Representation Stability

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Motivating example: configuration spaces

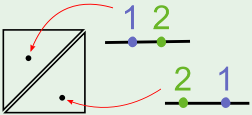
Definition (configuration space)

M – topological space

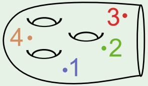
$F_n(M)$ – (ordered) configuration space of M on n points

$$F_n(M) := \{(m_1, m_2, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\} \subseteq M^n$$

$F_n(M) = M^n \setminus$ "fat diagonal"

Eg, $F_2([0,1]) =$ 

$$F_n(M) = \left\{ \begin{array}{c} \text{embeddings} \\ \{1, 2, 3, \ldots, n\} \hookrightarrow M \end{array} \right\}$$

Eg, 

$\in F_4(\Sigma)$

Motivating example: configuration spaces

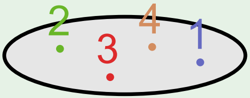
Definition (configuration space)

$F_n(M) := \{(m_1, m_2, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\} \subseteq M^n$

$\left\{ \begin{array}{c} \text{Connected components} \\ \text{of } F_n([0,1]) \end{array} \right\} \longleftrightarrow S_n$

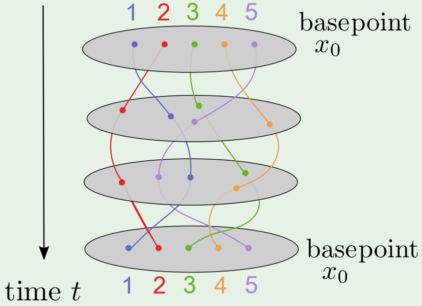
$\in F_4([0,1])$

$F_n(D^2)$ is connected



$\in F_4(D^2)$

$\pi_1(F_n(D^2)) =$ Artin's pure braid group PB_n

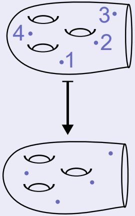


Unordered configuration spaces

$S_n \circlearrowleft F_n(M)$

Definition (Unordered configuration space)

$$\begin{array}{c} F_n(M) \\ \downarrow \\ C_n(M) := F_n(M)/S_n \end{array}$$



The *unordered configuration space of M on n points* is

$$C_n(M) = \left\{ \begin{array}{c} n\text{-element} \\ \text{subsets of } M \end{array} \right\}$$

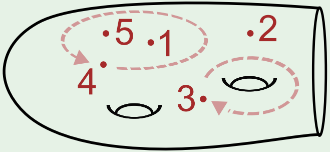
$\pi_1(C_n(D^2)) =$ Artin's braid group B_n

Fact: $F_n(D^2), C_n(D^2)$ are $K(\pi, 1)$ spaces for the (pure) braid groups

Homology of configuration spaces

Goal: Understand $H_*(F_n(M))$.

$S^1 \times S^1 \to F_5(M)$



$\rightsquigarrow$ A class in $H_2(F_5(M))$ (up to sign)

Hard problem: Understand additive relations between these classes.

Key: Fix M.

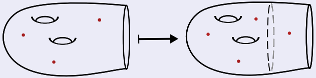
Package $\{H_*(F_n(M))\}_n$ into a single algebraic object, with additional structure coming from S_n–actions and topological operations.

Classical Homological Stablility for $C_n(M)$

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

$\exists$ *stabilization map*

$$t : C_n(M) \to C_{n+1}(M)$$



Theorem (McDuff, Segal, 70s))

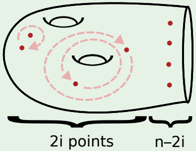
Fix M.

$\{C_n(M)\}_n$ *is* **homologically stable**.

For each i, the maps

$$t_* : H_i(C_n(M)) \to H_i(C_{n+1}(M))$$

is an isomorphism for $n \geqslant 2i$.

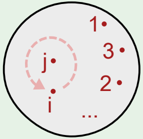
$H_i(C_n(M))$ is spanned by: 

Homological Stablility for $F_n(M)$?

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

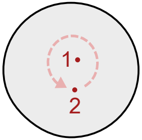
Question: Is $\{F_n(M)\}_n$ homologically stable?

Answer: No!

Eg, $H_1(F_n(D^2)) = \mathbb{Z}^{\binom{n}{2}}$, generators $\alpha_{i,j} =$ 

$\in H_1(F_n(D^2))$

Up to action of S_n
and stabilization map t,
$\{H_1(F_n(D^2))\}_n$ is generated by:

$\alpha_{1,2} =$ 

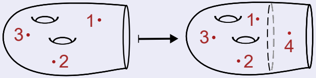
$\in H_1(F_2(D^2))$

Representation Stability for $F_n(M)$

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

$\exists$ stabilization map

$$t : F_n(M) \to F_{n+1}(M)$$

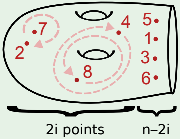


Theorem (Church–Ellenberg–Farb, Miller–W (non-orientable M))

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable***.*

$$\mathbb{Z}[S_{n+1}] \cdot t_*(H_i(F_n(M);\mathbb{Z})) = H_i(F_{n+1}(M);\mathbb{Z}) \qquad \text{for } n \geqslant 2i.$$

$H_i(F_n(M))$ is spanned by:



Further work

Original results: Church (2012), Church–Ellenberg–Farb (2015)

Generalizations, such as broader classes of spaces *M*, improved stable ranges, alternate stabilization maps, "higher-order" stability:

Church–Ellenberg–Farb–Nagpal (2014)
Ellenberg–Wiltshire-Gordon (2015)
Hersh–Reiner (2017)
Church–Miller–Nagpal–Reinhold (2017)
Moseley–Proudfoot–Young (2017)
Miller–W (2019, 2020)
Pawlowski–Ramos–Rhoades (2020)
Wawrykow (2022, preprint)
Bibby–Gadish (2023)
Lütgehetmann (preprint)
Wiltshire-Gordon (preprint)
Alpert–Manin (preprint)

Palmer (2013)
Kupers–Miller (2015)
Petersen (2017)
Schiessl (2017)
Gadish (2017)
Ramos (2017, 2018, 2020)
Bahran (preprint)
Ho (preprint)
Tosteson (preprint)
Alpert (preprint)
Himes (preprint)

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable***.*

- **finite generation**

$$\mathbb{Z}[S_{n+1}] \cdot t_*(H_i(F_n(M);\mathbb{Z})) = H_i(F_{n+1}(M);\mathbb{Z}) \qquad \text{for } n \geqslant 2i.$$

- **polynomial Betti numbers**

$$dim_{\mathbb{Q}} H_i(F_n(M);\mathbb{Q}) = \text{polynomial in } n \text{ of degree} \leqslant 2i$$

Eg, $\dim_{\mathbb{Q}} H_1(F_n(D^2);\mathbb{Q}) = \binom{n}{2} = \frac{(n)(n-1)}{2}$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable**.

- **multiplicity stability**

  The decomposition of $H_i(F_n(M);\mathbb{Q})$ into irreducible S_n–reps stabilizes for $n \geqslant 4i$.

$$
\begin{aligned}
\text{Eg, } H_1(F_2(D^2);\mathbb{Q}) &\cong V_{(2)} \\
H_1(F_3(D^2);\mathbb{Q}) &\cong V_{(3)} \oplus V_{(2,1)} \\
H_1(F_4(D^2);\mathbb{Q}) &\cong V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \\
H_1(F_5(D^2);\mathbb{Q}) &\cong V_{(5)} \oplus V_{(4,1)} \oplus V_{(3,2)} \\
H_1(F_6(D^2);\mathbb{Q}) &\cong V_{(6)} \oplus V_{(5,1)} \oplus V_{(4,2)} \\
H_1(F_7(D^2);\mathbb{Q}) &\cong V_{(7)} \oplus V_{(6,1)} \oplus V_{(5,2)}
\end{aligned}
$$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ *is* **representation stable**.

- **character polynomials**

  The character $\chi_{H_i(F_n(M);\mathbb{Q})}$ *is a polynomial in the "cycle-counting" functions, independent of n.*

Eg, $\chi_{H_1(F_n(D^2);\mathbb{Q})}(\sigma) = (\#\text{2-cycles in } \sigma) + \binom{\#\text{1-cycles in } \sigma}{2}$
for $\sigma \in S_n$, for all n.

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable**.

- **recursive resolutions**

  For $n \geqslant 2i+2$, the S_n–rep $H_i(F_n(M))$ is determined by a partial resolution by S_n–reps

$$Ind_{S_{n-2}}^{S_n} H_i(F_{n-2}(M)) \longrightarrow Ind_{S_{n-1}}^{S_n} H_i(F_{n-1}(M)) \longrightarrow H_i(F_n(M)) \longrightarrow 0$$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable**.

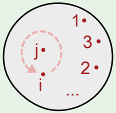
- **free module structure**

  $H_i(F_n(M))$ is an induced module of a certain form, induced specific from certain subreps of

$$H_i(F_0(M)), H_i(F_1(M)), \ldots, H_i(F_{2i}(M))$$

$$\text{Eg, } H_1(F_n(D^2)) = \bigoplus_{\{i,j\}\subseteq\{1,2,\ldots,n\}} \mathbb{Z}\alpha_{i,j}$$

$$\cong \mathrm{Ind}_{S_2\times S_{n-2}}^{S_n} H_1(F_2(D^2))$$

$\alpha_{i,j} =$ 

Other instances of representation stability

Analogous behaviour has been established in the (co)homology of:

- certain flag varieties (Weyl group reps)
- hyperplane arrangements associated to reflection groups W_n (W_n-reps)
- congruence subgroups $\mathrm{GL}_n(A, I) \subseteq \mathrm{GL}_n(A)$ (S_n- or $\mathrm{GL}_n(A/I)$-reps)
- mapping class groups and moduli spaces (S_n-reps)
- Torelli and related groups ($\mathrm{Sp}_{2g}(\mathbb{Z})$-reps, etc)

$\vdots$

Question: What underlying structure is driving these stability patterns?

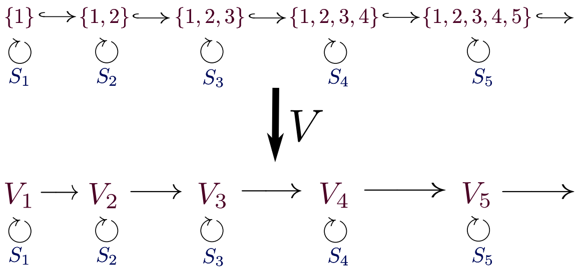
FI and FI-modules

Answer: They are finitely presented FI–modules.

Definition (FI and FI-modules)

Let FI denote the category of **F**inite sets and **I**njective maps.

An FI-*module* is a functor V : FI $\to$ Ab Gps.



FI and FI-modules

Examples of FI-modules

Example: $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$ trivial S_n-reps

Example: $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \hookrightarrow \cdots$ canonical S_n permutation reps

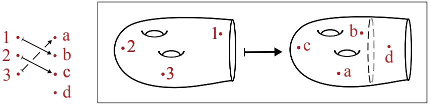
Example: $\mathbb{Z}[x_1] \hookrightarrow \mathbb{Z}[x_1, x_2] \hookrightarrow \mathbb{Z}[x_1, x_2, x_3] \hookrightarrow \cdots$ S_n permutes variables

Non-Example: $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$ alternating S_n-reps

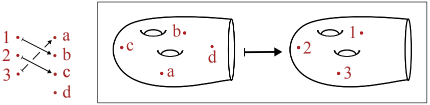
Non-Example: $\mathbb{Z}[S_1] \hookrightarrow \mathbb{Z}[S_2] \hookrightarrow \mathbb{Z}[S_3] \hookrightarrow \cdots$ left regular S_n-reps

Example: $H_i(F_1(M)) \to H_i(F_2(M)) \to H_i(F_3(M)) \to \ldots$

FI-action:



FI^{op}-action:



Finite generation

Finite generation

Homogeneous degree-2 polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

$$\begin{array}{ccccccc}
S_1 \circlearrowleft & & S_2 \circlearrowleft & & S_3 \circlearrowleft & & \\
R[x_1]_{(2)} & \hookrightarrow & R[x_1, x_2]_{(2)} & \hookrightarrow & R[x_1, x_2, x_3]_{(2)} & \hookrightarrow & \\
\| & & \| & & \| & & \\
\langle x_1^2 \rangle & & \langle x_1^2, x_2^2, x_1x_2 \rangle & & \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_3x_2 \rangle & & \\
\uplus & & \uplus & & & & \\
x_1^2 & & x_1x_2 & & & &
\end{array}$$

Generators

$\{\mathbb{Z}[x_1, \ldots, x_n]_{(2)}\}_n$ is *finitely generated in degree* $\leqslant 2$ by generators

$$x_1^2 \in V_1, \quad x_1x_2 \in V_2.$$

Current directions

Goals:

- Develop commutative algebraic tools for proving finiteness properties of FI-modules.
- Adapt tools to study other categories (eg) encoding actions of different families of groups.

Thank you!