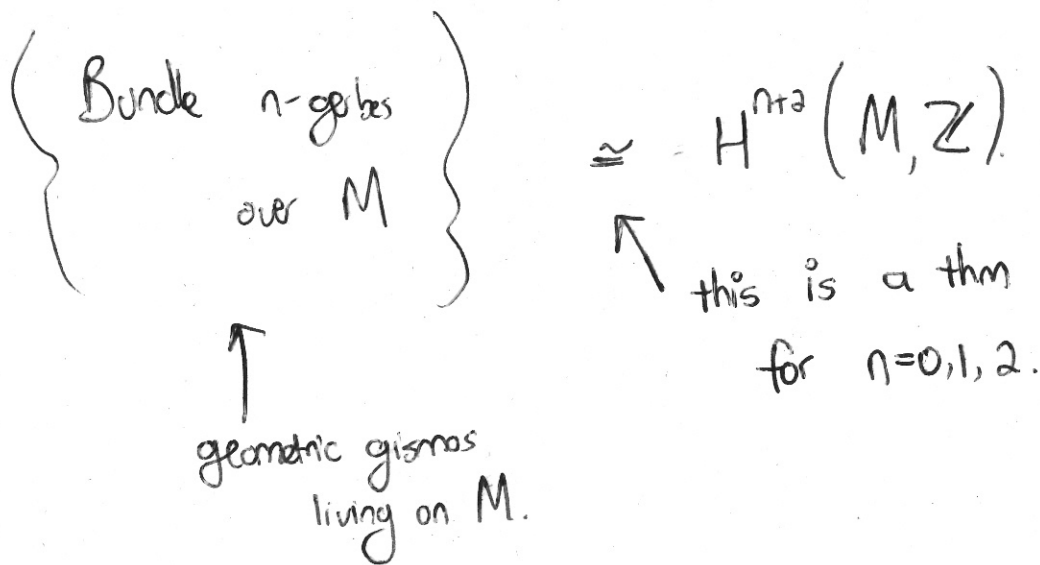


String connections and Chern-Simons 2-gerbes

(1)

In Corbett's talk: string structures up to homotopy.

In this talk: the actual string structures.



These actually arise canonically in nature.

(1) Some definitions

- (2) Examples :
- a) basic gerbe over compact simple Lie group
 - b) Lifting bundle gerbe
 - c) Chern-Simons bundle 2-gerbe.
- } bundle 1-gerbes

The Bundle gerbes are nice in that you can define them for all n by a recursive defn.

$n=0$ Bundle 0-gerbe on M :

- a) a surjective submersion $\pi: Y \rightarrow M$
- b) a smooth map $\mu: Y \times_M Y \rightarrow U(1)$

~~Coherence~~ Coherence:

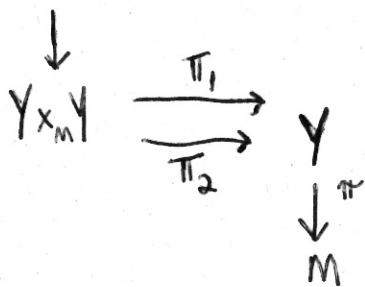
$$\pi_{12}^* \mu \cdot \pi_{23}^* \mu = * \pi_{13}^* \mu.$$

Exercise : Show that this forms a monoidal groupoid \cong to groupoid of Principal $U(1)$ -bundles over M .

Q: You don't want to assume your map is proper

$n=1$ Bundle 1-gerbe :

$P \leftarrow$ a $U(1)$ -bundle, i.e. a bundle 0-gerbe



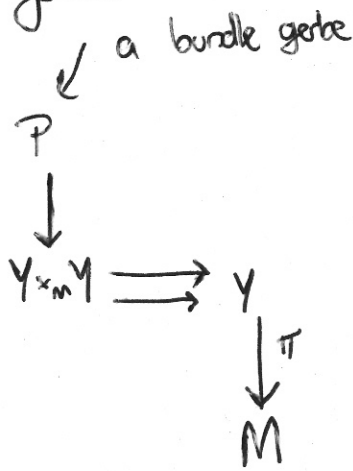
+ a morphism $\mu: \pi_{12}^* P \otimes_{\pi_{23}^* P} \rightarrow \pi_{13}^* P$ in that groupoid.

coherence : associativity.

Bundle gerbes form a monoidal 2-groupoid. (3)

Iso classes $\cong H^3(M, \mathbb{Z})$.

$n=2$ Bundle 2-gerbe.



+ a 1-morphism

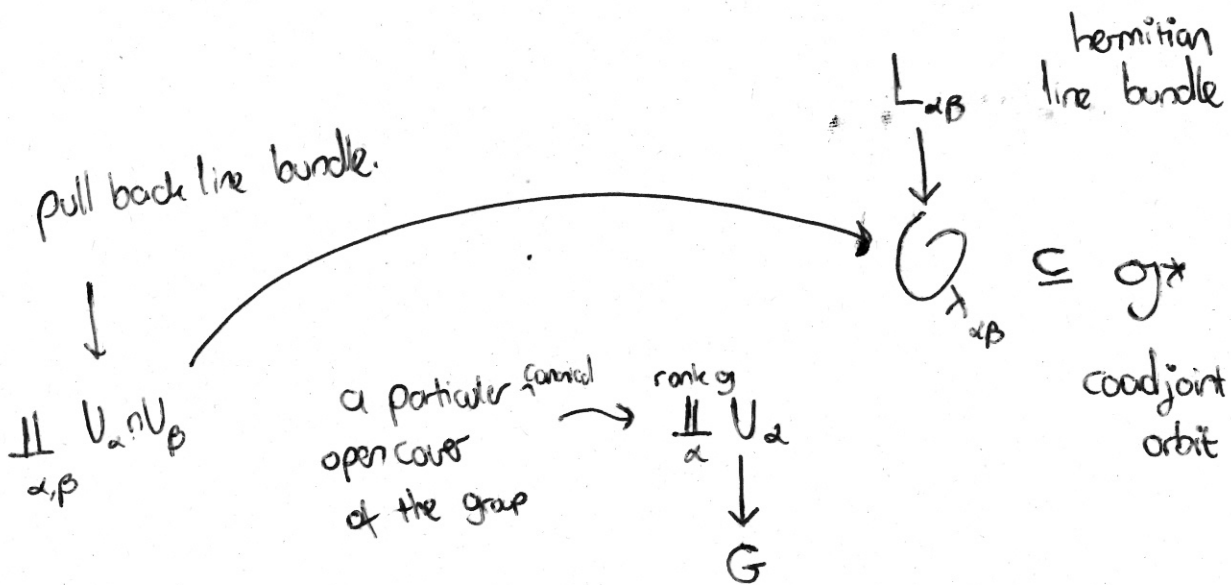
↗
+ 2-morphism (associator)

and new coherence axiom : pentagon.

a) Basic gerbe \mathcal{G}_G such that $[\mathcal{G}_G] = 1 \in H^3(G, \mathbb{Z})$.

G simply connected.

construction Meinrenken, Gawedzki-Reis.



Explicitly:

Conj. classes are parametrized by elements in the Weyl alcove.

It is a simplex ... pullback ... get cover.

Hopkins: The space of conj. classes is a simplex, and you're taking map ...

The iso μ comes from

$$\lambda_{\alpha\gamma} = \lambda_{\alpha\beta} + \lambda_{\beta\gamma}$$

This is the basic bundle gerbe.

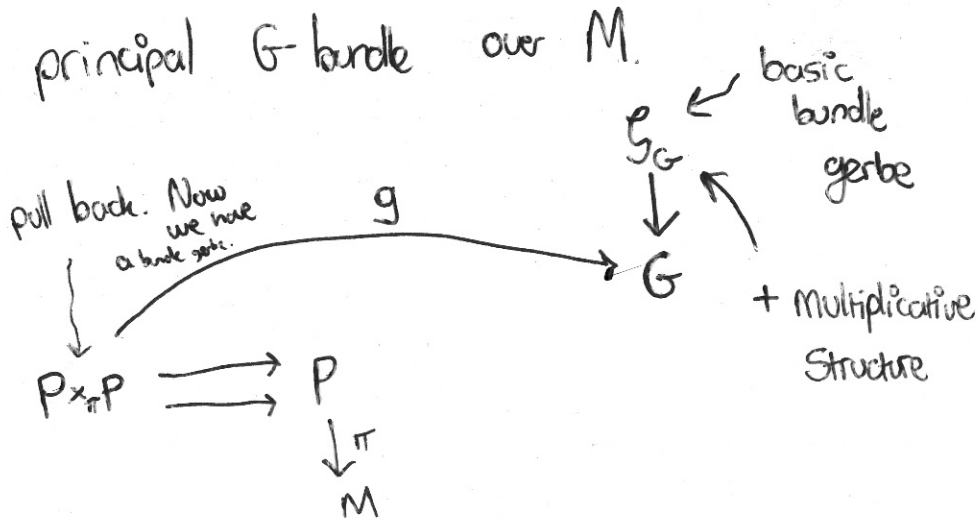
Possible to make it:

- G -equivariant under conjugation.

No time... Konrad angry ☹

c). Chern-Simons 2-gerbe $\mathbb{C}\mathbb{S}_P$

- Need:
- a compact simple simply connected Lie group G
 - a principal G -bundle over M .



Why interesting?
Lemma

(6)

If $G = \text{Spin}$, then

$$[\mathbb{C}\mathcal{S}_P] = \frac{1}{2} P_1(P) \in H^4(M, \mathbb{Z})$$

We know from Corbett's talk that this obstructs string structures on M .

The transgression of $\mathbb{C}\mathcal{S}_P$ is G_G ?

(3) String structures

A Trivialization of $\mathbb{C}\mathcal{S}_P$ is:

- a bundle gerbe \mathcal{S} over P
- $g^* \mathcal{G}_{\text{spin}} \otimes \pi_2^* \mathcal{S} \rightarrow \pi_1^* \mathcal{S}$
(a 1-isomorphism)
- a 2-morphism + coherence.

Lemma [Steverson] $[\mathbb{C}\$P]=0 \iff \mathbb{C}\P has a trivialization (7)

\iff
 P admits string structures.

So a trivialization is a 1-morphism to the trivial gerbe.

Trivializations:

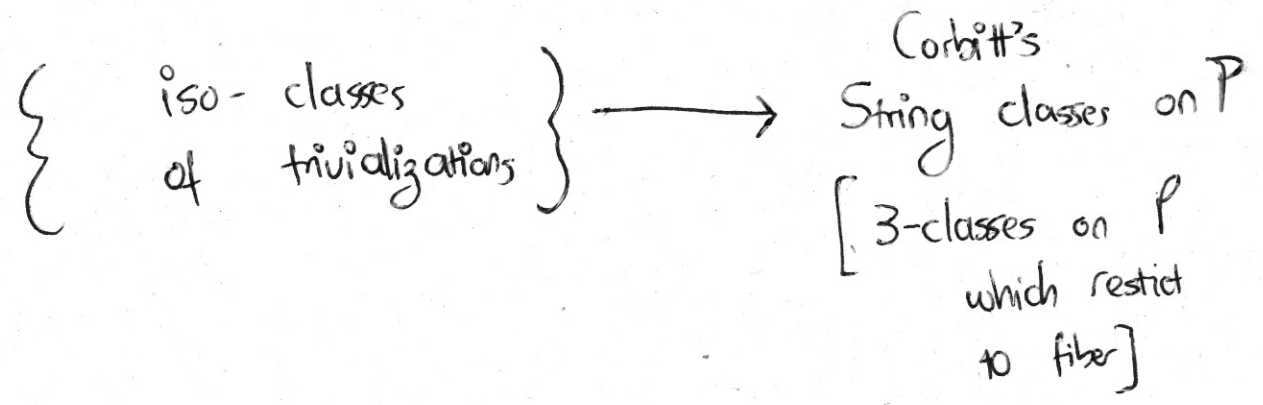
- form a 2-groupoid
- it is a module over the monoidal groupoid of bundle gerbes over M

(geometric counterpart of Cobitt's action of $H^3(M, \mathbb{Z})$ on string structures)

and is free and transitive.

These trivializations are a good model for the actual string structures;

Lemma The trivializations of CS_P are the string structures



$$S \longmapsto [S] \in H^3(P, \mathbb{Z})$$

That is, this map is a bijection.

No-one has defined string structures explicitly yet.

So we define a string structure as a trivialization of the Chern-Simons 2-gerbe

④ Connections on bundle n -gerbes.

⑨

Goal \circledast lift to differential cohomology:

$$\left\{ \begin{array}{l} \text{iso-classes} \\ \text{of bundle} \\ n\text{-gerbes with} \\ \text{connection over } M \end{array} \right\} \cong \hat{H}^{n+2}(M, \mathbb{Z}).$$

This is true for $n=0,1,2$.

Connections

A connection on a bundle 0 -gerbe is

$$A \in \Omega^1(Y)$$

st. $\pi_2^* A - \pi_1^* A = d \log(\mu)$.

curvature $\circledast = \pi_1^*(dA)$.

A connection on a bundle 1 -gerbe is

$$B \in \Omega^2(Y), \text{ and a connection on } P$$

st. $\pi_2^* B - \pi_1^* B = \text{curv}(B)$

μ preserves connection.

principal $U(1)$ -bund



P

$$\text{Curvature} = \pi_1^* (\text{dB})$$

↑
descends to base.

(6)

A connection on a 2-bundle gerbe is:

- $C \in \Omega^3(Y)$ + connection on bundle gerbe \mathcal{P}

st.

- $\pi_2^* C - \pi_1^* C = \text{curv}(\mathcal{P})$
- M, μ connection preserving (this is actually extra data!)

Examples

- a) Given basic gerbe G_G , has canonical connection. Characterized by

$$\text{curvature} = \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle$$

the canonical 3-form on G (the generator of H^3).

b) CS_P has a connection depending on a connection A on P .

(11)

This connection on CS_P has

• $C = CS_3(A) \in \Omega^3(P)$

↑ Chern-Simons 3-form.

• Connection on g^*S_{spin} is the pullback connection of the previous example.

We can ask that trivializations respect these connections.

(5) String connections

A geometric string structure on (P, A) consists of a bundle + connection
string structure S on P + a string connection for (S, A) .

Defn A string connection on (S, A) is a connection on S , such that the 1-morphism and 2-morphism from trivialization of CS_P respect the connection.

Relation to Urs' talk:

(12)

$$[C \mathcal{S}_P] \in \hat{H}^4(M, \mathbb{Z})$$

Use this class to twist ordinary differential cohomology.

A string structure + string connection is a class in the twisted differential cohomology of M twisted by

$[C \mathcal{S}_P]$ in Urs' sense.

Counting string connections

torsor

$$G \times M \rightarrow M \times M$$

equivalence of 2-categories

Thm [KW]

- Geometric string structures form a 2-groupoid.
- It is a module for bundle gerbes with connection over M .
- On iso classes, this is a free and transitive action of $\hat{H}^3(M, \mathbb{Z})$.
- For Every string structure S and every connection A , there exists a string connection. The possible choices form a contractible space.