SKEW-SIMPLICIAL GROUPS

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The present paper arose from an attempt to understand the connection between the phenomenon of a cyclic object and the action of the circle on a topological space. The concept of cyclic object lies at the base of the construction of cyclic homology, introduced independently by A. Connes [1] and B. L. Tsygan [2]. There have already appeared dihedral, quaternionic, and symmetric objects [3, 4]. It turns out that all of them can be unified in the framework of the general concept of the action of a skew-simplicial group on a simplicial object.

It is shown in the first section that the category of skew-simplicial groups has a nontrivial final object W., consisting of the Weyl groups of a system of roots of type B. Each skew-simplicial group is an extension of some simplicial group by one of the seven objects in W.. The second section is devoted to connections with equivariant topology. It is proved that the geometric realization |G.| of a skew-simplicial group G. is a topological group. Moreover, the homotopy category of simplicial sets with action of G. is equivalent to the homotopy category of topological |G.|-spaces.

1. Let Δ be the category of finite totally ordered sets [n] = {0, 1, ..., n} and nondecreasing maps. We consider the dual category Δ^0. It is well known that its morphisms are generated by the simplest morphisms of the form

$$d^i:[n]\to[n-1],\quad s^i:[n]\to[n+1],\quad 0\leqslant i\leqslant n,$$

which satisfy the following relations:

$$d^i d^j = d^{j-1} d^i,\quad i<j; \tag{1}$$

$$d^i s^j = \begin{cases} s^{j-1} d^i, & i<j, \\ id, & i=j, \quad i=j+1, \\ s^j d^{i-1}, & i>j+1; \end{cases} \tag{2}$$

$$s^i s^j = s^{j+1} s^i,\quad i\leqslant j. \tag{3}$$

Definition 1.1. A small category Σ is called a category of type Δ if it contains Δ as a subcategory and has the same objects. In addition, it is required that each morphism $f\in \mathrm{Hom}_\Sigma([n],[m])$ can be represented uniquely as a composition $f=\varphi\cdot g$, where $g\in \mathrm{Aut}_\Sigma[n]$, and $\varphi\in \mathrm{Hom}_\Delta([n],[m])$.

Thus, a category Σ of type Δ is determined completely by a collection of automorphisms of $\mathrm{Aut}_\Sigma[n]$, $n=0, 1, \ldots$, and rules for commutation with the morphisms of Δ. The collection of such Σ forms a category Cat Δ with functors preserving Δ as morphisms.

Example 1.2. The category ΔW. We define a sequence of groups $W_n=(Z/2)^{n+1}\times\Sigma_{n+1}$, $n=0, 1\ldots$ Here Σ_{n+1} is the symmetric group, acting on the right on the set [n] = {0, 1, ... n}

$$[n]\times\Sigma_{n+1}\to[n],\quad (i,\tau)\mapsto\tau^*(i),$$

and Z/2 is the group with two elements {+1, −1}. We define the multiplication

$$((\varepsilon_0,\ldots,\varepsilon_n),\tau)((\eta_0,\ldots,\eta_n),\sigma)=((\varepsilon_0\cdot\eta_{\tau^*(0)},\ldots,\varepsilon_n\cdot\eta_{\tau^*(n)}),\tau\cdot\sigma).$$

On W. we introduce the structure of a simplicial set, defining the face and degeneracy operators in the following way:

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$$d^i((\varepsilon_0,\ \ldots,\ \varepsilon_n),\ \tau)=((\varepsilon_0,\ \ldots,\ \hat{\varepsilon}_i,\ \ldots,\ \varepsilon_n),\ d^i\tau),$$

$$(d^i\tau)^*(j)=\begin{cases}\tau^*(j), & i>j,\quad \tau^*(i)>\tau^*(j);\\ \tau^*(j)-1, & i>j,\quad \tau^*(i)<\tau^*(j);\\ \tau^*(j+1), & i\leqslant j,\quad \tau^*(i)>\tau^*(j+1);\\ \tau^*(j+1)-1, & i\leqslant j,\quad \tau^*(i)<\tau^*(j+1);\end{cases} \tag{4}$$

$$s^i((\varepsilon_0,\ \ldots,\ \varepsilon_n),\ \tau)=((\varepsilon_0,\ \ldots,\ \varepsilon_i,\ \varepsilon_i,\ \ldots,\ \varepsilon_n),\ s^i\tau),$$

$$(s^i\tau)^*(j)=\begin{cases}\tau^*(j), & i>j,\quad \tau^*(i)>\tau^*(j);\\ \tau^*(j)+1, & i>j,\quad \tau^*(i)<\tau^*(j);\\ \tau^*(j-1), & i<j-1,\quad \tau^*(i)>\tau^*(j-1);\\ \tau^*(j-1)+1, & i<j-1,\quad \tau^*(i)<\tau^*(j-1);\\ \tau^*(i)+(1-\varepsilon_i)/2, & i=j;\\ \tau^*(i)+(1+\varepsilon_i)/2, & i=j+1.\end{cases} \tag{5, 6}$$

Finally, we define the category ΔW of type Δ by commutation relations in the dual form (i.e., for the category ΔW^0):

$$d^i\cdot\omega=(d^i\omega)\cdot d^{\omega^*(i)},\quad s^j\cdot\omega=(s^j\omega)\cdot s^{\omega^*(j)}. \tag{7}$$

Here ω^* denotes the action of the group W_n on the set $[n]$, induced by the obvious projection $W_n\to\Sigma_{n+1}$.

Definition 1.3. A skew-simplicial group is a pair $(G.,\ \gamma)$, consisting of a simplicial set $G.$ and a simplicial map $\gamma:G.\to W.$, for which the following conditions hold:

1) the set G_n is a group and the map γ_n a homomorphism for each n;
2) for each n and all $0\le i\le n$, $0\le j\le n$ one has

$$d^i(g_1g_2)=(d^ig_1)(d[\gamma(g_1)^*(i)]g_2),$$
$$s^j(g_1g_2)=(s^jg_1)(s[\gamma(g_1)^*(j)]g_2). \tag{8}$$

Here and below for awkward indices we use the abbreviations $d[i]=d^i$, $s[j]=s^j$.

By a morphism of skew-simplicial groups $(G.,\ \gamma)\to(G.',\ \gamma')$ is meant a simplicial map $f:G.\to G.'$ such that f_n is a homomorphism for each n, while $\gamma=\gamma'\cdot f$. We denote the category of skew-simplicial groups by Skew-SGr.

THEOREM 1.4. One has an isomorphism of categories

$$\mathrm{Cat}\,\Delta \xrightarrow{\cong} \mathrm{Skew-SGr}.$$

Proof. Let Σ be a category of type Δ. We construct a skew-simplicial group structure on $G.=\mathrm{Aut}^0_\Sigma[.]$ in four steps. For convenience we shall work in the dual category Σ^0. By the letter g (possibly with dashes) we shall denote the elements of G.

Step 1. Structure of simplicial set. According to Definition 1.1, the morphism $d^i\cdot g$ of the category Σ^0 can be written uniquely in the form $g'\cdot d^j$. The correspondence $g\mapsto g'$ defines operators $d^i:G_n\to G_{n-1}$. Analogously, we get operators $s^j:G_n\to G_{n+1}$. The simplicial identities (1)-(3) automatically hold by virtue of the uniqueness.

Step 2. Construction of homomorphisms $e_n,\ \tilde{e}_n:G_n\rightrightarrows\Sigma_{n+1}$. The equation $d^i\cdot g=g'\cdot d^j$ from the first step also gives a correspondence $(d^i,\ g)\mapsto d^j$, defining a right action of G_n on the set $[n]$: $g^*(i)=j$. This is equivalent to giving a homomorphism $e_n:G_n\to\Sigma_{n+1}$ (cf. Example 1.2). In exactly the same way, from the equations $s^i\cdot g=g''\cdot s^k$ we get a homomorphism $\tilde{e}_n:G_n\to\Sigma_{n+1}$.

Step 3. The homomorphisms e and $\tilde{e}$ are equal. We introduce the abbreviations $g^*=e(g)^*$, $\tilde{g}^*=\tilde{e}(g)^*$. As a result of steps 1 and 2, we have the equations

$$d^i\cdot g=(d^ig)\cdot d^{g^*(i)},\quad s^j\cdot g=(s^jg)\cdot s^{\tilde{g}^*(j)}, \tag{9}$$

which let us make the transformations

$$d^i\cdot s^j\cdot g=(d^is^jg)\cdot d[(s^jg)^*(i)]\cdot s[\tilde{g}^*(j)],$$

$$s^{j-1}\cdot d^i\cdot g=(s^{j-1}d^ig)\cdot s[(\widetilde{d^ig})^*(j-1)]\cdot d[g^*(i)].$$

When i < j, the left sides of these equations coincide according to (2). It follows from the uniqueness that

$$d[(s^j g)^*(i)]\cdot s[\tilde{g}^*(j)] = s[(\widetilde{d^i g})^*(j-1)]\cdot d[g^*(i)].$$

Since this is a relation of type (2), necessarily $g^*(i)\neq\tilde{g}^*(j)$.

If i > j + 1, analogously we get

$$d[(s^j g)^*(i)]\cdot s[\tilde{g}^*(j)] = s[(\widetilde{d^{i-1}g})^*(j)]\cdot d[g^*(i-1)].$$

Again directly $g^*(i-1)\neq\tilde{g}^*(j)$. Thus, if i ≠ j, then $g^*(i)\neq\tilde{g}^*(j)$ always. Consequently, $g^*=\tilde{g}^*$, i.e., e = $\tilde{e}$.

Step 4. Lifting the maps $e_n: G_n \to \Sigma_{n+1}$ to a simplicial map G. → W.. Applying the method of step 3, from (9) and (1) we express $(d^i g)^*$ in terms of g^*. We get exactly (4) with the letter τ replaced by g. This means that the map e commutes with face operators. We consider how matters stand with degeneracies. From (9) and (3) we get (5) with τ replaced by g. However, there remains an uncertainty: $(s^i g)^*(i)$ can be equal to $g^*(i)$ or $g^*(i) + 1$. In the first case we set $h_i(g) = +1$, and in the second, $h_i(g) = -1$. The map

$$\gamma: g \mapsto \big((h_0(g),\ \ldots,\ h_n(g)),\ e(g)\big)$$

is exactly the lift needed. Combining (2), (3), and (6), .e see finally that this is a simplicial map.

It is easy to see that the correspondence $\Sigma \mapsto \mathrm{Aut}^0_\Sigma[.]$ defines a functor CatΔ → Skew − SGr.

If a skew-simplicial group (G., γ) is given, one can construct a category ΔG of type Δ, defining the morphisms formally and imposing relations of the form (7). The existence of Example 1.2 guarantees that in the simplicial "part", new relations, different from the standard ones (1)-(3), do not appear. Thus, we have constructed an inverse functor. The theorem is completely proved.

We have reduced the study of categories of type Δ to the study of skew-simplicial groups. For the sake of brevity, we shall call them SS-groups and write G. = (G., γ).

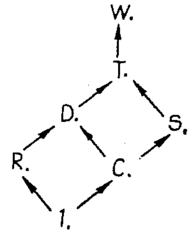
It follows from Definition 1.3 that W. is a final object, i.e., any SS-group maps uniquely into W.. It is easy to prove the following proposition.

Proposition 1.5. The SS-group W. contains exactly seven SS-subgroups. The complete list is given here (name, notation, generators, group of n-dimensional simplices):

Trivial	1.	∅	$1_n = 1$
Cyclic	C.	$(+1,\ +1;\ \tau)$	$C_n = Z/n+1$
Symmetric	S.	$(+1,\ +1,\ +1;\ \tau)$	$S_n = \Sigma^{n+1}$
Reflexive	R.	$(-1;\ id)$	$R_n = Z/2$
Dihedral	D.	$(-1,\ -1;\ id)$	$D_n = Z/n+1 \times Z/2$
Reflexosymmetric	T.	$(-1;\ id),\ (+1,\ +1,\ +1;\ \tau)$	$T_n = Z/2 \times \Sigma_{n+1}$
Weyl SS-group	W.	$(+1,\ -1;\ id)$	$W_n = (Z/2)^{n+1} \times \Sigma_{n+1}$

Here τ is transposition.

The diagram of inclusions has the form



Thus, all SS-groups fall into seven classes depending on the image in W.. Here the trivial class consists precisely of the simplicial groups. We shall call the SS-groups from the list 1.5 simple.

Proposition 1.6. Any SS-group can be represented as an extension

$$1_. \to H_. \to G_. \to L_. \to 1_.,$$

where H. is a simplicial group and L. is a simple SS-group.

Proof. We set $H_. = \ker(\gamma : G_. \to W_.)$, $L_. = \gamma(G_.)$. Then H. necessarily belongs to the trivial class. □

Example 1.7. (i) The multiple cyclic SS-groups $C_.^{(n)}$, $n=2, 3, \ldots, \infty$ (which correspond to the categories $\Lambda^{(n)}$ of [1]) can be represented as extensions

$$1_. \to (Z/_n)_. \to C_.^{(n)} \to C_. \to 1_., \ n < \infty;$$
$$1_. \to Z_. \to C_.^{(\infty)} \to C_. \to 1_.$$

(ii) The quaternionic object Q. [4] is the extension

$$1_. \to (Z/2)_. \to Q_. \to D_. \to 1_.$$

(iii) The SS-group of braids B. is an extension of the simplicial group of colored braids $A_. : 1_. \to A_. \to B_. \to S_. \to 1_.$ The groups B_n consist of the braids which are interlacings of n + 1 threads. Under the action of the operator d^i on a braid the i-th thread disappears and the operator s^j duplicates the j-th thread.

One defines the SS-group of braids of bands B^* analogously:

$$1_. \to A_.^* \to B_.^* \to W_. \to 1_..$$

The bands can be twisted. The number of revolutions modulo 2 gives the sign in W.. Under a degeneracy the corresponding band is cut into two halves, which are interlaced as many times as they themselves are twisted.

Now we consider SS-groups from the homotopy point of view.

Proposition 1.8. Simple SS-groups have the following homotopy type:

$$1_. \sim *, \ C_. \sim S^1, \ S_. \sim *, \ R_. \sim * \sqcup *, \ D_. \sim S^1 \sqcup S^1, \ T_. \sim * \sqcup *, \ W_. \sim *.$$

Here * denotes a point and S^1 a circle.

The proof for 1., C., R., D. involves no difficulties. In the case of the symmetric SS-group S. we firstly verify that $\pi_1(S.) = 0$, and then we construct a contracting homotopy in the integral chains by the formula

$$s(\tau)(i) = \begin{cases} 0, & i = 0; \\ \tau(i-1)+1, & i > 0; \end{cases}$$

where $\tau \in S_n$, $s : ZS_n \to ZS_{n+1}$.

A simplicial set is called finite if it contains only a finite number of nondegenerate simplices.

Proposition 1.9. The connected finite SS-groups are completely enumerated in the following sequence:

$$1_., \ C_., \ C_.^{(2)}, \ C_.^{(3)}, \ldots, \ C_.^{(n)}, \ldots$$

Proof. We consider a connected finite SS-group G.. If it is not trivial, then it belongs to the cyclic class. According to Proposition 1.6, one can construct an extension

$$1_. \to H_. \to G_. \to C_. \to 1_..$$

Here H. is a finite simplicial group. It follows from [5] that it satisfies the Kan condition. On the other hand, this is a finite simplicial set. Consequently, its connected component coincides with the standard simplicial simplex Δ[n] of some dimension n. There remains only a unique possibility H. is a discrete simplicial group. If we now pass to geometric realizations (cf. Sec. 2), then we get a finite-sheeted covering $|G_.| \to |C_.| = S^1$. It is clear that G. can only be equal to $C_., C_.^{(2)}, C_.^{(3)}, \ldots$ □

Now we make some inferences about homology theories. Such homology theories as cyclic, dihedral, and quaternionic, are based on finite SS-groups, whose geometric realizations are the Lie groups SO(2), O(2), and the normalizer of a circle in SU(2) respectively. It is natural to ask whether there exist analogous homology theories for the other Lie groups. Proposition 1.9 asserts that the component of the identity of such a group is necessarily a circle, i.e., there do not exist SO(3), SU(2), etc. homologies.

2. In this section we shall work with geometric realizations of simplicial sets. The results remain valid in a more general situation. Only a suitable realization functor of a simplicial object in a category is needed.

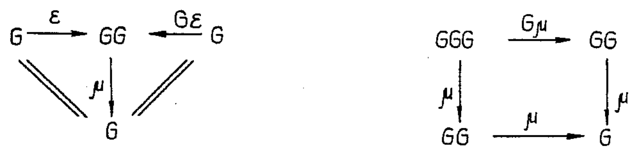
A simplicial set (or Δ^0-set) is a functor from the category Δ^0 to the category of sets Sets. For any category Σ of type Δ (cf. Definition 1.1) we define a Σ^0-set analogously as a functor $\Sigma^0 \to$ Sets. According to Theorem 1.4, the category Σ has the form ΔG for some skew-simplicial group G..

Definition 2.1. Let G. be a skew-simplicial group. We define a functor G inside the category of simplicial sets Δ^0Sets, letting $GX_n = G_n \times X_n$,

$$d^i(g, x)=(d^i g, d^{g^*(i)} x), \quad s^j(g, x)=(s^j g, s^{g^*(j)} x).$$

We note that G(*) = G.

We define another natural transformation $\varepsilon: 1 \to G$, $\mu: G G \to G$ by $\varepsilon(x)=(1, x)$, $\mu(g_1, (g_2, x))=(g_1 g_2, x)$. The collection (G, ϵ, μ) forms a monad (or "triple") in the category Δ^0Sets. This means that the following diagrams are commutative:



(10)

By a G-algebra we mean a pair (X., ξ) consisting of a Δ^0-set X. and a simplicial map $\xi: GX. \to X.$, such that the following diagrams are commutative:

(11)

A morphism of G-algebras $(X., \xi) \to (X'., \xi')$ is a simplicial map f:X. → X.', such that $f \cdot \xi = \xi' \cdot Gf$. It is easy to verify the following:

LEMMA 2.2. The categories of ΔG^0-sets and G-algebras are isomorphic. An isomorphism is given by associating with each ΔG^0-set X. a G-algebra (X., ξ) such that $\xi(g, x) = gx$.

Thus, a ΔG^0-set is a simplicial set with an "action" of a skew-simplicial group G..

Now we recall the definition of geometric realization and singular complex. We shall consider the category of Hausdorff, compactly generated, spaces, Spaces. Let Δ^n be a geometric simplex of dimension n: $\Delta^n=\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \Sigma t_i = 1\}$. We shall denote by δ^i and σ^j the standard maps of coboundary and codegeneracy. By geometric realization of a Δ^0-set X. is meant the quotient-space (with compactly generated topology)

$$|X.| = \Big(\coprod_{n \geq 0} X_n \times \Delta^n\Big) \Big/ \sim$$

with respect to the equivalence relation $(d^i x, u) \sim (x, \delta^i u)$, $(s^j x, u) \sim (x, \sigma^j u)$. The class of the element (x, v) will be denoted by [x, v].

We define the singular complex s.(Y) of the topological space Y to be the Δ^0-set represented as the composition of the covariant and contravariant functors

$$\Delta \to \text{Spaces} \to \text{Sets}, \qquad (12)$$
$$[n] \mapsto \Delta^n \mapsto \text{Map}(\Delta^n, Y).$$

It is known that the geometric realization functor is the left adjoint of the singular complex functor. We denote the natural transformations of the adjoint by

$$\alpha : 1 \to s_.|\,.\,|, \quad \beta : |s_.(\,.\,)| \to 1. \tag{13}$$

Let F be the category of finite sets [n] and any maps between them. Since the first arrow of (12) factors through the inclusion $\Delta \to F$, and the category ΔW "projects" to F, then s.(Y) is endowed with a natural structure as ΔW^0-set, hence also the structure of ΔG^0-set for any skew-simplicial group G.. According to Lemma 2.2 this defines a G-algebra (s.(Y), ω). Actually ω defines an action of the group G_n on the set $s_n(Y)$: each singular simplex $\sigma : \Delta^n \to Y$ is "mixed" inside itself according to the formulas

$$g(\sigma)(t) = \sigma\big(g^*(t)\big), \quad g^*(t) = (t_{(g^{-1})^*(0)}, \ \ldots, \ t_{(g^{-1})^*(n)}), \tag{14}$$

where $t = (t_0, \ \ldots, \ t_n) \in \Delta^n$.

<u>THEOREM 2.3.</u> Let G. be a skew-simplicial group.

A. For a Δ^0-set X. there is a natural homeomorphism $\Phi(X_.) : |GX_.| \to |G_.| \times |X_.|$, such that:

(i) the map $\mu^r = |\mu| \cdot \Phi(G_.)^{-1} : |G_.| \times |G_.| \to |G_.|$ turns $|G_.|$ into a topological group;

(ii) for the G-algebra ξ:GX. → X., the composition

$$\xi^r = |\xi| \cdot \Phi(X_.)^{-1} : |G_.| \times |X_.| \to |X_.|$$

defines an action of the group $|G_.|$ on the space $|X_.|$.

B. Let the group |G.| act continuously on the space Y, i.e., let Y be a |G.|-space. There exists a natural monomorphism of Δ^0-sets $\Psi(Y) : Gs_.(Y) \to s_.|G_.| \times s_.(Y)$, such that the map $\nu^s = s_.\nu \cdot \Psi(Y) : Gs_.(Y) \to s_.(Y)$ defines a G-algebra $\big(s_.(Y), \ \nu^s\big)$.

C. The constructions given above define adjoint functors

$$r^G : (X_., \ \xi) \mapsto (|X_.|, \ \xi^r), \quad s^G : (Y, \ \nu) \mapsto \big(s_.(Y), \ \nu^s\big),$$

$$r^G : \Delta G^0\,\text{Sets} \rightleftarrows |G_.| - \text{Spaces} : s^G.$$

Here we identify the category of G-algebras with the category ΔG^0Sets.

<u>Proof of A.</u> We define $p : GX_. \to G_.$ by the formula $(g, \ x) \mapsto g$. The map adjoint to the composition

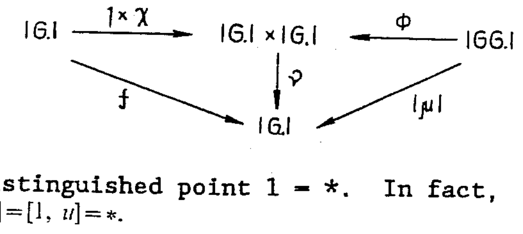
$$GX_. \xrightarrow{G\alpha} Gs_.|X_.| \xrightarrow{\omega} s_.|X_.|,$$

will be denoted by $\varphi : |GX_.| \to |X_.|$. Here ω is given by (14) and α is the morphism adjoint to (13). We set $\Phi(X_.) = (|p|, \ \varphi)$. It is also easy to extract the explicit formula $g \in G_n$, $x \in X_n$, $u \in \Delta^n$,

$$\Phi(X_.) : [(g, \ x), \ u] \mapsto \big([g, \ u], \ [x, \ g^*(u)]\big).$$

The proof that this is a homeomorphism goes in the same way as the formula $|Y \times Z_.| \cong |Y_.| \times |Z_.|$ in the classical case for Δ^0-sets [5]. The only difference is the appearance of "twisted" and not just standard partitions of the prism $\Delta^k \times \Delta^n$ into simplices of dimension $k \cdot n$.

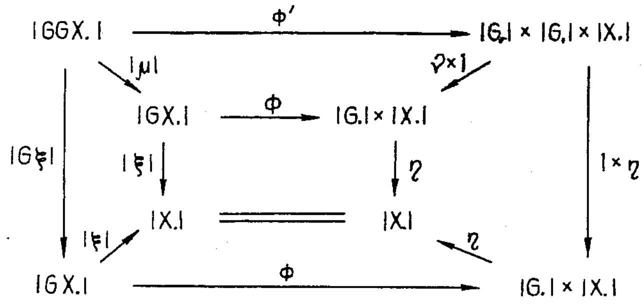
(i) We let $\nu = |\mu| \cdot \Phi^{-1} : |G_.| \times |G_.| \to |G_.|$. It is easy to see that the inclusion of the point $* \to |G_.|$ is a two-sided unit. We define the "inversion" map $\chi : |G_.| \to |G_.|$ by the formula $[g, \ u] \mapsto [g^{-1}, \ g^*(u)]$. We verify that it is well defined. Let us assume that $u = \delta^i$ so $(g, u) = (g, \ \delta^i v) \sim (d^i g, \ v)$. We apply the map χ to the last element $\chi(d^i g, v) = \big((d^i g)^{-1}, \ (d^i g)^*(v)\big)$. It follows from the equation $d^i(gg^{-1}) = (d^i g)(d[g^*(i)]g^{-1})$ that $(d^i g)^{-1} = \big(d[g^*(i)]g^{-1}\big)$. Hence we can write $\chi(d^i g, \ v) = \big(d[g^*(i)]g^{-1}, \ (d^i g)^*(v)\big) \sim \big(g^{-1}, \ \delta[g^*(i)](d^i g)^*(v)\big)$. But $g^* \delta^i = \delta[g^*(i)](d^i g)^*$, since these are images of equal morphisms from the category ΔG (cf. the dual equation (7)). One verifies analogously that it is well defined with respect to degeneracies. The map χ really inverts the group law. This follows from the commutativity of the following diagram:



where f is the map to the distinguished point 1 = *. In fact, $(1 \times \chi)[g, \ u] = ([g, \ u], \ [g^{-1}, \ g^*(u)] = \Phi[(g, \ g^{-1}), \ u]$, but $|\mu|[(g, \ g^{-1}), \ u] = [1, \ u] = *$.

The associativity of the multiplication ν follows from (ii) as a special case.

(ii) We prove that the map $\eta = |\xi| \cdot \Phi^{-1}: |G.| \times |X.| \to |X.|$ gives an action of the group $|G.|$ on the space $|X.|$. The associativity of the action is precisely the commutativity of the right square in the diagram



Here $\Phi' = (1 \times \Phi(X.)) \cdot \Phi(GX.)$ is a homeomorphism. Consequently, it suffices to establish the commutativity of all the other squares. For the central, lower, and left squares this follows from the definition of η and the fact that (X., ξ) is a G-algebra (cf. (11)). From the calculations $(1 \times \eta) \cdot \Phi' = (1 \times \eta)(1 \times \Phi) \cdot \Phi = (1 \times \eta\Phi) \cdot \Phi = (1 \times |\xi|) \cdot \Phi = \Phi \cdot |G\xi|$ the commutativity of the outer square of the diagram follows. It remains to verify the upper square. We compare the values on elements

$$(\Phi \cdot |\mu|)[(g, h, x), u] = ([gh, u], [x, (gh)^*(u)]),$$

$$((\nu \times 1) \cdot \Phi')[(g, h, x), u] = (|\mu| \cdot \Phi^{-1} \times 1)([g, u], [h, g^*(u)], [x, h^* g^*(u)]) =$$

$$= (|\mu| \times 1)([(g, h), u], [x, h^* g^*(u)]) = ([gh, u], [x, h^* g^*(u)]),$$

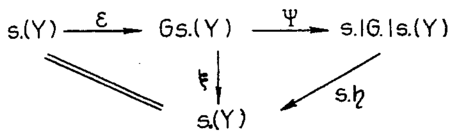
i.e., $\Phi \cdot |\mu| = (\nu \times 1) \cdot \Phi'$, which is what was required.

The validity of the equation $\eta(1, z) = z$, $z \in |X.|$ is guaranteed by the existence of a unit $\varepsilon: 1 \to G$ in the G-algebra (X., ξ). In fact, according to (11), $\xi \cdot \varepsilon = id.$ Hence $\eta(1, [x, u]) = |\xi| \cdot |\varepsilon| [x, u] = [x, u]$.

<u>Proof of B.</u> We define the map Ψ as the composition

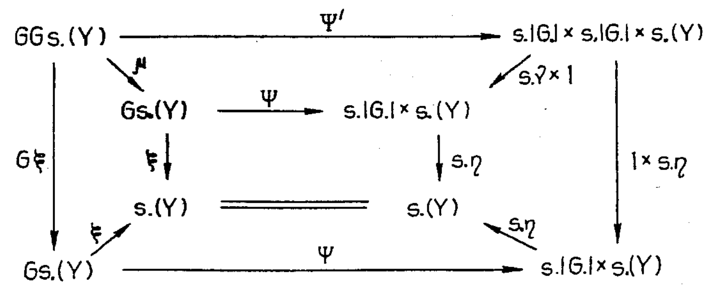
$$Gs.(Y) \xrightarrow{(p, \omega)} G. \times s.(Y) \xrightarrow{\alpha \times 1} s.|G.| \times s.(Y),$$

where p(g, x) = g, and α and ω are defined, respectively, in (13) and (14). We note that the second map is injective and the first is an isomorphism (the inverse map is given by the formula $(g, x) \mapsto (g, g^{-1}x)$). Consequently, Ψ is a monomorphism. Let us assume that the map $\eta: |G.| \times Y \to Y$ defines the action of the group |G.|. We set $\xi = s.\eta \cdot \Psi: Gs.(Y) \to s.(Y)$. The existence of a unit (cf. the first diagram of (11)) is equivalent with the commutativity of the left triangle in the following diagram:



But this follows from the commutativity of the large and right triangles.

It remains to verify the commutativity of the left square in this diagram (the monomorphism Ψ' is defined below)



Since Ψ and Ψ' are monomorphisms, it suffices to prove the commutativity of all the other squares. With respect to the central, lower, and right squares, everything is clear from the definitions. We set

$$\Psi' = \Psi(|G.| \times Y) \cdot G\Psi(Y).$$

Then the outer square splits into two commutative squares.

$$\begin{array}{ccccc} GGs.(Y) & \xrightarrow{G\Psi} & G(s.|G.| \times s.(Y)) & \xrightarrow{\Psi} & s.|G.| \times s.|G.| \times s.(Y) \\ \downarrow{\scriptstyle G\xi} & & \downarrow{\scriptstyle Gs.\eta} & & \downarrow{\scriptstyle 1\times s.\eta} \\ Gs.(Y) & = & Gs.(Y) & \xrightarrow{\Psi} & s.|G.| \times s.(Y) \end{array}$$

We check the commutativity of the upper square on elements

$$(\Psi\cdot\mu)(g, h, x) = \Psi(gh, x) = (\alpha(gh), ghx),$$

$$\Psi'(g, h, x) = \Psi\left(g, (\alpha(h), hx) = (\alpha(g), g\alpha(h), ghx)\right).$$

It remains to prove that $\alpha(gh) = s.\,\nu(\alpha(g), g\alpha(h))$. We verify this on values. Let $u \in \Delta^n$ for some n, so $\alpha(gh)(u) = [gh, u] = |\mu|[(g, h), u] = \nu\cdot\Phi[(g, h), u] = (\alpha(g), g\alpha(h))(u)$, which is what was needed.

Proof of C. To verify the adjointness of the functors

$$r^G : (X., \xi) \mapsto (|X.|, \xi^r), \quad s^G : (Y, \eta) \mapsto (s.(Y), \eta^s)$$

we construct natural transformations $\alpha^G : 1 \to s^G r^G$ and $\beta^G : r^G s^G \to 1$, using α and β from (13). According to [5], it only remains to prove that the compositions

$$s^G \xrightarrow{\alpha^G s^G} s^G r^G s^G \xrightarrow{s^G \beta^G} s^G,$$

$$r^G \xrightarrow{r^G \alpha^G} r^G s^G r^G \xrightarrow{\beta^G r^G} r^G$$

are identity transformations. But this follows directly from the analogous properties of the transformations α and β.

The proof of the theorem is completely finished. □

Now we define weak equivalences in the categories ΔG^0Sets and |G.|-Spaces as maps inducing isomorphisms in homotopy groups. Formally inverting (cf. [6]) weak equivalences, we get the homotopy categories HoΔG^0Sets and Ho|G.|-Spaces.

COROLLARY 2.4. The categories HoΔG^0Sets and Ho|G.|-Spaces are equivalent.

Proof. This is a direct consequence of part C of Theorem 2.3, if one uses the simplest test for equivalence of homotopy categories from [7]. □

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Remark on the Paper

In the paper of W. G. Dwyer, M. J. Hopkins, and D. M. Kan, "The homotopy theory of cyclic sets," Trans. Am. Math. Soc., 291, No. 1, 281-289 (1985), a stronger result is found. The categories of |G.|-Spaces and ΔG^0 Sets are endowed with structures of closed model categories in the sense of Quillen. It is proved that the equivalence of the corresponding homotopy theories preserves these structures.