Coherent sheaves on Hilbert schemes through the Coulomb lens

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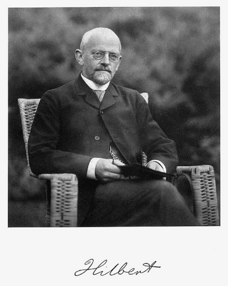
The cast of characters



Hilbert schemes for type A Kleinian singularities are Coulomb branches.

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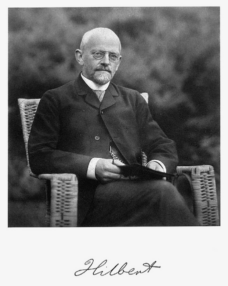
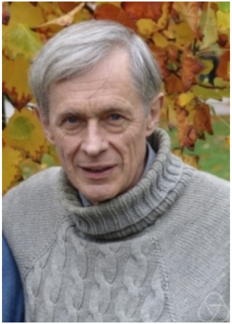


Hilbert schemes for type A Kleinian singularities are Coulomb branches.

As usual, the actual names are from people who had nothing to do with this stuff.

The cast of characters

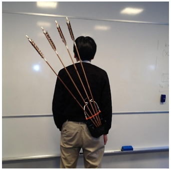




Hilbert schemes for type A Kleinian singularities are Coulomb branches.

Warmer....

The cast of characters




Hilbert schemes for type A Kleinian singularities are Coulomb branches.

OK, that's a bit more up to date.

Let me focus on the case of $\mathbb{C}^2$.

Definition

The Hilbert scheme $\mathsf{Hilb}^n(\mathbb{C}^2)$ *is the (fine) moduli space of ideals of codimension* n *in* $\mathbb{C}[x,y]$. *This is the unique crepant resolution of* $\mathbb{C}^{2n}/S_n$ *via the Hilbert-Chow map.*

There are two interesting physics-tinged constructions of this variety:

- It is a Nakajima quiver variety for the Jordan quiver, that it is, it is a symplectic reduction of the cotangent bundle $T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$ for the action of GL_n. Physicists would call this a **Higgs branch**.
- It is a BFN **Coulomb branch** associated to the same group and representation. These have a very fancy geometric definition, but I'll explain how combinatorialists should think about it in a moment.

To see this Coulomb presentation, start with the smash product $R\#S_n$ of the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ with the symmetric group S_n.

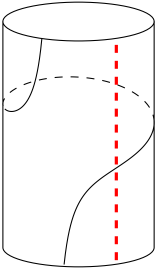
Let $e_+ \in \mathbb{C}S_n$ be the projection to the trivial and e_- the projection to the sign.

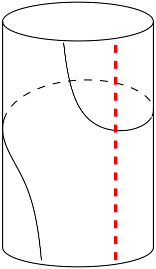
Proposition

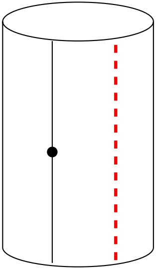
$\mathbb{C}[\mathsf{Hilb}^n(\mathbb{C}^2)] \cong R^{S_n} = e_+(R\#S_n)e_+$

Of course, the interesting structure in $\mathsf{Hilb}^n(\mathbb{C}^2)$ is in the projective coordinate ring. The Higgs description gives one version of this coordinate ring via GIT. The Coulomb gives another one that's less familiar.

I'll suggest a slightly odd representation for $\mathbb{C}[x, y]$:

$x =$ 

$y =$ 

$u = xy =$ 

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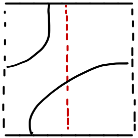
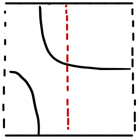
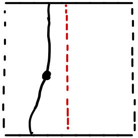
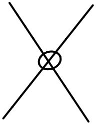
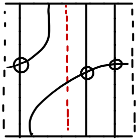
$x =$ 

$y =$ 

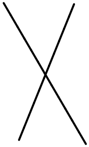
$u = xy =$ 

It's a bit easier on my drawing skills to cut and flatten the cylinder.

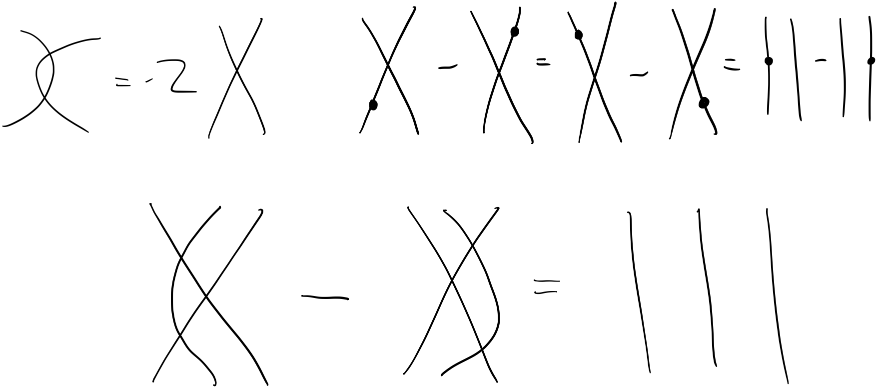
To increase n, we just have n of these strands, and we can incorporate the action of S_n by crossing them (with a circle whose significance will be clear later):

$s_i = (i, i+1) =$  $x_i =$ 

The circle is there because I want to save an undecorated crossing for $s_i - 1$.

$s_i - 1 =$ 

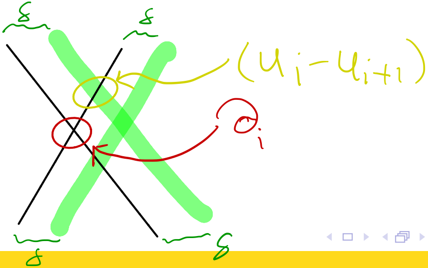
We can thus write $R\#S_n$ as the algebra given by n strand diagrams (which we can think of as smooth paths $[0,1] \to \mathrm{Sym}^n(\mathbb{R}/\mathbb{Z})$ which meet the big diagonal generically), modulo the local relations:



It is possible to upgrade this to the Hilbert scheme by noting that the action of $s_i - 1$ on $\mathbb{C}[u_1, \dots, u_n]$ can be factored:

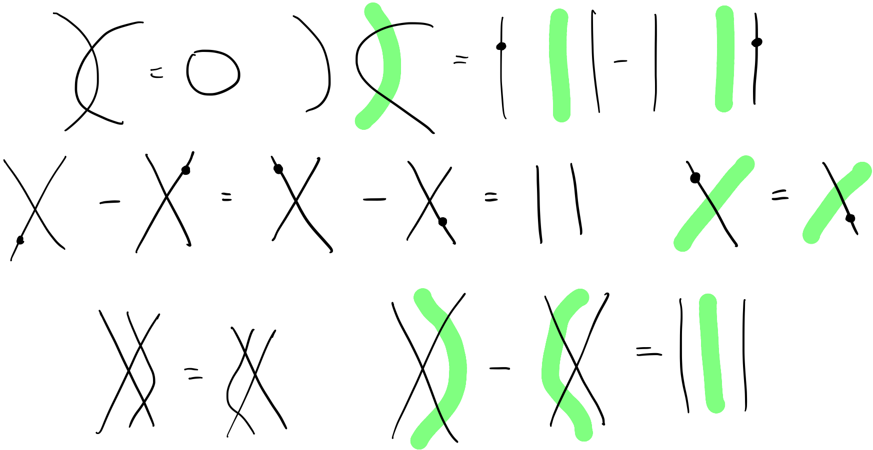
$$s_i - 1 = (u_i - u_{i+1})\partial_i \qquad \partial_i(f) = \frac{f^{s_i} - f}{u_i - u_{i+1}}$$

In terms of our diagrams, it seems we forgot to put on our glasses, and each of our strands doubles to an original strand and a ghost, which is δ steps to the right for some $0 < \delta \ll 1$.



Looking microscopically

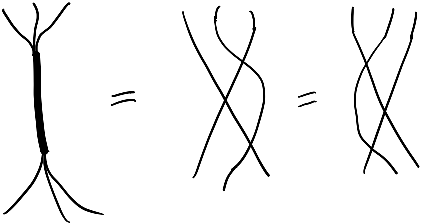
We still have local relations, which are not hard to work out:



Looking microscopically

It's also useful to let strands collide and stick together.

I won't write out all the relations around this, but the most important is that when strands collide, and then get peeled apart, that's the same as a half-twist.



We can view this as writing $R\#S_n$ as an endomorphism algebra in a much bigger category $\mathcal{C}'_\delta$.

- The objects of this category are n-element subsets of $\mathbb{R}/Z$.
- The morphisms $S \to T$ are diagrams in the cylinder whose bottom x-values are given by S and top ones by T, modulo the diagrams drawn earlier.

Note that the thick strands allow us to make sense of this category for multi-subsets.

Definition

Let $\mathcal{C}_\delta$ be the idempotent completion of this category.

In these terms, we can rewrite R^{S_n} a bit differently:

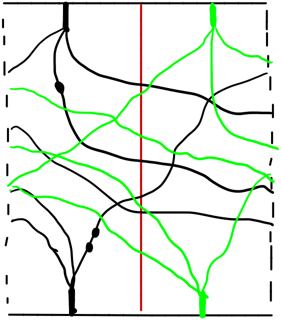
Proposition

Assume $0 < \delta < 1/n$. In $\mathcal{C}_\delta$, the object $\mathbf{a} = \{a, \dots, a\}$ for any $a \in (0,1)$ is isomorphic to the image of e_+ on

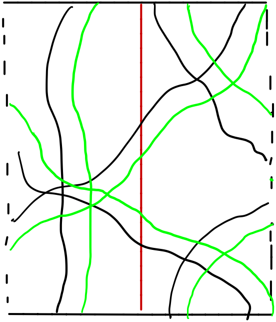
$$\mathbf{r} = \left\{\frac{1}{2n}, \frac{3}{2n}, \dots, \frac{2n-1}{2n}\right\}$$

The endomorphism ring of $\mathbf{a}$ doesn't depend on δ, so for arbitrary δ, we have

$$\mathrm{End}_{\mathcal{C}_\delta}(\mathbf{a}) \cong R^{S_n}.$$



An element of $\mathrm{End}_{\mathcal{C}_\delta}(\mathbf{a})$.



An element of $\mathrm{End}_{\mathcal{C}_\delta}(\mathbf{r})$.

Theorem

For arbitrary $\delta \notin \{\frac{a}{m} \mid 0 < m \leq n, a \in \mathbb{Z}\}$, the category $\mathcal{C}_\delta$ is (the category of projective modules over) a non-commutative resolution of singularities of $\mathrm{Sym}^n(\mathbb{C}^2)$.

Of course, this is the cheap kind of non-commutative resolution:

Theorem

For arbitrary $\delta \notin \Delta = \{\frac{a}{m} \mid 0 < m \leq n, a \in \mathbb{Z}\}$, the category $\mathcal{C}_\delta$ is the category of projective modules over $\mathrm{End}(\mathcal{T}_\delta)$ for a tilting bundle $\mathcal{T}_\delta$ on $\mathsf{Hilb}^n(\mathbb{C}^2)$.

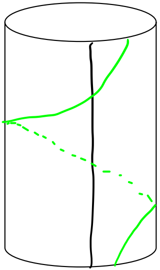
In particular, $D^b(\mathsf{Coh}(\mathsf{Hilb}^n(\mathbb{C}^2)) \cong K^b(\mathcal{C}_\delta)$.

How do we see this? We have to give a coherent sheaf on $\mathsf{Hilb}^n(\mathbb{C}^2)$ for each object in $\mathcal{C}_\delta$; we'll describe this using graded modules over the projective coordinate ring.

Definition

Let $\gamma\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/Z$ be a smooth function. Let a **γ-type diagram** *be a diagram with strands in the cylinder (as before) such that at height $y = t$, we place the ghost of a strand γ_t units to its right.*

As before, these diagrams are **generic** *if they avoid tangencies and triple points.*



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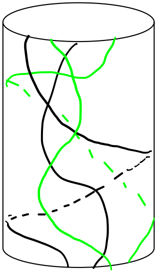


Definition

Let $\mathcal{D}_\gamma(S,T)$ be the $\mathbb{C}$-span of the space of generic γ-type diagrams with bottom S and top T modulo the local relations given before. Let $\mathcal{D}_{\delta,n}(S,T)$ be this space when $\gamma(x) = nx + \delta$.

Lemma

If γ and γ' are smoothly based homotopic, then $\mathcal{D}_\gamma \cong \mathcal{D}_{\gamma'}$. That is, $\mathcal{D}_\gamma \cong \mathcal{D}_{\delta,n}(S,T)$ with $\delta = \gamma_0 = \gamma_1$ and n the winding number of γ.

Stacking of cylinders gives an associative product

$$\mathcal{D}_{\delta,m}(S,T) \times \mathcal{D}_{\delta,n}(T,U) \to \mathcal{D}_{\delta,m+n}(S,U).$$

Theorem

The graded ring $\bigoplus_{n\geq 0}\mathcal{D}_{\delta,n}(\mathbf{a},\mathbf{a})$ *is the projective coordinate ring of* $\mathsf{Hilb}^n(\mathbb{C}^2)$.

For an n-tuple S of elements $\mathbb{R}/\mathbb{Z}$, and a fixed $\delta \in \mathbb{R}/\mathbb{Z}$, we let $\mathcal{T}_\delta(S)$ be the unique coherent sheaf satisfying

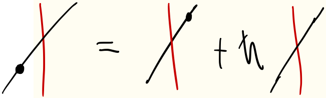
$$\Gamma(\mathsf{Hilb}^n(\mathbb{C}^2); \mathcal{T}_\delta(S) \otimes \mathcal{O}(n)) \cong \mathcal{D}_{\delta,n}(\mathbf{a}, T)$$

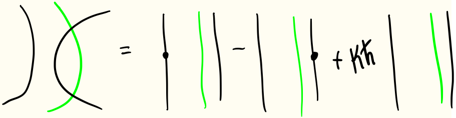
Theorem

The sheaves $\mathcal{T}_\delta(S)$ *for fixed* δ *generate* $\mathsf{Coh}(\mathsf{Hilb}^n(\mathbb{C}^2))$ *and* $\mathrm{Ext}^{>0}(\mathcal{T}_\delta(S), \mathcal{T}_\delta(S')) = 0$ *for all* S, S'.

How can we prove all of this? Use the Cherednik algebra $\mathsf{H} = \mathsf{H}_{1,\kappa}$.

We can quantize our relations by changing $\mathbb{C}[x,y]$ to the Weyl algebra where $[x,y] = \hbar$. We can modify our graphical relations to:





Theorem

In the deformed category $\mathcal{C}_\delta^\hbar$, we have

$$\mathrm{End}_{\mathcal{C}_\delta}(\mathbf{a}) \cong e_+ \mathsf{H} e_+ \qquad \delta \notin \Delta$$

$$\mathrm{End}_{\mathcal{C}_\delta}(\mathbf{r}) \cong \mathsf{H} \qquad 0 < \delta < \frac{1}{n}$$

In characteristic p, we get a Frobenius constant quantization of the Hilbert scheme, which gives an Azumaya algebra localizing H.

This Azumaya algebra isn't split, but it is after we base extend from the p-center of H by adding the dots u_i. Thus, the behavior of the splitting bundle is controlled by the interaction of different weight spaces for these dots.

Theorem

Let $\delta = \frac{\kappa}{p} + \frac{1}{2p}$. The Bezrukavnikov-Kaledin Azumaya algebra is split by the vector bundle

$$\bigoplus_{S \subset \frac{1}{p}\mathbb{Z}/\mathbb{Z}} \mathcal{T}_\delta(S).$$

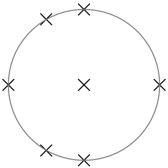
Question

Take your favorite construction in the theory of coherent sheaves on this Hilbert scheme, describe how it works in this language.

- For example, can one describe MacDonald polynomials in this language and give a proof of their positivity?
- What about triply graded knot homology, following the work of Gorsky, Negut and Rasmussen?

One interesting thing we can do is understand the wall-crossing functors.

Work of Bezrukavnikov defines an action of $\pi_1(\mathbb{C}^\times \setminus \{e^{2\pi i\alpha} \mid \alpha \in \Delta\})$ on $\mathsf{Coh}(\mathsf{Hilb}^n(\mathbb{C}^2))$.



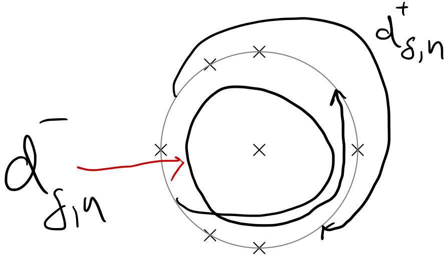
We can extend this to an action of the fundamental groupoid, sending

- elements with $|z| \neq 1$ to $D^b(\mathsf{Coh}(\mathsf{Hilb}^n(\mathbb{C}^2)))$
- elements with $z = e^{2\pi i\delta}$ with $\delta \in \mathbb{R} \setminus \Delta$ to $K^b(\mathcal{C}_\delta)$.

If $n \in \mathbb{Z} + \delta' - \delta$, then $\mathcal{D}_{\delta,n}$ is a bimodule over $\mathcal{C}_\delta$ and $\mathcal{C}_{\delta'}$.

Proposition

Derived tensor product with $\mathcal{D}_{\delta,n}$ defines a derived equivalence $K^b(\mathcal{C}_\delta) \to K^b(\mathcal{C}_{\delta+n})$ matching the action in the fundamental groupoid of the contours $d^+_{\delta,n}$ if $n > 0$ and $d^-_{\delta,n}$ if $n < 0$.



This is a special case of a much more general framework, due to Braverman, Finkelberg and Nakajima. They define a **Coulomb branch** attached to any connectected group G acting on a representation V.

Theorem (Kodera-Nakajima, Braverman-Etingof-Finkelberg)

The Hilbert scheme (with its Cherednik quantization) of $\widetilde{\mathbb{C}^2/\mathbb{Z}_\ell}$ is the Coulomb branch for GL_n acting on $\mathfrak{gl}_n \oplus (\mathbb{C}^n)^{\oplus \ell}$ (in a very strange presentation!).

Theorem (Bezrukavnikov-Kaledin, W.)

Any smooth Coulomb branch has a tilting generator with an explicit presentation generalizing the category $\mathcal{C}_\delta$ and similar description of wall-crossing.

Interesting special case: Slodowy slices in type A.

In this case, we replace $\mathcal{C}_\delta$ with a version of the KLR algebra of type A wrapped on a circle.

- From this description, we can check directly the Bezrukavnikov-Okounkov conjecture relating the wall-crossing functors to quantum cohomology.
- We can recover and generalize previous work of Anno and Nandamukar on this coherent sheaves.

Another interesting special case: when G is abelian.

Theorem (McBreen-W.)

The analogue of $\mathcal{C}_\delta$ in this case has a quadratic presentation, is Koszul, and is equivalent to the wrapped Fukaya category of the corresponding multiplicative hypertoric variety, as predicted by homological mirror symmetry.

We hope to push this result into other cases, but the Fukaya side is much more complicated for non-abelian groups.

Thanks for listening.