Superconnections and index theory

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1. Superconnections
2. Index theorem
3. Sketch some proofs

Defn (Quillen) A superconnection $\not{\nabla}$ on a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\begin{matrix} V \\ \downarrow \\ M \end{matrix}$ is an odd derivation on ~~the~~ $\Omega^{\cdot}(M, V)$.

(An ordinary connection is a degree 1 derivation on $\Omega^{\cdot}(M, V)$)

Superconnections form an affine space modeled on $\Omega(M, \mathrm{End}\, V)^{odd}$.

$\mathrm{End}(V)$ has a grading

$$\underset{\text{even}}{\begin{pmatrix} \times & \\ & \times \end{pmatrix}} \qquad \underset{\text{odd}}{\begin{pmatrix} & \times \\ \times & \end{pmatrix}}$$

So a superconnection can be written as

$$\not{\nabla} = \omega_0 + \nabla + \omega_2 + \omega_3$$

where ω_i in different degrees (?).

Why superconnections? For K-theory.

In K-theory w support,

$$V \xrightarrow{f} W$$

where f is nonsingular.

For unitary superconnection, have

$$\nabla\!\!\!/ = \begin{pmatrix} & f^* \\ f & \end{pmatrix} + \nabla + \cdots$$

so $V^0 \xrightarrow{f} V^1$

so nicely represents class.

$$ch(\nabla\!\!\!/) = \operatorname{Tr} e^{\nabla\!\!\!/^2}$$

Chern character form.

When $\not\nabla^2$ has support somewhere, then $\mathrm{Tr}\, e^{\not\nabla^2}$

decays exponentially fast off support ... good.

2. Index theory

Defn Let M be smooth, Riemannian, spin. The **Dirac operator** associated to the data of a complex unitary ~~sp~~ v. bundle w superconnection

$$\left(\begin{array}{c} V, \not\nabla \\ \downarrow \\ M \end{array}\right)$$

only difference is now I get forms in all degrees

is defined by:

Levi-Civita

$$\mathcal{D}(\not\nabla) : \Gamma(\$ \otimes V) \xrightarrow{\nabla \otimes 1 \oplus 1 \otimes \not\nabla} \Omega^\bullet(M, \$ \otimes V)$$

Clifford multiply

$$\xrightarrow{c(\cdot)} \Gamma(\$ \otimes V)$$

- If M is closed, $\mathcal{D}$ is elliptic, and formally self-adjoint

so looks like $\mathcal{D}(\not\nabla) = \left(\begin{array}{c|c} & \cancel{\mathcal{D}}(\not\nabla) \\ \hline \not{\mathcal{D}}(\not\nabla) & \end{array}\right)$

Expect a thm

$$\operatorname{index}\left(\not{D}(\nabla)\right) = \int_M \hat{A}(\Omega^M)\, ch(\nabla)$$

In fact, this is a corollary of the Atiyah-Singer index thm.

Because a superconnection can be homotoped to zero, so

$$\operatorname{index}(\not{D}(\nabla)) = \int_M \hat{A}(\Omega^M)\, ch(\nabla)$$

So they don't tell us about topology, but about geometry.

Local index theorems

Mckean-Singer:

$$\operatorname{Tr} e^{-t\mathcal{D}(\nabla)^2} = \operatorname{index}\left(\not{D}(\nabla)\right)$$

Heat kernel
$$e^{-t\mathcal{D}(\nabla)^2}\psi(x) = \int_M P_t(x,y)\,\psi(y)\,dy$$

$$\operatorname{Tr} e^{-t\, \mathcal{D}(\nabla\!\!/)^2} = \int_M \operatorname{Tr} P_t(x,x)\, d\mathrm{vol}_x .$$

<u>Thm</u> (Bott, Patodi, Singer, ...)

$$\lim_{t\to 0} \operatorname{Tr} P_t(x,x) \cdot d\mathrm{vol}_x = (2\pi i)^{-n/2} \left[\hat{A}(\Omega^M)\, ch(\nabla\!\!/)\right]$$

Might imagine with a bold nabla, it's true but no...
a superconnection diverges.

Need to introduce scaling (each form of different degree scales differently)

$$\nabla\!\!/^{\,s} = |s|^{-1/2}\,\omega_0 + \nabla + |s|^{1/2}\,\omega_2 + |s|\,\omega_3 + \cdots$$

Then it <u>does</u> hold (the local index thm) ... due to Getzler, using stochastic techniques.
He tried hard for one without stochastic techniques.

Families

A Riemannian map is a triple

$$(\pi, g, P)$$

where $\pi : M \longrightarrow B$ is a proper submersion of smooth manifolds

$g^{M/B}$ metric on $T(M/B)$

P is roughly a connection,

$$P : T(M) \longrightarrow T(M/B).$$

We ask that fibres be closed + spin.

Now we can make a family of Dirac operators.

$$\begin{array}{l} V, \nabla \\ \downarrow \\ M \\ \downarrow \pi \\ B \end{array}$$

fibrewise Dirac operator.

Recall top. index thm gives you class in K-theory of base.

Bismut did this at level of forms.

$\pi_*(V)$: the fibre at $y \in B$ is

smooth sections $\to \Gamma_y(S^{M/B} \otimes V)$

Bismut: If we have a connection on the top, you get a superconnection on the bottom.

$$\pi_! \mathbb{V} = \pi_! \nabla + \pi_! \omega.$$

Bismut told us what to do with ∇.

I can tell you what to do with ω !

$\therefore$ this is a main motivation for superconnections

$$\pi_* \omega(\xi_1, \ldots, \xi_i) = c^{M/B}\left(i(\tilde{\xi}_1), i(\tilde{\xi}_2), \ldots, i(\tilde{\xi}_i)\omega_i\right)$$

This is superconnection on base.

Can scale the fibers $\pi \mapsto \pi^t$. (multiply metric).

Theorem (Alex)

$$\lim_{t \to 0} \operatorname{ch} \pi_!^t \mathbb{V} = (2\pi i)^{-\dim M/B} \pi_*\left[\hat{A}(\Omega^{M/B}) \operatorname{ch}(\mathbb{V})\right]$$

$$\boxed{\pi_!^{\,t}(\not{D}) = \left[\pi_!\, \not{D}^{\frac{1}{t}}\right]^{t}}$$

I was interested in determinant line bundles.
You end up with η - invariants.

both defined from spectrum of Dirac operator.

With superconnections, spectral things not so easy!
(don't commute).

2 choices : Spectral defn $\longrightarrow$ no nice geometric theorems

or what I do:

"Geometric defn"

want to have curvature
= 2-form part, etc.

Look at $\nabla, \not{D}$
$\downarrow$
M

Inside superconnection, have $\left(\begin{array}{c|c} & f^* \\ \hline f & \end{array}\right)$.

Associate to this

$$\begin{array}{c} (\det V, \det, \nabla) \\ \big\downarrow \\ M, \end{array}$$

line bundle with connection, ∇, whose curvature is

$$\mathrm{curv}(\nabla) = \mathrm{ch}\, \nabla\!\!/ \; .$$