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Papendiek 14

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## A Note on Height Pairings, Tamagawa Numbers, and the Birch and Swinnerton-Dyer Conjecture

S. Bloch\*

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

### Introduction

Let  $G$  be an algebraic group defined over a number field  $k$ . By choosing a lifting of  $G$  to a group scheme over  $\mathcal{O}_S \subset k$ , the ring of  $S$ -integers for some finite set of places  $S$  of  $k$ , we may define  $G(\mathcal{O}_v)$ , where  $\mathcal{O}_v \subset k_v$  is the ring of integers in the  $v$ -adic completion of  $k$  for all non-archimedean places  $v \notin S$ . In this way, we can define the adelic points  $G(\mathbf{A}_k)$ . Since different choices of lifting will change  $G(\mathcal{O}_v)$  for only a finite number of  $v$ ,  $G(\mathbf{A}_k)$  is intrinsically defined independent of the choice of  $\mathcal{O}_S$ -scheme structure.

It may happen that  $G(k) \subset G(\mathbf{A}_k)$  is discrete. This will be the case, for example, if  $G$  is affine. If so, we may try to compute the volume of  $G(\mathbf{A}_k)/G(k)$ . Writing  $\mathbb{F}_v =$  residue field at  $v$ ,  $q_v = \#\mathbb{F}_v$ ,  $N_v = \#G(\mathbb{F}_v)$ , the natural volume form gives  $\text{Vol}(G(\mathcal{O}_v)) = N_v q_v^{-1}$  for all  $v \notin S$ . It can happen that  $\prod_v N_v q_v^{-1}$  does not

converge (example:  $G = G_m$ ), but in many cases there is an  $L$ -function  $L(G, s)$  available such that  $L(G, s) = \prod_{v \notin S} L_v(G, s)$  where the product converges absolutely

for  $\text{Re } s \gg 0$  and extends meromorphically to the whole plane with  $L_v(G, 1) = \frac{q_v}{N_v}$ .

Suppose  $\lim_{s \rightarrow 1} L(G, s)(s-1)^{-r} \neq 0, \infty$ . The Tamagawa number  $\tau(G)$  is defined by modifying the measure on  $G(\mathbf{A}_k)$  so  $\text{Vol}(G(\mathcal{O}_v)) = 1$ , all  $v \notin S$ , computing the measure of  $G(\mathbf{A}_k)/G(k)$ , and then multiplying by  $\lim_{s \rightarrow 1} L(G, s)(s-1)^{-r}$ . For more details, the reader should see [10].

The Tamagawa number has been computed for all except a few particularly stubborn affine algebraic groups, and takes the value (see [10, 4–6])

$$\tau(G) = \frac{\#\text{Pic}(G)}{\#\text{III}(G)},$$

where  $\text{Pic}(G) =$  Picard group, and  $\text{III}(G) = \text{Ker}(H^1(\bar{k}/k, G(\bar{k})) \rightarrow \prod_v H^1(\bar{k}_v/k_v, G(\bar{k})))$ . Moreover,  $r \leq 0$ , and  $r = 0$  if  $G(\mathbf{A}_k)/G(k)$  is compact.

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Suppose now that  $G$  is not necessarily affine, but that  $G(k)$  is discrete in  $G(\mathbf{A}_k)$ . One conjectures that  $\text{III}(G)$  is finite. (This is not known for a single abelian variety  $G$ !)  $\text{Pic}(X)$  may be infinite but  $\text{Pic}(X)_{\text{torsion}}$  is finite and one may

(0.1) *Conjecture.*  $\tau(G) = \frac{\#\text{Pic}(G)_{\text{tors}}}{\#\text{III}(G)}$ . Moreover,  $r \leq 0$  and  $r=0$  if and only if  $G(\mathbf{A}_k)/G(k)$  has finite volume.

We refer to this in the sequel as the Tamagawa number conjecture.

Consider now the case of an abelian variety  $A$ . Conjecture (0.1) makes sense only if  $A(k)$  is finite. The Hasse-Weil  $L$ -function  $L(A, s) = \prod_{v \notin S} L_v(A, s)$ , where  $S$  = set of bad reduction places, and

$$L_v(A, s) = \frac{1}{\det(1 - q_v^{-s} F_v | H_{\text{et}}^1(A_{\mathbb{F}_v}, \mathbb{Q}))} \quad (F_v = \text{geometric frobenius}).$$

Birch and Swinnerton-Dyer conjecture that  $L(A, s)$  has a zero of order  $r = rk A(k)$  at  $s=1$  (so  $r \geq 0$ ) and that

$$(0.2) \quad \lim_{s \rightarrow 1} L(A, s)(s-1)^{-r} = \frac{\#\text{III}(A) \cdot \langle \rangle \cdot V_{\infty} \cdot V_{\text{bad}}}{\#A(k)_{\text{tors}} \cdot \#\text{Pic}(A)_{\text{tors}}},$$

where  $V_{\infty} = \text{Volume } A(k \otimes_{\mathbb{Q}} \mathbb{R})$  and  $V_{\text{bad}} = \text{Volume } \prod_{v \in S} A(k_v)$ . Finally,  $\langle \rangle$  denotes the height pairing [1, 3]

$$\langle \rangle: A(k) \times A'(k) \rightarrow \mathbb{R}$$

with  $A'(k) = \text{Pic}^0(A)$ .

The purpose of this note is to deduce (0.2) from (0.1), and thus to give a purely volume-theoretic interpretation of Birch and Swinnerton-Dyer. An element  $\alpha \in \text{Pic}(A)$  corresponds to a  $\mathbb{G}_m$ -torsor  $X_{\alpha} \rightarrow A$ . If  $\alpha \in \text{Pic}^0(A) = A'(k)$ ,  $X_{\alpha}$  is a group extension of  $A$  by  $\mathbb{G}_m$ . We construct in this way an extension

$$(0.3) \quad 0 \rightarrow T \rightarrow X \rightarrow A \rightarrow 0$$

where  $T$  is the split torus with character group  $\cong A'(k)/\text{torsion}$ . An important point is that the "logarithmic modulus" map factors

$$\begin{array}{ccc} 0 \rightarrow T(\mathbf{A}_k) & \rightarrow & X(\mathbf{A}_k) \\ \text{log.} \downarrow & & \searrow \\ \text{mod.} & & \text{Hom}(A'(k), \mathbb{R}) \end{array}$$

The product formula shows  $\text{log.mod.}(T(k)) = (0)$ , so by restriction to global points, we obtain

$$A(k) \cong X(k)/T(k) \rightarrow \text{Hom}(A'(k), \mathbb{R}),$$

or again

$$(0.4) \quad A(k) \times A'(k) \rightarrow \mathbb{R}.$$

Using the axiomatic characterization of Neron's local pairings [1, 3], we show that (0.4) is the height pairing. From this it follows without difficulty that  $X(k)$  is discrete and cocompact in  $X(\mathbf{A}_k)$ , and that (0.1) for  $X$  implies (0.2) for  $A$ .

It seems likely that this technique will lead to height pairings in many new situations, e.g., for algebraic cycles other than zero cycles and divisors. I hope to return to this question in the future. I am indebted to W. Messing for several helpful discussions regarding the Neron model.

## 1. The Global Construction

Let  $A$  be an abelian variety over a number field  $k$ . Let  $N$  be the Neron model of  $A$  over the ring of integers  $\mathcal{O}_k$ ,  $N^0 \subset N$  the largest open subgroup scheme whose fibres are connected. Let  $A'$  be the dual abelian variety,  $N'$  = Neron model of  $A'$ . It is known (cf. [11], p. 53) that

$$(1.1) \quad N' \cong \mathbf{Ext}_{\mathcal{O}\text{-group scheme}}^1(N^0, \mathbf{G}_m).$$

In particular, if we fix once for all a splitting

$$(1.2) \quad A'(k) = B \oplus A'(k)_{\text{tors}}$$

and use  $A'(k) = N'(\mathcal{O}_k)$ , we can build an extension over  $\mathcal{O}_k$

$$(1.3) \quad 0 \rightarrow T \rightarrow X \rightarrow N^0 \rightarrow 0,$$

where  $T$  is the  $k$ -split torus with character group  $B$ . Let  $A_k$  denote the adeles of  $k$ . Since  $H^1(\text{Sp } R, \mathbf{G}_m) = (0)$  for  $R$  local, we get exact sequences

$$(1.4) \quad \begin{aligned} 0 \rightarrow T(k) \rightarrow X(k) \rightarrow A(k) \rightarrow 0 \\ 0 \rightarrow T(\mathbf{A}_k) \rightarrow X(\mathbf{A}_k) \rightarrow N^0(\mathbf{A}_k) \rightarrow 0. \end{aligned}$$

Define  $T^1 \subset T(\mathbf{A}_k)$  by the exact sequence

$$(1.5) \quad 0 \rightarrow T^1 \rightarrow T(\mathbf{A}_k) \xrightarrow{l} \text{Hom}(B, \mathbb{R}) \rightarrow 0$$

where  $l$  is induced from the usual logarithmic modulus map from the ideles to  $\mathbb{R}$ . Define further, for  $v$  a place of  $k$

$$(1.6) \quad X_v^1 = \begin{cases} X(\mathcal{O}_v) & v \text{ non-archimedean} \\ X(k_v)_{\text{max. compact}} & v \text{ archimedean} \end{cases}$$

$$\tilde{X}^1 = T^1 \cdot \prod_v X_v^1 \subset X(\mathbf{A}_k).$$

Finally, let  $X^1$  be the rational saturation of  $\tilde{X}^1$ , i.e.,

$$(1.7) \quad X^1 = \{a \in X(\mathbf{A}_k) \mid \exists n \geq 1, n \in \mathbb{Z}, na \in \tilde{X}^1\}.$$

(1.8) **Lemma.** *There exists a diagram with exact rows and columns*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T^1 & \longrightarrow & X^1 & \longrightarrow & N^0(\mathbf{A}_k) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & T(\mathbf{A}_k) & \longrightarrow & X(\mathbf{A}_k) & \longrightarrow & N^0(\mathbf{A}_k) \longrightarrow 0 \\
 & & \downarrow l & & \downarrow & & \\
 & & \text{Hom}(B, \mathbb{R}) & = & \text{Hom}(B, \mathbb{R}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

*Proof.* It suffices to show  $X^1 \cap T(\mathbf{A}_k) = T^1$ , and  $X^1 \twoheadrightarrow N^0(\mathbf{A}_k)$ . The first point is straightforward, using that  $T(\mathbf{A}_k)/T^1$  is torsion-free, and  $l$  is trivial on  $T(\mathcal{O}_v)$  and  $T(k_v)_{\max, \text{compact}}$ . For the second, note that the image of  $\tilde{X}^1$  in  $N^0(\mathbf{A}_k)$  contains  $A(k_v) = N^0(\mathcal{O}_v)$  for almost all  $v$  and the cokernel

$$W \stackrel{\text{def}}{=} N^0(\mathbf{A}_k) / \text{Im}(\tilde{X}^1 \rightarrow N^0(\mathbf{A}_k))$$

is finite. It follows easily that  $X(\mathbf{A}_k)/\tilde{X}^1 \cong \text{Hom}(B, \mathbb{R}) \oplus W$ . Replacing  $\tilde{X}^1$  by  $X^1$  eliminates torsion in the quotient, and the lemma follows by diagram chasing. Q.E.D.

Combining (1.4) and (1.8), and using the fact that  $T(k) \subset T^1$  (product formula) we get

$$A(k) \cong X(k)/T(k) \rightarrow X(\mathbf{A}_k)/T(k) \rightarrow \text{Hom}(B, \mathbb{R})$$

and hence a pairing

$$\langle \rangle: A(k) \times A'(k) \rightarrow \mathbb{R}.$$

(1.9) **Theorem.** *The above pairing coincides with the height pairing.*

We postpone the proof until the next section.

(1.10) **Theorem.**  *$X(k) \subset X(\mathbf{A}_k)$  is discrete and cocompact.*

*Proof.* Let  $U = X(k) \cap X^1 \subset X(\mathbf{A}_k)$ . Since the height pairing is perfect, we get  $0 \rightarrow T(k) \rightarrow U \rightarrow A(k)_{\text{tors}} \rightarrow 0$ , and hence exact sequences

$$\begin{array}{l}
 0 \rightarrow T^1/T(k) \rightarrow X^1/U \rightarrow N^0(\mathbf{A}_k)/A(k)_{\text{tors}} \rightarrow 0 \\
 (1.11) \quad 0 \rightarrow X^1/U \rightarrow X(\mathbf{A}_k)/X(k) \rightarrow \frac{\text{Hom}(B, \mathbb{R})}{\text{Image } A(k)} \rightarrow 0
 \end{array}$$

The image of  $A(k)$  in  $\text{Hom}(B, \mathbb{R})$  is known to be discrete and cocompact (perfectness of height pairings), and compactness is known for  $T^1/T(k)$  (classical theorem about ideles) and  $N^0(\mathbf{A}_k)$ . The assertions of the theorem follow. Q.E.D.

What about the Tamagawa number of  $X$ ? With notation as in the introduction, let  $r = rk A'(k)$ . We choose convergence factors in the sense of [10] for the measure on  $X(\mathbf{A}_k)$ :

$$(1.12) \quad \begin{array}{ll} (1 - q_v^{-1})^r L_v(A, 1)^{-1} & v \text{ non-archimedean, } A \text{ has good} \\ & \text{reduction at } v, \\ (1 - q_v^{-1})^r & v \text{ non-archimedean, } A \text{ does not} \\ & \text{have good reduction at } v, \\ 1 & v \text{ archimedean.} \end{array}$$

These correspond to convergence factors  $(1 - q_v^{-1})^r$  on  $T(\mathbf{A}_k)$  and  $L_v(A, 1)^{-1}$  on  $N^0(\mathbf{A}_k)$  ( $v$  good reduction place). Writing  $\zeta_k(s)$  for the zeta function of  $k$  we get from (1.11)

$$(1.13) \quad \begin{aligned} \text{Volume}(T^1/T(k)) &= \lim_{s \rightarrow 1} (\zeta_k(s)(s-1))^r \\ \text{Vol. } N^0(\mathbf{A}_k) &= \underbrace{\text{Vol.}(A(K \otimes_{\mathbb{Q}} \mathbb{R}))}_{V_{\infty}} \cdot \underbrace{\prod_{v \text{ bad}} \text{Vol. } A(k_v)}_{V_{\text{bad}}} \\ \text{Vol.}(X^1/U) &= \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \rightarrow 1} (\zeta_k(s)(s-1))^r V_{\infty} V_{\text{bad}} \\ \text{Vol.}(X(\mathbf{A}_k)/X(k)) &= \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \rightarrow 1} (\zeta_k(s)(s-1))^r V_{\infty} V_{\text{bad}} R \end{aligned}$$

where  $R$  is the absolute value of the discriminant of the height pairing.

We assume now that  $\lim_{s \rightarrow 1} \zeta_k(s)^r L(A, s) \neq 0, \infty$ , i.e., that the  $L$ -function of  $X$  has no zero or pole at  $s=1$  as predicted by the Tamagawa number conjecture, or equivalently that the  $L$ -function of  $A$  has a zero of order  $r = rk A(k)$  as predicted by Birch and Swinnerton-Dyer. To define the Tamagawa number  $\tau(X)$  we eliminate the (non-canonical) choice of convergence factors by dividing the volume computed above by  $\lim_{s \rightarrow 1} \zeta_k(s)^r L(A, s)$ , getting

$$(1.14) \quad \tau(X) = \frac{1}{\# A(k)_{\text{tors}}} \lim_{s \rightarrow 1} L(A, s)^{-1} (s-1)^r V_{\infty} V_{\text{bad}} R.$$

Conjecture (0.2) is thus equivalent to

$$(1.15) \quad \tau(X) \stackrel{?}{=} \frac{\# A'(k)_{\text{tors}}}{\# III(A)}.$$

$$(1.16) \quad \textbf{Lemma. } III(A) \cong III(X) \text{ and } A'(k) \cong \text{Pic}(X)_{\text{tors}}.$$

*Proof.* The first isomorphism follows from chasing the diagram

$$\begin{array}{ccccc}
& 0 & & 0 & \\
& \downarrow & & \downarrow & \\
& \text{III}(X) & \longrightarrow & \text{III}(A) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^1(\bar{k}/k, X) & \longrightarrow & H^1(\bar{k}/k, A) & \longrightarrow & H^2(\bar{k}/k, T) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \prod_v H^1(\bar{k}_v/k_v, X) & \longrightarrow & \prod_v H^1(\bar{k}_v/k_v, A) & \longrightarrow & \prod_v H^2(\bar{k}_v/k_v, T)
\end{array}$$

For the second isomorphism, note that if  $T$  is a split torus over a ring  $R$  with character group  $\hat{T}$ , then taking units in the ring of regular functions on  $T$  yields an exact sequence

$$0 \rightarrow R^* \rightarrow R[T]^* \rightarrow \hat{T} \rightarrow 0.$$

Let  $\pi: X \rightarrow A$  be the projection. The above sequence globalizes

$$0 \rightarrow \mathbf{G}_{m,A} \rightarrow \pi_* \mathbf{G}_{m,X} \rightarrow B_A \rightarrow 0$$

where  $B_A$  is the constant Zariski sheaf on  $A$  with stalk  $B$ . The boundary map

$$B = \Gamma(A, B_A) \rightarrow H^1(A, \mathbf{G}_m) = \text{Pic } A$$

is the natural inclusion, so we obtain

$$H^1(A, \pi_* \mathbf{G}_{m,X}) \cong (\text{Pic } A)/B.$$

Locally over  $A$ ,  $Z \cong \mathbf{G}_m^r \times A$ , so  $R^1 \pi_* \mathbf{G}_{m,X} = (0)$  and we find

$$\text{Pic } X \cong (\text{Pic } A)/B$$

and a similar result holds for torsion. Q.E.D.

Combining (1.15) and (1.16) yields

(1.17) **Theorem.** *The Birch and Swinnerton-Dyer conjecture holds for  $A$  if and only if*

$$\tau(X) = \frac{\# \text{Pic}(X)_{\text{tors}}}{\# \text{III}(X)}.$$

## 2. The Local Neron Pairing

The purpose of this section is to prove (1.9). Let  $k$  be a local field,  $A$  an abelian variety over  $k$ ,  $N = \text{Neron model of } A$ ,  $N^0 \subset N$  the subgroup scheme with connected fibres. The Néron model of the dual variety  $A' = \mathbf{Ext}^1(A, \mathbf{G}_m)$  is then  $N' = \mathbf{Ext}_{\mathcal{O}}^1(N^0, \mathbf{G}_m)$  ([11], p. 53). Thus given a divisor  $\Delta$  on  $A$  defined over  $k$  and algebraically equivalent to 0, we get a corresponding extension

$$(2.1) \quad 0 \rightarrow \mathbf{G}_m \rightarrow X_{\Delta} \rightarrow N^0 \rightarrow 0.$$

If  $\mathcal{L}_\Delta$  is the line bundle associated to  $\Delta$ ,

$$X_\Delta \cong V(\mathcal{L}_\Delta) - (0\text{-section})$$

as a  $G_m$ -torsour. The extension (2.1) depends only on the linear equivalence class of  $\Delta$ .

Restricting to  $\text{Sp } k$ , the extension (2.1) is split as a torsour over  $A - |\Delta|$  ( $|\Delta| = \text{Supp } \Delta$ )

$$(2.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbf{G}_{m,k} & \rightarrow & X_{\Delta,k} & \rightarrow & A \rightarrow 0 \\ & & & & \swarrow \sigma_\Delta & & \cup \\ & & & & & & A - |\Delta| \end{array}$$

where  $\sigma_\Delta$  is canonical up to translation by  $\mathbf{G}_{m,k}(k) = k^*$  (choosing  $\sigma_\Delta$  is tantamount to choosing a rational section of  $\mathcal{L}_\Delta$  corresponding to the divisor  $\Delta$ ). Let  $Z_{\Delta,k}$  = group of zero cycles  $\mathfrak{A} = \sum n_i(p_i)$  on  $A$  defined over  $k$  such that  $\sum n_i \deg p_i = 0$  and  $\text{Supp } \mathfrak{A} \subset A - |\Delta|$ . We get a homomorphism

$$(2.3) \quad \sigma_\Delta: Z_{\Delta,k} \rightarrow X_\Delta(k).$$

Define

$$X_\Delta^1 = \begin{cases} X_\Delta(\text{Sp } \mathcal{O}_k) & k \text{ non-archimedean} \\ X_\Delta(k)_{\text{max. compact}} & k \text{ archimedean,} \end{cases}$$

$$\mathbf{G}_m^1 = \begin{cases} \mathcal{O}_k^* & k \text{ non-archimedean} \\ (k^*)_{\text{max compact}} & k \text{ archimedean,} \end{cases}$$

$$F = \begin{cases} \mathbf{Z} & k \text{ non-archimedean, } N = N^0 \\ \mathbf{Q} & k \text{ non-archimedean, } N \neq N^0 \\ \mathbf{R} & k \text{ archimedean.} \end{cases}$$

(2.4) **Lemma.** *Assume either  $v$  archimedean or  $N = N^0$ . Then there is a diagram with exact rows and columns*

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{G}_m^1 & \longrightarrow & X_\Delta^1 & \longrightarrow & A(k) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & k^* & \longrightarrow & X_\Delta(k) & \longrightarrow & A(k) \longrightarrow 0 \\ & & \downarrow i & & \downarrow & & \\ & & F & = & F & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$



*Proof.* The map labeled  $l$  is either the logarithm or the valuation map. As in the proof of (1.8), the only thing we need to show is  $X_{\Delta}^1 \twoheadrightarrow A(k)$ . In the non-archimedean case we have  $A(k) \cong N(\mathcal{O}) = N^0(\mathcal{O})$ . Surjectivity  $X_{\Delta}^1 = X(\mathcal{O}) \rightarrow N^0(\mathcal{O})$  follows from  $H^1(\text{Sp } \mathcal{O}, \mathbf{G}_m) = (0)$ . In the archimedean case, the existence of an exponential implies the connected component of 0 in  $A(k)$  is contained in the image of  $X_{\Delta}^1$ . Factoring out by  $X_{\Delta}^1$ , we obtain an extension of a finite group by  $\mathbb{R}$ . Such an extension is necessarily split, so we get

$$0 \rightarrow X_{\Delta}^1 \rightarrow X_{\Delta}(k) \rightarrow \mathbb{R} \oplus (\text{finite}) \rightarrow 0.$$

Since  $X_{\Delta}^1$  is maximal compact,  $(\text{finite}) = (0)$ . Q.E.D.

Suppose now  $N \neq N^0$ , and let  $A(k)_0 = \text{Image}(X_{\Delta}^1 \rightarrow A(k))$ . Note  $A(k)/A(k)_0$  is finite, and we have a diagram (defining  $Y$ )

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{G}_m^1 & \longrightarrow & X_{\Delta}^1 & \longrightarrow & A(k)_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & k^* & \longrightarrow & X_{\Delta}(k) & \longrightarrow & A(k) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & Y & \longrightarrow & A(k)/A(k)_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

In particular  $Y \otimes \mathbb{Q} \cong \mathbb{Q}$  (canonically) so we get  $X_{\Delta}(k) \rightarrow \mathbb{Q}$ .

In any of the above cases, let  $\psi_{\Delta}: X_{\Delta}(k) \rightarrow F$  be the map just defined, and for  $\mathfrak{A} \in Z_{\Delta, k}$  define

$$(2.6) \quad \langle \Delta, \mathfrak{A} \rangle_{\text{local}} = \psi_{\Delta} \sigma_{\Delta}(\mathfrak{A}).$$

When the given ground field is the completion of a global field at some place  $v$ , we write  $\langle \rangle_v$  instead of  $\langle \rangle_{\text{local}}$ .

(2.7) **Theorem.** *Let  $k$  be a number field. Let  $a \in A(k)$ ,  $a' \in A'(k)$ . Let  $\Delta$  (resp.  $\mathfrak{A}$ ) be a divisor algebraically equivalent to zero defined over  $k$  (resp. a zero cycle of degree 0 defined over  $k$ ) on  $A$  such that  $[\Delta] = a'$  (resp.  $\mathfrak{A}$  maps to  $a \in A(k)$ ). Assume further that  $\text{Supp } \Delta$  and  $\text{Supp } \mathfrak{A}$  are disjoint. Then*

$$\langle a, a' \rangle = \sum_{\substack{v \text{ place} \\ \text{of } k}} \langle \Delta, \mathfrak{A} \rangle_v,$$

where  $\langle a, a' \rangle$  is defined as in (1.9).

*Proof.* Consider the global extension

$$0 \rightarrow T \rightarrow X \rightarrow N^0 \rightarrow 0$$

as in section 1 and push out along  $a' \in \hat{T}$  to get

$$0 \rightarrow \mathbf{G}_m \rightarrow X_\Delta \rightarrow N^0 \rightarrow 0.$$

We can think of  $\sigma_\Delta: Z_{\Delta,k} \rightarrow X_\Delta(k)$  just as in the local case. The problem therefore reduces to showing the map

$$(2.8) \quad X_\Delta(\mathbf{A}_k) \rightarrow \mathbb{R}$$

defined via the techniques of Sect. 1 coincides with the sum of the local maps

$$\psi_{\Delta,v}: X_\Delta(k_v) \rightarrow F = \begin{cases} \mathbb{Z} \\ \mathbb{Q} \\ \mathbb{R} \end{cases}$$

defined in the beginning of this paragraph.

Finally, this point is clear from the diagram

$$(2.9) \quad \begin{array}{ccccccc} \mathbf{G}_{m,k}^1 & \longrightarrow & X_\Delta(\mathbf{A}_k) / \prod_v X_{\Delta,v}^1 & \longrightarrow & X_\Delta(\mathbf{A}_k) / X_\Delta^1 & \longrightarrow & 0 \\ \downarrow \Sigma v & & \downarrow \Sigma \psi_{\Delta,v} & & \downarrow (2.8) & & \\ 0 \rightarrow \text{Ker}(\text{sum}) & \longrightarrow & \prod_{v \text{ arch.}} \mathbb{R} \times \prod_{v \text{ non-arch}} \mathbb{Q} & \xrightarrow{\text{sum}} & \mathbb{R} & \longrightarrow & 0 \end{array}$$

Néron has shown [3] that the height pairing can be written as a sum of local terms. With notation as in (2.7)

$$(2.10) \quad \langle a, a' \rangle = \sum_v \langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}}.$$

(2.11) **Proposition.**  $\langle \Delta, \mathfrak{A} \rangle_v = \langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}}.$

*Proof.* We write  $v: k_v^* \rightarrow \mathbb{R}$  for the logarithmic valuation, normalized in accordance with the global product formula. Let  $D_a(A)_k$  denote the group of divisors on  $A$  algebraically equivalent to zero and defined over  $k$ . The local Néron pairing is characterized by the following properties:

- (1)  $\langle \rangle_{v, \text{Neron}}: \{(\Delta, \mathfrak{A}) \in D_a(A)_k \times Z_k(A) \mid |\Delta| \cap |\mathfrak{A}| = \emptyset\} \rightarrow \mathbb{R}$
- (2)  $\langle \rangle_{v, \text{Neron}}$  is bilinear, assuming all terms in the desired equality are defined.
- (3) If  $\Delta = (f)$ , then  $\langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}} = v(f(\mathfrak{A}))$ , where for  $\mathfrak{A} = \sum n_i(p_i)$ ,  $f(\mathfrak{A}) = \prod_i f(p_i)^{n_i}$ .
- (4)  $\langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}} = \langle \Delta_a, \mathfrak{A}_a \rangle_{v, \text{Neron}}$ , where  $a \in A(k_v)$  and the subscript indicates translation by  $a$ .

(5) For  $\Delta \in D_a(A)_{k_v}$  and  $x_0 \in A(k_v) - |\Delta|$ , the map

$$x \mapsto \langle \Delta, (x) - (x_0) \rangle_{v, \text{Neron}}$$

is bounded on every  $v$ -bounded subset of  $A(k_v) - |\Delta|$ .

(Here  $v$ -bounded subset means subset of a coordinate neighborhood on which  $v$  (coordinate functions) are bounded.)

We show that the pairing  $(\Delta, \mathfrak{A}) \mapsto \langle \Delta, \mathfrak{A} \rangle_v$  satisfies condition (1)–(5), except that (4) will be proven only for  $a \in N^0(\mathcal{O}) \subset A(k)$ .

(2.12) **Lemma.**  $\langle \cdot \rangle_v$  satisfies (3).

*Proof.* Let  $\Delta = (f)$ . Then  $X_\Delta \cong \mathbf{G}_m \times N^0$  and

$$\begin{aligned} \sigma_\Delta: A - |\Delta| &\rightarrow \mathbf{G}_m \times A \\ \sigma_\Delta(a) &= (f(a), a). \end{aligned}$$

Since  $\psi_\Delta = v$  on  $\mathbf{G}_m(k_v) = k_v^*$ , the lemma follows. Q.E.D.

(2.13) **Lemma.**  $\langle \cdot \rangle_v$  satisfies (2), i.e., it is bilinear.

*Proof.* Bilinearity in  $\mathfrak{A} \in Z_k(A)$  holds by definition. We must show

$$\langle \mathfrak{A}, \Delta_1 \rangle + \langle \mathfrak{A}, \Delta_2 \rangle = \langle \mathfrak{A}, \Delta_1 + \Delta_2 \rangle$$

whenever  $\Delta_i \in D_a(A)$  and  $|\mathfrak{A}| \cap (|\Delta_1| \cup |\Delta_2|) = \emptyset$ . Note that  $\sigma_{\Delta_1 + \Delta_2}$  can be taken to be the “sum” in the sense of torseurs of  $\sigma_{\Delta_1}$  and  $\sigma_{\Delta_2}$ , i.e., the rational section of  $\mathcal{L}_{\Delta_1 + \Delta_2}$  can be taken to be the tensor or rational sections of  $\mathcal{L}_{\Delta_1}$  and  $\mathcal{L}_{\Delta_2}$ . The diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbf{G}_m & \longrightarrow & X_{\Delta_1 + \Delta_2} & \longrightarrow & N^0 & \rightarrow 0 \\ & \uparrow & & \searrow^{\sigma_{\Delta_1 + \Delta_2}} & & \uparrow & \\ & \text{multiply} & & & & & \\ & (\sigma_{\Delta_1}, \sigma_{\Delta_2}) & & & & & \\ 0 \rightarrow & \mathbf{G}_m \times \mathbf{G}_m & \longrightarrow & X_{12} & \longrightarrow & N^0 & \rightarrow 0 \\ & \parallel & & \searrow^{\text{Cartesian}} & & \uparrow & \\ & & & & & & \text{diagonal} \\ 0 \rightarrow & \mathbf{G}_m \times \mathbf{G}_m & \longrightarrow & X_{\Delta_1} \times X_{\Delta_2} & \longrightarrow & N^0 \times N^0 & \rightarrow 0 \end{array}$$

commutes, where  $X_{12}$  is the pullback as indicated. Defining  $X_{12}^1$  in the same way as  $X_\Delta^1$  above, one finds

$$(X_{12}(k)/X_{12}^1) \otimes \mathbf{Q} \cong [(k^*/\mathbf{G}_m^1) \times (k^*/\mathbf{G}_m^1)] \otimes \mathbf{Q}$$

and the map  $X_{12}(k)/X_{12}^1 \rightarrow X_{\Delta_1 + \Delta_2}(k)/X_{\Delta_1 + \Delta_2}^1$  corresponds to addition on  $k^*/\mathbf{G}_m^1$ . The assertion of the lemma now follows. Q.E.D.

(2.14) **Lemma.** Let  $a \in N^0(\mathcal{O}) \subset A(k)$ . Then  $\langle \Delta, \mathfrak{A} \rangle = \langle \Delta_a, \mathfrak{A}_a \rangle$ .

*Proof.* Let  $\delta_a: N^0 \rightarrow N^0$  be translation by  $a$ . There is a map of  $G_m$ -torsors  $\tau_a: X_\Delta \rightarrow X_{\Delta_a}$  such that the diagram

$$\begin{array}{ccc}
 X_\Delta & \xrightarrow{\tau_a} & X_{\Delta_a} \\
 \sigma_\Delta \swarrow & & \searrow \sigma_{\Delta_a} \\
 A - |\Delta| & \xrightarrow{\delta_a} & A - |\Delta_a| \\
 \swarrow & & \searrow \\
 A & \xrightarrow{\delta_a} & A
 \end{array}$$

commutes for suitable choice of  $\sigma_\Delta, \sigma_{\Delta_a}$ .

The key point is that we may choose  $\tau_a$  such that  $\tau_a(X_\Delta^1) \subset X_{\Delta_a}^1$ . This is clear in the non-archimedean case because  $X_\Delta^1 = X_\Delta(\mathcal{O})$  and it suffices to take  $\tau_a$  defined over  $\mathcal{O}$ . In the archimedean case, choose  $\tilde{a} \in X_{\Delta_a}^1$  lying over  $a$  and consider the composition

$$\begin{array}{ccccc}
 X_\Delta(k) & \xrightarrow{\tau_a} & X_{\Delta_a}(k) & \xrightarrow{\delta_{-\tilde{a}}} & X_{\Delta_a}(k) \\
 \downarrow & & \downarrow & & \downarrow \\
 A(k) & \xrightarrow{\delta_a} & A(k) & \xrightarrow{\delta_{-a}} & A(k)
 \end{array}$$

Modifying  $\tau_a$  by an element of  $k^*$  we may assume  $\delta_{-\tilde{a}} \circ \tau_a$  is the identity on  $k^*$ , whence an isomorphism of groups  $X_\Delta(k) \xrightarrow{\cong} X_{\Delta_a}(k)$ . Thus  $\delta_{-\tilde{a}} \circ \tau_a(X_\Delta^1) = X_{\Delta_a}^1$ .

Since  $\tilde{a} \in X_{\Delta_a}^1$  we get  $\tau_a(X_\Delta^1) = X_{\Delta_a}^1$ .

Since subtracting  $\tilde{a}$  does not change the image of a point in  $X_{\Delta_a}$  under  $\psi_{\Delta_a}$ , the above discussion actually shows that for any zero cycle  $z$  on  $X_\Delta$  defined over  $k$  we have  $\psi_\Delta(z) = \psi_{\Delta_a} \tau_a(z)$ . Thus

$$\begin{aligned}
 \langle \Delta_a, \mathfrak{A}_a \rangle_v &= \psi_{\Delta_a}(\sigma_{\Delta_a} \delta_a)(\mathfrak{A}) = \psi_{\Delta_a}(\tau_a \sigma_\Delta)(\mathfrak{A}) \\
 &= \psi_\Delta \sigma_\Delta(\mathfrak{A}) = \langle \Delta, \mathfrak{A} \rangle_v. \quad \text{Q.E.D.}
 \end{aligned}$$

(2.15) **Lemma.** *The pairing  $\langle \cdot, \cdot \rangle_v$  satisfies condition (5).*

*Proof.* The assignment  $x \mapsto \langle \Delta, (x) - (x_0) \rangle_v$  is continuous, and  $v$ -bounded sets are compact. Q.E.D.

*Proof. of (2.11).* Let  $\{\Delta, \mathfrak{A}\} = \langle \Delta, \mathfrak{A} \rangle_{v, \text{Neron}} - \langle \Delta, \mathfrak{A} \rangle_v$ . We have  $\{(f), \mathfrak{A}\} = 0$  so we may define

$$\{ \cdot \cdot \}: A'(k) \times Z_k(A) \rightarrow \mathbb{R}.$$

Let  $Z_k(A)^0 \subset Z_k(A)$  be those zero cycles  $\sum n_i(p_i)$  such that  $p_i \in N^0(\mathcal{O}) \subset A(k)$ . There is a natural surjection  $Z_k(A)^0 \rightarrow N^0(\mathcal{O})$  with kernel generated by elements  $(a_1 + a_2) - (a_1) - (a_2) + (0)$ ,  $a_i \in N^0(\mathcal{O})$ . Translation invariance implies  $\{ \cdot \cdot \}$  factors

through  $\{ \cdot \}: A'(k) \times N^0(\mathcal{O}) \rightarrow \mathbb{R}$ . The image under  $\{\Delta, \cdot\}$  of a subgroup of  $N^0(\mathcal{O})$  contained in a  $v$ -bounded neighborhood of 0 is trivial by (5). It follows that  $\{ \cdot \} = 0$ . Q.E.D.

## References

1. Lang, S.: Les formes bilinéaires de Néron et Tate. Sem. Bourbaki, no. 274, 1964
2. Manin, Ju., Zarkin, Y.G.: Heights on families of abelian varieties. Mat. Sbornik **89**, 171-181 (1972)
3. Néron, A.: Quasi-fonctions et hauteurs sur les variétés abéliennes. Annals of Math. **82**, 249-331 (1965)
4. Ono, T.: Arithmetic of algebraic tori. Ann. of Math., **74**, 101-139 (1961)
5. Ono, T.: On the Tamagawa number of algebraic tori. Ann. of Math., **78**, 47-73 (1963)
6. Sansuc, Thèse, Paris (1978)
7. Tate, J.: The arithmetic of elliptic curves. Invent. Math. **23**, 179-206 (1974)
8. Tate, J.: On the conjecture of Birch and Swinnerton-Dyer and a geometric analog. Sem. Bourbaki No. **306**, Feb. 1966
9. Tate, J.: Letter to Serre, June 21, 1968
10. Weil, A.: Adèles and algebraic groups. Institute for Advanced Study, Princeton, 1961
11. Mazur, B., Messing, W.: Universal extensions and one-dimensional crystalline cohomology. Lecture Notes in Math. No. **370**, Berlin-Heidelberg-New York: Springer 1974

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