CFT and algebra in braided tensor categories.

Part I: Christoph Schweigert.

Chapter 1. *Modular tensor categories and rational CFT.*

Rational semi-simple conformal vertex operator algebras. Every such algebra has a rich representation theory. Using the category of representations we can construct conformal blocks and use them to endow the category of representations with some additional structure. In particular it will be a modular tensor category.

Definition. A *modular tensor category* is an abelian complex-linear semi-simple noetherian strict tensor category such that the monoidal unit is simple. (Denote by I the set of representatives of classes of simple objects.) Modular tensor category is a also a ribbon category, in particular the left and the right duals of an object always coincide. Finally, the braiding must satisfy a non-degeneracy condition, which means that we have an isomorphism of algebras $K_0(C) \otimes_{\mathbf{Z}} \mathbf{C} \to \mathrm{End}(\mathrm{id}_C)$.

Fact (Reshetikhin and Turaev). For any modular tensor category C there is a tensor functor tft_C from the category of cobordism decorated over C to the category of finite-dimensional complex vector spaces. In particular, we have a factoriation homomorphism. [See Turaev's book for its properties.] We also have a representation of mapping class group $\mathrm{Map}(X)$ on $\mathrm{tft}_C(X)$.

Chapter 2. *CFT correlators.*

If X is a two-dimensional oriented manifold with boundary (there is also an unoriented version). X is called a worldsheet. Actually X is a topological manifold.

Strategy: Decorate X. Find an appropriate space of "functions" for correlators.

Holomorphic factorization. Associate to X its oriented cover $\hat{X}$ (twisted double) (glue two copies of X along their boundaries). For a disc we obtain a sphere, for a Möbius band we obtain a torus, for a torus we obtain two tori.

Step 1. Find a decoration for X such that $\hat{X} \in \mathrm{cobord}^C_{3,2}$. Step 2. Now $\mathrm{Cor}(X) \in \mathrm{tft}_C(\hat{X})$. (a) $\mathrm{Cor}(X)$ is invariant under $\mathrm{Map}(\hat{X})^\sigma$. (Modular invariance.) (b) Compatibility with factorization conditions.

Insight: Decoration data is bicategory of special symmetric Frobenius algebras in C. Definition of Frobenius algebra $(A, \eta, m, \Delta, \epsilon)$, where (A, η, m) is an algebra, Δ is a comultiplication which is a morphism of A-bimodules. Special property: $m \circ \Delta = \mathrm{id}_A$ and $\epsilon \circ \eta = (\dim A)\mathrm{id}_1$.

A typical worldsheet looks like a manifold with boundary and defect lines (embedded branching lines with marked points).

A decoration maps a 2-cell to some SSFA A, a 1-cell which is a boundary maps to a left or right A-module (depending on the orientation; A is the algebra associated to the neighboring 2-cell) and defect lines are mapped to A_1-A_2-bimodules, where A_1 and A_2 are the algebra associated to two neighboring 2-cells. Now we move to 0-cells. Branching points on defect lines are mapped to a morphism of A_1-A_3-bimodules from $D_1 \otimes_{A_2} D_2$ to D_3. [The other two cases of 0-cells I do not understand.]

Correlators from cobordisms. If M_X is a cobordism from $\emptyset$ to $\hat{X}$, then $\mathrm{Cor}(X) = \mathrm{tft}_C(M_X)1 \in \mathrm{tft}_C(\hat{X})$. Here $M_X = (\hat{X} \times [-1, 1])/(\sigma\colon t \mapsto -t)$. We have $\partial M_X = \hat{X}$.

Part II: Ingo Runkel.

Plan: (1) Bulk algebra. (2) Module categories. (3) Outlook of the logarithmic conformal field theory.

Recall: We have a modular tensor bicategory C. Objects are special symmetric Frobenius algebras, morphisms are bimodules, 2-morphisms are intertwiners.

We have a functor $R\colon C \to C_+ \times C_-$ that is an adjoint to the tensor product functors $C_+ \times C_- \to C$. Here $C_+ = C$ and C_- is C with the inverse braiding and the inverse twist.

Proposition: If A is a SSFA in C, then $R(A)$ is a SSFA in $C_+ \times C_-$.

Definition: The left center of an algebra A in C is the maximal subobject Z_l of A such that the obvious condition is satisfied. The right center is defined using the inverse braiding.

Definition: The full center of a SSFA $A \in C$ is $Z(A) = Z_l(R(A)) \in C_+ \times C_-$.

Proposition: $Z(A)$ is a commutative SSFA in $C_+ \times C_-$. $Z_l(A) \times 1$ and $1 \times Z_r(A)$ are subalgebras of $Z(A)$. $Z(A) = \bigoplus_{i,j \in I} (U_i^* \times U_j)^{\oplus z_{i,j}(A)}$.

Theorem: The number of isomorphism classes of simple A-left modules is equal to the trace of $z(A)$.

Theorem (Kong-Runkel, 2007): If C is a modular tensor category and A and B are Morita-equivalent simple SSFA, then $Z(A)$ is isomorphic to $Z(B)$ as algebra.

Theorem (Kong-Runkel, 2008): If C is a modular tensor category and F is a commutative simple SSFA in $C_+ \times C_-$ such that $\dim F = \dim C$, then there exists a SSFA $A \in C$ such that F is isomoprhic to $Z(A)$ and $T(F) = \oplus_i A_i$, where A_i are simple SSFA, Morita-equivalent to each other.

Chapter 2. *Module categories.*

Definition: If C and M are abelian complex linear categories and C is a tensor category, then M is a module category over C if we have a functor $M \times C \to M$ such that the obvious (weak) associativity conditions are satisfied.

Examples: A monoidal category is a module over itself. If A is an algebra in C and A-mod is the right A-module category, then we have an obvious functor given by tensor product.

Theorem (Ostrik, 2001): If C is a modular tensor category (without braiding), M is a semi-simple finite indecomposable (cannot be written as a direct sum of two other categories) module category over C, then M is equivalent to the category of A-modules for some algebra A in C.

Proof: Define the inner hom as a right adjoint to the monoidal product.