Fock Space: A Model of Linear Exponential Types

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Abstract

It has been observed by several people that, in certain contexts, the free symmetric algebra construction can provide a model of the linear modality **!** . This construction arose independently in quantum physics, where it is considered as a canonical model of quantum field theory. In this context, the construction is known as *(bosonic) Fock space*. Fock space is used to analyze such quantum phenomena as the annihilation and creation of particles. There is a strong intuitive connection to the principle of *renewable resource*, which is the philosophical interpretation of the linear modalities.

In this paper, we examine Fock space in several categories of vector spaces. We first consider vector spaces, where the Fock construction induces a model of the $\otimes, \&, !$ fragment in the category of symmetric algebras. When considering Banach spaces, the Fock construction provides a model of a *weakening cotriple* in the sense of Jacobs. While the models so obtained model a smaller fragment, it is closer in practice to the structures considered by physicists. In this case, Fock space has a natural interpretation as a space of holomorphic functions. This suggests that the "nonlinear" functions we arrive at *via* Fock space are not merely continuous but analytic.

Finally, we also consider fermionic Fock space, which corresponds algebraically to *skew* symmetric algebras. By considering fermionic Fock space on the category of finite-dimensional vector spaces, we obtain a model of full propositional linear logic, although the model is somewhat degenerate in that certain connectives are equated.

1 Introduction

Linear logic was introduced by Girard [G87] as a consequence of his decomposition of the traditional connectives of logic into more primitive connectives. The resulting logic is more resource sensitive; this is achieved by placing strict control over the structural rules of contraction and weakening,

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introducing a new "modal" operator OF COURSE (denoted $!$) to indicate when a formula may be used in a resource-*insensitive* manner—*i.e.* when a resource is renewable. Without the $!$ operator, the essence of linear logic is carried by the multiplicative connectives; at its most basic level, linear logic is a logic of monoidal-closed categories (in much the same way that intuitionistic logic is a logic of cartesian-closed categories). In modelling linear logic, one begins with a monoidal-closed category, and then adds appropriate structure to model linear logic's additional features. To model linear negation, one passes to the $*$-autonomous categories of Barr [B79]. To model the additive connectives, one then adds products and coproducts. Finally, to model the exponentials, and so regain the expressive strength of traditional logic, one adds a triple and cotriple, satisfying properties to be outlined below. This program was first outlined by Seely in [Se89].

Linear logic bears strong resemblance to linear algebra (from which it derives its name), but one significant difference is the difficulty in modelling $!$. The category of vector spaces over an arbitrary field is a symmetric monoidal closed category, indeed in some sense the prototypical monoidal category, and as such provides a model of the intuitionistic variant of multiplicative linear logic. Furthermore, this category has finite products and coproducts with which to model the additive connectives. It thus makes sense to look for models of various fragments of linear logic in categories of vector spaces. However, modelling the exponentials is more problematic. It is the primary purpose of this paper to present methods of modelling exponential types in categories arising from linear algebra. We study models of the exponential connectives in categories of linear spaces which have monoidal (but generally not monoidal-closed) structure. (We shall also include a model in finite-dimensional vector spaces which is closed.)

The construction which will be used to model the exponential formulas, although standard in algebra, arose independently in quantum field theory, and is known as *Fock space*. It was designed as a framework in which to consider many-particle states. The key point of departure for quantum field theory was the realization that elementary particles are created and destoyed in physical processes and that the mathematical formalism of ordinary quantum mechanics needs to be revised to take this into account. The physical intuitions behind the Fock construction will also be familiar to mathematicians in that it corresponds to the free symmetric algebra on a space. As such, it induces a pair of adjoint functors, and hence a cotriple in the algebra category. It is this cotriple which will be used to model $!$. It should be noted that this category of algebras inherits the monoidal structure from the underlying category of spaces, but there is no hope that this category could have a monoidal-closed structure.

While Fock space has an abstract representation in terms of an infinite direct sum, physicists such as Ashtekar, Bargmann, Segal and others, see [AM-A80, Ba61, S62] have analyzed concrete representations of Fock space as certain classes of holomorphic functions on the base space. Thus, these models further the intuition that the exponentials correspond to the analytic properties of the space. In fact, there is a clear sense in which morphisms in the Kleisli category for the cotriple can be viewed as generalized holomorphic functions. Thus, there should be an analogy to coherence spaces where the Kleisli category corresponds to the stable maps.

Fock space also has two additional features which correspond to additional structure, not expressible in the syntax of linear logic. These are the *annihilation* and *creation* operators, which are used to model the annihilation and creation of particles in a field. These may give a tighter control of resources not expressible in the pure linear logic. Thus, these models may be closer to the bounded linear logic of Girard, Scedrov and Scott [GSS91].

A possible application of this work is that the refined connectives of linear logic may lend insight into certain aspects of quantum field theory. For example, there are two distinct methods of combining particle states. One can superimpose two states onto a single particle, or one can have two particles coexisting. The former seems to correspond to additive conjunction and the latter to the multiplicative. This physical imagery is missing in quantum mechanics, which was specially designed to handle a single particle; it only shows up in quantum field theory.

In this paper, we begin by reviewing the categorical structure necessary to model linear logic, and specifically exponential types. We then describe the Fock construction on vector spaces and explain the properties of the resulting model. We next consider Fock space on normed vector spaces. While the model so obtained has weaker properties, this case is closer to that considered by physicists. In fact, in this case the Fock construction gives a model of a weakening cotriple in the sense of Jacobs [J93]. We next describe the interpretation of Fock space as a space of Holomorphic functions. Finally, the physical meaning of the Fock construction is discussed.

We wish to point out that this paper corrects an error in an earlier draft [BPS]. This is discussed in section 6.

2 Linear Logic and Monoidal Categories

We shall begin with a few preliminaries concerning linear logic. We shall not reproduce the formal syntax of linear logic, nor the usual discussion of its intuitive interpretation or utility—for this the reader is referred to the standard references, such as [G87]. We do recall [Se89] that a categorical semantics for linear logic may be based on Barr's notion of $*$-autonomous categories [B79]. If only to establish notation, here is the definition.

Definition 1 *A category $\mathcal{C}$ is $*$-autonomous if it satisfies the following:*

1. *$\mathcal{C}$ is symmetric monoidal closed; that is, $\mathcal{C}$ has a tensor product $A \otimes B$ and an internal hom $A \multimap B$ which is adjoint to the tensor in the second variable*
$$Hom(A \otimes B, C) \cong Hom(B, A \multimap C)$$
2. *$\mathcal{C}$ has a dualizing object $-$; that is, the functor $(\ )^{\perp} \colon \mathcal{C}^{op} \longrightarrow \mathcal{C}$ defined by $A^{\perp} = A \multimap -$ is an involution (*viz.* the canonical morphism $A \longrightarrow ((A \multimap -) \multimap -)$ is an isomorphism).*

In addition various coherence conditions must hold—a good account of these may be found in [M-OM89]. Coherence theorems may be found in [BCST, Bl92]. An equivalent characterization of $*$-autonomous categories is given in [CS91], based on the notion of weakly distributive categories. That characterization is useful in contexts where it is easier to see how to model the tensor $\otimes$, the "par" $⅋$ and linear negation, and the coherence conditions may be expressed in terms of those operations.

The structure of a $*$-autonomous category models the evident eponymous structure of linear logic: the categorical tensor $\otimes$ is the linear multiplicative $\otimes$ and the internal hom $\multimap$ is linear implication. The dualizing object $-$ is the unit for linear "par" $⅋$, or equivalently, is the dual of the unit I for the tensor[1].

There are a number of variants of linear logic whose categorical semantics is based on this. First is full "classical" linear logic, which includes the additive operations. These correspond to requiring that the category $\mathcal{C}$ have products and coproducts. (If $\mathcal{C}$ is $*$-autonomous, one of these will imply the other by de Morgan duality.) There is also Girard's notion of "intuitionistic" linear

[1]In other papers we have used the notation $\top$ for the unit for $\otimes$, and $\oplus$ instead of $⅋$. Here we shall try to avoid controversy by using notation traditional in the context of Banach spaces, and by generally ignoring the "par". So in this paper, $\oplus$ means direct sum, which coincides with Girard's notation. We use $\times$ for cartesian product, corresponding to Girard's &. And we shall use the usual notation for the appropriate spaces when referring to the units.

logic [GL87], which omits linear negation and "par"—this corresponds to merely requiring that $\mathcal{C}$ be *autonomous*, that is to say, symmetric monoidal closed (with or without products and coproducts, depending on whether or not the additives are wanted). There is an intermediate notion, "full intuitionistic linear logic" due to de Paiva [dP89], in which the morphism $A \longrightarrow A^{\perp\perp}$ need not be an isomorphism. And as mentioned above, there is the notion of weakly distributive category [CS91, BCST], where negation and internal hom are not required.

One important class of $*$-autonomous categories are the *compact* categories [KL80] where the tensor is self-dual: $(A \otimes B)^{\perp} \cong A^{\perp} \otimes B^{\perp}$. These categories form the basis for Abramsky's *interaction categories* [Abr].

In this paper we shall model various fragments of linear logic; we shall describe the fragments in terms of the categorical structure present, without explicitly identifying the fragments.

Finally, in order to be able to recapture the full strength of classical (or intuitionistic) logic, one must add the "exponential" $!$ (and its de Morgan dual $?$). (All our structures will model $!$.) We saw in [Se89] that this amounts to the following.

Definition 2 *A monoidal category $\mathcal{C}$ with finite products admits (Girard)* **storage** *if there is a cotriple $! : \mathcal{C} \longrightarrow \mathcal{C}$ (with the usual structure maps $A \xleftarrow{\epsilon_A} \,!A \xrightarrow{\delta_A} \,!!A$), satisfying the following:*

1. *for each object $A \in \mathcal{C}$, $!A$ carries (naturally) the structure of a (cocommutative) $\otimes$-comonoid $\top \xleftarrow{e_A} \,!A \xrightarrow{d_A} \,!A \otimes \,!A$ (and the coalgebra maps are comonoid maps), and*
2. *there are natural comonoidal isomorphisms*
$$\mathrm{I} \xrightarrow{\sim} \,!1 \quad \textit{and} \quad !A \otimes \,!B \xrightarrow{\sim} \,!(A \times B) \ .$$

Some remarks: First, it is not hard to see that the first condition above is redundant, the comonoidal structure on $!A$ being induced by the isomorphisms of the second condition. However, the first condition is really the key point here, as may be seen from several generalizations of this definition, to the intuitionistic case without finite products in [BBPH], and to the weakly distributive case, again without finite products, [BCS93]. The main point here is that without products one replaces the second condition with the requirement that the cotriple $!$ (and the natural transformations ϵ, δ) be comonoidal. And second, one ought not drown in the categorical terminology—terms like "comonoidal" in essence refer to various coherence (or commutativity) conditions which may be looked up when needed. Readers not interested in coherence questions can follow the discussion by just noting the existence of appropriate maps, and believe that all the "right" diagrams will commute. They can regard it as somebody else's business to ensure that this is indeed the case.

(In this vein we ought to cite [Bi94], where the definition above is improved by requiring that the induced adjunction between $\mathcal{C}$ and $\mathcal{C}_!$ is monoidal, in order to guarantee the soundness of term equalities.)

In the mid-1980's, Girard studied coherence spaces as a model of system F, and realized the following fact, which led directly to the creation of linear logic. Of course Girard did not put the matter in these categorical terms at the time, but the essential content remains the same—ordinary implication factors through linear implication *via* the cotriple $!$. (Another way of expressing this is to say that a model of full classical linear logic induces an interpretation of the typed λ-calculus.)

Theorem 1 *If $\mathcal{C}$ is a $*$-autonomous category with finite products admitting Girard storage $!$, then the Kleisli category $\mathcal{C}_!$ is cartesian closed.*

This result is virtually folklore, but a proof may be found in [Se89].

One of the problems with finding models of linear logic comes from the difficulty of finding well-behaved (in the above sense) cotriples on $*$-autonomous categories. For example, one of the main problems with vector spaces as a model of linear logic is the lack of any natural interpretation of $!$. (We shall soon return to this point, and indeed, in a sense this is the main point of this paper.) This question seems closely bound up with questions of completeness. Barr [B91] has shown how in certain cases one can get appropriate cotriples (*via* cofree coalgebras) from a subcategory of the Chu construction [B79]. One case where this route works out fairly naturally is if the $*$-autonomous category is compact. The following is proved in [B91].

Theorem 2 *Given a complete compact closed category, one can construct cofree coalgebras by the formula*

$$!\,A = \top \times A \times (A \otimes_s A) \times (A \otimes_s A \otimes_s A) \times \cdots$$

(where the tensors $\otimes_s$ are the symmetric tensor powers discussed below).

We observe that a compact category which is complete is also cocomplete, by self-duality. This theorem is the basis for Abramsky's modeling of the exponentials in *interaction categories* [Abr].

3 Fock Space

In this section, we describe the basic construction of Fock space; the exposition follows [Ge85] closely. Fock space is one of the crucial constructions of quantum field theory, and is designed to treat quantum systems of many identical, noninteracting particles. One of the crucial notions of quantum field theory is that particles may be created or annihilated, and Fock space will be equipped with canonical operators to model this phenomenon. We should observe that generally physicists consider Fock space on *Hilbert spaces*, but for the purposes of this discussion, vector spaces are sufficient.

The states of a quantum system form a complex vector space. Given two such systems, they may be combined *via* the operator $\otimes$. So if the first system is in state v_1 and the second in state v_2, then the combined system would be in state $v_1 \otimes v_2$. Note that when we say we are combining the systems, we are only viewing them as a single system. We are not allowing interaction. So, if V represents a one-particle system, then $V \otimes V$ represents a two-particle system. To model quantum field theory, one wishes to consider a system of many particles. A natural candidate would be:

$$\mathbf{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \ldots$$

However, this is not quite correct. We wish the particles of the system to be indistinguishable. This leads us to replace the abve tensor with either the symmetrized or antisymmetrized tensor.

3.1 Symmetric and Antisymmetric Tensors

First, we introduce the symmetric tensor product of a vector space with itself.

Definition 3 *Let A be a vector space. The vector space $A \otimes_s A$ is defined to be the following coequalizer:*

$$A \otimes A \underset{\tau}{\overset{id}{\rightrightarrows}} A \otimes A \longrightarrow A \otimes_s A$$

Note that τ is the twist map, $a \otimes b \mapsto b \otimes a$.

This is the general definition of symmetrized tensor. It turns out that in categories of vector spaces, this quotient is canonically isomorphic to the equalizer of these two maps, and that this equalizer is split by the map:

$$a \otimes b \mapsto \frac{1}{2}(a \otimes b + b \otimes a)$$

We will frequently use this representation in the sequel.

The n^{th} symmetric power is defined analogously. The vector space $\bigotimes^n A$ has $n!$ canonical endomorphisms, and the vector space $\bigotimes_s^n A$ is the coequalizer of all of these. Again, it is isomorphic to the equalizer, and there is a splitting, as above. A good way to view the symmetrized tensor is to observe that the symmetric group acts on the space $\bigotimes^n A$, and that the symmetrized tensor is the invariant subspace. As such, an appropriate notation for the symmetrized tensor is:

$$\frac{\bigotimes^n A}{n!}$$

We will also freely use this representation, as well.

The antisymmetric tensor will be defined in a similar fashion. Again, we first define the antisymmetric tensor of a vector space B with itself. It will be denoted $B \otimes_A B$. It is the coequalizer of the following diagram:

$$B \otimes B \overset{id}{\underset{-\tau}{\rightrightarrows}} B \otimes B \longrightarrow B \otimes_A B$$

Here, $-\tau$ is the map $a \otimes b \mapsto -b \otimes a$.

Members of this space can canonically be viewed as elements of the ordinary tensor product, of the form:

$$x = \tfrac{1}{2}(a \otimes b - b \otimes a)$$

The n^{th} antisymmetric power is defined analogously, and is denoted $\bigotimes_A^n$.

3.2 Bosonic and Fermionic Fock Space

Definition 4 *Let B be a complex vector space. The* **symmetric Fock space** *of B is the infinite direct sum of the spaces $\bigotimes_s^n B$, where, when n is zero we use the complex numbers. The* **antisymmetric Fock space** *of B is the infinite direct sum of the spaces $\bigotimes_A^n B$.*

$$\mathcal{F}(B) = \mathbf{C} \oplus B \oplus \cdots \oplus \bigotimes\nolimits_s^n B \oplus \cdots$$

$$\mathcal{F}_A(B) = \mathbf{C} \oplus B \oplus \cdots \oplus \bigotimes\nolimits_A^n B \oplus \cdots$$

The particles of symmetric Fock space are called *bosons*. Examples of such are photons. Particles of antisymmetric Fock space are called *fermions*. Examples of such are electrons and neutrinos.

An interesting property of fermions is revealed in the above construction. Suppose one had a system of two fermions, each in the same state v. This system would be represented in fermionic Fock space by the following expression:

$$(0, 0, \frac{1}{2}(v \otimes v - v \otimes v), 0, 0, \ldots)$$

This expression is clearly 0. This leads to the observation that one may not have two fermions existing in the same state. This is known as the *Pauli exclusion principle*.

Given the nature of infinite direct sums of vector spaces, it is reasonable to think of elements of Fock space as polynomials. Symmetrizing the tensor ensures that the variables commute. In the fermionic case, we get anticommuting variables. When we consider categories of normed vector spaces, this analogy becomes even clearer. Polynomials are replaced by convergent power series. We will show that the bosonic Fock space of a Banach space has a canonical representation as a space of holomorphic functions.

3.3 Annihilation and Creation of Particles

For ease of exposition, we consider the unsymmetrized Fock space:

$$\mathcal{U} = \mathbf{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \ldots$$

The operators we discuss are easily extended to the bosonic and fermionic cases. Given an arbitrary nonzero $v \in V$, we define a map

$$C_v : \mathcal{U} \to \mathcal{U}$$

$$C_v((v_0, v_1, v_2, \ldots)) = (0, v_0 v, \sqrt{2} v \otimes v_1, \sqrt{3} v \otimes v_2, \ldots)$$

In the above expression, $v_n \in V^{\otimes^n}$. This operator is thought of as "creating a particle in state v".

Similarly one may annihilate particles. Choose an element ϕ of the dual space V^*, and define:

$$A_\phi((v_0, v_1, v_2 \otimes v_3, v_4 \otimes v_5 \otimes v_6, \ldots)) = (\phi(v_1), \sqrt{2}\phi(v_2)v_3, \sqrt{3}\phi(v_4)v_5 \otimes v_6, \ldots)$$

This operation takes an n particle state to an $n-1$ particle state and so on. The square roots in the above two expressions are "normalization" factors, and are added to make the desired equations hold. In this expression, each v_i is an element of V. The equations expressing the interaction of the annihilation and creation operators are to be found in [Ge85]. A more complete discussion of the physical meaning of Fock space is contained in the penultimate section.

4 Fock Space as a Model of Storage

Now we check that the Fock space actually satisfies all the properties that need to be satisfied by an exponential type, *i.e.* satisfies the properties of [Se89], discussed in Section 2. This consists of two parts, verifying that Fock spaces form a cotriple on the category of symmetric algebras and verifying the so-called exponential law, *viz.* $!(A \times B) \cong \,!A \otimes \,!B$. We check the former by displaying a suitable adjunction in the next subsection. Note that in the category of vector spaces, we have $\times = \oplus$.

Proposition 3 *Let A and B be vector spaces.*

$$\mathcal{F}(A \times B) \cong \mathcal{F}(A) \otimes \mathcal{F}(B)$$

Proof –

For the purposes of this proof only, we will denote the n-th symmetric tensor by $\mathcal{S}^n$. Let V and W be vector spaces. We construct a morphism

$$\mathcal{S}^n(V)\otimes\mathcal{S}^m(W)\to\mathcal{S}^{n+m}(V\oplus W)$$

$$(v_1\otimes_s v_2\ldots)\otimes(w_1\otimes_s w_2\ldots)\mapsto(v_1\otimes_s v_2\ldots)\otimes_s(w_1\otimes_s w_2\ldots)$$

On the righthand side of the above expression, we are viewing each v_i and w_i as an element of $V\oplus W$.

This lifts to an isomorphism:

$$\mathcal{S}^n(V\oplus W)\cong\bigoplus_{a=0}^{n}\mathcal{S}^a(V)\otimes\mathcal{S}^{n\perp a}(W)$$

The inverse map is defined as follows. We now denote vectors in V or W, when considered in $V\oplus W$, by $(v_i,0)$ and $(0,w_i)$ respectively. (Remember that elements of this form generate $V\oplus W$.)

$$\begin{aligned}&(v_1,0)\otimes_s(v_2,0)\otimes_s\ldots\otimes_s(v_i,0)\otimes_s(0,w_{i+1})\otimes_s\ldots\otimes_s(0,w_n)\\ \mapsto\quad&(v_1\otimes_s v_2\otimes_s\ldots\otimes_s v_i)\otimes(w_{i+1}\otimes_s\ldots\otimes_s w_n)\end{aligned}$$

The naturality of these maps, and the fact that they are inverse are left to the reader. We note at this point that the symmetrization of the tensors is crucial for establishing that this is an isomorphism, as it was necessary to rearrange terms. Finally, it is straightforward to verify that this extends to an isomorphism of the desired form. Note that $\mathcal{F}(A\oplus B)$ is generated by pure tensors, *i.e.* expressions which are nonzero only on one term of the direct sum. We also note that the expression $\bigoplus_{a=0}^{n}\mathcal{S}^a(V)\otimes\mathcal{S}^{n\perp a}(W)$ corresponds to the finite rank part of $\mathcal{F}(A)\otimes\mathcal{F}(B)$. By the definition of the countable direct sum of vector spaces, all elements of $\mathcal{F}(A)\otimes\mathcal{F}(B)$ are contained in a finite rank piece. ∎

The above proof follows [FH], Appendix B, closely. In fact, this argument can be carried out at a categorical level, as is clear from the previously mentioned theorem of Barr [B91]. The above is also proved in [BSZ92] for Hilbert spaces.

In the next section, we will see that Fock space corresponds to the free symmetric algebra. It is also straightforward to verify that the isomorphism constructed in the above proof is in fact an algebra homomorphism.

Now we consider the antisymmetrized Fock space. We will show that one gets a model of the exponential types in the category of finite-dimensional vector spaces using the antisymmetrized Fock space.

Proposition 4 *If V is a finite-dimensional vector space of dimension n, then $\mathcal{F}_A(V)$ is also a finite-dimensional vector space with dimension 2^n.*

Proof – Consider the vector space $\bigotimes_A^p V$ with $p>n$. We claim that this space is the zero vector space. Since $\otimes$ is adjoint to internal hom in $\mathcal{VEC}_{fd}$, the space $\bigotimes_A^p V$ is isomorphic to the space of completely antisymmetric p-linear maps from V to the scalars. Let f denote such a map. Since V is only n-dimensional one cannot have p linearly independent arguments to such maps. Thus one of the arguments must be a linear combination of the others. Thus on any arguments f becomes a combination of terms of the form $f(\ldots, u, \ldots, u, \ldots)$ where two arguments must be equal. But antisymmetry makes such a term zero. Thus f is the zero vector and the vector space $\bigotimes_A^p V$ is the one-point space. Thus the infinite direct sum becomes a finite direct sum. Now consider $p \leq n$. It is clear that one can only choose C_p^n sets of p linearly independent vectors given a basis. Thus the dimensionality of the space $\bigotimes_A^p V$ is C_p^n and hence, adding the dimensions to get the dimension of the direct sum, we conclude that the dimension of $\mathcal{F}_A(V)$ is 2^n. ∎

The exponential law for the antisymmetric case can be argued similarly to the symmetric case. The detailed verification can be found in [BSZ92] in Section 3.2 on exponential laws, or in [FH] in appendix B.

4.1 Categories of algebras

In this section we shall review some basic facts about categories of algebras, and see in particular how these fit into the current context. (See [M71] for a review of the basic categorical facts, and [L65] for the basic algebra, for instance.) For reference, we do give the following definition here.

Definition 5 *A triple consists of a functor $F\colon \mathcal{B} \longrightarrow \mathcal{B}$, together with natural transformations $\eta\colon id \longrightarrow F$ and $\mu\colon FF \longrightarrow F$, such that $\mu \circ \eta F = \mu \circ F\eta = id$ and $\mu \circ \mu F = \mu \circ F\mu$.*

One simple point to recall is that categories of algebras and of coalgebras are closely connected to the existence of triples and cotriples. Given a triple $F\colon \mathcal{B} \longrightarrow \mathcal{B}$, (with structure morphisms η, μ), an F-algebra is an object B and a morphism $h\colon F(B) \longrightarrow B$ (subject to two commutativity conditions, corresponding to the associative and unit laws). (This notion can be generalized to arbitrary functors.) There is a canonical category of such algebras, the Eilenberg-Moore category $\mathcal{C}^F$, and an adjunction $\mathcal{C} \rightleftarrows \mathcal{C}^F$. Any adjunction canonically induces a triple, and this one canonically induces the original triple. The category of free F-algebras is the Kleisli category $\mathcal{C}_F$ of the triple; again, there is a canonical adjunction $\mathcal{C} \rightleftarrows \mathcal{C}_F$ which induces the original triple. Of course this dualizes for cotriples, with the corresponding notion of coalgebras. (We shall avoid the unpleasant use of terms like "coEilenberg-Moore" and "coKleisli".)

Usually mathematicians have been more interested in the Eilenberg-Moore category of a triple (or cotriple) than in the Kleisli category; although there has been some interest in Kleisli categories recently (for instance in the context of linear logic, as mentioned earlier in this paper), we shall follow this tradition and shall work in Eilenberg-Moore categories. Indeed, it is there that we shall find some of our models. One reason for this is quite practical: it is often simpler to recognize the category of algebras and so derive the triple (similarly, once one has a candidate for a triple, it is often simpler to construct the category of algebras and verify the adjunction than to directly show the original functor is a triple). But there is another reason: we want to show that the Fock space functor is a cotriple (so as to model **!**), but on the categories of spaces we consider, this is not the case—rather it is a triple. By passing to the algebras, we can fix this, because of the following fact:

Fact Given an adjunction $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv U$, the composite UF is a triple on $\mathcal{C}$, and so (dually) the composite FU is a cotriple on $\mathcal{D}$ [BW].

So we obtain our model of **!** on the category of algebras.

4.1.1 Algebras for the symmetric (bosonic) Fock space construction

We begin with a more traditional notion of algebra; the connection between these comes *via* the triple induced by the adjunction given by the free algebra construction, as outlined above. In other words, the category of (traditional) algebras is equivalent to the category of UF algebras.

Definition 6 *An algebra* A *is a space* A *equipped with morphisms*

$$m\colon A \otimes A \longrightarrow A \text{ and } i\colon \mathsf{C} \longrightarrow A$$

satisfying

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{C} \otimes A \arrow[r, "i \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathsf{C} \arrow[l, "id \otimes i"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

Here we are supposing the base field to be C; otherwise replace C with the base field k. If in addition the following diagram commutes, then the algebra A is said to be *symmetric* or commutative. (τ is the canonical "twist" morphism.)

$$\begin{tikzcd}
A \otimes A \arrow[r, "\tau"] \arrow[dr, "m"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

An example of such an algebra comes from the Fock space, the multiplication m is defined by "multiplication of polynomials" in an evident manner. The use of the symmetrized tensor in the definition of Fock space guarantees that this will indeed be a symmetric algebra, and it is standard that this description gives the free such algebra. In other words, we have the following proposition.

Proposition 5 *Given a vector space* B*, the Fock space* $\mathcal{F}(B)$ *canonically carries an algebra structure, and indeed is the free symmetric algebra generated by* B*.*

It follows from this that we have a cotriple on the category $\mathcal{SALG}$ of symmetric algebras, given by taking the Fock algebra on the underlying space of an algebra. As the details of this are both standard and similar to the case of the antisymmetric Fock space construction, which we shall discuss in more detail next, we shall leave the details here to the reader.

4.2 Algebras for the antisymmetric (fermionic) Fock space construction

Recall that we work in the context of finite dimensional vector spaces $\mathcal{VEC}_{fd}$ when considering the antisymmetric Fock construction. This category is self-dual, and is compact with biproducts: the product and coproduct coincide. This duality also implies that a triple is also a cotriple, so we can model **!** in the category of spaces. However, to show that the Fock space construction defines a triple (or cotriple), it is again simpler to consider the category of algebras. Although we are not familiar with any previous consideration of this category of algebras as such, the context is familiar: the antisymmetric Fock space construction is usually called (when thought of as an algebra) the Grassman algebra, or the "alternating" or "exterior" algebra; the multiplication defined on it is called the "wedge product" (a term derived from the usual notation for this product).

Definition 7 *An alternating algebra* A *is a graded algebra* A *(with unit) whose multiplication map satisfies the property that, if x, y are of degree m, n respectively, then $xy = (-1)^{nm}yx$ (which by the grading must be of degree $n + m$).*

Note that the unit must be of degree 0. Morphisms of alternating algebras are just homomorphisms as algebras.

Proposition 6 *There is a canonical alternating algebra structure on $\mathcal{F}_A(V)$, for any finite dimensional vector space V. The antisymmetric Fock construction is left adjoint to the forgetful functor $U\colon \mathcal{VEC}_{fd} \underset{U}{\overset{\mathcal{F}_A}{\rightleftarrows}} \mathcal{AALG}$, where $\mathcal{AALG}$ is the category of alternating algebras. As a consequence, $\mathcal{F}_A$ defines a triple (and so cotriple) on $\mathcal{VEC}_{fd}$.*

Proof – (Sketch) The multiplication on $\mathcal{F}_A(V)$ is the standard "wedge" product [L65], which to elements $x_1 \otimes_A \ldots \otimes_A x_n, y_1 \otimes_A \ldots \otimes_A y_m$ gives the product $x_1 \otimes_A \ldots \otimes_A x_n \otimes_A y_1 \otimes_A \ldots \otimes_A y_m$. Here $x \otimes_A y$ means the equivalence class of $x \otimes y$ in $A \otimes_A A$. (Essentially this is the same "multiplication of power series" we had in the symmetric case, with the alternating product used in place of the usual tensor.) For a vector space V, define $\eta\colon V \longrightarrow U\mathcal{F}_A(V)$ as the canonical injection. Given an alternating algebra A, define $\epsilon\colon \mathcal{F}_A(UA) \longrightarrow A$ by "adding the terms of the series": $\langle x_0, x_1, x_2^1 \otimes_A x_2^2, \ldots\rangle \mapsto i(x_0) + x_1 + m(x_2^1, x_2^2) + \cdots$, where i, m are the algebra maps.

To verify that we have an adjunction we must show the following commute:

$$\begin{tikzcd}
\mathcal{F}_A(V) \arrow[r, "\mathcal{F}_A(\eta)"] \arrow[dr, "1"'] & \mathcal{F}_A(U\mathcal{F}_A(V)) \arrow[d, "\epsilon\mathcal{F}_A"] \\
& \mathcal{F}_A(V)
\end{tikzcd}$$

$$\begin{tikzcd}
UA \arrow[r, "\eta U"] \arrow[dr, "1"'] & U\mathcal{F}_A(UA) \arrow[d, "U\epsilon"] \\
& UA
\end{tikzcd}$$

The second diagram is obvious; to verify the first, notice that $\mathcal{F}_A(\eta(x))$ maps

$$\begin{array}{rcl}\langle x_0, x_1, x_2^1 \otimes_A x_2^2, \ldots\rangle & \mapsto & \langle x_0, \\ & & \langle 0, x_1, 0, \ldots\rangle, \\ & & \langle 0, 0, x_2^1, 0, \ldots\rangle \otimes_A \langle 0, 0, x_2^2, 0, \ldots\rangle, \\ & & \vdots \\ & & \rangle\end{array}$$

and it is clear that "adding up this series" just returns the original term. ∎

It now follows that we can model **!** in $\mathcal{VEC}_{fd}$ with $\mathcal{F}_A$, *via* the formula $!V = (\mathcal{F}_A(V^\perp))^\perp$. In summary, we have the following theorem:

Theorem 7 *In the category of symmetric algebras, we have a model of the fragment* $!, \otimes, \&$. *In the category* $\mathcal{VEC}_{fd}$*, we have a model of full propositional classical linear logic.*

We mention that while $\mathcal{VEC}_{fd}$ models the full propositional calculus, it is somewhat degenerate being a compact category. We now consider Fock space on normed vector spaces. While we lose some of the expressive power in such models, (in particular, we are no longer able to model contraction) Fock space is particularly interesting in such categories. We will see that it has a canonical representation as a space of holomorphic functions.

5 Normed Vector Spaces

Vector spaces are, in some sense, intrinsically finitary structures. Every vector is a finite sum of multiples of basis vectors, and one is only allowed to take finite sums of arbitrary vectors. It seems reasonable that to model **!** and **?**, one should be able to take infinite sums of vectors, thereby capturing the idea of infinitely renewable resource. However, to do this, one needs a notion of convergence. And to define convergence, one needs a notion of norm. Once a space is normed, then it is possible to define limits and Cauchy sequences, and so on. Normed vector spaces, which are the principal objects of study in functional analysis, should be considered as the meeting ground of concepts from linear algebra and analysis. They are also an ideal place to model linear logic.

We will now briefly review the basic concepts of the subject. In this paper, we will focus on *Banach spaces*. Much of the discussion easily lifts to Hilbert spaces. We will introduce Banach spaces in detail, and refer the reader to [KR83] for a discussion of Hilbert spaces.

Henceforth all vector spaces are assumed to be over the complex numbers and are allowed to be infinite-dimensional. We will use Greek letters for complex numbers and lower-case Latin letters from the end of the alphabet for vectors.

Definition 8 *A* **norm** *on a vector space V is a function, usually written $\|\ \|$, from V to $\mathbf{R}$, the real numbers, which satisfies*

1. $\| v \| \geq 0$ *for all* $v \in V$,
2. $\| v \| = 0$ *if and only if* $v = 0$,
3. $\| \alpha v \| = | \alpha | \| v \|$,
4. $\| v + w \| \leq \| v \| + \| w \|$.

For finite dimensional vector spaces the norm usually used is the familiar Euclidean norm. As soon as one has a norm one obtains a metric by the equation $d(u,v) = \| u - v \|$. It turns out that the spaces that are complete with respect to this metric play a central role in functional analysis.

Definition 9 *A* **Banach space** *is a complete, normed vector space.*

Example 1 Consider the space of sequences of complex numbers. We write a for such a sequence, $a = \{a_n\}_{n=1}^{\infty}$ and we write $\| a \|_{\infty}$ for the supremum of the $| a_i |$.

$$l_{\infty} = \{a : \| a \|_{\infty} < \infty\}$$

This is a Banach space with $\| a \|_{\infty}$ as the norm.

Another norm is obtained on sequences as follows. Define:

$$\| a \|_1 = \Sigma_{n=1}^{\infty} | a_i |$$

Then let:

$$l_1 = \{a : \| a \|_1 < \infty\}$$

More generally, if $p \geq 0$, we may define:

$$l_p = \{a : \| a \|_p \equiv (\Sigma_{n=1}^{\infty} | a |^p)^{1/p} < \infty\}$$

All of these will be examples of Banach spaces. Furthermore, these can be defined not only for sequences of complex numbers, but for sequences obtained from any Banach space.

The following theorem shows one common way in which Banach spaces arise. First we need a definition.

Definition 10 *Suppose that B_1, B_2 are Banach spaces and that T is a linear map from B_1 to B_2. We say that T is* **bounded** *if $sup_{x \neq 0} \frac{\| Tx \|}{\| x \|}$ exists. We define the* **norm of** *T, written $\| T \|$, to be this number.*

If T is indeed bounded, then a standard argument [KR83], establishes

Lemma 8 $sup_{\|x\|=1} \| Tx \| = \| T \|$.

Thus one can use vectors of unit norm to calculate the norm of a linear function rather than having to look for the sup over all nonzero vectors. Linear maps from a Banach space to itself are traditionally called *operators*, and the norm of such maps is called the *operator norm.*

Since a Banach space is also a metric space under the induced metric described above, one wishes to characterize which linear maps are also continuous. In this regard, we have the following result.

Lemma 9 *A linear map from $f : A \to B$ is continuous if and only if it is bounded.*

The following theorem shows that the category of Banach spaces and bounded linear maps is enriched over itself.

Theorem 10 *If A is a normed vector space and B is a Banach space then the space of bounded linear maps with the norm above is a Banach space.*

We will denote this space $A \multimap B$.

There are several possible categories of interest with Banach spaces as the objects. The most obvious one is the category with bounded linear maps as the morphisms. However, it turns out that the category with *contractive maps*[2] is of greater interest and has nicer categorical properties.

Definition 11 *A* **contractive map***, T, from A to B is a bounded linear map satisfying the condition, $\| Tx \| \leq \| x \|$. Equivalently, the contractive maps are those of norm less than or equal to 1.*

We will write $\mathcal{BANCON}$ for the category of Banach spaces and contractive maps.

5.1 Monoidal Structure of $\mathcal{BANCON}$

We first point out that $\mathcal{BANCON}$ has a canonical symmetric monoidal closed structure. We begin by constructing a tensor product. Let A and B be objects in $\mathcal{BANCON}$: form the tensor of A and B, $A \otimes_{\mathbf{C}} B$, as complex vector spaces. We first define a partial norm for elements of the form $a \otimes b$ by the equation:

$$\| a \otimes b \| = \| a \| \| b \|$$

We would like to extend this partial norm to a norm on all of $A \otimes_{\mathbf{C}} B$. Such a norm is called a *cross norm.* It turns out that there are many such cross norms, a number of which were discovered by Grothendieck. The one we will use in this paper is called the *projective cross norm.* It is in some sense the least such. A detailed discussion of these issues is contained in [T79]. The projective cross norm is defined for an arbitrary element, x, of $A \otimes_{\mathbf{C}} B$ by the following formula:

$$\| x \| = inf\{\| a \| \| b \| \; such \; that \; x = \Sigma a \otimes b\}$$

One can verify that this is in fact a cross norm on $A \otimes_{\mathbf{C}} B$. Now, the resulting normed space will not be complete in general, so one obtains a Banach space by completing it. This will act as the tensor product in the category $\mathcal{BANCON}$. It will be denoted simply by $A \otimes B$. Furthermore, we have the following adjunction.

Lemma 11 *The functor $B \otimes (\ )$ is left adjoint to $B \multimap (\ )$.*

Corollary 12 *$\mathcal{BANCON}$ is a symmetric monoidal closed category.*

As such, $\mathcal{BANCON}$ is a model of (at least) the multiplicative fragment of intuitionistic linear logic.

5.2 Completeness Properties of $\mathcal{BANCON}$

We begin by constructing coproducts.

Definition 12 *Let A and B be Banach spaces. The direct sum, $A \oplus B$, is the Cartesian product equipped with the norm $\| a \oplus b \| = \| a \| + \| b \|$.*

Then we have the distributivity property of $\otimes$ over $\oplus$.

[2]Strictly speaking, they should be called "non-expansive" maps.

Proposition 13 *$A \otimes (B \oplus B') \cong (A \otimes B) \oplus (A \otimes B')$.*

We now discuss finite products.

Definition 13 *The product of two Banach spaces, $A \times B$, has as its underlying space $A \oplus B$, but now with norm given by:*

$$\| a \oplus b \| = max\{\| a \|, \| b \|\}$$

As a category of vector spaces, $\mathcal{BANCON}$ is fairly unique in this respect. While most such categories model the additive fragment of linear logic, they invariably equate the two connectives, since finite products and coproducts coincide. In other words, $\mathcal{BANCON}$ does not share the familiar property of being an additive category.

We now present countably infinite products and coproducts.

Definition 14 *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of Banach spaces. Define $\Pi(A_i)$ to be those sequences which converge in the l_∞ norm,* i.e. *bounded sequences equipped with the obvious norm.*

Define $\Sigma(A_i)$ to be all sequences which converge in the l_1 norm.

This gives countable products and coproducts in $\mathcal{BANCON}$. Similar constructions can be applied for uncountable products and coproducts.

Equalizers in $\mathcal{BANCON}$ correspond to equalizers in the underlying category of vector spaces. The fact that bounded maps are continuous implies that the subspace will be complete. Coequalizers are obtained as a quotient, with the induced norm being the infimum of the norms of the elements of the equivalence class. See [C90] for a discussion of quotients of Banach spaces.

Theorem 14 *$\mathcal{BANCON}$ is complete and cocomplete.*

6 $\mathcal{BANCON}$ as a Model of Weakening

In the next section, we will show that the Fock space of a Banach space has a canonical interpretation as a space of holomorphic functions. In this section, we explore the nature of $\mathcal{BANCON}$ as a model of linear logic. We obtain a somewhat weaker model for the following reason. In this category, as we have previously observed, products and coproducts do not coincide. Thus, the isomorphism:

$$!A \otimes\ !B \cong\ !(A \oplus B)$$

is not useful for modelling the additive fragment. We obtain instead a *weakening cotriple* in the sense of Jacobs [J93].

Jacobs denotes a weakening cotriple by $\underset{w}{!}$. Such a cotriple satisfies all of the axioms of modelling $!$, except that the coalgebras will not have the comonoid structure necessary to model contraction.

Thus, we have syntax of the following form:

$$\frac{\Gamma, \vdash A}{\Gamma, \underset{w}{!}B \vdash A}\ We \qquad\qquad \frac{\Gamma, B \vdash A}{\Gamma, \underset{w}{!}B \vdash A}\ Der \qquad\qquad \frac{\underset{w}{!}\Gamma \vdash A}{\underset{w}{!}\Gamma \vdash \underset{w}{!}A}\ Sto$$

One models these proof rules as in linear logic, following [Se89].

We point out that the map:

$$\mathbf{!}\,A \otimes \mathbf{!}\,B \longrightarrow \mathbf{!}\,(A \otimes B)$$

necessary to model storage is indeed a contraction. This map is obtained as

$$\mathbf{!}\,A \otimes \mathbf{!}\,B \cong \mathbf{!}\,(A \oplus B) \to \mathbf{!!}\,(A \oplus B) \cong \mathbf{!}\,(\,\mathbf{!}\,A \otimes \mathbf{!}\,B) \to \mathbf{!}\,(A \otimes B)$$

It is shown in an appendix that this map is contractive.

Theorem 15 *In the category $\mathcal{BANCON}$, we obtain a model of the fragment $\underset{\mathrm{w}}{\mathbf{!}}, \otimes, \&$.*

Remark 16 *We wish to point out an error in an earlier draft of this paper [BPS]. In that paper, it is stated that the Fock construction is functorial on the larger category of Banach spaces and bounded linear maps. In fact, when one applies the Fock construction to a map of norm greater than* 1*, one might obtain a divergent expression. Thus, we are forced to work in the smaller category of contractions.*

7 The Holomorphic-Function Representation of Fock Space

It has been observed by a number of people that the free symmetric algebra provides a model of the exponential type. But the observation that this construction corresponds to bosonic Fock space allows us to relate results in quantum physics to the model theory of linear logic. In this section, we present one such relationship. The symmetrized Fock space on a Banach space B, turns out to be a space of holomorphic functions (analytic functions) on B, properly defined. This hints at possible deeper connections between analyticity and computability which need to be explored.

The ideas here stem from early work by Bargmann [Ba61] on Hilbert spaces of analytic functions in quantum mechanics. This was extended by Segal [S62, BSZ92] to quantum field theory and Segal's extension was used by Ashtekar and Magnon [AM-A80] to develop quantum field theory in curved spacetimes. (A brief summary of the ideas is contained in an appendix to [P80] and in [P79].) The latter work involved making sense of the familiar Cauchy-Riemann conditions on infinite-dimensional spaces.

We quickly recapitulate the basic notion of analytic function in terms of one complex variable before presenting the infinite-dimensional case. A very good elementary reference is *Complex Analysis* by Ahlfors [Ah66]. Given the complex plane, $\mathbf{C}$, one can define functions from $\mathbf{C}$ to $\mathbf{C}$. Let z be a complex variable; we can think of it as $x + iy$ and thus one can think of functions from $\mathbf{C}$ to $\mathbf{C}$ as functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. An *analytic* or *holomorphic* function is one that is everywhere differentiable. In the notion of differentiation, the limit being computed, *viz.*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

allows h to be an arbitrary complex number and hence this limit is required to exist no matter in what direction h approaches 0. This much more stringent requirement makes complex differentiability much stronger than the usual notion of differentiability. If a complex function is differentiable at a point it can be represented by a convergent power series in a suitable open region about the point. If one uses the fact that h can approach zero along either axis one can derive the *Cauchy-Riemann* equations for a complex valued function $f = u(x,y) + iv(x,y)$ of the complex variable $z = x + iy$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

What is remarkable about complex functions is that this definition of analyticity yields the result that a complex-analytic function can be expressed by a convergent power-series in a region of the complex plane. This is remarkable because only one derivative is involved in the Cauchy-Riemann equations whereas the statement that a power-series representation exists is stronger, for real-valued functions, even than requiring infinite differentiability. In real analysis one has examples of functions that are infinitely differentiable at a point, but do not have a power series representation in any neighbourhood of that point. A function may have a power series representation that is valid everywhere, a so-called *entire* holomorphic function; the complex exponential function is an example.

There is a formal perspective, due to Wierstrass, that is rather more illuminating. Think of a complex variable $z = x + iy$ and its conjugate $\overline{z} = x - iy$ as being, formally, independent variables. A function could depend on z and on its complex conjugate, $\overline{z}$, for example, the function that maps each z to $z\overline{z} + iz\overline{z}$. An *analytic* or *holomorphic* function is one which has no dependance on $\overline{z}$. This is expressed formally by $df/d\overline{z} = 0$. When expressed in terms of the real and imaginary parts of f and z, this equation becomes the familiar Cauchy-Riemann equations. Thus this reinforces the view that a holomorphic function is properly thought of as a single complex-valued function of a single variable rather than as two real-valued functions of two real variables.

The theory of functions of finitely many complex variables is a nontrivial extension of the theory of functions of a single complex variable. Entirely new phenomena occur, which have no analogues in the theory of a single complex variable. An excellent recent text is the three volume treatise by Gunning [Gu90]. For our purposes we need only the barest beginnings of the theory. Given $\mathbf{C}^n$, we can have functions from $\mathbf{C}^n$ to $\mathbf{C}$. One can introduce complex coordinates on $\mathbf{C}^n$, $z_1, \ldots, z_n$. One can define a holomorphic function here as one having a convergent power-series expansion in $z_1, \ldots, z_n$. The key lemma that allows one to mimic some of the results of the one-dimensional case is Osgood's lemma[3].

Lemma 17 *If a complex-valued function is continuous in an open subset D of $\mathbf{C}^n$ and is holomorphic in each variable seperately, then it is holomorphic in D.*

From this one can conclude that a holomorphic function in n variables satisfies the Cauchy-Riemann equations $\frac{\partial f}{\partial \overline{z_i}} = 0$. One is free to take either one of (a) satisfying Cauchy-Riemann equations or (b) having convergent power-series representations as the definition of holomorphicity.

Now we describe how to define holomorphic functions on infinite-dimensional, complex, Banach spaces. The basic intuition may be summarized thus. One starts with subspaces of finite *codimension*. Thus the quotient spaces are isomorphic to some $\mathbf{C}^n$. One can define what is meant by a holomorphic function on these quotient spaces as in the preceding paragraph. By composing a holomorphic function with the canonical surjection from the original Banach space to the quotient space we get a function on the original Banach space. These functions can all be taken to be holomorphic.

[3]There is a considerably harder theorem, called Hartog's theorem, which drops the requirement of continuity.

$$\begin{tikzcd} & B \arrow[d] \arrow[dr, dotted] & \\ B/\sim \; = & \mathbf{C}^n \arrow[r] & \mathbf{C} \end{tikzcd}$$

Intuitively these are the functions that are constant along all but finitely many directions, and holomorphic in the directions along which they do vary. These functions are called *cylindric holomorphic functions.* Because the sequence of coefficients of a power-series is absolutely convergent, we can define an l_1 norm on these functions in terms of the power-series. Finally the collection of all holomorphic funcitons is defined by taking the l_1-norm completion of the cylindric holomorphic functions.

Given a Banach space B, let U be a subspace with finite codimension n, *i.e.* the quotient space B/U is an n complex-dimensional vector space. The space B/U is isomorphic to $\mathbf{C}^n$. Let $\phi : B/U \to \mathbf{C}^n$ be an isomorphism; such a map defines a choice of complex coordinates on B/U. Let π_U be the canonical surjection from B to B/U.

Definition 1 *A* **cylindric holomorphic** *function on B is a function of the form $f \circ \phi \circ \pi_U$, where U, π_U and ϕ are as above and f is a holomorphic function from $\mathbf{C}^n$ to $\mathbf{C}$.*

We need to argue that the choice of coordinates does not make a real difference. Of course which functions get called holomorphic does depend on the choice of coordinates, but the space of holomorphic functions has the same structure[4]. Suppose that U and V are both subspaces of B and that U is included in V. Suppose that both these spaces are spaces of finite codimension, say n and m respectively. Clearly $n \geq m$. Now we have a linear map $\pi_{UV} : B/U \to B/V$ given by $x + U \mapsto x + V$; clearly this is a surjection. Now given coordinate functions $\phi : B/U \to \mathbf{C}^n$ and $\psi : B/V \to \mathbf{C}^m$ we can define a function $\alpha : \mathbf{C}^n \to \mathbf{C}^m$, given by $\psi \circ \pi_{UV} \circ \phi^{-1}$, which makes the diagram commute. Thus we do not have to impose "coherence" conditions on the choice of coordinates, we can always translate back and forth between different coordinate systems.

We will suppress these translation functions in what follows and assume that the coordinates have been serendipitously chosen to make the form of the functions simple. In other words, we can fix a family of subspaces $\{W_n | n \in N\}$ with W_n having codimension n and $W_{n+1} \subset W_n$. The coordinates can be chosen so that the space B/W_n has coordinates $z_1, \ldots, z_n$.

Suppose that f is a cylindric holomorphic function on B. This means that there is a finite-codimensional subspace W, and a holomorphic function f_W, from W to $\mathbf{C}$, such that $f = f_W \circ \pi_W$. The function f_W regarded as a function of n complex variables has a power-series representation

$$f_W(z_1, \ldots, z_n) = \Sigma a_{i_1 \ldots i_k} z_1^{i_1} \ldots z_k^{i_k}$$

and furthermore we have the following convergence condition

$$\Sigma |a_{i_1 \ldots i_k}| < \infty.$$

Thus with each such cylindric holomorphic function we can define the sum of the absolute values of the coefficients in the power-series expansion as the norm of the function. Viewing the sequences of coefficients as the elements of a complex vector space, we have an l_1 norm. We write $\| f \|$ for this norm of a cylindric holomorphic function.

[4]This happens even in the one dimensional case. The function $\overline{z}$ is considered anti-holomorphic traditionally, but one could have called it holomorphic by interchanging the role of z and $\overline{z}$.

Definition 2 *An* l_1-**holomorphic** *function on* B *is the limit of a sequence of cylindric holomorphic function in the above norm.*

The l_1 emphasizes that the holomorphic functions are obtained by a particular norm completion. In the corresponding theory of holomorphic functions on Hilbert spaces, one uses the inner-product to define polynomials and then perform a completion in the L_2 norm. A key difference is that our norm is defined on the sequence of coefficients whereas in the Hilbert space case, one uses the L_2 norm which is defined in terms of integration.

In the resulting Banach space there are several formal entities that were adjoined as part of the norm-completion process. We need to discuss in what sense these formally-defined entities can be regarded as bona-fide functions. Let $W_1, \ldots, W_r, \ldots$ be an infinite sequence of subspaces of B, each embedded in the previous. Assume, in addition, that all these spaces have finite codimension. Now assume that there is a sequence of cylindric holomorphic functions, f_n, on B obtained from a holomorphic function, $f^{(n)}$ on each of the quotient spaces B/W_i. Finally, assume that the sequence $\| f_n \|$ of (real) numbers is convergent. Such a sequence of cylindric holomorphic functions defines a holomorphic function on B. We call this function f. We need to exhibit f as a map from B to $\mathbf{C}$. Accordingly, let x be a point of B. For each of the functions f_n we have $|f_n(x)| \leq \| f_n \|$. Since the sequence of norms converges we have the sequence $f_n(x)$ converges absolutely and hence converges. Thus the function f *qua* function is given at each x of B by $\lim_{n\to\infty} f_n(x)$. However, in order to use the word "function" we need to show that the power-series has a domain of convergence. Unfortunately, it may not have a non-trivial domain of convergence but, in a sense to be made precise, it comes close to having a non-trivial domain of convergence.

The power-series representation of the function f is given as follows. It depends, in general, on infinitely many variables but each term in the power series will be a monomial in finitely many variables. Consider the coefficient of $z_{i_1}^{j_1} \ldots z_{i_k}^{j_k}$ in the expansion of f. In all but finitely many of the f_n all the indicated variables will appear in their power-series expansions. Consider the coefficients of this term in each power series; this forms a sequence of complex numbers α_n where α_n is 0 if there is no such term in the expansion of f_n. Since $|\alpha_n| \leq \| f_n \|$ the sequence α_n converges absolutely and hence converges to, say, α. This is the coefficient of $z_{i_1}^{j_1} \ldots z_{i_k}^{j_k}$ in the power-series expansion of f.

Consider the coordinates $z_1, \ldots, z_n$. This defines an n-dimensional subspace of the Banach space, which we call U_n. Now consider the power-series for f. It defines a family of holomorphic functions f^n where f^n is defined on the subspace U_n and is obtained by retaining only those terms in the power-series expansion of f which involve variables among $z_1, \ldots, z_n$. These are analytic functions on the U_n and, as such, have non-trivial domains of convergence. However, as n increases the radii of convergence could tend to 0. So we have the slightly weaker statement than the usual finite-dimensional notion; instead of having a non-zero radius of convergence in the Banach space we have a non-zero radius of convergence on every finite-dimensional subspace. If one uses *entire* functions, rather than analytic functions, at the starting point of the construction, then one can show that the resulting functions are entire; see page 67, theorem 1.13, of the book by Baez, Segal and Zhou [BSZ92]. Unfortunately when using the representation of elements of Fock space one may carry out simple operations that do not produce entire functions so we cannot just choose to work with entire functions. Nevertheless, many common functions, most notably the exponential, are entire.

Given a bona fide holomorphic function one can express it as a power series. The coefficients are calculated in the usual way, *viz.* by using Taylor's theorem

$$f = \Sigma_n \Sigma_{i_1+\ldots+i_k=n} \frac{1}{i_1! \ldots i_k!} \left(\frac{\partial^n}{\partial^{i_1} z_1 \ldots \partial^{i_k} z_k} \right) z_1^{i_1} \ldots z_k^{i_k}$$

Since the mixed partial derivatives commute (the functions are holomorphic and hence certainly differentiable enough) the partial derivatives are, concretely speaking, symmetric arrays. Abstractly speaking this just means that they are elements of the symmetrized tensor product.

We can write this as follows.

Theorem 18 *A holomorphic function can be represented by its power-series expansion where the n^{th} term in the power-series expansion is a symmetrized n^{th} derivative:*

$$f = \Sigma(1/k!)D^{(k)}f$$

where the notation $D^{(k)}f$ means symmetrized k^{th} derivative of f.

The symmetrized derivatives live in the symmetrized tensor products of B with itself. One thus has a correspondence with the standard Fock representation and the notion of holomorphic function since in each case one has a string of symmetrized vectors.

8 The Physical Origin of Fock Space

The Fock space constructions described in the previous sections were independently invented by physicists and mathematicians. The symmetric Fock space (called the bosonic Fock space by physicists) is well known to mathematicians as the symmetric tensor algebra whereas the antisymmetric Fock space (fermionic Fock space) was invented by Grassman, at least in the finite-dimensional case, under the name of exterior algebra or alternating algebra. In this section we describe the role of Fock space in quantum field theory. In order to prevent intolerable regress in definitions we assume that the reader has an at least intuitive grasp of differential equations, the definition of a smooth manifold and associated concepts like that of a smooth vector field.

We begin with a brief discussion of quantum mechanics and classical mechanics. In classical mechanics one has systems which vary in time. The role of theory is to describe the temporal evolution of systems. Such temporal evolution is governed by a differential equation. The fact that one uses differential equations says something fundamental about the local nature of the dynamics of physical systems, at least according to conventional classical mechanics. In dealing with differential equations one has to distinguish between quantities that are determined and quantities that may be freely specified: the so called "initial conditions". Experiment tells one that systems are described by second-order differential equations and hence that the functions being described and their first derivatives, at a given point of time, are part of the initial conditions. The space of all possible initial conditions is called the space of possible states or "phase" space, and is the kinematical arena on which dynamical evolution occurs. The points of phase space are called states. If the system is a collection of, say 7, particles, the states will correspond to the 42 numbers required to specify the positions and the velocities of each of the particles in three-dimensional space.

Through each point in phase space is a vector giving rise to a smooth vector field called the Hamiltonian vector field. One can draw a family of curves such that at every point there is exactly one curve passing through that point and the Hamiltonian is tangent to the curve at that point. Roughly speaking, the vector field defines a differential equation and the curves represent the family of solutions where each point represents a possible specification of initial conditions. An *observable* is a physical quantity that is determined by the state. As such it corresponds to a real-valued function on phase space. A typical example is the total energy of a system. Most of experimental mechanics is aimed at determining the Hamiltonian. In the formal development of analytical mechanics there is a special antisymmetric 2-form called the *symplectic form* which plays a fundamental mathematical role but is hard to describe in an intuitive or purely physical way.

In quantum mechanics, the above picture changes in the following fundamental ways. The observables become the fundamental physical entities. These are defined to form a particular subalgebra of an algebraic structure called a $\mathbf{C}^*$-algebra. The key point is that this algebra is not commutative, unlike the algebra of smooth functions on a manifold. Furthermore, the failure of commutativity is directly linked to the symplectic form; this was Dirac's contribution to the theory of quantum mechanics. Thus, structures available at the classical level provide guidance as to what the "correct" $\mathbf{C}^*$-algebra should be.

There is a representation of this algebra as the algebra of operators on a Hilbert space. The space of states acquires the structure of a Hilbert space and becomes the carrier of the representation of the $\mathbf{C}^*$-algebra. One presentation of this abstract Hilbert space is as the space of square-integrable complex-valued functions on a suitable underlying space; for example the space of possible configurations of a system. The space of states has acquired linear structure; this means that one can add states reflecting the intuition that in quantum mechanics a system can be in the superposition of two (or more) states. The inner product measures the extent to which two states resemble each other. Finally the fact that one has complex functions is strongly suggested by the observation of interference phenomena in nature.

An observable is a self-adjoint operator. The link between the mathematics and experiment is the following. If one attempts to measure the observable O for a system in state ψ one will obtain an eigenvalue of O. Self-adjoint operators have real eigenvalues so we will get a real-valued result. If ψ is an eigenvector with eigenvalue α, then, with no indeterminacy or uncertainty, one will obtain the value α. If ψ is not an eigenvector, one can express ψ as a linear combination of eigenvectors in the form $\psi = \Sigma a_i \psi_i$ where the ψ_i are assumed to be eigenvectors with eigenvalues α_i. The result of measuring O will be α_i with probability $|a_i|^2$. It is important to keep in mind that the absolute squares of the a_i correspond to probabilities but it is the a_i themselves that enter into the linear combinations of states. This interplay between the complex coefficients and the interpretation of their squares as probabilities is what distinguishes the probabilistic aspects of quantum mechanics from statistical mechanics which also has a probabilistic aspect but where one directly manipulates probabilities.

The dynamics of systems is described by a *first-order* differential equation called Schroedinger's equation. Thus, the evolution of states in quantum mechanics is determinate, just as in classical mechanics. The indeterminacy usually associated with quantum mechanics appears in the fact that the state of a system may not be an eigenstate of the observable being measured so the outcome of the measurement may be indeterminate.

Quantum mechanics is designed to handle systems in which the number of interacting entities (usually called "particles") is fixed. On the other hand, experiment tells us that at sufficiently high energies particles may be created or destroyed. Quantum field theory was invented to account for such processes. The original formulations of this theory due to Dirac, Heisenberg, Fock, Jordan, Pauli, Wigner and many others was quite heuristic. Now a reasonably rigourous theory is available; see the book by Baez, Segal and Zhou [BSZ92] for a recent exposition of quantum field theory.

The first need in a many-particle theory is a space of states which can describe variable numbers of particles; this is what Fock space is [Ge85]. The second ingredient is the availability of operators that can describe the creation and annihilation of particles. Of course, there is much more that needs to be said in order to see how all this formalism translates into calculations of realistic physical processes but that would require a very thick book which, in any case, has been written many times over.

Given a Hilbert space H in quantum mechanics representing the states of a single particle one can construct a many-particle Hilbert space as $\mathcal{F}(H)$. Suppose that $\psi, \phi \in H$; one interprets the element $\psi \otimes_s \phi$ of $H \otimes_s H$ as a two-particle state with one particle in the state ψ and the other in the state ϕ. Similarly for the other summands of $\mathcal{F}(H)$. The reason for the symmetrization is that one is dealing with indistinguishable particles so that the n-particle states have to carry representations of the permutation group. Thus one could have particle states that were symmetric or antisymmetric under interchange leading to the bosonic or fermionic Fock spaces respectively. It is a remarkable fact that both types of particles are observed in nature. Notice that $\psi \wedge \psi$ is identically zero hence one cannot have many-particle states in the antisymmetric Fock space in which both particles are in the same one-particle state. This is observed in nature as the exclusion principle. Fock space is the space of states for quantum field theory and is constructed from the space of states for quantum mechanics.

The presentation of Fock space above emphasized the concept of many-particle states. Mathematically, however, $\mathcal{F}(H)$ is just a Hilbert space and can be presented differently. As we have shown in the last section, it can be presented as the space of holomorphic functions of a Hilbert space (the details are somewhat different from the Banach space case but the ideas are essentially the same). The space of holomorphic functions has as its inner product

$$\langle g, f\rangle = \frac{1}{2\pi i}\int f(z)g(\overline{z})e^{\perp z\overline{z}}dz\,d\overline{z}.$$

(See [IZ80] page 435, for example.) What do the creation and annihilation operators look like from this perspectiveΓ For simplicity, let us look at power series in a single variable z. This amounts to only looking at the many-particle states of the form σ tensored with itself. The creation operator is just $z * (.)$ while the annihilation operator is just $d(.)/dz$. One can easily check that $(AC - CA)f = d(z * f)/dz - z * df/dz = f$; in other words the basic algebraic relation holds. Furthermore one can ask what the eigenstates of A and C look like. Clearly the eigenstate of C is just the zero vector. The eigenstate of A is the state represented by the holomorphic function e^z. These states actually exist in nature and are called "coherent" states; they occur, for example, in lasers. The key point about coherent states is that they "look classical"; one can remove a particle without changing the state. As such they bear a resemblance to the role of **!** formulas in linear logic.

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Appendix-Contractibility of the Exponential Isomorphism

It is straightforward to verify that the isomorphism:

$$!\,(A \oplus B) \cong\ !\,A \otimes\ !\,B$$

constructed on vector spaces lifts to an isomorphism on Banach spaces. The only issue is whether the maps so constructed are contractive.

We now show that the two maps in this isomorphism are contractive. We need some preliminary facts about norms on tensor products of Banach spaces. Suppose that A and B are two Banach spaces. Let a be in A and b in B. The norm of $a \otimes b$ is $\| a \|.\| b \|$; the so-called "cross-norm". Not all elements of $A \otimes B$ are of the form $a \otimes b$ for some a in A and some b in B; in general an element of $A \otimes B$ will be a sum of such terms. Furthermore there is no unique representation as such a sum. The norm is then defined as follows

$$\| u \|_{A \otimes B} = inf\{\Sigma_i \| a_i \|.\| b_i \| : \Sigma a_i \otimes b_i = u, \forall i.a_i \in A, b_i \in B\}.$$

Note also that the norm is a continuous, but not linear function. Thus one cannot argue in terms of basis elements.

We consider $\beta : \mathcal{F}(A) \times \mathcal{F}(B) \longrightarrow \mathcal{F}(A \oplus B)$ first. Suppose that u and v are pure tensors in $\mathcal{F}(A)$ and $\mathcal{F}(B)$ respectively then clearly $\| \beta(u \otimes v) \| = \| u \|.\| v \|$ since β just takes the tensor of u and v and we have the above remark about the norm of such elements in the tensor product space. Now let $y \in \mathcal{F}(A)$ and $z \in \mathcal{F}(B)$ be arbitrary. We consider the action of β on $y \otimes z$; this is still not the most general situation. As usual, we write y_i for the ith component of y in the standard basis of Fock space; thus y_i is a pure tensor. Similarly for z_j.

$$\| \beta(y \otimes z) \| = \| \Sigma_{i,j} \beta(y_i \otimes z_j) \|$$

$$\leq \Sigma_{i,j} \| \beta(y_i \otimes z_j) \| = \Sigma_{i,j} \| (y_i \otimes z_j) \| = \Sigma_{i,j} \| y_i \|.\| z_j \| = \| y \|.\| z \| = \| y \otimes z \|.$$

The equalities are obvious, the inequality is the triangle inequality.

Now suppose that we have an element $u \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ and that u has the decomposition $\Sigma_i y^{(i)} \otimes z^{(i)}$, as well, of course, as other such decompositions. Now we have, using the linearity of β, the triangle inequality and the argument just above,

$$\| \beta(u) \| \leq \Sigma_i \| y^{(i)} \otimes z^{(i)} \|.$$

However, we do not know that the right hand side is less than $\| u \|$; in fact using the triangle inequality we would get the opposite inequality. But since the norm of u is defined as the infimum of the above sum of norms across all such decompositions. Now if the infimum is actually realized by such a decomposition then we still have the above inequality but now we know that the right hand side is indeed $\| u \|$. If the infimum is not realized, there is a sequence of decompositions with the right hand sides as above converging to the infimum and since $\| \beta(u) \|$ is less than all such sums it must be less than the infimum. Thus in all cases $\| \beta(u) \| \leq \| u \|$ and hence β is contractive.

To show that α is contractive we need a fact about how symmetrization affects norms. Suppose that we have $u, v \in B$, where B is any banach space. Now $u \otimes_s v = (1/2)(u \otimes v + v \otimes u)$. We claim that

$$\| u \otimes_s v \| = \| u \otimes v \|.$$

Clearly one decomposition of $u \otimes_s v$ is $(1/2)(u \otimes v + v \otimes u)$; if this were the one realizing the infimum in the definition of the norm we would be done. Now suppose that there were another decomposition, $\Sigma_i p_i \otimes q_i$, for $u \otimes_s v$ such that $\| u \otimes_s v \| = \Sigma_i \| p_i \otimes q_i \| < \| u \otimes v \|$. Consider the expression $(2\Sigma_i p_i \otimes q_i) - v \otimes u = u \otimes v$. Computing norms and using the triangle inequality we get

$$\| u \otimes v \| = \| 2\Sigma_i p_i \otimes q_i - v \otimes u \| \leq 2\Sigma_i \| p_i \otimes q_i \| + \| u \| \| v \|.$$

In other words we have, by simple arithmetic,

$$\| u \otimes v \| = \| u \| \| v \| \leq \Sigma_i \| p_i \otimes q_i \|.$$

This contradicts the assumption above. Thus symmetrization preserves norms. It is clear that the above argument could have been carried out for symmetrization over more than two elements.

The map α just undoes symmetrization thus, *on pure tensors*, α is norm preserving. Now consider an arbitrary element, x, of $\mathcal{F}(A \otimes_s B)$. We have

$$\| \alpha(x) \| = \| \Sigma_n \alpha \langle 0, 0, \ldots, 0, x_n, 0, 0, \ldots \rangle \|$$

$$\leq \Sigma_n \| \alpha(\langle 0, 0, \ldots, 0, x_n, 0, 0, \ldots \rangle) \|$$

$$= \Sigma_n \| \langle 0, 0, \ldots, 0, x_n, 0, 0, \ldots \rangle \|$$

$$= \Sigma_n \| x_n \| = \| x \|.$$

Thus α is contractive as well. It immediately follows that the morphism

$$\mathbf{!}\, A \otimes \mathbf{!}\, B \rightarrow \mathbf{!}\, (A \otimes B)$$

is a morphism in $\mathcal{BANCON}$ and satisfies the necessary properties.