

2-Covering spaces

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Topological groupoids and functors

A *topological groupoid* \mathbf{X} consists of:

- ▶ A space X_0 ('objects') and a space X_1 ('arrows' or 'morphisms')
- ▶ Maps $s, t: X_1 \rightarrow X_0$ ('source' and 'target') $\text{id}_{(-)}: X_0 \rightarrow X_1$ ('unit' or 'identity')
- ▶ Composition $X_1 \times_{X_0} X_1 \rightarrow X_1$ and inverse $(-)^{-1}: X_1 \rightarrow X_1$

Often will denote a groupoid \mathbf{X} by

$$X_1 \rightrightarrows X_0$$

Topological groupoids and functors

A *functor* between topological groupoids, $f: \mathbf{X} \rightarrow \mathbf{Y}$ consists of

- ▶ Maps $f_0: X_0 \rightarrow Y_0$ and $f_1: X_1 \rightarrow Y_1$ ('object component' and 'arrow component')

such that

- ▶ $sf_1 = f_0s$, $tf_1 = f_0t$ (respects source and target)
- ▶ $f(\gamma\eta) = f(\gamma)f(\eta)$, $f(id_x) = id_{f(x)}$ and $f(\gamma^{-1}) = f(\gamma)^{-1}$ (respects composition, identities and inverses)

We thus have a category TG of topological groupoids and functors.

Examples

- ▶ There is a functor $Top \hookrightarrow TG$ sending a space to the groupoid with no non-trivial arrows.
- ▶ There is a full subcategory $Gpd \hookrightarrow TG$ of groupoids with the discrete topology - these will be referred to as t-d (topologically discrete) groupoids.
- ▶ Let X be a locally connected, semilocally simply-connected topological space. The fundamental groupoid

$$\Pi_1(X) := (X' / \sim \rightrightarrows X)$$

can be given a topology such that $\Pi_1(X) \rightarrow X \times X$ is a covering space.

Examples

Let $p : E \rightarrow X$ be a map, and $\text{vert}(E') \subset E'$ the subset of *vertical* paths. Denote by \sim_v the equivalence relation 'vertically homotopic rel endpoints'.

Then

$$\Pi_1/X(E) := (\text{vert}(E')/\sim_v \rightrightarrows E^\delta)$$

is a (t-d) groupoid equipped with a functor

$$\Pi_1/X(E) \rightarrow X.$$

When p is a fibre bundle, or even locally homotopically trivial, and the fibres have universal covering spaces, we can make $\Pi_1/X(E)$ a topological groupoid.

Fibres are discrete. . . ish

- ▶ The fibres of a covering space are sets, or more accurately, spaces in the essential image of the functor $Set \rightarrow Top$.
- ▶ We think of a covering space as being a family of sets parameterised by the base space. The idea is that a 2-covering space is a family of groupoids, all of which are in the essential image of

$$Gpd \rightarrow TG'$$

where TG' is a category (actually bicategory) of groupoids which is just the naïve (2-)category TG with some formal inverses thrown in.

Brief detour into TG'

Definition

A *weak equivalence* (Everaert-Kieboom-van der Linden)

$f: \mathbf{X} \xrightarrow{\sim} \mathbf{Y}$ of topological groupoids is a functor which satisfies:



$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ X_0 \times X_0 & \xrightarrow{f_0 \times f_0} & Y_0 \times Y_0 \end{array}$$

is a pullback (f is *fully faithful*), and



$$\begin{array}{ccccc} X_0 & \longleftarrow & X_0 \times Y_0 & Y_1 & \\ f_0 \downarrow & & \downarrow & \searrow \rho & \\ Y_0 & \longleftarrow_s & Y_1 & \xrightarrow{t} & Y_0 \end{array},$$

ρ admits local sections (f is *essentially epi*).

Brief detour into TG'

- ▶ There is a 2-category (also denoted TG) of groupoids, functors and *natural transformations* (Ehresmann). Equivalences in this 2-category are weak equivalences but the converse is not true.
- ▶ The bicategory TG' is 'the' universal construction such that the functor $q: TG \rightarrow TG'$ sends weak equivalences to actual equivalences (Pronk). We can construct TG' such that q includes TG as a *strict* sub-bicategory with the same objects as TG' ([R.], drawing on Makkai, Bartels).
- ▶ 1-arrows in this bicategory, which are spans $\mathbf{X} \xleftarrow{\sim} \mathbf{X}[U] \rightarrow \mathbf{Y}$, will be denoted $\mathbf{X} \twoheadrightarrow \mathbf{Y}$ and equivalences will be denoted $\mathbf{X} \xrightarrow{\sim} \mathbf{Y}$.

Fibres are discrete. . . ish

Definition

A groupoid \mathbf{P} is *weakly discrete* if it is equivalent (in TG') to a t-d groupoid. That is, there is an equivalence $\mathbf{D} \xrightarrow{\sim} \mathbf{P}$.

If we let \mathbf{X}^δ denote a groupoid considered with the discrete topology, we have the following pleasing result:

Fibres are discrete. . . ish

Lemma

A groupoid \mathbf{P} is weakly discrete if and only if the canonical functor $\mathbf{P}^\delta \rightarrow \mathbf{P}$ is a weak equivalence.

The fundamental groupoid $\Pi_1(X)$ of a (locally nice) space is weakly discrete. If X is locally badly behaved this can fail to be so, or even fail to be topological e.g. Hawaiian earring (P. Fabel).

2-covering spaces

Definition

Let $\mathbf{Z} \rightarrow X$ be a functor. \mathbf{Z} is a *2-covering space* of X if there is an open cover $\{U_i\}$ of X such that for each pullback groupoid

$$\mathbf{Z}_{U_i} := (Z_1 \times_X U_i \rightrightarrows Z_0 \times_X U_i)$$

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2-covering spaces equipped with an equivalences like the \mathbf{Z}_{U_i} are will be called *trivialisable*.

An example

Let A be an abelian topological group. Recall (Murray) that an A -bundle gerbe (P, Y) on a space X consists of a map $Y \rightarrow X$ admitting local sections, a principal A -bundle $P \rightarrow Y \times_X Y$ together with a 'product' map

$$p_{12}^* P \otimes p_{23}^* P \rightarrow p_{13}^* P$$

of bundles over $Y^{[3]} := Y \times_X Y \times_X Y$ which is associative over $Y^{[4]}$.

Theorem

An A -bundle gerbe (P, Y) on X for discrete A determines a 2-covering space

$$(P \rightrightarrows Y) \rightarrow X.$$

An example, continued

As a concrete example, let (E, S^3) be the lifting bundle gerbe associated to the $U(1)$ -bundle $S^3 \rightarrow S^2$ and the central extension $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$.

$$E \rightarrow S^3 \times_{S^2} S^3$$

is a principal \mathbb{Z} -bundle and $E \rightrightarrows S^3$ is a 2-covering space of S^2 . It can be shown that this 2-covering space is not trivialisable.

Properties of 2-covering spaces

Let us now consider a general 2-covering space $\mathbf{Z} \rightarrow X$. We find the following results

Theorem

- ▶ *The map $Z_0 \rightarrow X$ admits local sections,*
- ▶ *$Z_1 \rightarrow Z_0 \times_X Z_0$ is a covering space,*
- ▶ *Each fibre Z_x is a weakly discrete groupoid,*
- ▶ *For path-connected X , all the fibres are equivalent objects in TG' .*

Note that for the second point we need to allow fibres to be empty, and if $Z_0 \times_X Z_0$ is not path-connected then the fibres need not all be isomorphic.

Generic example

Let us return to our example from earlier, $\Pi_1/X(E) \rightarrow X$. We need to use some easy properties of the functor $\Pi_1 : Top/X \rightarrow TG/X$:

- ▶ Π_1 commutes with pullbacks:
 $\Pi_1/X(E) \times_X A \simeq \Pi_1/A(E \times_X A),$
- ▶ $\Pi_1/X(X \times F) \simeq \Pi_1(F) \times X,$
- ▶ Vertically homotopic maps are sent to isomorphic functors

The properties are used to *define* the topology on $\Pi_1/X(E)$.

Generic example

Assume (wlog) X is path-connected.

- ▶ If $E \rightarrow X$ is a fibre bundle then as it is locally trivial, $\Pi_1/X(E)$ is locally trivialisable. All that is necessary for $\Pi_1/X(E)$ to be a 2-covering space is that the typical fibre has a universal covering space.
- ▶ If $E \rightarrow X$ is locally homotopy trivial, and the typical fibre (defined up to homotopy equivalence) has a universal covering space, then $\Pi_1/X(E)$ is again a 2-covering space.
- ▶ As a more specific example of this last case, assume that X is paracompact and has a nhd basis of inessential open sets (i.e. $U \hookrightarrow X$ is null-homotopic). Then any Hurewicz fibration over X is locally homotopy trivial (Dold).

Low-dimensional homotopy properties

- ▶ Recall that for a covering space $p: \tilde{X} \rightarrow X$, $\pi_1(p)$ is injective. We would like a similar result for 2-covering spaces, but now we do not consider the fundamental group, but the fundamental *2-group*. This is best explained with an example, which we shall need to use later
- ▶ Consider the fundamental groupoid of a (based) loop space, $\Pi_1(\Omega X)$. The H-space structure on ΩX induces an up-to-homotopy associative multiplication on the objects and arrows of the groupoid, and reversing paths gives up-to-homotopy inverses.

Low-dimensional homotopy properties

- ▶ $\Pi_2(X, *) := \Pi_1(\Omega X)$ is called the *fundamental 2-group* of X (e.g. Baez-Lauda). Groupoids with such a group-like structure are called 2-groups (they have a history under different names going back to the 1960s). Note that this construction is functorial for pointed spaces and maps.
- ▶ We do not have at present a completely satisfactory loop 'space' of a groupoid, but it is still possible [R.] to define the fundamental 2-group $\Pi_2(\mathbf{X}, *)$ of a pointed groupoid \mathbf{X} by other means.
- ▶ This reduces (up to equivalence) to the original construction when \mathbf{X} is a space, and is a functor of pointed groupoids and basepoint-preserving functors.

Low-dimensional homotopy properties

There are analogues of path- and homotopy-lifting theorems (more like Dold fibrations than Hurewicz or Serre). These help us to prove:

Theorem

*Given a 2-covering space $\mathbf{Z} \rightarrow X$ (and basepoints), the induced functor $\Pi_2(\mathbf{Z}, *) \rightarrow \Pi_2(X, *)$ is faithful.*

From work of M. Dupont we see that this should be taken as the definition of a sub-2-group.

A 2-connected cover

We say a groupoid \mathbf{X} is 2-connected if $\Pi_2(\mathbf{X}, *)$ is equivalent to the trivial groupoid.

Let (X, x) be a pointed space with a nhd basis of inessential open sets (not assuming X is paracompact) and let $PX \rightarrow X$ be the path fibration.

Theorem

$\mathbf{X}^{(2)} := \Pi_1 /_X(PX) \rightarrow X$ is a 2-covering space and the groupoid $\mathbf{X}^{(2)}$ is 2-connected.

Furthermore, this is functorial: a map $X \rightarrow Y$ gives a commuting square

$$\begin{array}{ccc} \mathbf{X}^{(2)} & \longrightarrow & \mathbf{Y}^{(2)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Comments

- ▶ The local contractibility condition on the base is too strong, but this is the best we can do with the current definition of 2-covering space. This points toward replacing TG' with a different localisation TG'' (replacing open covers with another Grothendieck topology).
- ▶ We can form (weak) quotients of $\mathbf{X}^{(2)}$ by sub-2-groups of $\Pi_2(X, *)$ – these will also be 2-covering spaces, giving us a functor from the 2-category of such sub-2-groups to that of (pointed, path-connected) 2-covering spaces. This is conjectured to be an equivalence of 2-categories.
- ▶ 2-covering spaces give us lots of explicit examples of 2-bundles (T. Bartels) which previously have been hard to come by.

References

- R. D.M. Roberts, *Fundamental bigroupoids and 2-covering spaces*, PhD thesis. Draft available from <http://ncatlab.org/david+roberts/show/HomePage> (Chapters 1 and 2).