Allegories and bicategories of relations

Definition 1 ([CW87]). A (locally ordered) *cartesian bicategory* is a locally partially ordered 2-category $\mathcal{C}$ satisfying the following:

1. $\mathcal{C}$ is symmetric monoidal: there is a pseudofunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ together with natural isomorphisms α, λ, ρ and σ satisfying the usual coherence conditions;

2. every object of $\mathcal{C}$ is a commutative comonoid, that is, comes equipped with maps
$$\Delta_X\colon X\to X\otimes X \qquad t_X\colon X\to I$$
whose right adjoints we write Δ_X^*, t_X^*, where I is the tensor unit, satisfying the obvious associativity, symmetry and unitality axioms, and this is the only such comonoid structure on X;

3. every morphism $r\colon X \nrightarrow Y$ is a lax comonoid morphism:
$$\Delta_Y\circ r\le (r\otimes r)\circ\Delta_X \qquad t_Y\circ r\le t_X$$

Proposition 2 ([CW87, theorem 1.6]). *A bicategory $\mathcal{C}$ is cartesian if and only if the following hold:*

1. $\mathrm{Map}(\mathcal{B})$ *has finite 2-products (given by $\otimes$ and I).*
2. *The hom-posets of $\mathcal{B}$ have finite products, and 1_I is the terminal object of $\mathcal{B}(I,I)$.*
3. *The tensor product defined as*
$$r\otimes s=(\pi_1^* r\pi_1)\cap(\pi_2^* s\pi_2)$$
where the π_i are the product projections, is functorial.

Definition 3 ([CW87, def. 2.1]). An object X in a cartesian bicategory is called *Frobenius* (Carboni–Walters say *discrete*) if it satisfies
$$\Delta\circ\Delta^*=(\Delta^*\otimes 1)\circ(1\otimes\Delta) \tag{1}$$
A *bicategory of relations* is a cartesian bicategory in which every object is Frobenius.

Proposition 4. *A bicategory of relations $\mathcal{B}$ is compact closed, that is, there is an identity-on-objects involution $(-)^\circ\colon \mathcal{B}^{\mathrm{op}}\to\mathcal{B}$ and a natural isomorphism*
$$\mathcal{B}(X\otimes Y,Z)\cong\mathcal{B}(X,Z\otimes Y)$$

Lemma 5 ([CW87, corollary 2.6]). *In a bicategory of relations*

1. *If f is a map then $f^*=f^\circ$.*
2. *If f and g are maps and $f\le g$ then $f=g$.*

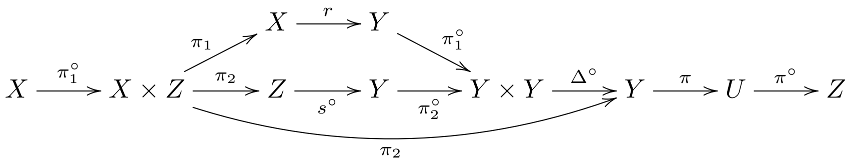
Proposition 6. *A bicategory of relations is the same thing as a unitary tabular allegory.*

Only one part of the proof is non-trivial, but we postpone the whole thing until after the necessary lemmas.

Freyd and Ščedrov give a construction [FŠ90, B.3] of the free allegory on a regular theory, which allows us to interpret any formula of regular logic in a unitary pre-tabular allegory (relative to some given interpretation of the basic sorts, terms and predicates). Suppose we have predicates $R(x, y)$ and $S(y, z)$, interpreted as $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$ respectively. Then their relational composite is given by the formula

$$SR(x, z) = \exists v.z^* R(x, v) \wedge x^* S(v, z)$$

This can be interpreted in two different ways: as the composite $sr\colon X \nrightarrow Z$, or more 'literally' as



where parallel morphisms are combined with $\cap$.

Proposition 7. *In a unitary pre-tabular allegory, the two interpretations above of a relational composite are equal; that is,*

$$sr = (\pi^\circ \pi \Delta^\circ (\pi_1^\circ r \pi_1 \cap \pi_2^\circ s^\circ \pi_2) \cap \pi_2)\pi_1^\circ$$

Proof. First note that on the left-hand side $sr = sr \cap \top = sr \cap \pi_2 \pi_1^\circ$, and that on the right

$$(\pi^\circ \pi \Delta^\circ (\pi_1^\circ r \pi_1 \cap \pi_2^\circ s^\circ \pi_2) \cap \pi_2)\pi_1^\circ = (\top_{YZ}(r\pi_1 \cap s^\circ \pi_2) \cap \pi_2)\pi_1^\circ$$

where $\top_{YZ} = \pi^\circ \pi\colon Y \nrightarrow U \nrightarrow Z$ is the top element. In one direction we have

$$\begin{aligned}
sr \cap \pi_2 \pi_1^\circ &= (sr\pi_1 \cap \pi_2)\pi_1^\circ && \text{modular law} \\
&= (sr\pi_1 \cap \pi_2 \cap \pi_2)\pi_1^\circ \\
&\leq (s(r\pi_1 \cap s^\circ \pi_2) \cap \pi_2)\pi_1^\circ && \text{modular law} \\
&\leq (\top_{YZ}(r\pi_1 \cap s^\circ \pi_2) \cap \pi_2)\pi_1^\circ
\end{aligned}$$

In the other,

$$\begin{aligned}
(\top_{YZ}(r\pi_1 \cap s^\circ \pi_2) \cap \pi_2)\pi_1^\circ &\leq (\top_{YZ} s^\circ (sr\pi_1 \cap \pi_2) \cap \pi_2)\pi_1^\circ && \text{modular law} \\
&\leq (\top_{YY}(sr\pi_1 \cap \pi_2) \cap \pi_2)\pi_1^\circ \\
&= (\pi_2 \pi_1^\circ (sr\pi_1 \cap \pi_2) \cap \pi_2)\pi_1^\circ && \top = \pi_2 \pi_1^\circ \\
&= (\pi_2 (\pi_1^\circ sr\pi_1 \cap \pi_1^\circ \pi_2) \cap \pi_2)\pi_1^\circ && \text{maps distribute} \\
&= \pi_2 (\pi_1^\circ sr\pi_1 \cap \pi_1^\circ \pi_2 \cap \pi_2^\circ \pi_2)\pi_1^\circ && \text{modular law} \\
&= \pi_2 (\pi_1^\circ sr\pi_1 \cap (\pi_1^\circ \cap \pi_2^\circ)\pi_2)\pi_1^\circ && \text{maps distribute} \\
&= \pi_2 (\pi_1^\circ sr\pi_1 \cap \Delta \pi_2)\pi_1^\circ && \text{see below} \\
&= \pi_2 \Delta (\Delta^\circ \pi_1^\circ sr\pi_1 \cap \pi_2)\pi_1^\circ && \text{modular law} \\
&= (sr\pi_1 \cap \pi_2)\pi_1^\circ && \Delta\pi_1 = \Delta\pi_2 = 1 \\
&= sr \cap \pi_2 \pi_1^\circ && \text{modular law}
\end{aligned}$$

In the fourth last line we used the fact that $\pi_1 \cap \pi_2 = \Delta^\circ$, which follows from lemma 8 below and the fact that $\Delta = \langle 1, 1 \rangle$. $\square$

Lemma 8. *Let $\mathcal{A}$ be a unitary pre-tabular allegory. If $A \overset{f}{\dashleftarrow} X \overset{g}{\dashrightarrow} B$ in $\mathrm{Map}(\mathcal{A})$, then $\langle f, g \rangle = \pi_1^\circ f \cap \pi_2^\circ g$ in $\mathcal{A}$.*

Proof. Write $r = \pi_1^\circ f \cap \pi_2^\circ g$. From the modular law and the fact that product cones tabulate top morphisms it follows that $\pi_1 r = f$ and $\pi_2 r = g$. Thus $r = \langle f, g \rangle$ if and only if r is a map.

For the counit inequality,

$$\begin{aligned} rr^\circ &= (\pi_1^\circ f \cap \pi_2^\circ g)(f^\circ \pi_1 \cap g^\circ \pi_2) \\ &\leq \pi_1^\circ f f^\circ \pi_1 \cap \pi_2^\circ g g^\circ \pi_2 && \text{distrib.} \\ &\leq \pi_1^\circ \pi_1 \cap \pi_2^\circ \pi_2 \\ &= 1 && \text{proj'ns tabulate} \end{aligned}$$

For the unit,

$$\begin{aligned} r^\circ r &= (f^\circ \pi_1 \cap g^\circ \pi_2)(\pi_1^\circ f \cap \pi_2^\circ g) \\ &= (f^\circ \pi_1 \cap g^\circ \pi_2)\pi_1^\circ f \cap (f^\circ \pi_1 \cap g^\circ \pi_2)\pi_2^\circ g && \text{distrib.} \\ &= (f^\circ \cap g^\circ \pi_2 \pi_1^\circ) f \cap (f^\circ \pi_1 \pi_2^\circ \cap g^\circ) g && \text{modular law} \\ &= f^\circ f \cap g^\circ g && g^\circ \pi_2 \pi_1^\circ = \top, \text{ etc.} \\ &\geq 1 \cap 1 = 1 \end{aligned}$$

$\square$

Lemma 9. *Let $t, u \colon A \times B \leftrightarrow B$. Then*

$$(t \cap \pi_2)\pi_1^\circ \cap (u \cap \pi_2)\pi_1^\circ = (t \cap u \cap \pi_2)\pi_1^\circ$$

Proof. By the modular law, the left-hand side is

$$((\pi_2 \cap u)\pi_1^\circ \pi_1 \cap t \cap \pi_2)\pi_1^\circ$$

It therefore suffices to show that

$$(\pi_2 \cap u)\pi_1^\circ \pi_1 \cap \pi_2 = u \cap \pi_2$$

By lemma 10 below, we have that $\pi_1^\circ \pi_1 = \pi_{13}\pi_{12}^\circ$, so the left-hand side above is

$$\begin{aligned} (\pi_2 \cap u)\pi_{13}\pi_{12}^\circ \cap \pi_2 &= ((\pi_2 \cap u)\pi_{13} \cap \pi_2 \pi_{12})\pi_{12}^\circ && \text{modular law} \\ &= (\pi_2 \pi_{13} \cap u\pi_{13} \cap \pi_2 \pi_{12})\pi_{12}^\circ && \text{maps distribute} \\ &= (u\pi_{13} \cap \pi_2 (A \times \Delta_B)^\circ)\pi_{12}^\circ && \text{lemma 11} \\ &= (u\pi_{13}(A \times \Delta_B) \cap \pi_2)(A \times \Delta_B)^\circ \pi_{12}^\circ && \text{modular law} \\ &= u \cap \pi_2 \end{aligned}$$

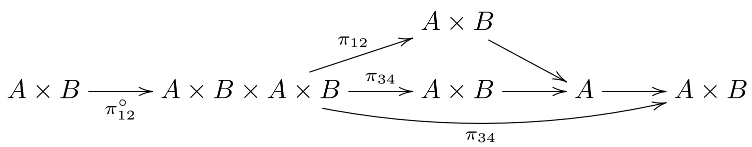
$\square$

Lemma 10. *Let $\pi_1 \colon A \times B \dashrightarrow B$ and take*

$$\pi_{12} = \langle \pi_1, \pi_2 \rangle, \pi_{13} = \langle \pi_1, \pi_3 \rangle \colon A \times B \times B \dashrightarrow A \times B$$

Then $\pi_1^\circ \pi_1 = \pi_{13}\pi_{12}^\circ$.

Proof. By prop. 7, the left-hand side is



Factoring the double (co)projections through the canonical isomorphism $A \times B \times A \times B \cong A \times A \times B \times B$, we get

$$\begin{aligned}
(\pi_1^\circ(\pi_1 \cap \pi_2) \cap \pi_{24})\pi_{13}^\circ &= (\pi_1^\circ\pi_1(\Delta \times B \times B)^\circ \cap \pi_{24})\pi_{13}^\circ && \text{lemma 11} \\
&= (\pi_1^\circ\pi_1 \cap \pi_{13})\pi_{12}^\circ && \text{mod., } \pi\Delta = 1 \\
&= (\pi_1^\circ\pi_1 \cap \pi_2^\circ\pi_3)\pi_{12}^\circ && \text{lemma 8} \\
&= \pi_{13}\pi_{12}^\circ && \text{idem}
\end{aligned}$$

□

Lemma 11. *Let* $\pi_2, \pi_3 \colon A \times B \times B \to B$*, and write* $\pi_{12} = \langle \pi_1, \pi_2 \rangle$*,* $\pi_{13} = \langle \pi_1, \pi_2 \rangle$ *as before. Then*

$$\pi_2 \cap \pi_3 = \pi_2(\pi_{12} \cap \pi_{13}) = \pi_2(1_A \times \Delta_B)^\circ$$

where in the middle and on the right $\pi_2 \colon A \times B \to B$*.*

Proof. For the left-hand equality we have

$$\begin{aligned}
\pi_2(\pi_{12} \cap \pi_{13}) &= \pi_2(\pi_1^\circ\pi_1 \cap \pi_2^\circ\pi_2 \cap \pi_2^\circ\pi_3) && \text{lemma 8} \\
&= \pi_2(\pi_1^\circ\pi_1 \cap \pi_2^\circ\pi_2) \cap \pi_3 && \text{modular law} \\
&= \pi_2\pi_1^\circ\pi_1 \cap \pi_2 \cap \pi_3 && \text{modular law} \\
&= \top \cap \pi_2 \cap \pi_3 \\
&= \pi_2 \cap \pi_3
\end{aligned}$$

For the right, $1_A \times \Delta_B = \langle \pi_1, \langle 1, 1 \rangle \pi_2 \rangle$, which is

$$\pi_1^\circ\pi_1 \cap \pi_2^\circ\pi_2 \cap \pi_3^\circ\pi_2$$

by lemma 8, the opposite of which composed with π_2 is the first line above. □

Now we may proceed with our postponed proof.

Proof of prop. 6. Both allegories and bicategories of relations are locally partially ordered 2-categories equipped with an identity-on-objects involution.

Suppose $\mathcal{B}$ is a bicategory of relations. The Frobenius law implies the modular law [CW87, remark 2.9(ii)]. The tensor unit I, the terminal object of $\mathrm{Map}(\mathcal{B})$, is a unit: there is a unique map $X \to I$ for any X, and 1_I is the top element of $\mathcal{B}(I, I)$ by prop. 2. The product projections tabulate the top elements, so $\mathcal{B}$ is pre-tabular.

Conversely, let $\mathcal{A}$ be a unitary pre-tabular allegory, and refer to prop. 2. $\mathrm{Map}(\mathcal{A})$ has finite products, and local finite products are given by the definition of an allegory and the presence of the unit; the identity on the unit is by definition the top element of the relevant hom set.

The functoriality of the tensor product

$$r \otimes s = \pi_1^* r \pi_1 \cap \pi_2^* s \pi_2$$

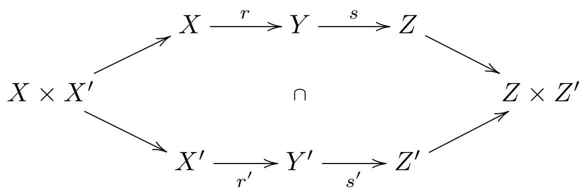
is the only difficult part. Firstly, on identities we have

$$1 \otimes 1 = \pi_1^* \pi_1 \cap \pi_2^* \pi_2 = 1$$

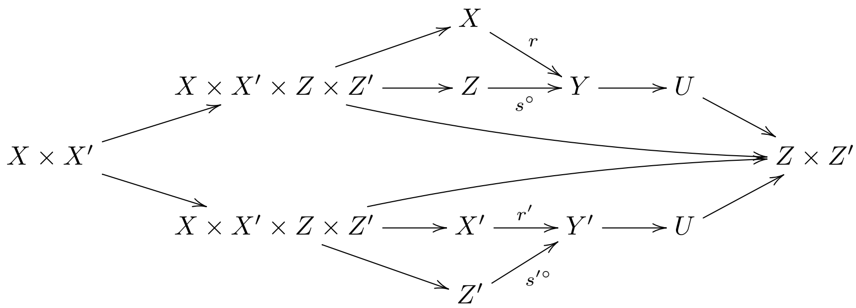
because the projections are a tabulation. Now consider

$$sr \otimes s'r' = \pi_1^* sr\pi_1 \cap \pi_2^* s'r'\pi_2$$

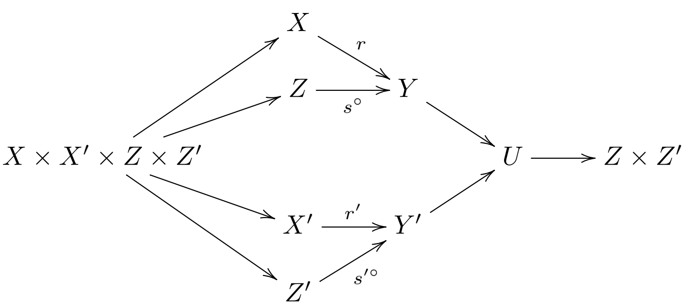
as in



By prop. 7, this is equal to



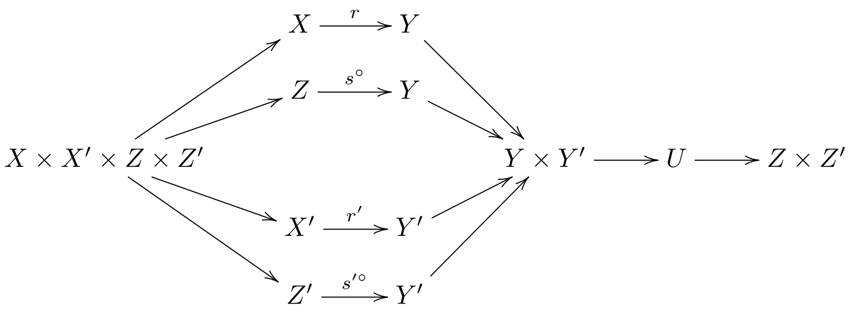
where unlabelled arrows are the obvious projections, and parallel arrows are to be combined with $\cap$ as before. By lemma 9 this is the result of precomposing with the coprojection $X \times X' \looparrowright X \times X' \times Z \times Z'$ the meet of the following morphism with π_2:



We may use the modular law at U, then the equality

$$Y' \dashrightarrow U \looparrowright Y = Y' \looparrowright Y \times Y' \dashrightarrow Y$$

and then the modular law again to turn the above into



But now we may use the symmetry of $\cap$ to swap the morphism containing s° with that containing r': the resulting morphism (after $\cap$ing with π_2 and composing with the coprojection out of $X \times X'$) is exactly the interpretation after prop. 7 of

$$(\pi_1^\circ s \pi_1 \cap \pi_2^\circ s' \pi_1) \circ (\pi_1^\circ r \pi_1 \cap \pi_2^\circ r' \pi_2)$$

Thus $\otimes$ is functorial. $\square$

References

[CW87] Aurelio Carboni and R. F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49:11–32, 1987.

[FŠ90] Peter Freyd and André Ščedrov. *Categories, Allegories.* North-Holland, 1990.