Equivariant K-theory: A Talk for the UChicago 2016 Summer School

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(1) In the past few talks we've gotten to know K-theory. It has many faces.

$$K\text{-theory is...}\begin{cases}\text{A way to study vector bundles}\\ \text{An invariant for spaces } (\textit{cohomology theory})\\ \text{An object in its own right } (\textit{a spectrum})\end{cases}$$

In this talk we're gonna put a (compact Lie group) G everywhere and see what happens. I'll try to say why this is a good idea in a bit (c.f. §2), but before I do let's just push on with the thought experiment of where we can stick a G. You can safely take $G = C_2$ and the whole talk will still be interesting (or as interesting as it would have been otherwise.)

Riffing off of the above:

$$\textit{Equivariant } K\text{-theory should be...}\begin{cases}\text{A way to study } G\text{-vector bundles}\\ \text{An invariant for spaces with a } G\text{-action } (G\textit{-cohomology theory})\\ \text{An action of } G \text{ on } K \ (\textit{a } G\textit{-spectrum?})\end{cases} \quad {}^{1}$$

Even though it's perhaps the least intuitive, the last notion is really the strongest. Let me justify that by showing how an action of G on K-theory gives us the other two interpretations of equivariant K-theory for free.

But first, there's a couple things I could mean by 'an action of G on K-theory'. For example, I could ask that every $g \in G$ gives rise to a natural transformation

$$g : K^0(-) \longrightarrow K^0(-).$$

between functors on the homotopy category of pointed spaces. Since $K^0(-) \cong [-, BU \times \mathbb{Z}]$, this amounts to asking for an action of G on the object $BU \times \mathbb{Z}$ *in the homotopy category of spaces.* I want to ask for something stronger: I want to ask for an action of G on the *space* $BU \times \mathbb{Z}$. In fact, I will assume we have a compatible action of G on each of the Grassmanians $BU(n)$. An action on the space lets me build an invariant for G-spaces:

$$[X, BU]^G := \text{equivariant maps up to equivariant homotopy equivalence.}^2$$

This is functorial for equivariant maps and doesn't mind equivariant homotopies between equivariant maps. I will explain later why trying to do this with just an action on the homotopy type is a bad idea (cf. Remark 1.13(d)).

[1] The question mark is because it's not currently in vogue to call a spectrum with an action of G a G-spectrum.

[2] I've dropped the $\mathbb{Z}$ at this point since it wasn't doing much.

We could also *define* a G-vector bundle to be an element of $[X, BU(n)]^G$. Since we have a map: $[X, BU(n)]^G \to [X, BU(n)]$ we can think of elements of this set as vector bundles of rank n together with some extra structure. This map is neither injective nor surjective in this generality (since our action G was totally random), so we can interpret that to mean that a given vector bundle may admit many or no structures of this kind.

If we have any hope of getting a nice theory, we should look for particularly nice actions of groups on the $BU(n)$. There are two examples that are lying around in nature (though the first is more natural than the second).

(1) The group C_2 acts on $\mathbb{C}^\infty$ by complex conjugation. This induces a map on $BU(n)$ which sends a rank n subspace $V \subset \mathbb{C}^\infty$ to its image in $\mathbb{C}^\infty$ after conjugation, $\overline{V}$.

(2) Let $\mathcal{V}$ denote the direct sum of all (isomorphism classes of) finite dimensional representations of G. Then G acts on $\mathcal{V}$ and hence on its Grassmanians, $Gr_n(\mathcal{V})$. Choosing a basis gives an action on the $BU(n) \cong Gr_n(\mathbb{C}^\infty)$.

Now we should ask: what structure on vector bundles do these actions yield?

Let's start with situation (1).

Definition 1.1 (Atiyah)**.** A **R**eal vector bundle over a space X with C_2-action $\tau : X \longrightarrow X$ is the data of a vector bundle $\pi : E \longrightarrow X$ and a C_2-action c on the space E such that:

(i) The projection π is equivariant, and

(ii) The map $\overline{\tau} : E \longrightarrow E$ is conjugate-linear on each fiber. That is, $c : E_x \longrightarrow E_{\tau(x)}$ satisfies $c(\lambda v) = \overline{\lambda} c(v)$ for $\lambda \in \mathbb{C}$.

Remark 1.2. If $X \subset \mathbb{P}^n(\mathbb{C})$ is the set of solutions to a system of homogeneous, polynomial equations with coefficients in $\mathbb{R}$ then complex conjugation provides an action of C_2 on X. The fixed points are called the 'real points' of X, denoted $X(\mathbb{R})$, because they are precisely the set of solutions with real coordinates. This is why we are stuck with Atiyah's horrible terminology.

Example 1.3. If, in the situation of 1.2, X happens to be smooth, then its tangent bundle is a complex vector bundle and the conjugation on X provides it with a **R**eal structure.

Now suppose we're in the second situation. Arguing as before, and using the fact that G acts *linearly* on $\mathbb{C}^\infty$ (as opposed to complex linearly), we come upon the following notion.

Definition 1.4. A G-vector bundle over a G-space X is the data of a vector bundle $\pi : E \longrightarrow X$ together with a G-action on E such that:

(i) The projection π is equivariant, and

(ii) The action map $E_x \longrightarrow E_{gx}$ is complex-linear.

Example 1.5. If X is a smooth manifold with a smooth action of G then the complexified tangent bundle $T_X \otimes \mathbb{C}$ gives an example of a G-vector bundle.

Now we can use the Grothendieck construction to define invariants $K\mathbf{R}$ and K_G called **R**eal K-theory and equivariant K-theory, respectively. I'll explain in a minute how these extend to cohomology theories and spectra, but first some examples and remarks.

Example 1.6. Let's see what happens when $X = *$ is a point.

Real case: A **R**eal vector bundle over a point is the data of a complex vector space V with a conjugate-linear involution, τ. The fixed points V^τ form a real vector space (no funky boldface) and the inclusion yields an isomorphism of complex vector space: $V^\tau \otimes_{\mathbb{R}} \mathbb{C} \overset{\cong}{\to} V$. So the category of **R**eal vector spaces is equivalent to the category of real vector spaces (phew) and we have

$$K\mathbf{R}^0(*) = KO^0(*) = \mathbb{Z}.$$

G case: A G-vector bundle over a point is just a complex vector space with a linear action of G. Such a thing is called a *representation* of G. The Grothendieck group of complex representations is called the representation ring and denoted $R(G)$, so we have

$$K_G^0(*) = R(G).$$

This example can be generalized:

Exercise 1.7. When G acts trivially on X we have: (i) for $G = C_2$, $K\mathbf{R}^0(X) = KO^0(X)$, and (ii) $K_G^0(X) = R(G) \otimes_{\mathbb{Z}} K^0(X)$.

Example 1.8. Now let's see what happens when $X = G$ with its usual action.

Real case: A **R**eal vector bundle over $C_2 = \{\pm 1\}$ is determined by the complex vector bundle over $+1$, since we are forced to define $E_{-1} = \overline{E_{+1}}$. Thus:

$$K\mathbf{R}^0(C_2) = K^0(*) = \mathbb{Z}.$$

G case: Any G-vector bundle E over G is determined by its fiber over the identity since the action of G provides an equivariant equivalence $G \times E_0 \longrightarrow E$ over G. Thus:

$$K_G^0(G) = K^0(*) = \mathbb{Z}.$$

Exercise 1.9. More generally, if G acts freely on X then $K_G^0(X) = K^0(X/G)$.

Exercise 1.10. The previous result is *not* true for $K\mathbf{R}$ (a counterexample is the space $S^1 \times S^1$ where C_2 acts by the antipodal action on the second factor; but it's hard to show this directly.) Nevertheless, prove that $K\mathbf{R}^0(X \times C_2) \cong K^0(X)$.

Example 1.11. Take $X = G/H$ where H is a closed subgroup. If E is a G-vector bundle then the fiber over the identity coset, E_0, is an H-representation. Denote by $G \times_H E_0$ the quotient of $G \times E_0$ by the relation $(gh, v) \sim (g, hv)$ for $h \in H$. Then the map $G \times_H E_0 \longrightarrow E$ turns out to be an isomorphism. If you start with an H-representation E_0 then you can prove that $G \times_H E_0$ is a G-vector bundle (this is where you need H to be a closed subgroup of a compact Lie group, or something close to it, to prove local triviality.) It follows that

$$K_G^0(G/H) = K_H^0(*) = R(H).$$

Exercise 1.12. If Y is a space with an action of H, define $G \times_H Y$ as the quotient of $G \times Y$ by the relation $(gh, y) \sim (g, hy)$. Then $K_G^0(G \times_H Y) \cong K_H^0(Y)$.

Remarks 1.13. (a) When $G = C_2$ we now have two objects of interest: K_{C_2} and $K\mathbf{R}$. They are *not the same*. In the former case the C_2-action on a bundle is complex linear, and in the latter it is conjugate linear.

(b) There is a common generalization of these theories called $K\mathbf{R}_G$ which has to do with **R**eal representations of G. This is used to interpolate between K_G and KO_G- we'll see an example of this below when G is the trivial group.

(c) [3] Notice that the fixed point space BU^{C_2} under the action of conjugation is BO. This is one proof of (1.7(i)). The fixed point space of BU under the action of G, however, is a bunch of copies of BU. This point is a little subtle, but it's possible to show that

$$(BU \times \mathbb{Z})^G \cong BU \times R(G)$$

where $R(G)$ is the group completion of the monoid of complex representations of G. You can use this to give a proof of (1.11), for example.This explains how $K\mathbb{R}$ is a mixture of real and complex K-theory, while K_G is a mixture of K-theory and the representation ring of G.

[3] I'd like to thank Peter May for correcting an earlier draft of these notes that made the incorrect claim that $BU^G = BU$.

(d) Homotopy theory doesn't see the action of G on K-theory, but it *does* see the action of C_2 given by conjugation. To be more precise, the action of G factors through the natural action of the space of linear isometries $\mathcal{L}(\mathbb{C}^\infty, \mathbb{C}^\infty)$ which acts on BU, and on K-theory. This space is, famously, contractible. So the action is trivial 'up to homotopy'. On the other hand, the space of isometries which are *either* linear or conjugate linear is homotopy equivalent to C_2, and this acts on BU and K-theory by conjugation as indicated. This action can be detected in homotopy theory. For example, on $\widetilde{K}(S^2) \cong \mathbb{Z}$ conjugation acts by -1.

(e) The previous item (d) justifies why we had to demand more than just an action of G on the homotopy type of BU. In the case of K_G, the action is trivial, so $[X, BU]^G$ is not an interesting invariant. In the case of $K\mathbf{R}$, we wouldn't get the link to KO. For example, the fixed points of the action of C_2 on $\widetilde{K}(S^2)$ are just 0- and that's less interesting than the computation $\widetilde{KO}(S^2) = \mathbb{Z}/2$.

(2) At this point I owe a debt: I've introduced a random generalization of some mathematical object without giving a reason to care. You should never do that. I should tell you how we can prove new theorems about old objects and how we can give new proofs of old theorems.

Let me start with the latter. It's pretty easy, with current technology, to give quick proofs of *complex* Bott periodicity. One version of Bott periodicity says there's a canonical homotopy equivalence $\Omega^2 BU \cong BU$. Translating this into a statement about K-theory

Theorem 2.14 (Bott)**.** *There is a natural isomorphism*

$$\widetilde{K}^0(\mathbb{C}P^1 \times X) \cong K^0(X)$$

In fact, $K^* \cong \mathbb{Z}[v^{\pm 1}]$ as a graded ring, where $v \in K^{-2} = \widetilde{K}^0(S^2)$ is the class corresponding to $[\mathcal{O}(1)] - 1$. (Here $\mathcal{O}(1)$ is the dual of the tautological bundle on $\mathbb{C}P^1 \cong S^2$).

Some proofs of complex Bott periodicity generalize to giving periodicity theorems for the two flavors of equivariant K-theory we've seen. One of these theorems looks the same as the nonequivariant one, but the other is more interesting:

Theorem 2.15 (Atiyah, Segal)**.** *There are natural isomorphisms*

$$K^0_G(X) \cong \widetilde{K}^0_G(S^2 \wedge X_+),$$

$$K\mathbf{R}^0(X) \cong K\mathbf{R}^0(\mathbb{C}P^1 \wedge X_+)$$

where we give $\mathbb{C}P^1$ the action by complex conjugation*.*

It's harder (and usually just omitted) to find quick proofs of *real* Bott periodicity, for KO. But it turns out you can use the second isomorphism above to *deduce* real Bott periodicity from complex Bott periodicity together with that conjugation action. Even better, we'll be able to describe the structure of the graded ring KO^*.

Let me show you show that goes. First I'll need a piece of notation. If V is a vector space, denote by S^V its one-point compactification, which has a natural basepoint at ∞. If V happens to be a representation of G, then we get a representation of G. Here are two pleasant exercises:

Exercise 2.16. Show that there is a natural homeomorphism $S^V \wedge S^W \cong S^{V \oplus W}$. In particular, you recover the fact that $S^n \wedge S^m \cong S^{n+m}$.

Exercise 2.17. Let ρ denote the regular representation of C_2 (i.e. the action on $\mathbb{R}^2$ given by permuting the basis vectors.) Show that there is an equivariant homeomorphism $S^\rho \cong \mathbb{C}P^1$ where C_2 acts by conjugation on $\mathbb{C}P^1$.

Let σ denote the sign representation, i.e. $\mathbb{R}$ with its action of -1, and write $a+b\sigma$ for the representation $\mathbb{R}^{\oplus a} \oplus \sigma^{\oplus b}$. Note that $\rho = 1+\sigma$.

Then the periodicity theorem allows us to define a cohomology theory on C_2-spaces by:

$$K\mathbf{R}^n(X) = \begin{cases} \widetilde{K\mathbf{R}}^0(S^{-n} \wedge X_+), & n \leq 0 \\ \widetilde{K\mathbf{R}}^0(S^{n\sigma} \wedge X_+), & n \geq 0 \end{cases}$$

Cohomology theories have long exact sequences associated to pairs. Here's a nice pair: $(I, \{\pm 1\})$ where I is the unit interval and C_2 acts by $\{\pm 1\}$. Notice that $I/\{\pm 1\} \cong S^\sigma$. The interval is *equivariantly* contractible with this action, so we get a long exact sequence that looks like:

$$\cdots \longrightarrow K\mathbf{R}^*(S^\sigma) \longrightarrow K\mathbf{R}^* \longrightarrow K\mathbf{R}^*(C_2) \longrightarrow K\mathbf{R}^{*+1}(S^\sigma) \longrightarrow \cdots$$

But now remember that $K\mathbf{R}^* = KO^*$ and $K\mathbf{R}^*(C_2) = K^*$, by our previous examples. So we can rewrite this sequence as:

$$\cdots \longrightarrow K\mathbf{R}^*(S^\sigma) \longrightarrow KO^* \longrightarrow K^* \longrightarrow K\mathbf{R}^{*+1}(S^\sigma) \longrightarrow \cdots$$

Exercise 2.18. Show that periodicity implies that $K\mathbf{R}^*(S^\sigma) = KO^{*+1}$.

Exercise 2.19. Show that the map $KO^* \longrightarrow K^*$ in the sequence above is induced by complexification of vector bundles.

So we end up with:

$$\cdots \longrightarrow KO^{*+1} \xrightarrow{\chi} KO^* \xrightarrow{c} K^* \xrightarrow{\delta} KO^{*+2} \longrightarrow \cdots$$

Bott periodicity then follows from two calculations. First we need to know what the maps in the above sequence are.

Lemma 2.20. *(a) The map χ is given by multiplication by η where $\eta \in \widetilde{KO}(\mathbb{R}P^1)$ is $[\gamma_1]-1$ where γ_1 is the tautological line bundle.*

(b) The map δ is given by Bott periodicity followed by the map induced by taking the underlying real bundle associated to a complex bundle:

$$K^{-n} \overset{\cong}{\longleftarrow} K^{-n+2} \overset{\text{real}}{\longrightarrow} KO^{-n+2}.$$

This already buys us a computation of KO^{-1} and KO^{-2}.

$$\begin{tikzcd} 0 \arrow[r] & KO^{-1} \arrow[r] \arrow[d, "\cong"] & KO^{-2} \arrow[r] \arrow[d, "\cong"] & K^{-2} \arrow[r, "\delta"] \arrow[d, "\cong"] & KO^0 \arrow[r] \arrow[d, "\cong"] & KO^{-1} \arrow[r] \arrow[d, "\cong"] & 0 \\ & \mathbb{Z}/2 \arrow[r, "\cong"'] & \mathbb{Z}/2 \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "\cdot 2"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & \end{tikzcd}$$

The bottom sequence follows from the top one by exactness as soon as we justify that multiplication by 2. But that's just because the underlying real bundle of $\mathbb{C}$ is $\mathbb{R}^2$.

The other main calculation that needs to be done is the following.

Lemma 2.21. *There is a relation $\eta^3 = 0 \in KO^{-3}$.*

Proof. One can compute directly that

$$\pi_3 BO = \pi_2 O = \pi_2 O(4) = \pi_2 SO(4)$$

But $SO(4)$ is double covered by $S^3 \times S^3$, and $\pi_2 S^3 = 0$, so the result follows. $\square$

That's kind of unsatisfying. It's possible, but hard without some sophistication, to give a direct proof.

Exercise 2.22 (Hard)**.** Provide an explicit trivialization of η^3 over S^3.

This lemma combined with the previously compute KO^{-2} implies that the next piece of the sequence looks like:

$$\begin{tikzcd}
0 \arrow[r] & KO^{-3} \arrow[r] \arrow[d, equal] & KO^{-4} \arrow[r] \arrow[d, "\cong"] & K^{-4} \arrow[r] \arrow[d, "\cong"] & KO^{-2} \arrow[r] \arrow[d, "\cong"] & KO^{-3} \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 &
\end{tikzcd}$$

Now, the map $K^{-6} \to KO^{-4}$ can be computed by including into K^{-4}. The composite $c \circ \mathrm{real}$ takes a vector bundle E to $E \oplus \overline{E}$. Now, the action of complex conjugation on $K^{-2} = \mathbb{Z}$ is by -1 on the generator v. But complex conjugation acts by ring maps on K^*, so we are forced to have a stable equivalence $(\overline{v})^2 = v^2$. So we have shown that the composite $K^{-6} \to KO^{-4} \to K^{-4}$ is multiplication by 2, which forces the first map to be an isomorphism. This implies the final pieces of the puzzle:

$$\begin{tikzcd}
0 \arrow[r] & KO^{-5} \arrow[r] \arrow[d, equal] & KO^{-6} \arrow[r] \arrow[d, equal] & K^{-6} \arrow[r] \arrow[d, "\cong"] & KO^{-4} \arrow[r] \arrow[d, "\cong"] & KO^{-5} \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cong"'] & \mathbb{Z} \arrow[r] & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & KO^{-7} \arrow[r] \arrow[d, equal] & KO^{-8} \arrow[r] \arrow[d, "\cong"] & K^{-8} \arrow[r] \arrow[d, "\cong"] & KO^{-6} \arrow[r] \arrow[d, equal] & KO^{-7} \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cong"'] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

So there is an element $\beta \in KO^{-8}$ whose complexification is v^4, where v gives complex Bott periodicity. Using the compatibility of our exact sequence with products (complexification is a ring map and δ acts like a derivation) we get:

Theorem 2.23 (Bott periodicity)**.** *As a ring,*

$$KO^* = \mathbb{Z}[\eta, b, \beta^{\pm 1}]/(\eta^3, 2\eta, b^2 = 4\beta).$$

Under complexification, b maps to $2v^2$ and β maps to v^4, where $v \in K^{-2}$ is the gives complex Bott periodicity.

Remark 2.24. Atiyah gives a prettier proof of this deduction of the 8-periodicity of KO from the 2-periodicity of K by proving an intermediate result about the 4-periodicity of a theory called *self-conjugate K theory* which is given by $K\mathbf{R}((-) \times S^\sigma)$.

(3) Ok, that's a new take on an old theorem. What about new results? Well let's take for granted that you're interested in computing $K^*(X)$ for your favorite spaces X. This seems like a generally good idea. Knowing just about the K-theory of spheres you can prove the Hopf invariant one theorem. Adams solved the vector fields on spheres problem by computing, among other things, the K-theory of real projective spaces.

Another good class of spaces that we like to compute invariants of are *classifying spaces BG*, where G is a compact Lie group.

(Here, by classifying space, I mean a space X with a principal G-bundle on it such that $[-, X]$ represents the functor assigning to a compact space the set of isomorphism classes of principal G-bundles on that space. Here is a concrete model for such a homotopy type: (i) choose a faithful representation $G \hookrightarrow U(n)$, (ii) this gives an action of G on the Stiefel manifold of bases for n-dimensional spaces inside $\mathbb{C}^\infty$, (iii) define BG to be the quotient by this action. Intuitively, BG classifies vector bundles of rank n where we can choose all the transition maps to lie in the image of the representation $G \subset U(n)$. When G is a finite group, the homotopy type of BG is uniquely determined by requiring $\pi_1 BG = G$ and $\pi_* BG = 0$ otherwise.)

Example 3.25. $\mathbb{R}P^\infty$ is a model for BC_2 and $\mathbb{C}P^\infty$ is a model for BS^1.

Example 3.26. $BSO(n)$ classifies oriented vector bundles of rank n.

Example 3.27. If $H \subset G$ is a closed subgroup then there is a fiber bundle:

$$G/H \longrightarrow BH \longrightarrow BG.$$

So knowledge of BH and BG can tell us about homogeneous spaces. Lots of spaces are homogeneous spaces: spheres, projective spaces, Grassmanians, flag manifolds.

Example 3.28. Homomorphisms of groups induce maps of classifying spaces. So homotopy invariants of classifying spaces give invariants of groups. For example, the ordinary cohomology of BG is called 'group cohomology' and has purely algebraic applications.

So a good question to ask is: How do we compute $K^*(BG)$ for our favorite groups G?

Construction 3.29. Let V be a representation of G. Let EG denote some contractible space on which G acts freely (like the Stiefel manifold from earlier). Then define $E_V := EG \times_G V$, which is the quotient of $EG \times V$ by the relation $(xg, v) \sim (x, gv)$. Then the projection $\pi : E_V \longrightarrow BG$ is a vector bundle with fiber V. This is called the *Borel construction.*

The Borel construction respects direct sums and tensor products and so gives a ring homomorphism:

$$R(G) \longrightarrow K^0(BG).$$

But recall that $R(G) = K^0_G$. In fact, this map is just the map induced from the equivariant map: $EG \longrightarrow *$, since $K^0_G(EG) = K^0(BG)$. This point of view gives us a map:

$$K^*_G \longrightarrow K^*(BG)$$

attempting to compute the whole ring $K^*(BG)$. This respects all your favorite structure, too, like Adams operations. One might hope that you could use equivariant homotopy theory to prove something about this map. The answer is yes: and all the known proofs use the fact that K^*_G is a cohomology theory, i.e. in the course of the proof one is forced to consider the cohomology of spaces other than a point or EG.

Theorem 3.30 (Atiyah-Segal). *Let $I \subset K^*_G$ denote the ideal generated by virtual representations of dimension 0. Then the Borel construction induces an isomorphism upon completion:*

$$(K^*_G)^{\widehat{}}_I \xrightarrow{\cong} K^*(BG).$$

That is to say: $K^{\text{even}}(BG) \cong R(G)^{\widehat{}}_I$ and $K^{\text{odd}}(BG) = 0$.

Henry will talk more about this theorem and its proof later, but for now let's record a few examples to get a feel for the theorem.

Example 3.31. If $G = C_2$, then the representation ring is pretty simple. (You should verify this as an exercise). $R(C_2) = \mathbb{Z}[\sigma]/(\sigma^2 - 1)$, where σ is the sign representation on $\mathbb{C}$. It will be more convenient to write this as $\mathbb{Z}[x]/2x + x^2$ where $x = \sigma - 1$. Then the augmentation ideal is just x and we get $K^0(\mathbb{R}P^\infty) = \mathbb{Z}[\![x]\!]/(2x + x^2)$.

Example 3.32. More generally, if $G = C_p$, then $R(C_p) = \mathbb{Z}[\lambda]/(\lambda^p - 1)$. Again it's convenient to write this in terms of $x = \lambda - 1$. Then $R(C_p) = \mathbb{Z}[x]/((x+1)^p - 1)$ and $K^0(BC_p) = \mathbb{Z}[\![x]\!]/((x+1)^p - 1) = \mathbb{Z}[\![x]\!]/(px + \cdots + x^p)$.

Example 3.33. If $G = S^1$ then $R(S^1) = \mathbb{Z}[t, t^{-1}]$ where t is the standard 1-dimensional representation of S^1. The augmentation ideal is generated by $x = t - 1$ so $K^0(\mathbb{C}P^\infty) = \mathbb{Z}[\![x]\!]$. (We don't need to invert $x + 1$ since this happens for free: $t^{-1} = \dfrac{1}{1+x} = \sum_{n \geq 0} (-1)^n x^n$.)

Remark 3.34. The hard groups are really $KO^*(BG)$. There is a completion theorem in this case as well, involving KO_G^*. Of course, for that to be useful we have to be able to compute KO_G^*- it's more complicated that just the real representation ring, as is evidenced by the case when $G = \{e\}$. Luckily, **R**eal K-theory comes again to save the day. One can use it similarly to how we used it in the previous section to show that KO_G^* is 8-periodic and one can compute each group in terms of $RO(G)$, $R(G)$, and the representation ring for *quaternionic* representations. If you only care about KO^0 the result is the same because $KO_G^0 = RO(G)$.

Let me finish by advertising a relatively recent application of equivariant methods. Just as there's a natural action of C_2 on K-theory yielding a cohomology theory $K\mathbf{R}$, there's also a natural action of C_2 on MU which yields a cohomology theory $MU\mathbf{R}$- **R**eal cobordism. Hill, Hopkins, and Ravenel used this cohomology theory (or rather, a C_8-equivariant version built from it) and some serious equivariant homotopy theory to prove the Kervaire invariant one theorem (except for a single dimension). This ushered in a new era of people getting excited about equivariant homotopy theory. Perhaps you will be one of these people too.