String connections and Chern-Simons 2-gerbes

In Corbitt's talk: string structures up to homotopy.

In this talk: the actual string structures

$$\left\{ \begin{array}{c} \text{Bundle } n\text{-gerbes} \\ \text{over } M \end{array} \right\} \cong H^{n+2}(M, \mathbb{Z})$$

geometric gismos living on M.

this is a thm for $n=0,1,2$.

These actually arise canonically in nature.

1. Some definitions
2. Examples: a) basic gerbe over compact simple Lie group
   b) Lifting bundle gerbe
   c) Chern-Simons bundle 2-gerbe.

bundle 1-gerbes

Bundle gerbes are nice in that you can define them for all n by a recursive defn.

<u>$n=0$</u> <u>Bundle 0-gerbe</u> on M :

a) a surjective submersion $\pi : Y \longrightarrow M$

b) a smooth map $\mu : Y \times_M Y \longrightarrow U(1)$

Coherence :

$$\pi_{12}^* \mu \cdot \pi_{23}^* \mu = \pi_{13}^* \mu .$$

Exercise : Show that this forms a monoidal groupoid $\cong$ to groupoid of Principal $U(1)$-bundles over M.

Q : You don't want to assume your map is proper

<u>$n=1$</u> Bundle 1-gerbe :

$$\begin{tikzcd}
P \arrow[d] & \\
Y \times_M Y \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

$P \leftarrow$ a $U(1)$-bundle, i.e. a bundle 0-gerbe

+ a morphism $\mu : \pi_{12}^* P \otimes \pi_{23}^* P \rightarrow \pi_{13}^*$ in that groupoid.

coherence : associativity.

Bundle gerbes form a monoidal 2-groupoid.

Iso classes $\cong H^3(M, \mathbb{Z})$.

$\underline{n=2}$ Bundle 2-gerbe.

a bundle gerbe

$$\begin{tikzcd}
P \arrow[d] & \\
Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

\+ a 1-morphism

\+ 2-morphism (associator)

and new coherence axiom : pentagon.

a) Basic gerbe $\mathcal{G}_G$ such that $[\mathcal{G}_G] = 1 \in H^3(G,\mathbb{Z})$.

G simply connected.

<u>construction</u> Meinrenken, Gawedzki-Reis.

pull back line bundle

hermitian line bundle

$$\begin{array}{ccc} & & L_{\alpha\beta} \\ & & \downarrow \\ \coprod_{\alpha,\beta} U_\alpha \cap U_\beta & \longrightarrow & \mathcal{O}_{\lambda_{\alpha\beta}} \subseteq \mathfrak{g}^* \end{array}$$

coadjoint orbit

a particular canonical open cover of the group $\longrightarrow$ rank $\mathfrak{g}$ $\coprod_\alpha U_\alpha \to G$

[<u>Explicitly</u>:
Conj. classes are parametrized by elements in the Weyl alcove.
It is a simplex ... pullback ... get cover.

<u>Hopkins</u>: The space of conj. classes is a simplex, and you're taking map ...

The iso μ comes from

$$\lambda_{\alpha\gamma} = \lambda_{\alpha\beta} + \lambda_{\beta\gamma}$$

This is the basic bundle gerbe.

Possible to make it:

- G-equivariant under conjugation.

No time... Konrad angry $\ddot{\cup}$

c). Chern-Simons 2-gerbe $\mathbb{CS}_P$

Need:
- a compact simple simply connected Lie group G
- a principal G-bundle over M.

pull back. Now we have a bundle gerbe.

basic bundle gerbe

+ multiplicative structure

$$\begin{tikzcd}
 & & \mathcal{G}_G \arrow[d] \\
P \times_\pi P \arrow[rr, bend left, "g"] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d, "\pi"] & G \\
 & M &
\end{tikzcd}$$

Why interesting?

Lemma

If $G = \mathrm{Spin}$, then

$$[\mathbb{CS}_P] = \frac{1}{2} p_1(P) \in H^4(M, \mathbb{Z}).$$

We know from Corbitt's talk that this obstructs string structures on M.

$$\boxed{\text{The transgression of } \mathbb{CS}_P \text{ is } \mathcal{G}_G \text{ ?}}$$

3 String structures

A Trivialization of $\mathbb{CS}_P$ is:

- a bundle gerbe $\mathbb{S}$ over P
- $g^* \mathcal{G}_{spin} \otimes \pi_2^* \mathbb{S} \to \pi_1^* \mathbb{S}$ (a 1-isomorphism)
- a 2-morphism + coherence.

Lemma [Stevenson] $[\mathbb{C}\mathcal{S}_P] = 0 \iff \mathbb{C}\mathcal{S}_P$ has a trivialization

$\Updownarrow$

P admits string structures

> So a trivialization is a 1-morphism to the trivial gerbe.

Trivializations:

- form a 2-groupoid
- it is a module over the monoidal groupoid of bundle gerbes over M

(geometric counterpart of Corbitt's action of $H^3(M,\mathbb{Z})$ on string stuctures)

and is free and transitive.

These trivializations are a good model for the actual string structure;

<u>Lemma</u> The trivializations of $\mathbb{C}\mathcal{S}_P$ <u>are</u> the string structures

$$\left\{ \begin{array}{l} \text{iso-classes} \\ \text{of trivializations} \end{array} \right\} \longrightarrow$$ Corbitt's String classes on P [3-classes on P which restrict to fiber]

$$S \longmapsto [S] \in H^3(P, \mathbb{Z})$$

That is, this map is a bijection.

No-one has defined string structures explicitly yet.

So we <u>define</u> a string structure as a trivialization of the Chern-Simons 2-gerbe

4 Connections on bundle n-gerbes:

Goal: lift to differential cohomology:

$$\left\{ \begin{array}{c} \text{iso-classes} \\ \text{of bundle} \\ n\text{-gerbes with} \\ \text{connection over } M \end{array} \right\} \cong \hat{H}^{n+2}(M, \mathbb{Z}).$$

This is true for $n = 0, 1, 2$.

Connections

A connection on a bundle 0-gerbe is

$$A \in \Omega^1(Y)$$

s.t. $\pi_2^* A - \pi_1^* A = d\log(\mu)$.

curvature $:= \pi_!(dA)$.

principal U(1)-bund

A connection on a bundle 1-gerbe is

$B \in \Omega^2(Y)$, and a connection on P

s.t.

- $\pi_2^* B - \pi_1^* B = \mathrm{curv}(B)$
- μ preserves connection.

$$\text{Curvature} = \pi_! \underbrace{(dB)}_{\uparrow}$$

descends to base.

A <u>connection</u> on a 2-bundle gerbe is:

- $C \in \Omega^3(Y)$ + connection on bundle gerbe P

s.t.

- $\pi_2^* C - \pi_1^* C = \text{curv}(P)$
- M, μ connection preserving (this is actually extra data!)

<u>Examples</u>

a) Given basic gerbe $\mathcal{G}_G$, has canonical connection. Characterized by

$$\text{curvature} = \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle$$

the canonical 3-form on G (the generator of H^3).

b) $\mathbb{CS}_P$ has a connection depending on a connection A on P.

This connection on $\mathbb{CS}_P$ has

- $C = CS_3(A) \in \Omega^3(P)$

  Chern-Simons 3-form.

- connection on $g^*\mathcal{G}_{spin}$ is the pullback connection of the previous example.

We can ask that trivializations respect these connections.

5) String connections

bundle + connection

A <u>geometric string structure</u> on (P, A) consists of a string structure S on P + a string connection for (S, A).

<u>Defn</u> A <u>string connection</u> on (S, A) is a connection on S, such that the 1-morphism and 2-morphism from trivialization of $\mathbb{CS}_P$ respect the connection.

Relation to Urs' talk:

$$[\mathbb{C}\mathcal{S}_p] \in \hat{H}^4(M,\mathbb{Z})$$

Use this class to <u>twist</u> ordinary differential cohomology.

A string structure + string connection is a class in the <u>twisted</u> differential cohomology of M twisted by $[\mathbb{C}\mathcal{S}_p]$ in Urs' sense.

<u>torsor</u>

$$G \times M \to M \times M$$

equivalence of 2-categories

Counting string connections

<u>Thm [KW]</u>

- Geometric string structures form a 2-groupoid.
- It is a module for bundle gerbes with connection over M.
- On iso classes, this is a free and transitive action of $\hat{H}^3(M,\mathbb{Z})$
- For every string structure S and every connection A, there exists a string connection. The possible choices form a contractible space.