n \to A$ regarded as an abelian group under value-wise addition. The ∂_i induce homomorphisms $\partial_i : C^n \to C^{n+1}$ which again satisfy the simplicial identities and these imply that the homomorphisms

$$\partial = \partial_0 - \partial_1 + \partial_2 - \cdots + (-1)^{n+1}\partial_{n+1} : C^n \to C^{n+1}$$

satisfy the condition

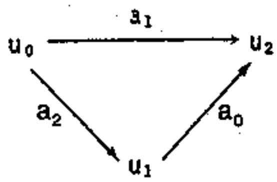
$$\partial\partial = 0.$$

The kernel of $\partial : C^n \to C^{n+1}$ is the abelian group $Z^n(X; A)$ of *n-cocycles on X with coefficients in A*. The image of $\partial : C^{n-1} \to C^n$ is the abelian group $B^n(X; A)$ of n-coboundaries on X with coefficients in A. So $B^n(X; A) \subset Z^n(X; A)$ and $H^n(X; A) = Z^n(X; A)/B^n(X; A)$ is the *n-cohomology group of X with coefficients in A*.

In summary, for abelian cohomology, *the geometric input* is the collection of facial relationships in simplexes, while *the algebraic input* is the alternating sum operation in an abelian group.

It is a well-known fact that it is not necessary for the group A to be abelian in order to obtain objects $H^0(X; A)$, $H^1(X; A)$ as above. This observation has useful consequences (see [11] for example). For example, there are applications with $A = \mathrm{GL}(n, k)$, the group of invertible $n \times n$-matrices with entries from k.

This can be generalized even further. The 1-cocycle condition in its alternating sum form $a_0 - a_1 + a_2 = 0$ rewrites as $a_1 = a_0 + a_2$. This eliminates the need for inverses and so A needs only to be a *monoid*. Monoids arise as sets of endomorphisms under composition. The restriction to *endo*morphisms is unnecessary: the natural context for 1-cohomology is to take A to be a category so that the 1-cocycle condition becomes commutativity of a *triangle*:



The 1-cohomology classes are isomorphism classes of such commutative triangles, although the correct object of study seems to be the category of commutative triangles itself.

Using this generalization, one can give a cohomological explanation for the transfer of properties of global algebraic structures (such as vector spaces) to local ones (such as vector bundles); see [13]. An example of such a category A is $\mathrm{Mat}(k)$ whose objects are natural numbers and whose arrows $n \to m$ are $m \times n$-matrices with entries from k. Cohomology with coefficients in $\mathrm{Mat}(k)$ contains all the information of that with coefficients in $\mathrm{GL}(k)$, and more.

There have been attempts to generalize 2-cohomology to allow A to be a general group. Guiding examples and applications seem hard to find. How then does the generalization to category for H^1 help for H^2? Is it not true that a group is a special category and so, if we could do H^2 for categories, could we not do H^2 for groups?

The answer is not to use categories as coefficients for H^2. The importance of the generalization for H^1 is that now we view the operation of A as *composition* which is defined only between elements whose sources and targets match-up.

When one looks at those examples of abelian 2-cohomology where we have a concrete interpretation, one sees the presence of *two operations*. For example, as well as its composition, the category $\mathrm{Mat}(k)$ has a functorial multiplication

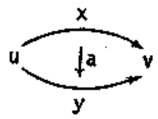
$$\mathrm{Mat}(k) \times \mathrm{Mat}(k) \to \mathrm{Mat}(k)$$

which takes (m, n) to $m + n$.

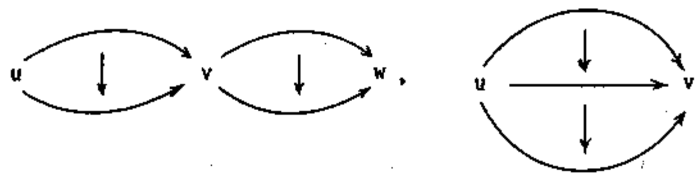
This suggests that 2-cohomology concerns two operations suitably related. Reinforcement for this view is the fact that a group A with a functorial unital associative operation $A \times A \to A$ is necessarily abelian and the operation is necessarily the group operation.

Just as the first operation does not need to be everywhere defined, neither does the second. This leads us to the view that *2-cohomology must be generalized to allow 2-categories as coefficient objects*. (Eventually, we will allow bicategories as coefficient objects; however, the definition of 'bicategory' involves the 3-cocycle condition which comes from the strict version presented here.)

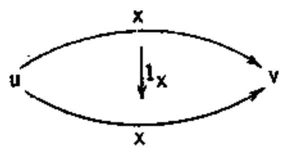
Although the present paper is self-contained, a less formal introduction to 2-categories is given in [6]. A 2-category A consists of *2-cells*



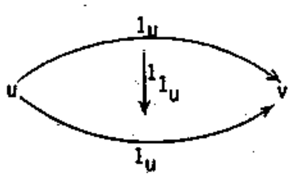
which can be composed horizontally and vertically.



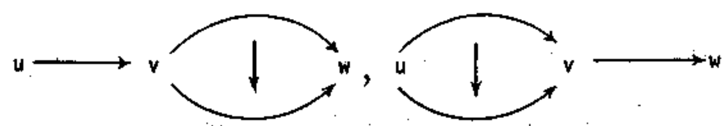
Identities for vertical composition have the form



and can be identified with *1-cells* (or *arrows*) $x: u \to v$. Identities for horizontal composition have the form

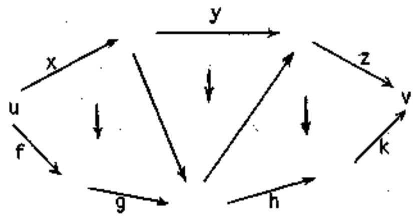


and can be identified with *0-cells* (or *objects*) u. Horizontal composition can be broken up into the two more basic forms

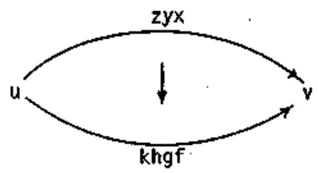


Iterated composition in a category leads to nothing more complicated than the fact that a string $\to \to \to \cdots \to$ has a unique composite.

Iterating the operations in a 2-category is called *pasting*. For example, a diagram such as



pastes (in many different ways) to a unique 2-cell



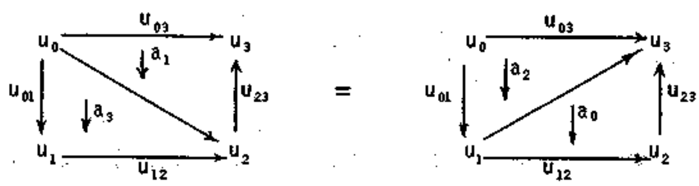
What then is the 2-cocycle condition in a 2-category? The alternating sum form

$$a_0 - a_1 + a_2 - a_3 = 0$$

rewrites as

$$a_3 + a_1 = a_0 + a_2$$

where the a_i should be seen as 2-cells. This suggests that the correct condition should be the equality of the two 2-cells obtained from pasting each side of the equation



The 0-dimensional form of this condition is the *2-cocycle equation*

$$(u_{23} *_0 a_3) *_1 a_1 = (a_0 *_0 u_{01}) *_1 a_2$$

where we use $*_0$ for horizontal composition and $*_1$ for vertical composition. The most appealing geometric form for the condition is in 3-dimensions where it becomes a *commutative tetrahedron*, see Fig. 1, where the arrow directions provide *orientations* for the 1- and 2-dimensional faces of the tetrahedron.

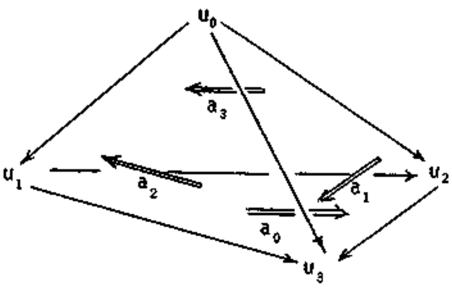


Fig. 1.

It is necessary to be systematic about the choice of orientation in each dimension (other choices can be accounted for by 2-category duality). Notice that, for the triangular faces of the tetrahedron, we have chosen the 2-cells to go from the single arrow to the composable pair. Thus before proceeding to the tetrahedron, we have considered an *oriented triangle*, not just a commutative one as for the 1-cocycle condition. Similarly, before proceeding to the 4-simplex, we must decide on an orientation of the tetrahedron itself, not just its faces. If we write u_{123}, u_{023}, u_{013}, u_{012} in the tetrahedron in place of a_0, a_1, a_2, a_3 (using what is present rather than what is missing), we can view the commutative tetrahedron as a 2-functor from the 2-category generated by the two diagrams in Fig. 2 into A which identifies the two 2-cells obtained by pasting. The general principle for the direction of an m-cell labelled $(x_0, x_1, \ldots, x_m)$ is that the $(m-1)$-cells involved in the source of $(x_0, x_1, \ldots, x_m)$ are the $(x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ with i odd and those involved in the target have the same form with i even.

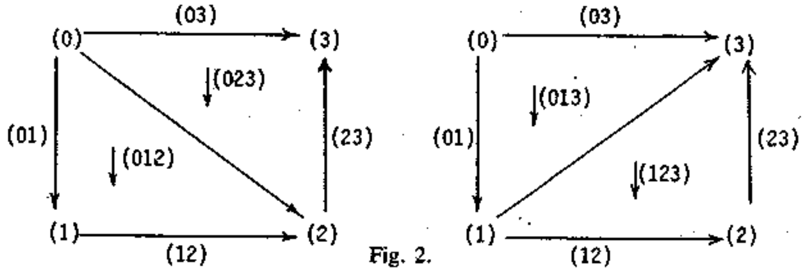


Fig. 2.

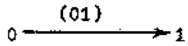
Following Roberts [8], we claim that n-cohomology should be developed using n-categories A as coefficient objects. For this, the geometric input is an oriented $(n+1)$-simplex while the algebraic input is the operation of pasting in an n-category.

This leads us to require a precise description of the "free n-category $\mathcal{O}_n$ on the oriented n-simplex $\Delta[n]$". These objects seem to be fundamental structures of nature so I decided they should have a short descriptive name of the ilk of 'cardinal' and 'ordinal': I settled on 'oriental'. The $(n-1)$-cocycle condition is expressed by an n-functor from the nth oriental $\mathcal{O}_n$ to an $(n-1)$-category A (regarded as an n-category with last composition discrete). The following diagrams provide the data which generate $\mathcal{O}_n$ for $n = 0, 1, \ldots, 6$ using pasting. The 3-, 4-, 5-cocycle equations can be obtained from these diagrams.

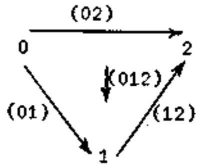
O_0


0

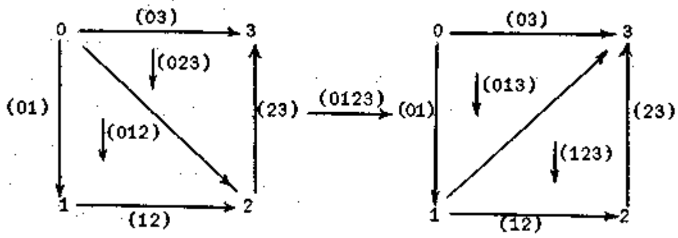
O_1


(01)
0
1

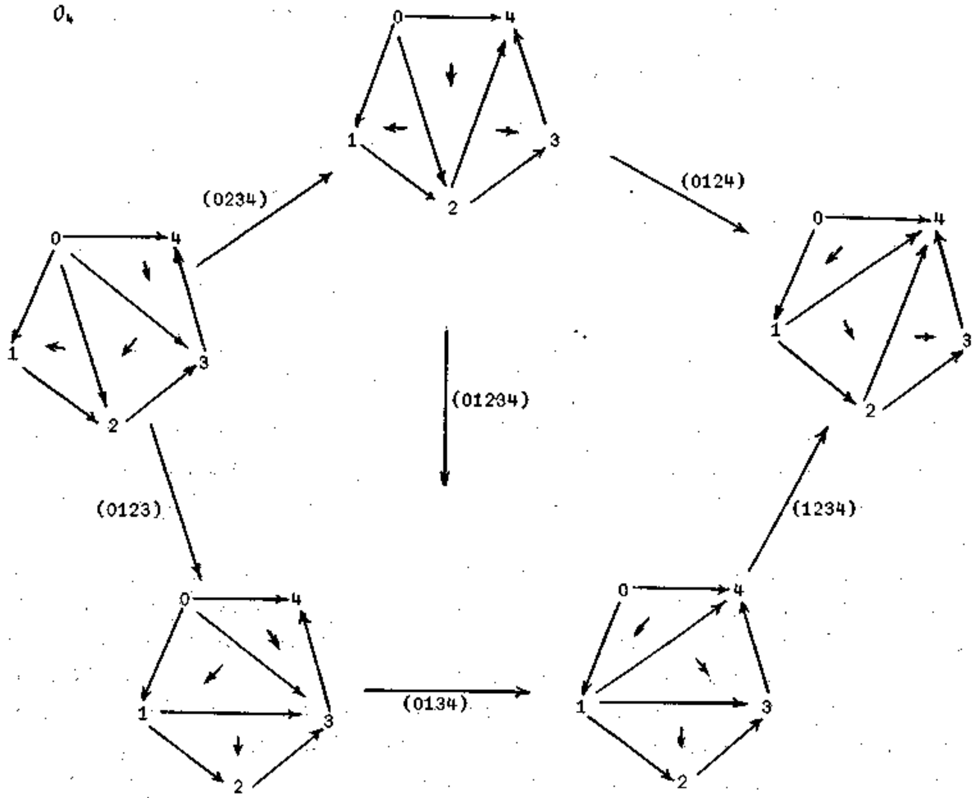
O_2

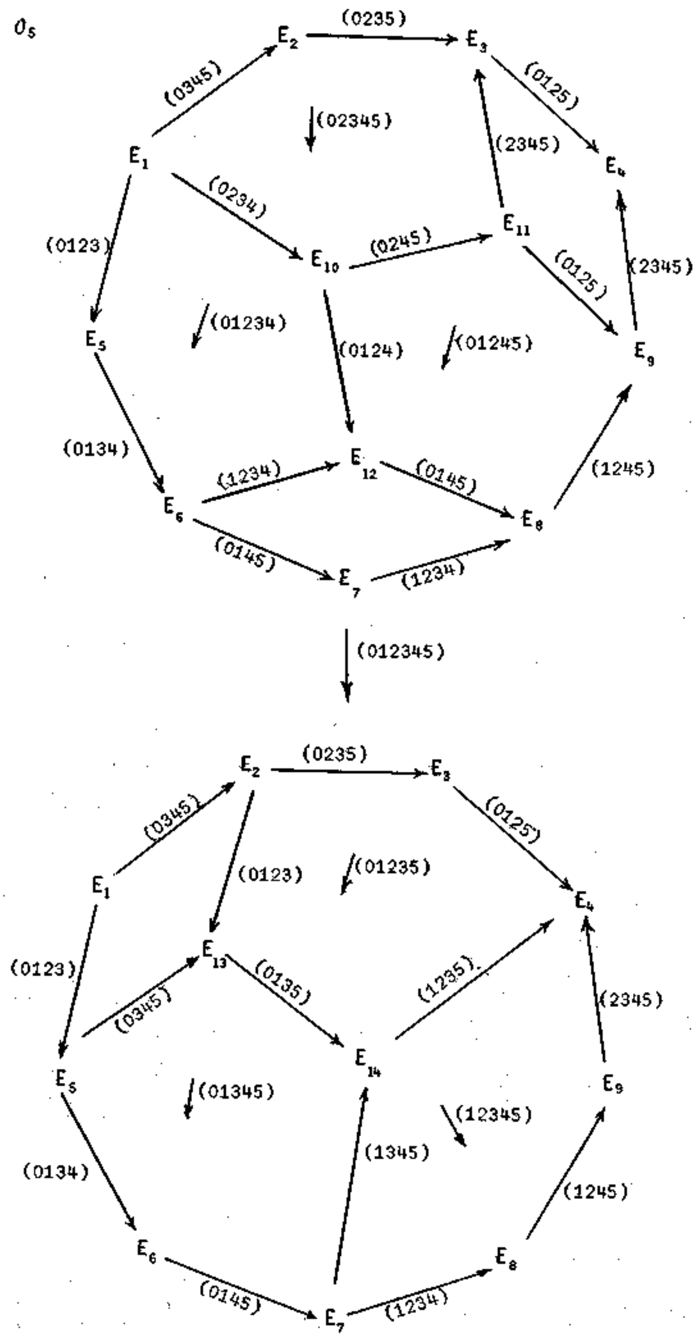

(02)
0
2
(012)
(01)
(12)
1

O_3


(03)
(023)
(01)
(012)
(23)
(12)
(0123)
(03)
(013)
(01)
(23)
(123)
(12)

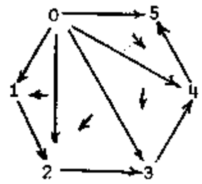
O_4


(0234)
(0124)
(01234)
(0123)
(1234)
(0134)

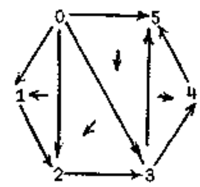


where the E_i, $i = 1, 2, \ldots, 14$, are given as follows:

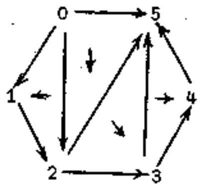
E_1



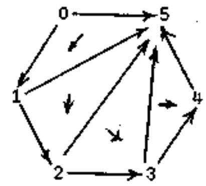
E_2



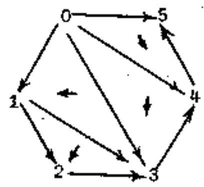
E_3



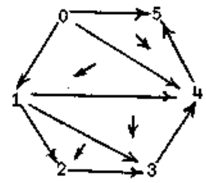
E_4



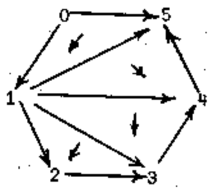
E_5



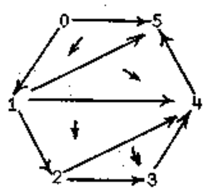
E_6



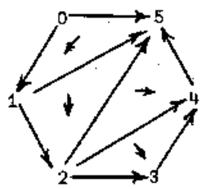
E_7



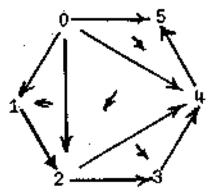
E_8



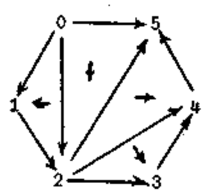
E_9



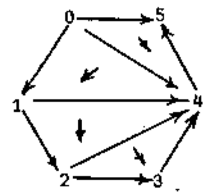
E_{10}



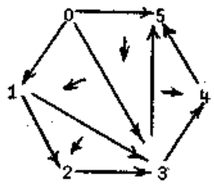
E_{11}



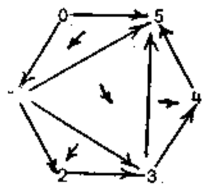
E_{12}



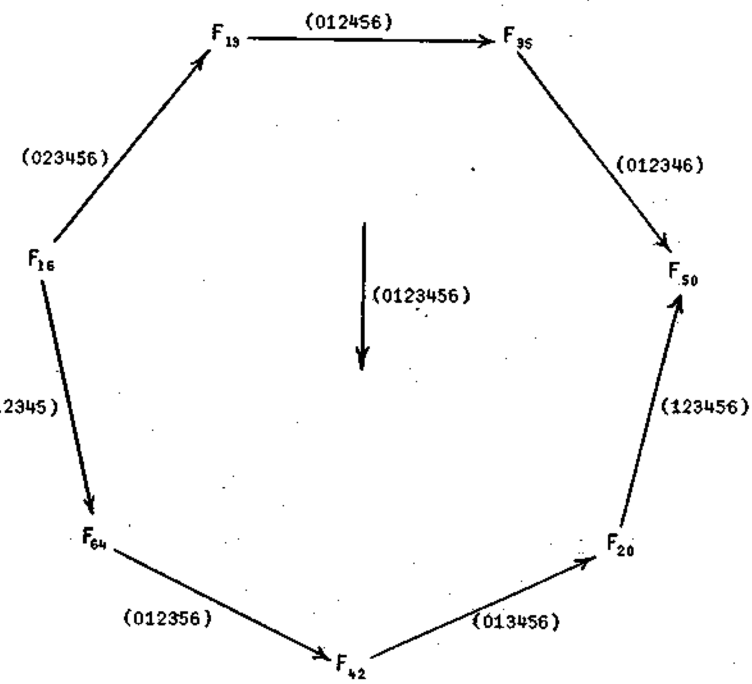
E_{13}



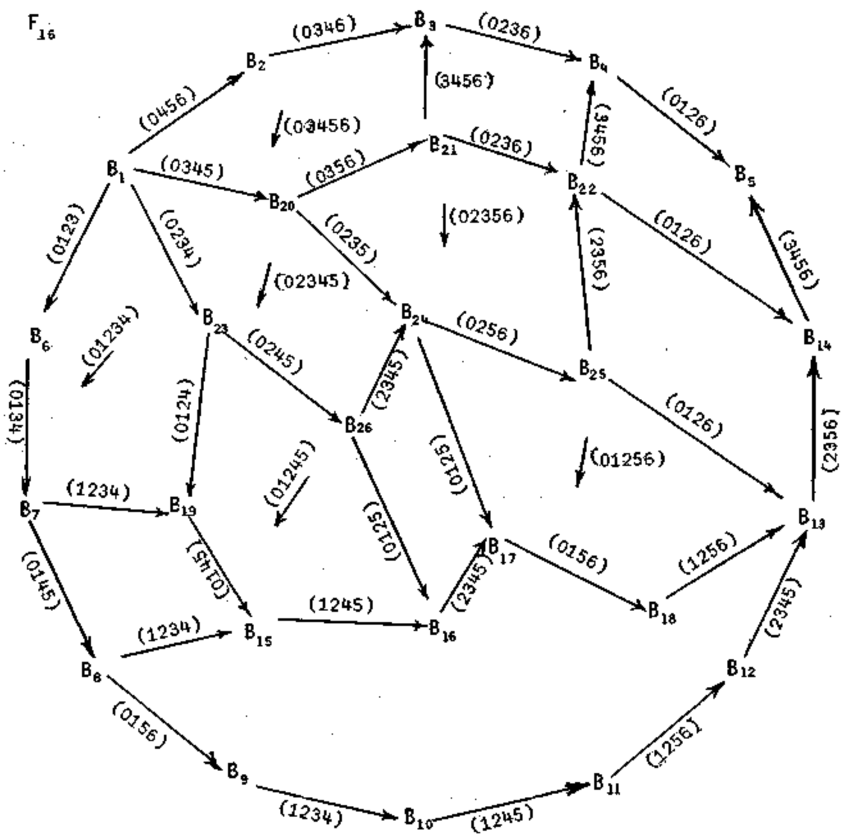
E_{14}



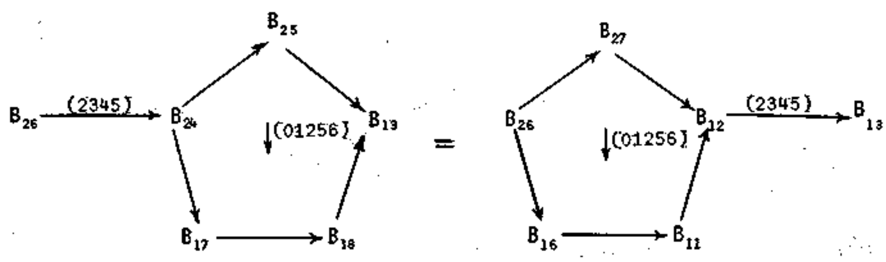
O_6

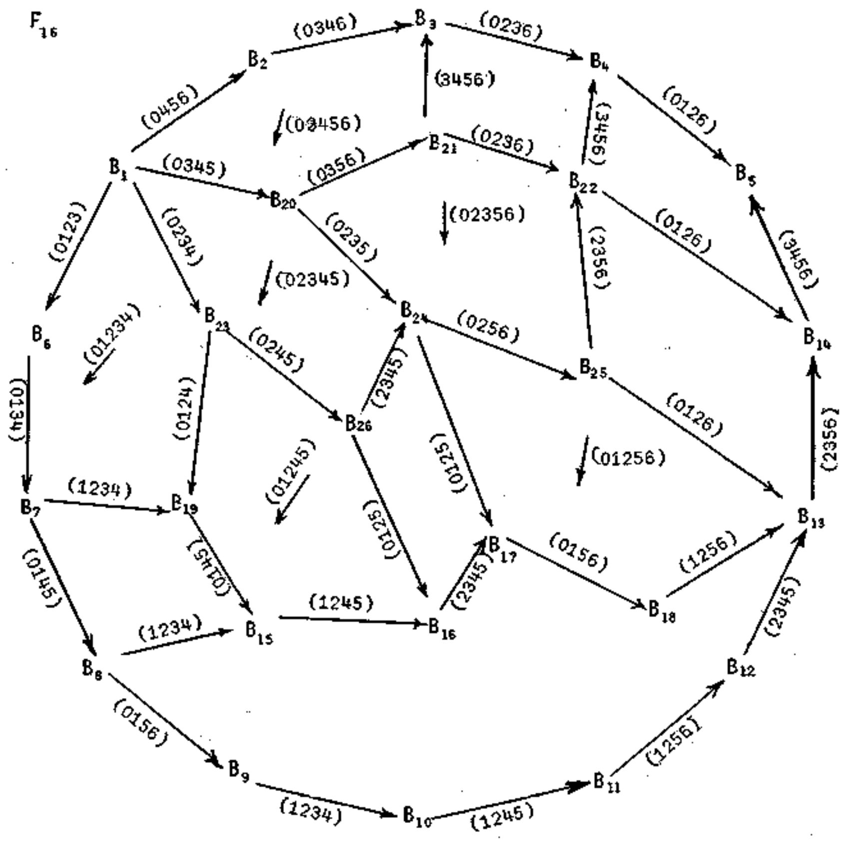


where the F_{ij} are given as follows:

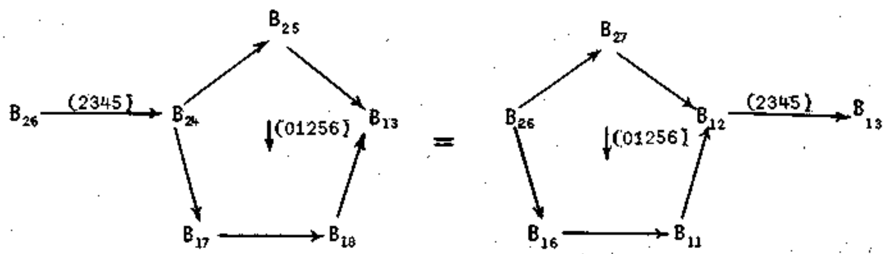

F16
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B15
B16
B17
B18
B19
B20
B21
B22
B23
B24
B25
B26
(0456)
(0346)
(0236)
(3456)
(0126)
(03456)
(0345)
(0356)
(0236)
(3456)
(02356)
(0126)
(3456)
(0123)
(0234)
(0235)
(2356)
(02345)
(01234)
(0256)
(0245)
(2345)
(0134)
(0124)
(0126)
(2356)
(01256)
(01245)
(0125)
(1234)
(0125)
(0145)
(0145)
(2345)
(0156)
(1256)
(1245)
(1234)
(2345)
(0156)
(1256)
(1234)
(1245)

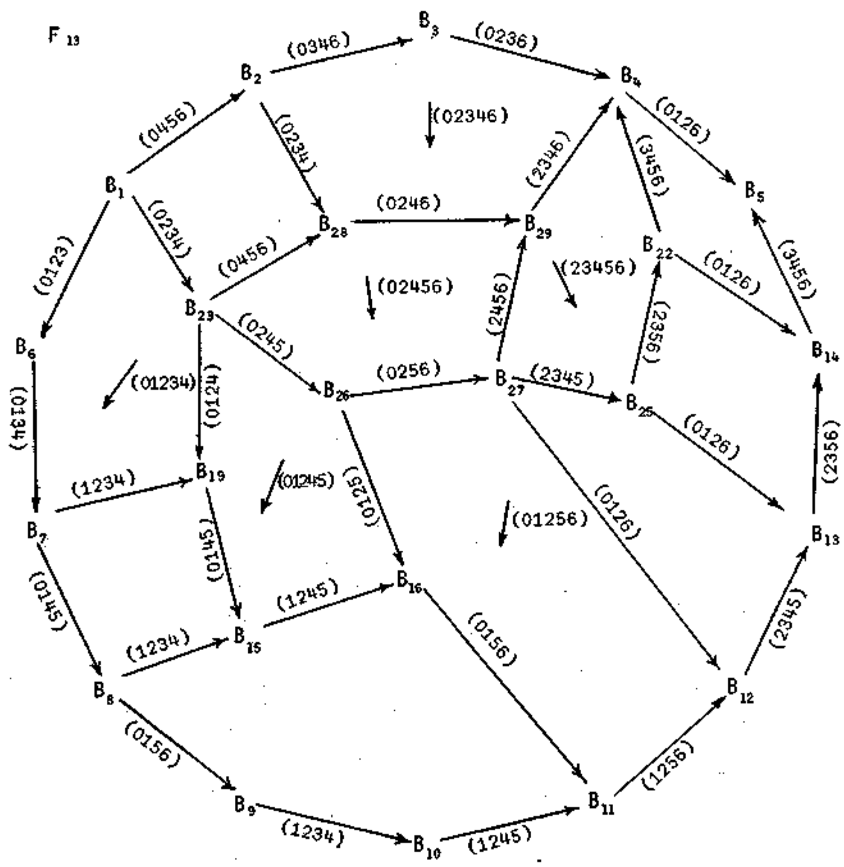
in which


B26
(2345)
B24
B25
(01256)
B13
B17
B18
=
B27
B26
(01256)
B12
(2345)
B13
B16
B11

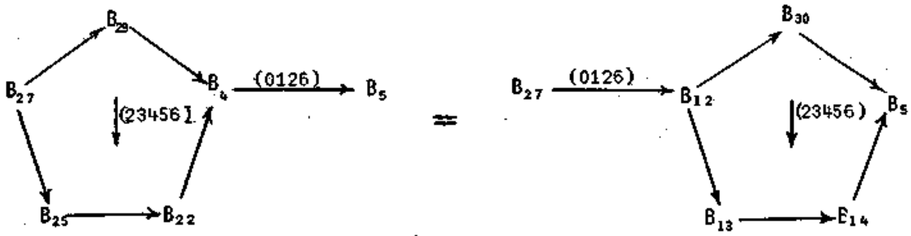

F16
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B15
B16
B17
B18
B19
B20
B21
B22
B23
B24
B25
B26
(0456)
(0346)
(0236)
(3456)
(03456)
(0126)
(0345)
(0356)
(0236)
(3456)
(0123)
(0234)
(02356)
(0235)
(2356)
(0126)
(3456)
(02345)
(01234)
(0256)
(0245)
(2345)
(0134)
(0124)
(0125)
(01256)
(0126)
(2356)
(01245)
(0125)
(1234)
(0145)
(2345)
(0156)
(1256)
(0145)
(1245)
(1234)
(2345)
(0156)
(1256)
(1234)
(1245)

in which

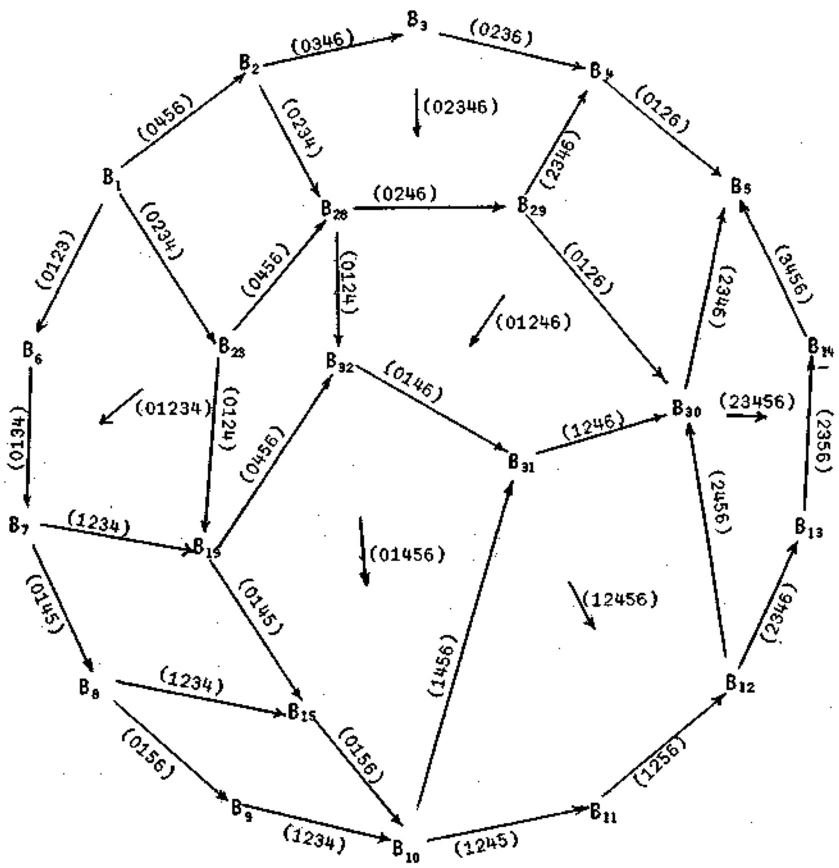

B26
(2345)
B24
B25
(01256)
B13
B17
B18
=
B26
B27
(01256)
B12
(2345)
B13
B16
B11


F 13
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B15
B16
B19
B22
B23
B25
B26
B27
B28
B29
(0346)
(0236)
(0456)
(0234)
(02346)
(0126)
(2346)
(3456)
(0246)
(0123)
(0234)
(0456)
(02456)
(23456)
(0126)
(3456)
(2456)
(2356)
(0245)
(0256)
(2345)
(01234)
(0124)
(0134)
(0126)
(1234)
(01245)
(0125)
(01256)
(0126)
(2356)
(0145)
(0145)
(1245)
(1234)
(0156)
(2345)
(0156)
(1256)
(1234)
(1245)

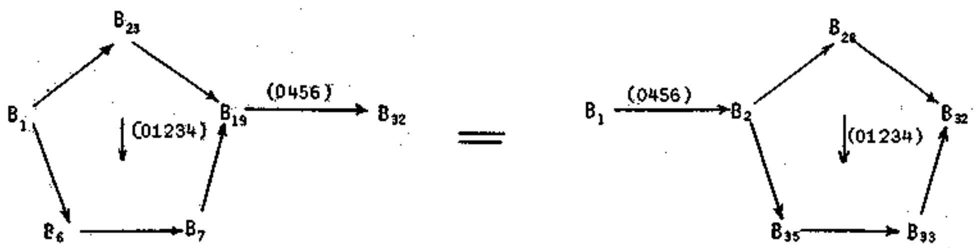
in which


B29
B27
(23456)
B4
(0126)
B5
B25
B22
=
B30
B27
(0126)
B12
(23456)
B5
B13
B14

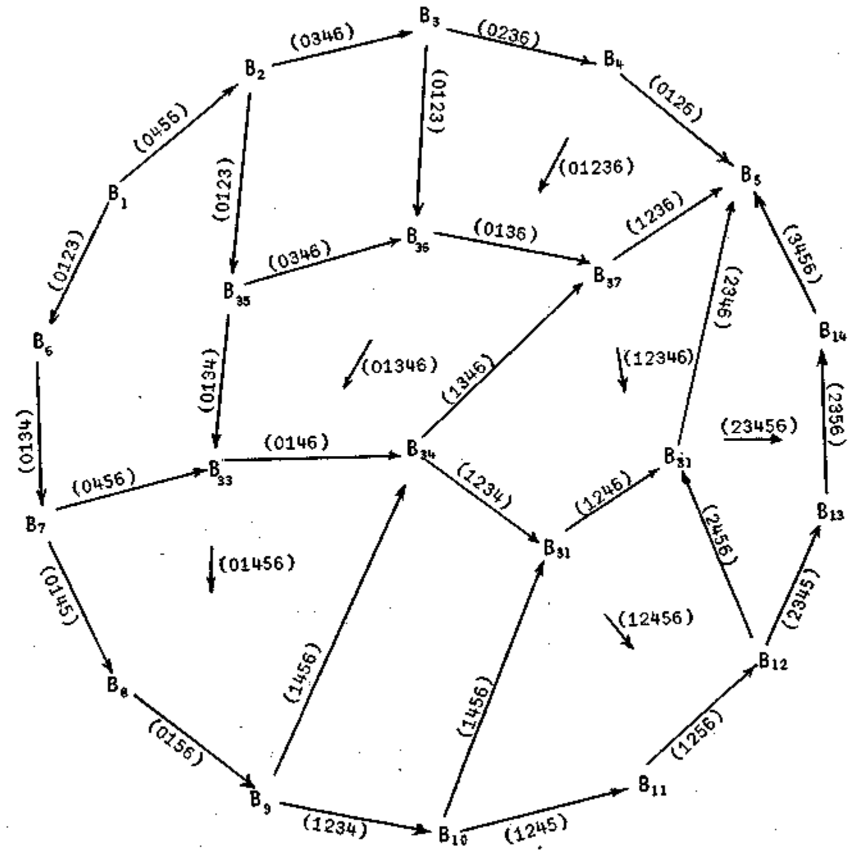
F_{35}


B3
(0346)
(0236)
B2
B4
(02346)
(0456)
(0234)
(0126)
(2346)
B1
(0246)
B28
B29
B5
(0234)
(0123)
(0456)
(0124)
(0126)
(3456)
(01246)
(2346)
B6
B23
B32
B14
(0146)
(01234)
(0124)
(23456)
(0134)
(0456)
(1246)
B30
(2356)
B31
(2456)
B7
(1234)
B19
B13
(01456)
(0145)
(0145)
(12456)
(1456)
(2346)
B8
(1234)
B12
B15
(0156)
(0156)
(1256)
B9
B11
(1234)
B10
(1245)

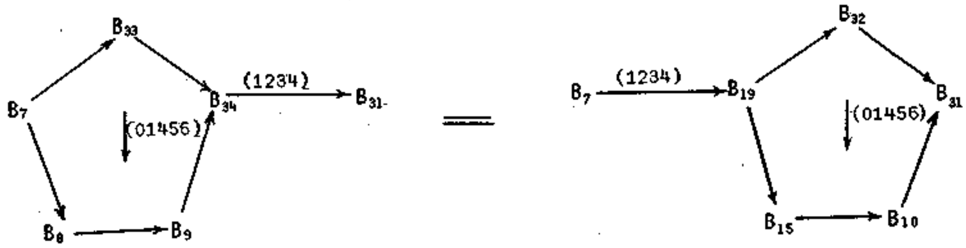
in which


B23
(0456)
B1
(01234)
B19
B32
B6
B7
=
B28
(0456)
B1
B2
B32
(01234)
B35
B33

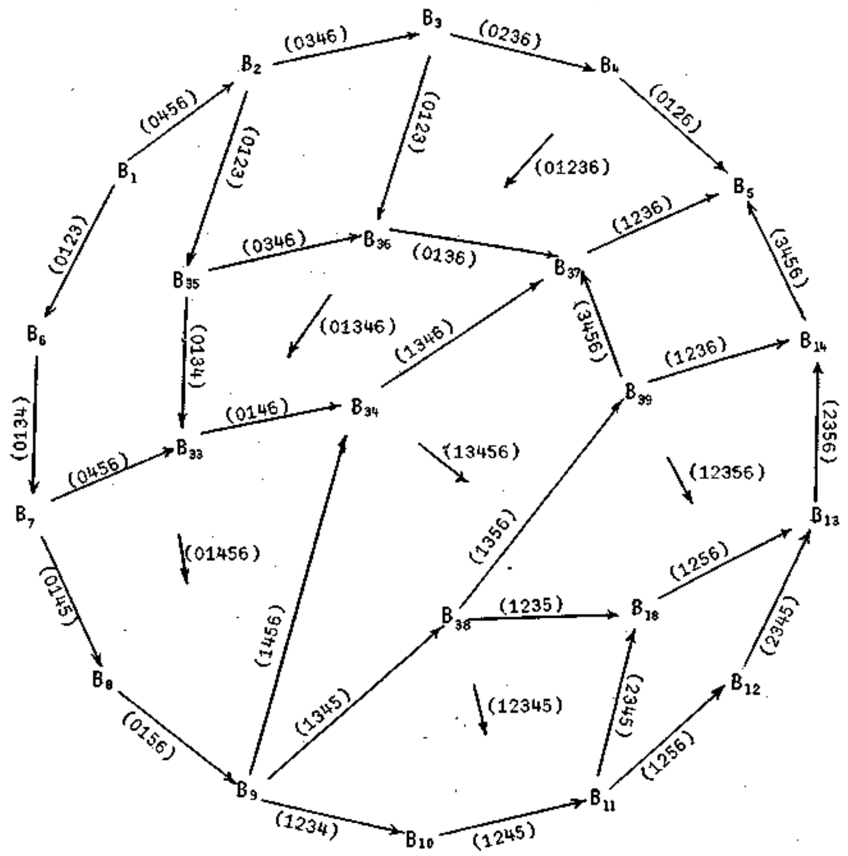
F_{50}


B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B31
B33
B34
B35
B36
B37
(0456)
(0346)
(0236)
(0126)
(0123)
(0123)
(0123)
(01236)
(0346)
(0136)
(1236)
(3456)
(2346)
(0134)
(0134)
(01346)
(1346)
(12346)
(2356)
(23456)
(0146)
(0456)
(1234)
(1246)
(2456)
(01456)
(0145)
(12456)
(2345)
(1456)
(1456)
(0156)
(1256)
(1234)
(1245)

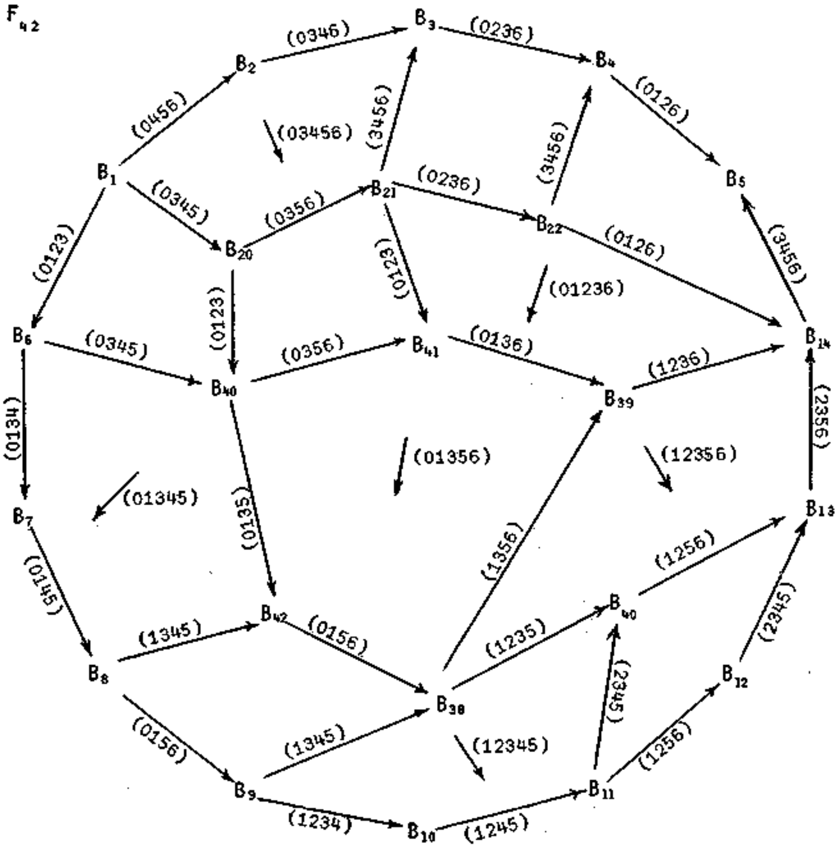
in which


B33
B7
B34
(1234)
B31
(01456)
B8
B9
B32
B7
(1234)
B19
B31
(01456)
B15
B10

F_{20}

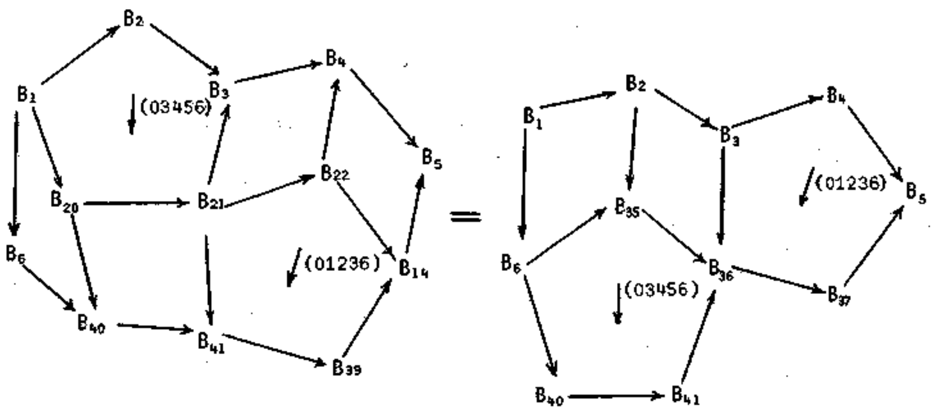

B3
(0346)
(0236)
B2
B4
(0456)
(0123)
(0123)
(01236)
(0126)
B1
B5
(0123)
(0346)
B36
(0136)
(1236)
B35
B37
(3456)
(0134)
(01346)
(1346)
(3456)
B6
(1236)
B14
(0134)
(0146)
B34
B39
(2356)
B33
(13456)
(12356)
(0456)
B7
B13
(1356)
(01456)
(1256)
(0145)
(1235)
B38
B18
(1456)
(2345)
B8
B12
(1345)
(12345)
(2345)
(0156)
(1256)
B9
B11
(1234)
B10
(1245)

F_{42}



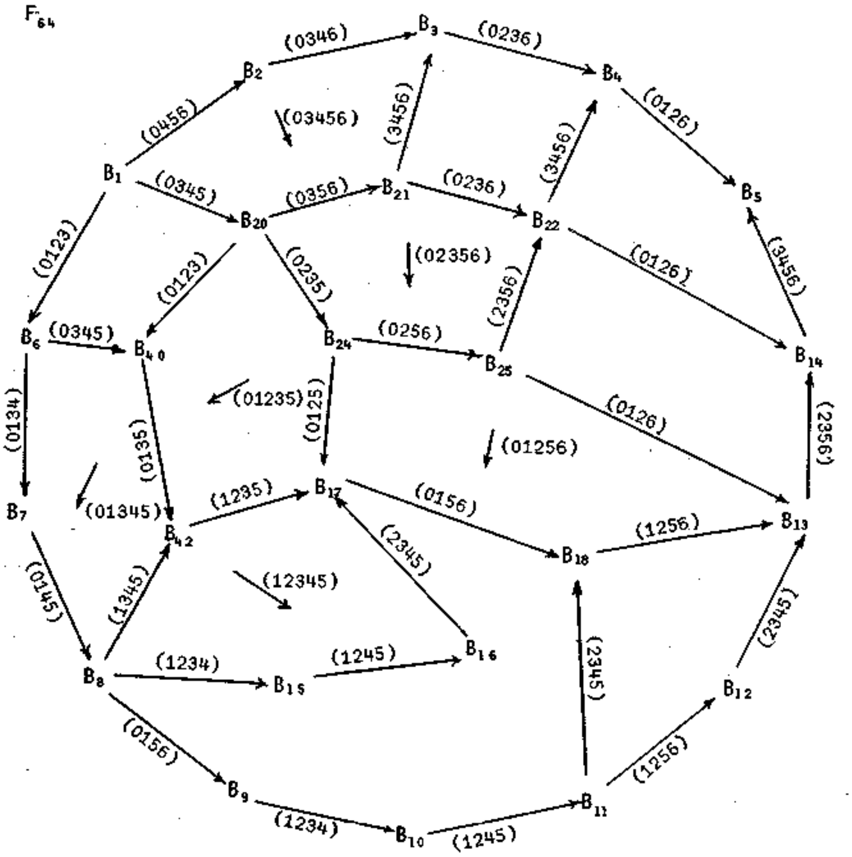
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B20
B21
B22
B38
B39
B40
B41
B42
(0456)
(0346)
(0236)
(0126)
(3456)
(2356)
(2345)
(1256)
(1245)
(1234)
(0156)
(0145)
(0134)
(0123)
(0345)
(0356)
(1345)
(1356)
(1235)
(1236)
(0136)
(0135)
(03456)
(01236)
(01356)
(01345)
(12356)
(12345)

in which

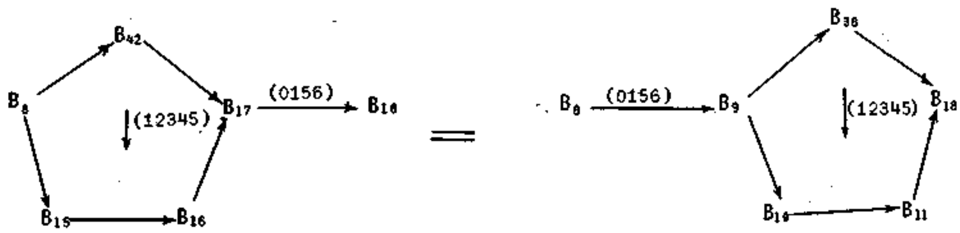


B1
B2
B3
B4
B5
B6
B14
B20
B21
B22
B39
B40
B41
(03456)
(01236)
=
B35
B36
B37

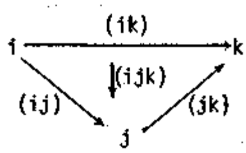
F_{64}

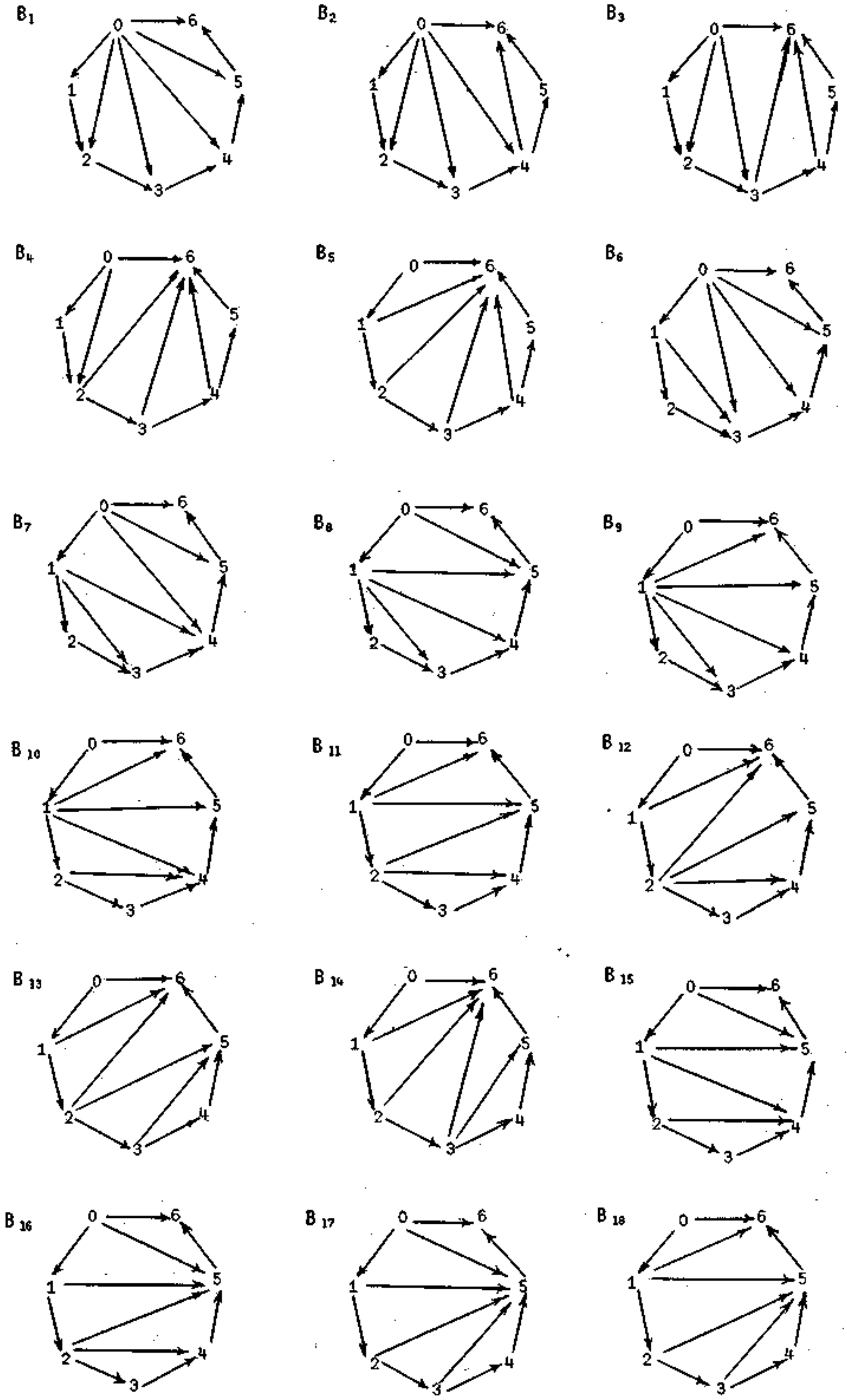


in which

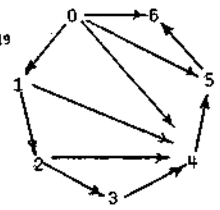


and where the B_i $(i = 1, 2, \dots, 42)$ are as follows (where the 2-cells $(i\ j\ k)$ are omitted from the triangles with vertices i, j, k since their directions are determined as in the diagram):

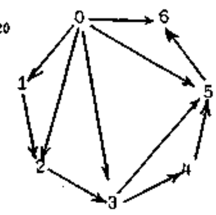



B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13
B14
B15
B16
B17
B18

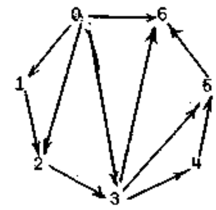
B_{19}



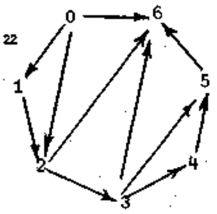
B_{20}



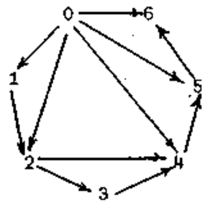
B_{21}



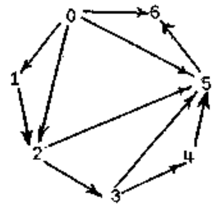
B_{22}



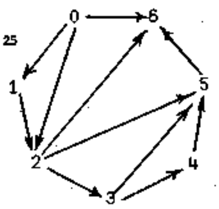
B_{23}



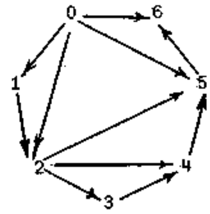
B_{24}



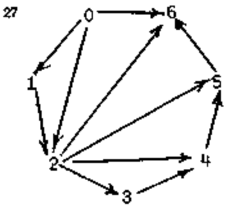
B_{25}



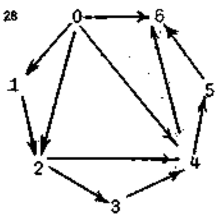
B_{26}



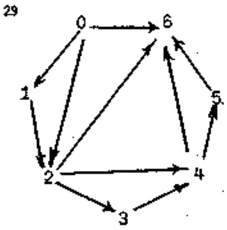
B_{27}



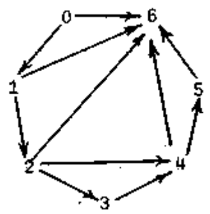
B_{28}



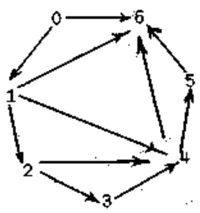
B_{29}



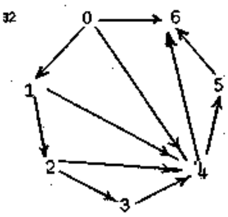
B_{30}



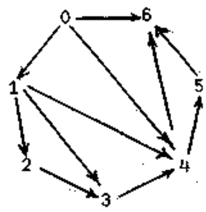
B_{31}



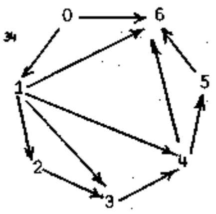
B_{32}



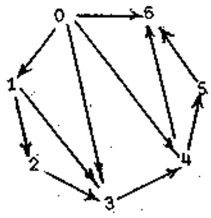
B_{33}



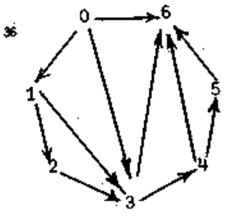
B_{34}



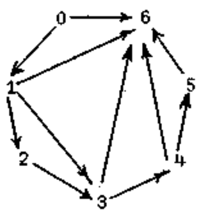
B_{35}



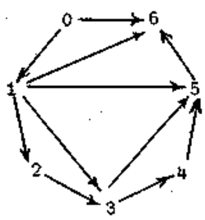
B_{36}



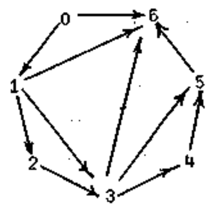
B 37



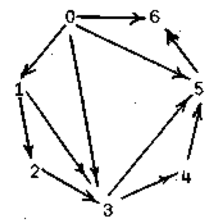
B 38



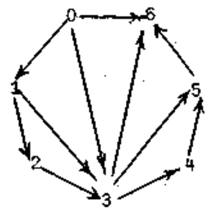
B 39



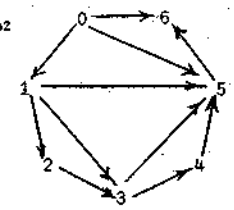
B 40



B 41



B 42



1. Higher dimensional categories

A *category* $(A, s, t, *)$ consists of a set A, functions $s, t: A \to A$ satisfying the equations

$$ss = ts = s\,, \qquad tt = st = t\,,$$

and a function $*: \{(a, b) \in A \times A \mid s(a) = t(b)\} \to A$, whose value $a * b$ at (a, b) satisfies the equations

$$s(a * b) = s(b)\,, \qquad t(a * b) = t(a)\,,$$

such that the following axioms hold:

$$s(a) = t(v) = v \quad \text{implies} \quad a * v = a\,; \qquad \text{(right identity)}$$

$$u = s(u) = t(a) \quad \text{implies} \quad u * a = a\,; \qquad \text{(left identity)}$$

$$s(a) = t(b),\ s(b) = t(c) \quad \text{imply} \quad a * (b * c) = (a * b) * c\,. \qquad \text{(associativity)}$$

The functions $s, t, *$ are respectively called *source, target, composition*; when they are understood the category $(A, s, t, *)$ is denoted by A. Elements of A are called *arrows* and the notation $a: u \to v$ is used to mean that $s(a) = u$ and $t(a) = v$. An arrow u is in the image of s if and only if it is in the image of t; such arrows u are called *identities* (or *objects*) and satisfy $s(u) = t(u) = u$. A pair (a, b) of arrows is called *composable* when $s(a) = t(b)$.

Example 1.1. (a) A *monoid* is a category for which the source and target functions are constant.

(b) An ordered set U determines a category $(A, s, t, *)$ where $A = \{(u, v) \in U \times U \mid u \leq v\}$, $s(u, v) = (u, u)$, $t(u, v) = (v, v)$, $(v, w) * (u, v) = (u, w)$. Notice that the set of identities in A is isomorphic to U. A category is called an *ordered set* when $a: u \to v$, $b: u \to v$ imply $a = b$.

(c) There are two cases of Example 1.1(b) which need to be distinguished. Let U be a set X with the *discrete* order: $u \leq v$ when $u = v$; the resulting category is denoted by X_d. Let U be a set X with the *chaotic* order: $u \leq v$ for all $u, v \in X$; the resulting category is denoted by X_c.

(d) A *graph* G consists of a pair of functions $s, t: P_1G \to P_0G$. Elements of P_0G are called 0-*paths* (or *vertices*) while elements of P_1G are called 1-*paths* (or *edges*). For $n > 1$, an n-path is an element $(a_1, \ldots, a_n)$ of $(P_1G)^n$ such that $s(a_i) = t(a_{i+1})$ for $i = 1, \ldots, n-1$. This gives a graph $s, t: P_nG \to P_0G$ where P_nG is the set of n-paths and $s(a_1, \ldots, a_n) = s(a_n)$, $t(a_1, \ldots, a_n) = t(a_1)$. *The free category* FG on the graph G is $(\sum_{n=0}^{\infty} P_nG, s, t, *)$ where s, t are the identity on P_0G and are given as above on n-paths, and

$$(a_1, \ldots, a_n) * (b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m) .$$

A graph G is called a *tree* when FG is an ordered set.

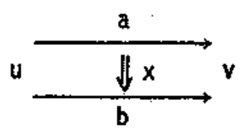
A 2-*category* $(A, s_0, t_0, *_0, s_1, t_1, *_1)$ consists of two categories $(A, s_0, t_0, *_0)$, $(A, s_1, t_1, *_1)$ satisfying the following conditions:

(i) $s_1s_0 = s_0 = s_0s_1 = s_0t_1$, $t_0 = t_0s_1 = t_0t_1$;

(ii) $s_0(a) = t_0(a')$ implies $s_1(a *_0 a') = s_1(a) *_0 s_1(a')$ and $t_1(a *_0 a') = t_1(a) *_0 t_1(a')$;

(iii) $s_1(a) = t_1(b)$, $s_1(a') = t_1(b')$, $s_0(a) = t_0(a')$ imply $(a *_1 b) *_0 (a' *_1 b') = (a *_0 a') *_1 (b *_0 b')$.

The identities for $*_0$ are called 0-*cells* and the identities for $*_1$ are called 1-*cells*. The notation



is used to mean $x \in A$, $s_1(x) = a$, $t_1(x) = b$, $s_0(x) = u$ and $t_0(x) = v$.

Example 1.2.(a) A 2-category for which s_1 t_1 are constant automatically has $s_0 = s_1$, $t_0 = t_1$, $*_0 = *_1$. A category $(A, s, t, *)$ is called *commutative* when $s(a) = t(b)$ implies $t(a) = s(b)$ and $a * b = b * a$; this holds precisely when $(A, s, t, *, s, t, *)$ is a 2-category. Hence *a 2-category for which* s_1, t_1 *are constant amounts to a commutative monoid*. (This is the essence of the argument of Eckmann–Hilton [2] which proves that the higher homotopy groups are commutative.) Clearly a category which is an ordered set is commutative if and only if it is discrete.

(b) Each category X yields a 2-category X_d on the same set with $s_0, t_0, *_0$ agreeing with the original structure on X and $s_1, t_1, *_1$ the discrete structure.

(c) Each category X yields a 2-category X_c on the set $\{(x, y) \in X \times X \mid s(x) = s(y), t(x) = t(y)\}$, and, identifying X with the diagonal in X_c, with $s_0(x, y) = s(x)$, $t_0(x, y) = t(x)$, $(x, y) *_0 (x', y') = (x * x', y * y')$, $s_1(x, y) = x$, $t_0(x, y) = y$, $(y, z) *_1 (x, y) = (x, z)$. If X is a monoid, then the underlying set of X_c is $X \times X$ and $s_1, t_1, *_1$ give the chaotic structure on X.

The first infinite ordinal is denoted by ω; that is, $\omega = \{0, 1, 2, \ldots\}$ as an ordered set.

An *ω-category* $(A, (s_n, t_n, *_n)_{n \in \omega})$ consists of categories $(A, s_n, t_n, *_n)$ for each $n \in \omega$ such that $(A, s_m, t_m, *_m, s_n, t_n, *_n)$ is a 2-category for all $m < n$. The identities for $*_n$ are called *n-cells*. A *cell* is an element of A which is an n-cell for some n. The notation $a : u \to_n v$ is used to mean that $s_n(a) = u$ and $t_n(a) = v$.

For $r \in \omega$, an *r-category* is an ω-category for which all elements are r-cells: this means that the structures $s_n, t_n, *_n$ are discrete for $n \geq r$. For $r = 2$ there is no conflict with the previous definition of a 2-category. A 1-category is a category, and a 0-category is a set.

An *ω-functor* $f : (A, (s_n, t_n, *_n)_{n \in \omega}) \to (A', (s'_n, t'_n, *'_n)_{n \in \omega})$ is a function $f : A \to A'$ such that $fs_n = s'_n f$, $ft_n = t'_n f$, and $s_n(a) = t_n(b)$ implies $f(a *_n b) = f(a) *'_n f(b)$, for all $n \in \omega$. In an obvious way, ω-functors between (small) ω-categories are the arrows for a category denoted by ω-Cat. Let r-Cat denote the sub-category of ω-Cat obtained by restricting to ω-functors between r-categories.

Example 1.3.(a) For any commutative monoid M and $n \in \omega$, there is an n-category $K(M, n)$ for which $s_r, t_r, *_r$ for all $r < n$ are the source, target, composition functions of M as a category. For $n = 0$ notice that M only needs to be a set and for $n = 1$, M does not need to be commutative.

(b) Each r-category X yields an $(r+1)$-category X_c on the set $\{(x, y) \in X \times X \mid s_{r-1}(x) = s_{r-1}(y), t_{r-1}(x) = t_{r-1}(y)\}$ with $s_n(x, y) = s_n(x)$, $t_n(x, y) = t_n(x)$, $(x, y) *_n (x, y') = (x *_n x', y *_n y')$ for $n < r$, and $s_r(x, y) = x$, $t_r(x, y) = y$, $(y, z) *_r (x, y) = (x, z)$.

(c) The categories ω-Cat, r-Cat are locally finitely presentable in the sense of Gabriel–Ulmer [4]. In particular, (small) limits and colimits exist. Thus new ω-categories can be constructed from diagrams of old ones. Furthermore the underlying set functor preserves limits and filtered colimits; so, for example, products are easily constructed on the products of the underlying sets.

(d) The constructions of the above (a)–(c) can be combined. Let M denote a sequence $M_0, M_1, M_2, \ldots$ where M_0 is a set, M_1 is a monoid and M_n is a commutative monoid for $n > 1$. There is an ω-category

$$\mathcal{N}(M) = \prod_{n\in\omega} K(M_n, n)_c$$

which, for a particular choice of the sequence M, will be used later; a more explicit description will be given at that time.

(e) *The free ω-category 2_ω on a singleton set* is the unique ω-category on the set

$$(2\times\omega)\cup\{\omega\} = \{(p,m) \mid p=0,1; m=0,1,2,\dots\}\cup\{\omega\}$$

for which $s_n(\omega)=(0,n)$ and $t_n(\omega)=(1,n)$, see Fig. 3. The remaining equations which fully describe 2_ω are

$$s_n(p,m)=t_n(p,m)=(p,m) \qquad \text{for } m\leq n;$$

$$s_n(p,m)=(0,n), \quad t_n(p,m)=(1,n) \qquad \text{for } m>n;$$

$$s_n(p,m)=t_n(p',m') \quad \text{implies}$$

$$(p,m)*_n(p',m') = \begin{cases}(p,m) & \text{for } m<n,\\ (p',m') & \text{for } m'\leq n;\end{cases}$$

and

$$\omega *_n (0,n) = \omega = (1,n) *_n \omega .$$

For any ω-category A, each $a\in A$ determines a unique ω-functor $\hat{a}: 2_\omega \to A$ taking ω to a; indeed, $\hat{a}(0,m)=s_m(a)$, $\hat{a}(1,m)=t_m(a)$. So there is a natural bijection

$$A \cong \omega\text{-Cat}(2_\omega, A).$$

In other words, the underlying set functor ω-Cat$\to$Set is represented by 2_ω.

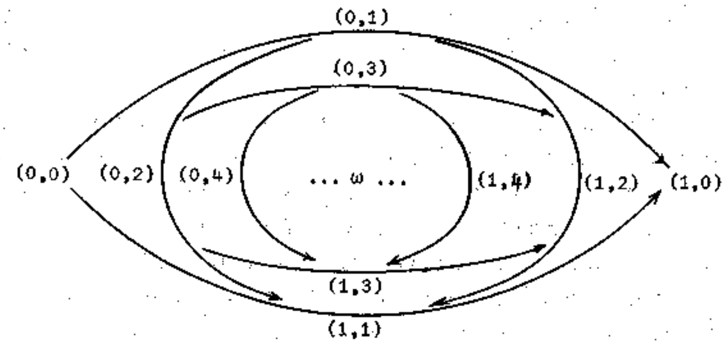


Fig. 3.

For ω-categories A, B, there is an *ω-functor ω-category* $[A, B]$ described as follows. An element of $[A, B]$ is an ω-functor $f: 2_\omega \times A \to B$. The nth source and target functions are given by

$$s_n(f)(p, m, a) = t_n(f)(p, m, a) = f(p, m, a) \quad \text{for } m \leq n\,,$$

and

$$s_n(f)(p, m, a) = s_n(f)(\omega, a) = f(0, n, a)\,,$$
$$t_n(f)(p, m, a) = t_n(f)(\omega, a) = f(1, n, a) \qquad \text{for } m > n\,.$$

The nth composition is, for $s_n(f) = t_n(g)$, given by

$$(f *_n g)(p, m, a) = \begin{cases} f(p, m, a) & \text{for } m \leq n\,, \\ f(p, m, a) *_n g(p, m, s_n(a)) & \text{for } m > n\,, \end{cases}$$
$$(f *_n g)(\omega, a) = f(\omega, a) *_n g(\omega, s_n(a))\,.$$

The asymmetry in the definition of composition is only apparent: the commutative diagram

$$\begin{array}{ccccc}
g(0, n, s_n(a)) & \xrightarrow{g(\omega, s_n(a))} & g(1, n, s_n(a)) = f(0, n, s_n(a)) & \xrightarrow{f(\omega, s_n(a))} & f(1, n, s_n(a)) \\
\downarrow{\scriptstyle g(0,n,a)} & \searrow{\scriptstyle g(\omega,a)} & \downarrow{\scriptstyle g(1,n,a) = f(0,n,a)} & \searrow{\scriptstyle f(\omega,a)} & \downarrow{\scriptstyle f(1,n,a)} \\
g(0, n, t_n(a)) & \xrightarrow[g(\omega, t_n(a))]{} & g(1, n, t_n(a)) = f(0, n, t_n(a)) & \xrightarrow[f(\omega, t_n(a))]{} & f(1, n, t_n(a))
\end{array}$$

yields the equality

$$f(\omega, a) *_n g(\omega, s_n(a)) = f(\omega, t_n(a)) *_n g(\omega, a)$$

from which also follows (by taking s_m, t_m) the same equality with ω replaced by (p, m) for $m > n$.

Theorem 1.4. *The category ω-Cat is Cartesian closed. Indeed, the natural bijection*

$$\omega\text{-Cat}(A \times B, C) \cong \omega\text{-Cat}(A, [B, C])$$

takes $f: A \times B \to C$ to $\hat{f}: A \to [B, C]$ where

$$\hat{f}(a) = (2_\omega \times B \xrightarrow{\hat{a} \times 1} A \times B \xrightarrow{f} C)\,.$$

Furthermore, if C is an r-category, then so is $[B, C]$. □

As a consequence of Theorem 1.4 it is possible to consider *categories $\mathscr{A}$ with homs enriched in* ω-Cat (in the sense of Eilenberg–Kelly [3]). Such an $\mathscr{A}$ consists of

- a set of objects;
- for objects u, v, an ω-category $\mathscr{A}(u, v)$;
- for objects u, v, w, a composition ω-functor $*: \mathscr{A}(v, w) \times \mathscr{A}(u, v) \to \mathscr{A}(u, w)$; and
- identity 0-cells $1_u \in \mathscr{A}(u, u)$;

such that $*$ is associative and the 1_u's are two-sided identities for $*$.

Each such $\mathscr{A}$ gives rise to an ω-category A defined as follows:

$$A = \{(u, a, v) \mid u, v \text{ are objects of } \mathscr{A}, a \in \mathscr{A}(u, v)\},$$
$$s_0(u, a, v) = u, \qquad t_0(u, a, v) = v,$$
$$(v, b, w) *_0 (u, a, v) = (u, b * a, v),$$
$$s_n(u, a, v) = s_{n-1}(a), \qquad t_n(u, a, v) = t_{n-1}(a),$$
$$(v, b, w) *_n (u, a, v) = (u, b *_{n-1} a, v).$$

Let (ω-Cat)-Cat denote the category whose arrows are enriched functors between small (ω-Cat)-enriched categories. Each enriched functor $F: \mathscr{A} \to \mathscr{B}$ clearly induces an ω-functor between the ω-categories obtained as above from $\mathscr{A}$, $\mathscr{B}$. This defines a functor

$$(\omega\text{-Cat})\text{-Cat} \to \omega\text{-Cat}.$$

Theorem 1.5. *The functor defined above is an equivalence*

$$(\omega\text{-Cat})\text{-Cat} \simeq \omega\text{-Cat}$$

which, for each cardinal r, restricts to an equivalence

$$(r\text{-Cat})\text{-Cat} \simeq (r+1)\text{-Cat}. \qquad \square$$

For 0-cells u, v in an ω-category A, Theorem 1.5 suggests the notation $A(u, v)$ for the ω-category whose underlying set is

$$\{a \in A \mid s_0(a) = u, t_0(a) = v\}$$

and whose compositions are the restrictions to this set of the compositions $*_1$, $*_2, \ldots$ of A (notice that $*_0$ is omitted).

In a set it is possible to discuss *equal* elements; in a category one has *isomorphic* objects; in a 2-category 0-cells can be *equivalent*. This leads to the recursive definition of what it means for 0-cells a, b in an ω-category A to be *r-equivalent*. For $r = 0$, this means $a = b$. For $r > 0$, this means there exist elements x, y of A satisfying the conditions

$$s_0(x) = t_0(y) = a\,, \qquad s_0(y) = t_0(x) = b\,,$$

$$y *_0 x,\ a \text{ are } (r-1)\text{-equivalent in } A(a, a),$$

and

$$x *_0 y,\ b \text{ are } (r-1)\text{-equivalent in } A(b, b).$$

From this the notion of r-equivalence is obtained. For this purpose it is illustrative to consider the diagram below, called *the exponential wedge*,

0
101
2120212
323132303231323
4342434143424340434243414342434
.
.
.

is which each row is obtained from the preceding row by wedging the next unused integer between the consecutive entries and putting it on each end. The ith entry in the rth row is $r(i) = r - e_i - 1$ where $i = 2^{e_i}k$ where k is odd.

Suppose u, v are 0-cells in A. An *r-equivalence* (x, y) *from* u *to* v is a pair of families of elements x_i, y_i of A, indexed by $i = 1, 2, 3, \ldots, 2^r - 1$, satisfying the following equations:

$$s_{r(i)}(x_i) = t_{r(i)}(y_i)\,, \qquad s_{r(i)}(y_i) = t_{r(i)}(x_i)\,,$$
$$u = y_1 *_{r(1)} x\,, \qquad x_i *_{r(i)} y_i = y_{i+1} *_{r(i+1)} x_{i+1}\,,$$
$$x_{2^r-1} *_{r(2^r-1)} y_{2^r-1} = v\,.$$

For example a 3-*equivalence from* u *to* v involves arrows

$$u \underset{y_4}{\overset{x_4}{\rightleftarrows}} v \qquad \text{in } (A, *_0)\,,$$

$$u \underset{y_2}{\overset{x_2}{\rightleftarrows}} y_4 *_0 x_4, \quad x_4 *_0 y_4 \underset{y_6}{\overset{x_6}{\rightleftarrows}} v \qquad \text{in } (A, *_1)\,,$$

$$u \underset{y_1}{\overset{x_1}{\rightleftarrows}} y_2 *_1 x_2, \quad x_2 *_1 y_2 \underset{y_3}{\overset{x_3}{\rightleftarrows}} y_4 *_0 x_4,$$

$$x_4 *_0 y_6 \underset{y_5}{\overset{x_5}{\rightleftarrows}} y_6 *_1 x_6, \quad x_6 *_1 y_6 \underset{y_7}{\overset{x_7}{\rightleftarrows}} v \qquad \text{in } (A, *_2)\,,$$

satisfying the eight equations

$$u = y_1 *_2 x_1, \qquad x_1 *_2 y_1 = y_2 *_1 x_2,$$

$$x_2 *_1 y_2 = y_3 *_2 x_3, \qquad x_3 *_2 y_3 = y_4 *_0 x_4,$$

$$x_4 *_0 y_4 = y_5 *_2 x_5, \qquad x_5 *_2 y_5 = y_6 *_1 x_6,$$

$$x_6 *_1 y_6 = y_7 *_2 x_7, \qquad x_7 *_2 y_7 = v.$$

Suppose (x, y) is an r-equivalence from u to v and (h, k) is an r-equivalence from v to w. An r-equivalence $(h, k) * (x, y)$ from u to w will be defined by recursion on r. For $r = 0$, the families x, y, h, k are empty and $u = v = w$, so $(h, k) * (x, y)$ is the unique 0-equivalence from u to w. For $r > 0$, put $p = 2^{r-1}$ and define $(a, b) = (h, k) * (x, y)$ as follows:

$$a_p = h_p *_0 x_p, \qquad b_p = y_p *_0 k_p,$$

$$(a_i, b_i)_{0<i<p} = (y_p *_0 h_i *_0 x_p, y_p *_0 k_i *_0 x_p)_{0<i<p} * (x_i, y_i)_{0<i<p},$$

$$(a_i, b_i)_{p<i<2^r} = (h_i, k_i)_{p<i<2^r} * (h_p *_0 x_i *_0 k_p, h_p *_0 y_i *_0 k_p)_{p<i<2^r}.$$

For any ω-category A, there is an *ω-category* r-eq A *of r-equivalences in* A defined as follows. The elements are quadruples (u, x, y, v) where u, v are 0-cells in A and (x, y) is an r-equivalence from u to v. Put $s_0(u, x, y, v) = u$, $t_0(u, x, y, v) = v$, and $(v, h, k, w) *_0 (u, x, y, v) = (u, (h, k) * (x, y), w)$. For $n > 0$, put $s_n(u, x, y, v) = (u, s_n(x), s_n(y), v)$, $t_n(u, x, y, v) = (u, t_n(x), t_n(y), v)$ and $(u, x', y', v) *_n (u, x, y, v) = (u, x' *_n x, y' *_n y, v)$ where $s_n(x)_i = s_n(x_i)$, $(x' *_n x)_i = x'_i *_n x_i$, and so on.

The function r-eq $A \to A$ which takes (u, x, y, v) to x_{2^r-1} is an ω-functor. An *r-groupoid* is an r-category A for which this function is surjective.

2. Orientals: the definition

As usual in algebraic topology, the ordinal $n + 1$ (which means $\{0, 1, \ldots, n\}$ as an ordered set) is denoted by $[n]$. Since ordered sets are examples of categories, there is a full subcategory Δ of Cat whose arrows are order-preserving functions between the ordered sets $[n]$ where $n \in \omega$. For each n there are $n + 1$ monics

$$\partial_0 \geq \partial_1 \geq \cdots \geq \partial_n : [n-1] \to [n]$$

such that the image of ∂_i does not contain i, and there are n epics

$$\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1} : [n] \to [n-1]$$

such that $\partial_i \dashv \sigma_i \dashv \partial_{i+1}$.

A simplicial set is a functor $X:\Delta^{op}\to \mathrm{Set}$. Each arrow $\alpha:[m]\to[n]$ in Δ gives a function $X_n\to X_m$ whose value at $x\in X_n$ is denoted by $x\alpha$. Elements of X_n are called *elements of* X *of dimension* n. The *ith face* of $x\in X_n$ is $x\partial_i$ which is called an *even* or *odd face* of x according as i is even or odd. (It is sometimes convenient to write $[-1]$ for the empty set and to take X_{-1} to be a singleton set, so that the unique function $X_0\to X_{-1}$ can be regarded as the value of X at the unique function $\partial_0:[-1]\to[0]$.) Call $x\in X_n$ *degenerate* when $x=y\sigma_i$ for some $y\in X_{n-1}$.

Each category A determines a simplicial set $\mathrm{Cat}(-,A):\Delta^{op}\to\mathrm{Set}$ called the *nerve* of A and denoted by ΔA. An element of ΔA of dimension n is a functor $a:[n]\to A$; for $\alpha:[m]\to[n]$, the element $a\alpha$ is the composite of a, α as functors. Such an a is non-degenerate if and only if it reflects identities; in the case where A is an ordered set this means a is monic.

The standard r-simplex is the nerve $\Delta[r]$ of the ordered set $[r]$. *The standard* ω*-simplex* is the nerve $\Delta\omega$ of the ordered set ω.

A simplicial map $f:X\to Y$ between simplicial sets X, Y is a natural transformation between the functors $X, Y:\Delta^{op}\to\mathrm{Set}$. Simplicial maps are the arrows for a category $[\Delta^{op},\mathrm{Set}]$.

The cardinality of a set z is denoted by $\#z$. For $x,y\subset z$, put $x-y=\{i\in x\,|\,i\not\in y\}$, and write $u=x+y$ when $u=x\cup y$ and $0=x\cap y$. The set of all finite subsets of z is denoted by $\mathcal{P}_f z$ and the set of subsets of z of cardinality n is denoted by $\binom{z}{n}$. There are obvious isomorphisms of sets

$$\mathcal{P}_f z\cong\sum_{n\in\omega}\binom{z}{n},\qquad \binom{x+y}{n}\cong\sum_{h+k=n}\binom{x}{h}\times\binom{y}{k}.$$

The set ω of finite ordinals is well ordered; each subset $u=\{u_0,u_1,\dots,u_n\}$ of ω can be organized so that $u_0<u_1<\cdots<u_n$ yielding an order-preserving monic $u:[n]\to\omega$ whose value at i is the element u_i of u with i predecessors; and write $u=(u_0,u_1,\dots,u_n)$. In this way the set $\binom{\omega}{n+1}$ is identified with the non-degenerate elements of $\Delta\omega$ of dimension n. For $u,v\in\binom{\omega}{n+1}$, write $u\le v$ when $u_i\le v_i$ for all $i\in[n]$.

For $r\subset\omega$ and $x\in\omega$, put $x/r=\{k\in x\,|\,k\le r\}$. In the case where $r\in x$, the set $x-\{r\}$ is identified with the monic $x\partial_i$ where $\#x/r=i+1$; that is $x\partial_i=(x_0,x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m-1},x_m)$ where $x=(x_0,x_1,\dots,x_m)$ and $r=x_i$.

A subset S of $\binom{\omega}{m+1}$ is called *well formed* when distinct elements of S have no common faces of the same parity; that is,

$$\text{(WF)}\qquad x,y\in S,\ x\partial_i=y\partial_j,\ i+j\ \text{even}\ \Rightarrow\ x=y\,.$$

For $m=0$ this is taken to mean that S is a singleton.

The set of faces from $S\subset\binom{\omega}{n+1}$ *of parity* p $(=0$ or $1)$ is defined by

$$\phi^p S=\{x\partial_i\,|\,x\in S,\ i+p\ \text{even}\}\,.$$

If S is well formed, each element of $\phi^1 S$ is an odd face of a unique element of S; however, odd faces can be even faces. Define

$$\rho^1 S = \phi^1 S - \phi^0 S \quad \text{and} \quad \rho^0 S = \phi^0 S - \phi^1 S .$$

So $\rho^1 S$ is the set of odd faces from S which do not occur as even faces.

The objective is to define the nerve of an ω-category; this should be a functor

$$\omega\text{-Cat} \to [\Delta^{\text{op}}, \text{Set}]$$

with a left adjoint. By a general categorical argument due to Kan [5], this amounts to defining a functor

$$\mathcal{O} : \Delta \to \omega\text{-Cat}$$

which will be the composite of the left adjoint with the Yoneda-embedding $\Delta \to [\Delta^{\text{op}}, \text{Set}]$. *The nth oriental* will be the n-category $\mathcal{O}_n$ obtained as the value of $\mathcal{O}$ at $[n] \in \Delta$. The technique will be to define a large ω-category $\mathcal{N}_\omega$ for which compositions are easily described, to cut down to a sub-ω-category $\mathcal{O}_\omega$ of $\mathcal{N}_\omega$, and to obtain $\mathcal{O}_n$ as a sub-n-category of $\mathcal{O}_\omega$.

The ω-category $\mathcal{N}_\omega$ is $\mathcal{N}(M)$ (as in Example 1.3(d)) where M_n is the commutative monoid $\mathcal{P}_{\text{f}}\binom{\omega}{n+1}$ whose composition is binary union. More explicitly, the elements of $\mathcal{N}$ are families

$$a = (a_n^1, a_n^0)_{n \in \omega}$$

where a_n^p is a finite subset of $\binom{\omega}{n+1}$. Sources and targets are given by the equations

$$s_n(a)_m^q = \begin{cases} a_m^q & \text{for } m < n , \\ a_n^1 & \text{for } m = n , \\ 0 & \text{for } m > n ; \end{cases} \qquad t_n(a)_m^q = \begin{cases} a_m^q & \text{for } m < n , \\ a_n^0 & \text{for } m = n , \\ 0 & \text{for } m > n . \end{cases}$$

For $s_n(a) = t_n(b)$, the composite $a *_n b$ is given by the rule

$$(a *_n b)_m^q = \begin{cases} a_m^q = b_m^q & \text{for } m < n , \\ b_n^1 & \text{for } q = 1 \text{ and } m = n , \\ a_n^0 & \text{for } q = 0 \text{ and } m = n , \\ a_m^q \cup b_m^q & \text{for } m > n . \end{cases}$$

Notice that an n-cell a is determined by $s_{n-1}(a)$, $t_{n-1}(a)$ and a single finite subset $a_n^1 = a_n^0$ of $\binom{\omega}{n+1}$, see Fig. 4.

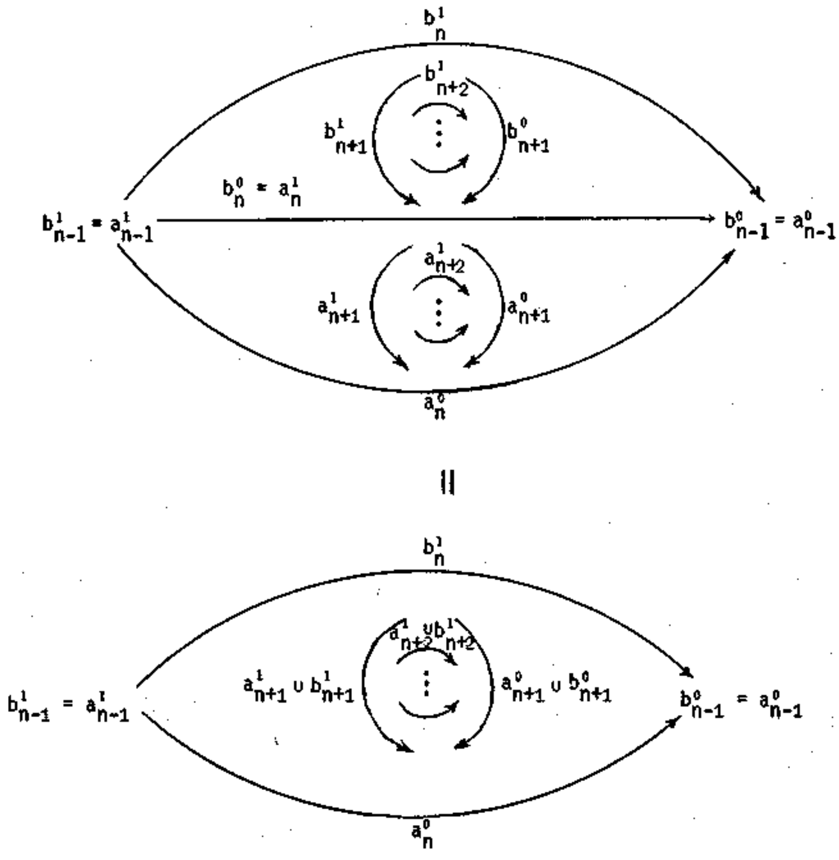


Fig. 4.

Define $\mathcal{O}_\omega$ to be the subset of $\mathcal{N}_\omega$ which consists of the cells a of $\mathcal{N}_\omega$ satisfying the following conditions for all $p, q \in [1]$, $m \in \omega$:

$$a_m^p \text{ is well formed;} \tag{1}$$

$$a_m^p = \rho^p a_{m+1}^q + a_m^0 \cap a_m^1 . \tag{2}$$

It is clear that the source and target functions of $\mathcal{N}_\omega$ restrict to $\mathcal{O}_\omega$. What is not at all obvious is that $\mathcal{O}_\omega$ is closed under the compositions of $\mathcal{N}_\omega$ and is hence an ω-category. The proof of this will be an excursion into combinatorics which will reveal the inner structure of $\mathcal{O}_\omega$. Before embarking, some remarks on the 2-category structure of $\mathcal{O}_\omega$ are in order.

To say $a \in \mathcal{N}_\omega$ is a 0-cell is to say $a_0^1 = a_0^0 \subset \binom{\omega}{1}$ and $a_m^p = 0$ for $m > 0$; to say a is also in $\mathcal{O}_\omega$ is to say a_0^1 is a singleton. So the 0-cells in $\mathcal{O}_\omega$ can be identified with elements of ω; they are in fact singleton sets of singleton sets of ordinals.

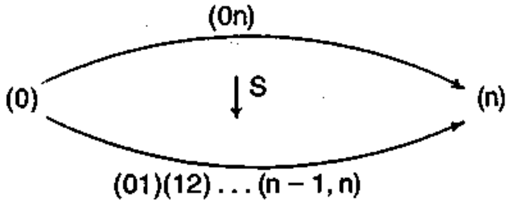


Fig. 5.

To say $a \in \mathcal{N}_\omega$ is a 1-cell is to say $a_1^1 = a_1^0 \subset \binom{\omega}{2}$ and $a_m^p = 0$ for $m > 1$; to say a is also in $\mathcal{O}_\omega$ is to say $a_0^1 = \{(h)\}$, $a_0^0 = \{(k)\}$, $a_1^1 = \{(h, h_1), (h_1, h_2), \ldots, (h_r, k)\}$ where $h < h_1 < h_2 \cdots < h_r < k$ in ω. So the 1-cells in $\mathcal{O}_\omega$ can be identified with arrows in the free category on the graph

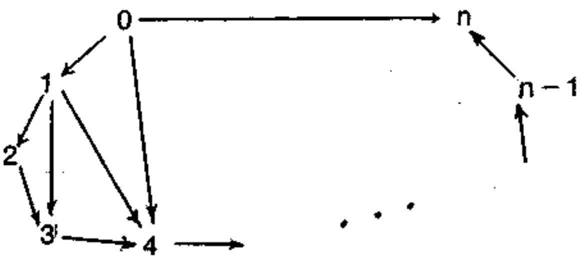
$$s, t : \binom{\omega}{2} \to \omega \quad \text{where } s(u) = u_0,\ t(u) = u_1 .$$

The 2-cells a of $\mathcal{N}_\omega$ which are in $\mathcal{O}_\omega$ and have $a_0^1 = \{(0)\}$, $a_0^0 = \{(n)\}$, $a_1^1 = \{(0, n)\}$, $a_1^0 = \{(0, 1), (1, 2), \ldots, (n-1, n)\}$ can be identified with *meaningful ways of inserting pairs of brackets in a word of n letters*, see Fig. 5.

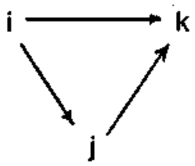
The set $S = a_2^1 = a_2^0$ which remains to be given must be well formed and satisfy

$$\rho^1 S = \{(0\, n)\}, \qquad \rho^0 S = \{(0\, 1), (1\, 2), \ldots, (n-1, n)\}.$$

Such a set S can be depicted by a diagram of the form



in which each element $(i\, j\, k)$ of S appears as a triangle



The arrows joining the outside consecutive ordinals are then labelled by the letters in the word. For example, take $n=6$ and $S=\{(0\,1\,6), (1\,2\,4), (1\,4\,6), (4\,5\,6)\}$. The diagram for S gives the associated way of bracketing the word ABCDEF, see Fig. 6.
The number of meaningful bracketings of an n-letter word is $(1/n!)2^{n-1}(2n-3)(2n-5)\cdots 3\cdot 1$ and, according to A. Joyal, has the name of Catalan associated with it. Starting with $n=2$, one obtains the sequence

$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots .$$

The elements of $\mathcal{O}_n$ are those $a \in \mathcal{O}_\omega$ for which $a^p_m \subset \binom{[n]}{m+1}$ for all $p \in [1]$ and $m \in \omega$ (although it suffices to have this for $m=0$ as will become clear).

3. Orientals: their structure

Each finite subset z of ω determines an element $\langle z\rangle$ of N_ω given by

$$\langle z\rangle^p_m = \left\{ x \in \binom{z}{m+1} \,\middle|\, r \in z - x \;\Rightarrow\; \#x/r \text{ has parity } p \right\}$$

for $p=0,1$ and $m \in \omega$. In particular, for $z=(z_0, z_1, \ldots, z_n)$, this means

$$\langle z\rangle^p_m = 0 \quad \text{for } m>n\,, \qquad \langle z\rangle^p_n = \{z\} \quad \text{for } p=0 \text{ and } p=1\,,$$

$$\langle z\rangle^p_{n-1} = \{z\partial_i \mid i+p \text{ is even}\}\,, \qquad \{z\}^1_0 = \{\{z_0\}\}\,, \qquad \{z\}^0_0 = \{\{z_n\}\}\,.$$

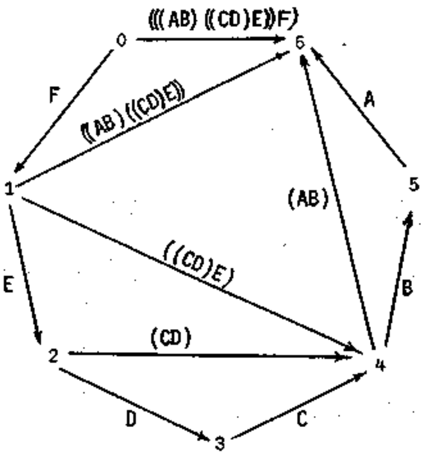


Fig. 6.

Proposition 3.1. *Each* $\langle z\rangle$ *is in* $\mathcal{O}_\omega$. *In fact,* $\langle z\rangle^p_m$ *is well formed and* $\rho^q\langle z\rangle^p_m = \langle z\rangle^q_{m-1}$ for $m \le n$.

Proof. Suppose $x, y \in \langle z\rangle^p_m$ are distinct, $x\partial_i = y\partial_j$ and $i<j$. Then $x_i \in z-y$, $y_j \in z-x$, $\#y/x_i = i$, $\#x/y_j = j+1$. So $i, j+1$ have parity p. So $i+j$ is odd. This proves that $\langle z\rangle^p_m$ is well formed.

Notice that $w+\{i\} \in \langle z\rangle^p_m$ if and only if for $k \in z-w$, the parity of $\#w/k$ is p for $k<i$ and not p for $k>i$.

Suppose p is not q. Take $w \in \rho^q\langle z\rangle^p_m$. If $\{k \in z-w \mid k<i\}$ is non-empty, then its largest element j has $w+\{j\} \in \langle z\rangle^p_m$ and $\#w/j$ of parity p contrary to $w \notin \phi^p\langle z\rangle^p_m$. So $\{k\in z-w \mid k<i\}$ is empty and $w \in \langle z\rangle^q_{m-1}$. Conversely, if $w \in \langle z\rangle^p_{m-1}$, then the first element i of $z-w$ is unique such that $w+\{i\} \in \langle z\rangle^p_m$; moreover, the parity of $\#w/i$ is p; so $w \in \rho^q\langle z\rangle^p_m$.

Suppose p is q. Take $w \in \rho^q\langle z\rangle^p_m$. If $\{k \in z-w \mid k>i\}$ is non-empty, then its smallest element j has $w+\{j\} \in \langle z\rangle^p_m$ and $\#w/j$ of parity not p contrary to $w \in \rho^q\langle z\rangle^p_m$. So $\{k \in z-w \mid k>i\}$ is empty and $w \in \langle z\rangle^q_{m-1}$. Conversely, if $w \in \langle z\rangle^q_{m-1}$, then the last element i of $z-w$ is unique such that $w+\{i\} \in \langle z\rangle^p_m$; moreover, $\#w/i$ has parity q; so $w \in \rho^q\langle z\rangle^p_m$. □

Consequently the set $\mathcal{O}_\omega$ is not trivial. The elements $\langle z\rangle$ are in fact the building blocks for $\mathcal{O}_\omega$: *it will turn out that* $\mathcal{O}_\omega$ *is the smallest sub-ω-category of* $\mathcal{N}_\omega$ *which contains all the elements* $\langle z\rangle$. At this stage it is not even clear that, for $z \in \binom{\omega}{n+1}$, the $(n-1)$-cell $s_{n-1}\langle z\rangle$ is a composite of cells $\langle u\rangle$ with $\#u \le n$.

Suppose S is a subset of $\binom{\omega}{m+1}$. Let $\lhd$ denote the smallest reflexive transitive relation on S which has $x<y$ if $x\partial_i = y\partial_j$ with i even and j odd. For $m=0$, this is understood to be the equality relation. For $m=1$, if $x \lhd y$ in S, then $x \le y$ in $\binom{\omega}{2}$ (that is, $x_0 \le y_0$ and $x_1 \le y_1$). For $m>1$ the relation $\lhd$ is more involved.

Lemma 3.2. (a) *The relation* $\lhd$ *on any* $S \subset \binom{\omega}{m+1}$ *is antisymmetric.*

(b) $x \lhd y$ *in* $\langle z\rangle^1_m$ *if and only if* $z-x \le z-y$.

(c) $x \lhd y$ *in* $\langle z\rangle^0_m$ *if and only if* $z-y \le z-x$.

Proof. (a) If $x \lhd y \lhd z$ in S, then the last element y_m of y is between (or equal to) the last elements x_m, z_m of x, z. If $x_m = y_m$, then $x \lhd y$ in S implies $x\partial_m \lhd y\partial_m$ in $\{u\partial_m \mid u \in S\}$. By induction on m, $x \lhd y \lhd x$ implies $x=y$.

(b) Suppose $x, y \in \{z\}^1_m$ with $u = x\partial_i = y\partial_j$ where i is even and j is odd. If $k \in z-u$, then $\#u/k$ is odd when either $k<x_i$ or $k>y_j$, and, $\#u/k$ is even when either $k>x_i$ or $k>y_j$. Thus $y_j<x_i$ and $z-u$ has no elements between y_j and x_i. Hence $z-x = (z-u)+\{y_j\} \le (z-u)+\{x_i\} = z-y$.

Conversely, assume $z-x \le z-y$. To prove $x \lhd y$ induction on $r = \sum_{k\in z-y}(\#(z-x)/k - \#(z-y)/k)$ will be used. If $r=0$, then $z-x = z-y$ so $x=y$. If $r>0$, the first element of $z-y$ such that any larger k has $\#(z-y)/k = \#(z-x)/k$ must be an element x_i of x. Put $u = x\partial_i$ and let $h \in z-x$ be such that

$\#(z-x)/h = \#(z-y)/x_i$. Then $w = u + \{h\} \in \langle z \rangle_m^1$, $x \lhd w$ and, by induction, $w \lhd y$. So $x \lhd y$.

(c) This is similar to (b). $\square$

The next combinatorial tool which must be introduced is the function

$$\mathrm{al}: \binom{z}{m+1} \to \mathscr{P}z$$

given by $\mathrm{al}(x) = \{x_i \mid m - i \text{ odd}\}$. Notice that for $u \in \binom{z}{m+2}$ and $m - j$ even, one has

$$\mathrm{al}(u\partial_j) = \mathrm{al}(u\partial_{j+1}).$$

A subset S of $\binom{z}{m+1}$ is said to satisfy the *alternating position condition* when the restriction of al to S is monic; that is

(AP) $\qquad x, y \in S, \mathrm{al}(x) = \mathrm{al}(y) \;\Rightarrow\; x = y$.

Lemma 3.3. *Each* $\langle z \rangle_m^p$ *satisfies* (AP).

Proof. Take distinct $x, y \in \langle z \rangle_m^p$ and assume $\mathrm{al}(x) = \mathrm{al}(y)$. Then there exists j with $x_j \neq y_j$; say $x_j < y_j$. Then $\#y/x_j = j$, $\#x/y_i = j+1$. But $x_j \in z - y$, $y_j \in z - x$ imply $\#y/x_j$ and $\#x/y_j$ have the same parity p, a contradiction. $\square$

The interplay between well formedness and the alternating position condition is made clear by the next two lemmas which concern the process of replacing faces of one parity by faces of the other.

Lemma 3.4. *Suppose* $S \subset \binom{z}{m+1}$, *satisfies* (AP) *and* $u \in \binom{z}{m+2}$ *has* $\langle u \rangle_m^0 \subset S$. *Then*

(a) $S \cap \langle u \rangle_m^1$ *is empty;*

(b) $T = (S - \langle u \rangle_m^0) + \langle u \rangle_m^1$ *satisfies* (AP);

(c) $\rho^p T = \rho^p S$ *for* $p = 0, 1$; *and*

(d) *if* S *is well formed and* $\rho^p S \subset \langle z \rangle_{m-1}^p$ *for* $m - p$ *odd, then* T *is well formed.*

Proof. (a) For m even, $\mathrm{al}: \langle u \rangle_m^0 \to \mathscr{P}z$ and $\mathrm{al}: \langle u \rangle_m^1 \to \mathscr{P}z$ have the same image. For m odd, the image of the former contains precisely one more element (namely, $\mathrm{al}(u\partial_0)$) than the image of the latter. So $y \in \langle u \rangle_m^1$ implies there exists $x \in \langle u \rangle_m^0$ with $\mathrm{al}(y) = \mathrm{al}(x)$; but then $x \in S$, so $y \notin S$ by (AP). So (a) follows.

(b) For m even, T has the same cardinality as S and $\mathrm{al}: T \to \mathscr{P}z$, $\mathrm{al}: S \to \mathscr{P}z$ have the same image. For m odd, T has cardinality one less than S and the image of al: $T \to \mathcal{P}z$ has one fewer element than the image of al: $S \to \mathcal{P}z$. So T and the image of al: $T \to \mathcal{P}z$ have the same cardinality. So T satisfies (AP).

(c) Assume $w \in \rho^p T - \rho^p S$. Either w is a p-parity face of $S - \langle u \rangle_m^0$ or of $\langle u \rangle_m^0$. In the former case w must also be a face of S of parity $q \neq p$ (or else $w \in \rho^p S$); since w is not a q-parity face of T, it must be a q-parity face of $\langle u \rangle_m^0$. If w is a p-parity face of $\langle u \rangle_m^0$, then the well-formedness of S is contradicted since w is a p-parity face of $S - \langle u \rangle_m^0$. So $w \in \rho^q \langle u \rangle_m^0 = \rho^q \langle u \rangle_m^1 \subset \phi^q T$ contrary to $w \in \rho^q T$. Thus w must be a p-parity face of $\langle u \rangle_m^1$. If w is a q-parity face of $\langle u \rangle_m^1$, then w is a q-parity face of T, contrary to $w \in \rho^p T$. So w is not a q-parity face of $\langle u \rangle_m^1$. So $w \in \rho^p \langle u \rangle_m^1 = \rho^p \langle u \rangle_m^0$. So w is a p-parity face of S. If w is a q-parity face of S then, since $w \in \rho^p T$, it would have to be a q-parity face of $\langle u \rangle_m^0$ contrary to $w \in \rho^p \langle u \rangle_m^0$. So $w \in \rho^p S$, a contradiction. This proves $\rho^p T \subset \rho^p S$. Since $S = (T - \langle u \rangle_m^1) + \langle u \rangle_m^0$, one obtains the reverse inclusion by interchanging $\langle u \rangle_m^0$ and $\langle u \rangle_m^1$ in the above argument.

(d) This part does not have such a direct proof. Suppose $a \in S$, $m - k$ is even, and p is not the parity of m. It will be required to know that, if $v \in \phi^p S$ satisfies the condition (*) below, then $v = a\partial_k$.

(*) For all $e \in [m]$ with $m - e$ even, if $e < k$, then $v_{e-1} = a_{e-1} < v_e \leq a_e$, while if $e > k$, then $v_{e-2} = a_{e-1} < v_{e-1} \leq a_e$.

What must be shown is that the set

$$X = \{x \in S \mid x\partial_i \text{ satisfies } (*) \text{ and } x\partial_i \neq a\partial_k \text{ for some odd } m - i\}$$

is empty. Since S is finite, X non-empty implies there is an $x \in X$ which can be taken to be minimal or maximal with respect to the order $\lhd$ on S; take the former for m even and the latter for m odd. So there is $m - i$ odd with $v = x\partial_i$ satisfying (*) and $v \not\in a\partial_k$. Since $a_k \in z - v$ and $\#v/a_k = k$ is not of parity p, one has $v \not\in \langle z \rangle_{m-1}^p$. So $v \not\in \rho^p S$. So $v = y\partial_j$ for some $y \in S$ with $m - j$ even. For m even, $y \lhd x$; while for m odd, $x \lhd y$. To obtain a contradiction to the choice of x it remains to show $y \in X$. If $j = k$, then $v = y\partial_k$ so (*) gives $\mathrm{al}(y) = \mathrm{al}(a)$; so (AP) gives $y = a$; so $v = a\partial_k$, a contradiction. For $j < k$, put $w = y\partial_{j+1}$. Then $v_e = w_e$ for $e \neq j$ while $w_j = y_j < y_{j+1} = v_j$, from which it follows that w satisfies (*); so $y \in X$. For $j > k$, put $w = y\partial_{j-1}$. Then $v_e = w_e$ for $e \neq j - 1$ while $v_{j-1} = y_j < y_{j+1} = w_{j-1}$, from which it follows that w satisfies (*); so $y \in X$.

The proof of (d) can now be given. To see that T is well formed, take $x, y \in T$ with $x\partial_i = y\partial_j$, $i + j$ even. Since S and $\langle u \rangle_m^1$ are well formed, it remains to consider the case where $x \in S - \langle u \rangle_m^0$ and $y = u\partial_r$ with r odd and to obtain a contradiction. In this case,

$$x\partial_i = y\partial_j = u\partial_r\partial_j = \begin{cases} u\partial_j\partial_{r-1} & \text{for } j < r\,, \\ u\partial_{j+1}\partial_r & \text{for } r \leq j\,. \end{cases}$$

If $j < r$ and j is even, then $i + r - 1$ is even so, since S is well formed, $x = u\partial_j$

contrary to $x \not\in \langle u \rangle^0_m$. If $r \le j$ and j is odd, then $i + r$ is even so, since S is well formed, $x = u\partial_{j+1}$ contrary to $x \not\in \langle u \rangle^0_m$. The remaining cases have $x\partial_i = u\partial_s\partial_h$ where $h < s$ and h, s are odd. If m and i are even, then $\#(x\partial_i/u_k) = h$ which is odd, so $x\partial_i \not\in \langle z \rangle^0_{m-1}$, so $x\partial_i \not\in \rho^0 S$, so $x\partial_i = x'\partial_{i'}$, for some $x' \in S$ and i' odd. If m and i are odd, then $\#(x\partial_i/u_s) = s - 1$ which is even, so $x\partial_i \not\in \langle z \rangle^1_{m-1}$, so $x\partial_i \not\in \rho^1 S$, so $x\partial_i = x'\partial_{i'}$ for some $x' \in S$ and i' even. Consequently, it can be assumed that $m - i$ is odd. For m even, put $a = u\partial_{h-1}$ and $k = s - 1$. For m odd, put $a = u\partial_{s+1}$ and $k = h$. Then $v = x\partial_i \in \phi^p S$ satisfies condition (*) above. So $u\partial_s\partial_h = x\partial_i = v = a\partial_k = u\partial_{s'}\partial_{h'}$ where $h' < s'$ and $s' - h'$ is odd; a contradiction. □

Lemma 3.5. *Suppose $S^0 \subset \binom{z}{m+1}$ satisfies* (AP) *and $u^1, u^2, \ldots, u^r \in \binom{z}{m+2}$ are such that $\langle u^i \rangle^0_m \subset S^{i-1}$ where $S^i = (S^{i-1} - \langle u^i \rangle^0_m) + \langle u^i \rangle^1_m$ for $i = 1, \ldots, r$. Then $\{u^1, u^2, \ldots, u^r\} \subset \binom{z}{m+2}$ is well formed.*

Proof. Induction will be used to prove $S^r \cap \langle u^1 \rangle^0_m = 0$ for $r > 0$. The case $r = 1$ is Lemma 3.4(a). Suppose $r > 1$ and $w \in S^r \cap \langle u^1 \rangle^0_m$. Then $w \in S^r = (S^{r-1} - \langle u^r \rangle^0_m) + \langle u^r \rangle^1_m$; so $w \in \langle u^r \rangle^1_m$ since $w \not\in S^{r-1}$ by the inductive assumption. So $w \in \langle u^1 \rangle^0_m \cap \langle u^r \rangle^1_m$. This implies $u^1 \lhd u^r$ in $\binom{z}{m+2}$. Since $\mathrm{al}(w) \in \mathrm{al}\, S^r \subset \mathrm{al}\, S^1$, $w \in \langle u^1 \rangle^0_m$, and S^1 satisfies (AP), there exists $v \in \langle u^1 \rangle^1_m$ with $\mathrm{al}(v) = \mathrm{al}(w)$. Since $\mathrm{al}\langle u^r \rangle^0_m \supset \mathrm{al}\langle u^r \rangle^1_m \ni \mathrm{al}(w)$, there exists $w' \in \langle u^r \rangle^0_m$ with $\mathrm{al}(w') = \mathrm{al}(w)$. So $\{s \mid 1 \le s,\ u^1 \lhd u^s, \exists a \in \langle u^s \rangle^0_m$ with $\mathrm{al}(a) = \mathrm{al}(w)\}$ contains r and so has a least element $s \le 1$ with $u^1 \lhd u^s$ and $a \in \langle u^s \rangle^0_m$ having $\mathrm{al}(a) = \mathrm{al}(w)$. If $a \in \langle u^t \rangle^1_m$ for some $t < s$, then $u^s \lhd u^t$ and there is $b \in \langle u^t \rangle^0_m$ with $\mathrm{al}(b) = \mathrm{al}(a)$; so $\mathrm{al}(b) = \mathrm{al}(w)$ and $u^1 \lhd u^t$ contrary to the choice of s^1. So $a \not\in \langle u^t \rangle^1_m$ for all $t < s$. But $a \in \langle u^s \rangle^0_m \subset S^{s-1} \subset S^1 + \langle u^2 \rangle^1_m + \cdots + \langle u^{s-1} \rangle^1_m$; so $a \in S^1 = (S^0 - \langle u^1 \rangle^0_m) + \langle u^1 \rangle^1_m$.

By (AP) for S^1, one obtains $a = v$. So $a \in \langle u^s \rangle^0_m \cap \langle u^1 \rangle^1_m$. So $u^s \lhd u^1$. So (Lemma 3.2(a)) $u^s = u^1$. So $a \in \langle u^1 \rangle^0_m \cap \langle u^1 \rangle^1_m$, a contradiction. Hence $S^r \cap \langle u^1 \rangle^0_m$ is empty, as asserted.

Induction will be used to prove $S^1 \cap \langle u^r \rangle^1_m = 0$ for $r > 1$. For $r = 2$, Lemma 3.4(a) applies. Suppose $r > 2$ and $w \in S^1 \cap \langle u^r \rangle^1_m$. Then $w \in S^1 = (S^2 - \langle u^2 \rangle^1_m) + \langle u^2 \rangle^0_m$. By induction, $w \not\in S^2$. So $w \in \langle u^r \rangle^1_m \cap \langle u^2 \rangle^0_m \subset \langle u^r \rangle^1_m \cap S^r = 0$ by the first result; a contradiction.

The above results together with $\langle u^r \rangle^0_m \subset S^r$, $\langle u^1 \rangle^1_m \subset S^1$ yield $\langle u^1 \rangle^p_m \cap \langle u^r \rangle^p_m = 0$ for $p = 0, 1$. The lemma follows. □

Lemma 3.6. *Suppose $U \subset \binom{z}{m+2}$ is well formed with $\rho^0 U \subset S^0 \subset \binom{z}{m+1}$. Then the elements of U can be listed $u^1, u^2, \ldots, u^r$ such that $\langle u^i \rangle^0_m \subset S^{i-1}$, $S^i = (S^{i-1} - \langle u^i \rangle^0_m) + \langle u^i \rangle^1_m$ for $i = 1, \ldots, r$, $S^r = (S^0 - \rho^0 U) + \rho^1 U$, and, $u^i \lhd u^j$ in U implies $j \le i$.*

Proof. This will be proved by induction on $\#U$. If U is empty, then $S^0 = (S^0 - \rho^0 U) + \rho^1 U$. Otherwise, choose $u \in U$ which is minimal with respect to the order $\lhd$ on U. Put $V = U - \{u\}$ and observe that, because U is well formed and u is minimal, one has $\rho^0 U = \rho^0 V + \langle u \rangle^0_m$, $\rho^1 U = (\rho^1 V - \langle u \rangle^0_m) + \langle u \rangle^1_m$. Since

$V \subset U$ is well formed and $\rho^0 V = \rho^0 U - \langle u \rangle_m^0 \subset S^0$, the inductive assumption yields a listing $u^1, u^2, \ldots, u^{r-1}$ of the elements of V such that $\langle u^i \rangle_m^0 \subset S^{i-1}$, $S^i = (S^{i-1} - \langle u^i \rangle_m^0) + \langle u^i \rangle_m^1$ for $i = 1, \ldots, r-1$, $S^{r-1} = (S^0 - \rho^0 V) + \rho^1 V$, and $u^i \lhd u^j$ in V implies $j \leq i$. But $\langle u \rangle_m^0 \subset \rho^1 V \subset S^{r-1}$, so put $u^r = u$ and $S^r = (S^{r-1} - \langle u^r \rangle_m^0) + \langle u^r \rangle_m^1$. Since u^r is minimal, $i < r$ implies $u^i \not\lhd u^r$. Also observe that

$$
\begin{aligned}
S^r &= (S^{r-1} - \langle u \rangle_m^0) + \langle u \rangle_m^1 \\
&= (((S^0 - \rho^0 V) + \rho^1 V) - \langle u \rangle_m^0) + \langle u \rangle_m^1 \\
&= (S^0 - (\rho^0 V + \langle u \rangle_m^0)) + (\rho^1 V - \langle u \rangle_m^1) + \langle u \rangle_m^1 \\
&= (S^0 - \rho^0 U) + \rho^1 U . \qquad \square
\end{aligned}
$$

Lemma 3.7. *If $S \subset T \subset \binom{z}{m+1}$ with T well formed and $\rho^p S = \rho^p T$ for $p = 0$ and 1, then $S = T$.*

Proof. Suppose $T - S$ is non-empty and let x be an element which is maximal with respect to $\lhd$. If $x\partial_0 \in \rho^0 T = \rho^0 S$, then there exists $y \in S$ with $x\partial_0$ an even face of y; since T is well formed, $x = y$ which is contrary to $x \not\in S$. Hence $x\partial_0 \not\in \rho^0 T$. So $x\partial_0 = u\partial_i$ for some $u \in T$ and i odd. So $x \lhd u$, $x \neq u$. By maximality of x, it follows that $u \in S$. Since T is well formed and $x \not\in S$, it follows that $u\partial_i \in \rho^1 S = \rho^1 T$ contrary to $u\partial_i = x\partial_0$. $\square$

For $z \in \mathcal{P}_f\omega$, subsets $C_m^k z$ or $\mathcal{P}\binom{z}{m+1}$ are recursively defined as follows:

$$
C_m^0 z = \{\langle z \rangle_m^0\},
$$

$$
C_m^k z = \left\{ (S - \langle u \rangle_m^0) + \langle u \rangle_m^1 \,\middle|\, u \in \binom{z}{m+2}, \langle u \rangle_m^0 \subset S \in C_m^{k-1} z \right\}.
$$

That this makes sense follows from Lemma 3.3 and Lemma 3.4(a), (b). Furthermore, define

$$
C_m z = \bigcup_k C_m^k z,
$$

and

$$
D_m z = \left\{ S \subset \binom{z}{m+1} \,\middle|\, S \text{ is well formed and } \rho^p S = \langle z \rangle_{m-1}^p \text{ for } p = 0, 1 \right\}.
$$

The relationship between $\mathcal{O}_\omega$ and $D_m z$ is clearly as follows:

an m-cell a of $\mathcal{N}_\omega$ with $a_k^p = \langle z \rangle_k^p$ for $k < m$ and $p = 0, 1$ is in $\mathcal{O}_\omega$ if and only if $a_m^1 = a_m^0 \in D_m z$.

The goal is to prove $D_m z = C_m z$ and so obtain a more constructive description of elements of $\mathcal{O}_\omega$.

Proposition 3.8. (a) *Each element $C_m z$ satisfies* (AP).

(b) $C_m z \subset D_m z$.

(c) $C_m z = \{(\langle z \rangle_m^0 - \rho^0 U) + \rho^1 U \mid U \subset (\,{}_{m+2}^{\,z}\,)$ *well formed*, $\rho^0 U \subset \langle z \rangle_m^0\}$.

(d) $\langle z \rangle_m^1 \in C_m z$.

(e) *If* $U, V \subset (\,{}_{m+2}^{\,z}\,)$ *are well formed with* $\rho^0 U \subset S \in C_m z$ *and* $\rho^0 V \subset (S - \rho^0 U) + \rho^1 U$, *then* U, V *are disjoint*, $U + V$ *is well formed, and* $u \in U$, $v \in V$ *imply* $v \not\lhd u$ *in* $U + V$.

Proof. (a) Lemma 3.3 and Lemma 3.4(b).

(b) Proposition 3.1 and Lemma 3.4(c) , (d).

(c) $C_m z$ is a subset of the right-hand side by Lemma 3.5. The reverse inclusion is obtained from Lemma 3.6.

(d) Recalling Proposition 3.1, take $U = \langle z \rangle_{m+1}^0$ (or $\langle z \rangle_{m+1}^1$) in (c).

(e) Lemmas 3.5 and 3.6. □

Lemma 3.9. *If* $S \in D_m z$ *and* $p \neq q$, *then there exists a well formed* $U \subset (\,{}_{m+2}^{\,z}\,)$ *such that* $\rho^p U \subset \langle z \rangle_m^p$ *and* $S = (\langle z \rangle_m^p - \rho^p U) + \rho^q U$.

Proof. It will be shown to begin with that the lemma follows from the truth of the following statement for given z, m:

(∗) if $S \in D_m z$ and $p \neq q$, then either $S \subset \langle z \rangle_m^p$ or $\langle u \rangle_m^q \subset S$ for some $u \in (\,{}_{m+2}^{\,z}\,)$.

Note that, by Lemma 3.7, the inclusion $S \subset \langle z \rangle_m^p$ in (∗) implies the equality $S = \langle z \rangle_m^p$.

Assume (∗) is true. To prove the lemma take $S \in D_m z$. By iterated application of (∗) with $p = 0$ one obtains $u^1, \ldots, u^r \in (\,{}_{m+2}^{\,z}\,)$ such that $(\cdots(S - \langle u^1 \rangle_m^1) + \cdots) + \langle u^r \rangle_m^0 = \langle z \rangle_m^0$ (recall Lemma 3.4(c)). So $S = (\cdots(\langle z \rangle_m^0 - \langle u^r \rangle_m^0) + \cdots) + \langle u^1 \rangle_m^1 \in C_m z$. So $U = \{u^1, \ldots, u^r\}$ is well formed by Lemma 3.5 and S satisfies (AP) by Proposition 3.8(a). Iteration of (∗) with $p = 1$ produces $v^1, \ldots, v^s \in (\,{}_{m+2}^{\,z}\,)$ such that $(\cdots(S - \langle v_m^1 \rangle^0) + \cdots) + \langle v^s \rangle_m^1 = \langle z \rangle_m^1$. Hence $S = (\cdots(\langle z \rangle_m^1 - \langle v^s \rangle_m^1) + \cdots) + \langle v^1 \rangle_m^0$ and $\langle z \rangle_m^1 = (\cdots(((\cdots(\langle z \rangle_m^0 - \langle u^r \rangle_m^0) + \cdots) + \langle u^1 \rangle_m^1) - \langle v^1 \rangle_m^0) + \cdots) + \langle v^s \rangle_m^1$. Put $V = \{v^1, \ldots, v^s\}$. By Lemma 3.5, $U + V$ (and hence V) is well formed. This gives Lemma 3.9 by taking $U \subset (\,{}_{m+2}^{\,z}\,)$ when $p = 0$ and $V \subset (\,{}_{m+2}^{\,z}\,)$ when $p = 1$.

It remains to prove (∗) by induction on $\#z = n + 1$. For $n = m$ one has $D_m z = \{\{z\}\}$ and $\langle z \rangle_m^0 = \langle z \rangle_m^1 = \{z\}$, so (∗) is trivially true. Inductively assume (∗) (and hence, by the above, the lemma) for $z\partial_n$ in place of z. Take $S \in D_m z$. Put $M = \{x \in S \mid z_n \not\in x\}$, $N = \{x \in S \mid z_n \in x\}$, $N\partial_m = \{x\partial_m \mid x \in N\}$; these are all well formed and $S = M + N$.

The next thing to observe is that $\rho^\pi(N\partial_m) = \langle z\partial_n \rangle_{m-2}^\pi$ for $\pi = 0, 1$. To see this for $\pi = 0$ (the other case is similar) take $v \in \rho^0(N\partial_m) = \phi^0(N\partial_m) - \phi^1(N\partial_m)$. Then $v = x\partial_m\partial_i$ where i is even, $i < m$ and $x \in N$; so $v = x\partial_i\partial_{m-1}$. If $x\partial_i \not\in \rho^0 S$, then $x\partial_i = y\partial_j$, j odd, $y \in S$; but $z_n \in x\partial_i$ implies $y \in N$ and $j < m$; so $v = y\partial_j\partial_{m-1} = y\partial_m\partial_j$ contrary to $v \notin \phi^1(N\partial_m)$. So $x\partial_i \in \rho^0 S = \langle z \rangle^0_{m-1}$ which implies $v \in \langle z\partial_n \rangle^0_{m-2}$ since $v = x\partial_i - \{z_n\}$. Conversely, if $v \in \langle z\partial_n \rangle^0_{m-2}$, then $v + \{z_n\} \in \langle z \rangle^0_{m-1} = \rho^0 S$; so $v + \{z_n\} = x\partial_i$ for some even i and $v + \{z_n\} \notin \phi^1 S$; so $x \in N$, $v = x\partial_m\partial_i \in \phi^0(N\partial_m) - \phi^1(N\partial_m) = \rho^0(N\partial_m)$.

Let p be the parity of m and let q be the parity of $m+1$. The following equations hold

$$\rho^p M = \langle z\partial_n \rangle^p_{m-1} - N\partial_m , \qquad \rho^q M = N\partial_m - \langle z\partial_n \rangle^p_{m-1}$$

as will now be proved; these proofs are lengthy but routine.

Take $v \in \rho^p M$. Then $v = x\partial_i \notin \phi^q M$ where $x \in M$ and i has parity p. If $v \in N\partial_m$, then $i = m$ and $x \in N$ (by (WF)) contrary to $x \in M$. So $v \notin N\partial_m$. If $v \in \phi^q S$, then $v = y\partial_j$ with j of parity q; but $v \notin \phi^q M$, so $y \in N$; but $j \neq m$, so $z_n \in y\partial_j = v \subset x$ contrary to $x \in M$. So $v \notin \phi^q S$. So $v \in \rho^p S = \langle z \rangle^p_{m-1}$. But $k \in z\partial_n - v$ implies $k \in z - v$, so $\#v/k$ has parity p. So $v \in \langle z\partial_n \rangle^p_{m-1} - N\partial_m$.

Take $v \in \langle z\partial_n \rangle^p_{m-1} - N\partial_m$. If $k \in z - v$, then either $k \in z\partial_n - v$, so that $\#v/k$ has parity p, or, $k = z_n$, so that $\#v/k = m$ which has parity p. So $v \in \langle z \rangle^p_{m-1} = \rho^p S$ and $z_n \notin v$. So $v = x\partial_i$ with $m - i$ even, $x \in S$, $v \notin \phi^q S \supset \phi^q M$. If $z_n \in x$, then $i = m$ and $v \in N\partial_m$, a contradiction. So $z_n \notin x$; so $x \in M$ and $v \in \rho^p M$ as required. This proves the first equation.

Take $v \in \rho^q M$. Then $v = x\partial_i$, $m - i$ odd, $x \in M$, $v \notin \phi^p M$. Since $\#v/z_n = m$ has parity p, one has $v \notin \langle z \rangle^q_{m-1} = \rho^q S$. So $v \in \phi^p S$; so $v = y\partial_j$, $m - j$ even, $y \in N$; so $j = m$ and $v \in N\partial_m$. If $v \in \langle z\partial_n \rangle^p_{m-1}$, then $\#v/x_i = i$ has parity p contrary to $m - i$ odd. So $v \in N\partial_m - \langle z\partial_n \rangle_{m-1}$.

Take $v \in N\partial_m - \langle z\partial_n \rangle^p_{m-1}$. Then $v = x\partial_m$, $x \in N$. By (WF) for S, $v \notin \phi^p M$. If $v \in \rho^p S$, then $v \in \langle z \rangle^p_{m-1}$, so $v \in \langle z\partial_m \rangle^p_{m-1}$, a contradiction. So $v \notin \rho^p S$. So $v \in \phi^q S$. So $v = y\partial_j$, $m - j$ odd. So $y \in M$. So $v \in \phi^q M$. So $v \in \rho^q M$. This proves the second equation.

As a consequence of these two equations,

$$\begin{aligned} N\partial_m &= N\partial_m \cap \langle z\partial_n \rangle^p_{m-1} + (N\partial_m - \langle z\partial_n \rangle^p_{m-1}) \\ &= (\langle z\partial_n \rangle^p_{m-1} - (\langle z\partial_n \rangle^p_{m-1} - N\partial_m)) + (N\partial_m - \langle z\partial_n \rangle^p_{m-1}) \\ &= (\langle z\partial_n \rangle^p_{m-1} - \rho^p M) + \rho^q M . \end{aligned}$$

Now it is possible to prove (*) for S with p the parity of m.

If M is empty, then $N\partial_m = \langle z\partial_n \rangle^p_{m-1}$, so $S = N \subset \langle z \rangle^p_m$.

If M is non-empty, then let x be a minimal, maximal element with respect to $\lhd$ for $p = 0, 1$, respectively. Then $\langle x \rangle^q_{m-1} \subset \rho^q M = N\partial_m - \langle z\partial_n \rangle^p_{m-1}$. Put $u = x + \{z_n\}$. Then $u\partial_{m+1} = x \in M \subset S$, while for $i \leq m$, $m - i$ odd, $u\partial_i = x\partial_i + \{z_n\} \in N \subset S$. So $\langle u \rangle^q_m \subset S$.

It remains to prove (*) for S with p, q interchanged (where p is still the parity of m).

For this, note that $N\partial_m \in D_{m-1}(z\partial_n)$. So, by the inductive hypothesis, there exists $L \subset \binom{z\partial_n}{m+1}$ disjoint from M with $L+M$ well formed, $\rho^q L \subset \langle z\partial_n\rangle^q_{m-1}$ and $N\partial_m = (\langle z\partial_n\rangle^q_{m-1} - \rho^q L) + \rho^p L$ (the lemma and Proposition 3.8(e)).

Suppose L is non-empty. Let x be maximal, minimal with respect to $\lhd$ in L for $p=0,1$, respectively. Then $\langle x\rangle^p_{m-1} \subset \rho^p L \subset N\partial_m$. Put $u = x + \{z_n\}$ (so $u_{m+1} = z_n$). Then $\langle u\rangle^p_m \subset N \subset S$. This gives (*) as required.

Suppose L is empty. Then $\rho^p M = \langle z\partial_n\rangle^q_{m-1}$ ($=N\partial_m$) and $\rho^p M = \langle z\partial_n\rangle^p_{m-1}$. So $M \in D_m(z\partial_n)$. By induction, either $M \subset \langle z\partial_n\rangle^q_m$ or $\langle u\rangle^p_m \subset M$ for some $u \in \binom{z\partial_n}{m+2}$. In the former case, $S \subset \langle z\rangle^q_m$ since $S = N + M$ and $N\partial_m = \langle z\partial_n\rangle^q_{m-1}$. In the latter case, $\langle u\rangle^p_m \subset M \subset S$. This again gives (*). □

Corollary 3.10. $C_m z = D_m z$. □

An ordered pair of subsets S, T of $\binom{z}{m+1}$ is said to have *oriented union* when $S \cap T = 0$, $S+T$ is well formed, and $x \in S$, $y \in T$ imply $x \not\lhd y$ in $S+T$. (The pair V, U of Proposition 3.8(e) is such.) An ordered triple of subsets R, S, T has oriented union when the pairs R, $S \cup T$ and $R \cup S$, T do.

Corollary 3.11 (Swelling up). *Suppose $a \in \mathcal{O}_\omega$ and put $z = \{k \in \omega \mid i \le k \le j$ where $a^1_0 = \{(i)\}$, $a^0_0 = \{(j)\}$. Then there exist $u^p_m \subset \binom{z}{m+1}$ for $p=0,1$ and $m=1,2,3,\ldots$ such that the following conditions hold:*

(0) $u^p_1 = 0$;

(i) *the triple u^0_m, a^p_m, u^1_m has oriented union;*

(ii) $\rho^p u^p_{m+1} \subset \langle z\rangle^p_m$ *and, for* $q \ne p$, $u^0 + a^p_m + u^1_m = (\langle z\rangle^p_m - \rho^p u^p_{m+1}) + \rho^q u^p_{m+1}$.

Proof. Define $u^p_1 = 0$ and suppose u^p_m have been constructed for all $m \le k$ such that (i), (ii) hold for $m < k$. It is required to construct u^p_{k+1} and verify the conditions for $m = k$. The basis for this is the calculation

$$
\begin{aligned}
&(((((\langle z\rangle^0_{k-1} - \rho^0 u^0_k) + \rho^1 u^0_k) - \rho^0 a^p_k) + \rho^1 a^p_k) - \rho^0 u^1_k) + \rho^1 u^1_k \\
&= ((((u^0_{k-1} + a^0_{k-1} + u^1_{k-1}) - \rho^0 a^p_k) + \rho^1 a^p_k) - \rho^0 u^1_k) + \rho^1 u^1_k \\
&= (((u^0_{k-1} + a^0_{k-1} \cap a^1_{k-1} + u^1_{k-1}) + \rho^1 a^p_k) - \rho^0 u^1_k) + \rho^1 u^1_k \\
&= ((u^0_{k-1} + a^1_{k-1} + u^1_{k-1}) - \rho^0 u^1_k) + \rho^1 u^1_k \\
&= \langle z\rangle^1_{k-1}\,.
\end{aligned}
$$

Proposition 3.8 precisely yields (i) with $m = k$. Put $v^p_k = u^0_m + a^p_m + u^1_m$ so that Lemmas 3.6 and 3.4(c) yield $\rho^0 v^p_k = \langle z\rangle^0_{k-1}$, $\rho^1 v^p_k = \langle z\rangle^1_{k-1}$. So $v^p_k \in D_m z$. Lemma 3.9 yields $u^p_{k+1} \subset \binom{z}{k+2}$ as in (ii) for $m = k$. □

Theorem 3.12. *$\mathcal{O}_\omega$ is a sub-ω-category of $\mathcal{N}_\omega$. Furthermore, the unions involved in the compositions of $\mathcal{O}_\omega$ are oriented unions.*

Proof. It suffices to show that, for $(n+1)$-cells $a, b, \in \mathcal{N}_\omega$ with $s_n(a) = t_n(b)$, if $a, b \in \mathcal{O}_\omega$, then $a *_n b \in \mathcal{O}_\omega$ and the second sentence of the theorem holds for $m = n+1$. Put $c = a^0_{n+1} = a^1_{n+1}$, $d = b^0_{n+1} = b^1_{n+1}$. Apply Corollary 3.11 to $a \in \mathcal{O}_\omega$ to obtain $u^p_m \subset \binom{z}{m+1}$. By Proposition 3.8, the calculation

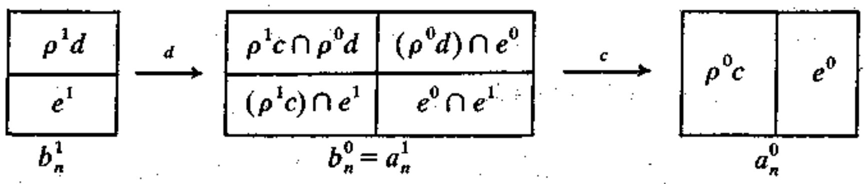
$$\begin{aligned}
&(((((\langle z\rangle^0_n - \rho^0 u^0_{n+1}) + \rho^1 u^0_{n+1}) - \rho^0 c) + \rho^1 c) - \rho^0 d) + \rho^1 d \\
&= ((((u^0_n + a^0_n + u^1_n) - \rho^0 c) + \rho^1 c) - \rho^0 d) + \rho^1 d \\
&= (((u^0_n + a^0_n \cap a^1_n + u^1_n) + \rho^1 c) - \rho^0 d) + \rho^1 d \\
&= ((u^0_n + a^1_n + u^1_n) - \rho^0 d) + \rho^1 d \\
&= ((u^0_n + b^0_n + u^1_n) - \rho^0 d) + \rho^1 d \\
&= u^0_n + b^1_n + u^1_n
\end{aligned}$$

implies u^0_{n+1}, c, d have oriented union. It remains to show that (2) holds for $a *_n b$.

Put $e^0 = a^0_n \cap a^1_n$, $e^1 = b^0_n \cap b^1_n$. By (2) for a, b, note that $a^0_n = \rho^0 c + e^0$, $b^1_n = \rho^1 d + e^1$, and $a^1_n = b^0_n = \rho^1 c \cap \rho^0 d + (\rho^0 d) \cap e^0 + (\rho^1 c) \cap e^1 + e^0 \cap e^1$. The definitions of ρ^0, ρ^1 give the equations

$$\rho^1(c+d) = \rho^1 d + (\rho^1 c) \cap e^1 = \rho^1(c+d)\,,$$

$$\rho^0(c+d) = \rho^0 c + (\rho^1 d) \cap e^0 = \rho^0(c+d)\,.$$



Hence $b^1_n = \rho^1(c+d) + e^0 \cap e^1$, $a^0_n = \rho^0(c+d) + e^0 \cap e^1$. So $a *_n b$ satisfies (2). □

Lemma 3.13. *Suppose $U \subset \binom{z}{m+2}$, $S \subset \binom{z}{m+1}$ are well formed with $\rho^0 U \subset S$ and $S = a^p_n$ for some $a \in \mathcal{O}_\omega$. If $x \lhd w \lhd y$ in S with $x, y \in \rho^0 U$, then $w \in \rho^0 U$.*

Proof. First consider the case where U has a single element u; so $\rho^0 U = \langle u\rangle^0_m \subset S$. Take $x \lhd w \lhd y$ in S where $x = u\partial_r$, $y = u\partial_s$ with r, s both even. It can be assumed that $w\partial_i = y\partial_j$ with i even and j odd. If $s \le j$, then $w\partial_i = y\partial_j = u\partial_s\partial_j =$

$u\partial_{j+1}\partial_s$ and, since S is well formed, $w = u\partial_{j+1} \in \langle u \rangle_m^0$. So suppose $j < s$. If $r < s$, then $u\partial_s\partial_r = u\partial_r\partial_{s-1}$; so $y \lhd x$, and Lemma 3.2(a) gives $x = w = y$; so $w \in \langle u \rangle_m^0$. So suppose $s \leq r$. Then $u_0, u_1, \ldots, u_{s-1} \in x \cap y$. So $u_0, u_1, \ldots, u_{s-1} \in w$ since $x \lhd w \lhd y$. In particular, $u_j \in w$ which, together with $w\partial_i = u\partial_s\partial_j$, implies $w_i = u_j$ and $w = u\partial_s \in \langle u \rangle_m^0$.

Now consider the case of a general U. Take $x \lhd w \lhd y$ in S with $x, y \in \rho^0 U$ and assume $w \not\in \rho^0 U$. Appeal now to Lemma 3.6 to obtain a listing of the elements of U. As each $\langle u \rangle_m^0$ is replaced by $\langle u \rangle_m^1$ in S, one cannot have both x, y in the same $\langle u \rangle_m^0$ by the first part of the proof. So it suffices to consider the situation where x is next to be removed and the situation where y is next to be removed.

In the first case one has $x = u\partial_i \in \langle u \rangle_m^0 \subset S$ with i even and it can be supposed that $x\partial_r = w\partial_s$ with r even and s odd. If $r < i$, then $w\partial_s = x\partial_r = u\partial_i\partial_r = u\partial_r\partial_{i-1}$ which, since S is well formed, implies $w = u\partial_r \in \langle u \rangle_m^0$, a contradiction. So $i \leq r$ and $w\partial_s = x\partial_r = u\partial_i\partial_r = u\partial_{r+1}\partial_i$ which means $u\partial_{r+1} \lhd w \lhd y$ in $(S - \langle u \rangle_m^0) + \langle u \rangle_m^1$. So the situation is maintained with U having one fewer element.

Similarly in the second case one has $y = u\partial_i \in \langle u \rangle_m^0 \subset S$ with i even and it can be supposed that $w\partial_r = y\partial_s$ with r even and s odd. If $i \leq s$, then $w\partial_r = u\partial_{s+1}\partial_i$ which, since S is well formed, implies $w = u\partial_{s+1} \in \langle u \rangle_m^0$, a contradiction. So $s < i$ and $w\partial_r = u\partial_s\partial_{i-1}$ which means $x \lhd w \lhd u\partial_s$ in $(S - \langle u \rangle_m^0) + \langle u \rangle_m^1$.

Since U is finite, the result follows. □

Corollary 3.14. *If $a \in \mathcal{O}_\omega$ is an n-cell and $u \in a_n^1 = a_n^0$ yet $a \neq \langle u \rangle$, then there exists $r \in [n-1]$ with $a = b *_r c$ for some $b, c \in \mathcal{O}_\omega$ which are not r-cells.*

Proof (*Excision of extremals*). Let r be the largest element of $[n-1]$ such that $a_{r+1}^p \neq \langle u \rangle_{r+1}^p$ for $p = 0$ or 1; this exists since $a \neq \langle u \rangle$. Property (2) and Proposition 3.1 give $a_{r+1}^q = \langle u \rangle_{r+1}^q + a_{r+1}^0 \cap a_{r+1}^1$ for $q = 0$ and 1. So $a_{r+1}^0 \cap a_{r+1}^1 \neq 0$. So choose $w \in a_{r+1}^0 \cap a_{r+1}^1$. Let $x, y \in a_{r+1}^0$ be minimal, maximal with respect to $\lhd$ such that $x \lhd w \lhd y$. By Lemma 3.13 either x or y is not in $\langle u \rangle_{r+1}^0$. So either x or y is in $a_{r+1}^0 \cap a_{r+1}^1$. Furthermore, since x is minimal, $\langle x \rangle_r^1 \subset \rho^1 a_{r+1}^0 \subset a_r^1$; and, since y is maximal, $\langle y \rangle_r^0 \subset \rho^0 a_{r+1} \subset a_r^0$.

In the case where $x \in a_{r+1}^0 \cap a_{r+1}^1$, the following equations define $b, c \in \mathcal{O}_\omega$ with $a = b *_r c$ (Corollaries 3.10 and 3.11):

$$c_m^q = 0\,, \quad b_m^q = a_m^q \qquad \text{for } m > r+1\,,$$
$$c_{r+1}^1 = c_{r+1}^0 = \{x\}\,, \qquad b_{r+1}^q = a_{r+1}^q - \{x\}\,,$$
$$c_r^1 = a_r^1\,, \qquad c_r^0 = (a_r^1 - \langle x \rangle_r^1) + \langle x \rangle_r^0 = b_r^1\,, \qquad b_r^0 = a_r^0\,,$$
$$c_m^q = b_m^q = a_m^q \quad \text{for } m < r\,.$$

In the case where $y \in a_{r+1}^0 \cap a_{r+1}^1$, the following equations define $b, c \in \mathcal{O}_\omega$ with $a = b *_r c$:

$$c_m^q = a_m^q, \quad b_m^q = 0 \qquad \text{for } m > r+1,$$
$$c_{r+1}^q = a_{r+1}^q - \{y\}, \qquad b_{r+1}^1 = b_{r+1}^0 = \{y\},$$
$$c_r^1 = a_r^1, \qquad c_r^0 = (a_r^0 - \langle y\rangle_r^0) + \langle y\rangle_r^1 = b_r^1, \qquad b_r^0 = a_r^0,$$
$$c_m^q = b_m^q = a_m^q \text{ for } m < r. \qquad \square$$

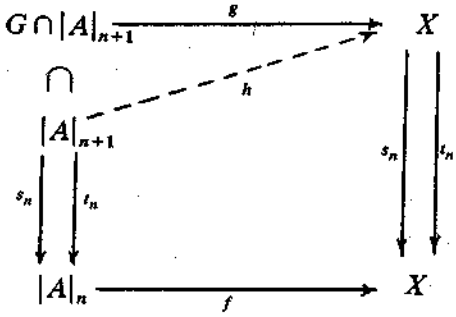
Theorem 3.15. *Every cell of $\mathcal{O}_\omega$ is an iterated composite of cells of the form $\langle z\rangle$. Hence $\mathcal{O}_\omega$ is the smallest sub-ω-category of $\mathcal{N}_\omega$ which contains all $\langle z\rangle$.*

Proof. Iterate Corollary 3.14 $\square$

4. Orientals: their freedom

For an ω-category A, let $|A|_n$ denote the n-category whose elements are the n-cells in A and whose compositions are the compositions $*_m$ of A for $m < n$.

An ω-category A is *freely generated* by a subset G of A when, for all ω-categories X, for all $n \in \omega$, for all n-functors $f: |A|_n \to X$, and for all functions $g: G \cap |A|_{n+1} \to X$ such that $s_n g = f s_n$, $t_n g = f t_n$, there exists a unique $(n+1)$-functor $h: |A|_{n+1} \to X$ whose restriction to $|A|_n$ is f and whose restriction to $G \cap |A|_{n+1}$ is g.



Theorem 4.1. *$\mathcal{O}_\omega$ is freely generated by the set of elements of the form $\langle z\rangle$ where z is a finite subset of ω.*

Proof. Since $|\mathcal{O}_\omega|_1$ is the free category on the graph $\binom{\omega}{2} \rightrightarrows \omega$ (as pointed out in Section 2), the universal property holds for $n = 0$. More generally, for any $n \in \omega$, suppose $f: |\mathcal{O}_\omega|_n \to X$ is an n-functor and $g: \binom{\omega}{n+2} \to X$ is a function such that f, g determine an arrow of graphs

$$\begin{array}{ccc}
\binom{\omega}{n+2} & \xrightarrow{\quad g \quad} & X \\
\big\downarrow \langle\ \rangle & & \\
|\mathcal{O}_\omega|_{n+1} & & s_n \Big\downarrow\ \Big\downarrow t_n \\
s_n \Big\downarrow\ \Big\downarrow t_n & & \\
|\mathcal{O}_\omega|_n & \xrightarrow{\quad f \quad} & X
\end{array}$$

It is required to define an $(n+1)$-functor $h:|\mathcal{O}_\omega|_{n+1} \to X$ which extends both f and g. By Theorem 3.15, if such an h exists, it is unique.

Let a be an $(n+1)$-cell in $\mathcal{O}_\omega$. Define $h(a) \in X$ by induction on the number k of elements in $a^1_{n+1} = a^0_{n+1}$. When $k = 0$, define $h(a) = f(a)$ since then a is an n-cell.

Suppose $k = 1$. Then $a^1_{n+1} = \{u\}$ for some $u \in \binom{\omega}{n+2}$. Then there exist unique n-cells d, e in $\mathcal{O}_\omega$ such that $a = d *_{n-1} \langle u \rangle *_{n-1} e$ and $e^1_n = e^0_n = \{v \in a^1_n \cap a^0_n \mid v \lhd u\partial_i$ in a^0_n for some even $i\}$. Define $h(a) = f(d) *_{n-1} g(u) *_{n-1} f(e)$ in X.

Suppose $k > 1$. Let u be the $\lhd$-minimal element of a^1_{n+1} which (to be specific) is first in the lexicographic order on $\binom{\omega}{n+2} \subset \omega^{n+2}$. Then $a = b *_n c$ for unique $(n+1)$-cells b, c in $\mathcal{O}_\omega$ such that $c^1_{n+1} = c^0_{n+1} = \{u\}$. Define $h(a) = h(b) *_n h(c)$ in X.

That h preserves the composition $*_n$ follows inductively from the fact that the compositions of $\mathcal{O}_\omega$ are oriented unions (Theorem 3.12). □

Corollary 4.2. *$\mathcal{O}_n$ is freely generated by the set of elements of the form $\langle z \rangle$ where $z \subset [n]$.* □

5. The nerve of an ω-category

Recall that the 0-cells of $\mathcal{O}_n$ can be identified with elements of $[n]$. For each order-preserving function $\zeta:[n] \to [m]$, there is a unique n-functor $\mathcal{O}_\zeta : \mathcal{O}_n \to \mathcal{O}_m$ which is given by ζ on 0-cells. Explicitly, for $a \in \mathcal{O}_n$,

$$\mathcal{O}_\zeta(a)^p_r = \left\{ y \in \binom{[m]}{r+1} \,\middle|\, \exists x \in a^p_r \text{ with } y = \zeta x \right\};$$

this follows from Corollary 4.2. Clearly this defines a functor

$$\mathcal{O} : \Delta \to \omega\text{-Cat}$$

which takes $\zeta:[n] \to [m]$ to $\mathcal{O}_\zeta : \mathcal{O}_n \to \mathcal{O}_m$.

The *nerve* A of an ω-category A is the simplicial set

$$\Delta^{\mathrm{op}} \xrightarrow{\mathcal{O}} \omega\text{-Cat}^{\mathrm{op}} \xrightarrow{\omega\text{-Cat}(-,\,A)} \mathrm{Set}\,.$$

An element of ΔA of dimension n is an n-functor $\mathcal{O}_n \to A$. Nerve is functorial in A using composition: this gives the *nerve functor*

$$\Delta - : \omega\text{-Cat} \longrightarrow [\Delta^{\mathrm{op}}, \mathrm{Set}]$$

which has a left adjoint

$$\Phi : [\Delta^{\mathrm{op}}, \mathrm{Set}] \longrightarrow \omega\text{-Cat}$$

whose value at a simplicial set X is given by the coend formula

$$\Phi X = \int^{[n]} X_n \times \mathcal{O}_n\,.$$

Yoneda's lemma provides an isomorphism $\Phi\Delta[n] \cong \mathcal{O}_n$.

Let $\dot{\Delta}[n]$ denote the subsimplicial set of $\Delta[n]$ whose elements of dimension m are the non-epic arrows $[m] \to [n]$ in Δ.

Lemma 5.1. $\Phi\dot{\Delta}[n] \cong |\mathcal{O}_n|_{n-1}$.

Proof. Each non-invertible monic $\mu : [m] \to [n]$ in Δ induces a simplicial map $\Delta[m] \to \dot{\Delta}[n]$. These simplicial maps form a colimit cone. Since Φ is a left adjoint, it preserves colimits. So the ω-functors $\mathcal{O}_m \to \Phi\dot{\Delta}[n]$ form a colimit cone through which the cone $\mathcal{O}_\mu : \mathcal{O}_m \to \mathcal{O}_n$ factors. The result follows. □

A simplicial set X is called *n-coskeletal* when, for all $k > n$ and all $x_0, x_1, \ldots, x_k \in X_{k-1}$ satisfying $x_j\partial_i = x_i\partial_{j-1}$ for $i < j$, there exists a unique $y \in X_k$ with $x_i = y\partial_i$ for all $i \in [k]$. In other words, each simplicial map $x : \dot{\Delta}[k] \to X$ for $k > n$ has a unique extension $y : \Delta[k] \to X$.

Theorem 5.2. *The nerve of an n-category is* $(n+1)$*-coskeletal.*

Proof. Suppose $k > n+1$ and A is an n-category. A simplicial map $x : \dot{\Delta}[k] \to \Delta A$ amounts to an ω-functor $x' : |\mathcal{O}_k|_{k-1} \to A$ by Lemma 5.1. Then $x's_{k-1}\langle [k] \rangle$, $x't_{k-1}\langle [k] \rangle$ are $(k-1)$-cells in A with the same $(k-2)$-source and target. Every element of A is an n-cell and $k-1 > n$, so $x's_{k-1}\langle [k] \rangle = x't_{k-1}\langle [k] \rangle$. By Corollary 4.2 there is a unique extension $y' : \mathcal{O}_k \to A$ of x' whose value at $\langle [k] \rangle$ is $x's_n\langle [k] \rangle$. This gives the unique extension $y : \Delta[k] \to \Delta A$ of x as required. □

A *simplicial set with hollowness* is a simplicial set X with certain elements distinguished and called *hollow* satisfying the following conditions:
- there are no hollow elements of dimension 0;
- hollow elements of dimension 1 are degenerate;
- all degenerate elements are hollow.

Let Ssh denote the category whose objects are simplicial sets with hollowness and whose arrows are hollowness-preserving simplicial maps.

For an ω-category A, an ω-functor $x:\mathcal{O}_n \to A$ is called *hollow* when $x\langle[n]\rangle$ is an $(n-1)$-cell in A. In this way, the nerve ΔA becomes an object NA of Ssh, and, the nerve functor lifts to a functor $N:\omega\text{-Cat} \to \text{Ssh}$. To construct the left adjoint for N, it is helpful to consider the category hΔ which contains Δ as a full subcategory and, for each $n>1$, contains an extra object h_n with extra arrows

$$[n] \xrightarrow{\ \varepsilon\ } h_n \ \ \begin{array}{c}\xrightarrow{\sigma'_0}\\ \vdots \\ \xrightarrow[\sigma'_{n-1}]{}\end{array}\ \ [n-1]$$

such that $\sigma'_i\varepsilon = \sigma_i$. Then Ssh is equivalent to the full subcategory of $[(\mathrm{h}\Delta)^{\mathrm{op}}, \mathrm{Set}]$ consisting of those functors $X:(\mathrm{h}\Delta)^{\mathrm{op}} \to \mathrm{Set}$ for which each $X\varepsilon: Xh_n \to X[n]$ is monic.

Let $\mathcal{O}_n^{\mathrm{h}}$ be the coequalizer in ω-Cat of the two ω-functors $2_\omega \to \mathcal{O}_n$ corresponding to $\langle[n]\rangle$, $s_{n-1}\langle[n]\rangle \in \mathcal{O}_n$. An ω-functor $\mathcal{O}_n^{\mathrm{h}} \to A$ amounts precisely to a hollow ω-functor $\mathcal{O}_n \to A$. It is now possible to define a functor $\mathrm{h}\mathcal{O}:\mathrm{h}\Delta \to \omega\text{-Cat}$ such that the diagram

$$\begin{array}{ccc}\Delta & \xrightarrow{\ \mathcal{O}\ } & \\ \downarrow & \searrow & \\ \mathrm{h}\Delta & \xrightarrow[\mathrm{h}\mathcal{O}]{} & \omega\text{-Cat}\end{array}$$

commutes and $\mathrm{h}\mathcal{O}$ takes $\varepsilon:[n] \to h_n$ to the quotient ω-functor $\mathcal{O}_n \to \mathcal{O}_n^{\mathrm{h}}$.

Let $\Psi':[(\mathrm{h}\Delta)^{\mathrm{op}}, \mathrm{Set}] \to \omega\text{-Cat}$ denote the left Kan extension of $\mathrm{h}\mathcal{O}:\mathrm{h}\Delta \to \omega\text{-Cat}$ along the Yoneda embedding $\mathrm{h}\Delta \to [(\mathrm{h}\Delta)^{\mathrm{op}}, \mathrm{Set}]$; the formula is:

$$\Psi' X = \int^{u \in \mathrm{h}\Delta} Xu \times (\mathrm{h}\mathcal{O})u\,.$$

The composite of Ψ' with the fully faithful $\mathrm{Ssh} \to [(\mathrm{h}\Delta)^{\mathrm{op}}, \mathrm{Set}]$ gives the left adjoint $\Psi:\mathrm{Ssh} \to \omega\text{-Cat}$ for N.

Let ω-Cath denote the full subcategory of ω-Cat consisting of those ω-categories A for which every element is a cell. Note that $\mathcal{O}_\omega$, $\mathcal{O}_n^{\mathrm{h}}$ are in ω-Cath and so Φ, Ψ land in ω-Cath.

For a simplicial set X, define Cell(X) to be the union, modulo $x = x\sigma_0$, of the sets

$$\mathrm{Cell}_n(X) = \{x \in X_n \mid x\partial_i = x\partial_i\partial_j\sigma_j \text{ for } j+1 < i\}\,.$$

One has $\mathrm{Cell}(\Delta A) \cong A$ in Set for all $A \in \omega$-Cath. It follows that $\Delta - : \omega$-Cath$\to [\Delta^{\mathrm{op}}, \mathrm{Set}]$, and hence $N : \omega$-Cath$\to$Ssh, is conservative (reflects isomorphisms).

In fact, I believe $N : \omega$-Cath$\to$Ssh to be fully faithful and I would now like to discuss my conjecture on characterizing the (replete) image of N.

To motivate this conjecture we must look at the nth cocycle condition which we can now write as an equation

$$x(s_n\langle 0\,1\ldots n+1\rangle) = x(t_n\langle 0\,1\ldots n+1\rangle)$$

involving an ω-functor $x : \mathcal{O}_{n+1} \to A$. Excision of extremals (Corollary 3.14) gives a method for decomposing the n-source and n-target of $\langle 0\,1\ldots n+1\rangle$ into composites of cells of the form $\langle z\rangle$. The result of this for $n = 0, 1, 2, 3, 4$ is given below using the *convention* of Roberts to omit brackets when the natural order of evaluation of the compositions $*_0, *_1, *_2, \ldots$ is intended.

$$s_0\langle 01\rangle = \langle 0\rangle, \qquad t_0\langle 01\rangle = \langle 1\rangle\,,$$

$$s_1\langle 012\rangle = \langle 02\rangle, \qquad t_1\langle 012\rangle = \langle 12\rangle *_0 \langle 01\rangle\,,$$

$$s_2\langle 0123\rangle = \langle 23\rangle *_0 \langle 012\rangle *_1 \langle 023\rangle, \qquad t_2\langle 0123\rangle = \langle 123\rangle *_0 \langle 01\rangle *_1 \langle 013\rangle\,,$$

$$s_3\langle 01234\rangle = \langle 234\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 0124\rangle *_2 \langle 34\rangle *_0 \langle 23\rangle *_0 \langle 012\rangle *_1 \langle 0234\rangle\,,$$

$$\begin{aligned} t_3\langle 01234\rangle = {} & \langle 1234\rangle *_0 \langle 01\rangle *_1 \langle 014\rangle *_2 \langle 34\rangle *_0 \langle 123\rangle *_0 \langle 01\rangle *_1 \langle 0134\rangle *_0 \langle 34\rangle \\ & *_0 \langle 0123\rangle *_1 \langle 034\rangle, \end{aligned}$$

$$\begin{aligned} s_4\langle 012345\rangle = {} & \langle 2345\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 125\rangle *_0 \langle 01\rangle *_1 \langle 015\rangle *_2 \langle 45\rangle *_0 \langle 234\rangle *_0 \langle 12\rangle \\ & *_0 \langle 01\rangle *_1 \langle 1245\rangle *_0 \langle 01\rangle *_1 \langle 015\rangle *_2 \langle 45\rangle *_0 \langle 234\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle \\ & *_1 \langle 45\rangle *_0 \langle 124\rangle *_0 \langle 01\rangle *_1 \langle 0145\rangle *_2 \langle 45\rangle *_0 \langle 01234\rangle *_1 \langle 045\rangle *_3 \langle 2345\rangle \\ & *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 125\rangle *_0 \langle 01\rangle *_1 \langle 015\rangle *_2 \langle 45\rangle *_0 \langle 234\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle \\ & *_1 \langle 01245\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 23\rangle *_0 \langle 012\rangle *_1 \langle 45\rangle *_0 \langle 0234\rangle *_1 \langle 045\rangle \\ & *_3 \langle 345\rangle *_0 \langle 23\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 235\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 0125\rangle \\ & *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 23\rangle *_0 \langle 012\rangle *_1 \langle 02345\rangle\,, \end{aligned}$$

$$\begin{aligned} t_4\langle 012345\rangle = {} & \langle 12345\rangle *_0 \langle 01\rangle *_1 \langle 015\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 123\rangle *_0 \langle 01\rangle *_1 \langle 45\rangle *_0 \langle 134\rangle \\ & *_0 \langle 01\rangle *_1 \langle 0145\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 123\rangle *_0 \langle 01\rangle *_1 \langle 45\rangle *_0 \langle 0134\rangle *_1 \langle 045\rangle \\ & *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 0123\rangle *_1 \langle 45\rangle *_0 \langle 034\rangle *_1 \langle 045\rangle *_3 \langle 345\rangle *_0 \langle 23\rangle \\ & *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 1235\rangle *_0 \langle 01\rangle *_1 \langle 015\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 123\rangle *_0 \langle 01\rangle \\ & *_1 \langle 01345\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 0123\rangle *_1 \langle 45\rangle *_0 \langle 034\rangle *_1 \langle 045\rangle *_3 \langle 345\rangle \\ & *_0 \langle 23\rangle *_0 \langle 12\rangle *_0 \langle 01\rangle *_1 \langle 01235\rangle *_2 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 23\rangle *_0 \langle 012\rangle \\ & *_1 \langle 45\rangle *_0 \langle 34\rangle *_0 \langle 023\rangle *_1 \langle 0345\rangle . \end{aligned}$$

Work of Roberts, Brown–Higgins, and myself has led us to expect that simplicial sets X with hollowness should be isomorphic to NA for some A when they satisfy some exactness condition. The clue to finding the correct condition (which is far less trivial here than in Brown's setting of ∞-groupoids where each horn should have a unique hollow filler) is to ask the following question:

in the nth cocycle condition above which cells must in general be identities in order to be able to say that $x\langle 0\,1\ldots\hat{k}\ldots n+1\rangle$ is uniquely determined?

For example when $n = 1$, we look at the equation

$$x_{02} = x_{12} *_0 x_{01}$$

and see that for $k = 1$ there is no condition since x_{02} is determined by the equation, whereas, for $k = 0$ we need x_{01} to be a 0-cell in order for the equation to determine x_{12}.

The case $n = 4$ gives enough data to make my conjecture convincing, see Table 1.

Table 1
$x(s_4\langle 012345\rangle) = x(t_4\langle 012345\rangle)$

k	m-cells which must become $(m-1)$-cells on applying x
0	⟨01345⟩⟨01235⟩⟨0145⟩⟨0134⟩⟨0123⟩⟨015⟩⟨01⟩
1	⟨01245⟩⟨01234⟩⟨0125⟩⟨012⟩
2	⟨12345⟩⟨01235⟩⟨0123⟩⟨1235⟩⟨123⟩
3	⟨01234⟩⟨02345⟩⟨2345⟩⟨0234⟩⟨234⟩
4	⟨12345⟩⟨01345⟩⟨0345⟩⟨345⟩
5	⟨01245⟩⟨02345⟩⟨0145⟩⟨1245⟩⟨2345⟩⟨045⟩⟨45⟩

The most striking thing is that in the row for k the m-cells listed all include k's neighbours ($k-1$ and $k+1$ unless one of these is impossible because $k = 0$ or 5). Roberts made a conjecture on the basis of this. However there are *other* cells which include the neighbours but are not in the list: Roberts requires far more cells to be identified than are really necessary. It still may be that the stronger condition which I am going to give is a consequence of that of Roberts; indeed, I have proved that in characterizing nerves of 2-categories the Roberts condition *is* enough.

Observe that the cells $\langle z\rangle$ listed for k have the property that $k-1, k, k+1$ are not in the complement $\neg z = \{r \in [5] \mid r \in z\}$ of z and, when the elements of $\neg z + \{k-1, k, k+1\}$ are written out in order, they alternate in parity.

This leads us to make the following definitions:

A finite subset $u = (u_0, u_1, \dots, u_m)$ of ω is called *alternating* when u_i, u_{i+1} have opposite parity for $i = 0, 1, \dots, m-1$. Call u *k-divided* when $k-1$, k, $k+1 \notin u$ and $u + \{k-1, k, k+1\}$ is alternating. Call a monic $\mu : [m] \rightarrowtail [n]$ in Δ a *k-monic* when the complement of its image is k-divided.

Recall that the *k-horn* $\Lambda^k[n]$ *in* $\Delta[n]$ is the sub-simplicial set of $\Delta[n]$ consisting of those $\alpha : [m] \to [n]$ whose image does not contain $[n] - \{k\}$. A simplicial map $\Delta[n] \to X$ amounts to an element of X_n (Yoneda lemma). A simplicial map $\Lambda^k[n] \to X$ amounts to a *k-horn in X*: that is, $x_0, \dots, \hat{x}_k, \dots, x_n \in X_{n-1}$ satisfying $x_j \partial_i = x_i \partial_{j-1}$ for $i < j$, $i \neq k$, $j \neq k$.

Let $\Delta_k[n]$ be the object of Ssh whose underlying simplicial set is $\Delta[n]$ and whose non-degenerate hollow elements are the k-monics. Regard $\Lambda^k[n]$ as in Ssh by taking as hollow the hollow elements of $\Delta_k[n]$ which are in $\Lambda^k[n]$. (So $\Lambda^k[n] \to \Delta_k[n]$ is a regular monic in Ssh.) A k-horn $(x_i)_{i\neq k}$ in $X \in$ Ssh is called *admissible* when it corresponds to an arrow $\Lambda^k[n] \to X$ in Ssh.

Conjecture 5.3. An object X of Ssh is isomorphic to NA for some $A \in \omega$-Cath if and only if it satisfies the following two conditions:

(1) for each admissible k-horn $(x_i)_{i\neq k}$ there exists a unique hollow x with $x\partial_i = x_i$ for all $i \neq k$; and,

(2) if x and $(x\partial_i)_{i\neq k}$ are hollow, then so is $x\partial_k$.

Note that (1) just says that X is orthogonal to (is a sheaf for) the inclusion $\Lambda^k[n] \to \Delta_k[n]$; that is

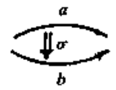
$$\mathrm{Ssh}(\Delta_k[n], X) \xrightarrow{\;\cong\;} \mathrm{Ssh}(\lambda^k[n], X)\,.$$

Also, (2) says that X is orthogonal to the identity map $\Delta[n] \to \Delta[n]$ with two different hollowness structures of $\Delta[n]$. The result would therefore carry over to ω-categories and simplicial objects in any finitely complete category.

What techniques are available to prove the conjecture? We already have an adjunction

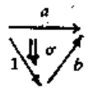
$$\omega\text{-Cath} \;\underset{N}{\overset{\Psi}{\begin{array}{c}\longleftarrow\\ \bot \\ \longrightarrow\end{array}}}\; \mathrm{Ssh}$$

and so 'all' we need to prove is that the unit $X \to N\Psi X$ is invertible when X satisfies (1) and (2). The description of Ψ as a coend does not seem very helpful in achieving this. The underlying set of ΨX must be isomorphic to Cell(X) as given above, but the asymmetry in the definition of Cell(X) makes it hard to work with. For example, 2-cells

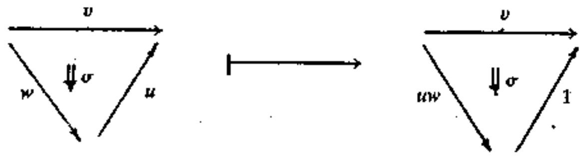


can be identified with

in Cell(ΔA), but they can also be identified with



In order to characterize the image of 2-Cat under N, I made use of a shift operation $X \to$ Cell(X) which amounts to



For higher dimensions there is an analogous shift operation which is defined by Roberts [9].

In previous work on characterizing nerves, Roberts and I were working without an explicit description of nerve. Roberts' approach was to use the characterization to construct it. Since we do now have the nerve functor $N: \omega\text{-Cath} \to \mathrm{Ssh}$, it seems possible to make use of this in the characterization. I propose the following approach.

Recall the ω-category 2_ω from Section 1 representing the underlying functor $\omega\text{-Cat} \to \mathrm{Set}$. For each n there is also the n-category 2_n representing $n\text{-Cat} \to \mathrm{Set}$. Since we believe N to be fully faithful, we should have isomorphisms of sets

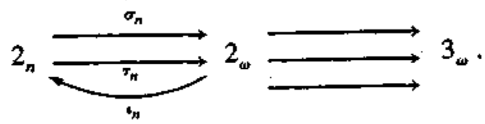
$$A \cong \omega\text{-Cat}(2_\omega, A) \cong \mathrm{Ssh}(N2_\omega, NA)\,.$$

Thus, a more symmetric definition of Cell(X) is

$$\mathrm{Cell}(X) \cong \mathrm{Ssh}(N2_\omega, X)$$

for $X \in \mathrm{Ssh}$. For X satisfying Conjecture 5.3(1), (2), it remains to produce compositions on Cell(X) making it an ω-category with $N\mathrm{Cell}(X) \cong X$.

Let σ_n, $\tau_n : 2_n \to 2_\omega$ denote the ω-functors corresponding to the n-cells $s_n(\omega) = (0, n)$, $t_n(\omega) = (1, n)$ of 2_ω. Let $\iota_n : 2_\omega \to 2_n$ correspond to the n-cell of 2_n which is not an $(n-1)$-cell. The idempotents $\sigma_n \iota_n$, $\tau_n \iota_n$ induce candidates for s_n, t_n on Cell(X): moreover, $\mathrm{Cell}_n(X) = \mathrm{Ssh}(N2_n, X)$ is the set of n-cells. By virtue of its representing ω-Cat$\to$Set, 2_ω is a co-ω-category in ω-Cat; the underlying co-category for the nth co-composition is:

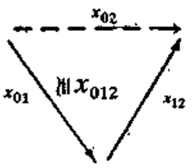


It must be shown that, for X satisfying Conjecture 5.3(1) and (2), the functor $\mathrm{Ssh}(N-, X)$ takes the above co-category to a category.

6. Bicategories and so on

Continuing in the speculative vein of the end of the last section, we offer the following remarks on bicategories and their higher dimensional analogues.

Let A be a bicategory in the sense of Bénabou (see [6]). The notion of a homomorphism of bicategories from a 2-category into A is easily generalized to homomorphism from an n-category into A: the 0-composition is associative up to invertible 2-cells in A which are coherent and natural in the m-cells for $m > 0$. The appropriate notion of *nerve* $N_b A$ *of* A is the object of Ssh whose underlying simplicial set is $\Delta_b A = \mathrm{Hom}(\mathcal{O}_*, A)$, whose elements $x : \mathcal{O}_2 \to A$ of dimension 2 are hollow when $x\langle 0\,1\,2\rangle$ is *invertible* (not necessarily an identity). This $N_b A$ satisfies Conjecture 5.3(1), (2) *except for uniqueness in property* (1). For example



where x_{02}, x_{012} are not *uniquely* determined by x_{01}, x_{12}.

This suggests the possibility of characterizing bicategories and (normal) homomorphisms as a full subcategory of Ssh. Even more boldly I propose objects of Ssh satisfying (1) without "uniqueness" and (2) as an accessible definition of *weak* ω*-categories* (the infinite generalization of bicategories).

References

[1] R. Brown and P.J. Higgins, The equivalence of crossed complexes and ∞-groupoids, Cahiers Topologie Géom. Différentielle 22 (1981) 371–386.

[2] B. Eckmann and P.J. Hilton, Group-like structures in general categories, I. Multiplications and comultiplications, Math. Ann. 145 (1962) 227–255.

[3] S. Eilenberg and G.M. Kelly, Closed categories, Proc. Conference on Categorical Algebra, La Jolla (Springer, Berlin, 1966) 421–562.

[4] P. Gabriel and F. Ulmer, Lokal präsentierbare Kategorien, Lecture Notes in Mathematics 221 (Springer, Berlin, 1971).

[5] D.M. Kan, Adjoint functors, Trans. Amer. Math. Soc. 87 (1958) 294–329.

[6] G.M. Kelly and R. Street, Review of the elements of 2-categories, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 75–103.

[7] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Mathematics 5 (Springer, Berlin, 1971).

[8] J.E. Roberts, Mathematical aspects of local cohomology, Proc. Colloquium on Operator Algebras and their Application to Mathematical Physics, Marseille (1977).

[9] J.E. Roberts, Complicial sets, handwritten manuscript, 1978.

[10] G. Segal, Classifying spaces and spectral sequences, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 105–112.

[11] J.-P. Serre, Local Fields, Graduate Texts in Mathematics 67 (Springer, Berlin, 1979).

[12] R. Street, Limits indexed by category-valued 2-functors, J. Pure Appl. Algebra 8 (1976) 148–181.

[13] R. Street, Characterization of bicategories of stacks, Lecture Notes in Mathematics 962 (Springer, Berlin, 1982) 282–291.