CHAPTER 1

Internal categories and anafunctors

In this chapter we consider anafunctors [Mak96, Bar06] as generalised maps between internal categories [Ehr63], and show they formally invert fully faithful, essentially surjective functors (this localisation was developed in [Pro96] without anafunctors). To do so we need our ambient category S to be a site, to furnish us with a class of arrows that replaces the class of surjections in the case $S = \mathbf{Set}$. The site comes with collections called covering families, or covers, and give meaning to the phrase "essentially surjective" when working internal to S. A useful analogy to consider is when $S = \mathbf{Top}$, and the covering families are open covers in the usual way. In that setting, 'surjective' is replaced by 'admits local sections', and the same is true for an arbitrary site - surjections are replaced by maps admitting local sections with respect to the given class of covers. The class of such maps does not determine the covers with which one started, and we use this to our advantage. A *superextensive* site[1] is a one where out of each covering family $\{U_i \to A | i \in I\}$ we can form a single map $\coprod_I U_i \to A$, and use these as our covers. A maps admits local sections over the original covering family if and only if it admits sections over the new cover, and it is with these we can define anafunctors. Finally we show that different collections of covers will give equivalent results if they give rise to the same collection of maps admitting local sections.

Most of the definitions in this chapter are standard. We draw without reference on the background to bicategories collected in Appendix A. The material on anafunctors and localising bicategories, although not new, do not seem to be widely known. Theorem 6.6 is new, but note analogues have appeared in the literature for Lie groupoids [] and étale Lie groupoids [Pro96] (which are not covered by this chapter), and for étale topological groupoids and algebraic groupoids (étale groupoids internal to schemes) [Pro96]. Theorem 6.8 and its corollaries are also new.

1. Internal categories and groupoids

Internal categories were introduced by Ehresmann [Ehr63], starting with differentiable and topological categories (i.e. internal to **Diff** and **Top** respectively). We collect here the necessary definitions and terminology without burdening the reader with pages of diagrams. For a thorough recent account, see [BL04] or [Bar06]. Familiarity with basic category theory [Mac71] is assumed.

Let S be a category with binary products and pullbacks. It will be referred to as the *ambient category*.

[1] This concept is due to Toby Bartels and Mike Shulman

Definition 1.1. An *internal category* X in a category S is a diagram

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \overset{s,t}{\rightrightarrows} X_0 \xrightarrow{e} X_1$$

in S such that the *multiplication* m is associative, the *unit map* e is a two-sided unit for m and s and t are the usual *source* and *target*.

The pullback in the diagram is

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r] \arrow[d] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "t"'] & X_0 \end{tikzcd} .$$

This, and pullbacks like this (where source is pulled back along target), will occur often. If confusion can arise, the maps in question will be noted down, as in $X_1 \times_{s,X_0,t} X_1$. Also, since multiplication is associative, there is a well-defined map $X_1 \times_{X_0} X_1 \times_{X_0} X_1 \to X_1$, which will also be denoted by m.

It follows from the definition[2] that there is a subobject $X_1^{iso} \hookrightarrow X_1$ through which e factors and an involution

$$(-)^{-1} \colon X_1^{iso} \to X_1^{iso}$$

sending arrows to their inverses such that the restriction of the structure maps to X_1^{iso} make $X_1^{iso} \rightrightarrows X_0$ an internal category, and that $(-)^{-1} \circ e = e$.

Often an internal category will be denoted $X_1 \rightrightarrows X_0$, the arrows m, s, t, e will be referred to as *structure maps* and X_1 and X_0 called the objects of arrows and objects respectively.

Remark 1.2. A very often used class of internal categories is that of Lie groupoids (e.g. [**Mac05**]). Since **Diff** doesn't have all pullbacks, modifications need to be made to the above definition. Since submersions admit pullbacks and are stable, s and t are assumed to be surjective submersions. Various other constructions involving pullbacks later on in this chapter also need care, and there is an established literature on the subject. More generally, one can consider internal category theory for ambient categories without pullbacks, given a class of maps analogous to submersion, but we will not do this in the present work.

Example 1.3. If M is a monoid object in S and $a \colon M \times X \to X$ is an action, there is a category $M \ltimes X \rightrightarrows X$, called the action category, where the source and target are projection and the action respectively. The subobject of invertible arrows is $M^* \ltimes X$. In particular, consider the case when X is the terminal object (assumed to exist so as to define the unit of the monoid). Then such a category is precisely a monoid.

Example 1.4. If $X \to Y$ is an arrow in S admitting iterated kernel pairs, there is a category $\check{C}(X)$ with $\check{C}(X)_0 = X$, $\check{C}(X)_1 = X \times_Y X$, source and target are projection on first and second factor, and the multiplication is projecting out the middle factor in $X \times_Y X \times_Y X$. The subobject of invertible arrows is all of $\check{C}(X)_1$.

[2] [**BP79**], but see [**EKvdL05**] for some more details.

A lot of interest in internal categories is for defining stacks over the ambient category (once it has the structure of a site, for which see below), and specifically, stacks of groupoids. These lead to considering internal groupoids as local models for the stack over the site (e.g. [BP79] in the case of a regular, finitely complete category).

DEFINITION 1.5. If an internal category X has $X_1^{iso} \simeq X_1$, then it is called an *internal groupoid*.

A lot of the terminology and machinery will be described here for internal categories, even though most of the examples of interest are internal groupoids.

EXAMPLE 1.6. Let S be a category. For each object $A \in S$ there is an internal groupoid $\mathrm{disc}(A)$ which has $\mathrm{disc}(A)_1 = \mathrm{disc}(A)_0 = A$ and all structure maps equal to id_A. Such a category is called *discrete*. It goes without saying that $\mathrm{disc}(A \times B) = \mathrm{disc}(A) \times \mathrm{disc}(B)$.
If S has binary products, there is an internal groupoid $\mathrm{codisc}(A)$ with $\mathrm{codisc}(A)_0 = A$, $\mathrm{codisc}(A)_1 = A \times A$ and where source and target are projections on the first and second factor respectively. The unit map is the diagonal and composition is projecting out the middle factor in $\mathrm{codisc}(A)_1 \times_{\mathrm{codisc}(A)_0} \mathrm{codisc}(A)_1 = A \times A \times A$. Such a groupoid is called *codiscrete*. Again, we have $\mathrm{codisc}(A \times B) = \mathrm{codisc}(A) \times \mathrm{codisc}(B)$.

EXAMPLE 1.7. The codiscrete groupoid is obviously a special case of example 1.4, which is called the Čech groupoid of the map $X \to Y$. The origin of the name is that in **Top**, for maps of the form $\coprod_I U_i \to Y$, the Čech groupoid $\check{C}(\coprod_I U_i)$ appears in the definition of Čech cohomology.

EXAMPLE 1.8. If G is a group object in a category S with finite products, the groupoid $\mathbf{B}G$ has $\mathbf{B}G_0 = *$, $\mathbf{B}G_1 = G$.

EXAMPLE 1.9. If C is a category with a set of objects enriched in **Top**, then let $C_0^{int} = \mathrm{Obj}(C)$ and $C_1^{int} = \coprod_{\mathrm{Obj}(C)^2} C(a,b)$. Then C^{int} is a category internal to **Top**. This example can be generalised to monoidal categories other than **Top** in which sufficient coproducts of the unit exist.

EXAMPLE 1.10. If X is a topological space which has a universal covering space (i.e. is path-connected, locally path-connected and semilocally simply connected), then the fundamental groupoid $\Pi_1(X)$ can be made into a groupoid internal to **Top**.

DEFINITION 1.11. Given internal categories X and Y in S, and *internal functor* $f : X \to Y$ is a pair of maps

$$f_0 \colon X_0 \to Y_0 \qquad f_1 \colon X_1 \to Y_1$$

called the object and arrow component respectively. The map f_1 restricts to a map $f_1 \colon X_1^{iso} \to Y_1^{iso}$ and both components commute with all the structure maps.

EXAMPLE 1.12. Given a homomorphism ϕ between monoids or groups, there is a functor between the categories/groupoids in example 1.3. More generally, given an equivariant map between objects with an M-action, it gives rise to a functor between the associated action categories.

Example 1.13. If $A \to B$ is a map in S, there are functors $\mathrm{disc}(A) \to \mathrm{disc}(B)$ and $\mathrm{codisc}(A) \to \mathrm{codisc}(B)$.

Example 1.14. If $A \to C$ and $B \to C$ are maps admitting iterated kernel pairs, and $A \to B$ is a map over C, there is a functor $\check{C}(A) \to \check{C}(B)$.

Example 1.15. A map $X \to Y$ in **Top** induces a functor $\Pi_1(X) \to \Pi_1(Y)$ (when these exist).

Definition 1.16. Given internal categories X, Y and internal functors $f, g\colon X \to Y$, an *internal natural transformation* (or simply transformation)

$$a\colon f \Rightarrow g$$

is a map $a\colon X_0 \to Y_1$ such that $s \circ a = f_0$, $t \circ a = g_0$ and the following diagram commutes

$$\begin{tikzcd} X_1 \arrow[r, "(g_1, a\circ s)"] \arrow[d, "(a\circ t, f_1)"'] & Y_1 \times_{Y_0} Y_1 \arrow[d, "m"] \\ Y_1 \times_{Y_0} Y_1 \arrow[r, "m"] & Y_1 \end{tikzcd}$$

expressing the naturality of a. If a factors through Y_1^{iso}, then it is called a *natural isomorphism*. Clearly there is no distinction between natural transformations and natural isomorphisms when Y is an internal groupoid.

We can reformulate the naturality diagram above in the case that a is a natural isomorphism. Denote by $-a$ the composite arrow

$$X_0 \xrightarrow{a} Y_1^{iso} \xrightarrow{(-)^{-1}} Y_1^{iso} \hookrightarrow Y_1.$$

Then the above diagram commuting is equivalent to this diagram commuting

$$\begin{tikzcd} X_0 \times_{X_0} X_1 \times_{X_0} X_0 \arrow[r, "-a\times f\times a"] \arrow[d, "\simeq"'] & Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \arrow[d, "m"] \\ X_1 \arrow[r, "g"'] & Y_1 \end{tikzcd} \tag{1}$$

which we will use repeatedly.

Example 1.17. Let $V_\rho, V_{\rho'}$ be the action groupoids associated to representations ρ, ρ' of G on V. They are given by functors from G to $GL(V)$ as described in example 1.12. A natural transformation between these functors is precisely an intertwiner.

Example 1.18. If X is a groupoid in S, A is an object of S and $f, g : X \to \mathrm{codisc}(A)$ are functors, there is a natural isomorphism $f \overset{\sim}{\Rightarrow} g$.

Internal categories (resp. groupoids), functors and transformations form a 2-category $\mathbf{Cat}(S)$ (resp. $\mathbf{Gpd}(S)$) [Ehr63]. There is clearly a 2-functor $\mathbf{Gpd}(S) \to \mathbf{Cat}(S)$. Also, disc and codisc, described in examples 1.6 and 1.13 are 2-functors $S \to \mathbf{Gpd}(S)$, whose underlying functors are left and right adjoint to the functor

$$(-)_0\colon \mathbf{Gpd}_1(S) \to S, \qquad (X_1 \rightrightarrows X_0) \mapsto X_0.$$

Here $\mathbf{Gpd}_1(S)$ is the category underlying the 2-category $\mathbf{Gpd}(S)$. Hence for an internal category X in S, there are functors $\mathrm{disc}(X_0) \to X$ and $X \to \mathrm{codisc}(X_0)$, the latter sending an arrow to the pair (source,target).

Definition 1.19. An *internal* or *strong equivalence* of internal categories is an equivalence in this 2-category: an internal functor $f\colon X \to Y$ such that there is a functor $f'\colon Y \to X$ and natural isomorphisms $f \circ f' \Rightarrow \mathrm{id}_Y$, $f' \circ f \Rightarrow \mathrm{id}_X$.

In all that follows, 'category' will mean 'internal category in S' and similarly for 'functor' and 'natural transformation/isomorphism'. We will not be considering here the effect a functor $S \to S'$ between ambient categories has on internal category theory.

2. Sites and covers

All the material in this section is standard. Even though we are assuming our ambient category has pullbacks, a lot of the definitions are made for more general categories.

Definition 2.1. A *Grothendieck pretopology* (or simply *pretopology*) on a category S is a collection J of families

$$\{(U_i \to A)_{i\in I}\}$$

for each object $A \in S$ satisfying the following properties

(1) $(\mathrm{id}\colon A \to A)$ is in J for every object A.
(2) Given a map $B \to A$, for every $(U_i \to A)_{i\in I}$ in J the pullbacks $B \times_A A_i$ exist and $(B \times_A A_i \to B)_{i\in I}$ is in J.
(3) For every $(U_i \to A)_{i\in I}$ in J and for a collection $(V_k^i \to U_i)_{k\in K_i}$ from J for each $i \in I$, the composites
$$(V_k^i \to A)_{k\in K_i, i\in I}$$
are in J.

Families in J are called *covering families*. A category S equipped with a pretopology is called a *site*, denoted (S, J).

Example 2.2. The basic example is the lattice of open sets of a topological space, seen as a category in the usual way, where a covering family of an open $U \subset X$ is an open cover of U by opens in X. This is to be contrasted with the pretopology $\mathcal{O}$ on **Top**, where the covering families of a space are just open covers of the whole space.

Example 2.3. On **Grp** the class of surjective homomorphisms form a pretopology.

Example 2.4. On **Top** the class of numerable open covers (i.e. those that admit a subordinate partition of unity [Dol63]) form a pretopology. Much of traditional bundle theory is carried out using this site, for example, the Milnor classifying space classifies bundles which are locally trivial over numerable covers **??**.

Definition 2.5. Let (S, J) be a site. The pretopology J is called a *singleton pretopology* if every covering family consists of a single arrow $(U \to A)$. In this case a covering family is called a *cover*.

Example 2.6. In **Top**, the classes of covering maps, local section admitting maps, surjective étale maps and open surjections are all examples of singleton pretopologies. The results of [Pro96] pertaining to topological groupoids were carried out using the site of open surjections.

Definition 2.7. A covering family $(U_i \to A)_{i\in I}$ is called *effective* if A is the colimit of the following diagram: the objects are the U_i and the pullbacks $U_i \times_A U_j$, and the arrows are the projections

$$U_i \leftarrow U_i \times_A U_j \to U_j.$$

If the covering family consists of a single arrow $(U \to A)$, this is the same as saying $U \to A$ is a regular epimorphism.

Definition 2.8. A site is called *subcanonical* if every covering family is effective.

Example 2.9. On **Top**, the usual pretopology of opens, the pretopology of numerable covers and that of open surjections are subcanonical.

Example 2.10. In a regular category, the regular epimorphisms form a subcanonical singleton pretopology.

In fact, the (pullback stable) regular epimorphisms in any category form the largest subcanonical topology, so it has its own name[3]

Definition 2.11. The *canonical singleton pretopology* R is the class of all regular epimorphisms which are pullback stable. It contains all the subcanonical singleton pretopologies.

Remark 2.12. If $U \to A$ is an effective cover, a functor $\check{C}(U) \to \mathrm{disc}(B)$ gives a unique arrow $A \to B$. This follows immediately from the fact A is the colimit of $\check{C}(U)$.

Definition 2.13. A *finitary* (resp. *infinitary*) *extensive* category is a category with finite (resp. small) coproducts such that the following condition holds: let I be a a finite set (resp. any set), then, given a collection of commuting diagrams

$$\begin{tikzcd} x_i \arrow[r] \arrow[d] & z \arrow[d] \\ a_i \arrow[r] & \coprod_{i\in I} a_i \end{tikzcd},$$

one for each $i \in I$, the squares are all pullbacks if and only if the collection $\{x_i \to z\}_I$ forms a coproduct diagram.

In such a category there is a strict initial object (i.e. given a map $A \to 0$, $A \simeq 0$).

Example 2.14. **Top** is infinitary extensive.

Example 2.15. $\mathbf{Ring}^{op}$ is finitary extensive.

In **Top** we can take an open cover $\{U_i\}_I$ of a space X and replace it with the single map $\coprod_I U_i \to X$, and work just as before using this new sort of cover, using the fact **Top** is extensive. The sort of sites that mimic this behaviour are called superextensive.

[3] of course, the nomenclature was decided the other way around - 'subcanonical' meaning 'contained in the canonical pretopology.'

Definition 2.16. (Bartels-Shulman) A *superextensive site* is an extensive category S equipped with a pretopology J containing the families

$$(U_i \to \coprod_I U_i)_{i\in I}$$

and such that all covering families are bounded. This means that for a finitely extensive site, the families are finite, and for an infinitary site, the families are small.

Example 2.17. Given an extensive category S, the *extensive pretopology* has as covering families the bounded collections $(U_i \to \coprod_I U_i)_{i\in I}$. The pretopology on any superextensive site contains the extensive pretopology.

Example 2.18. The category **Top** with its usual pretopology of open covers is a superextensive site.

Given a superextensive site, one can form the class $\amalg J$ of arrows $\coprod_I U_i \to A$.

Proposition 2.19. *The class $\amalg J$ is a singleton pretopology, and is subcanonical if and only if J is.*

Proof. Since identity arrows are covers for J they are covers for $\amalg J$. The pullback of a $\amalg J$-cover $\coprod_I U_i \to A$ along $B \to A$ is a $\amalg J$-cover as coproducts and pullbacks commute by definition of an extensive category. Now for the third condition. we use the fact that in an extensive category a map

$$f \colon B \to \coprod_I A_i$$

implies that $B \simeq \coprod_I B_i$ and $f = \coprod_i f_i$. Given $\amalg J$-covers $\coprod_I U_i \to A$ and $\coprod_J V_j \to (\coprod_I U_i)$, we see that $\coprod_J V_j \simeq \coprod_I W_i$. By the previous point, the pullback

$$\coprod_I U_k \times_{\coprod_I U_{i'}} W_i$$

is a $\amalg J$-cover of U_i, and hence $(U_k \times_{\coprod_I U_{i'}} W_i \to U_k)_{i\in I}$ is a J-covering family for each $k \in I$. Thus

$$(U_k \times_{\coprod_I U_{i'}} W_i \to A)_{i,k\in I}$$

is a J-covering family, and so

$$\coprod_J V_j \simeq \coprod_{k\in I}\left(\coprod_I U_k \times_{\coprod_I U_{i'}} W_i\right) \to A$$

is a $\amalg J$-cover.
The map $\coprod_I U_i \to A$ is the coequaliser of $\coprod_{I\times I} U_i \times_A U_j \rightrightarrows \coprod_I U_i$ if and only if A is the colimit of the diagram in definition 2.7. Hence $(\coprod_I U_i \to A)$ is effective if and only if $(U_i \to A)_{i\in I}$ is effective □

Notice that the original pretopology J is generated by the union of $\amalg J$ and the extensive pretopology.

Definition 2.20. Let (S, J) be a site. An arrow $P \to A$ in S is a *J-epimorphism* (or simply *J-epi*) if there is a covering family $(U_i \to A)_{i\in I}$ and a lift

$$\begin{tikzcd} & P \arrow[d] \\ U_i \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

for every $i \in I$. The class of J-epimorphisms will be denoted (J-epi).

This definition is equivalent to the definition in III.7.5 in [MM92]. The dotted maps in the above definition are called local sections, after the case of the usual open cover pretopology on **Top**. If the pretopology is left unnamed, we will refer to *local epimorphisms*.

One reason we are interested in superextensive sites is the following

Lemma 2.21. *If (S, J) is a superextensive site, the class of J-epimorphisms is precisely the class of $\amalg J$-epimorphisms.*

If S has all pullbacks then the class of J-epimorphisms form a pretopology. In fact they form a pretopology with an additional condition – it is *saturated*. The following is adapted from [BW84]:[4]

Definition 2.22. A singleton pretopology K is *saturated* if whenever the composite $V \to U \to A$ is in K, then $U \to A$ is in K.

In fact only a slightly weaker condition on S is necessary for (J-epi) to be a pretopology.

Example 2.23. Let (S, J) be a site. If pullbacks of J-epimorphisms exist then the collection (J-epi) of J-epimorphisms is a saturated pretopology.

There is a definition of 'saturated' for arbitrary pretopologies, but we will use only this one. Another way to pass from an arbitrary pretopology to a singleton one in a canonical way is this:

Definition 2.24. The *singleton saturation* of a pretopology on an arbitrary category S is the largest class $J_{sat} \subset$ (J-epi) of those J-epimorphisms which are pullback stable.

If J is a singleton pretopology, it is clear that $J \subset J_{sat}$. In fact J_{sat} contains all the covering families of J with only one element when J is any pretopology.

From lemma 2.21 we have

Corollary 2.25. *In a superextensive site (S, J), the saturations of J and $\amalg J$ coincide.*

One class of extensive categories which are of particular interest is those that also have finite/small limits. These are called *lextensive*. For example, **Top** is infinitary

[4]Note that in [BW84] what we are calling a Grothendieck *pretopology*, is called a Grothendieck *topology*.

lextensive, as is a Grothendieck topos. In contrast, a general topos is finitary lextensive. In a lextensive category

$$J_{sat} = (\amalg J)_{sat} = (J\text{-epi}).$$

Sometimes a pretopology J contains a smaller pretopology that still has enough covers to compute the same J-epis.

Definition 2.26. If J and K are two singleton pretopologies with $J \subset K$, such that $K \subset J_{sat}$, then J is said to be *cofinal* in K, denoted $J \leq K$.

Clearly $J \leq J_{sat}$.

Lemma 2.27. *If $J \leq K$, then $J_{sat} = K_{sat}$.*

3. Weak equivalences

For categories internal to **Set**, equivalences are precisely those fully faithful, essentially surjective functors. For internal categories, however, this is not the case.In addition, we need to make use of a pretopology to make the 'surjective' part of essentially surjective meaningful.

Definition 3.1. [**BP79, EKvdL05**] An internal functor $f : X \to Y$ in a site (S, J) is called

(1) *fully faithful* if

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "{(s,t)}"'] & Y_1 \arrow[d, "{(s,t)}"] \\ X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0 \end{tikzcd}$$

is a pullback diagram

(2) *essentially J-surjective* if the arrow labelled $\circledast$ is in (J-epi)

$$\begin{tikzcd} & X_0 \times_{Y_0} Y_1^{iso} \arrow[dl] \arrow[d] \arrow[ddr, bend left, "\circledast"] & \\ X_0 \arrow[d, "f_0"'] & Y_1^{iso} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ Y_0 & & Y_0 \end{tikzcd}$$

(3) a *J-equivalence* if it is fully faithful and essentially J-surjective.

The class of J-equivalences will be denoted W_J, and if mention of J is suppressed, they will be called *weak equivalences*.

Example 3.2. If $X \to Y$ is an internal equivalence, then it is a J-equivalence for all pretopologies J [**EKvdL05**]. In fact, if T denotes the trivial pretopology (only isomorphisms are covers) the T-equivalences are precisely the internal equivalences.

Example 3.3. If J is a singleton pretopology, and $U \to A$ is a J-cover (or more generally, is in J_{sat}), $\check{C}(U) \to \mathrm{disc}(A)$ is a J-equivalence.

Example 3.4. If $f : X \to Y$ is a functor such that f_0 is in (J-epi), then f is essentially J-surjective.

A very important example of a J-equivalence requires a little set up. The strict pullback of internal categories

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is the category with objects $X_0 \times_{Y_0} Z_0$, arrows $X_1 \times_{Y_1} Z_1$, and all structure maps given componentwise by those of X and Z.

Definition 3.5. Let S be a category with binary products, X a category internal to S and $p : M \to X_0$ an arrow in S. Define the *induced category* $X[M]$ to be the strict pullback

(2)
$$\begin{tikzcd} X[M] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \mathrm{codisc}(M) \arrow[r] & \mathrm{codisc}(X_0) \end{tikzcd}$$

with objects M and arrows $M^2 \times_{X_0^2} X_1$. The canonical functor in the top row has as object component p and is fully faithful.

It follows immediately from the definition that given maps $M \to X_0$, $N \to M$, there is a canonical isomorphism

(3)
$$X[M][N] \simeq X[N].$$

If we agree to follow the convention that $M \times_N N = M$ is the pullback along the identity arrow id_N, then $X[X_0] = X$. This also simplifies other results of this chapter, so will be adopted from now on. One consequence of this assumption is that the iterated fibre product

$$M \times_M M \times_M \ldots \times_M M,$$

bracketed in any order, is equal to M. We cannot, however, equate two bracketings of a general iterated fibred product – they are only canonically isomorphic.

Example 3.6. If $\check{C}(B)$ is the Čech groupoid associated to a map $j \colon B \to A$ in S, then $\mathrm{disc}(A)[B] \simeq \check{C}(B)$. Of special interest is the case when j is a cover for some pretopology on S.

Lemma 3.7. *If (S, J) is a site, X a category in S and $(U \to X_0)$ is a covering family, the functor $X[U] \to X$ is a J-equivalence.*

Proof. The object component of the canonical functor $X[U] \to X$ is $U \to X_0$ and since it is in J it is in J_{sat}. Hence $X[U] \to X$ is a J-equivalence. □

LEMMA 3.8. *Let X be an internal category in S, and $M \to X_0$, $N \to X_0$ arrows in S. Then the following square is a strict pullback*

$$\begin{tikzcd} X[M \times_{X_0} N] \arrow[r] \arrow[d] & X[N] \arrow[d] \\ X[M] \arrow[r] & X \end{tikzcd}$$

Proof. Consider the following cube

$$\begin{tikzcd} X[M \times_{X_0} N] \arrow[rr] \arrow[dr] \arrow[dd] & & X[N] \arrow[dr] \arrow[dd] & \\ & X[M] \arrow[rr] \arrow[dd] & & X \arrow[dd] \\ \mathrm{codisc}(M \times_{X_0} N) \arrow[rr] \arrow[dr] & & \mathrm{codisc}(N) \arrow[dr] & \\ & \mathrm{codisc}(M) \arrow[rr] & & \mathrm{codisc}(X_0) \end{tikzcd}$$

The bottom and sides are pullbacks, either by definition, or using (3), and so the top is a pullback. □

Fully faithful functors are stable under pullback, much like monomorphisms are.

LEMMA 3.9. *If $f\colon X \to Y$ is fully faithful, and $g\colon Z \to Y$ is any functor, $\hat{f}$ in*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "\hat{f}"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is fully faithful.

Proof. The following chain of isomorphisms establishes the claim

$$\begin{aligned} (Z_0 \times_{Y_0} X_0)^2 \times_{Z_0^2} Z_1 &\simeq X_0^2 \times_{Y_0^2} Z_1 \\ &\simeq (X_0^2 \times_{Y_0^2} Y_1) \times_{Y_1} Z_1 \\ &\simeq X_1 \times_{Y_1} Z_1, \end{aligned}$$

the last following from the fact f is fully faithful. □

4. Anafunctors

DEFINITION 4.1. [**Mak96**, **Bar06**] Let (S, J) be a site. An *anafunctor* in (S, J) from a category X to a category Y consists of a cover $(U \to X_0)$ and an internal functor

$$f\colon X[U] \to Y.$$

The anafunctor is a span in $Cat(S)$, and will be denoted

$$(U, f)\colon X \to\!\!\!\!\!+\, Y.$$

EXAMPLE 4.2. For an internal functor $f\colon X \to Y$ in the site (S, J), define the anafunctor $(X_0, f)\colon X \to\!\!\!\!\!+\, Y$ as the following span

$$X \xleftarrow{=} X[X_0] \xrightarrow{f} Y.$$

We will blur the distinction between these two descriptions. If $f = id\colon X \to X$, then (X_0, id) will be denoted simply by id_X.

EXAMPLE 4.3. If $U \to A$ is a cover in (S, J) and G is a group object in S, an anafunctor $(U, g)\colon \mathrm{disc}(A) \to\!\!\!\!\!+\, \mathbf{B}G$ is a Čech cocycle.

DEFINITION 4.4. [**Mak96**, **Bar06**] Let (S, J) be a site, $(U, f), (V, g)\colon X \to\!\!\!\!\!+\, Y$ anafunctors in S. A *transformation*

$$\alpha\colon (U, f) \to (V, g)$$

from (U, f) to (V, g) is an internal natural transformation

$$\begin{tikzcd} & X[U \times_{X_0} V] \arrow[dl] \arrow[dr] & \\ X[U] \arrow[dr, "f"'] & \overset{\alpha}{\Rightarrow} & X[V] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

If $\alpha\colon U \times_{X_0} V \to Y_1$ factors through Y_1^{iso}, then α is called an *isotransformation*. In that case we say (U, f) *is isomorphic to* (V, g). Clearly all transformations between anafunctors between internal groupoids are isotransformations.

EXAMPLE 4.5. Given functors $f, g\colon X \to Y$ between categories in S, and a natural transformation $a\colon f \Rightarrow g$, there is a transformation $a\colon (X_0, f) \Rightarrow (X_0, g)$ of anafunctors, given by $X_0 \times_{X_0} X_0 = X_0 \xrightarrow{a} Y_1$.

EXAMPLE 4.6. If $(U, g), (V, h)\colon \mathrm{disc}(A) \to\!\!\!\!\!+\, \mathbf{B}G$ are two Čech cocycles, a transformation between them is a coboundary on the cover $U \times_A V \to A$.

EXAMPLE 4.7. Let $(U, f)\colon X \to\!\!\!\!\!+\, Y$ be an anafunctor in S. There is an isotransformation $1_{(U,f)}\colon (U, f) \Rightarrow (U, f)$ called the identity transformation, given by the natural transformation with component

$$U \times_{X_0} U \simeq (U \times U) \times_{X_0^2} X_0 \xrightarrow{id_U^2 \times e} X[U]_1 \xrightarrow{f_1} Y_1 \tag{4}$$

EXAMPLE 4.8. [**Mak96**] Given anafunctors $(U, f)\colon X \to Y$ and $(V, f \circ k)\colon X \to Y$ where $k\colon V \simeq U$ is an isomorphism over X_0, a *renaming transformation* $(U, f) \Rightarrow (V, f \circ k)$ is an isotransformation with component

$$1_{(U,f)} \circ (k \times \mathrm{id}) : V \times_{X_0} U \to U \times_{X_0} U \to Y_1$$

k will be referred to as a renaming isomorphism.
More generally, we could let $k\colon V \to U$ be any refinement, and this prescription also gives an isotransformation $(U, f) \Rightarrow (V, f \circ k)$.

Example 4.9. As a concrete and relevant example of a renaming transformation we can consider the triple composition of anafunctors

$$\begin{aligned} (U, f)&\colon X \mathrel{+\!\!\!\longrightarrow} Y, \\ (V, g)&\colon Y \mathrel{+\!\!\!\longrightarrow} Z, \\ (W, h)&\colon Z \mathrel{+\!\!\!\longrightarrow} A. \end{aligned}$$

The two possibilities of composing these are

$$\big((U \times_{Y_0} V) \times_{Z_0} W, h \circ (gf^V)^W\big), \quad \big(U \times_{Y_0} (V \times_{Z_0} W), h \circ g^W \circ f^{V \times_{Z_0} W}\big)$$

The unique isomorphism $(U \times_{Y_0} V) \times_{Z_0} W \simeq U \times_{Y_0} (V \times_{Z_0} W)$ commuting with the various projections is then the required renaming isomorphism. The isotransformation arising from this renaming transformation is the *associator*.

We define the composition of anafunctors as follows. Let $(U, f)\colon X \mathrel{+\!\!\!\longrightarrow} Y$, $(V, g)\colon Y \mathrel{+\!\!\!\longrightarrow} Z$ be anafunctors in the site (S, J). Their composite $(V, g) \circ (U, f)$ is the composite span defined in the usual way.

$$\begin{tikzcd}
 & & X[U \times_{Y_0} V] \arrow[dl] \arrow[dr, "f^V"] & & \\
 & X[U] \arrow[dl] \arrow[dr, "f"'] & & Y[V] \arrow[dl] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

The pullback is as shown by lemma 3.8, and the resulting span is an anafunctor because $V \to Y_0$, and hence $U \times_{Y_0} V \to X_0$, is a cover, and using (3). We will sometimes denote the composite by $(U \times_{Y_0} V, g \circ f^V)$.

Remark 4.10. If one doesn't impose the existence of pullbacks on S (as in say **Diff**, see comment **??**), this composite span still exists, because $V \to Y_0$ is a cover

Consider the special case when $V = Y_0$, and hence (Y_0, g) is just an ordinary functor. Then there is a renaming transformation (the identity transformation!) $(Y_0, g) \circ (U, f) \Rightarrow (U, g \circ f)$, using the equality $U \times_{Y_0} Y_0 = U$. If we let $g = \mathrm{id}_Y$, then we see that (Y_0, id_Y) is a strict unit on the left for anafunctor composition. Similarly, considering $(V, g) \circ (Y_0, \mathrm{id})$, we see that (Y_0, id_Y) is a two-sided strict unit for anafunctor composition. In fact, we have also proved

Lemma 4.11. *Given two functors $f\colon X \to Y$, $g\colon Y \to Z$ in S, their composition as anafunctors is equal to their composition as functors:*

$$(Y_0, g) \circ (X_0, f) = (X_0, g \circ f).$$

A simple but useful criterion for describing isotransformations where either of the anafunctors is a functor is as follows.

LEMMA 4.12. *An anafunctor $(V, g)\colon X \mathrel{-\!\!\!+\!\!\!\to} Y$ is isomorphic to a functor $f\colon X \to Y$ if and only if there is a natural isomorphism*

$$\begin{tikzcd} & X[V] \arrow[dl] \arrow[dr, "g"] & \\ X \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\overset{\sim}{\Rightarrow}"] & & Y \end{tikzcd}$$

In a site (S, J) where the axiom of choice holds (that is, every epimorphism has a section), one can prove that every J-equivalence between internal categories is in fact an internal equivalence of categories. It is precisely the lack of splittings that prevents this theorem from holding in general sites. The best one can do in a general site is described in the the following two lemmas.

LEMMA 4.13. *Let $f\colon X \to Y$ be a J-equivalence in (S, J), and choose a cover $U \to Y_0$ and a local section $s\colon U \to X_0 \times_{Y_0} Y_1^{iso}$. Then there is a functor $Y[U] \to X$ with object component $s' := \mathrm{pr}_1 \circ s\colon U \to X_0$.*

Proof. The object component is given, we just need the arrow component. Denote the local section by $(s', \iota) : U \to X_0 \times_{Y_0} Y_1^{iso}$. Consider the composite

$$\begin{aligned} Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{(s',\iota)\times \mathrm{id} \times(-\iota,s')} (X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \hookrightarrow \\ X_0 \times_{Y_0} Y_3 \times_{Y_0} X_0 \xrightarrow{\mathrm{id}\times m \times \mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0 \simeq X_1 \end{aligned}$$

It is clear that this commutes with source and target, because these are projection on the first and last factor at each step. To see that it respects identities and composition, just use the fact that the ι component will cancel with the $-\iota$ component. □

Hence there is an anafunctor $Y \mathrel{-\!\!\!+\!\!\!\to} X$, and the next proposition tells us this is a pseudoinverse to f (in a sense to be made precise in proposition 4.19 below).

LEMMA 4.14. *Let $f\colon X \to Y$ be a J-equivalence in S. There is an anafunctor*

$$(U, \bar{f})\colon Y \mathrel{-\!\!\!+\!\!\!\to} X$$

and isotransformations

$$\iota\colon (X_0, f) \circ (U, \bar{f}) \Rightarrow id_Y$$
$$\epsilon\colon (U, \bar{f}) \circ (X_0, f) \Rightarrow id_X$$

Proof. We have the anafunctor $(U, \bar{f})$ from the previous lemma. Since the anafunctors id_X, id_Y are actually functors, we can use lemma 4.12. Using the special case of anafunctor composition when the second is a functor, this tells us that ι will be given by a natural isomorphism

$$\begin{tikzcd} & X \arrow[dr, "f"] \arrow[d, Rightarrow] & \\ Y[U] \arrow[ur, "\bar{f}"] \arrow[rr] & {} & Y \end{tikzcd}$$

This has component $\iota\colon U \to Y_1^{iso}$, using the notation from the proof of the previous lemma. Notice that the composite $f_1 \circ \bar{f}_1$ is just

$$Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\iota\times\mathrm{id}\times -\iota} Y_1^{iso} \times_{Y_0} Y_1 \times_{Y_0} Y_1^{iso} \hookrightarrow Y_3 \xrightarrow{m} Y_1.$$

Since the arrow component of $Y[U] \to Y$ is $U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\mathrm{pr}_2} Y_1$, ι is indeed a natural isomorphism using the diagram (1).

The other isotransformation is between $(X_0 \times_{Y_0} U, \bar{f} \circ \mathrm{pr}_2)$ and (X_0, id_X), and is given by the arrow

$$\epsilon\colon X_0 \times_{X_0} X_0 \times_{Y_0} U \simeq X_0 \times_{Y_0} U \xrightarrow{\mathrm{id}\times(s',a)} X_0 \times_{Y_0} (X_0 \times_{Y_0} Y_1) \simeq X_0^2 \times_{Y_0^2} Y_1 \simeq X_1$$

This has the correct source and target, as the object component of $\bar{f}$ is s', and the source is given by projection on the first factor of $X_0 \times_{Y_0} U$. This diagram

$$\begin{tikzcd}
(X_0 \times_{Y_0^2} U)^2 \times_{X_0^2} X_1 \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\simeq"'] & & X_1 \arrow[dd, "\simeq"] \\
U \times_{Y_0} X_1 \times_{Y_0} U \arrow[d, "-\iota\times f\times\iota"'] & & \\
(X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \arrow[rr, "\mathrm{id}\times m\times\mathrm{id}"'] & & X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0
\end{tikzcd}$$

commutes, and using (1) we see that ϵ is natural. ☐

Just as there is composition of natural transformations between internal functors, there is a composition of transformations between internal anafunctors [Bar06]. This is where the effectiveness of our covers will be used in order to construct a map locally over some cover. Consider the following diagram

$$\begin{tikzcd}
 & & X[U \times_{X_0} V \times_{X_0} W] \arrow[dl] \arrow[dr] & & \\
 & X[U \times_{X_0} V] \arrow[dl] \arrow[dr] & & X[V \times_{X_0} W] \arrow[dl] \arrow[dr] & \\
X[U] \arrow[drr, "f"'] & \overset{a}{\Rightarrow} & X[V] \arrow[d, "g"] & \overset{b}{\Rightarrow} & X[W] \arrow[dll, "h"] \\
 & & Y & &
\end{tikzcd}$$

from which we can form a natural transformation between the leftmost and the rightmost composites as functors in S. This will have as its component the arrow

$$\widetilde{ba}\colon U \times_{X_0} V \times_{X_0} W \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} U \times_{X_0} V \times_{X_0} V \times_{X_0} W \xrightarrow{a\times b} Y_1 \times_{Y_0} Y_1 \xrightarrow{m} Y_1$$

in S. Notice that the Čech groupoid of the cover

$$U \times_{X_0} V \times_{X_0} W \to U \times_{X_0} W \tag{5}$$

is

$$U \times_{X_0} V \times_{X_0} V \times_{X_0} W \rightrightarrows U \times_{X_0} V \times_{X_0} W,$$

using the two projections $V \times_{X_0} V \to V$. Denote this pair of parallel arrows by $s, t\colon UV^2W \rightrightarrows UVW$ for brevity. In [Bar06] we find this commuting diagram

(6)
$$\begin{tikzcd} UV^2W \arrow[r, "t"] \arrow[d, "s"'] & UVW \arrow[d, "\widetilde{ba}"] \\ UVW \arrow[r, "\widetilde{ba}"'] & Y_1 \end{tikzcd}$$

and so we have a functor $\check{C}(U \times_{X_0} V \times_{X_0} W) \to \mathrm{disc}(Y_1)$. Our pretopology J is assumed to be subcanonical, and using remark 2.12 this gives us a unique arrow $ba\colon U \times_{X_0} W \to Y_1$, the composite of a and b.

Remark 4.15. In the special case that $U \times_{X_0} V \times_{X_0} W \to U \times_{X_0} W$ is an isomorphism (or is even just split), the composite transformation has

$$U \times_{X_0} W \to U \times_{X_0} V \times_{X_0} W \xrightarrow{\widetilde{ba}} Y_1$$

as its component arrow. In particular, this is the case if one of a or b is a renaming transformation.

Example 4.16. Let $(U, f) : X \to\!\!\!\mapsto Y$ be an anafunctor and $U'' \xrightarrow{j'} U' \xrightarrow{j} U$ successive refinements of $U \to X_0$ (e.g isomorphisms). Let $(U', f_{U'}), (U'', f_{U''})$ denote the composites of f with $X[U'] \to X[U]$ and $X[U''] \to X[U]$ respectively. The arrow

$$U \times_{X_0} U'' \xrightarrow{\mathrm{id}_U \times j \circ j'} U \times_{X_0} U \to Y_1$$

is the component for the composition of the isotransformations $(U, f) \Rightarrow (U', f_{U'}), \Rightarrow (U'', f_{U''})$ described in example 4.8. Thus we can see that the composite of renaming transformations associated to isomorphisms ϕ_1, ϕ_2 is simply the renaming transformation associated to their composite $\phi_1 \circ \phi_2$.

Example 4.17. If $a\colon f \Rightarrow g,\ b\colon g \Rightarrow h$ are natural transformations between functors $f, g, h\colon X \to Y$ in S, their composite as transformations between anafunctors

$$(X_0, f), (X_0, g), (X_0, h)\colon X \to\!\!\!\mapsto Y.$$

is just their composite as natural transformations. This uses the equality

$$X_0 \times_{X_0} X_0 \times_{X_0} X_0 = X_0 \times_{X_0} X_0 = X_0.$$

Theorem 4.18. [Bar06] *For a site (S, J) where J is a subcanonical singleton pretopology, internal categories (resp. groupoids), anafunctors and transformations form a bicategory $\boldsymbol{AnaCat}(S, J)$ (resp. $\boldsymbol{Ana}(S, J)$).*

There is a strict 2-functor $\mathbf{Ana}(S, J) \to \mathbf{AnaCat}(S, J)$ which is the identity on 0-cells and induces isomorphisms on hom-categories. The following is the main result of this section, and allows us to relate anafunctors to the localisations considered in the next section.

PROPOSITION 4.19. *There are strict 2-functors*

$$\alpha_J\colon \boldsymbol{Cat}(S) \to \boldsymbol{AnaCat}(S,J),$$
$$\beta_J = \alpha_J\big|_{\boldsymbol{Gpd}(S)}\colon \boldsymbol{Gpd}(S) \to \boldsymbol{Ana}(S,J)$$

sending J-equivalences to equivalences such that

$$\begin{tikzcd}
\boldsymbol{Gpd}(S) \arrow[r, hook] \arrow[d, "\beta_J"'] & \boldsymbol{Cat}(S) \arrow[d, "\alpha_J"] \\
\boldsymbol{Ana}(S,J) \arrow[r, hook] & \boldsymbol{AnaCat}(S,J)
\end{tikzcd}$$

commutes.

Proof. We define α_J and β_J to be the identity on objects, and as described in examples 4.2, 4.5 on 1-cells and 2-cells (i.e. functors and transformations). We need first to show that this gives a functor $\mathbf{Cat}(S)(X,Y) \to \mathbf{AnaCat}(S,J)(X,Y)$. This is precisely the content of example 4.17. Since the identity 1-cell on a category X in $\mathbf{AnaCat}(S,J)$ is the image of the identity functor on S in $\mathbf{Cat}(S)$, α_J and β_J respect identity 1-cells. Also, lemma 4.11 tells us that α_J and β_J respect composition. That α_J and β_J send J-equivalences to equivalences is the content of lemma 4.14. $\square$

5. Localising bicategories at a class of 1-cells

Ultimately we are interesting in inverting all weak equivalences in $\mathbf{Gpd}(S)$, and so need to discuss what it means to add the formal pseudoinverses to a class of 1-cells in 2-category - a process known as *localisation*. This was done in [Pro96] for the more general case of a class of 1-cells in a bicategory, where the resulting bicategory is constructed and its universal properties (analogous to those of a quotient) examined. The application in *loc. cit.* is to showing the equivalence of various bicategories of stacks to localisations of 2-categories of groupoids. The results of this chapter can be seen as one-half of a generalisation of these results to an arbitrary site with pullbacks.

DEFINITION 5.1. Let E be a class of arrows in the ambient category S. E is called a *class of admissible maps for J* if it is a singleton pretopology in which a given singleton pretopology J is cofinal, and satisfying the following condition:

(S) E contains the split epimorphisms, and if $e\colon A \to B$ is a split epimorphism, and $A \xrightarrow{e} B \xrightarrow{p} C$ is in M, then $p \in M$.

EXAMPLE 5.2. If E is a saturated singleton pretopology, it is a class of admissible maps for itself, and (J-epi) is a class of admissible maps for J (they satisfy condition (S) because they are saturated). A singleton pretopology satisfying condition (S) is a class of admissible maps for itself, and will just be referred to as a *class of admissible maps*. In particular, E could be the class of J-epimorphisms for a non-singleton pretopology J.

Definition 5.3. [EKvdL05] Let E be some class of admissible maps in a category S. A functor $X \to Y$ in S is called an *E-equivalence* if it is fully faithful, and

$$X_0 \times_{Y_0} Y_1^{iso} \xrightarrow{t \circ \mathrm{pr}_2} Y_0$$

is in E. If this last condition holds we will say the functor is essentially E-surjective.

If $E = (J\text{-epi})$ for some pretopology J, we will still refer to J-equivalences. The class of E-equivalences will be denoted W_E.

Definition 5.4. [Pro96] Let B be a bicategory and $W \subset B_1$ a class of 1-cells. A *localisation of B with respect to W* is a bicategory $B[W^{-1}]$ and a weak 2-functor

$$U : B \to B[W^{-1}]$$

such that: U sends elements of W to equivalences, and is universal with this property i.e. composition with U gives an equivalence of bicategories

$$U^* : Hom(B[W^{-1}], D) \to Hom_W(B, D),$$

where Hom_W denotes the sub-bicategory of weak 2-functors that send elements of W to equivalences (call these *W-inverting*, abusing notation slightly).

The universal property means that W-inverting weak 2-functors $F : B \to D$ factor, up to a transformation, through $B[W^{-1}]$, inducing an essentially unique weak 2-functor $\widetilde{F} : B[W^{-1}] \to D$.

Definition 5.5. [Pro96] Let B be a bicategory B with a class W of 1-cells. W is said to *admit a right calculus of fractions* if it satisfies the following conditions

2CF1. W contains all equivalences
2CF2. a) W is closed under composition
b) If $a \in W$ and a iso-2-cell $a \overset{\sim}{\Rightarrow} b$ then $b \in W$
2CF3. For all $w : A' \to A$, $f : C \to A$ with $w \in W$ there exists a 2-commutative square

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "v"'] & A' \arrow[d, "w"] \arrow[dl, phantom, "\overset{\simeq}{\swarrow}"] \\ C \arrow[r, "f"'] & A \end{tikzcd}$$

with $v \in W$.
2CF4. If $\alpha : w \circ f \Rightarrow w \circ g$ is a 2-cell and $w \in W$ there is a 1-cell $v \in W$ and a 2-cell $\beta : f \circ v \Rightarrow g \circ v$ such that $\alpha \circ v = w \circ \beta$. Moreover: when α is an iso-2-cell, we require β to be an isomorphism too; when v' and β' form another such pair, there exist 1-cells u, u' such that $v \circ u$ and $v' \circ u'$ are in W, and an iso-2-cell

$\epsilon\colon v \circ u \Rightarrow v' \circ u'$ such that the following diagram commutes:

(7)
$$\begin{tikzcd}
f \circ v \circ u \arrow[r, Rightarrow, "\beta \circ u"] \arrow[d, Rightarrow, "f\circ\epsilon"', "\simeq"] & g \circ v \circ u \arrow[d, Rightarrow, "g\circ\epsilon", "\simeq"'] \\
f \circ v' \circ u' \arrow[r, Rightarrow, "\beta' \circ u'"'] & g \circ v' \circ u'
\end{tikzcd}$$

Remark 5.6. In particularly nice cases (as in the next section), the first half of 2CF4 holds due to left-cancellability of elements of W, giving us the canonical choice $v = I$.

Theorem 5.7. [Pro96] *A bicategory B with a class W that admits a calculus of right fractions has a localisation with respect to W.*

From now on we shall refer to a calculus of right fractions as simply a calculus of fractions, and the resulting localisation as a bicategory of fractions. Since $B[W^{-1}]$ is defined only up to equivalence, it is of great interest to know when a bicategory D in which elements of W are converted to equivalences is itself equivalent to $B[W^{-1}]$. In particular, one would be interested in finding such an equivalent bicategory with a simpler description than that which appears in [Pro96]. Thanks are due to Matthieu Dupont for pointing out (in personal communication) that the statement in loc. cit. actually only holds in one direction, as stated below, not in both.

Proposition 5.8. [Pro96] *A weak 2-functor $F : B \to D$ which sends elements of W to equivalences induces an equivalence of bicategories*

$$\widetilde{F}\colon B[W^{-1}] \xrightarrow{\sim} D$$

if the following conditions hold

EF1. *F is essentially surjective,*
EF2. *For every 1-cell $f \in D_1$ there is a $w \in W$ and a $g \in B_1$ such that $Fg \overset{\sim}{\Rightarrow} f \circ Fw$,*
EF3. *F is locally fully faithful.*

The following is useful in showing a weak 2-functor sends weak equivalences to equivalences, because this condition only needs to be checked on a class that is in some sense cofinal in the weak equivalences.

Theorem 5.9. *Let the bicategory B admit a calculus of fractions for W, and let $V \subset W$ be a class of 1-cells such that for all $w \in W$, there exists $v \in V$ and $s \in W$ such that there is an invertible 2-cell*

$$\begin{tikzcd}
 & a \arrow[d, "w"] \\
b \arrow[ur, "s"] \arrow[r, "v"'] \arrow[r, phantom, shift left=4, "\Downarrow"] & c\,.
\end{tikzcd}$$

Then a weak 2-functor $F\colon B \to D$ that sends elements of V to equivalences also sends elements of W to equivalences.

Proof. In the following the coherence cells will be implicit. First we show that Fw has a pseudosection in C for any $w \in W$. Let v, s be as above. Let $\widetilde{Fv}$ be a pseudoinverse of Fv, and let $j = Fs \circ \widetilde{Fv}$. Then there is the following invertible 2-cell

$$Fw \circ j \Rightarrow F(w \circ s) \circ \widetilde{Fv} \Rightarrow Fv \circ \widetilde{Fv} \Rightarrow I.$$

We now show that j is in fact a pseudoinverse for Fw. Since $s \in W$, there is a $v' \in V$ and $s' \in W$ and a 2-cell giving the following diagram

$$\begin{tikzcd}
d \arrow[r, "v'"] \arrow[d, "s'"'] \arrow[dr, phantom, "\Downarrow", near start] & a \arrow[d, "w"] \\
b \arrow[ur, "s"] \arrow[r, "v"'] \arrow[ur, phantom, "\Downarrow", near end] & c\,.
\end{tikzcd}$$

Apply the functor F, and denote pseudoinverses of Fv, Fv' by $\widetilde{Fv}, \widetilde{Fv'}$. Using the 2-cell $I \Rightarrow Fv' \circ \widetilde{Fv'}$ we get the following 2-cell

$$\begin{tikzcd}
Fd \arrow[d, "Fs"'] & Fa \arrow[l, "\widetilde{Fv'}"'] \arrow[d, "Fw"] \arrow[dl, phantom, "\Leftarrow"] \\
Fb \arrow[r, "Fv"'] & Fc
\end{tikzcd}$$

Then there is this composite invertible 2-cell

$$j \circ Fw \Rightarrow (Fs \circ \widetilde{Fv}) \circ (Fv \circ (Fs \circ \widetilde{Fv'})) \Rightarrow (Fs \circ Fs') \circ \widetilde{Fv'} \Rightarrow Fv' \circ \widetilde{Fv'} \Rightarrow I,$$

making Fw is an equivalence. Hence F sends all elements of W to equivalences. ☐

6. Anafunctors are a localisation

In this section we present the new result that $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ admit calculi of fractions for the weak equivalences, and the bicategory of anafunctors is an equivalent localisation.

Definition 6.1. (see, e.g. [**EKvdL05**]) The *isomorphism category* of an internal category X is the internal category denoted $X^{\mathbf{I}}$, with

$$X_0^{\mathbf{I}} = X_1^{iso}, \qquad X_1^{\mathbf{I}} = (X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

where the fibred product over X_1 arises by considering the composition maps

$$X_1 \times_{s,X_0,t} X_1^{iso} \to X_1$$
$$X_1^{iso} \times_{s,X_0,t} X_1 \to X_1.$$

Composition in $X^{\mathbf{I}}$ is the same as commutative squares in the case of ordinary categories. There are two functors $\mathbf{s}, \mathbf{t} \colon X^{\mathbf{I}} \to X$ which have the usual source and target maps of X as their respective object components.

This construction is internal version of the functor category $\mathbf{Cat}(\mathbf{I}, C)$, since the groupoid $\mathbf{I} = (\circ \xrightarrow{\simeq} \bullet)$ doesn't always exist internal to S.

Remark 6.2. There is an isomorphism $X_1^{\mathbf{I}} \simeq X_1^{iso} \times_{t,X_0,t} X_1 \times_{s,X_0,t} X_1^{iso}$ given by projecting out the last factor in

$$(X_1 \times_{s,X_0,t} X_1^{iso}) \times_{X_1} (X_1^{iso} \times_{s,X_0,t} X_1).$$

The astute reader will recognise the following as an internalisation of the usual notion of weak pullback

Definition 6.3. The *weak pullback* $X \tilde{\times}_Y Z$ of a diagram of internal categories

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is given by the pullback $X \times_{Y,\mathbf{s}} Y^{\mathbf{I}} \times_{\mathbf{t},Y} Z$. There is a 2-commutative square

$$\begin{tikzcd} X \tilde{\times}_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\simeq"] \\ X \arrow[r] & Y \end{tikzcd}$$

The following terminology is adapted from [EKvdL05], although strictly speaking this map is only a fibration when model structure from *loc. cit.* exists.

Definition 6.4. An internal functor $f \colon X \to Y$ is called a *trivial E-fibration* if it is fully faithful and $f_0 \in E$.

Lemma 6.5. *If a functor $f \colon X \to Y$ is an E-equivalence,*

$$X \times_Y Y^{\mathbf{I}} \xrightarrow{\mathbf{t} \circ \mathrm{pr}_2} Y$$

is a trivial E-fibration.

Proof. The object component of $\mathbf{t} \circ \mathrm{pr}_2$ is $t \circ \mathrm{pr}_2$, which is in E by definition if f is essentially E-surjective. Consider now the pullback

$$\begin{tikzcd} (X_0 \times_{Y_0} Y_1^{iso})^2 \times_{Y_0^2} Y_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ (X_0 \times_{Y_0} Y_1^{iso})^2 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

Remark 6.2 tells is that the pullback is isomorphic to $X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}}$ in the pullback

$$\begin{tikzcd}
X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} \arrow[r, "\mathrm{pr}_2"] \arrow[dd] & Y_1^{\mathbf{I}} \arrow[d, "\mathrm{pr}_1"] \\
 & Y_1 \arrow[d] \\
X_0^2 \arrow[r] & Y_0 \times Y_0
\end{tikzcd}$$

but if f is fully faithful,

$$\begin{aligned} X_0^2 \times_{Y_0^2} Y_1^{\mathbf{I}} &\simeq X_0^2 \times_{Y_0^2} Y_1 \times_{Y_1} Y_1^{\mathbf{I}} \\ &\simeq X_1 \times_{Y_1} Y_1^{\mathbf{I}}, \end{aligned}$$

hence $\mathbf{t} \circ \mathrm{pr}_2$ is fully faithful. □

The internal category $X \times_Y Y^{\mathbf{I}}$ is called the mapping path space construction in [EKvdL05]. If the model structure in *loc. cit.* exists, the above follows from cofibration-acyclic fibration factorisation.

THEOREM 6.6. *Let S be a category with pullbacks and a class E of admissible maps. The 2-categories $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ admit right calculi of fractions for the class W_E in each.*

Before we prove the theorem, we introduce a lemma

LEMMA 6.7. *Let $f, g\colon X \to Y$ be functors and $a\colon f \Rightarrow g$ a natural isomorphism. There is an isomorphism*

$$X_0^2 \times_{f^2, Y_0^2} Y_1 \simeq X_0^2 \times_{g^2, Y_0^2} Y_1$$

commuting with the projections to X_0^2.

Proof. Supressing the canonical isomorphisms $X_0^2 \times_{Y_0^2} Y_1 \simeq X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0$, the required isomorphism is

$$X_0 \times_{f,Y_0} Y_1 \times_{Y_0,f} X_0 \xrightarrow{(\mathrm{id},-a)\times\mathrm{id}\times(a,\mathrm{id})} X_0 \times_{g,Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \times_{Y_0,g} X_0 \xrightarrow{\mathrm{id}\times m\times\mathrm{id}} X_0 \times_{g,Y_0} Y_1 \times_{Y_0,g} X_0.$$

which is the identity map when restricted to the X_0 factors, from which the claim follows. □

Now the proof of theorem 6.6.

Proof. We show the conditions of definition 5.5 hold.

2CF1. Since E contains all the split epis, an internal equivalence is essentially E-surjective. Let $f\colon X \to Y$ be an internal equivalence, and $g\colon Y \to X$ a pseudoinverse. By definition there are natural isomorphisms $a\colon g \circ f \Rightarrow \mathrm{id}_X$ and $b\colon f \circ g \Rightarrow \mathrm{id}_Y$. To show that f is fully faithful, we first show that the map

$$q\colon X_1 \to X_0^2 \times_{Y_0^2} Y_1$$

is a split monomorphism over X_0^2. This diagram commutes

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, equal] & X_0^2 \times_{Y_0^2} Y_1 \arrow[d] \\ X_1 & X_0^2 \times_{gf,X_0^2} X_1 \arrow[l, "\simeq"'] , \end{tikzcd}$$

by the naturality of a, the marked isomorphism coming from lemma 6.7. The splitting commutes with projection to X_0^2 because the isomorphism does. Call the splitting s. The same argument implies that

$$Y_1 \to Y_0^2 \times_{X_0^2} X_1$$

is a split monomorphism over Y_0^2, and this implies the arrow

$$l\colon X_0^2 \times_{Y_0^2} Y_1 \to X_0^2 \times_{Y_0^2} Y_0^2 \times_{X_0^2} X_1 \simeq X_0^2 \times_{gf,X_0^2} X_1$$

is a split monomorphism. This diagram commutes

$$\begin{tikzcd} X_0^2 \times_{Y_0^2} Y_1 \arrow[rr, "l"] \arrow[d, "s"'] & & X_0^2 \times_{gf,X_0^2} X_1 \arrow[r, "\simeq"] & X_1 \arrow[d, equal] \\ X_1 \arrow[r, "q"'] & X_0^2 \times_{Y_0^2} Y_1 \arrow[r, "l"'] & X_0^2 \times_{gf,X_0^2} X_1 \arrow[r, "\simeq"] & X_1 \end{tikzcd}$$

using naturality again, and so $q \circ s = \mathrm{id}$. Thus q is an isomorphism, and f is fully faithful.

2CF2 a). That the composition of fully faithful functors is again fully faithful is trivial. To show that the composition of essentially E-surjective functors $f\colon X \to Y$, $g\colon Y \to Z$ is again so, consider the following diagram

$$\begin{tikzcd} & & Y_0 \times_{Z_0} Z_1 \arrow[r] \arrow[d] \arrow[rr, bend left, "t"] & Z_1 \arrow[r, "t"] \arrow[d, "s"] & Z_0 \\ X_0 \times_{Y_0} Y_1 \arrow[r] \arrow[d] \arrow[rr, bend left] & Y_1 \arrow[r, "t"] \arrow[d, "s"] & Y_0 \arrow[r, "g_0"'] & Z_0 & \\ X_0 \arrow[r, "f_0"'] & Y_0 & & & \end{tikzcd}$$

where the curved arrows are in E by assumption. The lower such arrow pulls back to an arrow $X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \to Y_0 \times_{Z_0} Z_1$ (again in E). Hence the composite

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \to Y_0 \times_{Z_0} Z_1 \xrightarrow{t \circ \mathrm{pr}_2} Z_0$$

is in E, and is equal to the composite

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id} \times g \times \mathrm{id}} X_0 \times_{Z_0} Z_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id} \times m} X_0 \times_{Z_0} Z_1 \xrightarrow{t \circ \mathrm{pr}_2} Z_0.$$

The map

$$X_0 \times_{Z_0} Z_1 \simeq X_0 \times_{Y_0} Y_0 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id} \times e \times \mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1$$

is a section of

$$X_0 \times_{Y_0} Y_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times g\times \mathrm{id}} X_0 \times_{Z_0} Z_1 \times_{Z_0} Z_1 \xrightarrow{\mathrm{id}\times m} X_0 \times_{Z_0} Z_1.$$

Now condition (S) tells us that $X_0 \times_{Z_0} Z_1 \xrightarrow{t\circ \mathrm{pr}_2} Z_0$ is in E, and $g \circ f$ is essentially E-surjective.

2CF2 b). We will show this in two parts: fully faithful functors are closed under isomorphism, and essentially E-surjective functors are closed under isomorphism. Let $w, f : X \to Y$ be functors and $a: w \Rightarrow f$ be a natural isomorphism. First, let w be essentially E-surjective. That is,

$$X_0 \times_{w,Y_0,s} Y_1 \xrightarrow{t\circ \mathrm{pr}_2} Y_0 \tag{8}$$

is in E. Now note that the map

$$X_0 \times_{f,Y_0,s} Y_1 \xrightarrow{(\mathrm{id},-a)\times \mathrm{id}} X_0 \times \times_{w,Y_0,s} Y_1 \times_{t,Y_0,s} Y_1 \xrightarrow{\mathrm{id}\times m} X_0 \times_{w,Y_0,s} Y_1 \tag{9}$$

is an isomorphism, and so the composite of 9 and 8 is in E. Thus f is essentially E-surjective.

Now let w be fully faithful. Thus

$$\begin{tikzcd} X_1 \arrow[r, "w"] \arrow[d] & Y_1 \arrow[d] \\ X_0 \times X_0 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

is a pullback square. Using lemma 6.7 there is an isomorphism

$$X_1 \simeq X_0 \times_{w,Y_0} Y_1 \times_{Y_0,w} X_0 \simeq X_0 \times_{f,Y_0} Y_1 \times_{Y_0,f} X_0.$$

The composite of this with projection on X_0^2 is $(s,t): X_1 \to X_0^2$, and the composite with

$$\mathrm{pr}_2 : X_0 \times_{f,Y_0} Y_1 \times_{Y_0,f} X_0 \to Y_1$$

is just f_1 by the diagram 1, and so this diagram commutes

$$\begin{tikzcd} X_1 \arrow[dr, "\simeq"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X_0^2 \times_{f^2,Y_0^2} Y_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ & X_0^2 \arrow[r, "f_0^2"'] & Y_0^2 \end{tikzcd}$$

i.e. f is fully faithful.

2CF3. The existence of a 2-commuting square is easy: take the weak pullback (definition 6.3). Since the weak pullback of an E-equivalence is the strict pullback of a trivial E-fibration (using lemma 6.5), we only need to show that the strict pullback of a trivial E-fibration is an E-equivalence. By lemma 3.9, the pullback of a trivial E-fibration is fully faithful. Since the object component of pulled back map is the pullback of the object component, which is in E, the pullback of the trivial E-fibration is again a trivial E-fibration.

2CF4. It is proved in [Pro96] that given a natural transformation

$$\begin{tikzcd} & Y \arrow[dr, "w"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\ & Y \arrow[ur, "w"'] & \end{tikzcd}$$

where w is fully faithful (e.g. w is in W_E), there is a unique $a' \colon f \Rightarrow g$ such that

$$\begin{tikzcd} & Y \arrow[dr, "w"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\ & Y \arrow[ur, "w"'] & \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y \arrow[r, "w"] & Z \end{tikzcd}.$$

This is the first half of 2CF4, where $v = \mathrm{id}_X$. If $v' \colon W \to X \in W_E$ such that there is a transformation

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

satisfying

$$\begin{tikzcd} & X \arrow[dr, "f"] & & \\ W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \arrow[r, "w"] & Z \\ & X \arrow[ur, "g"'] & & \end{tikzcd} = \begin{tikzcd} & & Y \arrow[dr, "w"] & \\ W \arrow[r, "v'"] & X \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow a"] & & Z \\ & & Y \arrow[ur, "w"'] & \end{tikzcd}$$

$$= \begin{tikzcd} W \arrow[r, "v'"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y \arrow[r, "w"] & Z \end{tikzcd},$$

we can choose a J-cover $U \to X_0$, a functor $u' \colon X[U] \to W$ and a natural isomorphism

$$\begin{tikzcd}
& X[U] \arrow[dl, "u'"'] \arrow[dr, "u"] & \\
W \arrow[rr, bend right, "v'"'] \arrow[rr, phantom, "\Leftarrow \epsilon"] & & X
\end{tikzcd}$$

where, since $J \subset E$, $u \in W_E$, and since $v' \circ u' \stackrel{\simeq}{\Rightarrow} u$, $v' \circ u' \in W_E$ by 2CF2 a) above. We can apply the first step again, using uniquess to get

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\
& X \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
W \arrow[r, "v'"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y
\end{tikzcd},$$

We paste this with ϵ,

$$\begin{tikzcd}
X[U] \arrow[rr, bend left, "u"] \arrow[dr, "u'"'] \arrow[rr, phantom, "\epsilon \Downarrow"] & & X \arrow[dr, "f"] & \\
& W \arrow[ur, "v'"] \arrow[dr, "v'"'] \arrow[rr, phantom, "\Downarrow b"] & & Y \\
& & X \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
X[U] \arrow[d] \arrow[dr, bend left] \arrow[dr, phantom, "\epsilon \Downarrow"] & & \\
W \arrow[r, "v'"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a'"] & Y
\end{tikzcd},$$

which is precisely the diagram (7). Hence 2CF4 holds. $\square$

If E is a class of admissible maps for J, E-equivalences are J-equivalences and so $W_E \subset W_J$. This means that the 2-functors α_J, β_J in proposition 4.19 send E-equivalences to equivalences. We use this fact and proposition 5.8 to show the following.

Theorem 6.8. *Let (S, J) be a site with a subcanonical singleton pretopology J and let E be a class of admissible maps for J. Then there are equivalences of bicategories*

$$\boldsymbol{AnaCat}(S, J) \simeq \boldsymbol{Cat}(S)[W_E^{-1}]$$

$$\boldsymbol{Ana}(S, J) \simeq \boldsymbol{Gpd}(S)[W_E^{-1}]$$

Proof. Let us show the conditions in proposition 5.8 hold. We will only supply the details for α_J, the same arguments clearly apply to β_J.

EF1. α_J (and β_J) are the identity on 0-cells, and hence surjective.

EF2. This is equivalent to showing that for any anafunctor $(U, f) \colon X \to\!\!\!\!\!\mapsto Y$ there are functors w, g such that w is in W_E and

$$(U, f) \stackrel{\simeq}{\Rightarrow} \alpha_J(g) \circ \alpha_J(w)^{-1}$$

where $\alpha_J(w)^{-1}$ is some pseudoinverse for $\alpha_J(w)$.

Let w be the functor $X[U] \to X$ – this has object component in $J \subset E$, hence an E-equivalence – and let $g = f\colon X[U] \to Y$. First, note that

$$\begin{tikzcd} & X[U] \arrow[dl] \arrow[dr, "="] & \\ X & & X[U] \end{tikzcd}$$

is a pseudoinverse for

$$\alpha_J(w) \quad = \quad \begin{tikzcd} & X[U][U] \arrow[dl, "\simeq"'] \arrow[dr] & \\ X[U] & & X \end{tikzcd} .$$

Then the composition $\alpha_J(f) \circ \alpha_J(w)^{-1}$ is

$$\begin{tikzcd} & X[U \times_U U \times_U U] \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which is isomorphic to (U, f) by the renaming transformation arising from the isomorphism $U \times_U U \times_U U \simeq U$.

EF3. If $a\colon (X_0, f) \Rightarrow (X_0, g)$ is a transformation of anafunctors for functors $f, g\colon X \to Y$, it is given by a natural transformation with component

$$X_0 \times_{X_0} X_0 \to Y_1.$$

Simply precompose with the isomorphism $X_0 \simeq X_0 \times_{X_0} X_0$ to get a unique natural transformation $a\colon f \Rightarrow g$ such that a is the image of a' under α_J. □

We now finish on a series of results following from this theorem, using basic properties of pretopologies from section 2.

Corollary 6.9. *When J and K are two subcanonical singleton pretopologies on S such that $J_{sat} = K_{sat}$, there is an equivalence of bicategories*

$$\boldsymbol{Ana}(S, J) \simeq \boldsymbol{Ana}(S, K)$$

Using corollary 6.9 we see that using a cofinal pretopology gives an equivalent bicategory of anafunctors.

If E is any class of admissible maps for subcanonical J, the bicategory of fractions inverting W_E is equivalent to that of J-anafunctors. Hence

Corollary 6.10. *Let E be a class of admissible maps for the subcanonical pretopology J. There is an equivalence of bicategories*

$$\boldsymbol{Cat}(S)[W_E^{-1}] \simeq \boldsymbol{Cat}(S)[W_J^{-1}]$$

*where of course $W_J = W_{J_{sat}}$. The same result holds with **Cat** replaced by **Gpd**.*

Finally, if (S, J) is a superextensive site (like **Top** with its usual pretopology of open covers), we have the following result which is useful when J is not a singleton pretopology.

COROLLARY 6.11. *Let (S, J) be a superextensive site where J is a subcanonical pretopology. Then*

$$\boldsymbol{Gpd}(S)[W_{J_{sat}}^{-1}] \simeq \boldsymbol{Ana}(S, \amalg J)$$

Proof. This essentially follows from the corollary to lemma 2.21. $\square$

Obviously this can be combined with previous results, for example if $K \leq \amalg J$, for J a non-singleton pretopology, K-anafunctors localise $\mathbf{Gpd}(S)$ at the class of J-equivalences.