A String Diagram Calculus for Predicate Logic and C. S. Peirce's System Beta

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Abstract

In the late nineteenth century, C. S. Peirce developed a graphical system for handling the first-order calculus of relations. This system, called Beta, is reformulated here in modern terms, using developments in categorical logic and geometrical representations of monoidal categories. First, we define the notion of (categorical predicate) theory as a particular case of Lawvere's hyperdoctrines (with Boolean fibers and satisfying the Beck-Chevalley condition), and express the syntax of first-order logic in terms of free theories on predicate languages. Observing that each (categorical predicate) theory gives rise to (and is embedded in) a monoidal 2-category of relations, the goal of the paper is to give a geometric presentation for the monoidal 2-category of relations of a free theory on a predicate language, obtaining Peirce's calculus as a result.

This geometric presentation is based on "string diagrams," akin to the Feynman diagrams in theoretical physics, and developed for calculations in general monoidal categories by Joyal-Street. The calculus of string diagrams is extended by adjoining rewrite rules, so that terms in first-order logic are represented by deformation equivalence classes of certain string diagrams, and inferences between terms are represented by certain rewrites of diagrams. The geometric calculus which so arises, which we identify with Peirce's system Beta, is shown to be sound and complete with respect to first-order logic.

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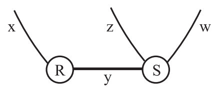
1 Introduction

Toward the end of the 19th century, the American logician C. S. Peirce developed a remarkable calculus of planar graphs for expressing logical formulae and inferences. Existential graphs, as he called them, evolved as a way of handling the "logic of relatives" (i.e., the calculus of relations) which Peirce, beginning in 1870, had invented as an outgrowth of his studies on Boolean algebra ([15]). Although little was published in Peirce's lifetime on existential graphs, he explicitly refers to this calculus as his "chef d'oeuvre," and as an outline of the logic of the future.

Existential graphs were developed in three stages of logical complexity. Alpha graphs, the initial part of his system, are geometrical representations of propositional assertions, i.e., Boolean combinations of propositional variables. (A rigorous modern formulation of system Alpha can be found in [4].) Next come Beta graphs, which geometrically represent first-order relational expressions. The basic operation, the relational product or composition, was seen by Peirce as analogous to Boolean conjunction (and so there is a close tie-in between his rules for Alpha and rules for Beta) but, as Peirce recognized, involved existential quantification as an essential component. Thus, as Peirce perceived, Beta could be used as a vehicle for expressing inferences in predicate logic, particularly those involving logical quantification, in the same way that first-order relational calculus adequately captures the language of first-order logic. Finally, there are Gamma graphs, which Peirce never brought to fruition and indeed were highly speculative by his own admission. Our reading is that Peirce was hinting at ideas found in modal logic, temporal logic, and "variable sets" (toposes), and that he suggests possibilities for "geometrizing" higher-order logic, but clarification of these points will have to await a future paper.

A remarkable feature about Beta graphs, the focus of the present paper, is their metaphorical background: Peirce, guided by analogies with chemistry, pictured an n-ary relation as something like an element with "valency n." Composition of relations R and S is then pictured as a "bonding" between two or more elements, as between atoms in a molecular configuration. For instance, if R is a predicate expression with free variables x, y, and S is a predicate expression with free variables y, z, w, then the relational composite expressed by $\exists_y R(x,y) \wedge S(y,z,w)$ is pictured by drawing a "line of identity" connecting R to S:



This metaphor of chemical bonding accords well with modern logical terminology, where we say for example that the variable y is "bound" in the composite expression.

Now, with a different set of scientific contexts in mind (namely, the use of graphs in particle physics and relativity à la Feynman and Penrose), Joyal and Street ([10]) have introduced a mathematical theory of so-called string diagrams, in order to make rigorous the use of such diagrams in generalized tensor calculus. The idea is that the lines or edges in a string diagram represent linear spaces (e.g., Hilbert space representations of Lie groups), and nodes where lines meet represent intertwining operators. (Thus an operator of the form

$$V_1 \otimes \ldots \otimes V_m \longrightarrow W_1 \otimes \ldots \otimes W_n$$

is represented by a node incident to m edges coming from above and n edges from below.) As Joyal and Street show, the appropriate language for formalizing string diagrams is monoidal (or tensor) category theory, whereby string diagrams are viewed as morphisms in a freely generated monoidal category.

The basic insight behind our approach to Peirce's system Beta is that the calculus of relations is profitably viewed as a certain type of monoidal category (or rather a monoidal bicategory; cf. [6]), where relations are certain morphisms between sets which compose by relational composition. (More exactly, we deal with formal relations viewed as arrows between types, working in a typed categorical representation of first-order logic.) Then, applying the machinery of string diagrams to present such monoidal categories, the graphs that result are essentially the same as Peirce's Beta graphs. The details of this approach require several ingredients not found in the string diagram calculus of [10], such as surgeries on string diagrams, and the use of Peirce's "sep lines" to handle logical negation.

This paper is organized as follows. In section 2, we give a purely categorical formulation of first-order logic based on Lawvere's notion of hyperdoctrine ([11]; we need a particular variant of hyperdoctrines: where the base need only be cartesian, not cartesian closed, and the fibers are Boolean algebras, and such that the Beck-Chevalley condition is satisfied). This gives a presentation of first-order logic which avoids the usual syntactic machinery of variables, in terms of a hyperdoctrine freely generated from a predicate language. In section 3, we sketch part of the construction of this free hyperdoctrine, emphasizing that it is embedded in a formal calculus of relations which is constructed geometrically as the paper progresses. In section 4, we introduce a variant of Joyal-Street string diagrams, and more especially the notion of surgery needed to give geometric presentations of monoidal categories. We use string diagrams and surgeries to construct the base category of the free hyperdoctrine, as well as relations which reflect the adjoint relationship between re-indexing and quantification, together with the Beck-Chevalley condition. In section 5, we "fold in" the rules of Peirce's system Alpha, which expresses the Boolean algebra structures on the fibers; this completes our description of Peirce's Beta. In section 6, we prove our main theorem: that the monoidal bicategory which was constructed geometrically in the preceding sections is isomorphic to the monoidal bicategory of relations generated by the free hyperdoctrine which represents first-order logic (relative to a given predicate language). That is to say, that Beta is sound and complete with respect to first-order logic.

2 Languages and theories

Definition 1 *A (predicate) language consists of a set of predicates P, a set of sorts S, and a typing function $\tau : P \to S^*$, where S^* is the free monoid on S.*

Viewing a set S as a discrete category and the free monoid on S^* as a monoidal category, a typing function $\tau : P \to S^*$ amounts to a functor $\lambda : S^* \to \mathcal{S}et$. To get λ from τ, define $\lambda(w)$ as the set $\tau^{-1}(\{w\})$. To get τ from

λ, define P as the disjoint sum $\Sigma_{w \in S^*} \lambda(w)$, and if $p \in P$, define $\tau(p)$ as w whenever $p \in \lambda(w)$. In what follows, we usually view a language as a pair $(S, \lambda : S^* \to \mathcal{S}et)$.

Definition 2 *A morphism or translation of languages $(S, \lambda) \to (S', \lambda')$ consists of a function $f : S \to S'$ together with a natural transformation $\theta : \lambda \to \lambda' f^*$.*

Equivalently, a morphism of languages $(S, P, P \xrightarrow{\tau} S^*) \to (S', P', P' \xrightarrow{\tau'} (S')^*)$ consists of a pair of functions $f : S \to S'$, $\phi : P \to P'$ compatible with the typing functions: if p has type $\langle s_1, \ldots, s_n \rangle$, then $\phi(p)$ has type $\langle fs_1, \ldots, fs_n \rangle$.

As an example of a morphism, consider a set-theoretic model of a predicate language. A model assigns to each sort $s \in S$ a set, say $F(s)$. This gives a function $F : S \to Ob(\mathcal{S}et)$, from S to the class of sets, also denoted $\mathcal{S}et_0$. The model then assigns to each predicate p of type $\langle s_1, \ldots, s_n \rangle$ a subset $\theta(p)$ of the product $Fs_1 \times \ldots \times Fs_n$. Now let Λ be the composite functor given by

$$\mathcal{S}et_0^* \xrightarrow{\Pi} \mathcal{S}et \xrightarrow{P} \mathcal{S}et,$$

where the first map sends a list of sets $S_1, \ldots, S_n$ to their cartesian product and the second map is the (contravariant) power set functor: this gives a large predicate language $(\mathcal{S}et_0, \Lambda : \mathcal{S}et_0^* \to \mathcal{S}et)$. Then a set-theoretic model of a language (S, λ) may be described as a morphism $(F, \theta) : (S, \lambda) \to (\mathcal{S}et_0, \Lambda)$ into the language of sets.

The language of sets carries a lot of extra structure, such as the Boolean algebra structures on power sets. A certain amount of this structure suffices to model the semantic aspects of first-order logic. We abstract this structure in the following definition, based on Lawvere's theory of hyperdoctrines [11]:

Definition 3 *A (categorical predicate) theory consists of a category C with finite products, together with a contravariant functor $T : C \to \mathcal{B}ool$ (or a covariant functor $C^{op} \to \mathcal{B}ool$) mapping to the category of Boolean algebras, such that*

(1) For each morphism $f : A \to B$ in C, $f^* = T(f) : TB \to TA$ has a left adjoint $\exists_f : TA \to TB$;

(2) Given a pullback

$$\begin{array}{ccc} P & \xrightarrow{k} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

in C, the following diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{k^*} & TP \\ {\scriptstyle \exists_f}\downarrow & & \downarrow{\scriptstyle \exists_h} \\ TC & \xrightarrow[g^*]{} & TB. \end{array}$$

The archetypal example of a theory is the theory of sets $(\mathcal{S}et, P : \mathcal{S}et^{op} \to \mathcal{B}ool)$ in which we regard the contravariant power set functor as valued in the category of Boolean algebras. In this case, $Pf : PB \to PA$ sends a subset $D \subseteq B$ to its inverse image $f^{-1}(D) \subseteq A$; the left adjoint $\exists_f : PA \to PB$ corresponds to taking direct images along f. The adjunction of (1) says

$$(*) \qquad f(C) \subseteq D \text{ if and only if } C \subseteq f^{-1}(D).$$

As is well known, direct images model existential quantification: if, for example, $\pi : A \times B \to B$ is projection to the second factor, and if $C \subseteq A \times B$ is the extension of a predicate $p(a, b)$ relative to some set-theoretic model, then the image

$$\pi(C) = \{b \mid \exists a \in A : p(a, b)\}$$

is the extension of the formula $\exists_a p(a, b)$. In order to capture the taking of direct images in Boolean algebras more general than power sets, one rewrites $(*)$ as

$$\exists_f C \leq D \quad \text{if and only if} \quad C \leq f^* D,$$

and so we require that each $f^* = Tf$ have a left adjoint $\exists_f$. It follows that universal quantification, defined by $\forall_f = \neg\exists_f\neg$, is right adjoint to f^*.

Condition (2), called the *Beck-Chevalley condition*, also holds in the theory of sets (as the reader can easily verify) and figures prominently in type-theoretic versions of first-order categorical logic. To translate between the version given here and standard presentations of first-order logic, the reader should bear in mind that here we never write an inference $p \vdash q$ between p, q unless they have the same *type*. At the semantic level, this ensures that their extensions $E(p)$ and $E(q)$, relative to a given model, belong to the same Boolean algebra, so that $E(p) \leq E(q)$ makes sense. (Syntactically, it means that p and q have the same free variables; however, in our approach, variables are not needed.) Thus, if we wish to compare the logical strengths of formulas p and q whose types do not match, we may retype them by pulling them back to expressions f^*p and g^*q over a common type T (syntactically: adjoin some dummy free variables), and compare the retyped expressions. The Beck-Chevalley condition ensures that the meaning or interpretation of quantification is preserved under such retyping operations.

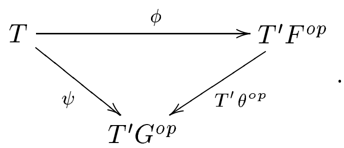
To translate between theories $(C, T : C^{op} \to \mathcal{B}ool)$, we introduce the following definition:

Definition 4 *A morphism or translation of theories $(C, T) \to (C', T')$ consists of a product-preserving functor $F : C \to C'$ together with a natural transformation $\phi : T \to T'F^{op}$ such that for all morphisms $f : A \to B$ in C, the following diagram commutes:*

$$\begin{array}{ccc} TA & \xrightarrow{\phi_A} & T'FA \\ {\scriptstyle \exists_f}\downarrow & & \downarrow{\scriptstyle \exists_{Ff}} \\ TB & \xrightarrow[\phi_B]{} & T'FB. \end{array}$$

By "product-preserving" we mean that the map $\langle F\pi_A, F\pi_B \rangle : F(A \times B) \to FA \times FB$ is an isomorphism. In our framework, products are not canonically given, but are defined, as usual, only up to isomorphism. In order to have adequate control over this feature, we need a further level of structure.

Definition 5 *Given two translations (F, ϕ), (G, ψ): $(C, T) \to (C', T')$, a modification $(F, \phi) \to (G, \psi)$ between them is a natural transformation $F \stackrel{\theta}{\to} G$ such that the following diagram commutes:*

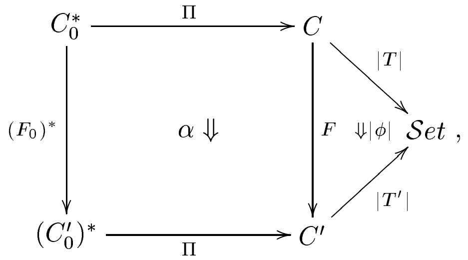


We thus have a 2-category of theories, translations between theories, and modifications between translations.

Each theory (C, T) has an underlying language, constructed as follows. The set of sorts of the language is the set of objects $C_0 = Ob(C)$. The functor $C_0^* \to \mathcal{S}et$ is defined up to isomorphism as a composite

$$C_0^* \stackrel{\Pi}{\to} C \stackrel{T}{\to} \mathcal{B}ool \stackrel{|\,|}{\to} \mathcal{S}et,$$

where $|\,| : \mathcal{B}ool \to \mathcal{S}et$ is the underlying-set functor and Π sends a finite list of objects in C to their product. Of course, there is no unique choice of product, so in general there are many underlying languages of a theory, but they are all canonically isomorphic. Thus, it is harmless to suppose that we have chosen an underlying language UT for every theory T. If $(F, \phi) : (C, T) \to (C', T')$ is a theory morphism, then we have a diagram



where $\alpha : F\Pi \stackrel{\sim}{\to} \Pi F_0^*$ is a canonical isomorphism which arises by virtue of preservation of products by F. Hence, the pasting of the 2-cells,

$$|T'|\alpha \cdot |\phi|\Pi : |T|\Pi \to |T'|\Pi F_0^*,$$

gives an underlying morphism between underlying languages, so we have an underlying functor

$$U : \mathcal{T}h \to \mathcal{L}ang$$

from the category of theories to the category of languages.

The syntax of first-order logic may be described as the construction of a theory FL which is freely generated from a language L. This means that given a modeling $\mu : L \to UT$ of a language L in a theory T, one can extend to a translation $\hat{\mu} : FL \to T$ between theories, uniquely up to invertible modification: the category of translations and modifications $\mathcal{T}h(FL, T)$ is equivalent to the discrete category or set of morphisms $\mathcal{L}ang(L, UT)$, in a 2-natural sense. The construction we give of FL avoids the usual variable-based syntactic machinery in favor of categorical and geometrically based constructions which are intimately connected with Peirce's system Beta.

3 The free theory of a language

The category of types C of FL is the "free category with finite products" generated by the set of sorts S of L, and may be described by means of "wreath products." Let $\mathcal{F}in$ be the category of finite cardinals $\{1, \ldots, n\}$ (empty if $n = 0$) and functions between them. If D is any category, then there is a category $\mathcal{F}in \int D$ whose objects are pairs $(m, \{1, \ldots, m\} \stackrel{x}{\to} D)$, where $m \in \mathcal{F}in_0$ and x is a functor on $\{1, \ldots, m\}$ as a discrete category; the morphisms $(m, x) \to (n, y)$ are pairs (f, ϕ), where $f : m \to n$ is a morphism in $\mathcal{F}in$ and $\phi : x \to yf$ is a natural transformation. An object (m, x) amounts to a list $\langle x_1, \ldots, x_m \rangle$ of objects in D, so that $(\mathcal{F}in \int D)_0 = D_0^*$; a morphism is a function f together with a list $\langle \phi_1, \ldots, \phi_m \rangle$ of morphisms in D, where $\phi_i : x_i \to y_{fi}$. In particular, for each object $\langle x_1, \ldots, x_m \rangle$ there are m morphisms $k_i : (1, x_i) \to (m, \langle x_1, \ldots, x_m \rangle)$, where the image $1 \to m$ is $\{i\}$ and where $k_i : x_i \to x_i$ is the identity. These morphisms are injections which realize $\langle x_1, \ldots, x_m \rangle$ as the coproduct $\Sigma_{i=1}^m x_i$ in $\mathcal{F}in \int D$. It is easy to show that $\mathcal{F}in \int D$ is the free category with finite sums $\Sigma(D)$ generated by D, in the sense that if E is any category with finite sums (coproducts), there is a (2-)natural equivalence

$$\mathcal{C}at(D, E) \simeq \mathcal{C}oprod(\mathcal{F}in \textstyle\int D, E)$$

between the functor category on the left and the category of coproduct-preserving functors on the right. The free category with products is similarly formed as $\Pi(D) = (\mathcal{F}in \int D^{op})^{op}$. If D is a discrete category S, then $\Sigma(S) = \mathcal{F}in \int S$ and $\Pi(S) = (\Sigma S)^{op}$. Given a predicate language $L = (S, S^* \stackrel{\lambda}{\to} \mathcal{S}et)$, the category of types of FL is defined to be $\Pi(S)$.

To complete the description of FL, we need to construct an appropriate functor $\Pi(S)^{op} \stackrel{\theta}{\to} \mathcal{B}ool$ (or $\Sigma(S) \to \mathcal{B}ool$). The reader should think of $(\Pi S)_0 =$

S_0^* as the collection of types of the free theory, and of

$$\begin{array}{ccc} \mathcal{F}orm & = & \sum_{T \in S_0^*} \theta(T) \\ \downarrow & & \downarrow {\scriptstyle \Sigma_{T\in S_0^*}!} \\ S_0^* & \cong & \sum_{T \in S_0^*} 1 \end{array}$$

as the collection of all formulas or terms in the free theory, fibered over the collection of types. $\Pi(S)$ must be allowed to act both contravariantly $\mathcal{F}orm \to S_0^*$ (by pullback operations f^*) and covariantly (by existentially quantifying), and of course locally (i.e., on a fiber $\mathcal{F}orm_T = \theta(T)$ over a type T) there must be an appropriate Boolean algebra structure.

To handle all this structure, it is useful to view the formulas p of type $\langle x_1, \ldots, x_n \rangle$ as formal n-ary relations of type $x_1 \times \ldots \times x_n$ and, following Peirce, to express the structure in terms of geometrized calculus of relations. Working in a two-sided relational calculus (e.g., of relations $R : A \to B$ between sets, where the relational composite of $A \overset{R}{\to} B \overset{S}{\to} C$ is defined by $RS(a,c) = \exists_b R(a,b) \wedge S(b,c)$), we recall that the category of relations carries a monoidal product $\otimes$. Namely, given $R : A \to B$ and $S : C \to D$, $R \otimes S : A \times C \to B \times D$ is defined by $(R \otimes S)(\langle a,c \rangle, \langle b,d \rangle) = R(a,b) \wedge S(c,d)$, and the monoidal unit is the terminal set 1.

Remark 1 If C is any regular category, then there is a monoidal category $\mathcal{R}el(C)$, where the objects are objects of C and whose morphisms $A \to B$ are relations, i.e., monic arrows $R \hookrightarrow A \times B$. In fact, this is a monoidal 2-category, since relations of type $A \to B$ may be partially ordered by inclusion. Similarly, there is a monoidal bicategory $\mathcal{S}pan(C)$ of spans in C, if C has finite limits.

Remark 2 $\mathcal{R}el(C)$ is compact closed: each functor $A \otimes -$ has a right adjoint of the form $A^* \otimes -$ (by taking $A^* = A$). We remark that $\otimes$ is *not* the cartesian product in $\mathcal{R}el(C)$ (in fact, compact closed categories and, more generally, $*$-autonomous categories whose monoidal product is the cartesian product are equivalent to posets). In fact, the cartesian product is given by taking coproducts in C, if and only if C is a lextensive category [5]).

Our strategy for constructing the theory FL on a language L (with S as set of sorts) will be as follows. We build a monoidal (2-) category of relations where the set of objects or 0-cells is S^*, generated from predicates $p \in \lambda(T)$ in L (viewed as morphisms [i.e., relations] $p : T \to 1$ to the unit 1 of S^*) and monoidal subcategories $\Pi(S)$ and $\Sigma(S)$. The theory FL is retrieved as follows. Formulas of type T in FL will be defined as morphisms $T \to 1$. Pullback operations and quantifications will then be defined as special cases of relational composition. Conjunction of two formulas $R : A \to 1$ and $S : A \to 1$ is defined as the composite

$$A \overset{\delta}{\to} A \otimes A \overset{R \otimes S}{\to} 1 \otimes 1 \cong 1$$

in which we pull back along the diagonal $\delta : A \to A \times A$ in $\Pi(S)$.

The desired monoidal category is formed by extending the method of string diagrams and their deformations, used in [11]. The extension is twofold. First, we give presentations of monoidal categories by means of *surgery rules* on string diagrams ([3]). Second, to incorporate negation operations, we follow Peirce and introduce so-called "sep lines": simple closed curves which surround subdiagrams of string diagrams (and regarded as negating these subdiagrams when viewed as subexpressions). The complete collection of deformations and surgery rules on diagrams is the geometric correlate of the rules of inference in the first-order relational calculus, and may be viewed as a modern formulation of Peirce's system Beta.

4 The geometry of positive formulas

In this section, we begin by introducing a variant of the string diagram calculus of [10] for symmetric monoidal categories, and a notion of surgery on diagrams needed to construct presentations of (symmetric) monoidal categories. After presenting the categories $\Pi(S)$ and $\Sigma(S)$, we construct a monoidal 2-category $\mathcal{R}el_+(L)$ which gives the desired relational calculus for positive first-order formulas. Throughout this section we assume familiarity with the terminology and results of [10], including the notions of topological graph with boundary and of tensor scheme.

4.1 Permutative diagrams

Definition 6 *A permutative diagram consists of a topological graph G with boundary $\partial G \subseteq G_0$ (i.e., a subset of G_0 consisting of nodes with valency 1) together with a continuous map $G \xrightarrow{\phi} [a_0, b_0] \times [a_1, b_1]$ such that*

(1) $\phi^{-1}(\{a_0, b_0\} \times [a_1, b_1])$ is empty;

(2) $\phi^{-1}([a_0, b_0] \times \{a_1, b_1\}) = \partial G$;

(3) The composite $G - G_0 \hookrightarrow G \xrightarrow{\phi} [a_0, b_0] \times [a_1, b_1] \xrightarrow{\pi_2} [a_1, b_1]$, when restricted to each connected component of $G - G_0$, is a smooth embedding;

(4) $\phi(x) = \phi(y) \Rightarrow x = y$ holds for $x \in \partial G$ and for all but finitely many $x \in G$.

These axioms have the following consequences: by (1) and (2), for each $x \in G_0 - \partial G$, $\phi(x) = (a, b)$ is interior to the rectangle $[a_0, b_0] \times [a_1, b_1]$. Let $I = G_0 - \partial G$ be the set of "interior" nodes. If K is a connected component of $G - I$, then by (3), K is a closed, open, or half-open line segment. If $x \in I$ is an endpoint of (the closure of) K, then $\phi(K)$ lies entirely above or entirely below $\phi(x)$, again by (3). We refer to the conditions of the preceding sentence by saying K lies above/below x ($\in K$). In that case, if $\phi(x) = (a, b)$, then for all sufficiently small $\epsilon > 0$, the line $y = b \pm \epsilon$ intersects $\phi(K)$ exactly once. Finally, by (4), if we consider the set of components K_i such that $x \in \partial K_i$ and K_i lies above/below x, then there exists $\epsilon > 0$ so small that for $0 < \delta < \epsilon$, $\phi(K_i) \cap \phi(K_j) \cap ([a_0, b_0] \times \{b \pm \delta\})$ is empty whenever $i \neq j$. Thus, for $x \in I$ there is a left-to-right order α_x/β_x on those components K_i lying above/below x, given by the order of the horizontal (abscissa) coordinates of $\phi(K_i) \cap ([a_0, b_0] \times \{b \pm \delta\})$.

Definition 7 *Let $T = (A, S, A \overset{\langle\sigma,\tau\rangle}{\to} S^* \times S^*)$ be a tensor scheme; let $D = (G, \partial G, \phi)$ be a permutative diagram, with $I = G_0 - \partial G$. A labeling of D in T consists of a pair of functions $\lambda_1 : I \to A$, $\lambda_2 : \pi_0(G - I) \to S$ such that for each $x \in I$, $\sigma\lambda_1 x = \langle \lambda_2 K_1, \ldots, \lambda_2 K_m \rangle$, where $\langle K_1, \ldots, K_m \rangle$ is the order α_x, and $\tau\lambda_1 x = \langle \lambda_2 L_1, \ldots, \lambda_2 L_n \rangle$, where $\langle L_1, \ldots, L_n \rangle$ is the order β_x.*

Each T-labeled permutative diagram is described by a quadruple $(\phi, \lambda_1, \lambda_2, \pi)$, where $\phi : G \to \mathbf{R}^2$ is a continuous map, $\lambda_1 : I \to A$ is a node-labeling function, $\lambda_2 : \pi_0(G - I) \to S$ is a string-labeling function, and π is a poset structure on the set of connected components of $G - I$ (given by taking the disjoint sum of the orders α_x, β_x, where x ranges over I). Thus, given a tensor scheme T and a graph with boundary $(G, \partial G)$, the set $\Lambda(G, \partial G; T)$ of all T-labeled permutative diagrams over $(G, \partial G)$ may be topologized as a subspace of

$$\mathrm{Map}(G, \mathbf{R}^2) \times A^I \times S^{\pi_0(G-I)} \times \mathrm{Pos}(\pi_0(G - I)),$$

where $\mathrm{Map}(G, \mathbf{R}^2)$, the set of continuous maps $\phi : G \to \mathbf{R}^2$, is given the compact-open topology, whereas A^I, the set of functions $I \to A$, $S^{\pi_0(G-I)}$, and $\mathrm{Pos}(\pi_0(G - I))$, the set of poset structures on $\pi_0(G - I)$, are given the discrete topologies.

Definition 8 *A deformation of T-labeled permutative diagrams is a path $\gamma : I \to \Lambda(G, \partial G; T)$.*

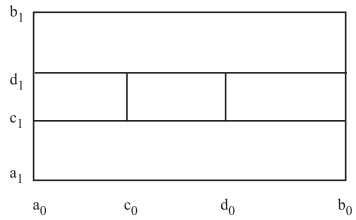
A deformation is thus a homotopy through T-labeled diagrams, where the node labelings and string labelings remain constant, as do the linear orders on the source and target strings of nodes $x \in I$. The left-right orders on points in $\gamma(t)(\partial G) \cap ([a_0, b_0] \times \{b_1\})$ and in $\gamma(t)(\partial G) \cap ([a_0, b_0] \times \{a_1\})$ remain constant as well by condition (4). Defining T-labeled diagrams to be deformation equivalent if there is a path between them in $\Lambda(G, \partial G; T)$, each deformation-equivalence class has a well-defined domain and codomain in S^*, defined as the ordered lists of labels of strings which abut these respective ordered lists of points.

Proposition 1 *(cf. [10]) Deformation-equivalence classes of T-labeled permutative diagrams are the morphisms of a symmetric strict monoidal category, naturally equivalent (as a symmetric monoidal category) to the free symmetric monoidal category on the tensor scheme T.*

To exhibit the symmetric monoidal structure, we need the following definition:

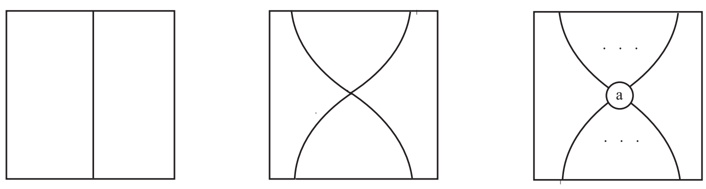
Definition 9 *Let $\phi : G \to [a_0, b_0] \times [a_1, b_1]$ be a permutative diagram. A subdiagram consists of the pullback of ϕ along a subrectangle $[c_0, d_0] \times [c_1, d_1] \hookrightarrow [a_0, b_0] \times [a_1, b_1]$ such that $\phi^{-1}(\{c_0, d_0\} \times [c_1, d_1])$ is empty and $\phi^{-1}([a_0, b_0] \times \{c_1, d_1\}) \subseteq \{x \in G - I \mid \phi(x) = \phi(y) \Rightarrow x = y\}$. The pullback or restriction is denoted res ϕ.*

It is straightforward that $H = \phi^{-1}([c_0, d_0] \times [c_1, d_1])$, with $\partial H = \phi^{-1}([c_0, d_0] \times \{c_1, d_1\})$, is a topological graph with boundary, and that $\phi : (H, \partial H) \to [c_0, d_0] \times [c_1, d_1]$ is a permutative diagram. Indeed, the restriction of ϕ to each of the rectangular sectors shown defines a permutative diagram in an analogous manner:



.

If $[c_0, d_0] = [a_0, b_0]$ and $c_1 = a_1$ or $d_1 = b_1$, then there are two vertically juxtaposed permutative subdiagrams of G, and we regard G as their vertical composite (i.e., morphism composition). Similarly, if $[c_1, d_1] = [a_1, b_1]$ and $c_0 = a_0$ or $d_0 = b_0$, then there are two horizontally juxtaposed permutative subdiagrams, and G is their horizontal composite (i.e., tensor product). Given two objects $u = \langle s_1, \ldots, s_m \rangle$ and $v = \langle t_1, \ldots, t_n \rangle$, the symmetry isomorphism $u \otimes v \longrightarrow v \otimes u$ is represented by a diagram in which each string labeled s_i crosses each string labeled t_j. Up to deformation, any permutative diagram G is composed of diagrams of the following types:



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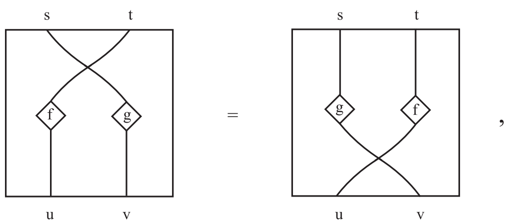
where the first diagram is a single S-labeled string, the second is a crossing of two (S-labeled) strings, corresponding to a symmetry isomorphism, and the third corresponds to an element $a \in A$ of the tensor scheme (with strings labeled appropriately by letters of $\sigma(a)$, $\tau(a)$).

In the sequel, we will use a schematic to indicate subdiagrams and their domains and codomains:

indicates a subdiagram f with domain u and codomain v; more precisely, $u, v \in S^*$ are word labels $\langle u_1, \ldots, u_m \rangle$ and $\langle v_1, \ldots, v_n \rangle$, where u_i labels the ith string from the left along the top edge of the subdiagram, etc. As an example of this usage, our notion of deformation equivalence entails equalities of the following type within a subdiagram:



where, for example, the crossing of the "strings" labeled s and t is a schematic shorthand for a multiple string crossing of multiplicity mn, if $s = \langle s_1, \ldots, s_m \rangle$ and $t = \langle t_1, \ldots, t_n \rangle$. The equality, of course, is the geometric reflection of the naturality of the symmetry isomorphism.

4.2 Surgery

Our theory also makes use of certain "surgeries" on permutative diagrams.

Definition 10 *Let $\phi : G \to [a_0, b_0] \times [a_1, b_1]$ be a permutative diagram, with subdiagram res $\phi : H \to [c_0, d_0] \times [c_1, d_1]$. Suppose $\psi : H' \to [c_0, d_0] \times [c_1, d_1]$ is a permutative diagram such that res ϕ and ψ have the same C^∞ germs on their boundaries: restrict to the same function on some neighborhood of $[c_0, d_0] \times \{c_1, d_1\}$ in $[c_0, d_0] \times [c_1, d_1]$. Define a new permutative diagram $\phi[H/H']$, with underlying graph G' defined by $(G - H) \cup H'$, $\partial G' = \partial G$, and $\phi[H/H'] : G' \to [a_0, b_0] \times [a_1, b_1]$ defined by*

$$\phi[H/H'](x) = \begin{cases} \phi(x) & \text{if } x \in G - H; \\ \psi(x) & \text{if } x \in H'. \end{cases}$$

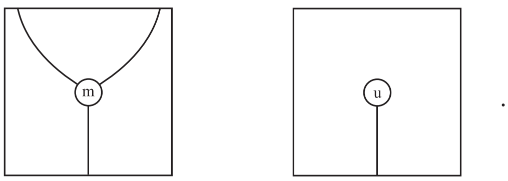
Abusing language, we call this the surgical replacement of H by H'. This construction permits consideration of equivalence relations on symmetric monoidal categories freely generated from a tensor scheme.

More formally, if $T = (A, S, A \to S^* \times S^*)$ is a tensor scheme, then a T-surgery rule is a pair of (deformation-equivalence classes of) T-labeled diagrams of the form $(\phi : H \to \mathbf{R}^2, \psi : H' \to \mathbf{R}^2)$ which have identical domain and codomain (as words in S^*). Given a set of T-surgery rules, define an equivalence relation on deformation-equivalence classes of T-diagrams, generated by the relation $G \rightsquigarrow G'$, where G' is obtained from G by surgical replacement of one T-labeled subdiagram $\phi : H \to \mathbf{R}^2$ by another $\psi : H' \to \mathbf{R}^2$, where (ϕ, ψ) is a pair of representatives of a T-surgery rule.

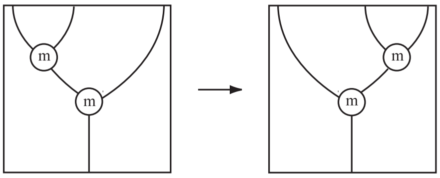
A *permutative diagram presentation* of a symmetric monoidal category M is defined to be a tensor scheme T together with a set of T-surgery rules such that the symmetric monoidal category of T-labeled permutative diagrams modulo deformation- and surgery-equivalence is isomorphic to M. In the sequel, T-surgeries are also used to construct certain symmetric monoidal categories enriched in the category of posets. Namely, if $\hom(u, v)$ denotes the set of deformation classes of T-diagrams with source u and target v, then instead of considering the equivalence relation generated by a set of T-surgeries, one may consider the poset structure on $\hom(u, v)$ induced by the reflexive transitive closure $\rightsquigarrow$ on the set of T-surgeries (identifying f and g if $f \rightsquigarrow g$ and $g \rightsquigarrow f$). The poset structure coincides with the equivalence relation in case the set of T-surgeries is already a symmetric relation on diagrams, i.e., if the inverse of every surgery rule is a surgery rule. Thus, we will sometimes stipulate that certain surgery rules are *invertible*.

The same idea applies to progressive string diagrams for monoidal categories in the sense of Joyal-Street, where crossings of strings are not allowed. We now give two examples for the monoidal case.

Example 1 Let T be the tensor scheme given by $S = 1$, $A = \{m, u\}$, and $A \stackrel{\langle\sigma,\tau\rangle}{\to} S^* \times S^*$ as indicated in the string diagrams



Let R be the equivalence relation generated by the surgery moves



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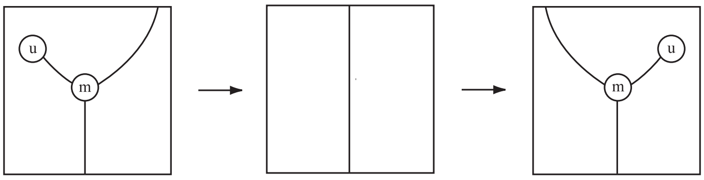

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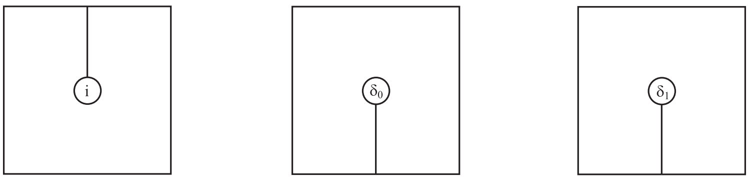
The monoidal category, generated by deformation classes of planar progressive diagrams modulo these surgery moves, is isomorphic to the simplical category $\mathbf{\Delta}$ (of finite ordinals and weakly increasing maps, including the empty ordinal).

Example 2 Let T be the tensor scheme given by $S = 1$, $A = \{i, \delta_0, \delta_1\}$, and $A \overset{\langle\sigma,\tau\rangle}{\to} S^* \times S^*$ as indicated in the string diagrams



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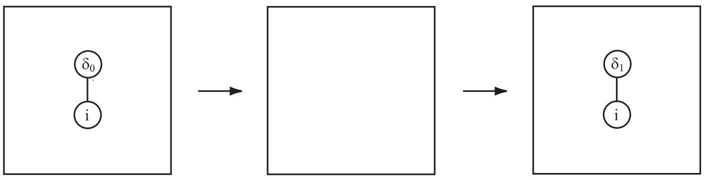
Let R be the equivalence relation generated by the surgery moves



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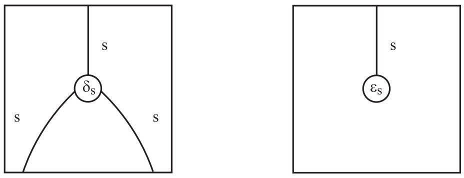
The monoidal category with presentation $\langle T, R\rangle$ is precisely the cubical category $\square$.

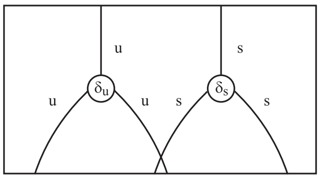
4.3 Presentation of $\Pi(S)$ and $\Sigma(S)$

Next, returning to the symmetric monoidal case, we present the category $\Pi(S)$. $\Pi(S)$ is generated by deformation classes of permutative diagrams on a tensor scheme $(A_\Pi, S, A_\Pi \overset{\langle\sigma,\tau\rangle}{\to} S^* \times S^*)$, with $A_\Pi = \{\delta_s \mid s \in S\} \cup \{\epsilon_s \mid s \in S\}$, and



to indicate the source-target function $\langle\sigma,\tau\rangle : A_\Pi \to S^* \times S^*$.

To describe the surgery moves, which are based partly on naturality requirements for diagonal maps δ and projection maps ϵ, we first need to describe derived arrows δ_w and ϵ_w for $w \in S^*$. If $u = \langle s_1, \ldots, s_n\rangle$ and if $v = us$, we define δ_v inductively as



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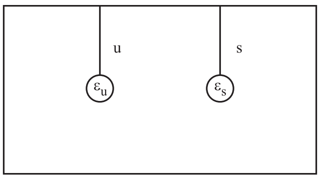
where it is understood that the source of δ_u consists of n strings with the label sequence $\langle s_1, \ldots, s_n\rangle$ and the target of $2n$ strings with the label sequence $s_1 \ldots s_n s_1 \ldots s_n$. Similarly, ϵ_v is defined inductively as



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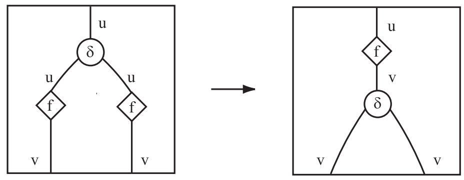
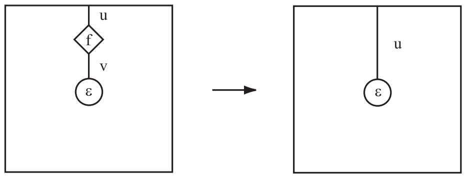
The schema for the surgery rules is as follows: given a subdiagram f, we have



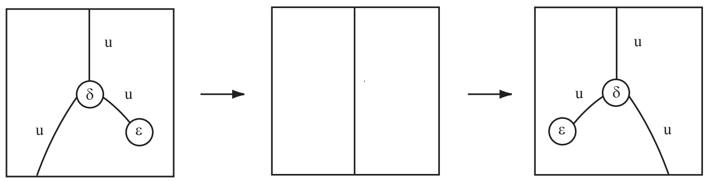
;

(S1)



;

which correspond to naturality of δ and ϵ, and we also have

(S1) 

.

which correspond to triangular equations for the adjoint pair $\Delta \dashv \Pi$, where Δ is the diagonal functor $\Pi(S) \to \Pi(S) \times \Pi(S)$ and Π is the product $\Pi(S) \times \Pi(S) \to \Pi(S)$. Displaying this set of surgery rules by (S1), we have the following theorem.

Theorem 1 *The symmetric monoidal category generated by the tensor scheme $A = \{\delta_s \mid s \in S\} \cup \{\epsilon_s \mid s \in S\}$, modulo the surgery rules (S1), is isomorphic to $\Pi(S)$.*

The opposite category $\Sigma(S)$ is obtained by inverting all diagrams of $\Pi(S)$. In practice, we drop the labels δ and ϵ since no ambiguity will result in our diagrams. Peirce calls a configuration of strings which meet at a δ a "ligature," and a string ending at an ϵ a "loose end"; we will use this terminology in the sequel.

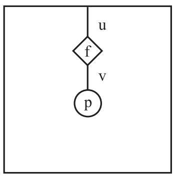
4.4 Pullbacks and quantifiers

As we remarked earlier, both $\Pi(S)$ and $\Sigma(S)$ act (contravariantly) on the set of formulas $\mathcal{F}orm(L)$ of the free theory FL of a language $L = (S, S^* \xrightarrow{\lambda} \mathcal{S}et)$. As also remarked, we follow Peirce and view all constructs of the theory (the elements of $\mathcal{F}orm(L)$, and also of $\Pi(S)$ and $\Sigma(S)$) as formal relations which can be composed horizontally (the relational or horizontal product $R \otimes S$) and vertically (the relational composite RS). The actions by $\Pi(S)$ and $\Sigma(S)$, by pulling back and existentially quantifying, emerge as operations derived from horizontal and vertical compositions, which are viewed here and by Peirce as more basic.

Thus, we will present a monoidal 2-category $\mathcal{R}el_+(L)$, taking the generating tensor scheme to be $A_\Pi \cup A_\Sigma \cup P \overset{\langle\sigma,\tau\rangle}{\to} S^* \times S^*$, where $A_\Pi \to S^* \times S^*$ and $A_\Sigma \to S^* \times S^*$ are the tensor schemes used to present $\Pi(S)$ and $\Sigma(S)$, where P is the set of predicates of L, and where $\langle\sigma,\tau\rangle(p) = (\text{type of } p, 1)$, with 1 the unit of S^*. Those (deformation classes of) permutative diagrams generated by this tensor scheme and with target 1 (i.e., the empty target) will be referred to as positive geometric formulas. The collection of diagrams with given source and target will be partially ordered, via a collection of surgery rules described under (S2) and (S3).

In addition to the surgery rules used to present $\Pi(S)$ and $\Sigma(S)$, we need surgery rules (S2) to reflect the adjoint relationship between pullback operations f^* and existential quantifiers $\exists_f$. Later in this paper, we will see that these rules are also derivable from Peirce's so-called Alpha rules.

If p is a positive geometric formula of source type v and if $f : u \to v$ is a morphism of $\Pi(S)$ viewed as a permutative diagram, we define f^*p as the diagram



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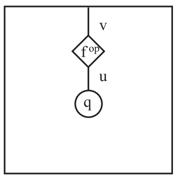
If q is of source type u and f is as before, then $f^{op} : v \to u$ in $\Sigma(S)$ is obtained by inverting the diagram for f, and we define $\exists_f q$ as



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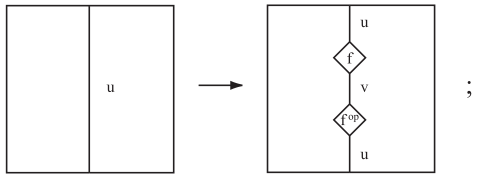
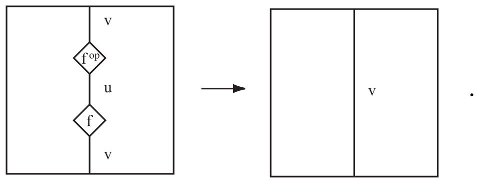
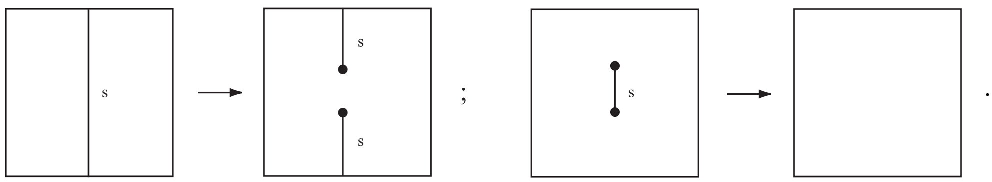
The unit and counit of the adjunction $\exists_f \dashv f^*$ are expressed by surgery rules (noninvertible in general):



(S2)



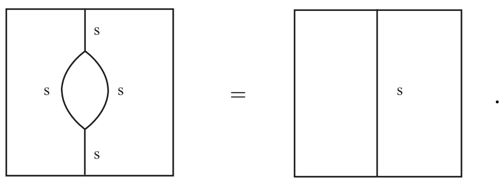
A basic example concerns existential quantification via the taking of direct images along a projection map (i.e., $\epsilon : s \to 1$ in $\Pi(S)$): the first move, for the unit $1 \to f^*\exists_f$, is already in Peirce's work and is called "breaking in positive regions":



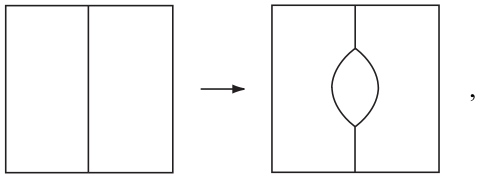
This is the case where $f = \epsilon_s : s \to e$.

Remark 3 These moves denote the unit and counit of $\exists_f \dashv f^*$ where $f = \epsilon_s : s \to 1$. Notice that we have dropped the label ϵ (as the label is superfluous). Both the unit and counit moves of the preceding example are instances of Peirce's weakening rule, given in §4.2.

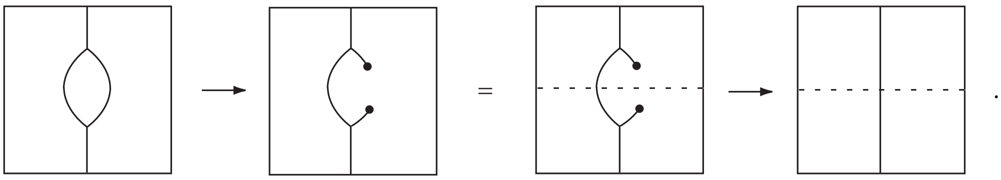
Lemma 1



Proof: The equality means that each diagram can be derived from the other through surgeries. To get from the right to the left, we apply a unit



where $f = \delta_s$. To get from left to right, apply the sequence

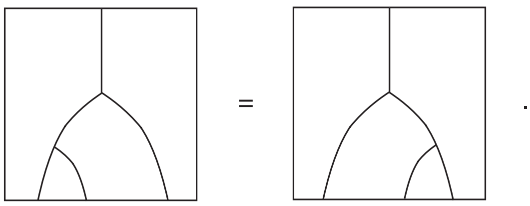


where the first step is a local breaking, the second step indicates a decomposition into subdiagrams, and the third applies surgery rules for $\Sigma(S)$ and $\Pi(S)$ to these subdiagrams.   q. e. d.

Remark 4 The equality of Lemma 1 may be viewed as an instance of Peirce's iteration rule, given in §4.2.

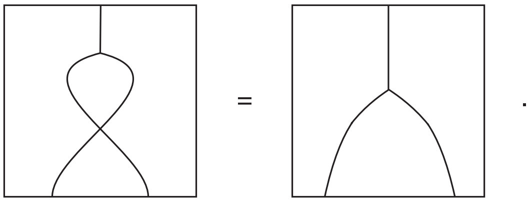
The next two lemmas follow from Theorem 1: we isolate them for future convenience.

Lemma 2



Cf. [5]: each object u has a comonoid structure. Lemma 3 asserts cocommutativity.

Lemma 3



4.5 Beck-Chevalley rules

Next, we introduce surgery rules (S3) which correspond to the Beck-Chevalley condition. These concern pullbacks in $\Pi(S)$ or pushouts in $\Sigma(S) = \mathcal{F}in \int S$; to visualize these pushouts geometrically, and ultimately to get an efficient set of rules which are necessary and sufficient for Beck-Chevalley, first we observe that a (permutative) diagram for a morphism in $\mathcal{F}in \int S$ has two ingredients:

(1) A diagram for the underlying map in $\mathcal{F}in$ (the case $S = 1$);

(2) A labeling of each connected component of a $\mathcal{F}in$-diagram by an element of S.

Condition (2) is clear: each string of a generating diagram δ_s^{op} or ϵ_s^{op} in the tensor scheme for $\Sigma(S)$ is labeled by the same element $s \in S$, and connected components of larger diagrams preserve this feature. The feature means essentially that the arguments below, written in the case for $\mathcal{F}in$, apply generally to $\mathcal{F}in \int S$. For example, since $\mathcal{F}in$ admits all pushouts, our next observation yields the following result:

Proposition 2 $\Sigma(S)$ $(\Pi(S))$ *admits all finite pushouts (pullbacks).*

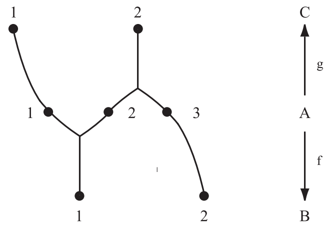
Second, we observe that a pushout P, as in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[h]{} & P \end{array},$$

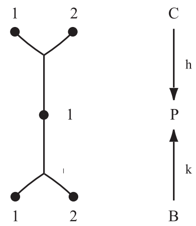
may be visualized as the set of connected components of a permutative diagram obtained by gluing together the diagrams for $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$ along the set of nodes corresponding to A. Topologically it makes no difference whether these diagrams read “top-down” or “bottom-up” (we get the same connected components regardless); we may choose, for example,

This is the diagram used for the composite operation $g^*\exists_f$, and we calculate the set of connected components of the composite diagram to get the pushout.

Example 3



has one connected component, so $P = 1$, and the remaining morphisms h, k of the pushout in $\mathcal{F}in$ are given as:



.

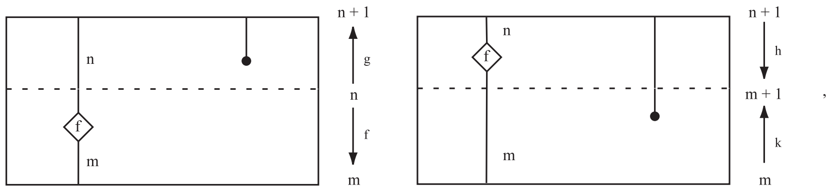
The second diagram represents the composite $\exists_h k^*$, where h and k are diagonal maps in $\Pi(S)$. Since $g^*\exists_f = \exists_h k^*$ by Beck-Chevalley, we are obliged to include a surgery rule which permits surgical replacement of each of the two preceding diagrams by the other.

To get a complete set of surgery rules which capture the Beck-Chevalley condition, we study geometric representations of pushouts in $\mathcal{F}in$ along lines similar to those of Example 3. We observe that epi-mono factorization in $\mathcal{F}in$ induces epi-mono factorization in $\mathcal{F}in \int S$, so that it suffices to analyze the possibilities in $\mathcal{F}in$ for the following three cases:

- pushing out a mono along a mono;

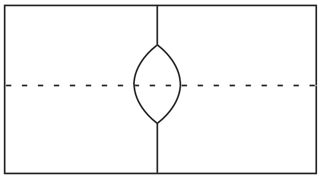
- pushing out a mono along an epi;
- pushing out an epi along an epi.

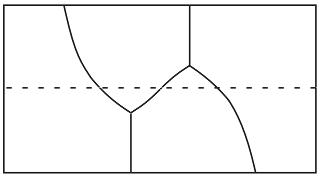
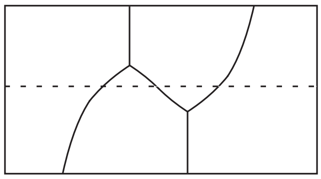
The first two possibilities are cases where Beck-Chevalley holds automatically: any monic in $\mathcal{F}in$ decomposes into a sequence of monics of type $n \to n+1$; thus, if $g : n \to n+1$ is a monic and $n \xrightarrow{f} m$ is any map in $\mathcal{F}in$, then (without loss of generality) the gluing of diagrams of f and g is represented by the left-hand diagram below:

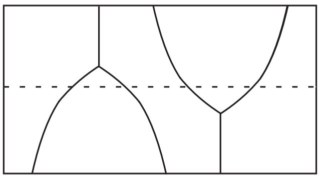
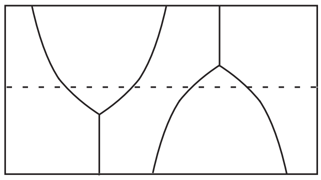


and now, in the second diagram, each of the connected components of f (one for each element in m) crosses the dotted line exactly once, so that the pushout P has cardinality $m+1$, and the Beck-Chevalley equality $g^*\exists_f = \exists_k h^*$, for g monic, follows from instances of deformation equivalence for permutative diagrams as displayed in the equation.

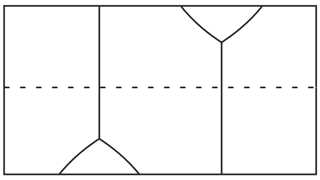
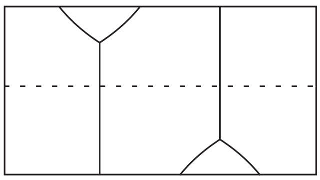
The last possibility is slightly more interesting, but simple to analyze nonetheless. Each epi decomposes into a sequence of epis of type $n+1 \to n$. In pushing out one epi of this type along another, one can arrange the diagrams (by manipulating symmetry isomorphisms and Lemma 3) until one has a subdiagram, involving a pushout of two instances of the epi $2 \to 1$ (possibly "expanded" by applications of $m \times -$ or $- \times m$), fitting into one of the following cases:

(1)  ;

(2)  or  ;

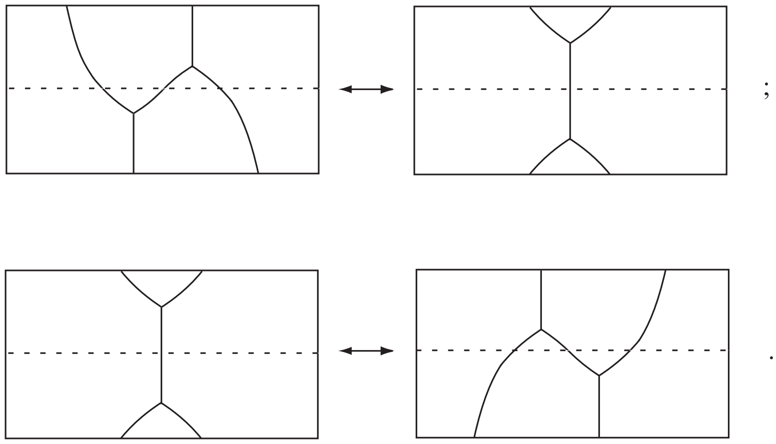
(3)  or  .

The Beck-Chevalley equality pertaining to case (1) follows from Lemma 1. For case (3), it follows from a deformation resulting in, respectively,

 or  .

This leaves case (2), discussed in the prior example. Summarizing, the Beck-Chevalley condition is equivalent to certain deformations plus (invertible) surgery moves

(S3)



Remark 5 An interesting alternative to the string diagram representations of Peirce's system Beta, as given here, involves consideration of "surface diagrams" (see [12, 1]). In this formulation, instead of representing morphisms in $\Pi(S)$ and $\Sigma(S)$ by planar immersions of one-dimensional topological graphs, one thickens the graphs out to two-dimensional surfaces with boundary (by embedding the graphs in $\mathbf{R}^3$ and taking their normal bundles; for example, the graph of δ is thickened out to a "pair of pants"). These surface diagrams give oriented cobordisms between sets of circles, and the surgery rules (S1) together with the Beck-Chevalley rules (S3) are captured precisely by cobordism equivalence. Thus, among the surgery rules discussed thus far, only the rules (S2) for adjunctions $\exists_f \dashv f^*$ change the cobordism type of such surface diagrams. Thus, if $\mathcal{C}_2$ denotes the symmetric monoidal category of two-dimensional cobordisms, then the symmetric monoidal category generated by $\Pi(S)$ and $\Sigma(S)$ (as symmetric monoidal subcategories), and subject to the Beck-Chevalley equations, may be presented as a wreath product $\mathcal{C}_2 \int S$, whose morphisms are two-dimensional cobordisms with connected components labeled in S. In the sequel, we refer to this fact by calling the rules of (S1) and (S3) **cobordism rules**.

5 Rules for first-order diagrams

In this section, we complete our formulation of Peirce's system Beta, first by expanding our class of diagrams so as to include "lines of negation," and correspondingly expanding our notion of deformation between such diagrams. Second, after giving rules of interpretation of these diagrams, we adjoin surgery rules to reflect Boolean algebra structure.

5.1 Negation

To incorporate negation, we essentially follow Peirce and introduce the following definition:

Definition 11 *A sep line (on a* $\mathcal{R}el_+(L)$*-diagram with underlying graph* $G \stackrel{\phi}{\to} [a_0, b_0] \times [a_1, b_1]$*) is the boundary of a rectangle* $[c_0, d_0] \times [c_1, d_1] \subseteq [a_0, b_0] \times [a_1, b_1]$ *such that*

(1) $\phi^{-1}(\{c_0, d_0\} \times [c_1, d_1])$ *is empty;*

(2) $\phi^{-1}([c_0, d_0] \times \{c_1\})$ *contains no node labeled by an element of* $A_{\Pi(S)} \cup P$*, and* $\phi^{-1}([c_0, d_0] \times \{d_1\})$ *contains no node labeled by an element of* $A_{\Sigma(S)} \cup P$*.*

A *first-order diagram* consists of a $\mathcal{R}el_+(L)$-diagram together with a finite collection of nonintersecting sep lines. A first-order form (or *form* for short) is a first-order diagram with empty target. A *first-order subdiagram* of a first-order diagram D consists of a subdiagram $\phi : H \to [c_0, d_0] \times [c_1, d_1]$ of the underlying $\mathcal{R}el_+(L)$-diagram of D (such that $[c_0, d_0] \times [c_1, d_1]$, relative to any given sep line of D, contains the sep line or is exterior to it), together with all of the sep lines contained in $[c_0, d_0] \times [c_1, d_1]$. A *subform* is such a subdiagram with empty target.

The intention is that a sep line negates the region it encloses. Keeping the notation used in the preceding definition, the rectangle $[c_0 + \delta, d_0 - \delta] \times [c_1 + \delta, d_1 - \delta]$ defines a subdiagram for all sufficiently small δ, and under some model $(S, S^* \stackrel{\Lambda}{\to} \mathcal{S}et) \to (\mathcal{S}et_0, \mathcal{S}et_0^* \stackrel{\Lambda}{\to} \mathcal{S}et)$, the subdiagram is interpretable (independently of sufficiently small δ) as a relation $R \hookrightarrow A \times B$ on sets A and B, where A is the interpretation of the source of the subdiagram, and B of its target. The semantics of placing a sep line around such a region, as in the prior definition, is to take the complement of R in $A \times B$. "Sepping" can be iterated, where one sep line is interior to another; if there are no nodes between two such sep lines, then the appropriate semantics of multiple seppings is the iterating of negations or complementations.

A further intention is that there is a space of first-order diagrams on a graph G labeled in a tensor scheme T (topologized as a subspace

$$\Delta(G, \partial G; T) \hookrightarrow \mathrm{Map}(G, \mathbf{R}^2) \times A^I \times S^{\pi_0(G-I)} \times \mathcal{P}os(\pi_0(G - I)) \times \exp(\mathbf{R}^4),$$

where the last factor denotes the free commutative topological monoid on the space of quadruples (c_0, d_0, c_1, d_1) representing sep lines), and that the logical interpretation of a first-order diagram is invariant under deformation (i.e., along a path in the space of first-order diagrams).

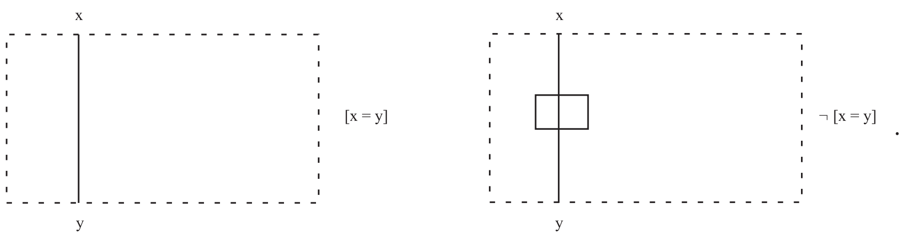
Definition 12 *A deformation of* T*-labeled first-order diagrams is a path* $\gamma : I \to \Delta(G, \partial G; T)$*.*

In practice, this means that the top of the sep line in a diagram undergoing deformation is allowed to pass back and forth over nodes labeled as δ, ϵ in $A_{\Pi(S)}$, and over "crossing points" (where $\phi : G \to \mathbf{R}^2$ is not injective). It corresponds to the fact that pullback operations f^*, for f a morphism in $\Pi(S)$, commute with negation $\neg$, as required in the definition of (categorical predicate) theory (where $f^* = T(f)$ is a morphism of Boolean algebras). Since the category of relations is self-dual, the bottoms of sep lines may pass back and forth across nodes labeled in $A_{\Sigma(S)}$ and over crossing points.

For the reader's convenience, we present a few examples of first-order diagrams, deformations, and their interpretations as formulas in first-order logic. In these examples, solid rectangles denote sep lines and a dotted line represents the boundary of a diagram or subdiagram, p and q are predicates or formulas labeling nodes, and the strings abutting these nodes have been labeled, not by the appropriate sorts but by variable terms of those sorts. (It is hoped that this last convention will assist the reader in making the translation into the standard variable-based notation which accompanies these diagrams, even though our approach eschews variables.)

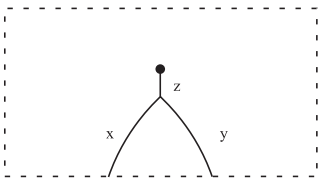
In interpreting subdiagrams, distinct variables are assigned to each sort of the source and each sort of the target, and are then assigned to each string whose closure is interior to the subdiagram; two distinct variables are explicitly asserted to be equal if they label the same strings, or strings which meet at a ligature. Horizontal juxtapositions of subdiagrams are interpreted as conjunctions. In interpreting vertical composites, variables which label strings which meet at a boundary between two subdiagrams are identified (i.e., should be literally the same).

Example 4



In the first diagram, the source variable x and the target variable y are asserted to be equal; in the second diagram, the equality is negated by the sep line.

Example 5

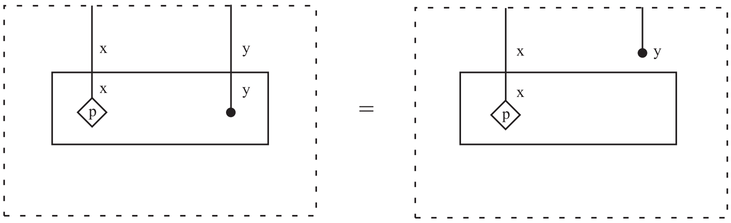


$$\exists z[z = x \wedge z = y].$$

As in Example 4, this is also an equality predicate (with two free variables x, y). However, Example 4 should be interpreted as an identity arrow $X \to X$ in the monoidal category of relations; Example 5, as an arrow of the form $1 \to X \times X$ mated with the identity $X \to X$.

In the next example, a deformation equivalence between diagrams is interpreted as an equality between formulas.

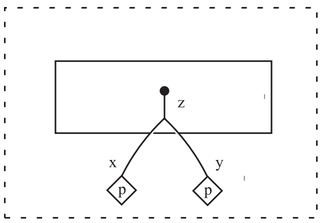
Example 6



$$\neg(p(x) \wedge [y = y]) = \neg p(x) \wedge [y = y].$$

When vertically composing two subdiagrams, variables are identified across boundaries (e.g., the instances of x or y in the example).

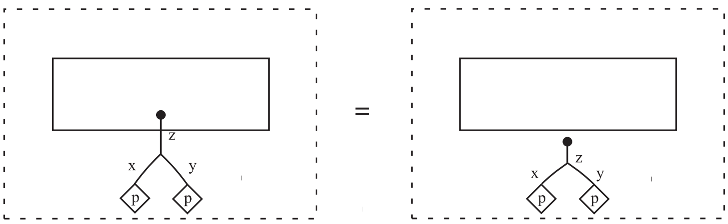
Example 7



$$\exists x \exists y \neg(\exists z[z = x \wedge z = y]) \wedge p(x) \wedge p(y).$$

This asserts that there exist two distinct elements x, y which satisfy p.

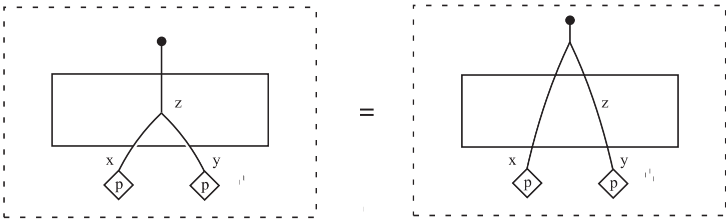
Example 8



$$\exists z \exists x \exists y (z = x \wedge z = y) \wedge p(x) \wedge q(y) \wedge \neg[z = z] =$$
$$(\exists x \exists y \exists z [z = x \wedge z = y] \wedge p(x) \wedge q(y)) \wedge \bot,$$

where $\bot$ denotes "false." Observe that the first of these expressions is self-contradictory.

Example 9

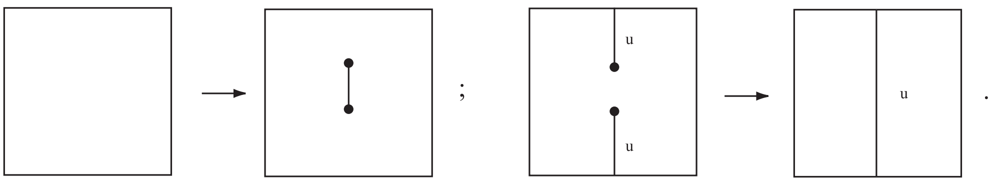


$$\exists x \exists y \exists z \neg[z = x \wedge z = y] \wedge p(x) \wedge p(y) =$$
$$\exists x \exists y \exists u \exists v \exists z [z = u \wedge z = v] \wedge \neg[u = x \wedge v = y] \wedge p(x) \wedge p(y).$$

This says that there exist two elements of a given type, and at least one of them satisfies p.

We extend the surgery rules (S1), (S2), and (S3) to the class of first-order diagrams as follows. First, such a surgery rule can only be applied to a subdiagram which, relative to any given sep line γ, is interior or exterior to γ. We say that such a subdiagram is *oddly enclosed* if it is in the open interior of an odd number of sep lines, and *evenly enclosed* otherwise. If a subdiagram is evenly enclosed, then (S1), (S2), and (S3) may be applied to it; if it is oddly enclosed, then the inverse of (S1), (S2), and (S3) may be applied to it. Since (S1) and (S3) are already invertible by definition, this convention has an effect only on the unit and counit rules (S2). However, as mentioned earlier, (S2) is derivable from the rules of "system Alpha," given below. If we drop (S2) for now and retrieve it later from Alpha, we may say that the sep-parity convention has no effect on the rules (S1), (S3) (i.e., "rules of cobordism"; cf. remark 5) given thus far.

Example 10 The following moves are valid inside a single sep line:

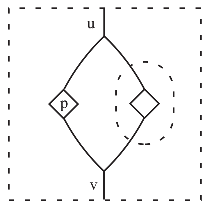


where in the graph at the beginning of the second move both strings are labeled by the same sort. Peirce would view this example as an instance of Alpha *weakening*: to oddly enclosed regions, an expression may be added (just as an inference may be weakened by adding an extra hypothesis); from evenly enclosed regions, an expression may be deleted (just as an inference is weakened by deleting a conclusion). For instance, in the second move above, one adds an assertion which equates two variables (cf. Example 4).

5.2 Boolean algebras and system Alpha

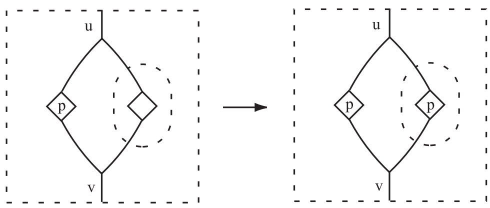
We formulate the remainder of Peirce's rules (based on his system Alpha [4]) as follows.

(1) Given a first-order diagram G and a first-order subdiagram p whose underlying $\mathcal{R}el_+(L)$-diagram is a subdiagram of a $\mathcal{R}el_+(L)$-subdiagram of G of the form



,

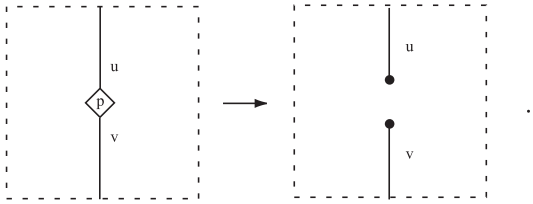
where the "empty" subdiagram D consists only of the loose ends labeled u and v, a first-order diagram deformation-equivalent to p may be (smoothly) attached to the u-string and v-string inside D, provided that D is enclosed by any sep line which encloses p:



.

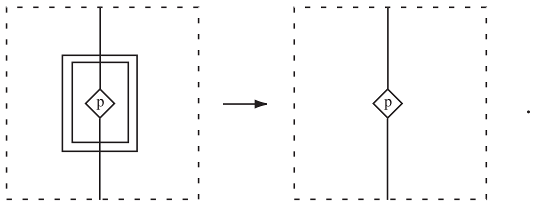
This surgery rule, called Alpha *iteration*, is invertible.

(2) The following directed move may be applied whenever a first-order subdiagram p is evenly enclosed:



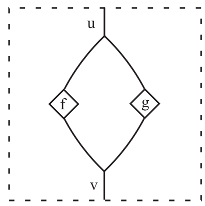
This surgery rule is reversed if p is oddly enclosed. This is the Alpha *weakening* rule.

(3) Two nested sep lines may be inserted or removed, provided that no nodes lie in the region between them, e.g.,



This invertible rule is called *double sep elimination/introduction*.

As shown in [4], the Alpha rules may be exploited to put a Boolean algebra structure on the set of (deformation classes of) first-order diagrams with given source and target. Given diagrams $f, g : u \to v$, their conjunction is

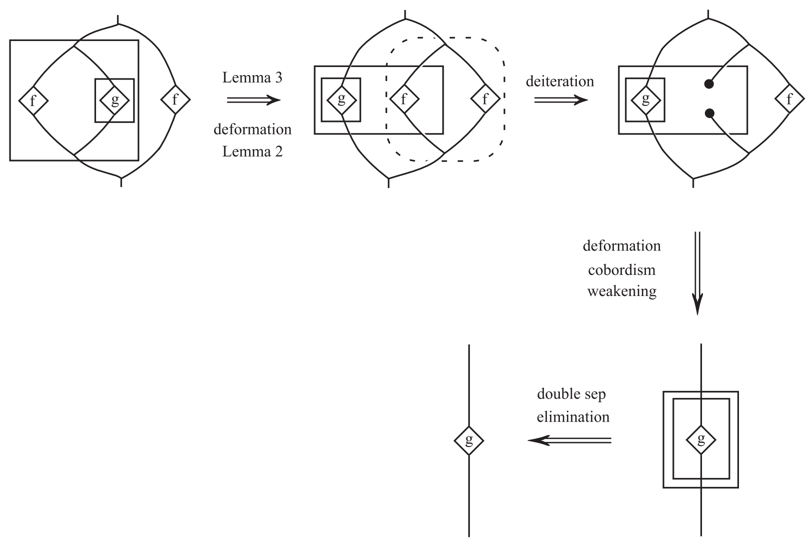


and the negation of f is given by surrounding the diagram f by a sep line.

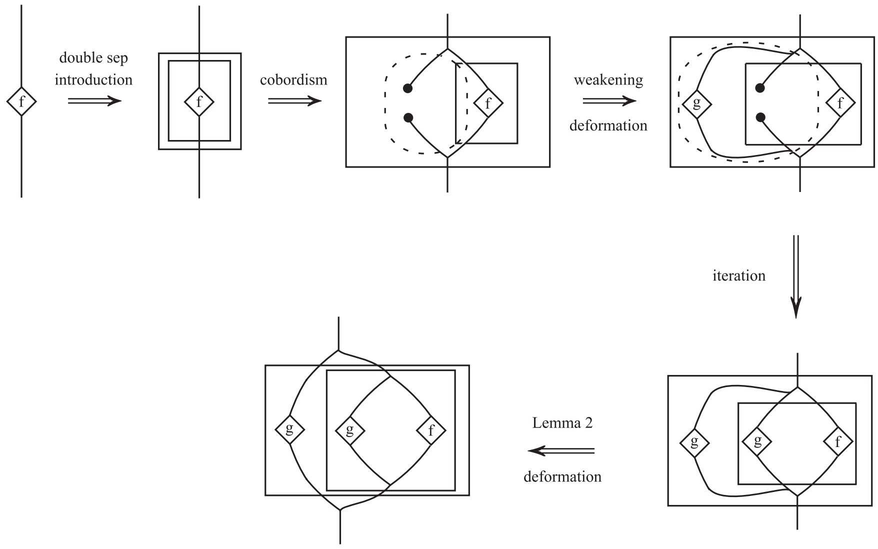
The actual case considered in [4] is, in the context of this paper, the case where one restricts to diagrams of the form $T \to 1$ which have no subdiagrams belonging to $\Sigma(S)$ (corresponding to quantifier-free formulas); we call these "propositional forms." This restriction puts some obvious restrictions on which surgeries are allowed (e.g., (S2) must be removed); the remaining set of surgeries we call propositional surgeries. Then deformation classes of propositional forms, modulo propositional surgeries, is essentially equivalent to a typed form of propositional logic. That is to say: if we define a propositional theory as in definition 3 but drop conditions (1) and (2) in that definition (i.e., if we drop quantification), and if we define a translation of propositional theories as in definition 4 but drop the quantification-preservation condition, then we have the following result:

Theorem 2 *(see [4]) Deformation classes of propositional forms on a predicate language L, modulo propositional surgery, is isomorphic (as a $\Pi(S)$-fibration) to the free propositional theory on L.*

It is straightforward to extend the methods of [4] to show that deformation classes of diagrams in $\hom(u, v)$, modulo all of the surgery rules, form a Boolean algebra. As an example, we present "modus ponens" $(f \Rightarrow g) \wedge f \rightsquigarrow g$ for $f, g \in \hom(u, v)$:



We also present the "dual" of modus ponens, $f \rightsquigarrow (g \Rightarrow (f \wedge g))$:



Similarly, it is easy to show that equivalence classes of diagrams $u \to v$ form a meet-semilattice (using iteration and weakening), a lattice (using the sep-parity convention and double-sep elimination/introduction), a Heyting algebra (using modus ponens and its dual above), and finally a Boolean algebra (Heyting algebra plus double-sep elimination/introduction).

6 Soundness and completeness of Beta

In this section, we show that the first-order relational calculus (and inferences therein) is expressed precisely by first-order diagrams and deformations and surgery rules thereon.

6.1 The monoidal 2-category $\mathcal{R}el(C, T)$

We begin by recalling the relationship between first-order logic and first-order relational calculus, whereby each theory (C, T) gives rise to a monoidal 2-category $\mathcal{R}el(C,T)$ of two-sided relations. As we shall see, each theory (C, T) can be retrieved in turn from $\mathcal{R}el(C, T)$.

For the theory of sets, $\mathcal{R}el(\mathcal{S}et, \mathcal{S}et^{op} \xrightarrow{P} \mathcal{B}ool)$ is constructed as follows. Objects of $\mathcal{R}el(\mathcal{S}et, P)$ are sets $A, B, \ldots$, and $\hom(A, B)$ is defined as $P(A \times B)$ (viewed as the set of relations from A to B). The monoidal product

$$\hom(A, B) \times \hom(C, D) \xrightarrow{\otimes} \hom(A \times C, B \times D)$$

is defined as a composite

$$P(A \times B) \times P(C \times D) \to P(A \times B \times C \times D) \xrightarrow{\sim} P(A \times C \times B \times D),$$

where the first map sends a pair of subsets to their product, and the second one arises from middle-four interchange. Composition of morphisms (i.e., relational product)

$$\begin{array}{rcl} \hom(A,B)\times\hom(B,C) & \to & \hom(A,C) \\ (R(a,b),S(b,c)) & \mapsto & \exists_b R(a,b)\wedge S(b,c) \end{array}$$

may be defined as a composite

$$\begin{array}{l} P(A\times B)\times P(B\times C) \stackrel{\otimes}{\to} P(A\times B^2\times C) \quad \stackrel{(1\times\delta_B\times 1)^*}{\to} P(A\times B\times C) \stackrel{\exists_b}{\to} P(A\times C) \\ \qquad\qquad\qquad (R(a,b),S(b',c)) \quad \mapsto R(a,b)\wedge S(b,c)\mapsto \exists_b R(a,b)\wedge S(b,c). \end{array}$$

The idea is to generalize this construction for the theory of sets to any theory (C,T), to obtain a formal calculus of relations $\mathcal{R}el(C,T)$.

Before establishing the monoidal category axioms on $\mathcal{R}el(C,T)$, we first generalize a few well-known results which concern the covariant power set functor $P:\mathcal{S}et\to\mathcal{S}et$.

Given a theory (C,T), let $\exists : C\to\mathcal{B}ool$ denote the covariant functor defined on objects by $\exists A = TA$ and on morphisms $f:A\to B$ by $\exists f=\exists_f : TA\to TB$. Let $M:C\to\mathcal{P}os$ be the composite

$$C\stackrel{\exists}{\to}\mathcal{B}ool\stackrel{|\,|}{\to}\mathcal{P}os,$$

where $|\,|$ is the underlying poset functor. We define structure maps $\eta : MX\times MY\to M(X\times Y)$ and $\tau : 1\to M1$; τ is the map from a singleton 1 which names the element "true": the maximal element in the Boolean algebra $T1$. The map η is given by a composite

$$|TX|\times|TY| \stackrel{\langle |T\pi_X|,|T\pi_Y|\rangle}{\longrightarrow} |T(X\times Y)|\times|T(X\times Y| \stackrel{\wedge}{\longrightarrow} |T(X\times Y)|.$$

Lemma 4 *The map η is natural.*

Proof: We must show that

$$\begin{array}{ccccc} TX\times TY & \xrightarrow{\pi_X^*\times\pi_Y^*} & T(X\times Y)\times T(X\times Y) & \xrightarrow{\wedge} & T(X\times Y) \\ \downarrow{\scriptstyle \exists_f\times\exists_g} & & & & \downarrow{\scriptstyle \exists_{f\times g}} \\ TX'\times TY' & \xrightarrow[\pi_{X'}^*\times\pi_{Y'}^*]{} & T(X'\times Y')\times T(X'\times Y') & \xrightarrow[\wedge]{} & T(X'\times Y') \end{array}$$

commutes, i.e., for $A\in TX$ and $B\in TY$, we have

$$\pi_{X'}^*\exists_f A\wedge\pi_{Y'}^*\exists_g B=\exists_{f\times g}(\pi_X^*A\wedge\pi_Y^*B).$$

To prove this, we apply Beck-Chevalley (BC) in conjunction with "Frobenius reciprocity (Frob),"

$$\text{(Frob)}\quad \exists_f(f^*B\wedge -)=B\wedge\exists_f -,$$

which obtains by taking left adjoints on the equation $f^*(B \Rightarrow -) = (f^*B \Rightarrow f^*-)$.

We have

$$\begin{aligned}
\pi^*_{X'}\exists_f A \wedge \pi^*_{Y'}\exists_g B &\overset{\text{BC}}{=} \exists_{f\times 1}\pi^*_X A \wedge \exists_{1\times g}\pi^*_Y B \\
&\overset{\text{Frob}}{=} \exists_{1\times g}\left((1\times g)^*\exists_{f\times 1}\pi^*_X A \wedge \pi^*_Y B\right) \\
&\overset{\text{BC}}{=} \exists_{1\times g}\left(\exists_{f\times 1}(1\times g)^*\pi^*_X A \wedge \pi^*_Y B\right) \\
&\overset{\text{Frob}}{=} \exists_{1\times g}\exists_{f\times 1}\left((1\times g)^*\pi^*_X A \wedge (f\times 1)^*\pi^*_Y B\right) \\
&\overset{\text{func}}{=} \exists_{f\times g}(\pi^*_X A \wedge \pi^*_Y B). \quad \text{q. e. d.}
\end{aligned}$$

Theorem 3 *The triple* $(M, \eta, \tau) : C \to \mathcal{P}os$ *is a lax monoidal functor.*

Proof: The statement is the conjunction of Lemma 4 together with the statement that the following diagrams commute:

$$\begin{array}{ccc}
TX \times TY \times TZ & \xrightarrow{\eta_{XY}\times 1} & T(X\times Y)\times TZ \\
{\scriptstyle 1\times\eta_{YZ}}\downarrow & & \downarrow{\scriptstyle \eta_{X,Y\times Z}} \\
TX \times T(Y\times Z) & \xrightarrow[\eta_{X\times Y,Z}]{} & T(X\times Y\times Z),
\end{array}$$

$$\begin{array}{ccc}
1\times TX & \xrightarrow{\tau\times 1} & T1\times TX \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \eta} \\
TX & \xleftarrow[M\lambda]{} & T(1\times X),
\end{array}$$

and

$$\begin{array}{ccc}
TX\times TY & \xrightarrow{\eta} & T(X\times Y) \\
{\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle T\sigma} \\
TY\times TX & \xrightarrow[\eta]{} & T(Y\times X).
\end{array}$$

The commutativity of these diagrams, which is left to the reader, is straightforward; e.g., for the first (second, third) diagram, one combines associativity (identity, symmetry, resp.) together with functoriality and the fact that maps of the form f^* preserve conjunction $\wedge$. q. e. d.

Lemma 5 *Given* $f : X \to X'$, $g : Y \to Y'$, *the following diagram commutes:*

$$\begin{array}{ccc}
TX'\times TY' & \xrightarrow{\eta} & T(X'\times Y') \\
{\scriptstyle f^*\times g^*}\downarrow & & \downarrow{\scriptstyle (f\times g)^*} \\
TX\times TY & \xrightarrow[\eta]{} & T(X\times Y).
\end{array}$$

Proof: The fact that the diagram

$$\begin{array}{ccccc}
TX' \times TY' & \xrightarrow{\pi^*_{X'} \times \pi^*_{Y'}} & T(X' \times Y') \times T(X' \times Y') & \xrightarrow{\wedge} & T(X' \times Y') \\
{\scriptstyle f^* \times g^*} \downarrow & & \downarrow {\scriptstyle (f \times g)^* \times (f \times g)^*} & & \downarrow {\scriptstyle (f \times g)^*} \\
TX \times TY & \xrightarrow[\pi^*_X \times \pi^*_Y]{} & T(X \times Y) \times T(X \times Y) & \xrightarrow[\wedge]{} & T(X \times Y)
\end{array}$$

commutes is trivial: commutativity of the left square follows from functoriality; the right square commutes because $(f \times g)^*$ is a Boolean map. q. e. d.

Now we define the monoidal 2-category $\mathcal{R}el(C, T)$. Objects of $\mathcal{R}el(C, T)$ are *types*, i.e., objects of C. Morphisms $A \stackrel{R}{\to} B$ are triples $\langle A, B, R \in M(A \times B)\rangle$; the hom set $\hom(A, B)$ is partially ordered by the relation $\leq$ coming from the Boolean algebra structure on $M(A \times B)$, and instances $R \leq S$ may be regarded as 2-cells. (Observe that 2-cell isomorphisms are equalities.) A monoidal product is defined on objects $A, B, \ldots$ by taking cartesian products in C. On morphisms, the monoidal product

$$\hom(A, B) \times \hom(C, D) \stackrel{\otimes}{\to} \hom(A \times C, B \times D)$$

is defined as the composite

$$M(A \times B) \times M(C \times D) \stackrel{\eta}{\to} M(A \times B \times C \times D) \stackrel{\sim}{\to} M(A \times C \times B \times D),$$

where the second map is M (or $|T|$) applied to a middle-four interchange. The unit of $\otimes$ is the object $1 \in C$. Composition

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C)$$

is defined by the composite

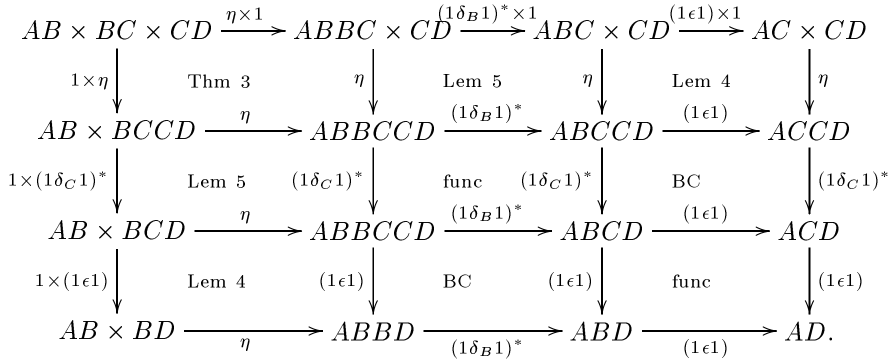
$$M(A{\times}B){\times}M(B{\times}C) \stackrel{\eta}{\to} M(A{\times}B^2{\times}C) \stackrel{(1\times\delta_B\times1)^*}{\to} M(A{\times}B{\times}C) \stackrel{M(1\times\pi_B\times1)}{\to} M(A{\times}C),$$

and the unit is the equality predicate

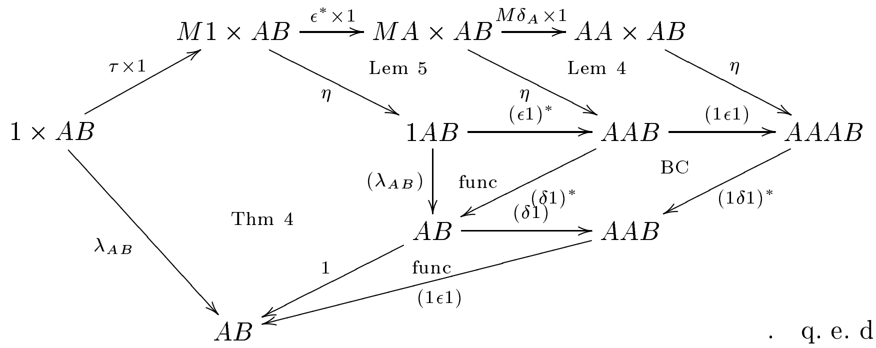
$$1 \stackrel{\tau}{\to} M1 \stackrel{\epsilon^*}{\to} MA \stackrel{M\delta_A}{\to} M(A \times A).$$

Lemma 6 *$\mathcal{R}el(C,T)$ is a (poset-enriched) category.*

Proof: In this, and in other proofs in this section, most details are left to the reader; we give a sketch as follows. To save space, we abbreviate $M(A \times B)$ to AB, and in a similar way $M(A_1 \times \ldots \times A_n)$ to $A_1 \ldots A_n$, unless $n = 1$ (where we write MA_1 instead). Associativity of vertical composition amounts to commutativity of a diagram of the form

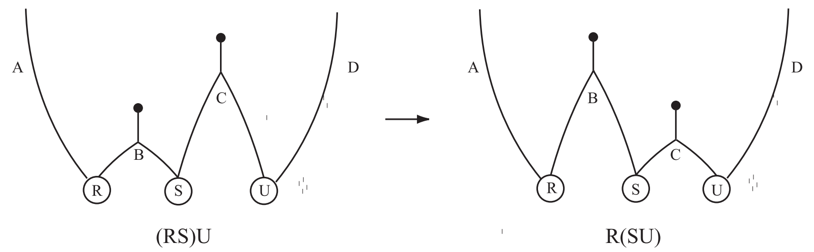


Similarly, one of the the unit laws for vertical composition follows from



. q. e. d.

Before proceeding further, we point out that both of the preceding commutative diagram proofs were discovered with the help of beta diagram moves, which were subsequently translated into commutative squares. Associativity, for example, corresponds to the move



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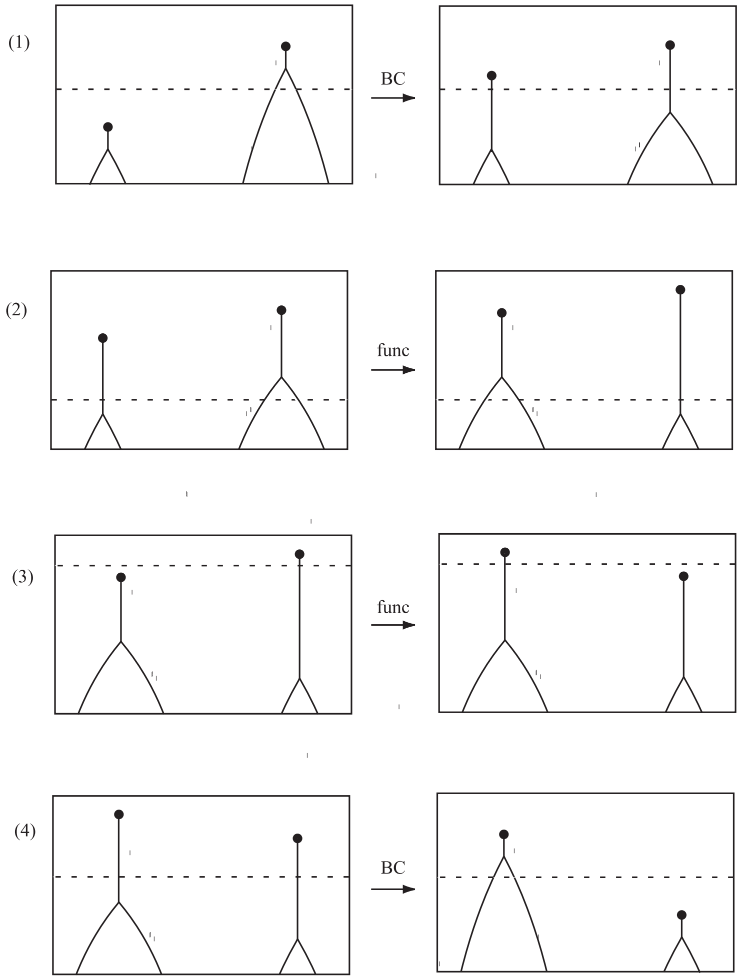
which may be decomposed into a sequence of four moves, which locally have the form:



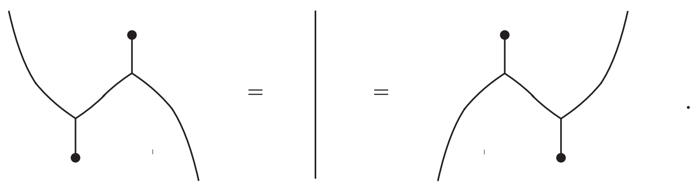
These moves are subsequently translated into the squares labeled "BC," "func" in the commutative diagram proof of associativity.

Thus, commutative diagram proofs (of this result and of others in this subsection) may be analyzed into two distinct components: one which involves instances of Theorem 3 and Lemmas 4 and 5 (and which are connected with assertions that interpretations of certain beta diagrams are independent of the order in which the diagrams are constructed), and another involving instances of functoriality of T and M and "BC" (which are connected with certain deformations and surgery rules applied to these beta diagrams).

In the spirit of the beta methodology, we therefore express proofs of certain results in this subsection using certain beta moves which are certainly shorter, and in our opinion easier to comprehend, than proofs involving large commutative diagrams. (And in these cases, the translation from Beta to commutative diagrams is straighforward enough to remove any objection that this procedure, prior to establishing that Beta is sound, is circular).

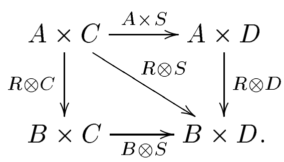
The next result expresses beta surgery equivalences which translate into parts of the diagrams for the unit laws in the course of proving Lemma 6 (parts labeled "func" and "BC").

Lemma 7

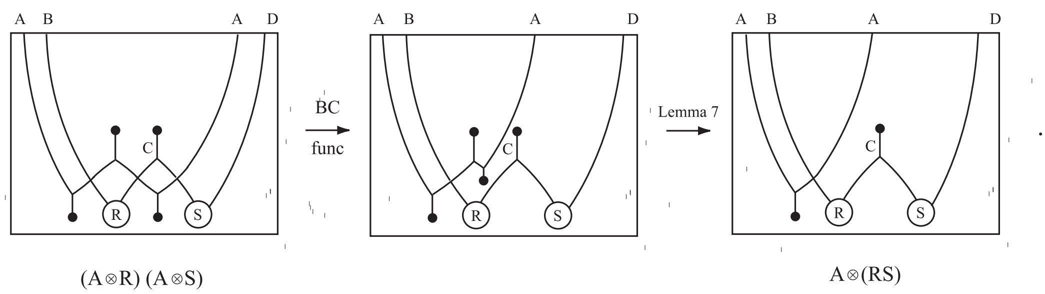


Lemma 8 *The monoidal product $\otimes$ of $\mathcal{R}el(C,T)$ is functorial with respect to vertical composition in $\mathcal{R}el(C,T)$.*

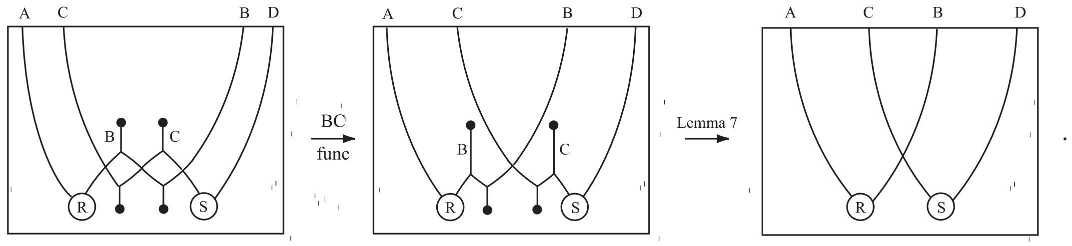
Proof: It suffices to show that (1) $A\otimes -$ and $-\otimes B$ are functorial for objects A, B; (2) the following interchange diagram commutes:



The proof that $A\otimes -$ preserves composition is given as follows:



The remaining details of (1) are left to the reader. The proof that the lower triangle of (2) commutes is given as follows:



Remaining details of (2) are left to the reader.

Finally, we observe that $\otimes$ is symmetric monoidal: the associativity, symmetry, and unit constraints may be defined as those induced from constraints for $(C, \times)$, via a functor $C \to \mathcal{R}el(C,T)$ given below (their naturality will be left as an exercise). q. e. d.

Theorem 4 *$\mathcal{R}el(C,T)$ is a compact symmetric monoidal 2-category.*

Proof: By the preceding results, $\mathcal{R}el(C,T)$ is a symmetric monoidal category enriched in the category of posets. The adjunction which expresses compactness,

$$\hom(X \otimes Y, Z) \cong \hom(X, (Y \otimes Z)),$$

follows immediately from $M((X \times Y) \times Z) \cong M(X \times (Y \times Z))$. q. e. d.

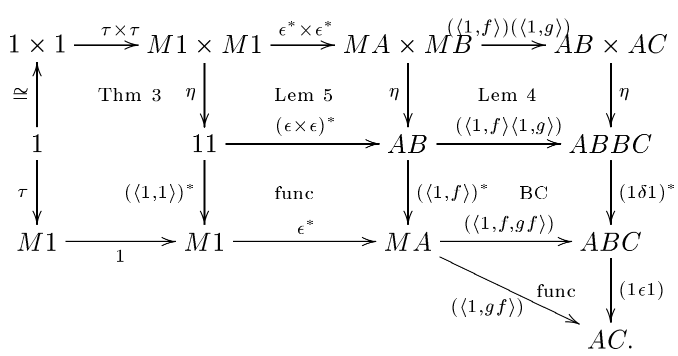
Next, we show how to retrieve the theory (C,T) from $\mathcal{R}el(C,T)$. Define functors $C \to \mathcal{R}el(C,T)$ and from $C^{op} \to \mathcal{R}el(C,T)$, both acting as identities on objects, and which send a morphism $f : A \to B$ to the value 1 under the composite given respectively as:

$$1 \overset{\tau}{\to} M1 \overset{\epsilon^*}{\to} MA \overset{M(\langle 1,f\rangle)}{\to} M(A \times B), \qquad 1 \overset{\tau}{\to} M1 \overset{\epsilon^*}{\to} MA \overset{M(\langle f,1\rangle)}{\to} M(B \times A).$$

It is trivial that each of these morphism assignments preserves identities.

Lemma 9 *$C \to \mathcal{R}el(C,T)$ and $C^{op} \to \mathcal{R}el(C,T)$ are functorial.*

Proof: We show $C \to \mathcal{R}el(C,T)$ preserves compositions in the following diagram; the case for $C^{op} \to \mathcal{R}el(C,T)$ is similar:



q. e. d.

It is now clear that both $C \to \mathcal{R}el(C,T)$ and $C^{op} \to \mathcal{R}el(C,T)$ are strict monoidal functors, in view of the observation preceding Theorem 4.

Let $\mathcal{F}orm$ denote the set of 1-cells of $\mathcal{R}el(C,T)$ which are of the form $A \to 1$. By definition of $\mathcal{R}el(C,T)$, there is a natural bijection

$$\mathcal{F}orm \cong \sum_{A \in C_0} TA.$$

Let $C_1 \times_{C_0} \mathcal{F}orm$ denote the pullback of the domain or typing function $\mathcal{F}orm \to C_0$ along the codomain function $C_1 \to C_0$. The functor $C \to \mathcal{R}el(C,T)$ induces a map

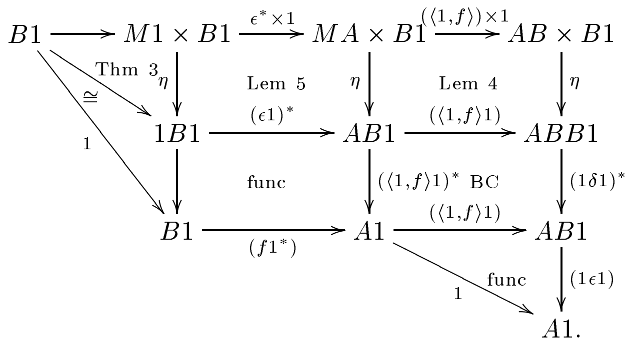
$$C_1 \times_{C_0} \mathcal{F}orm \to \mathcal{R}el(C,T)_1 \times_{C_0} \mathcal{F}orm,$$

which one may compose with $\mathcal{R}el(C,T)_1 \times_{C_0} \mathcal{F}orm \stackrel{\text{comp}}{\to} \mathcal{F}orm$, where comp is defined by composition in $\mathcal{R}el(C,T)$. The result is a map

$$C_1 \times_{C_0} \mathcal{F}orm \stackrel{P}{\to} \mathcal{F}orm.$$

Lemma 10 $P(A \stackrel{f}{\to} B, B \stackrel{p}{\to} 1) = A \stackrel{f^*p}{\to} 1.$

Proof:



q. e. d.

A similar construction results by pulling back $(C^{op})_1 \stackrel{\text{cod}}{\to} C_0$ along $\mathcal{F}orm \to C_0$ and forming the composite Q of

$$(C^{op})_1 \times_{C_0} \mathcal{F}orm \to \mathcal{R}el(C,T)_1 \times_{C_0} \mathcal{F}orm \stackrel{\text{comp}}{\to} \mathcal{F}orm,$$

where the first map is induced from $C^{op} \to \mathcal{R}el(C,T)$. If $A \stackrel{f^{op}}{\to} B$ denotes a morphism in C^{op}, then by imitating the proof of Lemma 10, one may show that $Q(B \stackrel{f}{\to} A, B \stackrel{q}{\to} 1) = A \stackrel{\exists_f g}{\to} 1$. In this way, the structure of a theory (C,T) is completely retrieved from the structure of the monoidal 2-category $\mathcal{R}el(C,T)$:

Theorem 5 *The functors $C^{op} \stackrel{T}{\to} \mathcal{B}ool$ and $C \stackrel{\exists}{\to} \mathcal{B}ool$ of a theory, viewed as fibrations, are isomorphic to the (split) fibrations (Form $\to C_0, P$), (Form $\to C_0, Q$).*

6.2 Soundness and Completeness

Let $\beta(L)$ denote the monoidal category of first-order diagrams modulo deformation equivalence. There is a reflexive transitive relation $\sim_\beta$ on $\beta(L)$, generated by surgery rules; let $\beta(L)/\sim_\beta$ denote the poset-enriched category where $\hom(u, v)$ carries the poset structure induced from $\sim_\beta$. The goal of this subsection is to prove

Theorem 6 *There is an isomorphism of (compact, monoidal) poset-enriched categories*

$$\beta(L)/\sim_\beta \xrightarrow{\cong} \mathcal{R}el(\Pi(S), \theta),$$

where $(\Pi(S), \theta : \Pi(S)^{op} \to \mathcal{B}ool)$ *is the free (categorical predicate) theory on* L.

It is in this sense that Peirce's Beta is isomorphic to first-order relational calculus.

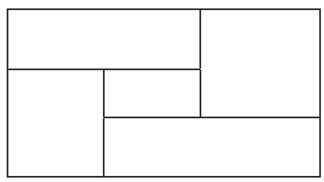
The proof proceeds as follows. First, we construct a functor $q : \beta(L) \to \mathcal{R}el(\Pi(S), \theta)$ which is the identity on objects and which is defined on morphisms by induction on the *rank* of representative first-order diagrams, where the rank is defined as the number of sep lines and interior and crossing nodes. We show that the functor q respects deformation equivalence; then we show that q respects the surgery relations on β, and then show that q is a surjection of 2-categories. The theorem follows easily from there.

Let D be a first-order diagram. To define $q(D)$, the idea is to partition D into first-order subdiagrams: to *tile* the rectangle R in which D is immersed into subrectangles, in such a way that D can be obtained through a succession of horizontal and vertical compositions. If each subdiagram has a lesser rank than D, then by induction the value of q on each subdiagram will have been defined, and we define $q(D)$ as the corresponding iterated composite in the monoidal category $\mathcal{R}el(\Pi(S), \theta)$.

The only trouble is that not all tilings can be sensibly interpreted as composites. Recall that two subdiagrams are (horizontally or vertically) composed by "erasing" an edge they have in common (given by a dotted line, as in Lemma 1).

Definition 13 *(see [7]): A tiling of a rectangle is composable if repeatedly replacing two tiles with a common edge by a single tile which is their union can eventually reduce the tiling to a single tile.*

The basic instance of an uncomposable tiling is the "pinwheel,"



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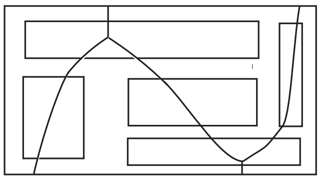
and we will take advantage of deformation equivalence to ensure that this will not occur.

For example, consider the first-order diagram D given by



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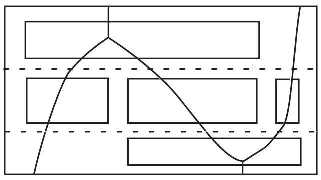
where the solid rectangular lines are sep lines. In order to evaluate $q(D)$, we deform two of the sep lines to obtain



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whereupon it becomes possible to tile D into composable subdiagrams as shown by the dotted lines.

Suppose D is a first-order diagram which is not surrounded by a sep line, up to deformation equivalence. [If this condition is not met, we can define $q(D)$ as $\neg q(D')$, where D' is obtained by removing the outermost sep line of D.] A sep line of D is *maximal* if it is not interior to any other sep line. Since D is not deformable to a diagram surrounded by a single sep line, one of the following cases holds:

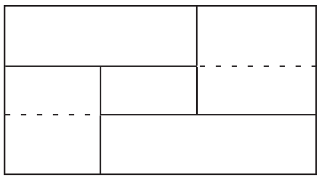
- D has two or more maximal sep lines;
- D has one maximal sep line and nodes exterior to that sep line;
- D has no sep lines.

In the latter two cases, it is relatively trivial to decompose D into a composable tiling, i.e., a decomposition into subdiagrams of lesser rank than D. In the first case: since the sep lines are disjoint, D can be tiled so that each tile defines a first-order subdiagram and contains at most one maximal sep line (which we assume to be in the tile's interior), and we just need to ensure that this tiling is composable. This necessitates the following abstract considerations on tilings, given in [7].

Given a tiling of a rectangle, we define partial orders $\leq_1$ and $\leq_2$ on the set of tiles $A, B, \ldots$, where $A \leq_1 B$ if the right edge of A meets the left edge of B in more than one point, and $A \leq_2 B$ if the top edge of A meets the bottom edge of B in more than one point. (The partial orders $\leq_1$ and $\leq_2$ are the reflexive transitive closures of these two relations.) Tilings of rectangles may be abstractly characterized in terms of such double orders (called tileorders), so that, for example, the pinwheel tiling given above may be referred to in terms of its tileorder.

Theorem 7 *(see [7]) A tileorder fails to be composable if and only if every sequence of compositions eventually yields a tileorder containing a pinwheel as a sub-double-order.*

If a pinwheel (arising from a tiling of a first-order diagram, as described above) is reached, we simply decompose it further, as we indicated earlier:



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The only obstruction occurs when one of the dotted lines of the decomposition passes through a (maximal) sep line of a tile. All one needs to do here is to apply an isotopy to the tile acting as the identity on a neighborhood of the tile's boundary, and which shrinks the sep line and the rectangle it surrounds into a smaller subrectangle. Then, without loss of generality, we may assume that the sep line lies above or below the dotted line, so that the decomposition can be carried out.

Now we define $q : \beta(L) \to \mathcal{R}el(\Pi(S), \theta)$ by induction, by tiling a (deformation equivalence class of a) first-order diagram D into subdiagrams D' of lesser rank in the manner given above, and composing their values $q(D')$ in the monoidal category $\mathcal{R}el(\Pi(S), \theta)$. If D has rank less than 2, then without loss of generality, D is either one of the primitive diagrams given in the discussion of §3.1 after definition 9, or the second diagram of Example 4. In the former case, D could be a single string or could have a single crossing node, where $q(D)$ is an identity or a symmetry isomorphism, or could have a single node labeled δ or ϵ, where $q(D)$ is the image of a diagonal or projection map under $\Pi(S) \to \mathcal{R}el(\Pi(S), \theta)$, or could have a node labeled δ^{op} or ϵ^{op}, where $q(D)$ is the image of $\Pi(S)^{op} \to \mathcal{R}el(\Pi(S), \theta)$, or could have a node labeled $p \in P$ in the predicate language, where $q(D)$ is the evident morphism $\langle \tau(p), 1, p \in \theta(\tau(p) \times 1) \rangle$ in $\mathcal{R}el(\Pi(S), \theta)$. In the case where D consists only of a sep line surrounding strings, $q(D)$ is $\langle A, A, \neg E_A \in \theta(A \times A) \rangle$, where E_A is the equality predicate:

$$1 \xrightarrow{\tau} \theta 1 \xrightarrow{\epsilon^*} \theta A \xrightarrow{\exists_\delta} \theta(A \times A),$$

and A is the source/target of D.

Lemma 11 *The functor $q : \beta(L) \to \mathcal{R}el(\Pi(S), \theta)$ is well-defined.*

Proof: The well-definedness of q may be analyzed into three distinct components: (1) the case where, during a deformation $t \mapsto D_t$, a node of D_t crosses a sep line of D_t; (2) the case of the deformation where two nodes interchange their relative heights. We must also show (3) that $q(D)$ is independent of the tiling of D. In cases (1) and (2), we must show that $q(D_t)$ is independent of t. But cases (2) and (3) are essentially consequences of the theory of [10] (plus the fact that $\mathcal{R}el(\Pi(S), \theta)$ is a symmetric monoidal category). As for case (1), we may consider separately the case (1a) where a node crosses the top of a sep line; the argument for this case essentially follows from Lemma 10 plus the fact that f^*, for f a morphism of $\Pi(S)$, preserves negation. Case (1b), where a node crosses the bottom of a sep line, is dual (here we invoke the fact that $A \otimes -$ is adjoint to itself, as in Theorem 4, so that $\mathcal{R}el(\Pi(S), \theta)$ is self-dual). Thus, q is well-defined. q. e. d.

Next, we must verify that q induces a well-defined map

$$\beta(L)/\sim_\beta \to \mathcal{R}el(\Pi(S), \theta)$$

by showing that if $f \rightsquigarrow_\beta g$, then $q(f) \leq q(g)$. Now the partial order $\rightsquigarrow_\beta$ is generated from surgery rules (S1), (S2), (S3) [taking into account the sep-parity convention] and the rules of Alpha. But q respects (S1) by Theorem 5 and construction of $\Pi(S)$ (Theorem 1), and similarly (S2) and (S3) by Theorem 5 and conditions (1) and (2) of Definition 3. The fact that the rules of Alpha are respected by q is covered under the Soundness Theorem of [4]: the hardest rule to verify is iteration, but this essentially follows from recursive application of the Boolean equations

$$\begin{aligned} f \wedge \neg(f \wedge g) &= f \wedge \neg g \\ f \wedge (g \wedge h) &= f \wedge (f \wedge g \wedge h). \end{aligned}$$

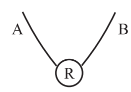
Details are left to the reader. Thus we have proved

Lemma 12 *(Soundness of Beta): The map $\tilde{q} : \beta(L)/\sim_\beta \to \mathcal{R}el(\Pi(S), \theta)$ induced from q is well-defined.*

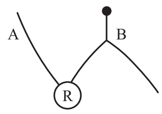
Lemma 13 *(Completeness of Beta): The poset-enriched functor $\tilde{q} : \beta(L)/\sim_\beta \to \mathcal{R}el(\Pi(S), \theta)$ is surjective on 1-cells and on 2-cells.*

Proof: Surjectivity on the level of 1-cells may be decomposed into two parts: (a) showing that relations of the form $A \to 1$ in $\mathcal{R}el(\Pi(S), \theta)$ are in the image of q; (b) all relations of the form $A \to B$ in $\mathcal{R}el(\Pi(S), \theta)$ are in the image of q.

Statement (b) follows easily from statement (a): from (a), we have that every relation of the form $\langle A \times B, 1, R \in T(A \times B) \rangle$ is the image of a corresponding beta diagram of the form



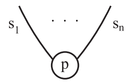
under q. The corresponding relation $\langle A, B, R \in T(A \times B)\rangle$ is then the image of



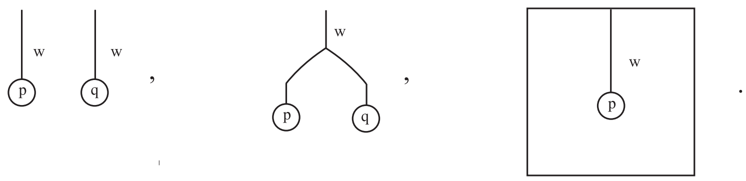
Here implicit use is made of compactness of $\mathcal{R}el(\Pi(S), \theta)$ (Theorem 4) and of $\beta(L)/\sim_\beta$ (see Lemma 7).

As for statement (a): Relations of the form $A \to 1$ correspond bijectively to elements in $\Sigma_{w\in S^*}\theta(w)$ by Theorem 5. Formulas, i.e., elements in $\Sigma_{w\in S^*}\theta(w)$, are formed by applying rules (i)–(iv), which are accompanied by the first-order diagrams whose image under q is the given formula:

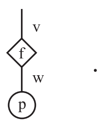
(i) $p \in \Sigma_{w\in S^*}\lambda(w)$ is an element of $\Sigma_{w\in S^*}\theta(w)$ (image of



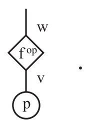
(ii) If $p, q \in \theta(w)$, then $p \wedge q \in \theta(w)$, $\neg p \in \theta(w)$ (image of



(iii) If $p \in \theta(w)$ and $V \xrightarrow{f} W$ in $\Pi(S)$, then $f^*p \in \theta(v)$ (image of



(iv) If $p \in \theta(v)$ and $V \xrightarrow{f} W$ in $\Pi(S)$, then $\exists_f p \in \theta(w)$ (image of



This completes the proof of statement (a), so that q is surjective on 1-cells.

Surjectivity on the level of 2-cells is proved by showing that every instance of the relation $p \leq q$ between 1-cells in $\mathcal{R}el(\Pi(S), \theta)$ follows from a relation $f \rightsquigarrow_\beta g$ in $\beta(L)/\sim_\beta$. It suffices to check the case where $p, q : A \to 1$, i.e., where p and q are formulas of the free theory FL (cf. the reduction of statement (b) by statement (a) earlier in this proof). But by freeness, instances of $p \leq q$ here follow purely from the axioms of theories and equations of $\Pi(S)$. As we saw in §3.3, the equations of $\Pi(S)$ are covered by (S1), the adjunctions $\exists_f \dashv f^*$ by (S2), Beck-Chevalley by (S3), and the Boolean algebra axioms are covered by the discussion at the end of §4.2. Thus the surjectivity at the 2-cell level is clear. q. e. d.

Proof of Theorem 6: Let $\rightsquigarrow_\beta \subseteq \beta(L)_1 \times \beta(L)_1$ and $\rightsquigarrow_\theta \subseteq \beta(L)_1 \times \beta(L)_1$ denote the reflexive and transitive relations on the 1-cells of $\beta(L)$ induced by pulling back the 2-cell relations $\leq$ in $\beta(L)/\sim_\beta$ and in $\mathcal{R}el(\Pi(S), \theta)$ along the respective quotient maps $\beta(L) \to \beta(L)/\sim_\beta$ and $\beta(L) \xrightarrow{q} \mathcal{R}el(\Pi(S), \theta)$. It suffices to show $\rightsquigarrow_\beta = \rightsquigarrow_\theta$. That $\rightsquigarrow_\beta \subseteq \rightsquigarrow_\theta$ follows from Lemma 12. That $\rightsquigarrow_\theta \subseteq \rightsquigarrow_\beta$ follows from Lemma 13. The proof is complete. q. e. d.

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