A MASTER COURSE ON ALGEBRAIC STACKS

BERTRAND TOËN

Contents

Translated by Ruian Chen.

Lecture 1. Reflections on the notion of topological spaces I

The purpose of these first few lectures is to understand the notion of manifolds in various contexts (topological, differentiable, analytic ...) For that we will start in this first lecture by considering the case of topological manifolds.

1.1. Review of the topological manifolds

Definition 1.1.1.

(1) *A* topological manifold *is a topological space X which possesses a open cover $\{U_i\}_{i\in I}$ such that for each $i \in I$ there exists a homeomorphism of U_i with an open set of $\mathbb{R}^{n_i}$ (for a certain integer $n_i \geq 0$ depending on i).*

(2) *The* category of topological manifolds *is the full subcategory of that of the topological spaces whose objects are topological manifolds. This category is denoted by* Top.

Let X be a topological manifold and $\{U_i\}_{i\in I}$ a open cover as in the Definition 1.1.1 (1) above. We put, for i and j in I, $U_{i,j} := U_i \cap U_j$. We have a diagram of topological spaces

$$\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i,$$

where the first morphism sends the component $U_{i,j}$ to U_i by the natural inclusion $U_{i,j} \subset U_i$, and the second morphism sends $U_{i,j}$ to U_j by the natural inclusion $U_{i,j} \subset U_j$. There also exists a natural morphism

$$\coprod_{i\in I} U_i \longrightarrow X$$

summing the inclusions $U_i \subset X$, which equalizes the two morphisms above. We thus obtain a well-defined morphism

$$\underrightarrow{\mathrm{colim}} \left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i \right) \longrightarrow X.$$

The important fact is the following.

Lemma 1.1.2. *The morphism*

$$\underrightarrow{\mathrm{colim}} \left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i \right) \longrightarrow X$$

is an isomorphism.

Proof. The lemma says that for a topological space Y, to give a morphism $f : X \to Y$ is the same as to give for any $i \in I$ a morphism $f_i : U_i \to Y$ such that $(f_i)|_{U_{i,j}} = (f_j)|_{U_{i,j}}$ for any $(i,j) \in I^2$. (Exercise: work out the details). □

It is necessary to interpret the preceding lemma as follows: any topological variety is obtained as a colimit of a diagram of open sets of $\mathbb{R}^n$ (for n varying). We draw from that the following principle:

The category Top *of topological manifolds is deduced from the category of open sets of $\mathbb{R}^n$ (and continues maps).*

We will clarify this principle later.

1.2. Manifolds and sheaves

Let C be a full subcategory of Top whose objects are open sets of $\mathbb{R}^n$ (for n varying). We denote by $\mathrm{Pr}(C)$ the category of presheaves of sets on C (also denoted by $\widehat{C}$). We consider the Yoneda embedding of C

$$\begin{array}{cccc} h_- & : & \mathrm{Top} & \longrightarrow & \mathrm{Pr}(C) \\ & & X & \longmapsto & h_X, \end{array}$$

where the presheaf h_X is defined by

$$h_X(Y) := \mathrm{Hom}_{\mathrm{Top}}(Y, X),$$

for any $Y \in C \subset \mathrm{Top}$.

Lemma 1.2.1. *The functor h_- above is fully faithful.*

Proof. The functor is faithful: for two morphisms $f, g : X \to X'$, we consider an open cover $\{U_i\}$ of X such that $U_i \in C$ (this exists since X is a manifold). If $h_f = h_g$, then for any i the two maps

$$h_f(U_i) = h_g(U_i) : \mathrm{Hom}(U_i, X) = h_X(U_i) \longrightarrow \mathrm{Hom}(U_i, X') = h_{X'}(U_i)$$

are equal. This means that $f|_{U_i} = g|_{U_i}$ for any i, and so that $f = g$.

The functor is full: Let X and X' be two topological manifolds and $u : h_X \to h_{X'}$ a morphism of $\mathrm{Pr}(C)$. Let $\{U_i\}$ be an open cover of X with $U_i \in C$. For any i, the morphism u induces a map

$$h_X(U_i) = \mathrm{Hom}(U_i, X) \longrightarrow h_{X'}(U_i) = \mathrm{Hom}(U_i, X').$$

The image of the inclusions $U_i \subset X$ under this map give morphisms $f_i : U_i \to X'$ for any i. For any i and j in I, the elements $(f_i)|_{U_{i,j}} \in h_{X'}(U_{i,j})$ and $(f_j)|_{U_{i,j}} \in h_{X'}(U_{i,j})$ are both images of the inclusion morphisms $U_{i,j} \subset X$ under the map $h_X(U_{i,j}) \to h_{X'}(U_{i,j})$, and are thus equal (recall: $U_{i,j} = U_i \cap U_j$). Thus, the morphisms $f_i : U_i \to X'$ glue together to define a morphism $f : X \to X'$. By construction, we have $h_f = u$. $\square$

Lemma 1.2.1 is a good starting point, we know that Top is identified (up to equivalence) with a full subcategory of $\mathrm{Pr}(C)$. We now try to characterize this subcategory.

We start by making C a Grothendieck site by declaring that a family of morphisms $\{U_i \to U\}_{i \in I}$ in C is a covering family if each morphism $U_i \to U$ is a open immersion, and if the total map $\coprod_{i \in I} U_i \to U$ is surjective. This defines a pre-topology on C (Exercise: check it), whose associated topology is denoted τ.

Lemma 1.2.2. *For any $X \in \mathrm{Top}$ the presheaf $h_X \in \mathrm{Pr}(C)$ is a sheaf with respect to the topology τ.*

Proof. This is another way of saying that for a topological variety Y and an open cover $\{U_i \to Y\}_{i \in I}$, to give a continuous map from Y to X is the same as to give continuous maps f_i from U_i to X such that f_i and f_j coincide on $U_i \cap U_j$. $\square$

Thus, Lemma 1.2.2 implies that we have a fully faithful functor

$$h_- : \mathrm{Top} \longrightarrow \mathrm{Sh}(C, \tau).$$

A sheaf isomorphism to h_X is called *representable by X*. Generally speaking we identify the category Top with its image in $\mathrm{Sh}(C, \tau)$.

To characterize the image we make the following definition.

Definition 1.2.3.

(1) *A morphism $f : F \to G$ in* $\mathrm{Sh}(C,\tau)$ *is a* local homeomorphism *if for any $X \in C$, and any morphism $h_X \to G$, the sheaf $F \times_G h_X$ is representable by $Y \in$* Top*, and the morphism $Y \to X$ induced from the projection $F \times_G h_X \cong h_Y \to h_X$ is a local homeomorphism of topological space.*[*]

(2) *A morphism in* $\mathrm{Sh}(C,\tau)$ *is an* open immersion *if it is a monomorphism and a local homeomorphism.*

We check easily that the open immersions in $\mathrm{Sh}(C,\tau)$ are stable under composition (Exercise: check it). We may also check that the local homeomorphisms are stable under composition, but this requires Corollary 1.2.5 below (Exercise: check it). We also show that a morphism of topological manifolds $X \to Y$ is a local homeomorphism of topological spaces if and only if $h_X \to h_Y$ is a local homeomorphism in the sense of the preceding definition (Exercise: check it).

We then have the following proposition.

Proposition 1.2.4. *A sheaf $F \in \mathrm{Sh}(C,\tau)$ is representable by a topological manifold (i.e., $F \cong h_X$ for a certain $X \in$* Top*) if there exist a family $\{U_i\}_{i\in I}$ of objects in C and a morphism of sheaves*

$$p : \coprod_{i\in I} h_{U_i} \longrightarrow F$$

satisfying the following two conditions.

(1) *The morphism p is an epimorphism of sheaves.*

(2) *For any $i \in I$ the morphism $U_i \to F$ is an open immersion (in the sense of Definition 1.2.3).*

Proof. Suppose F is representable by a topological manifold X. We choose an open cover $\{U_i\}_{i\in I}$ of X with $U_i \in C$, and we consider the natural morphism

$$p : \coprod_{i\in I} h_{U_i} \longrightarrow F \cong h_X$$

induced by the inclusions $U_i \subset X$. For $Y \in C$ and $f : Y \to X$, an element of $h_X(Y)$ we consider $\{f^{-1}(U_i)\}_{i\in I}$ which is an open cover of Y. Furthermore, for any $i \in I$ there exists a commutative diagram

$$\begin{tikzcd} f^{-1}(U_i) \arrow[r] \arrow[d] & Y \arrow[d] \\ U_i \arrow[r] & X, \end{tikzcd}$$

which shows that f is a locally the image of the morphism p. This implies that p is an epimorphism of sheaves. Moreover, for $Y \in C$ and for any morphism $h_Y \to h_X$ corresponding to an morphism $f : Y \to X$, we have

$$h_{U_i} \times_{h_X} h_Y \cong h_{U_i \times_X Y} = h_{f^{-1}(U_i)}.$$

As $f^{-1}(U_i) \to Y$ is an open immersion we see that each morphism $h_{U_i} \to F$ is an open immersion.

[*] Recall: a continuous map $f : X \to Y$ between topological spaces is a local homeomorphism if for any $x \in X$, there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y, such that f induces a homeomorphism of U with V.

Conversely, assume that F is a sheaf satisfying the two conditions in the proposition. We reconstruct a topological space X in the following way: let $\{U_i\}$ be a family of objects in C and $p : \coprod h_{U_i} \to F$ a morphism as in the statement in the proposition. We put $U = \coprod_i U_i \in \mathrm{Top}$. We note that the natural morphism

$$\coprod h_{U_i} \longrightarrow h_U$$

is an isomorphism in $\mathrm{Sh}(C,\tau)$ (Exercise: check this). We consider the two projections

$$h_U \times_F h_U \rightrightarrows h_U.$$

By hypothesis we have

$$h_U \times_F h_U \rightrightarrows h_R$$

where $R = \coprod_{i,j} U_{i,j}$ with $h_{U_i} \times_F h_{U_j}$ (thus each $U_{i,j}$ is isomorphic to an open set of U_i and of U_j). By Lemmas 1.2.1 and 1.2.2 the diagram

$$h_R \rightrightarrows h_U$$

is the image of h of a diagram of topological manifolds $R \rightrightarrows U$. We put

$$X \coloneqq \underrightarrow{\mathrm{colim}}(R \rightrightarrows U),$$

where the colimit is taken in the category of topological spaces. Note that R is an equivalence relation on U and that X is the quotient.

We notice that X is a topological space. For that, by definition the natural morphism $U \to X$ is surjective. Furthermore, $U_i \to X$ is an open immersion. In fact, since $h_{U_i} \to F$ is a monomorphism, we have $U_{i,i} = U_i$, which implies that $U_i \to X$ is injective (Exercise: check this). Moreover, a subset $V \subset X$ is open if and only if its preimage in U over the projection $U \to X$ is an open set. Yet, the inverse image of $U_i \subset X$ over this projection is the subset $\coprod_j U_{i,j}$ of U which is an open set. This shows that X is covered by open sets $U_i \in C$, and therefore is a topological manifold.

It remains to show that F and h_X are isomorphic. There exists a natural morphism of sheaves

$$\underrightarrow{\mathrm{colim}}(h_R \rightrightarrows h_U) \longrightarrow h_X.$$

Since $h_U \to F$ is an epimorphism, and that the epimorphism of sheaves are effective, we know that that the left-hand side is naturally isomorphic to F. It therefore remains to show that the morphism above is an isomorphism of sheaves. Since the morphism $h_U \to h_X$ is also an epimorphism of sheaves (Exercise: check it), we only need to show that the natural morphism

$$h_R \longrightarrow h_U \times h_X h_U$$

is an isomorphism. Since h is faithfully flat and commutes with limits it suffices to check that the natural morphism

$$R \longrightarrow U \times_X U$$

is an isomorphism. This is true because the natural morphism $U_{i,j} \to U_i \times_X U_j$ is an isomorphism (this is a bijective local homeomorphism). $\square$

Corollary 1.2.5. *Let $X \in \mathrm{Top}$, and $F \to X$ a morphism of sheaves. If there is an open cover $\{U_i\}_{i \in I}$ of X such that for any $i \in I$ the sheaf $F \times_{h_X} h_U$ is representable by a topological manifold, then F is representable by a topological manifold.*

Proof. For any $i \in I$, we choose $\{V_{i,j}\}_{j\in J}$ and $\coprod_j h_{V_{i,j}} \to F \times_{h_X} h_{U_i}$ as in Proposition 1.2.4. We then check that

$$\coprod_{i,j} h_{V_{i,j}} \longrightarrow F$$

is a morphism as in Proposition 1.2.4 (Exercise: check it). □

1.3. Quotient manifolds

Let G be a (discrete) group acting on a topological manifold $X \in \mathrm{Top}$. By functoriality the group G act on the sheaf h_X. Recall that the action of G on X is free if for any $x \in X$ and $g \in G$ we have $(g.x = x) \Rightarrow (g = e)$. Recall also that the action of G on X is *totally discontinuous* if any point $x \in X$ possesses an open neighborhood $U \subset X$ such that for any $g \in G$ we have

$$(g(U) \cap U \neq \varnothing) \Rightarrow (g = e).$$

In the following statement, we will be careful not to confuse the quotient sheaf h_X/G with the sheaf $h_{X/G}$ represented by the quotient space X/G.

Proposition 1.3.1.

(1) If the action of G on X is free then the quotient morphism

$$h_X \longrightarrow h_X/G$$

is a local isomorphism.

(2) If the action of G on X is totally discontinuous then the quotient sheaf $h_X/G \in \mathrm{Sh}(C,\tau)$ is [represented by] a topological manifold.

Proof. (1) Let $Y \in C$ and $h_Y \to h_Y/G$ a morphism of sheaves. We must show that $h_X \times h_X/G h_Y$ is a topological manifold. Since the quotient morphism $h_X \to h_X/G$ is an epimorphism, there exists an open cover $\{Y_i\}$ of Y and a commutative diagram

$$\begin{tikzcd} \coprod_i h_{Y_i} \arrow[r] \arrow[d] & h_Y \arrow[d] \\ h_X \arrow[r] & h_X/G. \end{tikzcd}$$

Corollary 1.2.5 applied to the morphism

$$h_X \times_{h_X/G} h_Y \longrightarrow h_Y$$

implies that it suffices to show that each $h_X \times_{h_X/G} h_{Y_i}$ is a manifold. We have

$$h_X \times_{h_X/G} h_{Y_i} \cong \big(h_X \times_{h_X/G} h_X\big) \times_{h_X} h_{Y_i}.$$

Yet, $h_X \times_{h_X/G} h_X \cong h_X \times h_G$ since it can be checked on the level of presheaves of sets (Exercise: check it). Thus, we have

$$h_X \times_{h_X/G} h_{Y_i} \cong h_{Y_i} \times h_G \cong h_{Y_i \times G}.$$

This finishes showing that $h_X \times_{h_X/G} h_Y$ is a topological manifold. Moreover, the morphism of manifolds

$$h_X \times_{h_X/G} h_Y \longrightarrow h_Y$$

is, after restriction to the cover $\{Y_i\}$, isomorphic to the projection

$$h_{Y_i} \times h_G \longrightarrow h_{Y_i}.$$

Since this morphism is a local homeomorphism, this implies that the morphism $h_X \times_{h_X/G} h_Y \longrightarrow h_Y$ is a local homeomorphism.

(2) For any $x \in X$ let U_x be an open neighborhood of x in X such that $g(U_x) \cap U_x = \varnothing$ for any $g \neq e$ in G. We consider the natural morphism

$$h_{U_x} \longrightarrow h_X/G.$$

By point (1), this morphism is a composition of local homeomorphisms ($h_{U_x} \to h_X \to h_X/G$) and therefore is a local homeomorphism. Furthermore, we have

$$h_{U_x} \times_{h_X/G} h_{U_x} \cong \left(h_{U_x} \times_{h_X/G} h_X\right)_{h_X} h_{U_x} \cong (h_{U_x} \times h_G)_{h_X} h_{U_x},$$

as we saw in the proof of point (1). Finally, we have

$$(h_{U_x} \times h_G)_{h_X} h_{U_x} \cong h_{(U_x \times G) \times_X U_x}.$$

From the choice of U_x, we have $(U_x \times G) \times_X U_x \cong U_x$ (Exercise: check it). Thus, we have shown that the diagonal morphism

$$h_{U_x} \longrightarrow h_{U_x} \times_{h_X/G} h_{U_x}$$

is an isomorphism, or in other words the morphism

$$h_{U_x} \longrightarrow h_X/G$$

is a monomorphism. This shows that for any $x \in X$ the morphism

$$h_{U_x} \longrightarrow h_X/G$$

is an open immersion. Since the total morphism

$$\coprod_{x \in X} h_{U_x} \longrightarrow h_X \longrightarrow h_X/G$$

is an epimorphism, this finishes the proof of the proposition. □

The preceding corollary implies that the quotient X/G exists also in the category Top (because h is fully faithful), but is a stronger statement.

1.4. Criticism of manifolds

Proposition 1.3.1 is a good way to construct examples of topological manifolds by totally discontinuous actions. However, when a group G acts freely on a manifold X but not totally discontinuously, the quotient space X/G is in general very pathological. The quotient sheaf h_X/G has better properties (e.g. point (1) of Proposition 1.3.1), even if it is not representable by a topological manifold.

A striking example is the following: we let the discrete group $\mathbb{Q}$ (for the additive law) act on the topological space $\mathbb{R}$ by the morphism

$$\mathbb{R} \times \mathbb{Q} \longrightarrow \mathbb{R}$$

given by $(x, t) \mapsto x + t$. We see that this action is free, but not totally discontinuous. Furthermore, the morphism $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is not a local homeomorphism because it is not locally injective. Finally, the quotient space $\mathbb{R}/\mathbb{Q}$ has the coarse [trivial] topology (because its open sets are the invariant open sets of $\mathbb{R}$). This shows that the quotient $\mathbb{R}/\mathbb{Q}$ is not a reasonable object from a geometric point of view. In return, the quotient sheaf $h_{\mathbb{R}}/\mathbb{Q}$ seems more interesting, because the morphism $h_{\mathbb{R}} \to h_{\mathbb{R}}/\mathbb{Q}$ being a local homeomorphism we can legitimately think that $h_{\mathbb{R}}/\mathbb{Q}$ is somehow locally isomorphic to an open set of $\mathbb{R}$. The sheaf $h_{\mathbb{R}}/\mathbb{Q}$ is the very first example of a *geometric space*.

Definition 1.4.1. *A sheaf $F \in \mathrm{Sh}(C, \tau)$ is a* geometric space *if there exist a family of objects $\{U_i\}_{i\in I}$ of C, and a morphism of sheaves*

$$p : \coprod_{i\in I} h_{U_i} \longrightarrow F$$

satisfying the following two conditions.

1. *The morphism p is an epimorphism of sheaves.*
2. *For any $i \in I$ the morphism $U_i \to F$ is a local homeomorphism (in the sense of Definition 1.2.3).*

Proposition 1.3.1 tells us for example that the quotient sheaf h_X/G, for a free action of G on a manifold X, is a geometric space. An example of geometric space is therefore $h_{\mathbb{R}}/\mathbb{Q}$ described above. One other example is $h_{\mathbb{R}}/(\mathbb{Z} + \alpha\mathbb{Z})$, with $\alpha \in \mathbb{R} - \mathbb{Q}$.

The geometric space form a full subcategory of $\mathrm{Sh}(C, \tau)$. This category will be studied in more details in the following lecture.

Lecture 2. Reflections on the notion of topological spaces II

In the previous lecture we saw how the category of topological manifolds could be reconstructed from the category C of open sets of $\mathbb{R}^n$. To be more precise we used the category C as well as two additional structures: the topology τ, and the notion of local homeomorphism. In this lecture we formalize this, and we explain how to define varieties/manifolds and geometric spaces in an abstract context.

2.1. Geometric contexts

Let C be a category equipped with a subcanonical (pre)topology τ. By the Yoneda lemma we identify C with a full subcategory of the category $\mathrm{Sh}(C, \tau)$ of sheaves on C. We denote the Yoneda embedding by

$$h : C \longrightarrow \mathrm{Sh}(C, \tau).$$

Let a class $\mathbb{P}$ of morphisms in C be given (i.e. $\mathbb{P}$ is a subset of the set C_1 of morphism of C). Recall the following notions:

(1) We say that a morphism $f : X \to Y$ in C is *carrable* if for any morphism $Z \to Y$ in C, the object $X \times_Y Z$ exists in C.
(2) We say that $\mathbb{P}$ is *stable under composition* if for any two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $(f \in \mathbb{P}$ and $g \in \mathbb{P}) \Rightarrow (g \circ f \in \mathbb{P})$.
(3) We say that $\mathbb{P}$ is *stable under base change* if for any cartesian square in C

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

we have $(f \in \mathbb{P}) \Rightarrow (f' \in \mathbb{P})$.
(4) We say that $\mathbb{P}$ *contains identities* if for any object $X \in C$ we have $\mathrm{id}_X \in \mathbb{P}$.

We note that if $\mathbb{P}$ is stable under base change and contains identities then $\mathbb{P}$ also contains isomorphisms (Exercise: check it. We use that commutative squares of isomorphisms are cartesian).

We also need the following notions.

(1) A family of morphisms $\{X_i \to X\}$ in C is a $\mathbb{P}$-*covering* if every morphism $X_i \to X$ belongs to $\mathbb{P}$ and furthermore the morphism

$$\coprod_i h_{X_i} \longrightarrow h_X$$

is an epimorphism of sheaves.
(2) A family of morphisms $\{X_i \to X\}$ in C is a $\mathbb{P}$-*open covering* if it is a $\mathbb{P}$-covering and furthermore every morphism $X_i \to X$ is a monomorphism.
(3) We say that $\mathbb{P}$ is *local to the topology* τ the following two conditions are satisfied.
    (a) Let $f : X \to Y$ be a morphism in C such that there exists a covering family $\{Y_i \to Y\}$ such that every induced morphism $f_i : X \times_Y Y_i \to Y$ belongs to $\mathbb{P}$. Then $f \in \mathbb{P}$.

   (b) Let $f : X \to Y$ be a morphism in C such that there exists a $\mathbb{P}$-covering family $\{X_i \to X\}$ such that every induced morphism $X_i \to [Y]$ belongs to $\mathbb{P}$. Then $f \in \mathbb{P}$.
(4) We say that the tripe $(C, \tau, \mathbb{P})$ is *compatible with finite sums* if the following three conditions are satisfied.
   (a) The category C possesses finite sums.
   (b) For any finite family of objects $\{X_i\}_{i\in I}$ in C the family of morphisms
$$\left\{X_j \longrightarrow \coprod_{i\in I} X_i\right\}_{j\in I}$$
   is a $\mathbb{P}$-covering.
   (c) The finite sums are disjoint in C: for any finite family of objects $\{X_i\}_{i\in I}$ in C, the sum $X := \coprod_{i\in I} X_i$, and for any $(j,k) \in I^2$ we have
$$X_j \times_X X_k \cong \varnothing \text{ if } j \neq k \qquad X_j \times_X X_j \cong X_j{}^*.$$
   (d) For any $X \in C$, if there exist two sheaves F and G such that $h_X \cong F \amalg G$, then there exists $Y \in C$ such $h_Y \cong F$.
(5) We say that the morphisms in $\mathbb{P}$ *have open images* if for any morphism $f : X \to Y$ of $\mathbb{P}$, there exists a family of morphisms $\{X_i' \to Y\}$ of $\mathbb{P}$ satisfying the following two conditions.
   (a) The morphisms $X_i' \to Y$ are monomorphisms.
   (b) The morphisms of sheaves $h_X \to h_Y$ and $\coprod_i h_{X_i'} \to h_Y$ have the same image.
(6) We say that the morphisms in $\mathbb{P}$ are *locally carrable* if for any morphism $f : X \to Y$ in $\mathbb{P}$, there exists a $\mathbb{P}$-open covering $\{X_i \to X\}_{i\in I}$ such that each morphism $X_i \to Y$ is carrable.

We finally arrive at the key definition of this lecture.

Definition 2.1.1. *A* geometric context *is given by the a category C, a (pre)topology τ on C, and a class of morphism $\mathbb{P}$ in C, which satisfy the following conditions.*
(1) The topology τ is subcanonical.
(2) The morphisms of $\mathbb{P}$ are locally carrable.
(3) $\mathbb{P}$ is stable under composition and base change, contains identities, and is local for the topology τ.
(4) The triple $(C, \tau, \mathbb{P})$ is compatible with finite sums.
(5) The morphisms of $\mathbb{P}$ have open images.

It is important to note right away the following fact that will be used implicitly in the sequel.

Lemma 2.1.2. *If $(C, \tau, \mathbb{P})$ is a geometric context, then the Yoneda embedding*
$$h : C \longrightarrow \mathrm{Sh}(C, \tau)$$
commutes with finite sums.

*The condition here means that the diagonal morphism of X_j in $X_j \times_X X_j$ is an isomorphism

Proof. For a family of finite objects $\{X_i\}_{i\in I}$ in C, the sum $X := \coprod_i X_i$, we must show that the natural morphism

$$\coprod_{i\in I} h_{X_i} \longrightarrow h_X$$

is an isomorphism of sheaves. For that we show that it is an epimorphism and a monomorphism.

To show that this is a monomorphism, we use that we have

$$\left(\coprod_i h_{X_i}\right) \times_{h_X} \left(\coprod_i h_{X_i}\right) \cong \coprod_{(i,j)\in I^2} \left(h_{X_i} \times_{h_X} h_{X_j}\right).$$

Since the sums are disjoint in C we find that

$$\left(\coprod_i h_{X_i}\right) \times_{h_X} \left(\coprod_i h_{X_i}\right) \cong \coprod_i h_{X_i},$$

which implies that the morphism $\coprod_{i\in I} h_{X_i} \to h_X$ is a monomorphism.

The fact that $\coprod_{i\in I} h_{X_i} \to h_X$ is an epimorphism is true by condition (b) of the definition of being *compatible with finite sums.* ☐

To fix the ideas, here are some examples of geometric contexts (Exercise: verify that they really satisfy the conditions of Definition 2.1.1).

(1) We take for C the category of topological spaces. The topology τ is the usual topology: a family of morphism of topological spaces $\{X_i \to X\}_{i\in I}$ is covering if for any i the morphism $X_i \to X$ is an open immersion and if the morphism $\coprod_i X_i \to X$ is surjective. For $\mathbb{P}$ we take the local homeomorphisms.

(2) We take for C the full subcategory of that of topological spaces consisting of finite disjoint sums of open sets of $\mathbb{R}^n$ (for n varying). The topology τ is the usual topology as above, and similarly $\mathbb{P}$ is like above the class of local homeomorphisms.

(3) We take C and τ as in Example (2) above. For $\mathbb{P}$ we take the class of topological submersions. We say that a continuous map $f : X \to Y$ is a topological submersion if there exist an open cover $\{U_i\}$ of X, open sets V_i of Y, and commutative diagrams

$$\begin{tikzcd} U_i \arrow[r, "u_i"] \arrow[d, "q_i"'] & V_i \times \mathbb{R}^{n_i} \arrow[d, "p_i"] \\ X \arrow[r] & Y, \end{tikzcd}$$

where u_i are homeomorphisms, and q_i and p_i are natural morphisms $U_i \subset X$ and $V_i \times \mathbb{R}^{n_i} \to V_i \subset Y$.

(4) We define a category C as follows: the objects are finite disjoint unions of open sets of $\mathbb{R}^n$ (for n varying). The morphisms are the maps $\mathcal{C}^\infty$ (i.e. infinitely differentiable). The topology τ on C is always the usual topology: a family of morphism of C, $\{X_i \to X\}_{i\in I}$ is covering if for any i the morphism $X_i \to X$ is $\mathcal{C}^\infty$ open immersion (i.e. is injective and the differential is bijection at any point of X_i) and if the morphism $\coprod_i X_i \to X$ is surjective. For $\mathbb{P}$ we take the class of local diffeomorphisms (i.e. the $\mathcal{C}^\infty$ map whose differential is bijection at any point).

(5) We take C and τ as in Example (4) above. For $\mathbb{P}$ we take the class of $\mathcal{C}^\infty$ submersions (i.e. $\mathcal{C}^\infty$ map which has a differential surjective at any point).

(6) We define a category C as follows: the objects are finite disjoint unions of open sets of $\mathbb{C}^n$ for n varying. The morphisms are the holomorphic functions (i.e. $\mathcal{C}^\infty$ and whose differential at any point is a $\mathbb{C}$-linear map). For $\mathbb{P}$ we take the smooth holomorphic maps (i.e. holomorphic and whose differential is surjective at any point).

2.2. Varieties

We fix a geometric context $(C, \tau, \mathbb{P})$ in the sense of Definition 2.1.1.

Definition 2.2.1.

(1) Let $X \in C$. A morphism of sheaves $f : F \to h_X$ is an open immersion *if it is a monomorphism and there exists a family of morphisms $\{X_i \to X\}$ in $\mathbb{P}$ such that the image of f is equal to that of the morphism $\coprod_i h_{X_i} \to h_X$.*

(2) A morphism of sheaves $f : F \to G$ is an open immersion *if for any $X \in C$ and any morphism $h_X \to G$, the induced morphism*

$$F \times_G h_X \longrightarrow h_X$$

is an open immersion in the sense above.

We note that the condition (5) of the definition of geometric contexts (cf. previous lecture) implies that one can assume that each morphism $X_i \to X$ is also a monomorphism in the preceding definition.

One shows that the open immersions are stable under composition and base change (Exercise). One also shows that for two sheaves F and G the natural morphism $F \to F \amalg G$ is an open immersion (Exercise: use that $(C, \tau, \mathbb{P})$ is compatible with finite sums).

Definition 2.2.2. *A sheaf $F \in \mathrm{Sh}(C, \tau)$ is a* variety *(with respect to the context $(C, \tau, \mathbb{P})$) if there exist a family of objects $\{U_i\}_{i \in I}$ in C and a morphism $p : \coprod_{i \in I} h_{U_i} \to F$ satisfying the following two conditions.*

(1) The morphism p is an epimorphism of sheaves.

(2) For any $i \in I$ the morphism $h_{U_i} \to F$ is an open immersion.

The data of objects U_i and of morphisms $p : \coprod_{i \in I} h_{U_i} \to F$ is called an open atlas *of F.*

The category of varieties *is the full subcategory of the category* $\mathrm{Sh}(C, \tau)$ *of sheaves consisting of varieties.*

Each geometric context described at the end of the previous section thus gives rise to a notion of variety. Here is what the notion of varieties in these contexts, numbered as in the previous section (Big exercise: show these assertion by drawing inspiration from the previous lecture which treats the context (2)).

(1) The category of varieties in this context is equivalent to that of topological spaces.
(2) The category of varieties in this context is equivalent to that of topological varieties (this has been shown in the previous lecture).
(3) The category of varieties in this context is equivalent to that of topological varieties (we note that the open immersions are the same as for the previous case, despite that the class $\mathbb{P}$ is different).
(4) The category of varieties in this context is equivalent to that of differentiable varieties (of $\mathcal{C}^\infty$-class).
(5) The category of varieties in this context is equivalent to that of differentiable varieties.
(6) The category of varieties in this context is equivalent to that of complex varieties.

2.3. Geometric spaces

To be able to define geometric spaces we must first extend the notion of morphisms in $\mathbb{P}$ of objects of C to all the sheaves, like we have done for the notion of local homeomorphism.

Definition 2.3.1.

(1) *A morphism $f : F \to G$ in $\mathrm{Sh}(C,\tau)$ is* representable *(by a variety) if for any object $X \in C$ and any morphism $h_X \to G$ the fiber product $F \times_G h_X$ is a variety.*

(2) *A morphism $f : F \to G$ in $\mathrm{Sh}(C,\tau)$* possesses the property $\mathbb{P}$ *(we also say* belong to $\mathbb{P}$*) if it is representable and if for any object $X \in C$ and any morphism $h_X \to G$ there exists an open atlas*

$$p : \coprod_{i\in I} h_{U_i} \longrightarrow F \times_G h_X$$

such that each induced morphism $h_{U_i} \to h_X$ corresponds to a morphism $U_i \to X$ belonging to $\mathbb{P}$.

We note the following important lemma. It shows in particular that the preceding notion of morphisms in $\mathbb{P}$ is compatible with the notion already existing in C.

Lemma 2.3.2.

(1) *An open immersion in $\mathrm{Sh}(C,\tau)$ is in $\mathbb{P}$.*

(2) *Let $X \in C$ and $F \to h_X$ a morphism of sheaves. We assume that there exists a covering family $\{X_i \to X\}$ satisfying the following conditions.*

(3) *(a) Each morphism $X_i \to X$ is a monomorphism and is in $\mathbb{P}$.*
(b) For any i the sheaf $F \times_{h_X} h_{X_i}$ is a variety.
Then the sheaf F is a variety.

(4) *The morphisms in $\mathbb{P}$ in the sense of Definition 2.3.1 are stable under composition and base change.*

(5) *Let $f : X \to Y$ be a morphism in C. Then f is in $\mathbb{P}$ if and only if the morphism $h_f : h_X \to h_Y$ is in $\mathbb{P}$ in the sense of Definition 2.3.1.*

(6) *A morphism $f : X \to Y$ induces an open immersion $h_f : h_X \to h_Y$ if and only if f is in $\mathbb{P}$ and if it is a monomorphism in C.*

Proof.

(1) Let $f : F \to G$ be an open immersion. Let $X \in C$, $h_X \to G$ a morphism, and consider the induced morphism

$$F' := F \times_G h_X \longrightarrow h_X.$$

There then exists a family of morphisms $\{X_i \to X\}$ in $\mathbb{P}$ such that F' is identified with the image of the morphism $\coprod_i h_{X_i} \to h_X$. Using the fact that the morphisms in $\mathbb{P}$ have open images, we may furthermore assume that each $h_{X_i} \to h_X$ is a monomorphism. Using the fact that the morphisms in $\mathbb{P}$ are locally carrable we may also assume that the morphisms $X_i \to X$ are carrable in C. The morphisms $h_{X_i} \to h_X$ factorize through F'. Moreover, one easily sees (Exercise) that each morphism $h_{X_i} \to F'$ is an open immersion. Since the morphism $p : \coprod_i h_{X_i} \to F'$ an epimorphism by hypothesis, this implies that F' is a variety. Moreover, $p : \coprod_i h_{X_i} \to F'$ is an open atlas such that $h_{X_i} \to h_X$ is in $\mathbb{P}$. This shows that f is in $\mathbb{P}$.

(2) Put $F_i := F \times_{h_X} h_{X_i}$. For any i there exists an open atlas $\coprod_j V_{i,j} \to F_i$. We then check that the total morphism
$$\coprod_{i,j} V_{i,j} \longrightarrow F$$
is an open atlas
(3) Exercise (using point (2)).
(4) Let $f : X \to Y$ be in C. First assume that f is in $\mathbb{P}$. Let $Z \to Y$ be a morphism in C. We must start by showing that the sheaf $h_X \times_{h_Y} h_Z$ is representable by a variety. For this we use that the morphisms in $\mathbb{P}$ are locally carrable. There thus exists a $\mathbb{P}$-open covering $\{X_i \to X\}$, such that each sheaf $h_{X_i} \times_{h_Y} h_Z$ is representable by an object U_i of C. By point (2) this shows that $h_X \times_{h_Y} h_Z$ is a variety. Moreover, the induced morphisms $U_i \to Z$ are in $\mathbb{P}$, because the morphisms of $\mathbb{P}$ are stable under base change. This shows that f is well in $\mathbb{P}$.
(5) can be deduced from (4) and the fact that a morphism f in C is a monomorphism if and only if the morphism h_f is a monomorphism in $\mathrm{Sh}(C, \tau)$. □

Definition 2.3.3. *A sheaf F is a* geometric space *if there exists a family of objects $\{U_i\}_{i\in I}$ in C and a morphism $p : \coprod_{i\in I} h_{U_i} \to F$ satisfying the following two conditions.*

(1) The morphism p is an epimorphism of sheaves.
(2) For any $i \in I$ the morphism $h_{U_i} \to F$ belongs to $\mathbb{P}$.

The data of objects U_i and of morphisms $p : \coprod_{i\in I} h_{U_i} \to F$ is called an atlas *of F.*

The category of geometric spaces *is the full subcategory of the category* $\mathrm{Sh}(C, \tau)$ *of sheaves consisting of geometric spaces.*

A variety is of course a geometric space (because the open immersions are also morphisms in $\mathbb{P}$).

We end with the following proposition, which presents geometric spaces as certain quotient of varieties by equivalence relations.

Proposition 2.3.4. *Let F be a geometric space. Then, there exist a sheaf X, and an equivalence relation $R \subset X \times X$ satisfying the following conditions.*

(1) The sheaves X and R are varieties. The sheaf X is a disjoint union of objects of C (i.e. $X \cong \coprod h_{U_i}$, with $U_i \in C$).
(2) The two morphisms $R \hookrightarrow X \times X \to X$ are in $\mathbb{P}$.
(3) The sheaf F is isomorphic to the quotient sheaf of X by the relation R
$$F \cong X/R := \varinjlim(R \rightrightarrows X).$$

Proof. We choose an atlas
$$p : \coprod_i h_{U_i} \longrightarrow F.$$
We set $X := \coprod h_{U_i}$ and $R := X \times_F X$. Since the morphism p is an epimorphism we have
$$F \cong \varinjlim(X \times_F X \rightrightarrows X),$$
which shows the condition (3). To show (1), we write
$$R = X \times_F X \cong \coprod_{i,j} h_{U_i} \times_F h_{U_j}.$$

This shows that R is a disjoint union of varieties and so a variety. Finally, since each morphism $h_{U_i} \to F$ is in $\mathbb{P}$, we have that each projection

$$h_{U_i} \times_F h_{U_j} \longrightarrow h_{U_i}$$

is in $\mathbb{P}$. This implies the condition (2). $\square$

Lecture 3. Schemes and algebraic spaces I

3.1. Reminders on rings and modules

[...]
The A-modules and morphisms of A-modules form a category denoted by A-Mod.
Let $f : A \to B$ be a morphism of rings, we define a [restriction of scalars] functor

$$c$$

[...] One can show that the functor F possesses a left adjoint

$$\begin{array}{ccc} A\text{-Mod} & \longrightarrow & B\text{-Mod} \\ M & \longmapsto & B \otimes_A M. \end{array}$$

[...]
We denote by Comm the category of commutative rings. [...]
We end with the notation

$$\mathrm{Aff} := \mathrm{Comm}^{\mathrm{op}}$$

Moreover, the identity functor $\mathrm{Aff} \to \mathrm{Comm}^{\mathrm{op}}$ is denoted by Spec. Thus, a morphism of commutative rings $A \to B$ is formally the same as a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ in Aff.

3.2. Flat morphisms

Definition 3.2.1. *A morphism of commutative rings $A \to B$ is* flat *if the functor*

$$B \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$$

is exact.

We start by noting that since the categories of modules are abelian categories and that the functor $B \otimes_A -$ is left adjoint (or right exact), the functor

$$B \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$$

is exact if and only if it preserves kernels. This is often the criterion that we use.
The general properties of flat morphisms are the following.

Lemma 3.2.2.

(1) The flat morphisms are stable under compositions.
(2) If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "f'"'] & B" \end{tikzcd}$$

is a cocartesian diagram in Comm*, and if f is flat then f' is flat (i.e. the flat morphisms are stable under cobase change).*

Proof. [...] □

Exercise: For any commutative ring A, show that the natural inclusion morphism $A \hookrightarrow A[X]$ is a flat morphism. The same for the morphism $A[X] \to A[X]$ that sends X to X^2. Show that the morphism $A[X] \to A$ which sends X to 0 is not flat.

Definition 3.2.3. *A morphism of commutative rings $A \to B$ is* faithfully flat *if the functor*

$$B \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$$

is exact and conservative (i.e. exact and furthermore $(B \otimes_A M \cong 0) \Rightarrow (M \cong 0)$).

The general properties of faithfully flat morphism are the same as those of flat morphisms.

Lemma 3.2.4.

1. *The faithfully flat morphisms are stable under compositions.*
2. *If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "f'"'] & B" \end{tikzcd}$$

is a cocartesian diagram in Comm*, and if f is faithfully flat then f' is faithfully flat (i.e. the flat morphisms are stable under cobase change).*

Proof. Same method as for Lemma 3.2.2 (Exercise: fill in the details). □

Exercise: For any commutative ring A, show that the natural inclusion morphism $A \hookrightarrow A[X]$ is a faithfully flat morphism. The same for $A[X] \to A[X]$ that sends X to X^2. Show that the inclusion morphism $\mathbb{Z} \to \mathbb{Q}$ is flat but not faithfully flat.

Definition 3.2.5. *A family of morphisms in* Aff

$$\{\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A\}_{i \in I}$$

is a faithfully flat and quasi-compact covering *(*fpqc *for short) if the following conditions are satisfied:*

1. *The set I is finite.*
2. *For any $i \in I$ the morphism $A \to A_i$ is a flat morphism.*
3. *The morphism $A \to \prod_{i \in I} A_i$ is faithfully flat.*

Exercise: Show that the condition (3) above implies the condition (2) (Exercise: one can use that there exists an equivalence of categories $\left(\prod_{i \in I} A_i\right)$-Mod $\cong \prod_{i \in I}(A_i\text{-Mod})$.

Lemma 3.2.6. *The fpqc coverings defined in Definition 3.2.5 defines a Grothendieck (pre)topology on* Aff*.*

Proof. This is deduced from Lemmas 3.2.2 and 3.2.4. □

Definition 3.2.7. *The Grothendieck topology on* Aff *whose covering families are the fpqc coverings is called the* fpqc topology*.*

We end the subject with the following lemma.

Lemma 3.2.8. *The fpqc topology is sub-canonical.*

Proof. We note that all representable presheaves on Aff are sheaves is equivalent to that for any faithfully flat morphism $A \to B$ the natural morphism

$$A \longrightarrow \varprojlim (B \rightrightarrows B \otimes_A B)$$

is an isomorphism (Exercise: check this).

So let $u : A \to B$ be a faithfully flat morphism. Since the functor $B \otimes_A -$ commutes with finite limits (by flatness), and is conservative (by faithful flatness), it is enough to show that the morphism

$$A \longrightarrow \varprojlim (B \rightrightarrows B \otimes_A B)$$

induces an isomorphism

$$B \otimes_A A \cong B \longrightarrow \varprojlim (B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B).$$

[...] □

3.3. Smooth and étale morphisms

For more details on this section we refer to the first exposé of [1], or in Paragraphs 2.1 and 2.2 of [2].

Let $C_0 \in \mathrm{Comm}$. We recall that a square zero extension of C_0 is the data of a morphism $p : C \to C_0$ in Comm which satisfies the following two conditions:

- The morphism p is surjective.
- If $x \in \mathrm{Ker}\, p$ then $x^2 = 0$.

The fundamental example of a square zero extension is the morphism

$$A[X]/(X^2) \longrightarrow A$$

which sends X to 0. It is called a trivial square zero extension. This is denoted by

$$A[\epsilon] \coloneqq A[X]/(X^2),$$

and $\epsilon \in A[\epsilon]$ is by definition the class of X. Thus, any element of $A[\epsilon]$ can be written in the form $a + b \cdot \epsilon$ with a and b in A.

To better understand the square zero extension we may consider the following example. Let $A \coloneqq \mathbb{C}[X_1, ..., X_n]/(P_1, ..., P_r)$ be a commutative $\mathbb{C}$-algebra of finite type. Let $x : A \to \mathbb{C}$ correspond to a point $x \in \mathbb{C}^m$ such that $P_j(x) = 0$ for any j. The data of a commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}[\epsilon] \arrow[d] \\ A \arrow[r, "x"'] \arrow[ur] & \mathbb{C} \end{tikzcd}$$

corresponding to the data of a vector $b \in \mathbb{C}^n$ such that $P_j(x + b \cdot \epsilon) = 0$ for any j. Now, since $\epsilon^2 = 0$ we have

$$P_j(x + b \cdot \epsilon) = P_j(x) + \sum_i b_i \cdot \frac{\partial P_j}{\partial X_i}(x) \cdot \epsilon = \sum_i b_i \cdot \frac{\partial P_j}{\partial X_i}(x) \cdot \epsilon.$$

Thus, the data of a commutative diagram as above is equivalent to the data of a vector $b \in \mathbb{C}^n$ satisfying

$$\sum_i b_i \cdot \frac{\partial P_j}{\partial X_i}(x) = 0,$$

or in turn to the data of a tangent vector to the variety of equations $P_1, ..., P_r$ at point x.

Definition 3.3.1.

(1) *A morphism $A \to B$ in* Comm *is* formally smooth *(resp.* formally étale*) if for any $C_0 \in$* Comm*, any square zero extension $C \to C_0$, and any commutative diagram in* Comm

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & C_0 \end{tikzcd}$$

there exists a (resp. a unique) morphism $B \to C$ in Comm *such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] \arrow[ur] & C_0 \end{tikzcd}$$

commutes.

(2) *A morphism in* Comm *is* smooth *(resp.* étale*) if it is formally smooth (resp. formally étale) and of finite presentation*.*

Lemma 3.3.2. *The smooth and étale morphisms are stable under composition and base change in* Aff*.*

Proof. Exercise. □

The following proposition gives one way to construct examples of étale morphisms.

Proposition 3.3.3. *Let $F \in A[X]$ be a polynomial, and consider the natural morphism*

$$A \longrightarrow A[X]/(F) = B.$$

Then, the morphism $A \to B$ is formally étale if and only if the derivative polynomial $F'(X)$ becomes invertible in B.

Proof. [...] □

An important corollary of the preceding proposition is that an algebraic extension of fields $K \to L$ is an étale morphism if and only if it is separable.

Another immediate consequence of the preceding proposition is that for any commutative ring A and any $f \in A$, the morphism $A \to A_f := A[X]/(f \cdot X - 1)$ is étale. Exercise: check this directly by applying the definition.

We also cite the following fact without proofs.

Proposition 3.3.4.

(1) *A morphism $A \to B$ is smooth if and only if B is isomorphic as an A-algebra to $A[X_1, ..., X_n]/(P_1, ..., P_m)$, where $P_i \in A[X_1, ..., X_n]$ are such that the ideal generated by the minors of order $m \times m$ of the matrix $(\partial P_i/\partial X_j(X))_{i,j}$ is B.*

---

* We recall that a morphism of commutative rings $A \to B$ is of finite presentation if B is isomorphic as A-algebra to an A-algebra of the form $A[X_1, ..., X_r]/(P_1, ..., P_r)$.

(2) A morphism $A \to B$ is smooth if and only if there exist an fpqc cover $\{u_i : B \to B_i\}$ and commutative diagrams

$$\begin{tikzcd} B \arrow[r, "u_i"'] \arrow[d, leftrightarrow] & B_i \\ A \arrow[r] & A[X_1, ..., X_{n_i}] \arrow[u, "v_i"'] \end{tikzcd}$$

where all morphisms u_i and v_i are étale.

(3) The smooth (and in particular étale) morphisms are flat.

To finish we pose the following definition.

Definition 3.3.5. *A morphism in* Aff *is an* Zariski open immersion *if it is an étale morphism and a monomorphism.*

We warn that we demand the morphism be monomorphism in Aff. In terms of rings this means that $A \to B$ is a Zariski open immersion if it is an étale morphism and if for any $C \in \mathrm{Comm}$ the induced morphism

$$\mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

is injective (i.e. that $A \to B$ is an epimorphism in Comm).

Exercise: show that a smooth monomorphism in Aff is also étale. Show that the étale morphism $A \to A_f$ considered above is a Zariski open immersion. One can also show that a flat monomorphism of finite presentation is also is also étale (we will not do it).

3.4. The algebro-geometric context

We now define a geometric context $(C, \tau, \mathbb{P})$ as follows.

(1) The category C is the category Aff opposite to the category of commutative rings.
(2) We define an étale topology (ét) on Aff by setting a family $\{\mathrm{Spec}\, A_i \to \mathrm{Spec}\, A\}$ of morphisms in Aff an étale covering if it is an fpqc covering and if all morphisms $A \to A_i$ are étale. (cf. Proposition 3.3.4 (3)).
(3) The class $\mathbb{P}$ is by definition the class of smooth morphisms (lisse).

Theorem 3.4.1. *The definition above defines a geometric context.*

Proof. [...] □

Definition 3.4.2. *The varieties (resp. geometric spaces) with respective to the geometric context* (Aff, ét, lisse) *of Theorem 3.4.1 are called* schemes *(resp. algebraic spaces).*

Lecture 4. Schemes and algebraic spaces II

Recall that the category Aff opposite to that of commutative rings is equipped with the étale topology ét. We simply denote by Sh(Aff) the category of sheaves on (Aff, ét). In the previous lecture we have defined three full subcategories

$$\{\text{Affine schemes}\} \subset \{\text{Schemes}\} \subset \{\text{Algebraic spaces}\} \subset \mathrm{Sh}(\mathrm{Aff}).$$

In this lecture we give some examples of affine schemes, schemes, and algebraic spaces. We also give examples of sheaves which are not algebraic spaces, but are geometric in some sense.

4.1. Some properties of morphisms

We recall the notions of smooth morphisms and ètale between affine schemes given in Lecture 3. The following definition gives some notions of morphisms between schemes and algebraic spaces. We will see more of them in the following course on stacks and algebraic spaces.

Definition 4.1.1.

(1) A morphism $F \to G$ in Sh(Aff) *is* representable by a scheme *(resp. by an algebraic space, resp., by an affine scheme) if for any affine scheme X and any morphism $X \to G$, the sheaf $F \times_G X$ is a scheme (resp. an algebraic space, resp. an affine scheme). A morphism representable by an affine scheme is also called an* affine morphism*.*

(2) A morphism $F \to G$ in Sh(Aff) *is* smooth *(resp.* étale*) if it is representable by an algebraic space and if furthermore for any affine scheme X and any morphism $X \to G$, there exist an atlas $\{U_i\}$ of $F \times_G X$ such that the composite morphisms $U_i \to X$ are smooth (resp. étale) morphisms.*

(3) A morphism $F \to G$ in Sh(Aff) *is an* open immersion *if it is an étale monomorphism.*

(4) A morphism $F \to G$ in Sh(Aff) *is a* closed immersion *if it is an affine morphism and if furthermore for any affine scheme $X = \operatorname{Spec} A$ and any morphism $X \to G$, the morphism of affine schemes*

$$F \times_G X \cong \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

corresponds to a surjective morphism of rings $A \to B$.

One can verify that the preceding notions are stable under base change and composition (Exercise. One will use Proposition 4.1.2 (1) below to show that the representable morphisms are stable under composition). We will also see that the notion of étale morphism above is compatible with the definition in Lecture 3. (Exercise. One will be inspired by Lemma 2.3.2 (4) about the case of smooth morphisms).

Proposition 4.1.2.

(1) Let X be a scheme (resp. an algebraic space) and $F \to X$ a morphism in Sh(Aff)*. Suppose that there exists an open atlas (resp. an atlas) $\{U_i\}$ of X such that for any i the sheaf $U_i \times_X F$ is a scheme (resp. an algebraic space). Then F is a scheme (resp. an algebraic space).*

(2) All three full subcategories of Sh(Aff) *consisting of affine schemes, of schemes, and of algebraic spaces are stable under finite limits and arbitrary sums.*

(3) *Let $F \to G$ be a morphism representable by a scheme (resp. an algebraic space, resp. an affine scheme). If G is a scheme (resp. an algebraic space, resp. an affine scheme), then so is F.*

Proof. Exercise. $\square$

4.2. Examples of schemes

4.2.1. **The affine line.** We consider the functor

$$\begin{array}{ccc} \mathrm{Aff}^{\mathrm{op}} = \mathrm{Comm} & \longrightarrow & \mathrm{Set} \\ A & \longmapsto & A, \end{array}$$

which sends an affine scheme $X = \operatorname{Spec} A$ to the underlying set of the ring A (again denoted by A). This functor is representable by the affine scheme

$$\mathbb{A}^1 := \operatorname{Spec} \mathbb{Z}[T]$$

called the *affine line*. In fact, for any commutative ring A we have a bijection functorial in A

$$A \cong \mathrm{Hom}_{\mathrm{Aff}}(\operatorname{Spec} A, \mathbb{A}^1) = \mathrm{Hom}_{\mathrm{Comm}}(\mathbb{Z}[T], A).$$

Explicitly, [...]. We also use the following notations (for $A \in \mathrm{Comm}$)

$$\mathbb{A}^n := (\mathbb{A}^1)^n \cong \operatorname{Spec} \mathbb{Z}[T_1, ..., T_n],$$

$$\mathbb{A}^n_A := \mathbb{A}^n \times \operatorname{Spec} A \cong \operatorname{Spec} A[T_1, ..., T_n].$$

The scheme $\mathbb{A}^n_A$ is called the *affine space of dimension n over A*. If we denote by by $X := \operatorname{Spec} A$ we also use the following notation

$$\mathbb{A}^n_X := \mathbb{A}^n \times X = \mathbb{A}^n_A.$$

4.2.2. **The multiplicative group.** We consider the functor

$$\begin{array}{ccc} \mathrm{Aff}^{\mathrm{op}} = \mathrm{Comm} & \longrightarrow & \mathrm{Set} \\ A & \longmapsto & A^\times, \end{array}$$

which sends an affine scheme $X = \operatorname{Spec} A$ to the set $A^\times$ of invertible elements of the ring A. This functor is representable by the affine scheme

$$\mathbb{G}_m := \operatorname{Spec} \mathbb{Z}[T, T^{-1}],$$

and is called the *multiplicative group* (Exercise: check that $\mathbb{G}_m$ indeed represents the functor above.)

4.2.3. **The linear group.** We consider the functor

$$\begin{array}{ccc} \mathrm{Aff}^{\mathrm{op}} = \mathrm{Comm} & \longrightarrow & \mathrm{Set} \\ A & \longmapsto & A^\times, \end{array}$$

which sends an affine scheme $X = \operatorname{Spec} A$ to the set $\mathrm{GL}_n(A)$ of $n \times n$ invertible matrices with coefficients in A. This functor is representable by the affine scheme

$$\mathrm{GL}_n := \operatorname{Spec} \mathbb{Z}[T_{i,j}] \left[\det(T_{i,j})^{-1}\right]_{1 \le i,j \le n},$$

which is called the *general linear group* (of rank n). We have $\mathrm{GL}_1 = \mathbb{G}_m$.

4.2.4. **Complementary open.** Let $A \in \mathrm{Comm}$ be a fixed commutative ring, I an ideal of A, $X := \operatorname{Spec} A$, and $Y := \operatorname{Spec} A/I$ a closed subscheme of X. We define a subfunctor U of X as follows: for any $B \in \mathrm{Comm}$, $U(B)$ is the subset of $X(B)$ consisting of morphisms $A \to B$ such that $B \otimes_A A/I = B/IB = 0$. The functor U is representable by a scheme, which we call the complementary open of X and denote symbolically by $X - Y$. In fact, let $\{f_i\}$ be a family of generator of I. For any i, we consider $U_{f_i} = \operatorname{Spec} A_{f_i}$ together with the morphism

$$U_{f_i} \longrightarrow X$$

induced by the natural morphism $A \to A_{f_i}$. We then see that each morphism $U_{f_i} \to X$ factorizes through $U \subset X$. We moreover show that the total morphism

$$\coprod_i U_{f_i} \longrightarrow U$$

is an open atlas of U. Indeed, the fact that $U_i \to X$ is an open immersion implies that the morphism $U_i \to U$ is also an open immersion. It therefore remains to show that the morphism

$$\coprod_i U_{f_i} \longrightarrow U$$

is an epimorphism of sheaves. For that, let $B \in \mathrm{Comm}$, and $\operatorname{Spec} B \to U$ a morphism. We need to find a covering family $\{\operatorname{Spec} B_j \to \operatorname{Spec} B\}$, and for any j an index i (which depends on j) and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} B_j \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
U_i \arrow[r] & U.
\end{tikzcd}$$

The morphism $\operatorname{Spec} B \to U$ correspond to a morphism of rings $u : A \to B$ such that $IB = B$. We can therefore write $1 = \sum_i b_i \cdot u(f_i)$, where the b_i are in B and all but a finite number of them are zero. We then consider the family

$$\left\{\operatorname{Spec} B_{u(f_i)} \longrightarrow \operatorname{Spec} B\right\}$$

where i runs through the subset of indices i such that $b_i \neq 0$. We clearly have the commutative diagrams

$$\begin{tikzcd}
\operatorname{Spec} B_j \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
U_i \arrow[r] & U.
\end{tikzcd}$$

Each morphism $SpecB_{u(f_i)} \to \operatorname{Spec} B$ is an open immersion, and it therefore remains to show that the functor

$$B\text{-Mod} \longrightarrow \prod \left(B_{u(f_i)}\text{-Mod}\right)$$

is conservative. For that, let M be a B-module such that $M \otimes_B B_{u(f_i)} = 0$ for any i. For $m \subset B$ a maximal ideal, there exists an index i_0 such that $u(f_{i_0}) \notin m$. Therefore, the localization morphism $B \to B_m$ factorizes through $B_{u(f_{i_0})}$. We then have

$$M \otimes_B B_m \cong \left(M \otimes_B B_{u(f_{i_0})}\right) \otimes_{B_{u(f_{i_0})}} B_m = 0.$$

This being true for any maximal ideal m of B, we have $M = 0$.

Note that the proof above shows that $X - Y$ is the union of subsheaves $U_{f_i} \subset X$, and in particular is an open subsheaf of X. Therefore, the inclusion morphism $X - Y \hookrightarrow X$ is an open immersion.

The existence of the scheme $X - Y$ is generalized when X is an algebraic space and $Y \hookrightarrow X$ a closed immersion: we define $X - Y$ to be the subfunctor of X consisting of morphisms $\operatorname{Spec} B \to X$ which are such that $\operatorname{Spec} B \times_X Y = \varnothing$. We then check, with the aid of the case where X is an affine scheme, that the inclusion morphism $X - Y \hookleftarrow X$ is an open immersion and in particular is representable. This implies that $X - Y$ is an algebraic space.

We will be careful that even if X and Y are affine, the scheme $X - Y$ is in general non-affine. For example, $\mathbb{A}^2 - 0$ (which corresponds to $A = \mathbb{Z}[X, Y]$ and $I = (X, Y)$) is not affine. We can see this in the following manner. For any scheme X, we can consider the set $\mathcal{O}(X) := \operatorname{Hom}(X, \mathbb{A}^1)$. Since the scheme $\mathbb{A}^1$ represents a sheaf of commutative rings, the set $\mathcal{O}(X)$ naturally inherits a structure of commutative rings. Furthermore, if $X = \operatorname{Spec} A$ is affine, then we have a natural isomorphism $A \cong \mathcal{O}(X)$. We can write $\mathbb{A}^2 - 0$ as an amalgamated sum in $\operatorname{Sh}(\mathrm{Aff})$

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{G}_m \times \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \times \mathbb{G}_m \arrow[r] & \mathbb{A}^2 - 0.
\end{tikzcd}$$

(Exercise: check this.) Applying the functor $X \mapsto \mathcal{O}(X)$ we find a cartesian diagram of rings

$$\begin{tikzcd}
\mathcal{O}(\mathbb{A}^2 - 0) \arrow[r] \arrow[d] & \mathbb{Z}[X, X^{-1}, Y] \arrow[d] \\
\mathbb{Z}[X, Y, Y^{-1}] \arrow[r] & \mathbb{Z}[X, X^{-1}, Y, Y^{-1}].
\end{tikzcd}$$

which implies that $\mathcal{O}(\mathbb{A}^2 - 0) \cong \mathbb{Z}[X, Y]$. Therefore, if $\mathbb{A}^2 - 0$ was affine, we would have $\mathbb{A}^2 - 0 \cong \mathbb{A}^2$, which is not true (we can for example compare their complex points and use the fact that $\mathbb{C}^2 - 0$ is not homeomorphic to $\mathbb{C}^2$).

4.2.5. **Projectives spaces.** We fix an integer $n \geq 1$. For a commutative ring $A \in \mathrm{Comm}$, we define $\mathbb{P}^n(A)$ to be the set of A-submodules $L \subset A^{n+1}$ which satisfies the following two conditions:

(1) There exists an A-submodule $P \subset A^{n+1}$ such that $A^{n+1} = L \oplus P$.
(2) For any field K and any morphism $A \to K$ we have $\dim_K(L \otimes_A K) = 1$.

For an morphism $A \to B$ in Comm we define a map $\mathbb{P}^n(A) \to \mathbb{P}^n(B)$ which, to an A-submodule $L \subset A^{n+1}$, associates $L \otimes_A B \subset B^{n+1}$ (note that the condition (a) above implies that $L \otimes_A B$ is a B-submodule of B^{n+1}). We must think of $L \in \mathbb{P}^n(A)$ as a *family of lines in* $\mathbb{A}^{n+1}$ *parametrized by* $\operatorname{Spec} A$.

We now show that the functor $\mathbb{P}^n$ thus defined is a scheme. To do that, for any integer $1 \leq j \leq n+1$, we consider $U_j(A) \subset \mathbb{P}^n(A)$ the subset of $L \subset A^{n+1}$ which are such that the projection onto the jth factor induces an isomorphism $L \cong A$. As A traverses through Comm, this defines a subfunctor $U_j \subset \mathbb{P}^n$.

We start by showing that for j fixed the inclusion morphism $U_j \hookrightarrow \mathbb{P}^n$ is an open immersion. For that, let $\operatorname{Spec} A \to \mathbb{P}^n$ be a morphism corresponding to an A-submodule $L \subset A^{n+1}$. The subfunctor $V_j := \operatorname{Spec} A \times_{\mathbb{P}^n} U_j \subset \operatorname{Spec} A$ is described as follows: an element of $\operatorname{Spec} A(B)$ is in $V_j(B)$ if it corresponds to a morphism $A \to B$ such that the j-th projection $L \otimes_A B \to B$ is an isomorphism. Note that as $L \to_A B$ and B are projective B-modules of rank 1, the morphism $p : L \otimes_A B \to B$ is an isomorphism if and only if it is a surjective morphism. In fact, if it is surjective, we have $L \otimes_A B \cong B \oplus \mathrm{Ker}(p)$, and the condition (b) above implies that $\mathrm{Ker}(p) \otimes_B B/m = 0$ for any maximal ideal m of B. The Nakayama lemma then implies that $\mathrm{Ker}(p) = 0$. Thus, $V_j := \mathrm{Spec}\, A \times_{\mathbb{P}^n} U_j \subset \mathrm{Spec}\, A$ is identified with the subfunctor of $\mathrm{Spec}\, A$ consisting of morphisms $A \to B$ such that the j-th projection $L \otimes_A B \to B$ is surjective. Write $I \subset A$ for the image of the j-th projection $L \to A$. This is an ideal of A, corresponding to a closed subscheme of $\mathrm{Spec}\, A$, and we see by definition that its complementary open is V_j. From this we see that V_j is a scheme, and furthermore $V_j \subset \mathrm{Spec}\, A$ is an open immersion. This shows that the morphism $U_j \hookrightarrow \mathbb{P}^n$ is an open immersion.

It remains to show that the total morphism

$$\coprod_j U_j \longrightarrow \mathbb{P}^n$$

is an epimorphism of sheaves. For that, we use the following lemma.

Lemma 4.2.1. *Let $\{U_j \to F\}$ be a finite family of open immersions in* $\mathrm{Sh(Aff)}$. *Then, the morphism*

$$\coprod_i U_i \longrightarrow F$$

is an epimorphism if and only if for any field K the induced morphism

$$\coprod_i U_i(K) \longrightarrow F(K)$$

is surjective.

Proof. We first show that the necessity. Let K be a field and $\mathrm{Spec}\, K \to F$ a morphism. There then exists an open atlas $\{\mathrm{Spec}\, K_i \to \mathrm{Spec}\, K\}$, and we have commutative diagrams

$$\begin{tikzcd} U_i \arrow[r] & F \\ \mathrm{Spec}\, K_i \arrow[u] \arrow[r] & \mathrm{Spec}\, K. \arrow[u] \end{tikzcd}$$

Now, since $K \to K_i$ is a monomorphism in Comm, the morphism $x \mapsto x \otimes 1$ induces an isomorphism $K_i \cong K_i \otimes_K K_i$. This implies that K_i is of dimension 1 as K-vector space, and thus $K \cong K_i$.

Next we show the sufficiency. We start by using that a morphism $f : F \to G$ is an epimorphism if and only if for any $X \in \mathrm{Aff}$ and any morphism $X \to G$ the morphism $F \times_G X \to X$ is an epimorphism (Exercise: check that.) This reduces the lemma to the case where F is an affine scheme $\mathrm{Spec}\, A$. In this case U_j are schemes (as open sets of $\mathrm{Spec}\, A$) (we use here the necessity already demonstrated). Now write

$$\{U_i = \mathrm{Spec}\, A_i \longrightarrow F = \mathrm{Spec}\, A\}$$

for the family. Let M be a nonzero A-module. The A-module M contains an A-submodule of the form A/I for a proper ideal I. Let m be a maximal ideal of A containing I, and note that the hypothesis implies that the morphism $A \to A/m$ factorizes through $A_i \to A/m$ for a certain index i. Write m_i for a maximal ideal of A_i that contains the kernel of $A_i \to A/m$, write the morphism $u : A \to A_i$, and therefore $u^{-1}(m_i) = m$. Thus, the ideal IA_i is contained in m_i and therefore $A_i/IA_i \neq 0$. Since $A \to A_i$ is flat we have a monomorphism

$$A/I \otimes_A A_i = A/IA_i \hookrightarrow M \otimes_A A_i$$

and thus $M \otimes_A A_i \neq 0$. This finishes showing that $\{U_i = \operatorname{Spec} A_i \to F = \operatorname{Spec} A\}$ is a covering family. □

According to the lemma it suffices to show that for any field K the morphism

$$\coprod_i U_i(K) \longrightarrow \mathbb{P}^n(K)$$

is surjective. But a point of $\mathbb{P}^n(K)$ corresponds to a vector subspace $L \subset K^{n+1}$ of dimension 1. There therefore exists an index j such that the j-th projection $L \to K$ is nonzero and therefore an isomorphism.

This ends the proof that $\mathbb{P}^n$ is a scheme. We remark that each open subscheme $V_j \subset \mathbb{P}^n$ is isomorphic to $\mathbb{A}^n$ (Exercise: check it. To a point $L \subset A^{n+1}$ of V_j we will correspond the image of $1 \in A \cong L$ in A^n under the projection that forgets the j-th factor). We can also see that $\mathbb{P}^n$ is not affine. In fact, for $n = 1$ we have a cocartesian diagram in $\mathrm{Sh}(\mathrm{Aff})$

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & V_1 \cong \mathbb{A}^1 \arrow[d] \\ V_2 \cong \mathbb{A}^1 \arrow[r] & \mathbb{P}^1, \end{tikzcd}$$

which by applying the functor $X \mapsto \mathcal{O}(X)$ gives a cartesian diagram of rings

$$\begin{tikzcd} \mathcal{O}(\mathbb{P}^1) \arrow[r] \arrow[d] & \mathbb{Z}[X] \arrow[d, "b"] \\ \mathbb{Z}[Y] \arrow[r, "a"'] & \mathbb{Z}[T, T^{-1}], \end{tikzcd}$$

where the morphism a sends Y to T and b sends X to T^{-1}. We also find $\mathcal{O}(\mathbb{P}^1) \cong \mathbb{Z}$. If $\mathbb{P}^1$ was affine it would be isomorphic to $\operatorname{Spec} \mathbb{Z}$, which is not the case because for example $\mathbb{P}^1(\mathbb{Z})$ is not reduced to a point.

4.3. Examples of algebraic spaces

A general procedure of construction of algebraic spaces is given by the following proposition. For that, we recall that a scheme in groups is a group object in the category of schemes (i.e. a sheaf in groups on Aff whose sheaf in underlying sets is a scheme). Similarly, for S a scheme, a scheme in groups on S is a group object in the category of schemes over S (it is therefore a sheaf in groups on the site Aff/S whose sheaf of underlying sets is a scheme).

Proposition 4.3.1. *Let k be a commutative ring and $S := \operatorname{Spec} k$. Let G be a scheme in groups that is affine and smooth over S, which acts on a scheme X. We assume that the morphism*

$$G \times X \longrightarrow X \times X$$

produces by the action by the second projection is an isomorphism (i.e. that the action is fixed-point free). Then the quotient sheaf X/G is representable by an algebraic space.

Proof. We will show that the natural morphism

$$X \longrightarrow X/G$$

is representable and smooth. This implies the result because an atlas $\{U_i\}$ of X will give by composition an atlas of X/G.

Let $\operatorname{Spec} A \to X/G$ be a morphism. There exists an étale covering $\{A \to A_i\}$, and we have commutative diagrams

$$\begin{tikzcd} X \arrow[r] & X/G \\ \operatorname{Spec} A_i \arrow[u] \arrow[r] & \operatorname{Spec} A. \arrow[u] \end{tikzcd}$$

We have

$$X \times_{X/G} \operatorname{Spec} A_i \cong \big(X \times_{X/G} X\big) \times_X \operatorname{Spec} A_i \cong (G \times X) \times_X \operatorname{Spec} A_i \cong G \times \operatorname{Spec} A_i.$$

By Proposition 4.1.2 (1) this implies that $X \times_{X/G} \operatorname{Spec} A$ is an algebraic space. Furthermore, we see that the projection $X \times_{X/G} \operatorname{Spec} A \to \operatorname{Spec} A$ becomes, after base change to $\operatorname{Spec} A_i$, isomorphic to the natural projection

$$G \times \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A_i.$$

This shows that the morphism $X \to X/G$ is representable by an algebraic space and is smooth. We end by applying the technical lemma below, which we will prove later in the course on algebraic stacks. □

Lemma 4.3.2. *Let $X = \operatorname{Spec} A$, and $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ an étale covering. Let $F \to X$ be a morphism in* $\operatorname{Sh}(\mathrm{Aff})$ *such that for any i the sheaf $F \times_X \operatorname{Spec} A$ is an affine scheme. Then F is representable by an affine scheme.*

Although Proposition 4.3.1 gives a way simply enough to construct examples of algebraic spaces, it is not easy to explicitly construct an algebraic space which is not a scheme. We can consult Example 3.4.2 of Appendix B of [3] for one such example.

4.4. Non-representable examples

We have seen that the quotient of a scheme by an affine group scheme and smooth étale group scheme and representable by an algebraic space when the action is fixed-point free. We may legitimately ask the question of the representability of such a quotient when the action is no longer fixed-point free. It turns out that, as we expect, a quotient by an action with fixed point is no longer representable in general. To convince ourselves we consider the very simple example below.

We take $X := \mathbb{A}^1$ the affine line and $G = \mathbb{Z}/2$ which acts on $\mathbb{A}^1$ by $x \mapsto -x$. Clearly, for $A \in \mathrm{Comm}$, the action of G on $\mathbb{A}^1(A) = A$ is $a \mapsto -a$. We claim that the quotient sheaf X/G is not a scheme nor an algebraic space. In fact, we could show (we will not do so) that if X/G is an algebraic space then it is necessarily an affine scheme. This affine scheme satisfies then

$$X/G \cong \operatorname{Spec} \mathcal{O}(X/G).$$

Now, since $\mathcal{O}(X/G) = \operatorname{Hom}(X/G, \mathbb{A}^1)$, the universal property of quotient gives

$$\mathcal{O}(X/G) = \{P \in \mathbb{Z}[T] | P(T) = P(-T)\} \subset \mathbb{Z}[T].$$

In other words, the morphism of affine schemes $X \to X/G$ corresponds tu the morphism of rings

$$\mathbb{Z}[U] \longrightarrow \mathbb{Z}[T]$$

which sends U to T^2. Thus, if X/G is an algebraic space we find that the morphism

$$X = \mathbb{A}^1 \longrightarrow X/G \cong \mathbb{A}^1$$

induced by $x \mapsto x^2$ locally has sections in the étale topology.

To see that it is not the case we proceed as follows. For $x : \operatorname{Spec} \mathbb{Z} \to X$ a point of X, we define

$$TxX \coloneqq \operatorname{Hom}_x(\operatorname{Spec} \mathbb{Z}[\epsilon], X),$$

where the right-hand side designates the subset of morphisms $\operatorname{Spec} \mathbb{Z}[\epsilon] \to X$ whose composition with $\operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{Z}[\epsilon]$ is equal to the morphism x. The set T_xX is called the tangent of X at x (we will talk about this later). We can easily check that if $f : X \to Y$ is an étale morphism then the induced morphism $T_xX \to T_{f(y)}Y$ is bijective (Exercise). This implies in particular that our morphism

$$X = \mathbb{A}^1 \longrightarrow X/G \cong \mathbb{A}^1$$

which locally has sections for the étale topology is such that the induced morphism

$$T_0\mathbb{A}^1 \longrightarrow T_0\mathbb{A}^1$$

is surjective. Now, we can explain this morphism as being the zero morphism between $T_0\mathbb{A}^1 \cong \mathbb{Z}$ (Exercise). This leads to a contradiction.

Lecture 4½. Supplements on schemes and algebraic spaces

In this lecture we cite in bulk some facts concerning schemes and algebraic spaces. We will not give proofs which are in line with the statements we have seen but which would take too much time. We refer to [1, 3, 5] for the details and for further references.

4½.1. The underlying Zariski space

In this section we fix an algebraic space X.

4½.1.1. **Back to the Zariski open sets.** Recall we have showed that for all closed immersion $Y \hookrightarrow X$ the subfunctor $X - Y$ of X was an open subfunctor and therefore defining a Zariski open set $X - Y \hookrightarrow X$ (see Example 4 of Lecture 4).

Proposition 4½.1.1. *For all open immersion $U \hookrightarrow X$ there exists a closed immersion $Y \hookleftarrow X$ such that $U = X - Y$.* □

The preceding proposition seems natural but the proof is not immediate given the definition of open immersions that we have adopted.

Exercise: Use Proposition 4½.1.1 to show that a Zariski open set U of $X := \operatorname{Spec} A$ is a (possibly infinite) union of open sets of the form $X_f := \operatorname{Spec} A_f$ (where A_f is the ring A localized at $f \in A$).

4½.1.2. **The set of points.** We consider the set $E(X)$ of pairs (K, x), where K is a field and $x \in X(K)$ is a point of X with values in K. We define an equivalence relation on $E(X)$ as follows: (K, x) is equivalent to (K', x') if there exist a field L and morphisms $i : K \hookrightarrow L$, $j : K' \hookrightarrow L$ such that $i(x) = j(x')$ (as elements of $X(L)$). Exercise: check that this defines an equivalence relation on $E(X)$.

Definition 4½.1.2. *We define the* set of points of X *as the quotient set $E(X)/R$ where R is the relation defined above. We denote it by $|X|$.*

Exercise: show that when $X = \operatorname{Spec} A$ the set $|X|$ is in natural bijection with that of the prime ideals of A.

The construction $X \mapsto |X|$ is (contravariantly) functorial in X, and naturally defines a functor from the category of algebraic spaces to that of sets.

Exercise: Show that if $|X| = \varnothing$ then $X = \varnothing$ (we will show that if $\{U_i \to X\}$ is an atlas then the induced map

$$\coprod_i |U_i| \longrightarrow |X|$$

is surjective, and then we will treat the case where X is an affine scheme).

4½.1.3. **The locally ringed space $(|X|, \mathcal{O}_X)$.** We define a topology on the set $|X|$ as follow: a subset $U \subset |X|$ is an open set if there exists an open subfunctor $V \subset X$ such that $U = |V|$ (as subset of $|X|$). In other words, U is the set of points of X represented by pairs (K, x) with $x \in V(K) \subset X(K)$.

Exercise: check that this defines a topology on $|X|$. When $X = \operatorname{Spec} A$ show that the open sets of the form $X_f := \operatorname{Spec} A_f$ form a base of this topology.

Definition 4½.1.3. *The set $|X|$ equipped with the topology defined above is called the* Zariski space underlying X. *It is simply denoted by $|X|$.*

Remark: Proposition 4½.1.1 shows that the closed sets in $|X|$ are of the form $|Y| \subset |X|$ for a closed immersion $Y \hookrightarrow X$.

We assume for the moment that X is a scheme. We remark then that a base of the topology on $|X|$ is given by the open sets of the form $|U| \subset |X|$, where $U \hookrightarrow X$ is an open immersion with U an affine scheme (Exercise: check it, we note that this is false for X an algebraic space that is note a scheme). We then define a sheaf $\mathcal{O}_X$ of commutative rings on $|X|$ by putting

$$\mathcal{O}_X(U) \coloneqq A$$

where $A \in \mathrm{Comm}$ is such that $U \cong \operatorname{Spec} A$. This defines a sheaf $\mathcal{O}_X$ on the open sets of a base of the topology on $|X|$ and therefore by extension on $|X|$ entirely.

Definition 4½.1.4. *The topological space $|X|$ equipped with sheaf $\mathcal{O}_X$ is called the* locally ringed space underlying the scheme *X.*

We note that the preceding definition is valid only when X is a scheme. When $X = \operatorname{Spec} A$ we can show that the ringed space $(|X|, \mathcal{O}_X)$ is isomorphic to the space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ as defined in [3] for example.

4½.2. The small étale topos

We return to the general case where X is an algebraic space. As Definition 4½.1.4 no longer makes sense, we will look for a palliative to the ringed space $(|X|, \mathcal{O}_X)$.

4½.2.1. **The locally ringed topos $(X_{\text{ét}}, \mathcal{O}_X)$.** We denote Et/X the category of étale morphisms $U \to X$ with U an affine scheme (and morphisms which commute with the projection onto X). This is a full subcategory $\mathrm{Sh}(\mathrm{Aff})/X$ of sheaves on X, and it inherits a Grothendieck topology induced by that on $\mathrm{Sh}(\mathrm{Aff})$. Clearly, a family of morphisms

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

is covering in Et/X if the family of morphisms of schemes $\{U_i \to V\}$ is an étale covering.

Definition 4½.2.1. *The site* Et/X *is called the* small étale site of *X. Its associated topos* $\mathrm{Sh}(\mathrm{Et}/X)$ *is called the* small étale topos of *X. It is also denoted by $X_{\text{ét}}$.*

We have on Et/X a sheaf of commutative rings, denoted by $\mathcal{O}_X$, that to an object $U = \operatorname{Spec} A \to X$ associates the ring A (Exercise: check that this defines a sheaf of rings on Et/X). We thus obtain a ringed topos $(X_{\text{ét}}, \mathcal{O}_X)$, which is a palliative to ringed space $(|X|, \mathcal{O}_X)$ when X is a general algebraic space. It should be noted, however, that when X is a scheme, $(|X|, \mathcal{O}_X)$ and $(X_{\text{ét}}, \mathcal{O}_X)$ are not equivalent objects.

The following example shows that the topos $X_{\text{ét}}$ is in general not equivalent to the topos of a topological space (and therefore not equivalent to topos of sheaves on $|X|$).

We consider $X = \operatorname{Spec} k$, where k is a field. We denote k^{sep}/k its separable closure and $G \coloneqq \mathrm{Aut}(k^{\text{sep}}/k)$ its Galois group. The group G is a topological group with the Krull topology, for which the open subgroups are of the form $\mathrm{Aut}(k^{\text{sep}}/k') \subset G$ for k' a separable finite extension of k. So we can show that the topos $\mathrm{Sh}(\mathrm{Et}/X)$ is equivalent to the category of *continuous G-sets*, that is the sets E with an action of G such that the morphism $G \times E \to E$ is continuous (or equivalently such that the stabilizer of any point $x \in E$ is an open subgroup in G). For example, when $k = \mathbb{R}$, we find that $\mathrm{Sh}(\mathrm{Et}/X)$ is equivalent to the category of sets with a $\mathbb{Z}/2$ action (i.e. the involution). This shows that the topos $\mathrm{Sh}(\mathrm{Et}/X)$ and $\mathrm{Sh}(|X|)$ are already not equivalent when X is the spectrum of a field, because we always have $|X| = *$ and therefore $\mathrm{Sh}(|X|) \cong \mathrm{Set}$ (and we can see that Set is equivalent to the category of continuous G-sets for G trivial).

4½.3. Quasi-coherent sheaves

We just saw the definition of ringed topos $(X_{\text{ét}}, \mathcal{O}_X)$. We can then consider the category of $\mathcal{O}_X$-modules on Et/X, which we will denote by $\mathcal{O}\text{-Mod}(X_{\text{ét}})$. Similarly, when X is a scheme we have the category of $\mathcal{O}_X$-modules on $|X|$, which we denote by $\mathcal{O}\text{-Mod}(X)$. By restriction of the site Et/X to the site of the space $|X|$, we find a restriction functor

$$\mathcal{O}\text{-Mod}(X_{\text{ét}}) \longrightarrow \mathcal{O}\text{-Mod}(X)$$

(Exercise: fill in the details). This functor is not an equivalence in general. There, however, are subcategories on which it induces an equivalence.

Recall that an object $M \in \mathcal{O}\text{-Mod}(X_{\text{ét}})$ is given by a $\mathcal{O}_X(U)$-module $M(U)$ for any object $U \to X$, and a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr] & & U \arrow[dl] \\ & X & \end{tikzcd}$$

for any morphism $M(U) \to M(V)$ that is $\mathcal{O}_X(U)$-linear and functorial in f

Definition 4½.3.1. *An object $M \in \mathcal{O}\text{-Mod}(X_{\text{ét}})$ is called* quasi-coherent *if for any morphism*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[rr, "f"] \arrow[dr] & & \operatorname{Spec} A \arrow[dl] \\ & X & \end{tikzcd}$$

in Et/X the induced morphism

$$M(\operatorname{Spec} A) \otimes_A B \longrightarrow M(\operatorname{Spec} B)$$

is an isomorphism.

The full subcategory of $\mathcal{O}\text{-Mod}(X_{\text{ét}})$ consisting of quasi-coherent objects are denoted by $\mathrm{QCoh}(X)$.

Exercise: show that for any commutative ring A the category $\mathrm{QCoh}(X)$, where $X = \operatorname{Spec} A$, is naturally equivalent to the category A-Mod of A-modules.

Proposition 4½.3.2. *The restriction functor*

$$\mathcal{O}\text{-Mod}(X_{\text{ét}}) \longrightarrow \mathcal{O}\text{-Mod}(X)$$

induces an equivalence between $\mathrm{QCoh}(X)$ and its essential image in $\mathcal{O}\text{-Mod}(X)$. □

4½.4. Properties of morphisms

4½.4.1. Properties of finiteness.

Definition 4½.4.1.

(1) An algebraic space X is quasi-compact *if there exists an atlas $\{U_i \to X\}_{i\in I}$ with I finite.*

(2) A morphism of algebraic spaces $f : X \to Y$ is quasi-compact *if for any affine scheme Z and any morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is quasi-compact.*

(3) A morphism of algebraic spaces $f : X \to Y$ is locally finitely presented *if for any affine scheme Z and any morphism $Z \to Y$, there exists an atlas $\{U_i \to X \times_Y Z\}$ such that each induced morphism $U_i \to Z$ is a finitely presented morphism (of affine schemes).*

(4) A morphism of algebraic spaces $f : X \to Y$ is finitely presented *if it is locally finitely presented and quasi-compact.*

(5) An algebraic space X is locally noetherian *if there exists an atlas $\{U_i \to X\}$ with $U_i = \operatorname{Spec} A_I$ and each A_i is a noetherian ring.*

(6) An algebraic space X is noetherian *if it is locally noetherian and quasi-compact.*

Exercise: show that the classes of morphisms defined above are stable under base change and composition. Also show that if $f : X \to Y$ is locally finitely presented and if Y is locally noetherian, then X is also locally noetherian.

One important property of a noetherian algebraic space X is that its underlying topological space is noetherian (i.e. that each descending chain of closed sets are stationary). (Exercise: check it). In particular, we can deduce that any closed set of $|X|$ is a finite union of irreducible closed sets (a closed set is irreducible if it is not a nontrivial union of two closed sets). Thus, $|X|$ is itself a finite union of irreducible closed sets, which we call the *irreducible components* of X.

4½.4.2. Separated and proper morphisms.

Definition 4½.4.2.

(1) A morphism of algebraic spaces $f : X \to Y$ is separated *if the diagonal morphism $X \to X \times X$ is a closed immersion.*

(2) A morphism of algebraic spaces $f : X \to Y$ is proper *if it is finitely presented, separated, and if for any affine scheme Z and any morphism $Z \to X \times_Y Z$, the induced morphism*

$$|X \times_Y Z| \longrightarrow |Z|$$

is a closed morphism of topological spaces.

Exercise: check that the proper and separated morphisms are stable under base change and composition.

Proposition 4½.4.3. *The morphism $\mathbb{P}^n \to \operatorname{Spec} \mathbb{Z}$ is proper.* □

The preceding proposition allows us to construct of proper morphisms. In fact, any morphism $X \to Y$ factorizes into $X \xrightarrow{j} Y \times \mathbb{P}^n \xrightarrow{p} Y$, where j is a (finitely presented) closed immersion and p the natural projection is a proper morphism. Such morphisms are called *projective.*

4½.4.3. **Flat, smooth and étale morphisms.**

Proposition 4½.4.4. *A finitely presented morphism $f : X \to Y$ between algebraic spaces is smooth (resp. étale) if and only if it is flat, and if for any field k and any morphism $\operatorname{Spec} k \to Y$ the induced morphism $X \times_Y \operatorname{Spec} k$ is smooth (resp. étale).* □

We will also note the following complementary fact.

Proposition 4½.4.5. *Let k be a field. A morphism $k \to A$ is étale if and only if A is isomorphic, as a k-algebra, to a finite product of fields $\prod k_i$, where k_i/k is a finite separable extension.* □

4½.4.4. **Covering criteria.** The following criteria is very useful to construct atlas. We have already used it to show that $\mathbb{P}^n$ is a scheme.

Proposition 4½.4.6. *Let $\{f_i : X_i \to Y\}$ be a family of locally finite presented morphisms between algebraic spaces. Assume that each f_i is an open immersion (resp. a smooth morphism). Then, for the induced morphism*

$$\coprod_i f_i : \coprod_i X_i \longrightarrow Y$$

to be an epimorphism of sheaves (for the étale topology) it is necessary and sufficient that for any field k (resp. for any separably closed field k), the induced map

$$\coprod_i f_i(k) : \coprod_i X_i(k) \longrightarrow Y(k)$$

is surjective. □

4½.5. Algebraic varieties over a field

For this section we denote by k a field.

Definition 4½.5.1. *An* algebraic variety over k *is a scheme X equipped with a morphism $X \to \operatorname{Spec} k$ that is finitely presented.*

For any scheme X, there exists a closed subscheme $X_{\mathrm{red}} \hookrightarrow X$, called the *reduced subscheme of* X. This is the smallest closed subscheme Y of X such that $X - Y = \varnothing$. When $X = \operatorname{Spec} A$ we have $X_{\mathrm{red}} := \operatorname{Spec} A_{\mathrm{red}}$, where $A_{\mathrm{red}} = A/\operatorname{rad}(A)$, where $\operatorname{rad}(A)$ is the ideal of nilpotent elements in A.

Suppose for now that X is an algebraic variety over k. We consider that X_{red} is again an algebraic variety. The scheme X_{red} is noetherian, and therefore is a finite union of irreducible components $X_{\mathrm{red}} = \bigcup Z_i$. Each Z_i being irreducible, we see that any nonempty Zariski open set of Z_i is dense. Furthermore, if $\operatorname{Spec} A_i \hookrightarrow Z_i$ is one such Zariski open set, the ring A_i is integral (we use here that Z_i is reduced and irreducible). We consider its field of fractions

$$K(Z_i) := \operatorname{Frac}(A_i)$$

which we see is independent of the choice of open set $\operatorname{Spec} A_i \hookrightarrow Z_i$.

Definition 4½.5.2. *Let X be an algebraic variety and keep the notations above. The integer*

$$d_i := \operatorname{DimTr}_k K(Z_i)$$

is called the dimension of X along the component Z_i.

We recall that $\mathrm{DimTr}_k\, K(Z_i)$ designates the transcendence degree of the field extension $K(Z_i)/k$.

Exercise: show that $\mathbb{P}^n$ is irreducible and that its dimension is n.

Lecture 5. **Stacks I**

We recall that for a category C and a subset $W \subset C_1$ of morphisms in C, a localization of C along W is a category $W^{-1}C$ together with a functor $l : C \to W^{-1}C$ such that for all category D, the functor

$$l^* : \underline{\mathrm{Hom}}(W^{-1}C, D) \longrightarrow \underline{\mathrm{Hom}}(C, D)$$

is fully faithful and its essential image consists of functors $C \to D$ sending W to the isomorphisms in D. We show that a localization always exists and is unique up to equivalence.

5.1. Homotopy theory of groupoids

We consider the case where $C = \mathrm{Gpd}$ is the category of groupoids (i.e. its objects are the groupoids and its morphisms are functors between groupoids). We denote by W the subset of equivalences of groupoids (i.e. the functors which are equivalences of categories) and we seek to describe the category $W^{-1}\mathrm{Gpd}$.

Definition 5.1.1. *The* homotopy category of groupoids *is the localized category* $W^{-1}\mathrm{Gpd}$. *It is denoted by* $\mathrm{Ho}(\mathrm{Gpd})$. *The set of morphisms in* $\mathrm{Ho}(\mathrm{Gpd})$ *between two objects A and B will be denoted by* $[A, B]$.

Denote by [Gpd] for now the category whose objects are groupoids, and for two groupoids A and B the set of morphisms from A to B in [Gpd] is by definition the set of isomorphism classes of functors from A to B (i.e. the set of isomorphisms of the category $\underline{\mathrm{Hom}}(A, B)$). We have a natural projection $p : \mathrm{Gpd} \to [\mathrm{Gpd}]$, which is identities on objects and the canonical projection on the sets of morphisms. Exercise:* describe the composition of morphisms in [Gpd].

Theorem 5.1.2. *The natural projection*

$$p : \mathrm{Gpd} \longrightarrow [\mathrm{Gpd}]$$

is a localization of Gpd *along W. Thus, p induces a natural equivalence*

$$\mathrm{Ho}(\mathrm{Gpd}) \simeq [\mathrm{Gpd}].$$

Proof. We start by noting that for all category D the functor

$$p^* : \underline{\mathrm{Hom}}([\mathrm{Gpd}], D) \longrightarrow \underline{\mathrm{Hom}}(\mathrm{Gpd}, D)$$

is fully faithful. In fact, as p is surjective on objects the functor p^* is faithful (Exercise:† check that). Furthermore, as p is surjective on the sets of morphisms the functor p^* is also full (Exercise:‡ check that).

---

* [RC]: Let $[f] : A \to B$ and $[g] : B \to C$ be morphisms in [Gpd], represented by $f : A \to B$ and $g : B \to C$, respectively. Then we define $[g] \circ [f] = [g \circ f]$. It remains to show that if $f \cong f'$ and $g \cong g'$, then $g \circ f \cong g \circ f' \cong g' \circ f'$; this is easily checked.

† [RC]: Fix $F, G : [\mathrm{Gpd}] \to D$. Let $\tau, \sigma : F \to G$ be two natural transformations such that $\tau \circ p = \sigma \circ p$, and we want to show that $\tau = \sigma$. But any object in [Gpd] has the form $p(A)$ for some groupoid A, and $\tau_{p(A)} = \tau_A \circ p = \sigma_A \circ p = \sigma_{p(A)}$, so $\tau = \sigma$.

‡ [RC]: Let us adopt the notations of the previous footnote. Given a natural transformation $\tau : F \circ p \to G \circ p$, we want to define a natural transformation $\overline{\tau} : F \to G$ such that $\overline{\tau} \circ p = \tau$. Given any object $p(A)$ in [Gpd], define $\overline{\tau}_{p(A)}$ to be τ_A; the surjectivity of p on the sets of morphisms guarantees that $\overline{\tau}$ defines a natural transformation.

It remains to show that the functor $F : \mathrm{Gpd} \to D$ which sends the morphisms in W to isomorphisms in D is in the essential image of p^*. By definition of the category [Gpd], the functor F factorizes through the canonical projection $\mathrm{Gpd} \to [\mathrm{Gpd}]$ if and only if $F(f) = F(g)$ for two isomorphic functors $f, g : A \to B$ between groupoids. Therefore, let f and g be two such functors, and we show that $F(f) = F(g)$.

Denote by $\overline{\Delta}^1$ the groupoid with two objects 0 and 1 and a unique isomorphism between 0 and 1. We see that there exists a functorial bijection between the set of functors $\overline{\Delta}^1 \to A$ and the set of isomorphisms in A. Choose a natural isomorphism $\gamma : f \to g$. The natural transformation γ defines a morphism of groupoids

$$h : A \times \overline{\Delta}^1 \longrightarrow B$$

so that there exists a commutative diagram of groupoids

$$\begin{tikzcd} A \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ A \times \overline{\Delta}^1 \arrow[r, "h"] & B \\ A \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where $i_0 : A \to A \times \overline{\Delta}^1$ is the functor $\mathrm{id} \times \{0\}$ and $i_1 : A \to A \times \overline{\Delta}^1$ is the functor $\mathrm{id} \times \{1\}$. (Exercise:* describe in detail the functor h in terms of γ). Denote by $q : A \times \overline{\Delta}^1 \to A$ the projection onto the first factor. We have $q \circ i_0 = q \circ i_1 = \mathrm{id}$. Furthermore, $i_0 \circ q$ and $i_1 \circ q$ are both naturally isomorphic to the identity functor of $A \times \overline{\Delta}^1$. The functors q, i_0 and i_1 are therefore equivalences of categories. By hypothesis on the object $F \in \underline{\mathrm{Hom}}(\mathrm{Gpd}, D)$, we therefore find $F(i_0) = F(i_1) = F(q)^{-1}$. Thus, we have

$$F(f) = F(h) \circ F(i_0) = F(h) \circ F(q)^{-1} = F(h) \circ F(i_1) = F(g). \qquad \square$$

Corollary 5.1.3. *The natural functor*

$$j : \mathrm{Set} \longrightarrow \mathrm{Gpd} \longrightarrow \mathrm{Ho}(\mathrm{Gpd}),$$

which sends a set to the corresponding discrete groupoid, is fully faithful and has a left adjoint

$$\pi_0 : \mathrm{Ho}(\mathrm{Gpd}) \longrightarrow \mathrm{Set}.$$

Proof. We define $\pi_0(A)$ as the set of isomorphism classes of objects in A. Theorem 5.1.2 allows us to easily check that j is fully faithful and that π_0 is its left adjoint. $\square$

Exercise:† Let G and H be two groups. We denote by BG (resp. BH) the groupoid with a unique object and G (resp. H) the automorphism group of that object. Describe the set $[BG, BH]$ of morphisms from BG to BH in the category Ho(Gpd).

---

* [RC]: Write η for the unique isomorphism in $\overline{\Delta}^1$ from 0 to 1. Fix a morphism $\theta : a \to a'$ in A, then h sends $(\theta, 0)$ to $f(\theta)$, $(\theta, 1)$ to $g(\theta)$, and (θ, η) to $g(\theta) \circ \gamma_a = \gamma_{a'} \circ f(\theta)$.

† [RC]: A morphism in $\underline{\mathrm{Hom}}(BG, BH)$ between $f, f' \in \mathrm{Hom}_{\mathrm{Gpd}}(BG, BH) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, H)$, which is a natural transformation that is necessarily a natural isomorphism, consists of a morphism in BH, which corresponds to an element $h \in H$, such that $f'(g) \circ h = h \circ f(g)$ for any $g \in G$. Therefore, $[BG, BH] \cong \mathrm{Hom}_{\mathrm{Grp}}(G, H)/\sim_H$, where $\sim_H$ denotes the conjugation by elements in H.

5.2. Homotopy theory of diagrams of groupoids

Let I be a category, and consider the category $\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$ of functors from I to Gpd (also called the category of I-diagrams of Gpd). Let $F, G : I \to \mathrm{Gpd}$ be two I-diagrams. We say that a morphism

$$f : F \longrightarrow G$$

in $\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$ is an equivalence if for any object $i \in I$, the induced morphism

$$f_i : F(i) \longrightarrow G(i)$$

is an equivalence of groupoids. The notion of equivalence defines a subset W_I of morphisms in $\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$.

Definition 5.2.1. *The* homotopy category of I-diagrams of groupoids *is* $W_I^{-1}\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$. *It is denoted* $\mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$. *The set of morphisms between F and G in* $W_I^{-1}\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$ *is denoted* $[F, G]$.

The generalization of Theorem 5.1.2 to the case of diagrams of groupoids requires the introduction of the notion of *weak morphisms* between objects of $\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$. For that, we denote, for any $F \in \underline{\mathrm{Hom}}(I, \mathrm{Gpd})$ and any morphism $u : i \to j$ in I, $u_*^F : F(i) \to F(j)$ the functor induced by u. Note that by definition we have $(u^F \circ v^F)_* = u_*^F \circ v_*^F$ and $\mathrm{id}_*^F = \mathrm{id}$.

Definition 5.2.2. *Let $F, G : I \to \mathrm{Gpd}$ be two I-diagrams of groupoids. A* weak morphism *$f : F \to G$ consists of the following data:*

(1) For any object $i \in I$, a functor of groupoids

$$f_i : F(i) \longrightarrow G(i).$$

(2) For any morphism $u : i \to j$ in I, a natural transformation

$$\gamma_u^f : u_*^G \circ f_i \longrightarrow f_j \circ u_*^F.$$

We assume furthermore that $\gamma_{\mathrm{id}}^f = \mathrm{id}$.

We demand moreover that these data satisfies the following condition: for any pair of morphisms $i \xrightarrow{u} j \xrightarrow{v} k$ in I, the two natural transformations

$$\left(v_*^G \circ \gamma_u^f\right) \circ \left(\gamma_v^f \circ u_*^F\right) : (v \circ u)_*^G \circ f_i = v_*^G \circ u_*^G \circ f_i \xrightarrow{v_*^G \circ \gamma_u^f} v^G \circ f_j \circ u_*^F \xrightarrow{\gamma_v^f \circ u_*^F} f_k \circ v_*^F \circ u_*^F = f_k \circ (v \circ u)_*^F$$

$$\gamma_{v \circ u}^f : (v \circ u)_*^G \circ f_i \longrightarrow f_k \circ (v \circ u)_*^F$$

*are equal**

$$\left(v_*^G \circ \gamma_u^f\right) \circ \left(\gamma_v^f \circ u_*^F\right) = \gamma_{v \circ u}^f.$$

* [RC]: In other words, the diagram of functors of groupoids

$$\begin{tikzcd}
F(i) \arrow[r, "f_i"] \arrow[d, "u_*^F"'] & G(i) \arrow[d, "u_*^G"] \\
F(j) \arrow[r, "f_j"] \arrow[d, "v_*^F"'] & G(j) \arrow[d, "v_*^G"] \\
F(k) \arrow[r, "f_k"] & G(k)
\end{tikzcd}$$

commutes up to unique natural transformations.

Let F, G and H be three I-diagrams of groupoids, and $f : F \to G$ and $g : G \to H$ two weak morphisms. We define the composite weak morphism $g \circ f : F \to H$ by the following data:

(1) For any object $i \in I$
$$(g \circ f)_i \coloneqq g_i \circ f_i : F(i) \longrightarrow H(i).$$

(2) For any morphism $u : i \to j$ in I, we have natural transformation $\gamma_u^{g \circ f}$ is defined as the composition
$$\gamma_u^{g\circ f} : u_*^H \circ g_i \circ f_i \xrightarrow{\gamma_u^g \circ f_i} g_j \circ u_*^G \circ f_i \xrightarrow{g_j \circ \gamma_u^f} g_j \circ f_j \circ u_*^F.$$

Clearly,
$$\gamma_u^{g\circ f} = \left(g_j \circ \gamma_u^f\right) \circ \left(\gamma_u^g \circ f_i\right).$$

Exercise: check that this composition is associative and unitary.

Definition 5.2.3. *Let $F, G : I \to \mathrm{Gpd}$ be two I-diagrams of groupoids, and $f, g : F \to G$ two weak morphisms. A* natural transformation *between f and g is the data of, for each object $i \in I$, a natural transformation*
$$\phi_i : f_i \longrightarrow g_i$$
such that for any morphism $u : i \to j$ in I we have
$$u_*^G \circ \phi_i = \phi_j \circ u_*^F.$$

The preceding definition allows us to define for two I-diagrams of groupoids F and G as above a category $\underline{\mathrm{Hom}}^{\mathrm{lax}}(F, G)$, whose objects are the weak morphisms and the morphisms are the natural transformations (Exercise: describe the composition of morphisms in this category). We see that the category $\underline{\mathrm{Hom}}^{\mathrm{lax}}(F, G)$ is a groupoid (Exercise: check it). The composition of weak morphisms described above defines a functor
$$\underline{\mathrm{Hom}}^{\mathrm{lax}}(F, G) \times \underline{\mathrm{Hom}}^{\mathrm{lax}}(G, H) \longrightarrow \underline{\mathrm{Hom}}^{\mathrm{lax}}(F, H).$$

For F and G fixed, we denote by $\underline{\mathrm{Hom}}(F, G)$ the full subcategory of $\underline{\mathrm{Hom}}^{\mathrm{lax}}(F, G)$ consisting of weak morphisms f such that $\gamma_u^f = \mathrm{id}$ for any morphism u in I. These morphisms are also the morphisms of I-diagrams (i.e. the morphisms between F and G in the category $\underline{\mathrm{Hom}}(I, \mathrm{Gpd})$), and will be called the *strict morphisms*.

We define a category $[\underline{\mathrm{Hom}}(I, \mathrm{Gpd})]$, whose objects are the I-diagrams of groupoids, and whose morphisms are the isomorphism classes of weak morphisms. Since a morphism of I-diagrams of groupoids is also a weak morphism (for which $\gamma_u = \mathrm{id}$ for any u), we have a natural functor
$$\underline{\mathrm{Hom}}(I, \mathrm{Gpd}) \longrightarrow [\underline{\mathrm{Hom}}(I, \mathrm{Gpd})],$$
which is the identity on the set of objects.

Theorem 5.2.4. *The natural functor*
$$\underline{\mathrm{Hom}}(I, \mathrm{Gpd}) \longrightarrow [\underline{\mathrm{Hom}}(I, \mathrm{Gpd})]$$
induces an equivalence
$$\mathrm{Ho}\left(\underline{\mathrm{Hom}}(I, \mathrm{Gpd})\right) \longrightarrow [\underline{\mathrm{Hom}}(I, \mathrm{Gpd})].$$

Proof. This is a similar proof, although a little more complicated than that of Theorem 5.1.2. We will not give it. ☐

Exercise: Let I be a groupoid and $F : I \to \mathrm{Gpd}$ a constant functor with value in a fixed groupoid $A \in \mathrm{Gpd}$. Determine the groupoids $\underline{\mathrm{Hom}}(*, F)$ and $\underline{\mathrm{Hom}}^{\mathrm{lax}}(*, F)$ of strict and weak morphisms of constant I-diagrams from a point to F. Thereof deduce that the natural functor

$$\underline{\mathrm{Hom}}(I, \mathrm{Gpd}) \longrightarrow \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$$

is not surjective on morphisms (i.e. that there exist weak morphisms non-isomorphic to strict morphisms).

We end this section with the following important fact.

Proposition 5.2.5. *A weak morphism $F \to G$ of I-diagrams of groupoids represents an isomorphism in $\mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$ if and only if for any $i \in I$ the functor*

$$f_i : F(i) \longrightarrow G(i)$$

is an equivalence of groupoids.

Proof. We choose for any $i \in I$ an inverse functor $g_i : G(i) \to F(i)$, and natural isomorphisms

$$\alpha_i : f_i \circ g_i \Rightarrow \mathrm{id}, \qquad \beta_i : g_i \circ f_i \Rightarrow \mathrm{id}\,.$$

We then show that there exists a unique structure of weak morphisms on g_i (i.e. γ_g) and unique isomorphisms of weak morphisms

$$\alpha : f \circ g \cong \mathrm{id}, \qquad \beta : g \circ f \cong \mathrm{id}$$

compatible with those already chosen for any i (Exercise: fill in the details). □

5.3. Homotopy limits

Let I be a category and denote by $\mathrm{Gpd} \to \underline{\mathrm{Hom}}(I, \mathrm{Gpd})$ the functor that sends a groupoid A to the constant I-diagram with value A. It induces a functor on the localizations

$$c : \mathrm{Ho}(\mathrm{Gpd}) \longrightarrow \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd})).$$

The following statement is a corollary to Theorem 5.1.2.

Corollary 5.3.1. *The preceding functor $c : \mathrm{Ho}(\mathrm{Gpd}) \longrightarrow \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$ has a right adjoint*

$$\underleftarrow{\mathrm{Holim}}_I : \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd})) \longrightarrow \mathrm{Ho}(\mathrm{Gpd}).$$

Proof. For $F \in \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$ we put

$$\underleftarrow{\mathrm{Holim}}_I F = \underline{\mathrm{Hom}}^{\mathrm{lax}}(c(*), F),$$

where $*$ is the groupoid that is reduced to a point and $\underline{\mathrm{Hom}}^{\mathrm{lax}}(c(*), F)$ is the groupoid of weak morphisms from $c(*)$ to F.

Clearly, an object x of $\underleftarrow{\mathrm{Holim}}_I F$ is the following data:

(1) For any object $i \in I$ an object $x_i \in F(i)$.
(2) For any morphism $u : i \to j$ in I an isomorphism $\gamma_u^x : u_*^F(x_i) \to x_j$. We assume that $\gamma_{\mathrm{id}}^x = \mathrm{id}$.

□

We demand further that for two morphisms $i \xrightarrow{u} j \xrightarrow{v} k$ in I, we have

$$\gamma_v^x \circ v_*^F(\gamma_u^x) = \gamma_{v\circ u}^x$$

(as morphisms $(v \circ u)_x^F(x_i) \to x_k$). The morphisms between x and y in $\underleftarrow{\mathrm{Holim}}_I F$ are simply the family of morphisms $\phi_i : x_i \to y_i$ in $F(i)$ such that $\phi_j \circ \gamma_u^x = \gamma_u^y \circ u_*^F(\phi_i)$. We check that the groupoid of functors $\underline{\mathrm{Hom}}\left(A, \underleftarrow{\mathrm{Holim}}_I F\right)$ is in bijection with the groupoid $\mathrm{Hom}^{\mathrm{lax}}(c(A), F)$ and in a way that is functorial in $A \in \mathrm{Gpd}$. By passing to the isomorphism classes of objects we find a functorial bijection in $A \in \mathrm{Ho}(\mathrm{Gpd})$

$$\left[A, \underleftarrow{\mathrm{Holim}}_I F\right] \cong [c(A), F].$$

This finishes the proof of the corollary.

Definition 5.3.2. *Denote by I the category of the form*

$$\begin{tikzcd} & 1 \arrow[d] \\ 2 \arrow[r] & 0. \end{tikzcd}$$

Let $F \in \mathrm{Ho}(\underline{\mathrm{Hom}}(I, \mathrm{Gpd}))$ be represented by a diagram of groupoids

$$\begin{tikzcd} & A_1 \arrow[d, "p"] \\ A_2 \arrow[r, "q"'] & A_0. \end{tikzcd}$$

The object $\underleftarrow{\mathrm{Holim}}_I F$ is called the homotopy fiber product of A_1 and A_2 over A_0. *It is denoted*

$$\underleftarrow{\mathrm{Holim}}_I FF =: A_1 \times_{A_0}^h A_2 \in \mathrm{Ho}(\mathrm{Gpd}).$$

Exercise: Show that the groupoid $A_1 \times_{A_0}^h A_2$ is naturally equivalent to the groupoid whose objects are the triples (a_1, a_2, u), with $a_i \in A_i$ and $u : p(a_1) \to q(a_2)$ an isomorphism in A_0, and the morphisms are pairs of morphisms $a_1 \to a_1'$, $a_2 \to a_2'$ which commute with u and u'.

Exercise*: Let H be a group and BH its classifying groupoid. Let $* \to BH$ be the unique functor. Calculate the groupoid $* \times_{BH}^h *$.

Corollary 5.3.1 has also the following generalization. We consider two categories I and J, and the functor

$$c_J : \underline{\mathrm{Hom}}(J, \mathrm{Gpd}) \longrightarrow \underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd})$$

which to a J-diagram of groupoids corresponds the $I \times J$-diagram "constant with respect to I". Again it induces a functor on the localizations

$$c_J : \mathrm{Ho}(\underline{\mathrm{Hom}}(J, \mathrm{Gpd})) \longrightarrow \mathrm{Ho}(\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd})).$$

* [RC]: By the previous exercise, the object of $* \times_{BH}^h *$ is the set of morphisms of BH, i.e. the underlying set of H, and the only morphisms are the identities. That is, $* \times_{BH}^h *$ is isomorphic to the discrete groupoid corresponding to the underlying set of H.

Corollary 5.3.3. *The preceding functor* $c_J : \mathrm{Ho}(\underline{\mathrm{Hom}}(J, \mathrm{Gpd})) \to \mathrm{Ho}(\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd}))$ *has a right adjoint*

$$\underset{\overleftarrow{I}}{\mathrm{Holim}} : \mathrm{Ho}(\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd})) \longrightarrow \mathrm{Ho}(\underline{\mathrm{Hom}}(J, \mathrm{Gpd})).$$

Proof. Let $F \in \mathrm{Ho}(\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd}))$. For any $j \in J$, we consider the groupoid $\underline{\mathrm{Hom}}^{\mathrm{lax}}(c(*), F(-, j))$. As j traverses the category J, this defines a J-diagram of groupoids which we denote $\underleftarrow{\mathrm{Holim}}_I F$. We check that, with the help of Theorem 5.1.2 that this object $\underleftarrow{\mathrm{Holim}}_I$ has the required universal property

$$[c_J(G), F] \cong \left[G, \underset{\overleftarrow{I}}{\mathrm{Holim}}\, F \right]. \qquad \square$$

Notice that the objects of $\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd})$ are the I-diagrams in the category of J-diagrams. When I is the category

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \longrightarrow & 0, \end{array}$$

an object of $\underline{\mathrm{Hom}}(I \times J, \mathrm{Gpd})$ is given by a diagram

$$\begin{array}{ccc} & & F_1 \\ & & \downarrow \\ F_2 & \longrightarrow & F_0 \end{array}$$

in $\underline{\mathrm{Hom}}(J, \mathrm{Gpd})$. The object $\underleftarrow{\mathrm{Holim}}_I F$ is then denoted $F_1 \times^h_{F_0} F_2 \in \mathrm{Ho}(\underline{\mathrm{Hom}}(J, \mathrm{Gpd}))$.

Definition 5.3.4. *For a diagram*

$$\begin{array}{ccc} & & F_1 \\ & & \downarrow \\ F_2 & \longrightarrow & F_0, \end{array}$$

in $\underline{\mathrm{Hom}}(J, \mathrm{Gpd})$*, the object* $F_1 \times^h_{F_0} F_2$ *is called the* homotopy fiber product of F_1 and F_2 over F_0.

We notice, according to the proof of Corollary 5.3.3, that the J-diagram $F_1 \times^h_{F_0} F_2$ is described explicitly as follows:

$$\begin{array}{cccccc} F_1 \times^h_{F_0} F_2 & : & J & \longrightarrow & \mathrm{Gpd} \\ & & j & \longmapsto & F_1(j) \times^h_{F_0(j)} F_2(j). \end{array}$$

5.4. The homotopy categories of prestacks and of stacks

For now, let C be a Grothendieck site.

Definition 5.4.1. *The* homotopy category of prestacks on C *is* $\mathrm{Ho}(\underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathrm{Gpd}))$*. It is denoted*

$$\mathrm{Ho}(\mathrm{PrCh}(C)) := \mathrm{Ho}(\underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathrm{Gpd})).$$

The category of prestacks Ho((PrCh(C)) is a generalization of the category of presheaves in the following sense: there exists a functor $\underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathrm{Set}) \to \underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathrm{Gpd})$, obtained by composition with the functor Set $\to$ Gpd which sends a set to the corresponding discrete groupoid. This induces a functor

$$i : \mathrm{Pr}(C) \longrightarrow \mathrm{Ho}(\mathrm{PrCh}(C))$$

of the category of presheaves of sets on C to the homotopy category of prestacks.

Proposition 5.4.2. *The functor*

$$i : \mathrm{Pr}(C) \longrightarrow \mathrm{Ho}(\mathrm{PrCh}(C))$$

is fully faithful and has a left adjoint

$$\pi_0^{\mathrm{pr}} : \mathrm{Ho}(\mathrm{PrCh}(C)) \longrightarrow \mathrm{Pr}(C).$$

Proof. We set for $F \in \mathrm{Ho}(\mathrm{PrCh}(C))$

$$\pi_0^{\mathrm{pr}}(F) : C^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

defined by $\pi_0^{\mathrm{pr}}(F)(X) := \pi_0(F(X))$, where $\pi_0(F(X))$ is the set of isomorphism classes of objects in $F(X)$. Using Theorem 5.2.4 we can show that

$$\mathrm{Hom}\,(\pi_0^{\mathrm{pr}}(F), G) \cong [F, i(G)].$$

□

By composing the functor i of Proposition 5.4.2 and the Yoneda embedding we find a fully faithful functor (again called the Yoneda embedding)

$$C \longrightarrow \mathrm{Ho}(\mathrm{PrCh}(C)),$$

by which we identify the category C with its image in Ho(PrCh(C)). The version of the Yoneda lemma for prestacks is stated below:

Proposition 5.4.3. *For any $F \in \mathrm{Ho}(\mathrm{PrCh}(C))$ and $X \in C$, there exists a functorial bijection in F and X*

$$[X, F] \cong \pi_0(F(X)) = \pi_0^{\mathrm{pr}}(F)(X),$$

where $\pi_0(F(X))$ is the set of isomorphism classes of objects of $F(X)$).

Proof. It is again an application of Theorem 5.1.2. We know that $[X, F]$ is in natural bijection with the set of isomorphism classes of objects in the groupoid $\underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F)$ of weak morphisms of X and F. We start by checking, with the usual help of Yoneda lemma (applied to presheaves of objects and of morphisms of F), that there exists a natural isomorphism

$$\underline{\mathrm{Hom}}(X, F) \cong F(X).$$

We must therefore show that the natural inclusion

$$j : \underline{\mathrm{Hom}}(X, F) \longrightarrow \underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F)$$

is an equivalence of groupoids. For that, we start by noticing that the groupoid $\underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F)$ is described as follows: An object of $\underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F)$ is the data for any morphism $u : Y \to X$ in C an object $x_u \in F(Y)$, and for any commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "v"'] & Y \arrow[d, "u"] \\ & X \end{tikzcd}$$

an isomorphism $\gamma_f : x_v \to f^*(x_u)$ in $F(Z)$, so that γ_f satisfy the usual *cocycle condition*: $g^*(\gamma_f)\gamma_g = \gamma_{f\circ g}$ and $\gamma_{\mathrm{id}} = \mathrm{id}$. We then define a functor

$$\phi : \underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F) \longrightarrow \underline{\mathrm{Hom}}(X, F)$$

which to an object of $\underline{\mathrm{Hom}}^{\mathrm{lax}}(X, F)$ associates $x_{\mathrm{id}} \in F(X)$. We easily see that ϕ is a functor inverse to j (Exercise: check the details). $\square$

We now see to generalize the notion of sheaves. For that, note that for any $F \in \underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathrm{Gpd})$, any object $X \in C$ and any covering family $\{U_i \to X\}$, we have a diagram of groupoids

$$\begin{tikzcd} F(X) \arrow[r] & \prod_i F(U_i) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \prod_{i,j} F(U_{i,j}) \arrow[l, "s_0" description] \arrow[r, shift left=4, "e_0"] \arrow[r, "e_1" description] \arrow[r, shift right=4, "e_2"'] & \prod_{i,j,k} F(U_{i,j,k}) \arrow[l, shift left=2, "t_1"'] \arrow[l, shift right=2, "t_0"] \end{tikzcd}$$

where the morphisms are defined as follows. [...]

Definition 5.4.4.

(1) *A prestack* $F \in \mathrm{Ho}(\mathrm{PrCh}(C))$ *is a* stack *if for any* $X \in C$ *and any covering family* $\{U_i \to X\}$ *the natural morphism*

$$F(X) \longrightarrow \underleftarrow{\mathrm{Holim}} \left[\begin{tikzcd} F(X) \arrow[r] & \prod_i F(U_i) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \prod_{i,j} F(U_{i,j}) \arrow[l, "s_0" description] \arrow[r, shift left=4, "e_0"] \arrow[r, "e_1" description] \arrow[r, shift right=4, "e_2"'] & \prod_{i,j,k} F(U_{i,j,k}) \arrow[l, shift left=2, "t_1"'] \arrow[l, shift right=2, "t_0"] \end{tikzcd} \right]$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Gpd})$.

(2) *The full subcategory of* $\mathrm{Ho}(\mathrm{PrCh}(C))$ *consisting of stacks is denoted* $\mathrm{Ho}(\mathrm{Ch}(C))$.

We end with a decannulated version of the preceding definition.

Proposition 5.4.5. *A prestack* $F \in \mathrm{Ho}(\mathrm{PrCh}(C))$ *is a stack if and only if it satisfies the following two conditions:*

(1) *For any* $X \in C$ *and any pair of objects* (a, b) *in* $F(X)$*, the presheaf*

$$\begin{array}{cccc} \underline{\mathrm{Iso}}(a,b) & : & (C/X)^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ & & (u : Y \to X) & \longmapsto & \mathrm{Hom}_{F(Y)}(u^*(a), u^*(b)) \end{array}$$

is a sheaf on the set C/X.

(2) *For any* $X \in C$*, any covering family* $\{U_i \to X\}$*, any family of objects* $a_i \in F(U_i)$*, and any family of isomorphisms in* $F(U_{i,j})$

$$\phi_{i,j} : (a_i)|_{U_{i,j}} \cong (a_j)|_{U_{i,j}}$$

satisfying

$$(\phi_{j,k})|_{U_{i,j,k}} \circ (\phi_{i,j})|_{U_{i,j,k}} = (\phi_{i,k})|_{U_{i,j,k}}, \qquad (\phi_{i,i})|_{U_i} = \mathrm{id},$$

there exists an object $a \in F(X)$ *and isomorphisms* $\alpha_i : a|_{U_i} \cong a_i$ *such that*

$$\phi_{i,j} = (\alpha_j)|_{U_{i,j}} \circ (\alpha_i)^{-1}|_{U_{i,j}}.$$

Idea of proof. We start by describing the homotopy limit involved in Definition 5.4.4 by using the formula

$$\underset{I}{\underleftarrow{\mathrm{Holim}}}\, F \simeq \underline{\mathrm{Hom}}^{\mathrm{lax}}(*, F)$$

for any I-diagram of groupoids F. We then notice that condition (1) of the proposition is equivalent to the fact that the functor

$$F(X) \longrightarrow \underleftarrow{\mathrm{Holim}} \left[F(X) \longrightarrow \prod_i F(U_i) \mathrel{\substack{\xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1}}} \prod_{i,j} F(U_{i,j}) \mathrel{\substack{\xrightarrow{e_0} \\ \xleftarrow{t_1} \\ \xrightarrow{e_1} \\ \xleftarrow{t_0} \\ \xrightarrow{e_2}}} \prod_{i,j,k} F(U_{i,j,k}) \right]$$

is fully faithful. Similarly, condition (2) of the proposition is equivalent to the fact that this functor is essentially surjective. □

Remarks:

(1) In practice it is the criteria given in Proposition 5.4.5 that we use to show that a prestack is a stack. In the literature these criteria are often taken as the definition of stack.

(2) A data $\{a_i \in F(U_i), \phi_{i,j}\}$ as in Proposition 5.4.5 is called the *descent data* for F relative to the covering $\{U_i \to X\}$. Condition (2) of Proposition 5.4.5 is also said "any descent data for F is effective".

(3) In condition (2), $(\phi_{i,i})|_{U_i}$ is the restriction of $\phi_{i,i}$ along the diagonal

$$U_i \longrightarrow U_{i,i} = U_i \times_X U_i.$$

Lecture 6. Stacks II

6.1. Examples of stacks

In this section we give some fundamental examples of stacks, as well as two procedures of general constructions. Many examples of stacks can be constructed by using these basic examples and these two procedures.

6.1.1. **The stack of sheaves.** We consider a Grothendieck site C. For any $X \in C$, we consider the groupoid $\mathrm{Fais}(X)^*$ of sheaves on the site C/X. For a morphism $u : Y \to X$ in C we have a restriction functor

$$u^* : \mathrm{Fais}(X) \longrightarrow \mathrm{Fais}(Y)$$

induced by the natural functor $C/Y \to C/X$. We clearly have $(u \circ v)^* = v^* \circ u^*$, and therefore $X \mapsto \mathrm{Fais}$ defines a prestack that we naturally call the prestack of sheaves on C.

Proposition 6.1.1. *The prestack* Fais *is a stack.*

Proof. We use the criteria given by Proposition 5.4.5. Let $X \in B$, and F and G two objects of $\mathrm{Fais}(X)$, that is two sheaves on C/X. The presheaf $\underline{\mathrm{Iso}}(F,G)$ is the presheaf of isomorphisms from F to G, which to $(u : Y \to X)$ corresponds the set of isomorphisms between $u^*(F)$ and $u^*(G)$ as sheaves on C/Y. The fact that $\underline{\mathrm{Iso}}(F,G)$ is a sheaf follows from the following lemma.

Lemma 6.1.2. *Let C be a Grothendieck site, and F and G two sheaves on C. Then the presheaf $\underline{\mathrm{Hom}}(F,G)$ of morphisms from F to G is a sheaf. Furthermore, the sub-presheaf $\underline{\mathrm{Iso}}(F,G) \subset \underline{\mathrm{Hom}}(F,G)$ consisting of the isomorphisms is a subsheaf.*

Proof of Lemma. The fact that $\underline{\mathrm{Hom}}(F,G)$ is a sheaf is well known. It just remains to show that $\underline{\mathrm{Iso}}(F,G) \subset \underline{\mathrm{Hom}}(F,G)$ is a subsheaf. But this is deduced easily from the fact that *being an isomorphism* is a local condition (Exercise: fill in the details). $\Diamond$

Now let $X \in C$ and $\{U_i \to X\}$ a covering family. Let F_i be the sheaves on C/U_i, and

$$\phi_{i,j} : (F_i)|_{U_{i,j}} \cong (F_j)|_{U_{i,j}}$$

isomorphisms satisfying

$$(\phi_{i,i})|_{U_i} = \mathrm{id}, \qquad (\phi_{j,k})|_{U_{i,j,k}} \circ (\phi_{i,j})|_{U_{i,j,k}} = (\phi_{i,k})|_{U_{i,j,k}}.$$

We define a sheaf F on C/X as follows. Let $Y \to X$ be an object of C/X. We put $Y_i := Y \times_X U_i$ and $Y_{i,j} := Y \times_X U_{i,j}$. The natural morphisms $Y_i \to U_i$ and $Y_{i,j} \to U_{i,j}$ determine objects of C/U_i and of $C/U_{i,j}$. By definition, the set $F(Y)$ is the set of families $\{a_i \in F_i(Y_i)\}$ satisfying

$$\phi_{i,j}\left((a_i)|_{Y_{i,j}}\right) = (a_j)|_{Y_{i,j}}$$

for any i, j. For a morphism $Z \xrightarrow{f} Y \to X$ in C/X, we define a map

$$f^* : F(Y) \longrightarrow F(Z)$$

by the formula $f^*(a)_i := f_i^*(a_i)$ where

$$f_i : Z_i = Z \times_X U_i \longrightarrow Y_i = Y \times_X U_i$$

is the morphism induced by f. We easily check that $Y \mapsto F(Y)$ defines a presheaf on C/X.

---

* [RC]: The largest subcategory of $\mathrm{Sh}(C/X)$ containing all objects which is a groupoid. (?)

Let $Y \to U_i$ be an object of C/U_i. The set $F(Y)$ is by definition the set of families of elements $\{a_j \in F_j(Y \times_{U_i} U_{i,j})\}$ which satisfies

$$\phi_{j,k}\left((a_j)|_{Y\times_{U_i}U_{i,j,k}}\right) = (a_k)|_{Y\times_{U_i}U_{i,j,k}}$$

for any j, k. We define a map $f_{i,Y}$ from $F(Y)$ to $F_i(Y)$ by putting

$$f_{i,Y}(a) := (a_i)|_Y \in F_i(Y),$$

where we use the natural morphism $Y \to Y_i = Y \times_X U_i$ to restrict a_i to Y. We define a converse map $g_{i,Y}$ from $F_i(Y)$ to $F(Y)$ by putting

$$g_{i,Y}(a)_j := \phi_{i,j}\left(a|_{Y\times_{U_i}U_{i,j}}\right) \in F_j(Y \times_{U_i} U_{i,j}).$$

The element $g_{i,Y}(a)$ is well defined due to the condition $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$ being satisfied on $C/U_{i,j,k}$. Since $(\phi_{i,i})|_{U_i} = \mathrm{id}$, we see that $f_{i,Y} \circ g_{i,Y} = \mathrm{id}$. Conversely, for $a \in F(Y)$, we have

$$g_{i,Y}(f_{i,Y}(a))_j = \phi_{i,j}\left((a_i)|_{Y\times_{U_i}U_{i,j}}\right) = a_j.$$

Thus, we have $g_{i,Y} \circ f_{i,Y} = \mathrm{id}$.

As Y varies in the category C/U_i, the map $g_{i,Y}$ and $f_{i,Y}$ therefore define isomorphisms inverse to each other between $F|_{U_i}$ and F_i. We consider the isomorphism

$$f_i : F|_{U_i} \cong F_i$$

of presheaves on C/U_i thus defined. Now let i and j be two indices, and $Y \to U_{i,j}$ an object of $C/U_{i,j}$. We consider the diagram

$$\begin{tikzcd} F(Y) \arrow[r, "f_{i,Y}"] \arrow[d, "f_{j,Y}"'] & F_i(Y) \arrow[dl, "\phi_{i,j}"] \\ F_j(Y). & \end{tikzcd}$$

By definition we have

$$\phi_{i,j}(f_{i,Y}(a)) = \phi_{i,j}((a_i)_Y) = a_j.$$

This almost shows condition (2) of Proposition 5.4.5, except that F must also be a sheaf. But this is deduced immediately from the following lemma.

Lemma 6.1.3. *The presheaf F thus defined is a sheaf.*

Proof of Lemma. Denote by $p_i : C/U_i \to C/X$ the natural functor. It induces by direct image a functor between the categories of sheaves

$$(p_i)_* : \mathrm{Sh}(C/U_i) \longrightarrow \mathrm{Sh}(C/X)$$

by the formula

$$(p_i)_*(F')(Y \to X) = F'\left(Y \times_X U_i \to U_i\right).$$

Similarly, we have a direct image functor

$$(p_{i,j})_* : \mathrm{Sh}(C/U_{i,j}) \longrightarrow \mathrm{Sh}(C/X).$$

By construction, we see that F is the equalizing presheaf of the two morphisms

$$\prod_i (p_i)_*(F_i) \rightrightarrows \prod_{i,j} (p_{i,j})_*\left((F_i)|_{U_{i,j}}\right)$$

where the first morphism is, on component (i,j), the projection onto $(p_i)_*(F_i)$ followed by the restriction morphism $(p_i)_*(F_i) \to (p_{i,j})_*\left((F_i)|_{U_{i,j}}\right)$ (induced by the morphism $U_{i,j} \to U_i$). The second morphism is, on component (i,j), the projection onto $(p_j)_*(F_j)$ followed by the restriction morphism $(p_j)_*(F_j) \to (p_{i,j})_*\left((F_i)|_{U_{i,j}}\right)$ and then the isomorphism

$$(p_{i,j})_*(\phi_{j,i}) : (p_{i,j})_*\left((F_j)|_{U_{i,j}}\right) \longrightarrow (p_{i,j})_*\left((F_i)|_{U_{i,j}}\right).$$

Now, all the presheaves $(p_i)_*(F_i)$ and $(p_{i,j})_*((F_i)|_{U_{i,j}})$ are the direct images of sheaves and therefore themselves sheaves. Thus, F is a limit of sheaves and therefore it is a sheaf. $\Diamond$

The preceding lemma shows that F is a sheaf (because we saw that $F|_{U_i}$ is isomorphic to F_i), and therefore defines an element of $\mathrm{Fais}(X)$ which satisfies condition (2) of Proposition 5.4.5. $\square$

6.1.2. **The stack of algebraic spaces.** Let Aff be the category of affine schemes, which we regard as a Grothendieck site equipped with the étale topology. For $X \in \mathrm{Aff}$, we consider $\mathrm{EspAlg}(X)$ the full sub-groupoid of $\mathrm{Fais}(X)$ consisting of algebraic spaces. The correspondence $X \mapsto \mathrm{EspAlg}(X)$ is a sub-prestack of the stack of sheaves $\mathrm{Fais}(X)$.

Proposition 6.1.4. *Let C be a Grothendieck site and F a stack on C. Let $F_0 \subset F$ be a full sub-prestack of F (i.e. a sub-presheaf in groupoids such that $F_0(X)$ is full in $F(X)$). We assume that the following property is satisfied (we then say that* being in F_0 is local*): if x is an object of $F(X)$ such that there exists a covering family $\{U_i \to X\}$ with $x|_{U_i}$ isomorphic to an object in $F_0(U_i)$ for any i, then $x \in F(X)$. Then F_0 is a stack.*

Proof. This is an easy application of Proposition 5.4.5. $\square$

The preceding proposition applies to $\mathrm{EspAlg} \subset \mathrm{Fais}$, and Proposition 4.1.2 implies that the prestack EspAlg is a stack.

Corollary 6.1.5. *The prestack* EspAlg *is a stack.*

We notice that the full sub-prestack $\mathrm{Sch} \subset \mathrm{EspAlg}$, of schemes, does not satisfy the condition of Proposition 6.1.4. We can in fact show that Sch is not a stack for the étale topology (i.e. that there exist descent data in Sch which are not effective in Sch, although they are effective in EspAlg).

6.1.3. **The stacks of quasi-coherent modules.** We now place ourselves in the case where $C = \mathrm{Aff}$ is the site of affine schemes equipped with the étale topology. We want to define a stack which to an affine scheme $\mathrm{Spec}\, A$ associates the groupoid $A--\mathrm{Mod}$ of A-modules. For a morphism of affine schemes $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ corresponding to a morphism of rings $A \to B$, we would put

$$f^* := B \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}.$$

However, with these definitions, for two morphisms $\mathrm{Spec}\, C \xrightarrow{g} \mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A$, the two functors $(f \circ g)^*$ and $g^* \circ f^*$ are only naturally isomorphic but not equal. In fact, for an A-module M, the C-module $M \otimes_A C$ is only naturally isomorphic to $(M \otimes_A B) \otimes_B C$ but not equal. Thus, $A \mapsto A\text{-Mod}$ *is not* a presheaf in groupoids.

There are two solutions to resolve this problem. We may generalize the notion of presheaf in groupoids to a notion of weak presheaf in groupoids for which we are given isomorphisms $\gamma_{f,g} : (f \circ g)^* \cong g^* \circ f^*$ satisfying a certain cocycle condition (analogous to the definition of weak morphisms between prestacks). This is the point of view that is often considered in the literature (see for example [4] and [1]). A second solution is to modify the groupoid of A-modules to a groupoid which is equivalent to it but is functorial in A (a general theorem, called the strictification, says that this is always possible). It is this second point of view that we will adopt.

For a commutative ring A, we therefore define a groupoid $\underline{A}$-Mod as follows. An object M consists of the following data:

- For any morphism of commutative rings $u : A \to B$, a B-module $M_u \in B$-Mod.
- For any commutative diagram of commutative rings
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[dl, "f"] \\ C & \end{tikzcd}$$
an isomorphism of C-modules
$$\gamma_f^M : M_u \otimes_B C \longrightarrow M_v$$
(simply denoted by γ_f when M is clear).

We furthermore demand that these data satisfy the following two conditions:

- $\gamma_{\mathrm{id}} = \mathrm{id}$.
- For any commutative diagram of commutative rings
$$\begin{tikzcd} & A \arrow[dl, "w"'] \arrow[d, "v"] \arrow[dr, "u"] & \\ D & C \arrow[l, "g"] & B \arrow[l, "f"] \end{tikzcd}$$
the following diagram commutes
$$\begin{tikzcd} (M_u \otimes_B C) \otimes_C D \arrow[r, "\gamma_f \otimes_C D"] \arrow[d, "\alpha"'] & M_v \otimes_C D \arrow[d, "\gamma_g"] \\ M_u \otimes_B D \arrow[r, "\gamma_{g \circ f}"'] & M_w, \end{tikzcd}$$
where $\alpha : (M_u \otimes_B C) \otimes_C D \cong M_u \otimes_B D$ is the natural isomorphism of simplification of tensor products (which sends $(m \otimes c \otimes d)$ to $(m \otimes g(c) \cdot d)$).

A morphism $\phi : M \to N$ in the groupoid $\underline{A}$-Mod consists of the data of an isomorphism
$$\phi_u : M_u \longrightarrow N_u$$
for any $u : A \to B$, such that for any commutative diagram
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[dl, "f"] \\ C & \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} M_u \otimes_B C \arrow[r, "\phi_u \otimes_B C"] \arrow[d, "\gamma_f"'] & N_u \otimes_B C \arrow[d, "\gamma_f"] \\ M_v \arrow[r, "\phi_f"'] & N_v. \end{tikzcd}$$

Now let $f : A \to B$ be a morphism of commutative rings. We define a functor

$$f^* : \underline{A}\text{-Mod} \longrightarrow \underline{B}\text{-Mod}$$

in the following manner. For an object M and morphism $u : B \to B'$, we put

$$f^*(M)_u := M_{u \circ f}.$$

Similarly, for a commutative diagram

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "v"'] & B' \arrow[dl, "g"] \\ B'' & \end{tikzcd}$$

we put

$$\gamma_g^{f^*(M)} = \gamma_g^M : f^*(M_u) \otimes_{B'} B'' = M_{u \circ f} \otimes_{B'} B'' \longrightarrow f^*(M)_v = M_{v \circ f}.$$

Thus defined, $A \mapsto \underline{A}$-Mod defines a presheaf in groupoids

$$\underline{\text{-Mod}} : \text{Aff}^{\text{op}} \longrightarrow \text{Gpd}$$

and therefore a prestack on Aff. This prestack is called the *prestack of modules* or still the *presheaf of quasi-coherent modules.*

Lemma 6.1.6. *The functor*

$$\underline{A}\text{-Mod} \longrightarrow A\text{-Mod}$$

which to an $\underline{A}$-module M associates the A-module M_{id}, is an equivalence of groupoids.

Proof. [...] □

Proposition 6.1.7. *The prestack $\underline{\text{-Mod}}$ is a stack.*

Proof. We check the conditions (1) and (2) of Proposition 5.4.5. [...]

Lemma 6.1.8. *For any commutative ring A, any fpqc covering family $\{A \to A_i\}$, and any A-module N, the diagram*

$$N \to \prod_i (N \otimes_A A_i) \rightrightarrows \prod_{i,j} (N \otimes_A A_{i,j})$$

identifies N to the equalizer of two morphisms

$$\prod_i (N \otimes_A A_i) \rightrightarrows \prod_{i,j} (N \otimes_A A_{i,j}) .$$

Proof of Lemma. We already see this lemma in the case where $N = A$ (see Lemma 3.2.8). The proof of the general case is the same. ◊

[...] □

6.1.4. **Two general criteria.** We return to the case of a general Grothendieck site C.

Proposition 6.1.9. *Let*

$$\begin{tikzcd} & F_2 \arrow[d] \\ F_1 \arrow[r] & F_0 \end{tikzcd}$$

be a diagram of presheaves in groupoids. If all F_i are stacks then $F_1 \times^h_{F_0} F_2$ is a stack.

Proof. We apply still Proposition 5.4.5. Denote by

$$p : F_1 \longrightarrow F_0, \qquad q : F_1 \longrightarrow F_0$$

two morphisms. Then the prestack $F_1 \times^h_{F_0} F_2$ is given by the functor which to $X \in C$ associates the groupoid $F_1(X) \times^h_{F_0(X)} F_2(X)$ whose objects are the triples (x, y, u), with x an object of $F_1(X)$, y and object of $F_2(X)$, and $u : p(x) \cong q(y)$ an isomorphism in $F_0(X)$. The morphisms between (x, y, u) and (x', y', u') are the pairs of morphisms $f : x \to x'$, $g : y \to y'$ such that $u' \circ p(f) = u \circ q(g)$.

Let $X \in C$ and $a = (x, y, u)$ and $b = (x', y', u')$ two objects of $F_1(X) \times^h_{F_0(X)} F_2(X)$. We then have an isomorphism of presheaves on C/X

$$\underline{\mathrm{Iso}}(a, b) \cong \underline{\mathrm{Iso}}(x, x') \times_{\underline{\mathrm{Iso}}(p(x),q(y'))} \underline{\mathrm{Iso}}(y, y').$$

Thus, $\underline{\mathrm{Iso}}(a, b)$ is a fiber product of sheaves and therefore is a sheaf.

Now let $X \in C$ and $\{U_i \to X\}$ is a covering family. Let $(x_i, y_i, u_i) \in \left(F_1 \times^h_{F_0} F_2\right)(U_i)$ and $\phi_{i,j} = (f_{i,j}, g_{i,j})$ a descent data for $F_1 \times^h_{F_0} F_2$. We see that $x_i \in F_1(U_i)$ and $f_{i,j}$ define a descent data for F_1 and therefore glue into an object $x \in F_1(X)$ equipped with isomorphisms $\alpha_i : x|_{U_i} \cong x_i$ with $f_{i,j} = \alpha_j \circ \alpha_i^{-1}$ on $U_{i,j}$. Similarly, y_i and $g_{i,j}$ glue into a $y \in F_2(X)$ equipped with isomorphisms $\beta_i : y|_{U_i} \cong y_i$ with $g_{i,j} = \beta_j \circ \beta_i^{-1}$ on $U_{i,j}$. Finally, the local morphisms

$$v_i : q(\beta_i^{-1}) \circ u_i \circ q(\alpha_i) : p(x)|_{U_i} \longrightarrow q(y)|_{U_i}$$

which glue into an isomorphism $v : p(x) \cong q(y)$. We then see that $(x, y, v) \in F_1(X) \times^h_{F_0(X)} F_2(X)$, and the isomorphisms

$$\gamma_i := (\alpha_i, \beta_i) : (x, y, v)|_{U_i} \cong (x_i, y_i, u_i)$$

satisfying $\phi_{i,j} = \gamma_j \circ \gamma_i^{-1}$ on $U_{i,j}$. □

Proposition 6.1.10. *Let F and G be two stacks on C. We consider the prestack* $\mathrm{Map}(F, G)$ *defined by*

$$\begin{array}{cccc} \mathrm{Map}(F, G) & : & C^{\mathrm{op}} & \longrightarrow & \mathrm{Gpd} \\ & & X & \longmapsto & \underline{\mathrm{Hom}}^{\mathrm{lax}}(F \times X, G). \end{array}$$

Then $\mathrm{Map}(F, G)$ *is a stack which we call the* stack of morphisms from F to G.

Proof. Exercise. □

Lecture 7. Stacks III

Throughout this lecture we fix an arbitrary Grothendieck site.

7.1. Local equivalences and associated stacks

We recall that we have a fully faithful functor

$$\mathrm{Pr}(C) \longrightarrow \mathrm{Ho}(\mathrm{PrCh}(C))$$

of category of presheaves of sets on C to the homotopy category of prestacks (see Proposition 5.4.2). This inclusion functor also has a left adjoint

$$\pi_0^{\mathrm{pr}} : \mathrm{Ho}(\mathrm{PrCh}(C)) \longrightarrow \mathrm{Pr}(C).$$

We also recall that for a prestack F, the presheaf $\pi_0^{\mathrm{pr}}(F)$ sends $X \in C$ to the set $\pi_0(F(X))$ of isomorphism classes of objects of $F(X)$. The Yoneda lemma (see Proposition 5.4.3) for prestacks therefore implies that there exist functorial isomorphisms

$$\pi_0^{\mathrm{pr}}(F)(X) \cong [X, F].$$

Proposition 7.1.1.

(1) *A presheaf F, considered as a prestack, is a stack if and only if a sheaf.*
(2) *The inclusion functor*

$$\mathrm{Sh}(C) \longrightarrow \mathrm{Ho}(\mathrm{Ch}(C))$$

is fully faithful and has a left adjoint

$$\pi_0 : \mathrm{Ho}(\mathrm{Ch}(C)) \longrightarrow \mathrm{Ho}(\mathrm{Sh}(C)).$$

Proof. (1) is deduced easily from Proposition 5.4.5. We define the functor π_0 to be the functor π_0^{pr} composed with the associated sheaf functor. □

From now on, we will always identify a presheaf with prestack it defines. Similarly, a sheaf will be identified with the stack it defines.

Definition 7.1.2. *Let F be a prestack, $X \in C$, and $s \in F(X)$ an object. We define a presheaf in groups on C/X as following*

$$\begin{array}{cccc} \pi_1^{\mathrm{pr}}(F,s) & : & (C/X)^{\mathrm{op}} & \longrightarrow & \mathrm{Grp} \\ & & (u : Y \to X) & \longmapsto & \mathrm{Aut}_{F(Y)}(u^*(s)). \end{array}$$

The sheaf associated to $\pi_1^{\mathrm{pr}}(F,s)$ is denoted

$$\pi_1(F,s) := a(\pi_1^{\mathrm{pr}}(F,s)).$$

We immediately notice that the presheaf $\pi_1^{\mathrm{pr}}(F,s)$ does not depend, up to isomorphism, on the isomorphism class of the object $s \in F(X)$ (Exercise: show that an isomorphism $\gamma : s \cong s'$) induces an isomorphism of presheaves $\pi_1^{\mathrm{pr}}(F,s) \cong \pi_1^{\mathrm{pr}}(F,s')$). Thus, up to isomorphism, the sheaf $\pi_1(F,s)$ does not depend on $s \in \pi_0(F(X)) = [X,F]$.

We also notice that the construction $(F,s) \mapsto \pi_1(F,s)$ is functorial in the pair (F,s). More precisely, if $f : F \to G$ is a weak morphism, we have a well-defined morphism that is functorial in f

$$\pi_1(f,s) : \pi_1(F,s) \longrightarrow \pi_1(G,f(s)).$$

Definition 7.1.3. *A weak morphism $f : F \to G$ of prestacks is a* local equivalence *if it satisfies the following two conditions:*

(1) The induced morphism $\pi_0(F) \longrightarrow \pi_0(G)$ is an isomorphism of sheaves.
(2) For any $X \in C$ and any object $s \in F(X)$, the induced morphism $\pi_1(F, s) \longrightarrow \pi_1(G, f(s))$ is an isomorphism of sheaves.

Exercise: check that the local equivalences are stable under composition. Also show that if f and g are two isomorphic weak morphisms, then f is a local equivalence if and only if g is.

Proposition 7.1.4. *Let $f : F \to G$ be a weak morphism of stacks. The following conditions are equivalent.*

(1) The morphism f is a local equivalence.
(2) For any $X \in C$, the induced morphism

$$f_X : F(X) \longrightarrow G(X)$$

is an equivalence of groupoids.
(3) The morphism f is an isomorphism in $\mathrm{Ho}(\mathrm{Ch}(C))$.

Proof. We show that (1) implies (2).

We start by showing that f_X is fully faithful. First of all, as F and G are stacks, for any $s \in F(X)$ the presheaf $\pi_1^{\mathrm{pr}}(F, s)$ and $\pi_1^{\mathrm{pr}}(G, f(s))$ are sheaves (see the first condition of Proposition 5.4.5). Thus, by hypothesis on f the functor f_X induces an isomorphism $\mathrm{Aut}_{F(X)}(s) \cong \mathrm{Aut}_{G(X)}(f(s))$. As $F(X)$ and $G(X)$ are groupoids this implies that

$$\mathrm{Iso}_{F(X)}(s, t) \longrightarrow \mathrm{Iso}_{G(X)}(f(s), f(t))$$

is a bijection provided $\mathrm{Iso}(s, t)$ is nonempty. (Exercise: check this assertion). Thus, f is fully faithful if and only if

$$(\mathrm{Iso}(f(s), f(t)) \neq \varnothing) \Rightarrow (\mathrm{Iso}(s, t) \neq \varnothing)$$

for any $s, t \in F(X)$.

Therefore let $v : f(s) \cong f(t)$ be an isomorphism in $G(X)$, and we show that s and t are isomorphic in $F(X)$. As $\pi_0(F) \to \pi_0(G)$ is an isomorphism, there exists a covering family $\{U_i \to X\}$ and isomorphisms $u_i : s|_{U_i} \cong t|_{U_i}$ in $F(U_i)$. From what we saw above, we can also choose u_i such that $f(u_i) = v|_{U_i}$. On $U_{i,j}$ we have

$$f(u_i)|_{U_{i,j}} = v|_{U_{i,j}} = f(u_j)|_{U_{i,j}}.$$

Again from what we saw above, this implies that $(u_i)|_{U_{i,j}} = (u_j)|_{U_{i,j}}$. Since ı$\mathrm{Iso}(s, t)$ is a sheaf (because F is a stack), the local isomorphisms u_i glue into an isomorphism $u : s \cong t$. This finishes showing that f_X is fully faithful for any $X \in C$.

Now let $t \in G(X)$. By hypothesis, there exists a covering family $\{U_i \to X\}$, objects $s_i \in F(U_i)$, and isomorphisms $u_i : f(s_i) \cong t|_{U_i}$. Consider for any i and j the isomorphism

$$(u_j \circ u_i^{-1})|_{U_{i,j}} : f(s_i)|_{U_{i,j}} \cong f(s_j)|_{U_{i,j}}.$$

By full faithfulness of f (by what we have already seen) there exist isomorphisms

$$\phi_{i,j} : (s_i)|_{U_{i,j}} \cong (s_j)|_{U_{i,j}}$$

in $F(U_{i,j})$ such that $f(\phi_{i,j}) = (u_j \circ u_i^{-1})|_{U_{i,j}}$. Still by full faithfulness of f, we see that s_i and $\phi_{i,j}$ define a descent data for F, and thus Proposition 5.4.5 implies the existence of $s \in F(X)$ that they glue. By construction, we see that $f(s)|_{U_i}$ is naturally isomorphic to $t|_{U_i}$, and that the local isomorphisms glue into an isomorphism between $f(s)$ and t.

Finally, (2) implies (3) by Proposition 5.2.5, and (3) implies (1) by functoriality of the constructions $F \mapsto \pi_0(F)$ and $(F, s) \mapsto \pi_1(F, s)$. $\square$

Definition 7.1.5. *Let F be a prestack. An* associated stack to F *is the data of a stack $a(F)$ and a local equivalence $F \to a(F)$.*

We cite the following theorem without proof.

Theorem 7.1.6.

(1) For any prestack F an associated stack $F \to a(F)$ exists.

(2) If $F \to a(F)$ is an associated stack, then for any stack G the induced morphism

$$[a(F), G] \longrightarrow [F, G]$$

is bijective.

(3) For any diagram of prestacks

$$F_1 \longleftarrow F_0 \longrightarrow F_2$$

there exists a natural isomorphism in $\mathrm{Ho}(\mathrm{Ch}(C))$

$$a\left(F_1 \times^h_{F_0} F_2\right) \cong a(F_1) \times^h_{a(F_0)} a(F_2).$$

$\square$

We deduce the following corollary from the theorem.

Corollary 7.1.7. *The inclusion functor*

$$\mathrm{Ho}(\mathrm{Ch}(C)) \longrightarrow \mathrm{Ho}(\mathrm{PrCh}(C))$$

admits a left adjoint

$$a : \mathrm{Ho}(\mathrm{PrCh}(C)) \longrightarrow \mathrm{Ho}(\mathrm{Ch}(C))$$

which to F associates its associated stack $a(F)$. $\square$

We notice that the stack associated to a presheaf F is simply given by its associated sheaf (since seen directly from Definition 7.1.5). Furthermore, for any prestack F and $i : F \to a(F)$ its associated stack we have

$$\pi_1(F, s) \cong \pi_1^{\mathrm{pr}}(a(F), i(s)).$$

7.2. Quotient stacks

We now use the notion of associated stack to construct new examples of stacks: the quotient stacks.

We start by considering a sheaf of groups G on C. We construct a prestack $K(G, 1)$ by putting

$$\begin{array}{rcccc} K(G,1) & : & C^{\mathrm{op}} & \longrightarrow & \mathrm{Gpd} \\ & & X & \longmapsto & B(G(X)), \end{array}$$

where we recall that for a group H, $B(H)$ is the group having a unique object denoted $*$ and with $\mathrm{Aut}_{B(H)}(*) = H$. Equivalently, for any groupoid A, the functors $B(H) \to A$ are in bijection with the pairs (x, u), where x is an object of A and $u : H \to \mathrm{Aut}_A(x)$ is a morphism of groups.

We propose to describe the stack associated to $K(G, 1)$ which we will denote BG (this is really confusing but it is a standard notation in the literature).

We recall that a G-torsor is a sheaf E equipped with an action (from the left) of G (i.e. a morphism $\mu : G \times E \to E$ which satisfies obvious axioms) which satisfies the two following conditions:

(1) For any object $X \in C$, there exists a covering $\{U_i \to X\}$ such that each $E(U_i)$ is nonempty (in other words the morphism $E \to *$ is an epimorphism of sheaves).
(2) The morphism
$$\mu \times \mathrm{id} : G \times E \longrightarrow E \times E$$
is an isomorphism.

The G-torsors on C form a category G-Tors(C), for which the morphisms are simply the morphisms of sheaves compatible with the action of G. We then note the following facts (Exercise: show them).

(1) For any $X \in C$ there exists a restriction functor
$$G\text{-Tors}(C) \longrightarrow G|_X\text{-Tors}(C),$$
where $G|_X$ is the sheaf of groups restricted to the site C/X.
(2) If E is a G-torsor, then for any $X \in C$, there exists a covering family $\{U_i \to X\}$ such that each restricted sheaf $E|_{U_i}$ is isomorphic, as $G|_{U_i}$-torsors, to $G|_{U_i}$ equipped with its action by left translation.
(3) Let E be a G-sheaf such that for any $X \in C$, there exists a covering family $\{U_i \to X\}$ such that each restricted sheaf $E|_{U_i}$ is isomorphic, as $G|_{U_i}$-torsors, to $G|_{U_i}$ equipped with its action by left translation. Then E is a G-torsor.
(4) Any morphism of G-torsors is an isomorphism.
(5) If G is the trivial G-torsor (i.e. G equipped with its action by left translation), then for any $X \in C$ there exists a functorial isomorphism in X
$$G(X) \cong \mathrm{Aut}_{G|_X\text{-Tors}(C/X)}(G|_X).$$

We consider the prestack of G-torsors defined as follows.
$$\begin{array}{cccc} G\text{-Tors} & : & C^{\mathrm{op}} & \longrightarrow & \mathrm{Gpd} \\ & & X & \longmapsto & G\text{-Tors}(X) := G|_X\text{-Tors}(C/X). \end{array}$$
There exists a morphism of prestacks
$$K(G,1) \longrightarrow G\text{-Tors}$$
which on $X \in C$ is given by the trivial torsor $G|_X$ and the natural isomorphism
$$G(X) \cong \mathrm{Aut}_{G|_X\text{-Tors}(C/X)}(G|_X).$$

Theorem 7.2.1. *The morphism*
$$K(G,1) \longrightarrow G\text{-Tors}$$
is an associated stack.

Proof. We must show on one hand that G-Tors is a stack, and on the other hand that the morphism $K(G,1) \to G$-Tors is a local equivalence.

Lemma 7.2.2. *The prestack* G-Tors *is a stack*

Proof of Lemma. We start by showing that condition (1) of Proposition 5.4.5 is satisfied. Let $X \in C$, and E and F two objects of G-Tors(X), that is two $G|_X$-torsors on C/X. The sheaf $\underline{\mathrm{Iso}}_G(E,F)$ of isomorphisms of $G|_X$-torsors is naturally identified to the equalizer of two morphisms
$$\underline{\mathrm{Iso}}(E,F) \rightrightarrows \underline{\mathrm{Iso}}(G \times E, F).$$

The first of these two morphisms sends an isomorphism of sheaves $f : E \to F$ to the composite

$$G \times E \xrightarrow{\mu} E \xrightarrow{f} F,$$

and the second sends f to the composite

$$G \times E \xrightarrow{\mathrm{id} \times f} G \times F \xrightarrow{\mu} F.$$

Since the prestack of sheaves is a stack we already know that $\underline{\mathrm{Iso}}(E,F)$ and $\underline{\mathrm{Iso}}(G \times E, F)$ are sheaves on C/X. Therefore, $\underline{\mathrm{Iso}}_G(E,F)$ is a limit of sheaves and is therefore a sheaf.

We now show that condition (2) of Proposition 5.4.5 is satisfied. For that let $X \in C$ and $\{U_i \to X\}$ a covering family. We are given a descent data $E_i \in G\text{-Tors}(U_i)$ and $\phi_{i,j} : (E_i)|_{U_{i,j}} \cong (E_j)|_{U_{i,j}}$ for G-Tors. By forgetting the action of G on E_i we find a descent data for the stack of sheaves. Therefore, there exist a sheaf E on C/X and isomorphisms of sheaves on C/U_i

$$\alpha_i : E|_{U_i} \cong E_i$$

such that

$$\phi_{i,j} = (\alpha_j)|_{U_{i,j}} \circ (\alpha_i)^{-1}|_{U_{i,j}}.$$

We define morphisms

$$a_i : G|_{U_i} \times E|_{U_i} \longrightarrow E|_{U_i}$$

by demanding that the following diagrams commute

$$\begin{tikzcd}
G|_{U_i} \times E|_{U_i} \arrow[r, "a_i"] \arrow[d, "\mathrm{id} \times \alpha_i"'] & E|_{U_i} \arrow[d, "\alpha_i"] \\
G|_{U_i} \times E_i \arrow[r, "\mu_i"'] & E_i,
\end{tikzcd}$$

where μ_i are the morphisms of the action of G. We then notice that a_i glue into a unique morphism of sheaves on C/X

$$a : G|_X \times E \longrightarrow E.$$

This defines a structure of $G|_X$-sheaf on E so that the isomorphisms $\alpha_i : E|_{U_i} \cong E_i$ are isomorphisms of $G|_{U_i}$-sheaves. Thus, E is a $G|_X$-sheaf locally isomorphic to a G-torsor, and is therefore a $G|_X$-torsor itself. It therefore defines an object $E \in G\text{-Tors}(X)$, which with which the isomorphism α_i glue into the descent data. This ends the proof of the lemma. $\Diamond$

Lemma 7.2.3. *The natural morphism*

$$K(G,1) \longrightarrow G\text{-Tors}$$

is a local equivalence.

Proof of Lemma. For any $X \in C$ the morphism

$$K(G,1)(X) \longrightarrow G\text{-Tors}(X)$$

identifies $K(G,1)(X)$ with the full sub-groupoid of $G\text{-Tors}(X)$ which only contains the trivial $G|_X$-torsor $G|_X$. This implies that the induced morphism

$$\pi_1(K(G,1),*) = G|_X \longrightarrow \pi_1\left(G\text{-Tors}, G|_X\right)$$

is an isomorphism. Moreover, since all objects of G-Tors(X) are locally isomorphic to each other (because they are all locally isomorphic to the trivial torsor), we have $\pi_0(G\text{-Tors}) \cong *$. Thus, the induced morphism

$$\pi_0(K(G,1)) = * \longrightarrow \pi_0(G\text{-Tors}) \cong *$$

can only be an isomorphism. $\Diamond$

Lemma 7.2.2 and Lemma 7.2.3 prove the theorem. $\square$

We now propose to give a generalization of the stack BG for which we add an action of G on a given sheaf E (the case of BG is recovered for $E = *$).

We therefore fix a sheaf E equipped with an action of G. We define a prestack $K(G,E,1)$ as follows. For $X \in C$, the set of objects of the groupoid $K(G,E,1)(X)$ is the set $E(X)$. A morphism from x to y in $K(G,E,1)(X)$ is the data of a $g \in G(X)$ such that $g \cdot x = y$. The composition of moprhisms is then defined by multiplication in $G(X)$. For $u : Y \to X$ a morphism in C, we have a functor

$$u^* : K(G,E,1)(X) \longrightarrow K(G,E,1)(Y).$$

This functor is just $u^* : E(X) \to E(Y)$ on the set of objects and is induced by $u^* : G(X) \to G(Y)$ on the set of morphisms.

Definition 7.2.4. *The stack associated to $K(G,E,1)$ is called the* quotient stack of E by G. *It is denoted $[E/G]$.*

Exercise: show that $\pi_0^{\mathrm{pr}}(K(G,E,1))$ is the quotient presheaf of E by G. Furthermore, show that if G acts fixed-point free on E then the natural projection $K(G,E,1) \to \pi_0^{\mathrm{pr}}(K(G,E,1))$ is an equivalence (i.e. an isomorphism in $\mathrm{Ho}(\mathrm{PrCh}(C))$. Thereof deduce that in this case the stack $[E/G]$ is identified with the quotient sheaf of E by G.

We end with a description of the stack associated to $K(G,E,1)$. So given a sheaf E equipped with an action of G. We define a prestack $B(G,E)$ as follows. For $X \in C$, the objects of the groupoid $B(G,E)(X)$ are the pairs (E_0,u), where $E_0 \in G\text{-Tors}(X)$ is a G-torsor on X, and $u : E_0 \to E|_X$ is a morphism of sheaves on C/X compatible with the action of $G|_X$. A morphism $(E_0,u) \to (E_0',u')$ is the data of a morphism $f : E_0 \to E_0'$ of G-torsors on X, such that $u' \circ f = u$. Moreover, for $f : Y \to X$ a morphism in C, the functor

$$f^* : B(G,E)(X) \longrightarrow B(G,E)(Y)$$

sends by definition (E_0,u) to $((E_0)|_Y, u|_Y)$, where the morphism $u|_Y$ is the morphism induced by restriction of C/X to C/Y.

There exists a morphism of prestacks

$$j : K(G,E,1) \longrightarrow B(G,E)$$

defined as follows. For $X \in C$ the functor

$$j_X : K(G,E,1)(X) \longrightarrow B(G,E)(X)$$

an object $x \in E(X)$ to the unique G-invariant morphism of sheaves on C/X

$$G|_X \longrightarrow E|_X$$

(which sends the element the neutral of $G|_X$ to the point x). By considering $G|_X$ as the trivial G-torsor on X, this gives an object of $B(G,E)(X)$. This defines the functor j_X at the level of objects. We leave as an exercise the definition of j_X at the level of morphisms (we use that $G(X)$ is identified with the automorphism group of trivial G-torsors on X).

Theorem 7.2.5. *The morphism*

$$K(G, E, 1) \longrightarrow B(G, E)$$

is an associated stack. Thus, we have

$$[E/G] \cong B(G, E).$$

Proof. This is essentially the same as that for Theorem 7.2.1. We leave it as an exercise. $\square$

Lecture 8. Stacks IV

In this lecture we place ourselves in the Grothendieck site Aff of affine schemes equipped with the étale topology. We saw in the previous lecture that there exist a homotopy category of prestacks Ho(PrCh(Aff)) and fully faithful functors

$$\mathrm{Sh}(\mathrm{Aff}) \hookrightarrow \mathrm{Ho}(\mathrm{Ch}(\mathrm{Aff}) \hookrightarrow \mathrm{Ho}(\mathrm{PrCh}(\mathrm{Aff})).$$

With these functors we will thereafter identify the categories Sh(Aff) and Ho(Ch(Aff)) with the full subcategories of Ho(PrCh(Aff)). Thus, the subcategories of Sh(Aff) consisting of algebraic spaces, of schemes, and of affine schemes will also be identified with their essential images in Ho(Ch(Aff)). Thus, we say that a stack $F \in \mathrm{Ho}(\mathrm{Ch}(\mathrm{Aff}))$ is an algebraic space, a scheme, or an affine scheme, if it is isomorphic in Ho(Ch(Aff)) to the image of an algebraic space, a scheme, or an affine scheme, via the functor $\mathrm{Sh}(\mathrm{Aff}) \hookrightarrow \mathrm{Ho}(\mathrm{Ch}(\mathrm{Aff}))$.

In general, we will use the phrase a *morphism of stacks* (or still *morphism of prestacks*, or even simply *morphism*) to mean a *morphism* in Ho(Ch(Aff)) (or even in Ho(PrCh(Aff))). Thus, unless otherwise specified, a *morphism* will be an isomorphism class of weak morphisms (see Theorem 5.2.4).

8.1. Algebraic stacks

Let $f : F \to G$ be a morphism in Ho(Ch(Aff)), $X \in \mathrm{Aff}$, and $x : X \to G$ a morphism. The *fiber of f at x* is by definition the stack $F \times^h_G X$. Assume that the morphism $x : X \to G$ corresponds, via the isomorphism $[X, G] \cong \pi_0(G(X))$, to an object $x \in G(X)$ (well-defined up to isomorphism). Then the stack $F \times^h_G X$ is described as follows. For $Y \in \mathrm{Aff}$, an object of $(F \times^h_G X)(Y)$ is given by a triple (z, u, α), where z is an object in $F(Y)$, $u : Y \to X$ is a morphism in Aff, and $\alpha : f(z) \cong u^*(x)$ is an isomorphism in $G(Y)$. A morphism $(z, u, \alpha) \to (z', u, \alpha')$ is given by a morphism $z \to z'$ in $F(Y)$, such that the following diagram commutes

$$\begin{tikzcd} f(z) \arrow[r] \arrow[d] & f(z') \arrow[dl] \\ u^*(x). & \end{tikzcd}$$

There is no morphism $(z, u, \alpha) \to (z', u, \alpha')$ for $u \neq u'$.

Definition 8.1.1.

(1) *A morphism $f : F \to G$ of stacks is* representable *(resp.* representable by a scheme*, resp.* representable by an affine scheme*) if for any $X \in \mathrm{Aff}$ and for any $x : X \to G$ the fiber of f at x, $F \times^h_G X$ is an algebraic space (resp. a scheme, resp. an affine scheme).*

(2) *A representable morphism $f : F \to G$ of stacks is* smooth *(resp.* étale*, resp.* quasi-compact*, resp.* locally finitely presented*, resp.* a closed immersion*, resp.* an open immersion *...) if for any $X \in \mathrm{Aff}$ the induced morphism of algebraic spaces is smooth (resp. étale, resp. quasi-compact, resp. locally finitely presented, resp. a closed immersion, resp. an open immersion ...) in the sense of algebraic spaces.*

We notice that the representable morphisms (resp. smooth representables, étale ...) are stable under composition (Exercise). They are also stable under base change in the sense of homotopy fiber products.

An important property of representable morphisms is the following (it is useful to show that a morphism is not representable).

Proposition 8.1.2. *If $f : F \to G$ is a representable morphism, then for any $X \in \mathrm{Aff}$ and $x \in [X, F]$, then induced morphism*
$$f : \pi_1(F, x) \longrightarrow \pi_1(G, f(x))$$
is a monomorphism of sheaves.

Proof. If f is representable, then for any $Y \to X$ the groupoid $(F \times^h_G X)(Y)$ is equivalent to a set. In particular, the automorphism groups of objects in $(F \times^h_G X)(Y)$ are trivial.

Let $X \in \mathrm{Aff}$ and $x \in [X, F]$. Assume that there exist $u : Y \to X$ and $a \in \pi_1(F, x)(Y) = \mathrm{Aut}_{F(Y)}(u^*(x))$ a nontrivial automorphism in the kernel of the morphism
$$\pi_1(F, x)(Y) \longrightarrow \pi_1(G, f(x)).$$
We consider the object $(u^*(x), u, \mathrm{id})$, which is an object in $(F \times^h_G X)(Y)$ (for the morphism $f(x) \in [X, G]$). Furthermore, the automorphism a of $u^*(x)$ induced by definition a nontrivial automorphism of the object $(u^*(x), u, \mathrm{id})$ in the groupoid $(F \times^h_G X)(Y)$. This implies that f is not representable. □

Before giving the definition of algebraic stacks we need the following definition.

Definition 8.1.3. *A morphism of stacks $f : F \to G$ is an* epimorphism *if the induced morphism $\pi_0(F) \to \pi_0(G)$ is an epimorphism of sheaves.*

We notice that $f : F \to G$ is an epimorphism is an epimorphism if and only if for any $X \in \mathrm{Aff}$ and any object $x \in G(X)$ there exist a covering family $\{U_i \to X\}$ and objects $y_i \in F(U_i)$ such that $x|_{U_i}$ are isomorphic to $f(y_i)$ for any i (i.e. any object of G is locally in the essential image of f).

Definition 8.1.4. *A stack F is* algebraic* *if it satisfies the following two conditions.*

(1) The diagonal morphism
$$F \longrightarrow F \times F$$
is representable.

(2) There exist affine schemes $\{U_i\}$ and a smooth and representable epimorphism
$$p : U = \coprod_i U_i \longrightarrow F.$$
Such a morphism p is called an atlas *for F.*

We will see that condition (2) implies condition (1). We present, however, the notion of algebraic stacks in this form which is what we will encounter in the literature. Moreover, it is often a custom to replace condition (1) by the following stronger condition.

(1') The diagonal morphism
$$F \longrightarrow F \times F$$
is representable, quasi-compact and separated.

As far as we are concerned, we will primarily focus on algebraic stacks F whose diagonal morphism $F \to F \times F$ is representable and affine.

We end this section with the following finiteness notions.

* In the literature we also find the terminology of *Artin stack*.

Definition 8.1.5. *Let F be an algebraic stack.*

(1) *We say that F is* quasi-compact *if there exists an atlas*
$$p : U \longrightarrow F$$
with U an affine scheme, and if furthermore the diagonal morphism $F \to F \times F$ is quasi-compact.

(2) *Let $X \in \mathrm{Aff}$ and $F \to X$ a morphism. We say that F is* locally finitely presented on X *if there exists an atlas*
$$p : U = \coprod_i U_i \longrightarrow F$$
with each U_i finitely presented on X.

(3) *Let $X \in \mathrm{Aff}$ and $F \to X$ a morphism. We say that the stack F is* finitely presented on X *if it is locally finitely presented on X and quasi-compact.*

(4) *We say that F is* locally noetherian *if there exists an atlas*
$$p : U = \coprod_i U_i \longrightarrow F$$
with each U_i noetherian.

(5) *We say that the stack F is* noetherian *if it is locally noetherian and quasi-compact.*

8.2. Some elementary properties

We start with a proposition which makes Definition 8.1.4 precise.

Proposition 8.2.1. *Let F be a stack. The following three conditions are equivalent.*

(1) *The diagonal morphism $F \to F \times F$ is representable.*

(2) *For any affine scheme X and any $x, y \in F(X)$, the sheaf $\underline{\mathrm{Iso}}(x, y)$ is representable by an algebraic space.*

(3) *For any affine scheme X and Y, and any morphisms $X, Y \to F$ the stack $X \times^h_F Y$ is representable.*

Proof. To say that $F \to F \times F$ is representable is equivalent to saying that for any affine scheme X and any morphism $(x, y) : X \to F \times F$, the stack $X \times^h_{F \times F} F$ is representable. We see that the stack $X \times^h_{F\times F} F$ is isomorphic to the sheaf $\underline{\mathrm{Iso}}(x, y)$. Thus (1) is equivalent to (2).

For any affine schemes X and Y, and any morphisms $X, Y \to F$, we have
$$X \times^h_F Y \cong (X \times Y) \times^h_{F\times F} F.$$
Thus, (1)⇒(3). Conversely, any morphism $X \times F \times F$ factorizes as
$$X \longrightarrow X \times X \longrightarrow F \times F.$$
Thus, we have
$$X \times^h_{F\times F} F \cong X \times^h_{X\times X} \left(X \times^h_F X\right),$$
and therefore (3)⇒(1). □

Proposition 8.2.2. *Let F be a stack. The following two conditions are equivalent.*

(1) *The stack F is algebraic.*

(2) There exist affine schemes $\{U_i\}$ and a representable smooth epimorphism

$$p : U = \coprod_i U_i \longrightarrow F.$$

Proof. By definition (1)⇒(2). We show that (2)⇒(1). For that we use Proposition 8.2.1 (3).

Let X and Y be two affine schemes, and $x : X \to F$ and $y : Y \to F$ two points. We consider $X \times^h_F Y \to X$, which is a morphism of sheaves (Exercise: check that the stack $X \times^h_F Y$ is equivalent to a sheaf), and we want to show that it is an algebraic space. We then apply Proposition 4.1.2 which allows us to pass to an étale covering of X. Thus, upon replacing X by an étale covering we may assume that the morphism $X \to F$ factorizes through $x' : X \to U$. We then have

$$X \times^h_F Y \cong X \times^h_U \left(U \times^h_F Y\right).$$

Since by hypothesis $U \times^h_F Y$ is an algebraic space we see that $X \times^h_F Y$ is an algebraic space. □

We now give some general criteria to construct algebraic stacks.

Proposition 8.2.3.

(1) An algebraic space is an algebraic stack.

(2) Let $f : F \to G$ be a morphism of stacks with G an algebraic stack. We assume that there exists an atlas

$$\{U_i \longrightarrow G\}$$

such that each $F \times^h_G U_i$ is an algebraic stack. Then F is an algebraic stack.

(3) Let $F_1 \to F_0 \leftarrow F_2$ be a diagram of algebraic stacks. Then $F_1 \times^h_{F_0} F_2$ is an algebraic stack.

(4) Let $f : F \to G$ be a representable morphism. If G is algebraic then so is F.

Proof. Exercise (inspired by the analogous facts for algebraic spaces). □

We end this section with the construction of algebraic stacks as quotient stacks.

Proposition 8.2.4. *Let G be an algebraic space smooth over $* = \operatorname{Spec} \mathbb{Z}$. Let X be an algebraic space on which G acts. Then the quotient stack $[X/G]$ is an algebraic stack.*

Proof. We start with a general lemma on quotient stacks. For that we recall that there exist natural morphisms

$$X \longrightarrow K(G, X, 1) \longrightarrow [X/G].$$

Lemma 8.2.5. *Let $Y \to [X/G]$ be a morphism with Y an affine scheme, corresponding to a diagram*

$$Y \longleftarrow E \longrightarrow X$$

with $E \to Y$ a $G|_Y$-torsor. Then the stack $Y \times_{[X/G]} X$ is identified with the sheaf E.

Proof. Let $Z \in \mathrm{Aff}$. The groupoid $(Y \times_{[X/G]} X)(Z)$ is identified with the set of pairs (a, u), with $a : Z \to Y$ and $u : G|_Z \cong E|_Z$ an isomorphism from $E|_Z$ to the trivial $G|_Z$-torsor (Exercise: check it). The image of the unit of $G(Z)$ under the isomorphism u gives a point in $E(Z)$. As Z varies in the affine schemes this defines a morphism $Y \times_{[X/G]} X \to E$ which we see is an isomorphism (Exercise). □

We return to the proof of the proposition. For that we show that the morphism

$$X \longrightarrow [X/G]$$

is a representable and smooth epimorphism. This implies that $[X/G]$ is an algebraic stack, because by composing with an atlas for X we find a smooth epimorphism $\coprod U_i \to [X/G]$ with U_i affine schemes, and we can apply Proposition 8.2.2.

The fact that $X \to [X/G]$ is an epimorphism is general. In fact, the morphism induced on the sheaves π_0 is the natural projection

$$X \longrightarrow \pi_0([X/G]) = X/G,$$

where X/G is the quotient sheaf of X by G. This is therefore an epimorphism of sheaves.

Now let $Y \in \mathrm{Aff}$ and $Y \to [X/G]$ a morphism. By Lemma 8.2.5 we find that $Y \times_{[X/G]} X \to Y$ is a $G|_Y$-torsor. Thus, through an étale covering of Y this morphism is isomorphic to the natural projection

$$Y \times G \longrightarrow Y.$$

It is therefore representable and smooth (because G is smooth). $\square$

Proposition 8.2.4 also has the following relative version. Given an affine scheme $S = \operatorname{Spec} k$, X and algebraic space over S (i.e. equipped with a morphism $X \to S$), and G is a group algebraic space over S (i.e. a group object in the algebraic spaces over S) which acts on X. We may then define the quotient stack $[X/G]$ which is a stack equipped with a morphism to S. Then the conclusion of Proposition 8.2.4 remains valid, and the proof is the same: the stack $[X/G]$ is algebraic.

Corollary 8.2.6. *Let G be a smooth group algebraic space. Then the stack BG is an algebraic stack.*

Proof. Indeed, we have $BG = [*/G]$. $\square$

Corollary 8.2.6 shows in particular that BGL_n is an algebraic stack (we recall that GL_n is the group scheme $A \mapsto \mathrm{GL}_n(A)$, and is smooth because it is an open subscheme of $\mathbb{A}^{n^2}$).

8.3. Two examples

We will start by defining, for any integer $n > 0$, a sub-prestack $\underline{\mathrm{Vect}}_n$ of the stack of modules $\underline{\text{-Mod}}$ (see Lecture 6). For that, we recall that an A-module M is projective if there exists an A-module N such that $M \oplus N$ is a free A-module. We say furthermore that M is projective of rank n (we also say *is a vector bundle of rank n*) if for any field K and any morphism $A \to K$, we have $\dim_K M \otimes_A K = n$. Similarly, we say that an $\underline{A}$-module M is projective of rank n if the A-module M_{id} is.

By definition $\underline{\mathrm{Vect}}_n(A)$ is the full sub-groupoid on $\underline{\text{-Mod}}(A)$ of projective A-modules of rank n. It is clear that for a morphism of rings $A \to B$ the functor

$$\underline{\text{-Mod}}(A) \longrightarrow \underline{\text{-Mod}}(B)$$

sends the sub-groupoid $\underline{\mathrm{Vect}}_n(A)$ into $\underline{\mathrm{Vect}}_n(B)$. Thus, $\underline{\mathrm{Vect}}_n$ is a full sub-stack of $\underline{\text{-Mod}}$.

Proposition 8.3.1. *The prestack $\underline{\mathrm{Vect}}_n$ is an algebraic stack.*

Proof. [...] $\square$

Let A be a commutative ring. Recall that an A-algebra (associative and unitary) is given by an A-module R equipped with two morphisms of A-modules

$$\mu : R \times_A R \longrightarrow R,$$
$$e : A \longrightarrow R$$

which satisfy the evident associative and unitary axioms.

Similarly, an $\underline{A}$-algebra R consists of the following data.

- For any morphism of commutative rings $u : A \to B$, a B-algebra $R_u \in B\text{-Alg}$ (associative and unitary).
- For any commutative diagram of commutative rings

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[dl, "f"] \\ C & \end{tikzcd}$$

  an isomorphism of C-algebras

$$\gamma_f^R : R_u \otimes_B C \longrightarrow R_v$$

  (simply denoted by γ_f when R is clear).

We furthermore demand that these data satisfy the following two conditions:

- $\gamma_{\mathrm{id}} = \mathrm{id}$.
- For any commutative diagram of commutative rings

$$\begin{tikzcd} & A \arrow[dl, "w"'] \arrow[d, "v"] \arrow[dr, "u"] & \\ D & C \arrow[l, "g"] & B \arrow[l, "f"] \end{tikzcd}$$

  the following diagram commutes

$$\begin{tikzcd} (R_u \otimes_B C) \otimes_C D \arrow[r, "\gamma_f \otimes_C D"] \arrow[d, "\alpha"'] & R_v \otimes_C D \arrow[d, "\gamma_g"] \\ R_u \otimes_B D \arrow[r, "\gamma_{g\circ f}"'] & R_w, \end{tikzcd}$$

  where $\alpha : (R_u \otimes_B C) \otimes_C D \cong R_u \otimes_B D$ is the natural isomorphism of simplification of tensor products (which sends $(m \otimes c \otimes d)$ to $(m \otimes g(c) \cdot d)$).

A morphism $\phi : R \to R'$ in $\underline{A}$-algebras consists of the data of an isomorphism

$$\phi_u : R_u \longrightarrow R'_u$$

for any $u : A \to B$, such that for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[dl, "f"] \\ C & \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} R_u \otimes_B C \arrow[r, "\phi_u \otimes_B C"] \arrow[d, "\gamma_f"'] & R'_u \otimes_B C \arrow[d, "\gamma_f"] \\ R_v \arrow[r, "\phi_f"'] & R'_v. \end{tikzcd}$$

Thus defined, the $\underline{A}$-algebras form a groupoid denoted by $\underline{A}$-Alg. Furthermore, just as for the case of $\underline{A}$-modules, we have for any morphism of commutative rings $A \to B$ a functor

$$\underline{A}\text{-Alg} \longrightarrow \underline{B}\text{-Alg}$$

which makes $A \mapsto \underline{A}$-Alg a prestack on Aff. This prestack is denoted $\underline{\mathrm{Ass}}$ (for *ass*ociative algebras).

Lemma 8.3.2. *The prestack* $\underline{\mathrm{Ass}}$ *is a stack.*

Proof. [...] □

We have a forgetful morphism of the algebra structure

$$f : \underline{\mathrm{Ass}} \longrightarrow \underline{\text{-Mod}}.$$

We then define the substack $\underline{\mathrm{Ass}}^n$ of $\underline{\mathrm{Ass}}$ by

$$\underline{\mathrm{Ass}}^n := \underline{\mathrm{Ass}} \times^h_{\underline{\text{-Mod}}} \mathrm{Vect}_n\, .$$

In other words, the substack $\underline{\mathrm{Ass}}^n \subset \underline{\mathrm{Ass}}$ consists of the $\underline{A}$-algebras whose underlying $\underline{A}$-modules is projective of rank n.

Theorem 8.3.3. *For any $n > 0$ the stack $\underline{\mathrm{Ass}}^n$ is an algebraic stack.*

Proof. [...] □

An important corollary of the proof of Theorem 8.3.3 is the following.

Corollary 8.3.4. *The stack $\underline{\mathrm{Ass}}^n$ is finitely presented over* $\mathrm{Spec}\,\mathbb{Z}$ *(and therefore noetherian).*

Proof. In the proof of the theorem we saw that an atlas of $\underline{\mathrm{Ass}}^n$ is given by

$$\underline{\mathrm{Ass}}^n \times^h_{\underline{\mathrm{Vect}}_n} * \longrightarrow \underline{\mathrm{Ass}}^n,$$

and that $\underline{\mathrm{Ass}}^n \times^h_{\underline{\mathrm{Vect}}_n} *$ is furthermore the spectrum of a $\mathbb{Z}$-algebra of finite type. Moreover, it is easy to see that the diagonal morphism $\underline{\mathrm{Ass}}^n \to \underline{\mathrm{Ass}}^n \times \underline{\mathrm{Ass}}^n$ is representable affine, and therefore quasi-compact (Exercise: prove it). □

References

[1] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin].

[2] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[3] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[4] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

[5] James S. Milne, *étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.