Deformation quantization.

Classical mechanics: $A^c$, a commutative $\mathbf{R}$-algebra with a Poisson bracket.

To quantize this we need to find an associative algebra $A^q$ over the ring $\mathbf{R}[[\hbar]]$ such that $A^q/\hbar A^q = A^c$ and for all $a$ and $b$ in $A^c$ and for all $\tilde{a}$ and $\tilde{b}$ in $A^q$ that are lifts of $a$ and $b$ we have $\{a, b\} = \hbar^{-1}[\tilde{a}, \tilde{b}] \bmod \hbar$.

These lectures will give an analogue of this picture for QFT.

(1) Need to explain what plays the role of commutative Poisson associative algebras.

(2) Explain how classical field theory is encoded in commutative Poisson language.

(3) Explain how to quantize classical field theory.

The structure that plays the role of associative algebra is called a *factorization algebra*. It is a $C^\infty$ of chiral algebra in the sense of Beilinson and Drinfeld.

Let $M$ be a manifold (spacetime on which we do QFT). Let $B(M)$ be the infinite-dimensional smooth manifold of smooth closed balls in $M$. Let $B_n(M)$ be the set of $n$ disjoint balls embedded in a larger ball. There are canonical maps $q$ from $B_n(M)$ to $B(M)$ (the larger ball) and $p$ to $B(M)^n$ (smaller balls).

A factorization algebra is a vector bundle $F$ on $B(M)$ equipped with $S_n$-equivariant maps $p^*(F^{\otimes n}) \to q^*(F)$ satisfying some evident compatibility. Here $\otimes$ denotes the external tensor product. Concretely: $F$ assigns a vector space to every ball in $M$. If we have some configuration of balls embedded into a larger ball, then we get a map $F(B_1) \otimes \cdots \otimes F(B_n) \to F(B)$. They must vary smoothly as the configuration of the balls varies.

The compatibility condition is an obvious operad-like condition. This is an algebra over colored operad, colors are $B(M)$, $n$-ary operations are $B_n(M)$ with an extra condition such that the vector space we assign to each color forms a smooth vector bundle.

Vector bundles come in 3 natural flavors: $C^\infty$, holomorphic, and locally constant sheaves. Factorization algebras exist in all 3 settings.

Definition: A locally constant factorization algebra is like a factorization algebra except that $F$ is a locally constant sheaf on $B(M)$ and the structure maps are maps of locally constant sheaves. (Everything is a cochain complex.)

Let $F$ be a locally constant factorization algebra on $\mathbf{R}^n$. Since $B(\mathbf{R}^n)$ is contractible, $F$ is equivalent (quasi-isomorphic) to a trivial sheaf with fiber $V$, a cochain complex. If we have an embedding of two balls into a larger ball, we get a map $V \otimes V \to V$. As the configurations of discs vary, products change by homotopies. So a locally constant factorization algebra on $\mathbf{R}^n$ is an $E_n$ algebra.

Next specialization: Holomorphic factorization algebras. Let $\Sigma$ be a Riemann surface. We know what it means for a map from a complex manifold to $B(\Sigma)$ to be holomorphic (use the standard construction for moduli space.)

Let us consider a holomorphic factorization algebra on $\mathbf{C}$ that is translation invariant and dilation invariant. Let $V$ be $F$ applied to any round disc. As in the previous paragraph we have a map $m_z\colon V \otimes V \to V$ that varies holomorphically as the balls vary with a fixed radii. Hence $m_z$ is a holomorphic map from an annulus to $\mathrm{Hom}(V \otimes V, V)$. So it has a Laurent expansion $m_z \tilde{\sum}_{k\in\mathbf{Z}} z^k a_k$, where $a_k$ is in some completion of $\mathrm{Hom}(V \otimes V, V)$. Remniscent of VOA. Beilinson and Drinfeld make the same definition in the algebraic setting. They show axioms for a chiral algebra on $\mathbf{C}$ are essentially equivalent to those of a vertex algebra.

We can replace $B(M)$ be the space of embeddings of closed $n$-dimensional ball into $M$. This does not change the validity of the theory. Then the space of configurations is a product.

What is the classical analogue of the factorization algebra? The basic idea: Factorization algebras on $M$ form a symmetric monoidal category. If $F$ and $F'$ are factorization algebras, then $(F \otimes F')(B) = F(B) \otimes F'(B)$.

Definition: A classical factorization algebra is a commutative algebra in the category of factorization algebras. (Recall that an $E_\infty$-object in a category of $E_n$-algebras is an $E_\infty$-algebra.)

How to associate a classical factorization algebra to a classical field theory? Suppose we have a classical field theory, e.g., a space of field is the space of sections of vector bundle $E \to M$. $S\colon \Gamma(M, E) \to \mathbf{R}$ is the classical action. $S$ is *local* if it is obtained by an integral of a Lagrangian. If $B \subset M$ is a ball, let $\mathrm{EL}(B)$ be the space of section of $\Gamma$ over the interior of $B$ that satisfy the Euler-Lagrange equations.

Rough idea: The classical factorization algebra $F_S$ associated to $S$ assignes to $B$ the algebra $O(\mathrm{EL}(B))$ of functions on the set of solutions to Euler-Lagrange equations. We want maps $F_S(B_1) \otimes \cdots \otimes F_S(B_n) \to F_S(B)$

if $B_i$ is embedded into $B$ for all $i$. We have a map $\mathrm{EL}(B) \to \mathrm{EL}(B_1) \otimes \cdots \otimes \mathrm{EL}(B_n)$. This yields a map $O(\mathrm{EL}(B_1) \otimes \cdots \otimes O(\mathrm{EL}(B_n)) \to O(\mathrm{EL}(B))$ as desired.

Example: Fields are smooth functions on $M$, $S(\phi) = \int_M \phi\Delta\phi$. The Euler-Lagrange equation is $\Delta\phi = 0$. Hence $\mathrm{EL}(B)$ is the space of harmonic functions on the interior of $B$. We let by definition $O(\mathrm{EL}(B)) := \prod_{n\geq 0} \mathrm{Hom}(\mathrm{EL}(B)^{\otimes n}, \mathbf{R})^{S_n}$. Hom means continuous linear maps and $\otimes$ is completed tensor product. Later we will see that we really need to take the derived space of solutions to Euler-Lagrange equations. Why doe sthis classical factorization algebra want to become just a factorization algebra.

Factorization algebra is a symmetric monoidal category. More precisely, the $E_0$ operad is defined by $E_0(n) = \emptyset$ for $n \geq 1$ and $E_0(0)$ is a point. An $E_0$-algebra in vector spaces is a vector space with an element.

Recall [retroactive change] that factorization algebras have a unit, which is a section of $F$ on $B(M)$ that is a unit for the products.

So an $E_0$-algebra in factorization algebras is just a factorization algebra.

| Classical | Quantum |
|---|---|
| Commutative algebras with a Poisson bracket of degree 1 | $E_0$-algebras |
| Poisson algebras | $E_1$-algebras (associative algebras) |
| Commutative algebras with a Poisson bracket of degree $-1$ | $E_2$-algebras |
| Commutative algebras with a Poisson bracket of degree $-2$ | $E_3$-algebras |

Beilinson and Drinfeld: Define an operad over $\mathbf{R}[[\hbar]]$ as follows: It is generated by a commutative product, a Poisson bracket of degree 1 with the differential of product being equal to the Poisson bracket times $\hbar$. Call this the BD operad. Modding out by $\hbar$ gives the operad of commutative algebras with Poisson bracket of degree 1.

Definition: The $P_0$ (P for Poisson) is the operad of Poisson commutative unital algebras with a bracket of degree 1.

General fact: Let $M$ be a manifold and $f$ a funcion on $M$. Then functions on the derived critical locus of $f$ form a $P_0$-algebra. Critical locus of $f$ is the set of zeros of $df$. Functions on the critical locus are $O(M)$ divided over the image of $df$ regarded as the map from $\Gamma(M, TM)$ to $O(M)$. Functions on the derived critical locus form a dga $\Gamma(M, \Lambda TM)$ with $\Lambda^k TM$ in the degree $-k$ and with differential given by $df$. $\Gamma(M, \Lambda TM)$ has Schouten bracket, which is of degree 1. This wants to become $E_0$. The graph of $df$ intersected with $M$ is the critical locus. Making intersection derived gives us the derived critical locus.

Observation: If $M$ has a measure, then $O(\mathrm{Crit}^n(f))$ has a canonical quantization to an $E_0$-algebra. Quantization is $(\Gamma(M, \Lambda TM), df + \hbar\Delta)$.

So the derived critical locus of $f$ is a $P_0$-algebra so it wants to quantize to $E_0$-algebra. If we have a classical field theory, then the derived space of solutions to Euler-Lagrange equations yields a $P_0$-algebra in the category of factorization algebras.

Example: $\phi \in C^\infty(M)$, $S(\phi) = \int \phi\Delta\phi$. Derived space of solutions to Euler-Lagrange equations is the complex $C^\infty(M) \to C^\infty(M)$ in degrees 0 and 1, the map between components being $\Delta$. If $B \subset M$ is a ball, then $O(\mathrm{EL}^n(B)) = \Pi_{n\geq 0}(\mathrm{Hom}((C^\infty(\mathrm{Int}(B)) \to C^\infty(\mathrm{Int}(B)))^{\otimes n}, \mathbf{R})^{S_n}$. This is a commutative dga. It defines a commutative factorization algebra. If $S(\phi) = \int \phi\Delta\phi + \phi^3$, we get the same algebra of functions.

Yang-Mills: First we consider the derived quotient of $\Omega^1(M) \otimes g$ by $\Omega^0(M) \otimes g$, then take the derived critical locus of the Yang-Mills actions.

What we get, when linearized, looks like $E = (\Omega^0(M) \to \Omega^1(M) \to \Omega^3(M) \to \Omega^4(M)) \otimes g$.

Theorem: If we take the derived space of solutions to the Euler-Lagrange equation, looking infinitesimally near a fixed solution, then we find a $P_0$-algebra in the category of factorization algebras on $M$.

We would like to quantize one of these. This amount to quantizing the action $S$ into the solution of the quantum master equation. This requires machinery of counter-terms, Wilsonian effective actions, to even define the QME.

Theorem (naïve version): Consider the scalar field theory with an action $S(\phi) = \int \phi(\Delta\phi + m^2\phi)$ plus arbitrary local cubic and higher-order terms. Let $F_S$ be the classical factorization algebra associated to it.

Let $Q^{(n)}(F_S)$ be the set of quantization defined mod $\hbar^{n+1}$, which is the space of the lifts of $F_S$ to an algebra over BD mod $\hbar^{n+1}$. There is a sequence $T^{(n)} \to T^{(n-1)} \to \cdots \to T^{(1)} \to \bullet$, where $T^{(n)}$ maps to $Q^{(n)}(F_S)$. Here $T^n \to T^{n-1}$ is a torsor for the abelian group of local functionals of the field $\phi$. $T^{(\infty)} = \lim T^n$, then $T^{(\infty)} = \sum_{k\geq 1} \hbar^k S^{(k)}$, where $S^{(k)}$ is a local functional. But this is non-canonical.

More sophisticated version. Consider any reasonable classical theory yielding classical factorization algebra $F$. Let $Q^{(n)}(F)$ be the simplicial set of quantizations defined mod $\hbar^{n+1}$. $\mathrm{Der}_{\mathrm{loc}}(F)$ is the cochain of complex derivations of $F$ preserving $P_0$-structure (local functions on an "extended" space of fields).

Theorem: There is a sequence of simplicial sets $\cdots \to T^{(n)} \to T^{(n-1)} \to \cdots \to T^{(1)} \to \bullet$ with maps $T^{(n)} \to Q^{(n)}(F_S)$ such that $T^{(n)}$ fits into a homotopy fiber diagram $T^{(n)} \to 0 \to \mathrm{Der}_{\mathrm{loc}}(F)[2]$ and $T^{(n)} \to T^{(n-1)} \to \mathrm{Der}_{\mathrm{loc}}(F)[2]$.

Theorem: Let $g$ be a simple Lie algebra. Then there is a quantization of Yang-Mills on $\mathbf{R}^4$ that is "renormalizable" (behaves well under scaling). The set of all such quantizations is $\hbar\mathbf{R}[[\hbar]]$.

If $F$ is a factorization algebra on $M$ corresponding to some QFT, then $F(B)$ is the set of observation we can make on $B$. If $B_1$ and $B_2$ are disjoint the map $F(B_1) \otimes F(B_2) \to F(B)$ is defined by doing both observations.

Correlation functions should be cochain maps $F(B_1) \otimes \cdots \otimes F(B_n) \to \mathbf{R}$. We also have obvious compatibility conditions.

We can consider correlation functions with coefficients in any cochain complex, we require that they must satisfy this equation.

Definition: (Beilinson-Drinfeld) $\mathrm{CH}_*(M, F)$ is the homotopy universal recipient of correlation functions. It is also equal to the colimit of $F(B_1) \otimes \cdot \otimes F(B_n)$ for all disjoint $B_i$.

Lemma: For a massive scalar field $\mathrm{CH}_*(M, F)$ is isomorphic to $\mathbf{R}[[\hbar]]$. In general, $\mathrm{CH}_*(M, F)$ looks like measures on the space of critical points of the classical action.

If we perturb around isolated critical point, $\mathrm{CH}_*(M, F) = \mathbf{R}[[\hbar]]$.

In this situation, correlation functions exist and are unique.

General program: Correlation functions define a measure on the space of classical solutions which we integrate.