WEIL–DELIGNE REPRESENTATIONS I
LOCAL LANGLANDS SEMINAR

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1. Notation

- $p :=$ a fixed prime
- $K :=$ a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$
- $\overline{K} :=$ an algebraic closure of K
- $O_K :=$ the ring of integers of K
- $\kappa :=$ the residue field of $\mathcal{O}_K$
- $q :=$ the cardinality of κ
- $G_K := \mathrm{Gal}(\overline{K}/K)$

2. The Weil group

- Let σ_K be the arithmetic Frobenius automorphism $(x \mapsto x^q)$ and $\phi_K = \sigma_K^{-1}$ the geometric Frobenius.
- Let I_K be the inertia group of K, i.e. $I_K := \ker(\pi : G_K \to G_\kappa)$.
- Note that $G_\kappa = \mathrm{Gal}(\overline{K}/K^{nr}) \cong \hat{\mathbb{Z}}$ where K^{nr} is the maximal unramified extension.
- The Weil group W_K is the inverse image of $\langle \sigma_K \rangle$ under π,

$$0 \to I_K \to W_K \to \langle \sigma_K \rangle \to 0,$$

endowed with the topology of a locally compact group such that $W_K \to \langle \sigma_K \rangle \cong \mathbb{Z}$ is continuous where $\mathbb{Z}$ has the discrete topology and I_K has

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the profinite topology from G_K. This is not the subspace topology. The canonical injective homomorphism $\Phi_K : W_K \hookrightarrow G_K$ is continuous and from the inclusion.

- Alternatively, $W_K \cong \text{Proj}\ \lim_L W_{L/K}$, where $W_{L/K} := W_K / W_L^c$ and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L^\times & \longrightarrow & W_{L/K} & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle N_{L'/L}} & & & & \uparrow & & \\ 1 & \longrightarrow & L'^\times & \longrightarrow & W_{L'/K} & \longrightarrow & \mathrm{Gal}(L'/K) & \longrightarrow & 1, \end{array}$$

  so the canonical injective homomorphism $\Phi_K : W_K \hookrightarrow G_K$ with dense image is the projective limit of homomorphisms $W_{L/K} \hookrightarrow \mathrm{Gal}(L/K)$.

3. Representations of the Weil group

Definition 3.1. Let $\mathrm{Rep}(G)$ denote the category of representations of G.

Remark 3.2. Since Φ_K is injective with dense image, we can identify $\mathrm{Rep}(G_K)$ with a sub-category of $\mathrm{Rep}(W_K)$.

Definition 3.3. A representation of W_K that lies in the subcategory corresponding to $\mathrm{Rep}(G_K)$ is called *Galois-type.*

Example 3.4. Via $r_K : K^\times \cong W_K^{ab}$, the absolute value $|\cdot|_K$ on $K^\times$ gives the absolute value character $\omega : W_K \to \mathbb{C}^\times$ sending $x \mapsto |x|_K$. This has infinite image and therefore is not a character of G_K.

Proposition 3.5. *A representation* ρ *of* W_K *is of Galois-type if and only if* $\rho(W_K)$ *is finite.*

Proof. The open subgroups of W_K of finite index are the W_L for finite L/K. Their intersection is $\ker \Phi = 1$. $\square$

Definition 3.6. Denote $\omega_s : W_K \to \mathbb{C}^\times$ the quasi-character sending $x \mapsto |x|^s$ for $s \in \mathbb{C}$.

Proposition 3.7 ([Tat67, Lemma 2.3.1.])**.** *Every one-dimensional representation of W_K that is unramified (i.e. trivial on I_K) is of the form ω_s for some $s \in \mathbb{C}$.*

Proof. Any unramified quasi-character χ of W_K will depend only on ω and, as a function of ω, is itself a quasi-character χ' of the value group $\{N(\mathfrak{p})^n \mid n \in \mathbb{Z}\}$ of W_K. This is given by $s = -\frac{\log(\chi' N(\mathfrak{p}))}{\log N(\mathfrak{p})}$. □

Theorem 3.8 ([Del73, Section 4.10])**.** *Every irreducible representation of W_K is of the form $r = r' \otimes \omega_s$ for some $s \in \mathbb{C}$ and representation r' of Galois-type. In fact, this is true for any extension of $\mathbb{Z}$ by a profinite group.*

Proof. Every representation of W_K is trivial on a finite-index subgroup J of I_K. Since I_K/J is finite, ϕ^n acts trivially on I_K/J by conjugation for some $n > 0$ and so is central in W_K/J. Each power π^m of ϕ^n has exactly one eigenvalue a_m if the representation is irreducible. Then each irreducible representation has a type given by

$$(a_m) \in \varinjlim_{n|m} \{X_m, \phi_{n,m}\}_m,$$

where $X_m = \mathbb{C}^\times$ and $\phi_{n,m}(x) = x^{\frac{m}{n}}$.

The representations of W_K of type s form an abelian category $M_s(W_K)$, and

$$\mathrm{Rep}(W_K) = \bigoplus_{s \in \mathbb{C}} \mathrm{Rep}_s(W_K)$$

The representations of type 1 are precisely the Galois-type representations $\mathrm{Rep}(G_K)$. Then we have an isomorphism

$$\cdot \otimes \omega_s : \mathrm{Rep}(G_K) \to \mathrm{Rep}_s(W_K).$$

□

Proposition 3.9. *A Galois-type representation of W_K is irreducible iff it is irreducible as a G_K-representation. Furthermore, if ρ is any irreducible W_K-representation, it is of Galois-type iff the image of* $\det \circ \rho$ *is a subgroup of* $\mathbb{C}^\times$ *of finite order.*

Definition 3.10. For any finite extension L/K, let $W_L := \phi_K^{-1}(G_L) \subset W_K$ where $G_L := \mathrm{Gal}(\overline{K}/L)$. Note: $W_K/W_L \cong G_K/G_L \cong \hom_K(L, K)$ is finite.

Then we have the restriction functor

$$\mathrm{res}_{L/K} : \mathrm{Rep}(W_K) \to \mathrm{Rep}(W_L)$$

given by $\rho \mapsto \rho|_{W_L}$. The induction functor

$$\mathrm{ind}_{L/K} : \mathrm{Rep}(W_L) \to \mathrm{Rep}(W_K)$$

is given by $(\rho, V) \mapsto (\tau, \{f : W_K \to V \mid f(xw) = \rho(x)f(w) \text{ for all } x \in W_L, w \in W_K\})$. These functors satisfy Frobenius reciprocity.

4. Weil–Deligne Representations

Definition 4.1. A Weil–Deligne representation of W_K is a triple (ρ, V, N) where (ρ, V) is a representation of W_K and N is a nilpotent $\mathbb{C}$-linear endomorphism of V such that

$$\rho(x)N\rho(x)^{-1} = |x|\, N.$$

It is called Frobenius semisimple if ρ is semisimple.

Definition 4.2. Let (ρ_1, V_1, N_1) and (ρ_2, V_2, N_2) be two Weil–Deligne representations.

Define the representation $(\rho, V, N) = (\rho_1, V_1, N_1) \otimes (\rho_2, V_2, N_2)$ by $V = V_1 \otimes V_2$ and, for $x \in W_K$ and $v_i \in V_i$,

$$\rho(x)(v_1 \otimes v_2) := \rho_1(x)v_1 \otimes \rho_2(x)v_2$$
$$N(v_1 \otimes v_2) := N_1 v_1 \otimes v_2 + v_1 \otimes N_2 v_2.$$

The formula is a result of:

$$\log(\rho_1(x) \otimes \rho_2(x)) = \log(\rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)).$$

Define the representation $(\rho, V, N) = \hom\big((\rho_1, V_1, N_1), (\rho_2, V_2, N_2)\big)$ by $V = \hom(V_1, V_2)$ and, for $\phi \in \hom(V_1, V_2)$, $x \in W_K$ and $v_i \in V_i$,

$$(\rho(x)\phi)(v_1) := \rho_2(x)(\phi(\rho_1(x)^{-1}v_1))$$
$$(N\phi)(v_1) := N_2(\phi(v_1) - \phi(N_1 v_1)).$$

The contragredient $\rho^\vee$ of a Weil–Deligne representation is $\hom(\rho, 1)$ where 1 is the trivial one-dimensional representation.

Remark 4.3. If $x \in W_K$ corresponds to the uniformizer π_K via the Artin reciprocity map $\mathrm{Art}_K : K^\times \to G_K^{ab}$, then N is conjugate to qN and hence has no nonzero eigenvalues, i.e. N is automatically nilpotent.

Remark 4.4. The kernel of N is stable under W_K, so (ρ, V, N) is irreducible iff (ρ, V) is irreducible and $N = 0$. So the irreducible Weil–Deligne representations of W_K are the irreducible representations of W_K.

Remark 4.5. The category of $\mathrm{WDRep}_k(W_K)$ does not depend on the topology on k. Thus, we can identify $\mathrm{WDRep}_{\mathbb{C}}(W_K)$ with $\mathrm{WDRep}_{\overline{\mathbb{Q}_\ell}}(W_K)$.

Example 4.6. If $n = 1$, then N is nilpotent and 1-by-1 and hence zero. Then a Weil–Deligne representation is just a continuous homomorphism $W_K \to \mathbb{C}^\times$.

Definition 4.7. The Weil–Deligne group W_K' is the group scheme $W_K \ltimes \mathbb{G}_a$ over $\mathbb{Q}$ given by the action

$$wxw^{-1} = |w|\, x,$$

for all $w \in W_K$. Composition is given by

$$(w_1, x_1)(w_2, x_2) = (w_1 w_2, |w_2|^{-1}\, x_1 + x_2).$$

Remark 4.8. A Weil–Deligne representation of W_K is the same as a representation of W_K'. This arises from the fact that finite-dimensional representations of the additive group $\mathbb{G}_a$ correspond to nilpotent endomorphisms.

5. L-adic representations

Theorem 5.1 (Grothendieck's l-adic monodromy theorem)**.** *Let* F *be an* ℓ*-adic field, where* $\ell \neq p$ *is prime. Let* (ρ, V) *be a finite-dimensional representation of* W_K *over* F*. Then there exists a finite-index open subgroup* $H \subset I_K$ *such that* $\rho(x)$ *is unipotent for all* $x \in H$*.*

Remark 5.2. A similar theorem is true if we replace W_K by G_K because unipotent subgroups are closed in the image of G_K and $W_K \subset G_K$ is dense.

Definition 5.3. Let $t_\ell : I_K \to \mathbb{Q}_\ell$ be a nonzero homomorphism. (This exists and is unique up to a constant multiple because the wild ramification group P_K is a pro-p-group and $I_K/P_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$).

We have $t_\ell(xyx^{-1}) = |x|\, t_\ell(y)$ for all $x \in W_K$, $y \in I_K$ (because conjugation by x induces raising to the $|x|$ power in I_K/P_K).

Corollary 5.4. *There exists a unique nilpotent operator* N *of* V *such that* $\rho(x) = \exp(t_\ell(x)N)$ *for all* $x \in H$ *in some open subgroup of* I_K*. (This is* N *from now on.)*

Proof. Nilpotency and uniqueness follow directly from writing $N = t_\ell(x_0)^{-1}\log(\rho(x_0))$ for some $x_0 \in H \cap I_K$ such that $t_\ell(x_0)$ is nontrivial (using the ℓ-adic monodromy theorem for nilpotency).

Existence follows because $\rho|_{H\cap I_K}$ factors through t_ℓ as some continuous representation of $Z_\ell(1)$ which coincides with the continuous representation $\mathbb{Z}_\ell(1) \to \mathrm{GL}_F(V), x \mapsto \exp(xN)$ on $t_\ell(x_0)$ and hence on $t_\ell(x_0)\mathbb{Z}_\ell(1)$ for all $x_0 \in H \cap I_K$ such that $t_\ell(x_0)$ is nontrivial. Thus, they coincide on $H \cap I_K$. □

Remark 5.5. Corollary 5.4 allows us to attach a Weil–Deligne representation to each representation of W_K. But we cannot naively use $(\rho, V) \mapsto (\rho, V, N)$ since (ρ, V) is not smooth in general.

Theorem 5.6 ([Del73, Section 8]). *There is an equivalence of categories between finite dimensional continuous representations of* W_K *and the Weil–Deligne representations of* W_K

$$\begin{aligned}(--)_{WD} :& \mathrm{Rep}_k(W_K) \to \mathrm{WDRep}_k(W_K)\\ &(\rho, V) \mapsto (\rho_\phi, V, N)\\ &\rho_\phi(\phi^n x) = \rho(\phi^n x)\exp(-t_\ell(x)N).\end{aligned}$$

Proof. The condition

$$\rho_\phi(x) N \rho_\phi(x)^{-1} = |x|\, N,$$

holds because the exponential commutes with N. Exercise: show that (ρ_ϕ, V) is a continuous representation of W_K.

For a map $f : (\rho_1, V_1) \to (\rho_2, V_2$, the uniqueness of the N_i gives

$$f \circ N_1 = N_2 \circ f.$$

So $(--)_{WD}$ is a faithful functor.

The uniqueness of the monodromy operator implies that N is the monodromy operator associated to (ρ_ϕ, V) $\square$

Remark 5.7. The functor depends on our choice of ϕ and t_ℓ, but only up to a natural automorphism of the identity functor.

Remark 5.8. We can view the ℓ-adic representations of G_K over $\mathbb{Q}_\ell$ as a subcategory of Weil–Deligne representations over $\mathbb{C}$ (or over $\overline{\mathbb{Q}}_\ell$) via Theorem 5.6, Remark 4.5, and Remark 3.2.

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