INTERNAL CATEGORIES, ANAFUNCTORS AND LOCALISATIONS

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Abstract.

We show that given a category S with binary products and a 2-category $\mathbf{Cat}'(S) \subset \mathbf{Cat}(S)$ of internal categories in S, closed under some natural operations, the bicategorical localisation $\mathbf{Cat}'(S)[W_E^{-1}]$ exists for W_E a class of weak equivalences analogous to fully faithful and essentially surjective functors. Secondly we show that this localisation is given by a bicategory with anafunctors as 1-arrows when S is a site, and give conditions when various such bicategories of anafunctors are equivalent. While the connections to stacks are not pursued here, this work provides a precise setting to the claim that stacks on arbitrary sites are internal groupoids up to essential equivalence. Finally, we make some conjectural comments about localisation of bicategories qua $(\infty, 2)$-categories.

Contents

1. Introduction

Pronk, in her work on stacks [Pronk, 1996], introduced the concept of localising a bicategory at a class of 1-arrows. She gave axioms that are analogues of the familiar Gabriel-Zisman axioms for a category of fractions [Gabriel-Zisman, 1967]. All this was in order to prove that a certain 2-category of topological stacks is a localisation of a certain 2-category of topological groupoids. In the same paper parallel results on differentiable and algebraic stacks also appear. This work refined the results in [Moerdijk, 1988].

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In the years since, a number of papers (e.g. [Landsman, 2001, Noohi, 2005b, Lerman, 2008, Carchedi, 2009, Breckes, 2009, Vitale, 2010, Abbad et al, 2010]) have appeared dealing with localising 2-categories of internal groupoids at a class of *weak equivalences.*[1] Weak equivalences, in this sense, were introduced by Bunge and Paré [Bunge-Paré, 1979] for groupoids in a regular category (e.g. a topos), and are an internal version of fully faithful, essentially surjective functors between internal categories. Since 'surjective' only makes sense in concrete categories, and even then it is not always useful, we need to introduce a class E of maps with which to replace the surjection part of 'essentially surjective'. For example, in the literature on Lie groupoids, surjective submersions are universally used. With the class E specified, we refer to weak equivalences as E-equivalences.

We note in passing that the articles [Joyal-Tierney, 1991, Everaert et al, 2005, Colman-Costoya, 2009], under various assumptions on S, construct model category structures on 1-categories of internal categories or groupoids for which the weak equivalences are E-equivalences for various ambient categories. In fact for S only finitely complete the model structure of [Everaert et al, 2005] may not exist, but the slightly weaker structure of a category of fibrant objects will.

In this paper we generalise the half of Pronk's result that says a full sub-2-category $\mathbf{Cat}'(S) \subset \mathbf{Cat}(S)$ of categories in S, with certain properties, admits a localisation at the class W_E of E-equivalences. More formally, let $\mathbf{Cat}'(S)$ be a full sub-2-category of $\mathbf{Cat}(S)$ with objects internal categories such that all pullbacks of the source and target maps exist, and let $\mathbf{Gpd}'(S) \hookrightarrow \mathbf{Cat}'(S)$ be the sub-2-category of groupoids. We say $\mathbf{Cat}'(S)$ has enough groupoids if the sub-(2,1)-category $\mathbf{Gpd}'(S) \hookrightarrow \mathbf{Cat}'(S)$ is coreflective. Recall that $\mathbf{I}$ is the groupoid with two objects and a unique isomorphism between them.

1.1. THEOREM. *If $\mathbf{Cat}'(S)$ admits base change along arrows in E, a class of admissible maps*[2] *in S, admits cotensors with $\mathbf{I}$ and has enough groupoids then $\mathbf{Cat}'(S)$ admits a calculus of fractions for W_E.*

The construction in [Pronk, 1996], while canonical, is not very efficient, as 2-arrows are equivalence classes of diagrams, and the hom-categories are *a priori* large in the technical sense. While largeness of its own is not detrimental, it would be desirable to show that the hom-categories are at least essentially small. These points are partial motivation for our second result, which we shall shortly describe.

In the case that maps belonging to E are refined by covers from a subcanonical singleton[3] pretopology J, then we can compare the localisation from theorem 1.1 to the bicategory $\mathbf{Cat}'_{ana}(S, J)$ with the same objects as $\mathbf{Cat}'(S)$ and J-*anafunctors* for 1-arrows ([Makkai, 1996, Bartels, 2006], see definition 5.1 below). Put simply, anafunctors are spans

$$X \leftarrow X[U] \xrightarrow{f} Y$$

[1] Other examples of early work on localising 1-categories of groupoids are [Hilsum-Skandalis, 1987, Pradines, 1989]

[2] See definitions 2.22 and 7.1 for details on base change and admissible maps respectively.

[3] A singleton Grothendieck pretopology is one where all the covering families consist of a single map.

of internal categories where the left 'leg' is a resolution of X by taking the base change $X[U]$ along a J-cover $U \longrightarrow X_0$ (as as such is fully faithful). Anafunctors will not be completely unfamiliar beasts, in that when X is an object of S and Y is a group object in S, considered as a groupoid with one object, anafunctors from the former to the latter are precisely Čech cocycles, and maps of anafunctors are coboundaries.

The second main result of this paper is the following. Note that $\mathbf{Cat}'(S)$ is a sub-bicategory of both $\mathbf{Cat}'_{ana}(S, J)$ and $\mathbf{Cat}'(S)[W_E^{-1}]$.

1.2. Theorem. *Let* $\mathbf{Cat}'(S)$ *and* E *be as in Theorem 1.1 and let* J *be a subcanonical singleton pretopology on* S *which is cofinal in* E. *Then there is an equivalence of bicategories*

$$\mathbf{Cat}'_{ana}(S, J) \simeq \mathbf{Cat}'(S)[W_E^{-1}]$$

which is, up to equivalence, the identity on $\mathbf{Cat}'(S)$.

When a weak size axiom holds for the category of J-covers of each object in S, then it is easy to show that $\mathbf{Cat}'_{ana}(S, J)$ is locally essentially small. This axiom holds for any reasonable category of geometric objects, for example manifolds, spaces, schemes, topoi with enough projectives.

As far as the author is aware, theorem 1.1 should cover all cases of bicategorical localisation of categories of groupoids in the literature (but is happy to be corrected!). What is new here is that these results work for internal categories proper, not just internal groupoids. For example, while the $(2, 1)$- category of Lie groupoids is well studied, and various models for the localisation considered here exist, the $(2, 2)$-category of Lie categories does not seem to have been considered since the work of Ehresmann and students.

As pointed out to me by Urs Schreiber, the technique of localising $(\infty, 1)$-categories (presented by simplicially-enriched categories) has been around for a long time, going back to work of Dwyer and Kan on simplicial localisation [Dwyer-Kan, 1980a, Dwyer-Kan, 1980b, Dwyer-Kan, 1980c]. In that context, one can consider a $(2, 1)$-category of internal groupoids as being contained in the $(\infty, 1)$-category of sheaves of $(\infty, 0)$-categories on the ambient category, and apply the results of Dwyer-Kan. However, dealing with non-invertible 2-arrows in the ∞-category context seems to be a lot more difficult, and requires much more machinery (see [Lurie, 2009]). Hence Theorem 1.1 could be viewed as localising an $(\infty, 2)$-category of presheaves of $(\infty, 1)$-categories. There is clearly a huge gap between this result, and one comparable to Dwyer-Kan's for $(\infty, 2)$-categories, but it is perhaps a small clue as to what may be achievable in the future, in particular, presenting $(\infty, 2)$-categories by bicategories with a chosen class of weak equivalences.

A remark is necessary about the relation between anafunctors and distributors = profunctors [Bénabou, 1973]. As mentioned in [Makkai, 1996], anafunctors are similar to profunctors. It was pointed out by Jean Bénabou in January 2011 on the 'categories' mailing list that anafunctors correspond to representable profunctors (in that there is an equivalence of bicategories with representable profunctors and anafunctors as 1-arrows between categories respectively), although this fact seems to have been independently discovered by others. However, anafunctors are not precisely profunctors, though they are similar in description, and although one can define internal profunctors (see, e.g. [Johnstone, 2002], section B2.7), these latter require more of the ambient category than we admit here merely in order to state the equivalence between internal profunctors and internal anafunctors, and also to define composition of internal profunctors. We thus use the terminology of anafunctors in this paper.

In private communication, J. Bénabou says that the concept of *calibration* was introduced by him ([Bénabou, 1975]) to extend internal profunctors beyond regular categories. I have not been able to ascertain the extent to which this paper overlaps with the said work of Bénabou.

We now outline the contents of the paper, which is intended to be self-contained. Sections 2 and 3 cover necessary background on internal categories and Grothendieck pretopologies, all of which would be familiar to experts. Section 4 covers weak equivalences between internal categories, while section 5 reviews the theory of internal anafunctors from [Bartels, 2006]. Section 6 covers the localisation theory for bicategories from [Pronk, 1996], before section 7 proves the main results of the paper. The last section gives a sufficient condition for the localised bicategories to be locally essentially small.

2. Internal categories

Internal categories were introduced in [Ehresmann, 1963], starting with differentiable and topological categories (i.e. internal to **Diff** and **Top** respectively). We collect here the necessary definitions, terminology and notation. For a thorough recent account, see [Baez-Lauda, 2004] or the encyclopedic [Johnstone, 2002].

Fix a category S. It will be referred to as the *ambient category*. We will assume throughout that S has binary products.

2.1. Definition. An *internal category* X in a category S is a diagram

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \overset{s,t}{\rightrightarrows} X_0 \xrightarrow{e} X_1$$

in S such that the *multiplication* m is associative (we also demand the triple pullback $X_1 \times_{X_0} X_1 \times_{X_0} X_1$ exists), the *unit map* e is a two-sided unit for m and s and t are the usual *source* and *target*. An *internal groupoid* is an internal category with an involution

$$(-)^{-1} \colon X_1 \longrightarrow X_1$$

satisfying the usual diagrams for an inverse.

Since multiplication is associative, there is a well-defined map $X_1 \times_{X_0} X_1 \times_{X_0} X_1 \longrightarrow X_1$, which will also be denoted by m. The pullback in the diagram in definition 2.1 is

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r] \arrow[d] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "t"'] & X_0 \end{tikzcd}\ .$$

This, and pullbacks like this (where source is pulled back along target), will occur often. If confusion can arise, the maps in question will be explicity written, as in $X_1 \times_{s,X_0,t} X_1$. One usually sees the requirement that S is finitely complete in order to define internal categories. This is not strictly necessary, and not true in the well-studied case of $S = \mathbf{Diff}$, the category of smooth manifolds.

Often an internal category will be denoted $X_1 \rightrightarrows X_0$, the arrows m, s, t, e (and $(-)^{-1}$) will be referred to as *structure maps* and X_1 and X_0 called the object of arrows and the object of objects respectively. For example, if $S = \mathbf{Top}$, we have the space of arrows and the space of objects, for $S = \mathbf{Grp}$ we have the group of arrows and so on.

2.2. Example. If $X \longrightarrow Y$ is an arrow in S admitting iterated kernel pairs, there is an internal groupoid $\check{C}(X)$ with $\check{C}(X)_0 = X$, $\check{C}(X)_1 = X \times_Y X$, source and target are projection on first and second factor, and the multiplication is projecting out the middle factor in $X \times_Y X \times_Y X$.

2.3. Example. Let S be a category. For each object $A \in S$ there is an internal groupoid $\mathrm{disc}(A)$ which has $\mathrm{disc}(A)_1 = \mathrm{disc}(A)_0 = A$ and all structure maps equal to id_A. Such a category is called *discrete*. We have $\mathrm{disc}(A \times B) \simeq \mathrm{disc}(A) \times \mathrm{disc}(B)$.
There is also an internal groupoid $\mathrm{codisc}(A)$ with

$$\mathrm{codisc}(A)_0 = A,\ \mathrm{codisc}(A)_1 = A \times A$$

and where source and target are projections on the first and second factor respectively. Such a groupoid is called *codiscrete*. Again, we have $\mathrm{codisc}(A \times B) \simeq \mathrm{codisc}(A) \times \mathrm{codisc}(B)$.

2.4. Example. The codiscrete groupoid is obviously a special case of example 2.2, which is called the Čech groupoid of the map $X \longrightarrow Y$. The origin of the name is that in **Top**, for maps of the form $\coprod_I U_i \longrightarrow Y$ (arising from an open cover), the Čech groupoid $\check{C}(\coprod_I U_i)$ appears in the definition of Čech cohomology.

2.5. Definition. Given internal categories X and Y in S, an *internal functor* $f : X \longrightarrow Y$ is a pair of maps

$$f_0 \colon X_0 \longrightarrow Y_0 \quad \text{and} \quad f_1 \colon X_1 \longrightarrow Y_1$$

called the object and arrow component respectively. Both components are required to commute with all the structure maps.

2.6. Example. If $A \longrightarrow B$ is a map in S, there are functors $\mathrm{disc}(A) \longrightarrow \mathrm{disc}(B)$ and $\mathrm{codisc}(A) \longrightarrow \mathrm{codisc}(B)$.

2.7. Example. If $A \longrightarrow C$ and $B \longrightarrow C$ are maps admitting iterated kernel pairs, and $A \longrightarrow B$ is a map over C, there is a functor $\check{C}(A) \longrightarrow \check{C}(B)$.

2.8. Definition. Given internal categories X, Y and internal functors $f, g\colon X \longrightarrow Y$, an *internal natural transformation* (or simply *transformation*)

$$a\colon f \Rightarrow g$$

is a map $a\colon X_0 \longrightarrow Y_1$ such that $s \circ a = f_0,\ t \circ a = g_0$ and the following diagram commutes

$$\begin{tikzcd} X_1 \arrow[r, "(g_1, a\circ s)"] \arrow[d, "(a\circ t, f_1)"'] & Y_1 \times_{Y_0} Y_1 \arrow[d, "m"] \\ Y_1 \times_{Y_0} Y_1 \arrow[r, "m"] & Y_1 \end{tikzcd} \tag{1}$$

expressing the naturality of a. If a factors through the 'object Y^{iso} of invertible arrows' (to made precise below), then it is called a *natural isomorphism*. Clearly there is no distinction between natural transformations and natural isomorphisms when Y is an internal groupoid.

We can reformulate the naturality diagram above in the case that a is a natural isomorphism. Denote by $-a$ the composite arrow

$$X_0 \xrightarrow{a} Y_1^{iso} \xrightarrow{(-)^{-1}} Y_1^{iso} \hookrightarrow Y_1.$$

Then the diagram (1) commuting is equivalent to this diagram commuting

$$\begin{tikzcd} X_0 \times_{X_0} X_1 \times_{X_0} X_0 \arrow[r, "-a\times f_1 \times a"] \arrow[d, "\simeq"'] & Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \arrow[d, "m"] \\ X_1 \arrow[r, "g_1"'] & Y_1 \end{tikzcd} \tag{2}$$

a fact we will use repeatedly.

2.9. Example. If X is a category in S, A is an object of S and $f, g : X \longrightarrow \mathrm{codisc}(A)$ are functors, there is a unique natural isomorphism $f \overset{\sim}{\Rightarrow} g$.

Internal categories (resp. groupoids), functors and transformations form a 2-category $\mathbf{Cat}(S)$ (resp. $\mathbf{Gpd}(S)$) [Ehresmann, 1963]. There is clearly a 2-functor $\mathbf{Gpd}(S) \longrightarrow \mathbf{Cat}(S)$. Also, disc and codisc, described in examples 2.3 and 2.6 are 2-functors $S \longrightarrow \mathbf{Gpd}(S)$, whose underlying functors are left and right adjoint to the functor

$$\mathrm{Obj}\colon \underline{\mathbf{Cat}(S)} \longrightarrow S, \qquad (X_1 \rightrightarrows X_0) \mapsto X_0.$$

Here $\underline{\mathbf{Cat}(S)}$ is the 1-category underlying the 2-category $\mathbf{Cat}(S)$. Hence for an internal category X in S, there are functors $\mathrm{disc}(X_0) \longrightarrow X$ and $X \longrightarrow \mathrm{codisc}(X_0)$, the arrow component of the latter being $(s, t) : X_1 \longrightarrow X_0^2$.

For some calculations we need access to the object of isomorphisms of an internal category. In ordinary category theory, given a category we can consider its *core* – the largest subcategory that is a groupoid. When S is finitely complete, the category of internal groupoids is a coreflective subcategory of the category of internal categories (morphisms are internal functors, see definition 2.5) [Bunge-Paré, 1979],[4] and so for every internal category $X_1 \rightrightarrows X_0$ there is a maximal subobject $X_1^{iso} \hookrightarrow X_1$ such that $X_1^{iso} \rightrightarrows X_0$ is an internal groupoid. The object X_1^{iso} of isomorphisms can in this case be constructed as a finite limit. However, the reverse implication (coreflective subcategory implies finitely complete) is not true, as shown by the following lemma, essentially proved in [Ehresmann, 1959].

2.10. Lemma. *Let* $\underline{\mathbf{LieCat}}$ *be the category of categories internal to* **Diff***, the category of finite-dimensional smooth manifolds, such that the source and target maps are submersions, and* $\underline{\mathbf{LieGpd}}$ *the corresponding category of internal groupoids. The inclusion* $\underline{\mathbf{LieGpd}} \hookrightarrow \underline{\mathbf{LieCat}}$ *makes* $\underline{\mathbf{LieGpd}}$ *a coreflective subcategory.*

In general, we shall consider sub-2-categories $\mathbf{Cat}'(S) \hookrightarrow \mathbf{Cat}(S)$ such that $\mathbf{Gpd}'(S) = \mathbf{Cat}'(S) \cap \mathbf{Gpd}(S) \hookrightarrow \mathbf{Cat}'(S)_{(2,1)}$ is coreflective, where $\mathbf{C}_{(2,1)} \hookrightarrow \mathbf{C}$ is the maximal sub-(2,1)-category. If this condition is fulfilled we shall say that $\mathbf{Cat}'(S)$ *has enough groupoids.*

2.11. Example. The following ambient 2-categories admit enough groupoids:

- $\mathbf{Cat}(S)$ for S finitely complete
- $\mathbf{Gpd}(S)$ for S with binary products
- **LieCat**, where the objects are categories in **Diff** such that the source and target maps are submersions.
- $\mathbf{Cat}(\mathbf{TopManif})_{reg}$, where the objects are categories in topological manifolds such that the source and target maps define foliations on a small enough neighbourhood of each point in the space of arrows (this result is also in [Ehresmann, 1959]).
- The 2-category of smooth monoid actions by a smooth monoid of finite type in the category of schemes over a field k (thanks to Matt E on math.stackexchange for showing this [Matt E, 2011]). That is, the full sub-2-category of categories in $\mathbf{Sch}/k$ which arise from the smooth action of a smooth monoid in schemes of finite type on a scheme.

It is a reasonable conjecture that the smoothness assumption on the monoid can be dropped, and one could go so far as to try to extend this from schemes over a field to schemes over a more general ring.

Over the course of this section we shall consider additional properties we want to impose on $\mathbf{Cat}'(S)$ in order to prove our result about localisation.

[4] In fact, the (2,1)-category of internal groupoids is coreflective in the (2,1)-category of internal categories, functors and natural *isomorphisms*.

2.12. Definition. An *internal* or *strong equivalence* of internal categories is an equivalence in the 2-category of internal categories. That is, an internal functor $f\colon X \longrightarrow Y$ such that there is a functor $f'\colon Y \longrightarrow X$ and natural isomorphisms $f \circ f' \Rightarrow \mathrm{id}_Y$, $f' \circ f \Rightarrow \mathrm{id}_X$.

Many constructions involving internal categories require pullbacks of the source and target maps. To this end, we shall be interested in a full sub-2-category $\mathbf{Cat}'(S) \subset \mathbf{Cat}(S)$ consisting of objects – internal categories X – such that all pullbacks of $s, t\colon X_1 \longrightarrow X_0$ and $s, t\colon X_1^{iso} \longrightarrow X_0$ exist. More precisely, we suppose that these maps belong to a class of maps in S of pullbacks exist. The reason for this is that when considering presentations of stacks by groupoids, where stacks are considered as generalisations of spaces/schemes/manifolds, the source and target maps arise as pullbacks of the 'cover' of the stack by a representable stack, and these are, in practice, required to be well-behaved. That being said, for finitely complete S one can always take the class of maps to be $\mathrm{Mor}(S)$.

2.13. Example. The prototypical example is that of Lie groupoids, where the source and target maps are required to be submersions. Restricting further we could consider étale Lie groupoids, where the source and target maps are local diffeomorphisms.

2.14. Example. In [Noohi, 2005a] Noohi considers a class LF of *local fibrations* which, amongst other properties, are stable under pullback, closed under composition and contain the open embeddings. The source and target maps of topological groupoids presenting topological stacks are then local fibrations. The 2-category of these topological groupoids will be denoted $\mathbf{Cat}^{LF}(\mathbf{Top})$.

2.15. Example. Let $\mathcal{C}$ be a Serre class in an abelian category A. Then consider the 2-category of groupoids X in A such that the kernel and cokernel of the associated crossed module $s^{-1}(0) \longrightarrow X_0$ are in $\mathcal{C}$. One could denote the resulting 2-category by $\mathbf{Gpd}^{\mathcal{C}}(S)$. When $S = \mathbf{Ab}$, then these groupoids represent pointed connected 2-types with abelian fundamental group and homotopy groups in $\mathcal{C}$.

Any class of algebraic stacks should also provide more examples of groupoid schemes with a restricted class of source and target maps.

The strict pullback of internal categories

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

when it exists, is the internal category with objects $X_0 \times_{Y_0} Z_0$, arrows $X_1 \times_{Y_1} Z_1$, and all structure maps given componentwise by those of X and Z. Often we will be able to prove that certain pullbacks exist because of conditions on various component maps in S. We do not assume that all strict pullbacks of internal categories exists.

There is a weaker notion of pullback, which is of much more interest in this 2-categorical setting, known as a *bipullback* (or in some places weak pullback – [Moerdijk-Mrčun, 2003] for example – but this has a separate meaning when in a 1-category, so we will not use it). This can be defined via considering an internal version of the arrow category of a category, or more precisely, the *isomorphism* category.

2.16. DEFINITION. (see, e.g. [Everaert et al, 2005]) Assume for an internal category X the object X_1^{iso} of isomorphisms exists. The *isomorphism category* of X is the internal category denoted $X^{\mathbf{I}}$, with

$$X_0^{\mathbf{I}} = X_1^{iso}, \qquad X_1^{\mathbf{I}} = (X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

where the fibred product over X_1 arises by considering the composition maps

$$\begin{aligned} X_1 \times_{s,X_0,t} X_1^{iso} &\longrightarrow X_1 \\ X_1^{iso} \times_{s,X_0,t} X_1 &\longrightarrow X_1. \end{aligned}$$

Composition in $X^{\mathbf{I}}$ is the same as commutative squares in the case of ordinary categories. There are two functors $\mathbf{s}, \mathbf{t} \colon X^{\mathbf{I}} \longrightarrow X$ which have the usual source and target maps of X as their respective object components.

This construction is an internal version of the functor category $\mathbf{Cat}(\mathbf{I}, C)$, since there is not always a groupoid analogous to $\mathbf{I} = (\circ \xrightarrow{\simeq} \bullet)$ internal to S. In other words, $X^{\mathbf{I}}$ is the *cotensor* (or *power*) of X with $\mathbf{I}$ (see e.g. [Kelly, 2005]).

2.17. DEFINITION. If for all objects X of $\mathbf{Cat}'(S)$ the cotensor $X^{\mathbf{I}}$ exists, we say $\mathbf{Cat}'(S)$ *admits cotensors with* $\mathbf{I}$.

2.18. REMARK. There is an isomorphism $X_1^{\mathbf{I}} \simeq X_1^{iso} \times_{t,X_0,t} X_1 \times_{s,X_0,t} X_1^{iso}$ given by projecting out the last factor in

$$(X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

It is easy to see in this form that this pullback exists given our assumptions on pullbacks of the source and target maps. In fact, we have an isomorphism $X_1^{\mathbf{I}} \simeq X[X_1^{iso}]$.

The following lemma is a simple exercise in keeping track of pullbacks.

2.19. LEMMA. *Assume* $\mathbf{Cat}'(S)$ *has enough groupoids. If the source and target maps of* X*, an object of* $\mathbf{Cat}'(S)$*, belong to a class of maps stable under pullback and closed under composition, then the source and target maps of* $X^{\mathbf{I}}$ *also belong to that class.*

As an example, if the 2-category of Lie categories (or groupoids) is defined to have objects those categories (resp. groupoids) with submersions for source and target maps, then **LieCat** and **LieGpd** have cotensors with **I**, by lemma 2.10.

2.20. REMARK. We shall thus assume from now on that $\mathbf{Cat}'(S)$ is defined so that $X^{\mathbf{I}} \in \mathbf{Cat}'(S)$ whenever $X \in \mathbf{Cat}'(S)$. This is true if we define $\mathbf{Cat}'(S)$ to be all internal categories with source and target maps belonging to a specified class of arrows from S of which pullbacks exist.

The astute reader will recognise the following as an internalisation of the usual notion of bipullback.

2.21. Definition. The *bipullback* $X\tilde{\times}_Y Z$ of a diagram

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

of internal categories is the 2-universal filler for this diagram. It is given, if it exists, by the strict pullback $X \times_{Y,\mathbf{s}} Y^{\mathbf{I}} \times_{\mathbf{t},Y} Z$. There is a 2-commutative square

$$\begin{tikzcd} X\tilde{\times}_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\overset{\simeq}{\Leftarrow}"] \\ X \arrow[r] & Y \end{tikzcd}$$

the universal property of which can be found in [Moerdijk-Mrčun, 2003] (we will not need to use the universal property).

If the ambient category S has pullbacks, then all bipullbacks exist in $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ (see e.g. [Vitale, 2010] for a proof). However it is not immediate that all bipullbacks exist in $\mathbf{Cat}'(S)$, even if S is finitely complete. Lemma 2.19 ensures that if $\mathbf{Cat}'(S)$ has strict pullbacks and cotensors with $\mathbf{I}$, then it has bipullbacks by assumption 2.20.

Recall that there is a functor $\mathrm{Obj}\colon \underline{\mathbf{Cat}'(S)} \longrightarrow S$, sending an internal category to its object of objects. Given a category X and a map $p : M \longrightarrow X_0$ in S, a cartesian lift of p is, amongst other things, a functor with object component p.

2.22. Definition. For a category X and a map $p : M \longrightarrow X_0$ in S, the domain $X[M]$ of a cartesian lift $X[M] \longrightarrow X$ of p will be called the *base change of X along p*.

If the base change along any map in a given class K of maps exists for all objects of $\mathbf{Cat}'(S)$, then we say $\mathbf{Cat}'(S)$ *admits base change along maps in K*. The base change, if it exists, is given by taking the strict pullback[5]

$$\begin{tikzcd} X[M] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \mathrm{codisc}(M) \arrow[r] & \mathrm{codisc}(X_0) \end{tikzcd} \tag{3}$$

in $\mathbf{Cat}(S)$. The canonical functor in the top row has p as its object component. If desired we could probably choose a cartesian lift for each map in S (using Choice) and get a weak 2-functor with object component

$$\begin{array}{ccc} \mathrm{Obj}(\mathbf{Cat}'(S)) \times_{\mathrm{Obj}(S)} \mathrm{Mor}(S) & \longrightarrow & \mathrm{Obj}(\mathbf{Cat}'(S)) \\ (X, M \longrightarrow X_0) & \mapsto & X[M], \end{array}$$

[5]Note that codisc may not land in $\mathbf{Cat}'(S)$, so we work in $\mathbf{Cat}(S)$, then check if the pullback is in $\mathbf{Cat}'(S)$. See example 2.27 for cases when this happens.

but we will not be using this, so will not make it precise.

It follows immediately from the definition that given maps $N \longrightarrow M$ and $M \longrightarrow X_0$, there is a canonical isomorphism

$$X[M][N] \simeq X[N]. \tag{4}$$

with object component the identity map.

2.23. Remark. If we agree to follow the convention that $M \times_N N = M$ is the pullback along the identity arrow id_N, then $X[X_0] = X$. This also simplifies other results of this paper, so will be adopted from now on.

One consequence of this assumption is that the iterated fibre product

$$M \times_M M \times_M \ldots \times_M M,$$

bracketed in any order, is *equal* to M. We cannot, however, equate two bracketings of a general iterated fibred product; they are only canonically isomorphic.

In all that follows, 'category' will mean object of $\mathbf{Cat}'(S)$ and similarly for 'functor' and 'natural transformation/isomorphism'.

2.24. Lemma. *Let X be a category and $M \longrightarrow X_0$, $N \longrightarrow X_0$ arrows in S such that $M \times_{X_0} N$ exists. Then the following square is a strict pullback*

$$\begin{tikzcd}
X[M \times_{X_0} N] \arrow[r] \arrow[d] & X[N] \arrow[d] \\
X[M] \arrow[r] & X
\end{tikzcd}$$

when the various base changes exist.

Proof. Consider the following cube

$$\begin{tikzcd}
X[M \times_{X_0} N] \arrow[rr] \arrow[dr] \arrow[dd] & & X[N] \arrow[dr] \arrow[dd] & \\
& X[M] \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
\mathrm{codisc}(M \times_{X_0} N) \arrow[rr] \arrow[dr] & & \mathrm{codisc}(N) \arrow[dr] & \\
& \mathrm{codisc}(M) \arrow[rr] & & \mathrm{codisc}(X_0)
\end{tikzcd}$$

The bottom and sides are pullbacks, either by definition, or using (4), and so the top is a pullback. ∎

The following technical lemma will be useful later. Even though Obj does not extend to a 2-functor, it captures some of the interaction between the fibrational nature of Obj and the 2-category nature of $\mathbf{Cat}'(S)$.

2.25. Lemma. *Let $f, g\colon disc(M) \longrightarrow Y$ be functors and $a\colon f \Rightarrow g$ be a natural isomorphism. Assume further that the base change of Y along f exists. Then the base change of Y along g exists, there is an isomorphism*

$$M^2 \times_{f^2, Y_0^2} Y_1 \simeq M^2 \times_{g^2, Y_0^2} Y_1$$

commuting with the projections to M^2, and in fact the categories $Y[M \xrightarrow{f} Y_0]$ and $Y[M \xrightarrow{g} Y_0]$ are isomorphic via a functor which is the identity on objects.

Proof. Supressing the canonical isomorphisms $X_0^2 \times_{Y_0^2} Y_1 \simeq X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0$, the required isomorphism is

$$\begin{aligned} X_0 \times_{f,Y_0} Y_1 \times_{Y_0,f} X_0 &\xrightarrow{(\mathrm{id},-a)\times \mathrm{id}\times(a,\mathrm{id})} X_0 \times_{g,Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0,g} X_0 \\ &\xrightarrow{\mathrm{id}\times m \times \mathrm{id}} X_0 \times_{g,Y_0} Y_1 \times_{Y_0,g} X_0. \end{aligned}$$

which is the identity map when restricted to the X_0 factors, from which the claim follows. ■

2.26. Remark. We will need to prove several times that certain pullbacks in S exist. We shall do this by implicitly exploiting the fact the presheaf category $\widehat{S} := \mathbf{Cat}(S^{op}, \mathbf{Set})$ is the completion of S together with the Yoneda lemma. In essence, one takes the pullback of a diagram in $\widehat{S}$, shows it is isomorphic to a representable presheaf, hence the pullback exists. We shall do this implicitly, by assuming that the pullback exists in S and showing it is isomorphic to a given pre-existing object in S (and all the requisite diagrams commute) by maps constructed out of given maps in S.

We are interested in 2-categories $\mathbf{Cat}'(S)$ which admits base change along a given class of maps E from S. The main examples for our purposes are when S is one of: **Top** (a category of topological spaces, convenient or otherwise), **Diff**, a category of manifolds (finite or infinite dimensional – the latter Fréchet for choice), **Grp** (or more generally a semi-abelian category), **Sch** (some category of schemes) or an abelian category $\mathcal{A}$, or a topos $\mathcal{E}$ (or more generally a Barr-exact category). The reader is invited to consider their own examples.

2.27. Example. If **LieGpd** denotes the 2-category of Lie groupoids, which are defined such that the source and target maps are submersions, then consider the following sub-2-categories:

- Proper Lie groupoids: those X where the map $(s, t)\colon X_1 \longrightarrow X_0^2$ is proper,
- étale Lie groupoids: those X where the maps $s, t\colon X_1 \longrightarrow X_0$ are local diffeomorphisms (i.e. étale),

- Proper étale Lie groupoids – also known as orbifolds [Moerdijk, 2002].

All of these are good examples of $\mathbf{Cat}'(\mathbf{Diff})$. Differentiable stacks of varying sorts are modelled by groupoids in these 2-categories.Note that codisc(M) is not an object of the 2-category of proper étale groupoids, but the base change along any submersion exists in that 2-category (see lemma 7.7).

The third item in example 2.27 illustrates an important point: the intersection of any two sub-2-categories of $\mathbf{Cat}(S)$ that satisfy our assumptions will also satisfy the assumptions.

2.28. Example. In [Noohi, 2005a] we find the definition of a class LF of *local fibrations*, which are maps in **Top** satisfying the properties:

1. Open embeddings are in LF,
2. LF is closed under composition,
3. LF is stable under pullback and local on the target,[6]
4. LF is stable under coproducts when considered as elements of a slice category $\mathbf{Top}/Y$.

We then consider the 2-categories $\mathbf{Cat}^{LF}(\mathbf{Top})$, $\mathbf{Gpd}^{LF}(\mathbf{Top})$ of topological categories and groupoids where the source and target maps belong to LF. Examples of classes LF include: open maps, local homeomorphisms, locally split maps, local Serre/Hurewicz fibrations.

Groupoids in $\mathbf{Gpd}^{LF}(\mathbf{Top})$ model topological stacks with varying properties, and all admit base change along arrows of the form $\coprod U_i \longrightarrow X$ where $\{U_i\}$ is an open cover of X (see lemma 7.7).

These two examples have a 'geometric' flavour, but we are not restricted to examples that look like stacks. Since **Grp** is a Mal'cev category, categories in **Grp** are all groupoids, so we only consider groupoids in **Grp**. It is well known that groupoids in **Grp** model pointed, connected 2-types, and $\pi_1(X) := Aut(1_{X_0})$ and $\pi_0(X) := X_0/X_1$ correspond to the second and first homotopy groups of the homotopy type corresponding to the groupoid X. It is therefore an interesting question to consider models of localisations of 2-types, for example at a set $\mathcal{P}$ of primes, or models of nilpotent 2-types. More generally, consider a class of groups $\mathcal{C}$ satisfying: if given an epimorphism of groups, $p\colon G \longrightarrow H$, then $G \in \mathcal{C}$ if and only if $H, \ker(p) \in \mathcal{C}$. For example, we could take the classes of p-groups, finitely generated p-groups, nilpotent groups. We can then consider sub-2- categories of $\mathbf{Gpd}(\mathbf{Grp})$ where π_1 and π_0 are either elements of $\mathcal{C}$, or are localised at $\mathcal{C}$. We shall not pursue this further here.

[6] A class of maps is *local on the target* if given an open cover $U \longrightarrow Y$, and if the projection $U \times_Y X \longrightarrow U$ is in that class, then $X \longrightarrow Y$ is in that class.

One last example we consider is groupoid schemes, which correspond to algebraic stacks. Without going into details, algebraic stacks are stacks of groupoids

$$\mathcal{X} \colon \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Gpd}$$

on the category of schemes, satisfying two generic conditions:

1. There is a representable cover $X \longrightarrow \mathcal{X}$, from a representable stack X, defined via some pretopology J on **Sch** (in particular, it is a map belonging to a pullback stable class);

2. The diagonal $\mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is representable and belongs to a class of maps which is pullback stable.

We gloss over the details about what it means for these various classes of maps to work and so on. The important point for us is that these two conditions imply that there is a groupoid $X^{[2]} := X \times_{\mathcal{X}} X \rightrightarrows X$ internal to **Sch** where the source and target maps belong to J, and where $(s,t) \colon X^{[2]} \longrightarrow X \times X$ belongs to a pullback-stable class of maps. Since **Sch** is finitely complete, we do not need to worry about pullbacks existing, only about the base change existing in the 2-category $\mathbf{Cat}'(\mathbf{Sch})$ we are interested in. (Note that this sort of base change is not the same as base change as considered in algebraic geometry – *caveat lector.*) Since (s,t) belongs to a pullback-stable class, this obviously presents no obstruction to the existence of base change along any map. The only thing we need to check is that source and target maps are what they need to be. See example 7.9.

3. Sites and covers

The idea of *localness* is inherent in many constructions in algebraic topology and algebraic geometry. For an abstract category the concept of 'local' is encoded by a Grothendieck pretopology. Localness is needed to be able to talk about local sections of a map in a category – a concept that will replace surjectivity when moving from **Set** to more general categories. This section gathers definitions and notations for later use.

3.1. Definition. A *Grothendieck pretopology* (or simply *pretopology*) on a category S is a collection J of families

$$\{(U_i \longrightarrow A)_{i\in I}\}_{A\in \mathrm{Obj}(S)}$$

of morphisms for each object $A \in S$ satisfying the following properties

1. $(\mathrm{id} \colon A \longrightarrow A)$ is in J for every object A.

2. Given a map $B \longrightarrow A$, for every $(U_i \longrightarrow A)_{i\in I}$ in J the pullbacks $B \times_A A_i$ exist and $(B \times_A A_i \longrightarrow B)_{i\in I}$ is in J.

3. For every $(U_i \longrightarrow A)_{i\in I}$ in J and for a collection $(V_k^i \longrightarrow U_i)_{k\in K_i}$ from J for each $i \in I$, the family of composites

$$(V_k^i \longrightarrow A)_{k\in K_i, i\in I}$$

are in J.

Families in J are called *covering families*. A category S equipped with a pretopology J is called a *site*, denoted (S, J).

3.2. Example. The basic example is the lattice of open sets of a topological space, seen as a category in the usual way, where a covering family of an open $U \subset X$ is an open cover of U by opens in X. This is to be contrasted with the pretopology $\mathcal{O}$ on **Top**, where the covering families of a space are just open covers of the whole space.

3.3. Example. On **Grp** the class of surjective homomorphisms form a pretopology.

3.4. Example. On **Top** the class of numerable open covers (i.e. those that admit a subordinate partition of unity [Dold, 1963]) form a pretopology. Much of traditional bundle theory is carried out using this site, for example, the Milnor classifying space classifies bundles which are locally trivial over numerable covers [Mil56, Dold, 1963, tom Dieck, 1966].

3.5. Definition. Let (S, J) be a site. The pretopology J is called a *singleton pretopology* if every covering family consists of a single arrow $(U \longrightarrow A)$. In this case a covering family is called a *cover*.

3.6. Example. In **Top**, the classes of covering maps, local section admitting maps, surjective étale maps and open surjections are all examples of singleton pretopologies. The results of [Pronk, 1996] pertaining to topological groupoids were carried out using the site of open surjections.

3.7. Example. The class *Subm* of surjective submersions in **Diff**, the category of smooth manifolds, is a singleton pretopology.

There are many different and useful pretopologies on the category **Sch** of schemes, such as the Zariski, étale, *fpqc* and Nisnevich pretopologies. Only the author's lack of familiarity with these will prevent these from playing much of a rôle in this paper. The knowledgeable reader is invited to try out their own examples from algebraic geometry in parallel to those given here.

3.8. Definition. A covering family $(U_i \longrightarrow A)_{i\in I}$ is called *effective* if A is the colimit of the following diagram: the objects are the U_i and the pullbacks $U_i \times_A U_j$, and the arrows are the projections

$$U_i \leftarrow U_i \times_A U_j \longrightarrow U_j.$$

If the covering family consists of a single arrow $(U \longrightarrow A)$, this is the same as saying $U \longrightarrow A$ is a regular epimorphism.

3.9. Definition. A site is called *subcanonical* if every covering family is effective.

3.10. Example. On **Top**, the usual pretopology of opens, the pretopology of numerable covers and that of open surjections are subcanonical.

3.11. Example. In a regular category, the regular epimorphisms form a subcanonical singleton pretopology.

In fact, the (pullback stable) regular epimorphisms[7] in any category form the largest subcanonical singleton pretopology, so it has its own name.

3.12. Definition. The *canonical singleton pretopology* R is the largest class of regular epimorphisms which are pullback stable. It contains all the subcanonical singleton pretopologies.

3.13. Remark. If $U \longrightarrow A$ is an effective cover, a functor $\check{C}(U) \longrightarrow \mathrm{disc}(B)$ gives a unique arrow $A \longrightarrow B$. This follows immediately from the fact A is the colimit of $\check{C}(U)$.

3.14. Definition. A *finitary* (resp. *infinitary*) *extensive* category is a category with finite (resp. small) coproducts such that the following condition holds: let I be a a finite set (resp. any set), then, given a collection of commuting diagrams

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & Z \arrow[d] \\ A_i \arrow[r] & \coprod_{i\in I} A_i \end{tikzcd} ,$$

one for each $i \in I$, the squares are all pullbacks if and only if the collection $\{X_i \longrightarrow Z\}_I$ forms a coproduct diagram.

In such a category there is a strict initial object (i.e. given a map $A \longrightarrow 0$ with 0 initial, we have $A \simeq 0$).

3.15. Example. **Top** is infinitary extensive.

3.16. Example. $\mathbf{Ring}^{op}$ is finitary extensive.

In **Top** we can take an open cover $\{U_i\}_I$ of a space X and replace it with the single map $\coprod_I U_i \longrightarrow X$, and work just as before using this new sort of cover, using the fact **Top** is extensive. The sort of sites that mimic this behaviour are called *superextensive*.

[7] Of course, the nomenclature was decided the other way around; 'subcanonical' meaning 'contained in the canonical pretopology'.

3.17. Definition. (Bartels-Shulman) A *superextensive site* is an extensive category S equipped with a pretopology J containing the families

$$(U_i \longrightarrow \coprod_I U_i)_{i\in I}$$

and such that all covering families are bounded; this means that for a finitely extensive site, the families are finite, and for an infinitary site, the families are small. The pretopology in this instance will also be called superextensive.

3.18. Example. Given an extensive category S, the *extensive pretopology* has as covering families the bounded collections $(U_i \longrightarrow \coprod_I U_i)_{i\in I}$. The pretopology on any superextensive site contains the extensive pretopology.

3.19. Example. The category **Top** with its usual pretopology of open covers is a superextensive site.

3.20. Example. A topos with the coherent pretopology is finitary superextensive, and a Grothendieck topos with the canonical pretopology is infinitary superextensive.

Given a superextensive site, one can form the class $\amalg J$ of arrows $\coprod_I U_i \longrightarrow A$.

3.21. Proposition. *The class $\amalg J$ is a singleton pretopology, and is subcanonical if and only if J is.*

Proof. Since identity arrows are covers for J they are covers for $\amalg J$. The pullback of a $\amalg J$-cover $\coprod_I U_i \longrightarrow A$ along $B \longrightarrow A$ is a $\amalg J$-cover as coproducts and pullbacks commute by definition of an extensive category. Now for the third condition we use the fact that in an extensive category a map

$$f\colon B \longrightarrow \coprod_I A_i$$

implies that $B \simeq \coprod_I B_i$ and $f = \coprod_i f_i$. Given $\amalg J$-covers $\coprod_I U_i \longrightarrow A$ and $\coprod_J V_j \longrightarrow (\coprod_I U_i)$, we see that $\coprod_J V_j \simeq \coprod_I W_i$. By the previous point, the pullback

$$\coprod_I U_k \times_{\coprod_I U_{i'}} W_i$$

is a $\amalg J$-cover of U_i, and hence $(U_k \times_{\coprod_I U_{i'}} W_i \longrightarrow U_k)_{i\in I}$ is a J-covering family for each $k \in I$. Thus

$$(U_k \times_{\coprod_I U_{i'}} W_i \longrightarrow A)_{i,k\in I}$$

is a J-covering family, and so

$$\coprod_J V_j \simeq \coprod_{k\in I} \left(\coprod_I U_k \times_{\coprod_I U_{i'}} W_i \right) \longrightarrow A$$

is a $\amalg J$-cover.

The map $\coprod_I U_i \longrightarrow A$ is the coequaliser of $\coprod_{I\times I} U_i \times_A U_j \rightrightarrows \coprod_I U_i$ if and only if A is the colimit of the diagram in definition 3.8. Hence $(\coprod_I U_i \longrightarrow A)$ is effective if and only if $(U_i \longrightarrow A)_{i\in I}$ is effective ■

Notice that the original superextensive pretopology J is generated by the union of $\amalg J$ and the extensive pretopology.

3.22. Definition. Let (S, J) be a site. An arrow $P \longrightarrow A$ in S is called a *J-epimorphism* (or simply *J-epi*) if there is a covering family $(U_i \longrightarrow A)_{i \in I}$ and a lift

$$\begin{tikzcd} & P \arrow[d] \\ U_i \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

for every $i \in I$. The class of J-epimorphisms will be denoted (J-epi).

This definition is equivalent to the definition in III.7.5 in [Mac Lane-Moerdijk, 1992]. The dotted maps in the above definition are called local sections, after the case of the usual open cover pretopology on **Top**. If the pretopology is left unnamed, we will refer to *local epimorphisms*.

One reason we are interested in superextensive sites is the following

3.23. Lemma. *If (S, J) is a superextensive site, the class of J-epimorphisms is precisely the class of* $\amalg J$*-epimorphisms.*

If S has all pullbacks then the class of J-epimorphisms form a pretopology. In fact they form a pretopology with an additional property – it is *saturated*. The following is adapted from [Barr-Wells, 1984].[8]

3.24. Definition. A singleton pretopology J is *saturated* if whenever the composite $V \longrightarrow U \longrightarrow A$ is in J, then $U \longrightarrow A$ is in J.

It should be pointed out that a saturated singleton pretopology is called a *calibration* in [Bénabou, 1975]. Note that only a slightly weaker condition on S is necessary for (J-epi) to be a pretopology.

3.25. Example. Let (S, J) be a site. If pullbacks of J-epimorphisms exist then the collection (J-epi) of J-epimorphisms is a saturated pretopology.

There is a definition of 'saturated' for arbitrary pretopologies, but we will use only this one. Another way to pass from an arbitrary pretopology to a singleton one in a canonical way is this:

3.26. Definition. The *universal J-epimorphisms* $J_{un} \subset$ (J-epi) associated to a pretopology J on an arbitrary category S is the largest class of those J- epimorphisms which are pullback stable.

It is clear that $(J_{un})_{un} = J_{un}$, and that when pullbacks exist, (J-epi) $= J_{un}$.

[8]Note that what we are calling a Grothendieck *pretopology* is referred to as a Grothendieck *topology* in [Barr-Wells, 1984].

3.27. EXAMPLE. The universal $\mathcal{O}$-epimorphisms for the class of open covers $\mathcal{O}$ in **Diff** is *Subm*, the class of surjective submersions. Notice that all surjective submersions admit local sections (essentially by the implicit function theorem), whereas not all maps in ($\mathcal{O}$-epi) are submersions, so that $\mathcal{O}_{un} \neq (\mathcal{O}\text{-epi})$.

If J is a singleton pretopology, it is clear that $J \subset J_{un}$. In fact J_{un} contains all the covering families of J with only one element when J is any pretopology.

From lemma 3.23 we have

3.28. COROLLARY. *In a superextensive site* (S, J)*, we have* $J_{un} = (\amalg J)_{un}$.

One class of extensive categories which are of particular interest is those that also have finite/small limits. These are called *lextensive*. For example, **Top** is infinitary lextensive, as is a Grothendieck topos. In contrast, a general topos is finitary lextensive. In a lextensive category

$$J_{un} = (\amalg J)_{un} = (J\text{-epi}) = (\amalg J\text{-epi}).$$

Sometimes a pretopology J contains a smaller pretopology that still has enough covers to compute the same J-epis.

3.29. DEFINITION. If J and K are two singleton pretopologies with $J \subset K$, such that $K \subset J_{un}$, then J is said to be *cofinal* in K, denoted $J \leq K$.

Clearly $J \leq J_{un}$ for any singleton pretopology J.

3.30. LEMMA. *If* $J \leq K$*, then* $J_{un} = K_{un}$.

4. Weak equivalences

Equivalences in **Cat**—assuming the axiom of choice—are precisely the fully faithful, essentially surjective functors. For internal categories, however, this is not the case. In addition, we need to make use of a pretopology to make the 'surjective' part of essentially surjective meaningful. To start with we shall just assume that our ambient category is equipped with a class E of morphisms which is pullback stable. We shall also assume throughout this section that our 2-category of internal categories has enough groupoids, so that we can use the object of isomorphisms of an internal category.

The following definition first made its appearence in [Bunge-Paré, 1979] for S finitely complete and regular, and E the class of regular epimorphisms, in the context of stacks and indexed categories.

4.1. DEFINITION. [Bunge-Paré, 1979, Everaert et al, 2005] Let S be a category with a specified class E of morphisms. An internal functor $f : X \longrightarrow Y$ in S is called

1. *fully faithful* if

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "(s,t)"'] & Y_1 \arrow[d, "(s,t)"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

   is a pullback diagram

2. *essentially E-surjective* if the arrow labelled $\circledast$ is in E

$$\begin{tikzcd}
 & & X_0 \times_{Y_0} Y_1^{iso} \arrow[dll] \arrow[d] \arrow[ddrr, bend left, "\circledast"] & & \\
X_0 \arrow[d, "f_0"'] & & Y_1^{iso} \arrow[dll, "s"'] \arrow[drr, "t"] & & \\
Y_0 & & & & Y_0
\end{tikzcd}$$

3. an *E-equivalence* if it is fully faithful and essentially E-surjective.

The class of E-equivalences will be denoted W_E.

If (S, J) is a site, then we are interested in the class $E = J_{un}$. The class of J_{un}-equivalences will be denoted W_J and they will, following [Everaert et al, 2005], be referred to as J-equivalences. If mention of J is suppressed, they will be called *weak equivalences.* This usage differs from *loc. cit.* where the class of (J-epi)-equivalences are referred to as J-equivalences. In a finitely complete category there is no difference, but this definition allows later proofs to hold for non-finitely complete categories.

4.2. Example. The canonical functor $X[M] \longrightarrow X$ is always fully faithful, by definition.

4.3. Example. If $X \longrightarrow Y$ is an internal equivalence, then it is a J-equivalence for all pretopologies J such that split epimorphisms are contained in J_{un} [Everaert et al, 2005]. In fact, if T denotes the trivial pretopology (only isomorphisms are covers) on a finitely complete category, the T-equivalences are precisely the internal equivalences.

4.4. Remark. This example does not include Lie groupoids as $\mathcal{O}_{un} = \mathit{Subm}$ does not contain the split epimorphisms. Internal equivalences *are* $\mathcal{O}$-equivalences, but this is a result that uses the structure of Lie groupoids in an essential way. In fact we have chosen to take J_{un}-equivalences as standard for non-finitely complete categories as this reflects the usage in the Lie groupoid literature.

4.5. Lemma. *If $f : X \longrightarrow Y$ is a functor such that f_0 is in J_{un}, then f is essentially J_{un}-surjective.*

4.6. Corollary. *If (S, J) is a site, X a category in S and $(U \longrightarrow X_0)$ is a covering family (e.g. J is a singleton pretopology), the functor $X[U] \longrightarrow X$ is a J-equivalence.*

Proof. The object component of the canonical functor $X[U] \longrightarrow X$ is $U \longrightarrow X_0$ and since it is in J it is in J_{un}. Hence as $X[U] \longrightarrow X$ is fully faithful it is a J-equivalence. ■

We now consider some easy results on the behavious of weak equivalences under pullbacks, both strict and weak. First, fully faithful functors are stable under strict pullback.

4.7. Lemma. *If $f\colon X \longrightarrow Y$ is fully faithful, and $g\colon Z \longrightarrow Y$ is any functor,* pr_1 *in*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is fully faithful whenever the strict pullback exists.

Proof. Assuming the pullback exists, the following chain of isomorphisms establishes the claim

$$\begin{aligned} (Z_0 \times_{Y_0} X_0)^2 \times_{Z_0^2} Z_1 &\simeq X_0^2 \times_{Y_0^2} Z_1 \\ &\simeq (X_0^2 \times_{Y_0^2} Y_1) \times_{Y_1} Z_1 \\ &\simeq X_1 \times_{Y_1} Z_1, \end{aligned}$$

the last following from the fact f is fully faithful. ■

The following terminology is adapted from [Everaert et al, 2005], although strictly speaking this map is only a fibration when model structure from *loc. cit.* exists, or more generally when the structure of a category of fibrant objects exists.

4.8. Definition. An internal functor $f\colon X \longrightarrow Y$ is called a *trivial E-fibration* if it is fully faithful and $f_0 \in E$.

4.9. Lemma. *Let* $\mathbf{Cat}'(S)$ *admit base change along arrows in E and cotensors with* $\mathbf{I}$. *If a functor $f\colon X \longrightarrow Y$ is an E-equivalence, the strict pullback $X \times_Y Y^{\mathbf{I}}$ exists and*

$$X \times_Y Y^{\mathbf{I}} \xrightarrow{\mathbf{t}\circ\mathrm{pr}_2} Y$$

is a trivial E-fibration.

Proof. The object component of $\mathbf{t} \circ \mathrm{pr}_2$ is $t \circ \mathrm{pr}_2$, which is in E by definition as f is essentially E-surjective. Consider now the pullback

$$\begin{tikzcd} (X_0 \times_{Y_0} Y_1^{iso})^2 \times_{Y_0^2} Y_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ (X_0 \times_{Y_0} Y_1^{iso})^2 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

Remark 2.18 tells us that the pullback is isomorphic to $X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}}$ in the pullback square

$$\begin{tikzcd} X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} \arrow[r, "\mathrm{pr}_2"] \arrow[dd] & Y_1^{\mathbf{I}} \arrow[d, "\mathrm{pr}_1"] \\ & Y_1 \arrow[d] \\ X_0^2 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

but if f is fully faithful,

$$\begin{aligned} X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} &\simeq X_0^2 \times_{Y_0^2} Y_1 \times_{Y_1} Y_1^{\mathbf{I}} \\ &\simeq X_1 \times_{Y_1} Y_1^{\mathbf{I}}, \end{aligned}$$

hence $\mathbf{t} \circ \mathrm{pr}_2$ is fully faithful. ■

The internal category $X \times_Y Y^{\mathbf{I}}$ is called the mapping path space construction in [Everaert et al, 2005] and if the model structure therein exists, the above follows from cofibration-acyclic fibration factorisation.

4.10. Corollary. *If the bipullback of an E-equivalence exists, it is again an E-equivalence.*

Proof. Trivial E-fibrations are stable under strict pullback and a bipullback of an E-equivalence is given by a strict pullback of a trivial E-fibration. ■

4.11. Lemma. *Let $\mathbf{Cat}'(S)$ admit base change along a class E of arrows in S and admit cotensors with $\mathbf{I}$. Then bipullbacks of E-equivalences exist in $\mathbf{Cat}'(S)$.*

Proof. Suppose we want to form the bipullback of

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

where f is an E-equivalence. It is an easy exercise to see that this is given by the base change of Z along $Z_0 \times_{Y_0} (Y_1 \times_{Y_0,f} X_0) \xrightarrow{\mathrm{pr}_1} Z_0$, which is in E, as it is the pullback of $Y_1 \times_{Y_0,f} X_0 \longrightarrow Y_0$, which is in E. ■

5. Anafunctors

We now let J be a subcanonical singleton pretopology on the ambient category S. In this section we assume that $\mathbf{Cat}'(S)$ admits base change along arrows in the given pretopology J. This is a slight generalisation of what is considered in [Bartels, 2006], where only $\mathbf{Cat}'(S) = \mathbf{Cat}(S)$ is considered.

5.1. Definition. [Makkai, 1996, Bartels, 2006] An *anafunctor* in (S, J) from a category X to a category Y consists of a cover $(U \longrightarrow X_0)$ and an internal functor

$$f\colon X[U] \longrightarrow Y.$$

Since $X[U]$ is an object of $\mathbf{Cat}'(S)$, an anafunctor is a span in $\mathbf{Cat}'(S)$, and can be denoted

$$(U, f)\colon X \twoheadrightarrow\!\!\!\!\!\mapsto Y.$$

5.2. Example. For an internal functor $f\colon X \longrightarrow Y$ in S, define the anafunctor $(X_0, f)\colon X \twoheadrightarrow\!\!\!\!\!\mapsto Y$ as the following span

$$X \xleftarrow{=} X[X_0] \xrightarrow{f} Y.$$

We will blur the distinction between these two descriptions. If $f = id\colon X \longrightarrow X$, then (X_0, id) will be denoted simply by id_X.

5.3. Example. If $U \longrightarrow A$ is a cover in (S, J) and $\mathbf{B}G$ is a groupoid with one object in S (i.e. a group), an anafunctor $(U, g)\colon \mathrm{disc}(A) \twoheadrightarrow\!\!\!\!\!\mapsto \mathbf{B}G$ is the same thing as a Čech cocycle.

5.4. Definition. [Makkai, 1996, Bartels, 2006] Let (S, J) be a site and let

$$(U, f), (V, g)\colon X \twoheadrightarrow\!\!\!\!\!\mapsto Y$$

be anafunctors in S. A *transformation*

$$\alpha\colon (U, f) \Rightarrow (V, g)$$

from (U, f) to (V, g) is an internal natural transformation

$$\begin{tikzcd}
& X[U \times_{X_0} V] \arrow[dl] \arrow[dr] & \\
X[U] \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & X[V] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

If α is a natural isomorphism, then α will be called an *isotransformation*. In that case we say (U, f) *is isomorphic to* (V, g). Clearly all transformations between anafunctors between internal groupoids are isotransformations.

5.5. Example. Given functors $f, g\colon X \longrightarrow Y$ between categories in S, and a natural transformation $a\colon f \Rightarrow g$, there is a transformation $a\colon (X_0, f) \Rightarrow (X_0, g)$ of anafunctors, given by the component $X_0 \times_{X_0} X_0 = X_0 \xrightarrow{a} Y_1$.

5.6. Example. If $(U, g), (V, h)\colon \mathrm{disc}(A) \twoheadrightarrow\!\!\!\!\!\mapsto \mathbf{B}G$ are two Čech cocycles, a transformation between them is a coboundary on the cover $U \times_A V \longrightarrow A$.

5.7. Example. Let $(U,f)\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ be an anafunctor in S. There is an isotransformation $1_{(U,f)}\colon (U,f) \Rightarrow (U,f)$ called the *identity transformation*, given by the natural transformation with component

$$U \times_{X_0} U \simeq (U \times U) \times_{X_0^2} X_0 \xrightarrow{id_U^2 \times e} X[U]_1 \xrightarrow{f_1} Y_1 \tag{5}$$

5.8. Example. [Makkai, 1996] Given anafunctors $(U,f)\colon X \longrightarrow Y$ and $(V, f \circ k)\colon X \longrightarrow Y$ where $k\colon V \simeq U$ is an isomorphism over X_0, a *renaming transformation*

$$(U,f) \Rightarrow (V, f \circ k)$$

is an isotransformation with component

$$1_{(U,f)} \circ (k \times \mathrm{id}) : V \times_{X_0} U \longrightarrow U \times_{X_0} U \longrightarrow Y_1.$$

The isomorphism k will be referred to as a *renaming isomorphism*.
More generally, we could let $k\colon V \longrightarrow U$ be any refinement, and this prescription also gives an isotransformation $(U,f) \Rightarrow (V, f \circ k)$.

See example 5.10 below for another useful example of an isotransformation.

We define (following [Bartels, 2006]) the composition of anafunctors as follows. Let

$$(U,f)\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y \quad \text{and} \quad (V,g)\colon Y \longrightarrow\!\!\!\!\!\!\mapsto Z$$

be anafunctors in the site (S,J). Their composite $(V,g) \circ (U,f)$ is the composite span defined in the usual way. It is again a span in $\mathbf{Cat}'(S)$.

$$\begin{tikzcd}
 & & X[U \times_{Y_0} V] \arrow[dl] \arrow[dr, "f^V"] & & \\
 & X[U] \arrow[dl] \arrow[dr, "f"'] & & Y[V] \arrow[dl] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

The square is a pullback by lemma 2.24 (which exists because $V \longrightarrow Y_0$ is a cover), and the resulting span is an anafunctor because $V \longrightarrow Y_0$, and hence $U \times_{Y_0} V \longrightarrow X_0$, is a cover, and using (4). We will sometimes denote the composite by $(U \times_{Y_0} V, g \circ f^V)$.

Consider the special case when $V = Y_0$, and hence (Y_0, g) is just an ordinary functor. Then there is a renaming transformation (the identity transformation!) $(Y_0, g) \circ (U,f) \Rightarrow (U, g \circ f)$, using the equality $U \times_{Y_0} Y_0 = U$ (by remark 2.23). If we let $g = \mathrm{id}_Y$, then we see that (Y_0, id_Y) is a strict unit on the left for anafunctor composition. Similarly, considering $(V,g) \circ (Y_0, \mathrm{id})$, we see that (Y_0, id_Y) is a two-sided strict unit for anafunctor composition. In fact, we have also proved

5.9. Lemma. *Given two functors $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ in S, their composition as anafunctors is equal to their composition as functors:*

$$(Y_0, g) \circ (X_0, f) = (X_0, g \circ f).$$

5.10. Example. As a concrete and relevant example of a renaming transformation we can consider the triple composition of anafunctors

$$\begin{aligned} (U, f)\colon X &\nrightarrow Y, \\ (V, g)\colon Y &\nrightarrow Z, \\ (W, h)\colon Z &\nrightarrow A. \end{aligned}$$

The two possibilities of composing these are

$$\left((U \times_{Y_0} V) \times_{Z_0} W, h \circ (gf^V)^W\right), \quad \left(U \times_{Y_0} (V \times_{Z_0} W), h \circ g^W \circ f^{V \times_{Z_0} W}\right)$$

The unique isomorphism $(U \times_{Y_0} V) \times_{Z_0} W \simeq U \times_{Y_0} (V \times_{Z_0} W)$ commuting with the various projections is then the required renaming isomorphism. The isotransformation arising from this renaming transformation is called the *associator*.

A simple but useful criterion for describing isotransformations where one of the anafunctors involved is a functor is as follows.

5.11. Lemma. *An anafunctor $(V, g)\colon X \nrightarrow Y$ is isomorphic to a functor $(X_0, f)\colon X \nrightarrow Y$ if and only if there is a natural isomorphism*

$$\begin{tikzcd}
 & X[V] \arrow[dl] \arrow[dr, "g"] & \\
X \arrow[rr, bend right, "f"'] & \overset{\sim}{\Rightarrow} & Y
\end{tikzcd}$$

Just as there is composition of natural transformations between internal functors, there is a composition of transformations between internal anafunctors [Bartels, 2006]. This is where the effectiveness of our covers will be used in order to construct a map locally over some cover. Consider the following diagram

$$\begin{tikzcd}
 & & X[U \times_{X_0} V \times_{X_0} W] \arrow[dl] \arrow[dr] & & \\
 & X[U \times_{X_0} V] \arrow[dl] \arrow[dr] & & X[V \times_{X_0} W] \arrow[dl] \arrow[dr] & \\
X[U] \arrow[drr, "f"'] & \overset{a}{\Rightarrow} & X[V] \arrow[d, "g"] & \overset{b}{\Rightarrow} & X[W] \arrow[dll, "h"] \\
 & & Y & &
\end{tikzcd}$$

from which we can form a natural transformation between the leftmost and the rightmost composites as functors in S. This will have as its component the arrow

$$\widetilde{ba} \colon U \times_{X_0} V \times_{X_0} W \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} U \times_{X_0} V \times_{X_0} V \times_{X_0} W \xrightarrow{a\times b} Y_1 \times_{Y_0} Y_1 \xrightarrow{m} Y_1$$

in S. Notice that the Čech groupoid of the cover

$$U \times_{X_0} V \times_{X_0} W \longrightarrow U \times_{X_0} W \tag{6}$$

is

$$U \times_{X_0} V \times_{X_0} V \times_{X_0} W \rightrightarrows U \times_{X_0} V \times_{X_0} W,$$

using the two projections $V \times_{X_0} V \longrightarrow V$. Denote this pair of parallel arrows by $s, t \colon UV^2W \rightrightarrows UVW$ for brevity. In [Bartels, 2006], section 2.2.3, we find the commuting diagram

$$\begin{tikzcd} UV^2W \arrow[r, "t"] \arrow[d, "s"'] & UVW \arrow[d, "\widetilde{ba}"] \\ UVW \arrow[r, "\widetilde{ba}"'] & Y_1 \end{tikzcd} \tag{7}$$

(again, this can be checked by using elements) and so we have a functor $\check{C}(U \times_{X_0} V \times_{X_0} W) \longrightarrow \mathrm{disc}(Y_1)$. Our pretopology J is assumed to be subcanonical, and using remark 3.13 this gives us a unique arrow $ba \colon U \times_{X_0} W \longrightarrow Y_1$, the composite of a and b.

5.12. Remark. In the special case that $U \times_{X_0} V \times_{X_0} W \longrightarrow U \times_{X_0} W$ is an isomorphism (or is even just split), the composite transformation has

$$U \times_{X_0} W \longrightarrow U \times_{X_0} V \times_{X_0} W \xrightarrow{\widetilde{ba}} Y_1$$

as its component arrow. In particular, this is the case if one of a or b is a renaming transformation.

5.13. Example. Let $(U, f) : X \longrightarrow\!\!\!\!\mapsto Y$ be an anafunctor and $U'' \xrightarrow{j'} U' \xrightarrow{j} U$ successive refinements of $U \longrightarrow X_0$ (e.g isomorphisms). Let $(U', f_{U'})$ and $(U'', f_{U''})$ denote the composites of f with $X[U'] \longrightarrow X[U]$ and $X[U''] \longrightarrow X[U]$ respectively. The arrow

$$U \times_{X_0} U'' \xrightarrow{\mathrm{id}_U \times j \circ j'} U \times_{X_0} U \longrightarrow Y_1$$

is the component for the composition of the isotransformations $(U, f) \Rightarrow (U', f_{U'}), \Rightarrow (U'', f_{U''})$ described in example 5.8. Thus we can see that the composite of renaming transformations associated to isomorphisms ϕ_1, ϕ_2 is simply the renaming transformation associated to their composite $\phi_1 \circ \phi_2$.

5.14. EXAMPLE. If $a\colon f \Rightarrow g$, $b\colon g \Rightarrow h$ are natural transformations between functors $f, g, h\colon X \longrightarrow Y$ in S, their composite as transformations between anafunctors

$$(X_0, f), (X_0, g), (X_0, h)\colon X \nrightarrow Y.$$

is just their composite as natural transformations. This uses the equality

$$X_0 \times_{X_0} X_0 \times_{X_0} X_0 = X_0 \times_{X_0} X_0 = X_0,$$

which is due to our choice in remark 2.23 of canonical pullbacks

The first half of the following theorem is proposition 12 in [Bartels, 2006], and the second half follows because all the constructions of categories involved in dealing with anafunctors outlined above are still objects of $\mathbf{Cat}'(S)$.

5.15. THEOREM. *[Bartels, 2006] For a site (S, J) where J is a subcanonical singleton pretopology, internal categories, anafunctors and transformations form a bicategory $\mathbf{Cat}_{\mathrm{ana}}(S, J)$. If we restrict attention to a sub-2-category $\mathbf{Cat}'(S)$ which admits base change for arrows in J, we have an analogous full sub-bicategory $\mathbf{Cat}'_{\mathrm{ana}}(S, J)$.*

There is a strict 2-functor $\mathbf{Cat}'_{\mathrm{ana}}(S, J) \longrightarrow \mathbf{Cat}_{\mathrm{ana}}(S, J)$ which is the identity on 0-cells and induces isomorphisms on hom-categories. The following is the main result of this section, and allows us to relate anafunctors to the localisations considered in the next section.

5.16. PROPOSITION. *There is a strict 2-functor*

$$\alpha_J\colon \mathbf{Cat}'(S) \longrightarrow \mathbf{Cat}'_{\mathrm{ana}}(S, J)$$

sending J-equivalences to equivalences, and commuting with the respective inclusions into $\mathbf{Cat}(S)$ and $\mathbf{Cat}_{\mathrm{ana}}(S, J)$.

PROOF. We define α_J to be the identity on objects, and as described in examples 5.2, 5.5 on 1-cells and 2-cells (i.e. functors and transformations). We need first to show that this gives a functor $\mathbf{Cat}'(S)(X, Y) \longrightarrow \mathbf{Cat}'_{\mathrm{ana}}(S, J)(X, Y)$. This is precisely the content of example 5.14. Since the identity 1-cell on a category X in $\mathbf{Cat}'_{\mathrm{ana}}(S, J)$ is the image of the identity functor on S in $\mathbf{Cat}'(S)$, α_J respects identity 1-cells. Also, lemma 5.9 tells us that α_J respects composition. That α_J sends J-equivalences to equivalences is the content of lemma 5.18. ∎

In a site (S, J) where the axiom of choice holds—every J-epimorphism has a section[9]—one can prove that every J-equivalence between internal categories is in fact an internal equivalence of categories. It is precisely the lack of splittings that prevents this theorem from holding in general sites. The best one can do in a general site is described in the the following two lemmas, where we make the additional assumption that our 2-category of internal categories has enough groupoids.

[9] In other words, existence of local sections is enough to guarantee a global section.

5.17. LEMMA. *Let* $f\colon X \longrightarrow Y$ *be a* J*-equivalence, and choose a cover* $U \longrightarrow Y_0$ *and a local section* $s\colon U \longrightarrow X_0 \times_{Y_0} Y_1^{iso}$*. Then there is a functor* $Y[U] \longrightarrow X$ *with object component* $s' := \mathrm{pr}_1 \circ s\colon U \longrightarrow X_0$*.*

PROOF. The object component is given, we just need the arrow component. Denote the local section by $(s', \iota) : U \longrightarrow X_0 \times_{Y_0} Y_1^{iso}$. Consider the composite

$$\begin{aligned} Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U &\xrightarrow{(s',\iota)\times\mathrm{id}\times(-\iota,s')} (X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \\ &\hookrightarrow X_0 \times_{Y_0} Y_3 \times_{Y_0} X_0 \xrightarrow{\mathrm{id}\times m\times\mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0 \simeq X_1 \end{aligned}$$

where the last isomorphism arises from f being full faithful. It is clear that this commutes with source and target, because these are projection on the first and last factor at each step. To see that it respects identities and composition, just use the fact that the ι component will cancel with the $-\iota$ component. ∎

Hence we have an anafunctor $Y \multimap X$, and the next proposition tells us this is a pseudoinverse to f.

5.18. LEMMA. *Let* $f\colon X \longrightarrow Y$ *be a* J*-equivalence in* S*. There is an anafunctor*

$$(U, \bar{f})\colon Y \multimap X$$

and isotransformations

$$\begin{aligned} \iota&\colon (X_0, f) \circ (U, \bar{f}) \Rightarrow id_Y \\ \epsilon&\colon (U, \bar{f}) \circ (X_0, f) \Rightarrow id_X \end{aligned}$$

PROOF. We have the anafunctor $(U, \bar{f})$ from lemma 5.17. Since the anafunctors id_X, id_Y are actually functors, we can use lemma 5.11. Using the special case of anafunctor composition when the second is a functor, this tells us that ι will be given by a natural isomorphism

$$\begin{tikzcd} & X \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] & \\ Y[U] \arrow[ur, "\bar{f}"] \arrow[rr] & {} & Y \end{tikzcd}$$

This has component $\iota\colon U \longrightarrow Y_1^{iso}$, using the notation from the proof of the previous lemma. Notice that the composite $f_1 \circ \bar{f}_1$ is just

$$Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\iota\times\mathrm{id}\times-\iota} Y_1^{iso} \times_{Y_0} Y_1 \times_{Y_0} Y_1^{iso} \hookrightarrow Y_3 \xrightarrow{m} Y_1.$$

Since the arrow component of $Y[U] \longrightarrow Y$ is $U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\mathrm{pr}_2} Y_1$, ι is indeed a natural isomorphism using the diagram (2).

The other isotransformation is between $(X_0 \times_{Y_0} U, \bar{f} \circ \mathrm{pr}_2)$ and (X_0, id_X), and is given by the arrow

$$\epsilon\colon X_0 \times_{X_0} X_0 \times_{Y_0} U \simeq X_0 \times_{Y_0} U \xrightarrow{\mathrm{id}\times(s',a)} X_0 \times_{Y_0} (X_0 \times_{Y_0} Y_1) \simeq X_0^2 \times_{Y_0^2} Y_1 \simeq X_1$$

This has the correct source and target, as the object component of $\bar{f}$ is s', and the source is given by projection on the first factor of $X_0 \times_{Y_0} U$. This diagram

$$\begin{tikzcd}
(X_0 \times_{Y_0^2} U)^2 \times_{X_0^2} X_1 \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\simeq"'] & & X_1 \arrow[dd, "\simeq"] \\
U \times_{Y_0} X_1 \times_{Y_0} U \arrow[d, "-\iota\times f\times\iota"'] & & \\
(X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \arrow[rr, "\mathrm{id}\times m\times\mathrm{id}"'] & & X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0
\end{tikzcd}$$

commutes (a fact which can be checked using elements), and using (2) we see that ϵ is natural. ∎

6. Localising bicategories at a class of 1-cells

Ultimately we are interesting in inverting all weak equivalences in $\mathbf{Cat}'(S)$ and so need to discuss what it means to add the formal pseudoinverses to a class of 1-cells in a 2-category – a process known as *localisation*. This was done in [Pronk, 1996] for the more general case of a class of 1-cells in a bicategory, where the resulting bicategory is constructed and its universal properties (analogous to those of a quotient) examined. The application in *loc. cit.* is to showing the equivalence of various bicategories of stacks to localisations of 2-categories of smooth, topological and algebraic groupoids. The results of this article can be seen as one-half of a generalisation of these results to more general sites.

6.1. Definition. [Pronk, 1996] Let B be a bicategory and $W \subset B_1$ a class of 1-cells. A *localisation of B with respect to W* is a bicategory $B[W^{-1}]$ and a weak 2-functor

$$U\colon B \longrightarrow B[W^{-1}]$$

such that U sends elements of W to equivalences, and is universal with this property i.e. composition with U gives an equivalence of bicategories

$$U^*\colon Hom(B[W^{-1}], D) \longrightarrow Hom_W(B, D),$$

where Hom_W denotes the sub-bicategory of weak 2-functors that send elements of W to equivalences (call these *W-inverting*, abusing notation slightly).

The universal property means that W-inverting weak 2-functors $F\colon B \longrightarrow D$ factor, up to a transformation, through $B[W^{-1}]$, inducing an essentially unique weak 2-functor $\widetilde{F}\colon B[W^{-1}] \longrightarrow D$.

6.2. Definition. [Pronk, 1996] Let B be a bicategory B with a class W of 1-cells. W is said to *admit a right calculus of fractions* if it satisfies the following conditions

2CF1. W contains all equivalences

2CF2. a) W is closed under composition
b) If $a \in W$ and there is an iso-2-cell $a \overset{\sim}{\Rightarrow} b$ then $b \in W$

2CF3. For all $w\colon A' \longrightarrow A$, $f\colon C \longrightarrow A$ with $w \in W$ there exists a 2-commutative square

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "v"] & A' \arrow[d, "w"] \arrow[dl, phantom, "\overset{\simeq}{\swarrow}"] \\ C \arrow[r, "f"] & A \end{tikzcd}$$

with $v \in W$.

2CF4. If $\alpha\colon w \circ f \Rightarrow w \circ g$ is a 2-cell and $w \in W$ there is a 1-cell $v \in W$ and a 2-cell $\beta\colon f \circ v \Rightarrow g \circ v$ such that $\alpha \circ v = w \circ \beta$. Moreover: when α is an iso-2-cell, we require β to be an isomorphism too; when v' and β' form another such pair, there exist 1-cells u, u' such that $v \circ u$ and $v' \circ u'$ are in W, and an iso-2-cell $\epsilon\colon v \circ u \Rightarrow v' \circ u'$ such that the following diagram commutes:

$$\begin{tikzcd} f \circ v \circ u \arrow[r, Rightarrow, "\beta \circ u"] \arrow[d, Rightarrow, "f \circ \epsilon"', "\simeq"] & g \circ v \circ u \arrow[d, Rightarrow, "g \circ \epsilon", "\simeq"'] \\ f \circ v' \circ u' \arrow[r, Rightarrow, "\beta' \circ u'"'] & g \circ v' \circ u' \end{tikzcd} \tag{8}$$

6.3. Remark. In particularly nice cases (as in the next section), the first half of 2CF4 holds due to left-cancellability of elements of W, giving us the canonical choice $v = I$.

6.4. Theorem. *[Pronk, 1996] A bicategory B with a class W that admits a calculus of right fractions has a localisation with respect to W.*

From now on we shall refer to a calculus of right fractions as simply a calculus of fractions, and the resulting localisation constructed by Pronk as a bicategory of fractions. Since $B[W^{-1}]$ is defined only up to equivalence, it is of great interest to know when a bicategory D in which elements of W are converted to equivalences is itself equivalent to $B[W^{-1}]$. In particular, one would be interested in finding such an equivalent bicategory with a simpler description than that which appears in [Pronk, 1996]. Thanks are due to Matthieu Dupont for pointing out (in personal communication) that proposition 6.5 actually only holds in one direction, not in both, as claimed in *loc. cit.*

6.5. Proposition. *[Pronk, 1996] A weak 2-functor $F : B \longrightarrow D$ which sends elements of W to equivalences induces an equivalence of bicategories*

$$\widetilde{F}\colon B[W^{-1}] \xrightarrow{\simeq} D$$

if the following conditions hold

EF1. F is essentially surjective,

EF2. For every 1-cell $f \in D_1$ there are 1-cells $w \in W$ and $g \in B_1$ such that $Fg \overset{\simeq}{\Rightarrow} f \circ Fw$,

EF3. F is locally fully faithful.

The following is useful in showing a weak 2-functor sends weak equivalences to equivalences, because this condition only needs to be checked on a class that is in some sense cofinal in the weak equivalences.

6.6. Theorem. *In the bicategory B and let $V \subset W$ be two classes of 1-cells such that for all $w \in W$, there exists $v \in V$ and $s \in W$ such that there is an invertible 2-cell*

$$\begin{tikzcd} & a \arrow[d, "w"] \\ b \arrow[ur, "s"] \arrow[r, "v"'] \arrow[ur, phantom, shift right=4, "\Downarrow\simeq"] & c\,. \end{tikzcd}$$

Then a weak 2-functor $F\colon B \longrightarrow D$ that sends elements of V to equivalences also sends elements of W to equivalences.

Proof. In the following the coherence arrows will be implicit. First we show that Fw has a pseudosection in D for any $w \in W$. Let v, s be as above. Let $\widetilde{Fv}$ be a pseudoinverse of Fv, and let $j = Fs \circ \widetilde{Fv}$. Then there is the following invertible 2-cell

$$Fw \circ j \Rightarrow F(w \circ s) \circ \widetilde{Fv} \Rightarrow Fv \circ \widetilde{Fv} \Rightarrow I.$$

We now show that j is in fact a pseudoinverse for Fw. Since $s \in W$, there is a $v' \in V$ and $s' \in W$ and an invertible 2-cell giving the following diagram

$$\begin{tikzcd} d \arrow[r, "v'"] \arrow[d, "s'"'] \arrow[dr, phantom, shift left=4, "\Downarrow"] & a \arrow[d, "w"] \\ b \arrow[ur, "s"] \arrow[r, "v"'] \arrow[ur, phantom, shift right=4, "\Downarrow"] & c\,. \end{tikzcd}$$

Apply the functor F, and denote pseudoinverses of Fv, Fv' by $\widetilde{Fv}, \widetilde{Fv'}$. Using the 2-cell $I \Rightarrow Fv' \circ \widetilde{Fv'}$ we get the following invertible 2-cell

$$\begin{tikzcd} Fd \arrow[d, "Fs"'] & Fa \arrow[l, "\widetilde{Fv'}"'] \arrow[d, "Fw"] \arrow[dl, phantom, "\Leftarrow"] \\ Fb \arrow[r, "Fv"'] & Fc \end{tikzcd}$$

Then there is this composite invertible 2-cell

$$j \circ Fw \Rightarrow (Fs \circ \widetilde{Fv}) \circ (Fv \circ (Fs \circ \widetilde{Fv'})) \Rightarrow (Fs \circ Fs') \circ \widetilde{Fv'} \Rightarrow Fv' \circ \widetilde{Fv'} \Rightarrow I,$$

making Fw is an equivalence. Hence F sends all elements of W to equivalences. ■

7. Anafunctors are a localisation

In this section we prove the result that $\mathbf{Cat}'(S)$ admits a calculus of fractions for the E-equivalences, and the bicategory of anafunctors is a localisation. Note that E is not required to be subcanonical, but rather that it satisfies a weak saturation condition.

7.1. Definition. Let E be a class of arrows in the ambient category S. E is called a *class of admissible maps* if it is a singleton pretopology containing the split epimorphisms which satisfies the following condition:

(S) If $e\colon A \longrightarrow B$ is a split epimorphism, and $A \xrightarrow{e} B \xrightarrow{p} C$ is in E, then $p \in E$.

7.2. Example. If E is a saturated singleton pretopology, it is a class of admissible maps. In particular, E could be $J_{un} = (J\text{-epi})$ for a non-singleton pretopology J on a finitely complete category.

7.3. Example. The singleton pretopology *Subm* of surjective submersions on **Diff** is subcanonical and satisfies condition (S), but does not contain the split epimorphisms, so is not admissible.

7.4. Remark. Recall that $\mathbf{Cat}'(S)$ is assumed to be a sub-2-category of $\mathbf{Cat}(S)$ defined such that all pullbacks of source and target maps exist. We do not assume that $\mathbf{Cat}'(S)$ admits finite (strict) limits or even all bipullbacks.

7.5. Theorem. *Let S be a category with a class E of admissible maps. Assume the 2-category $\mathbf{Cat}'(S)$ admits base change along maps in E, admits cotensors with $\mathbf{I}$ and has enough groupoids. Then $\mathbf{Cat}'(S)$ admits a right calculus of fractions for the class W_E of E-equivalences.*

Proof. We show the conditions of definition 6.2 hold.

2CF1. Since E contains all the split epis, an internal equivalence is essentially E-surjective (c.f. example 4.3). Let $f\colon X \longrightarrow Y$ be an internal equivalence, and $g\colon Y \longrightarrow X$ a pseudoinverse. By definition there are natural isomorphisms $a\colon g\circ f \Rightarrow \mathrm{id}_X$ and $b\colon f\circ g \Rightarrow \mathrm{id}_Y$. To show that f is fully faithful, we first show that the map

$$q\colon X_1 \longrightarrow X_0^2 \times_{f,Y_0^2} Y_1$$

is a split monomorphism over X_0^2. This diagram commutes

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, equal] & X_0^2 \times_{f,Y_0^2} Y_1 \arrow[d] \\ X_1 & X_0^2 \times_{gf,X_0^2} X_1 \arrow[l, "\simeq"'] \end{tikzcd},$$

by the naturality of a, the marked isomorphism coming from lemma 2.25, giving the desired splitting (call it s). The splitting commutes with projection to X_0^2 because the isomorphism does. The same argument implies that

$$Y_1 \longrightarrow Y_0^2 \times_{X_0^2} X_1$$

is a split monomorphism over Y_0^2, and this implies the composite arrow

$$l\colon X_0^2 \times_{Y_0^2} Y_1 \longrightarrow X_0^2 \times_{Y_0^2} Y_0^2 \times_{X_0^2} X_1 \simeq X_0^2 \times_{gf,X_0^2} X_1$$

is a split monomorphism. This diagram commutes

$$\begin{tikzcd} X_0^2 \times_{Y_0^2} Y_1 \arrow[rr, "l"] \arrow[d, "s"'] & & X_0^2 \times_{gf,X_0^2} X_1 \arrow[r, "\simeq"] & X_1 \arrow[d, equal] \\ X_1 \arrow[r, "q"'] & X_0^2 \times_{Y_0^2} Y_1 \arrow[r, "l"'] & X_0^2 \times_{gf,X_0^2} X_1 \arrow[r, "\simeq"] & X_1 \end{tikzcd}$$

using naturality again, and so $q\circ s = \mathrm{id}$, using the fact l is a monomorphism. Thus q is an isomorphism, and f is fully faithful.

2CF2 a). That the composition of fully faithful functors is again fully faithful is trivial. To show that the composition of essentially E-surjective functors $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ is again so, consider the following diagram

$$\begin{tikzcd} & & Y_0 \times_{Z_0} Z_1^{iso} \arrow[r] \arrow[d] \arrow[rr, bend left] & Z_1^{iso} \arrow[r, "t"] \arrow[d, "s"] & Z_0 \\ X_0 \times_{Y_0} Y_1^{iso} \arrow[r] \arrow[d] \arrow[rr, bend left] & Y_1^{iso} \arrow[r, "t"] \arrow[d, "s"] & Y_0 \arrow[r, "g_0"'] & Z_0 & \\ X_0 \arrow[r, "f_0"'] & Y_0 & & & \end{tikzcd}$$

where the curved arrows are in E by assumption. The lower such arrow pulls back to an arrow $X_0 \times_{Y_0} Y_1^{iso} \times_{Z_0} Z_1^{iso} \longrightarrow Y_0 \times_{Z_0} Z_1^{iso}$ (again in E). Hence the composite

$$X_0 \times_{Y_0} Y_1^{iso} \times_{Z_0} Z_1^{iso} \longrightarrow Y_0 \times_{Z_0} Z_1^{iso} \xrightarrow{t \circ \mathrm{pr}_2} Z_0$$

is in E, and is equal to the composite

$$X_0 \times_{Y_0} Y_1^{iso} \times_{Z_0} Z_1^{iso} \xrightarrow{\mathrm{id} \times g \times \mathrm{id}} X_0 \times_{Z_0} Z_1^{iso} \times_{Z_0} Z_1^{iso} \xrightarrow{\mathrm{id} \times m} X_0 \times_{Z_0} Z_1^{iso} \xrightarrow{t \circ \mathrm{pr}_2} Z_0.$$

The map

$$X_0 \times_{Z_0} Z_1^{iso} \simeq X_0 \times_{Y_0} Y_0 \times_{Z_0} Z_1^{iso} \xrightarrow{\mathrm{id} \times e \times \mathrm{id}} X_0 \times_{Y_0} Y_1^{iso} \times_{Z_0} Z_1^{iso}$$

is a section of

$$X_0 \times_{Y_0} Y_1^{iso} \times_{Z_0} Z_1^{iso} \xrightarrow{\mathrm{id} \times g \times \mathrm{id}} X_0 \times_{Z_0} Z_1^{iso} \times_{Z_0} Z_1^{iso} \xrightarrow{\mathrm{id} \times m} X_0 \times_{Z_0} Z_1^{iso}.$$

Now condition (S) tells us that $X_0 \times_{Z_0} Z_1^{iso} \xrightarrow{t \circ \mathrm{pr}_2} Z_0$ is in E, hence $g \circ f$ is essentially E-surjective.

2CF2 b). We will show this in two parts: fully faithful functors are closed under isomorphism, and essentially E-surjective functors are closed under isomorphism. Let $w, f \colon X \longrightarrow Y$ be functors and $a \colon w \Rightarrow f$ be a natural isomorphism. First, let w be essentially E-surjective. That is,

$$X_0 \times_{w,Y_0,s} Y_1^{iso} \xrightarrow{t \circ \mathrm{pr}_2} Y_0 \tag{9}$$

is in E. Now note that the map

$$X_0 \times_{f,Y_0,s} Y_1^{iso} \xrightarrow{(\mathrm{id},-a) \times \mathrm{id}} X_0 \times \times_{w,Y_0,s} Y_1^{iso} \times_{t,Y_0,s} Y_1^{iso} \xrightarrow{\mathrm{id} \times m} X_0 \times_{w,Y_0,s} Y_1^{iso} \tag{10}$$

is an isomorphism, and so the composite of (10) and (9) is in E. Thus f is essentially E-surjective.

Now let w be fully faithful. Thus

$$\begin{tikzcd} X_1 \arrow[r, "w_1"] \arrow[d] & Y_1 \arrow[d] \\ X_0 \times X_0 \arrow[r, "w_0 \times w_0"'] & Y_0 \times Y_0 \end{tikzcd}$$

is a pullback square. Using lemma 2.25 there is an isomorphism

$$X_1 \simeq X_0 \times_{w,Y_0,s} Y_1 \times_{t,Y_0,w} X_0 \simeq X_0 \times_{f,Y_0,s} Y_1 \times_{t,Y_0,f} X_0.$$

The composite of this with projection on X_0^2 is $(s,t) \colon X_1 \longrightarrow X_0^2$, and the composite with

$$\mathrm{pr}_2 \colon X_0 \times_{f,Y_0,s} Y_1 \times_{t,Y_0,f} X_0 \longrightarrow Y_1$$

is just f_1 by the diagram (2), and so this diagram commutes

$$\begin{tikzcd}
X_1 \arrow[dr, "\simeq"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_0^2 \times_{f^2, Y_0^2} Y_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
& X_0^2 \arrow[r, "f_0^2"'] & Y_0^2
\end{tikzcd}$$

i.e. f is fully faithful.

2CF3 follows from lemma 4.11, as $\mathbf{Cat}'(S)$ is assumed to admit base change along arrows in E.

2CF4. Section 4.1 in [Pronk, 1996] shows that given a natural transformation

$$\begin{tikzcd}
& Y \arrow[dr, "w"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\
& Y \arrow[ur, "w"'] &
\end{tikzcd}$$

where w is fully faithful (e.g. w is in W_E), there is a unique $a' \colon f \Rightarrow g$ such that

$$\begin{tikzcd}
& Y \arrow[dr, "w"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\
& Y \arrow[ur, "w"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y \arrow[r, "w"] & Z\,.
\end{tikzcd}$$

This is the first half of 2CF4, where $v = \mathrm{id}_X$. If $v' \colon W \longrightarrow X \in W_E$ such that there is a transformation

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\
& X \arrow[ur, "g"'] &
\end{tikzcd}$$

satisfying

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & & \\
W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \arrow[r, "w"] & Z \\
 & X \arrow[ur, "g"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & Y \arrow[dr, "w"] & \\
W \arrow[r, "v'"] & X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\
 & & Y \arrow[ur, "w"'] &
\end{tikzcd}$$

$$= \begin{tikzcd} W \arrow[r, "v'"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y \arrow[r, "w"] & Z \end{tikzcd}, \tag{11}$$

then as v' is an E-equivalence and $\mathbf{Cat}'(S)$ admits base change along arrows in E, we have a functor $u' \colon X[W_0 \times_{X_0} X_1] \longrightarrow W$ and a natural isomorphism ϵ:

$$\begin{tikzcd}
 & X[W_0 \times_{X_0} X_1] \arrow[dl, "u'"'] \arrow[dr, "u"] & \\
W \arrow[rr, bend right, "v'"'] \arrow[rr, phantom, "\Leftarrow \epsilon"] & & X
\end{tikzcd}$$

where $u \in W_E$, and since $v' \circ u' \overset{\cong}{\Rightarrow} u$, we have $v' \circ u' \in W_E$ by 2CF2 a) above as required by 2CF4. The uniqueness result from Pronk's argument, together with equation (11) to give us

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\
 & X \arrow[ur, "g"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd} W \arrow[r, "v'"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y \end{tikzcd}.$$

We paste this with ϵ,

$$\begin{tikzcd}
X[U] \arrow[rr, bend left, "u"] \arrow[dr, "u'"'] \arrow[rr, phantom, "\epsilon \Downarrow"] & & X \arrow[dr, "f"] & \\
 & W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\
 & & X \arrow[ur, "g"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X[U] \arrow[d, "u'"'] \arrow[dr, bend left, "u"] \arrow[d, phantom, shift left=4, "\overset{\epsilon}{\Downarrow}"] & & \\
W \arrow[r, "v'"'] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y
\end{tikzcd},$$

which is precisely the diagram (8) with $v = \mathrm{id}_X$. Hence 2CF4 holds. ∎

7.6. Remark. If we replace the assumption that E contains the split epimorphisms by the slightly weaker assumption that all internal equivalences in $\mathbf{Cat}'(S)$ are E-equivalences, then theorem 7.5 still holds as split epimorphisms are only used to prove 2CF1. By the result of [Moerdijk-Mrčun, 2005] that internal equivalences of Lie groupoids are $Subm$-equivalences, we recover the result that theorem 7.5 holds for Lie groupoids (and Lie categories more generally) and the class of $Subm$-equivalences, as well as for various sub-2-categories, such as proper étale Lie groupoids aka orbifolds.

We thus need to furnish examples of 2-categories $\mathbf{Cat}'(S)$ that admit base change along some class of arrows. The following lemma give a sufficiency condition for this to be so.

7.7. Lemma. *Let* $\mathbf{Cat}^{\mathcal{M}}(S)$ *be defined as the full sub-2-category of* $\mathbf{Cat}(S)$ *with objects those categories such that the source and target maps belong to a class* $\mathcal{M} \subset \mathrm{Mor}(S)$ *containing the isomorphisms which closed under pullback and composition (of course, this assumes pullbacks of maps in* $\mathcal{M}$ *exist). Then* $\mathbf{Cat}^{\mathcal{M}}(S)$ *admits base change along arrows in* $\mathcal{M}$*, as does the corresponding 2-category of groupoids.*

Proof. Let X be an object of $\mathbf{Cat}^{\mathcal{M}}(S)$ and $f\colon M \longrightarrow X_0 \in \mathcal{M}$. In the following two diagrams, the upper and lower squares are pullbacks (which exist by definition of $\mathcal{M}$)

$$
\begin{tikzcd}
X[M]_1 \arrow[rr] \arrow[d] \arrow[dd, bend right, "s'"'] & & M \arrow[d] \\
M \times_{X_0} X_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \\
M \arrow[r] & X_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X[M]_1 \arrow[r] \arrow[dd] \arrow[rr, bend left, "t'"] & X_1 \times_{X_0} M \arrow[r] \arrow[d] & M \arrow[d] \\
 & X_1 \arrow[r] \arrow[d] & X_0 \\
M \arrow[r] & X_0 &
\end{tikzcd}
$$

The maps marked s', t' are the source and target maps for the base change along f, and are obviously in $\mathcal{M}$, so $X[M]$ is in $\mathbf{Cat}^{\mathcal{M}}(S)$. The same argument holds for groupoids. ∎

In practice one often only wants base change along a subclass of $\mathcal{M}$, such as (open covers) $\subset$ (open maps) as classes of maps in **Top**.

7.8. Example. Take $S = \mathbf{Diff}$ and $\mathcal{M} = Subm$, the class of surjective submersions. Or from example 2.28, where $\mathcal{M} =$LF is a class of local fibrations as defined by Noohi, and we consider base change along maps of the form $\coprod U_i \longrightarrow X_0$, which are open maps, local homeomorphisms, and also local Serre/Hurewicz fibrations.

7.9. Example. Consider the example of the 2-category of groupoid schemes X with source and target maps étale (and satisfying some condition on $(s,t)\colon X_1 \longrightarrow X_0 \times X_0$, assumed to be a pullback-stable property). These correspond (up to details on (s,t)) to Deligne-Mumford stacks. By lemma 7.7 the base change along étale maps exists in the 2-category of such groupoids.

We now want to say when this bicategory of fractions is given by a bicategory of anafunctors.

7.10. Definition. Given a singleton pretopology J and a class E of admissible maps, we say E is *admissible for* J if $J \leq E$.

7.11. Example. J_{un} is a class of admissible maps for J if J_{un} contains the split epimorphisms. A saturated singleton pretopology is a class of admissible maps for itself.

If E is a class of admissible maps for J, E-equivalences are J-equivalences and so $W_E \subset W_J$. This means that the 2-functor α_J in proposition 5.16 sends E-equivalences to equivalences. We use this fact and proposition 6.5 to show the following.

7.12. Theorem. *Let* (S, J) *be a site with a subcanonical singleton pretopology* J*, let* E *be a class of admissible maps for* J *and let* $\mathbf{Cat}'(S)$ *be as in theorem 7.5. Then there is an equivalence of bicategories*

$$\mathbf{Cat}'_{\mathrm{ana}}(S, J) \simeq \mathbf{Cat}'(S)[W_E^{-1}]$$

Proof. Let us show the conditions in proposition 6.5 hold.

EF1. α_J is the identity on 0-cells, and hence surjective on objects.

EF2. This is equivalent to showing that for any anafunctor $(U, f)\colon X \looparrowright Y$ there are functors w, g such that w is in W_E and

$$(U, f) \overset{\sim}{\Rightarrow} \alpha_J(g) \circ \alpha_J(w)^{-1}$$

where $\alpha_J(w)^{-1}$ is some pseudoinverse for $\alpha_J(w)$.

Let w be the functor $X[U] \longrightarrow X$ (this has object component in $J \subset E$, hence is an E-equivalence) and let $g = f\colon X[U] \longrightarrow Y$. First, note that

$$\begin{tikzcd} & X[U] \arrow[dl] \arrow[dr, "="] & \\ X & & X[U] \end{tikzcd}$$

is a pseudoinverse for

$$\alpha_J(w) \quad = \quad \begin{tikzcd} & X[U][U] \arrow[dl, "="'] \arrow[dr] & \\ X[U] & & X \end{tikzcd} \quad .$$

Then the composition $\alpha_J(f) \circ \alpha_J(w)^{-1}$ is

$$\begin{tikzcd} & X[U \times_U U \times_U U] \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which is isomorphic to (U, f) by the renaming transformation arising from the isomorphism $U \times_U U \times_U U \simeq U$.

EF3. If $a\colon (X_0, f) \Rightarrow (X_0, g)$ is a transformation of anafunctors for functors $f, g\colon X \longrightarrow Y$, it is given by a natural transformation with component

$$X_0 \times_{X_0} X_0 \longrightarrow Y_1.$$

But we have declared $X_0 \times_{X_0} X_0 = X_0$. Hence we get a unique natural transformation $a\colon f \Rightarrow g$ such that a is the image of a' under α_J. ∎

We now give a series of results following from this theorem, using basic properties of pretopologies from section 3. Standing assumptions are as for theorem 7.5 and for theorem 7.12.

7.13. Corollary. *When J and K are two subcanonical singleton pretopologies on S such that $J_{un} = K_{un}$, there is an equivalence of bicategories*

$$\mathbf{Cat}'_{\text{ana}}(S, J) \simeq \mathbf{Cat}'_{\text{ana}}(S, K).$$

Using corollary 7.13 we see that using a cofinal pretopology gives an equivalent bicategory of anafunctors.

If E is any class of admissible maps for subcanonical J, the bicategory of fractions inverting W_E is equivalent to that of J-anafunctors. Hence

7.14. Corollary. *Let E be a class of admissible maps for the subcanonical pretopology J. There is an equivalence of bicategories*

$$\mathbf{Cat}'(S)[W_E^{-1}] \simeq \mathbf{Cat}'(S)[W_J^{-1}]$$

where of course $W_J = W_{J_{un}}$.

Finally, if (S, J) is a superextensive site (like **Top** with its usual pretopology of open covers), we have the following result which is useful when J is not a singleton pretopology.

7.15. Corollary. *Let (S, J) be a superextensive site where J is a subcanonical pretopology. Then*

$$\mathbf{Cat}'(S)[W_{J_{un}}^{-1}] \simeq \mathbf{Cat}'_{\text{ana}}(S, \amalg J).$$

Proof. This essentially follows from the corollary to lemma 3.23. ∎

Obviously this can be combined with previous results, for example if $K \leq \amalg J$, for J a non-singleton pretopology, K-anafunctors localise $\mathbf{Cat}'(S)$ at the class of J-equivalences.

8. Size considerations

The 2-category $\mathbf{Cat}'(S)$ is locally small, similar to the case of the 2-category of small categories (and in fact the latter is cartesian closed). However the construction of $B[W^{-1}]$ given by Pronk, even for a locally small bicategory B is *a priori* not necessarily locally small (or even locally essentially small). Recall that the axiom of choice for a site (S, J) is that for all J-epimorphisms $p\colon P \longrightarrow A$ there exists a section of p. This is too strong an assumption in practice. In many algebraic situations one has projective covers, for instance in **Grp** (every group has an epimorphism from a free group). We can rephrase this by saying the full subcategory of $\mathbf{Grp}/G$ consisting of the epimorphisms has a weakly initial object. More generally one could ask only that the category of all singleton covers of an object (see definition 8.3 below) has a *set* of weakly initial objects. This is the content of the axiom WISC below. We first give some more precise definitions.

8.1. Definition. A category C has a *weakly initial set* $\mathcal{I}$ of objects if for every object A of C there is an arrows $O \longrightarrow A$ from some object $O \in \mathcal{I}$.

Every small category has a weakly initial set, namely its set of objects.

8.2. Example. The category *Field* of fields has a weakly initial set, consisting of the prime fields $\{\mathbb{Q}, \mathbb{F}_p | p \text{ prime}\}$. To contrast, the category of sets with surjections for arrows doesn't have a weakly initial set of objects.

8.3. Definition. Let (S, J) be a site. For any object A, the *category of covers of* A, denoted J/A has as objects the covering families $(U_i \longrightarrow A)_{i\in I}$ and as morphisms $(U_i \longrightarrow A)_{i\in I} \longrightarrow (V_j \longrightarrow A)_{j\in J}$ tuples consisting of a function $r\colon I \longrightarrow J$ and arrows $U_i \longrightarrow V_{r(i)}$ in S/A.

When J is a singleton pretopology this is simply a full subcategory of S/A. We now define the axiom WISC (Weakly Initial Set of Covers), due to Mike Shulman, which in a sense limits how much Choice fails to hold. Let (S, J) be a site.

8.4. Definition. The site (S, J) is said to *satisfy WISC* if for every object A of S, the category J/A has a weakly initial set of objects.

When S is **Set** with surjections as covers, this is implied by the axiom COSHEP (Category Of Sets Has Enough Projectives). Without the condition that this is a *set* of objects (as opposed to a class or large set) then this would be true of all sites.

8.5. Example. Any regular category with enough projectives with the regular pretopology satisfies WISC.

8.6. Example. Assuming Choice in the metalogic—that is, in **Set**—then $(\mathbf{Top}, \mathcal{O})$ and $(\mathbf{Diff}, \mathcal{O})$ satisfy WISC.

Choice may be more than is necessary here; it would be interesting to see if WISC in $(\mathbf{Set}, \mathit{surjections})$ is enough to prove WISC in these cases, analogous to how enough injectives in a topos proves enough injectives for abelian group objects therein.

8.7. Lemma. *If (S, J) satisfies WISC, then so does (S, J_{un}).*

8.8. Lemma. *If (S, J) is a superextensive site, (S, J) satisfies WISC if and only if $(S, \amalg J)$ does.*

8.9. Proposition. *Let (S, J) be a site with a subcanonical singleton pretopology J, satisfying WISC and let $\mathbf{Cat}'(S)$ admit base change along arrows in S. Then $\mathbf{Cat}'_{ana}(S, J)$ is locally essentially small.*

Proof. Let $I(A)$ be a weakly initial set for J/A. Consider the locally full sub-2-category of $\mathbf{Cat}'_{ana}(S, J)$ with the same objects, and arrows those anafunctors $(U, f) : X \to\!\!\!\mapsto Y$ such that $U \longrightarrow X_0$ is in $I(X_0)$. Every anafunctor is then isomorphic, by the generalisation of example 5.8, to one in this sub-2-category. ∎

8.10. Corollary. *Assume $\mathbf{Cat}'(S)$ and J are as in theorem 7.12 (with $E = J_{un}$). Then any localisation $\mathbf{Cat}'(S)[W_{J_{un}}^{-1}]$ is locally essentially small for (S, J) satisfying WISC.*

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